Dynamics of Pseudo-Automorphisms of Blow-ups of $\mathbb{P}^k$

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Abstract

We explore the project of finding complex manifolds that can carry (pseudo)-automorphisms which have interesting dynamical behavior. We will discuss possible meanings of "interesting".

Rational surfaces

A compact, complex algebraic surface X is said to be *rational* if it is birationally equivalent to $\mathbb{CP}^2$. We start by giving the classical example of the Cremona Involution, which on $\mathbb{C}^2$ is given by $(x, y) \mapsto (1/x, 1/y)$, and on projective space we write it as a mapping of degree 2:

$$J : [x_0 : x_1 : x_2] \mapsto [1/x_0 : 1/x_1 : 1/x_2] = [x_1x_2 : x_0x_2 : x_0x_1]$$

This acts as an inversion in the coordinate triangle:

indeterminate point $e_0 = [1 : 0 : 0] \leftrightarrow \Sigma_0 = \{x_0 = 0\}$ exceptional curve

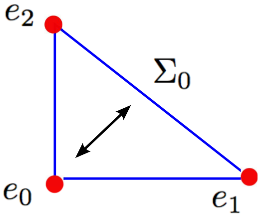


Figure: Inversion in coordinate triangle

The blowup of the point $e_0 = [1:0:0]$ is a new manifold X with local coordinates (ξ_1, ξ_2) and projection

$$\pi : X \to \mathbb{P}^2, \qquad \pi(\xi_1, \xi_2) = [1 : \xi_1 : \xi_1\xi_2] = [x_0 : x_1 : x_2]$$

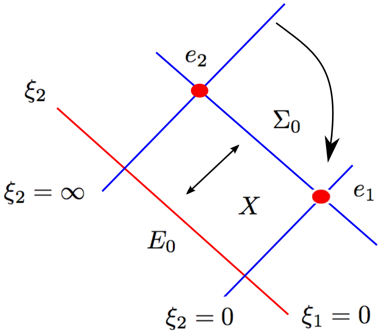


Figure: Indeterminate point e_0 replaced by exceptional curve E_0. There are still 2 indeterminate points and two exceptional curves, but for the induced $J_X : E_0 \leftrightarrow \Sigma_0$, Σ_0 is no longer exceptional.

Blow up e_0, e_1, e_2, and J becomes an automorphism

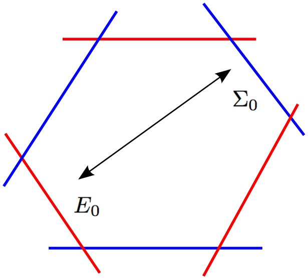


Figure: All indeterminate points e_0, e_1, and e_2 are replaced by exceptional curves E_0, E_1, and E_2. The map is now an automorphism, and it interchanges $E_j \leftrightarrow \Sigma_j$.

Dynamical complexity measured by *dynamical degree*

If X is a manifold of dimension k, and if $1 \leq \ell \leq k$, then the dynamical degree in codimension ℓ is the exponential rate of growth of the induced map f^* on $H^{\ell,\ell}(X)$:

$$\delta_\ell(f) := \lim_{n\to\infty} ||f^{n*} : H^{\ell,\ell}(X) \to H^{\ell,\ell}(X)||^{1/n}$$

This is independent of the choice of norm on $H^{\ell,\ell}$. For an automorphism, we have $(f^n)^* = (f^*)^n$, so δ_ℓ is the same as the spectral radius of f^* acting on $H^{\ell,\ell}(X)$.

This definition also works if f is merely rational.

Theorem (J-L Lin, Favre & Wulcan)

Let $f_A(x) = x^A$ be a monomial map. Then for each $\ell \geq 1$, $\delta_\ell(f_A)$ is the spectral radius of the ℓ-th exterior power $\wedge^\ell A$ of A.

This is nontrivial.

Problem: Determine $\delta_\ell(f)$ for $\ell > 1$ for other nontrivial f.

Heuristic: Dynamical complexity vs degree complexity (dynamical degree) in dimension 1

If $f : \mathbb{P}^1 \to \mathbb{P}^1$ is rational, then degree$(f) = \delta(f) = \deg_{top}(f)$. If $\delta(f) > 1$ (and thus f is not invertible), then for almost all starting points z_0 there is a limiting distribution of point masses over the preimages of z_0:

$$\mu_f := \lim_{n\to\infty} \frac{1}{d^n} \sum_{\{a : f^n(a) = z_0\}} \delta_a$$

This measure is *balanced*, which means that, locally, $f^*\mu_f = d \cdot \mu_f$. Thus, as we consider backward iteration, the different branches of f^{-1} give $\mu_f(S_j) = \mu_f(S)/d$. The effect is like Bernoulli trials:

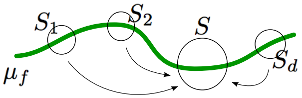


Figure: Choosing preimages of a point is like flipping a d-sided coin.

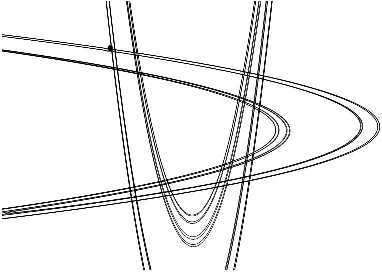
Theorem (Stable Manifold)

If x_0 is a saddle fixed point, the stable set

$$W^s(x_0) := \{x : \lim_{n\to\infty} dist(f^n x, x_0) = 0\}$$

is a manifold.

$W^s(x_0)$ is "almost never" proper, in which case the current of integration $[W^s(x_0)]$ is not well-defined. Stable/unstable manifolds for the Horseshoe Map $f_{c,a}(x,y) = (c + ay - x^2, -x)$ with $c = 6.0$, $a = 0.8$



Dynamics of the Horseshoe Map

- All points outside $\overline{W^s(x_0)} \cup \overline{W^u(x_0)}$ escape to infinity.
- $\overline{W^s(x_0)} \cap \overline{W^u(x_0)} \cong$ Cantor set $\times$ Cantor set
- Saddle (periodic) points are dense in Cantor set $\times$ Cantor set
- Dynamics on Cantor set $\times$ Cantor set is conjugate to $\{0,1\}^{\mathbb{Z}}$
- The closure $\overline{W^s(z_0)}$ is the same for all saddles z_0.

Heuristic: Dynamical complexity in terms of stable current in dimension 2

If f is an automorphism of a complex surface with $\delta_1(f) > 1$, then there is an invariant current T^s which satisfies $f^*T^s = \delta_1 T^s$. This is constructed from the currents of integration $[W^s_{\mathrm{loc}}]$ over the (local) stable manifolds W^s, and it is "measured" by a family of transversal measures.

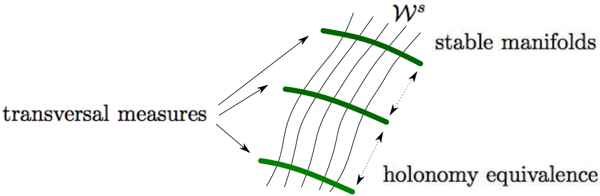


Figure: Mapping by f^* acts on the family of transversal measures like Bernoulli trials in 1-dimensional case.

Real example: illustrates connection between dynamical degree and length growth

Example with $\delta = 1.17628$, showing image of a line after 10 iterations. Green curve is an invariant cubic. Red points are fixed, which indicates some numerical instability. Numbered points are the orbits of the orbit of the exceptional curve Σ_γ. The real points of the manifold X make a covering of $\mathbb{RP}^2$, and fibers (copies of $\mathbb{P}^1$) lie above each labeled point.

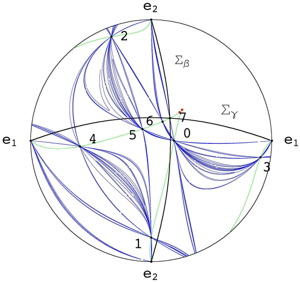


Figure: $f(x, y) = \dfrac{y + a}{x + b}$, $a = -0.499497$, $b = -0.415761$

The family of maps $f_{a,b}(x,y) = \dfrac{y+a}{x+b}$ may be conjugated to a map $L \circ J : \mathbb{P}^2 \to \mathbb{P}^2$. The triangle of exceptional curves is mapped as:

$$\Sigma_\beta \to e_2 \dashrightarrow \Sigma_0 \to e_1 \dashrightarrow \Sigma_\gamma \to Q$$

In this case, we may blow up e_2 and e_1 and obtain a new space Y such that f_Y has one point exceptional curve Σ_γ and one point of indeterminacy P. If the orbit of Q lands on P, then we may blow up that orbit to obtain an automorphism.

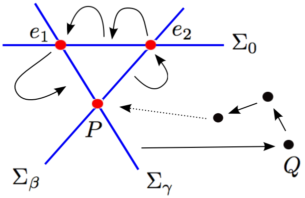
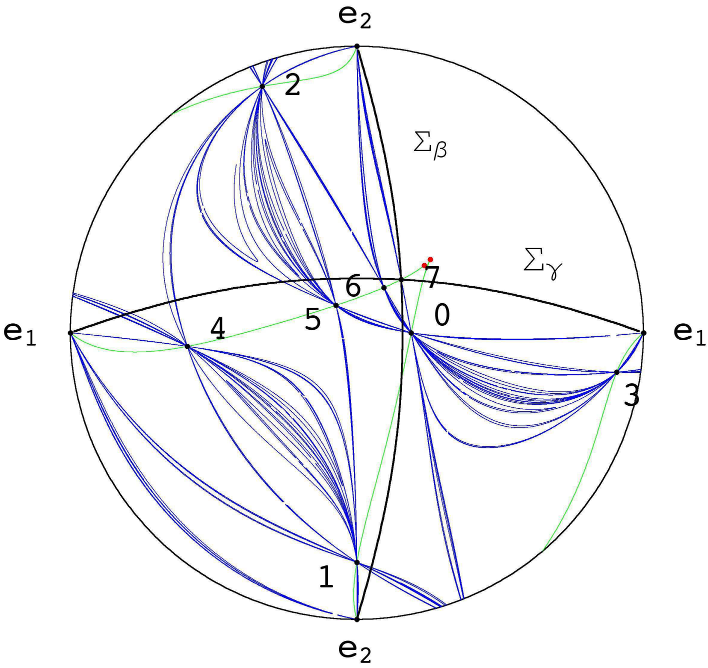


Figure: If the orbit of Q (black) lands on P, we may blow up the points of the orbit to obtain an automorphism.



e_2
2
Σ_β
Σ_γ
6
7
5
0
e_1
4
e_1
3
1
e_2

What are the possible compact surfaces X which have carry an $f \in Aut(X)$ with $\delta_1(f) > 1$?

Theorem (Cantat)

If X is a compact complex surface of (complex) dimension 2, and if there is an automorphism f of X with $\delta(f) > 1$, then one of the following holds:

- *X is a torus, and f is a "standard" torus automorphism.*
- *X is a $K3$ surface, or a finite quotient*
- *X is a rational surface*

The proof relies heavily on the Kodaira classification of surfaces.

Question: What are the possibilities for surfaces and maps can actually occur for $K3$ or rational surfaces?

This question is quite open. The $K3$ surfaces or rational surfaces which can carry nontrivial automorphisms are quite special and hard to find. The set of all $K3$ surfaces has dimension 20, and the ones that carry automorphisms would have smaller dimension. On the other hand, rational surfaces with automorphisms with $\delta_1 > 1$ can occur in families of arbitrarily large dimension.

Theorem (Nagata)

In the case of a rational surface, we may suppose that $\pi : X \to \mathbb{P}^2$ is an (iterated) blowup of the projective plane.

For such an iterated blowup, we may represent the cohomology $H^2 \cong H^{1,1}$ with respect to convenient bases, and this is useful for computing f^*.

Question: What happens in higher dimension?

Given that the case of blowups of $\mathbb{P}^2$ yield interesting automorphisms in dimension 2, it makes sense to ask:

Question: Are there 3-folds X which are obtained as blowups of $\mathbb{P}^3$ and which carry automorphisms f with $\delta(f) > 1$?

Theorem (T.T. Truong)

If X is obtained from $\mathbb{P}^3$ by blowing up points and curves satisfying a certain condition, and if f is an automorphism of X, then $\delta_1(f) = \delta_2(f)$.

Theorem (Bayraktar, Cantat)

If X is obtained from $\mathbb{P}^k$ by blowing up points, then any automorphism f of X satisfies $\delta_\ell(f) = 1$ for all ℓ.

Back to the Cremona involution; now dimension 3.

The Cremona involution on $\mathbb{P}^3$ is the cubic map given by

$J : [x_0 : x_1 : x_2 : x_3] \mapsto$

$$[1/x_0 : 1/x_1 : 1/x_2 : 1/x_3] = [x_1x_2x_3 : x_0x_2x_3 : x_0x_1x_3 : x_0x_1x_2]$$

This acts as an involution on the coordinate tetrahedron $e_j \leftrightarrow \Sigma_j$. We now see a new phenomenon: the *flip*, which takes any point of an edge of the tetrahedron and blows it up to the skew edge.

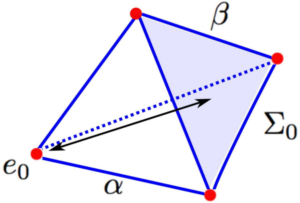


Figure: Cremona involution blows up any point of edge α to all of edge β.

Blow up vertex e_0

Let $\pi : X \to \mathbb{P}^3$ be the blowup at e_0, and let $J_X : X \to X$ be the induced map. Then J_X maps $E_0 \leftrightarrow \Sigma_0$ and "looks like" the 2D map J mapping $\mathbb{P}^2$ to itself: the black triangle inside E_0 is exceptional, and the "old edges of the tetrahedron" (in green) are still indeterminate.

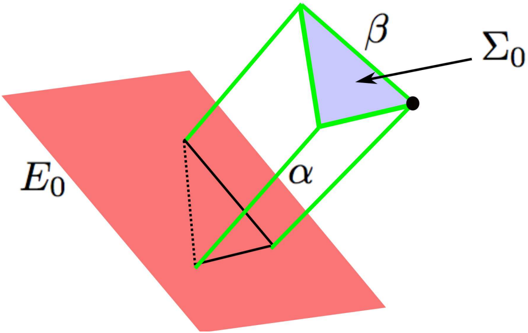


Figure: Cremona involution blows up any point of edge α to all of edge β.

The indeterminacy locus $\mathcal{I}(f)$ of any rational map $f : X \dashrightarrow Y$ has codimension at least 2. Thus if H is any hyper surface, we may define the image (proper transform) of H as the closure of $f(H - \mathcal{I}(f))$. We say that H is *exceptional* if the codimension of the proper transform of H is ≥ 2. A birational $f : X \dashrightarrow X$ is a *pseudo-automorphism* if neither f nor f^{-1} has an exceptional hyper surface.

In dimension 2, all pseudo-automorphisms are in fact automorphisms.

Pseudo-automorphisms behave very much in the spirit of automorphisms, and we expand our search to include this richer source of interesting maps.

Let $\pi : X \to \mathbb{P}^3$ be the blowup of e_0, e_1, e_2, e_3. Then the induced map J_X is still indeterminate at the "old edges of the tetrahedron" but is now a pseudo-automorphism.

We next explore the problem of finding pseudo-automorphisms of blowups of $\mathbb{P}^3$. Motivated by the 2-dimensional case, we start by looking at maps of the form $L \circ J$ (linear composed with Cremona involution).

Strategy for finding pseudo-automorphisms

The exceptional locus consists of the hyperplanes Σ_j, which are mapped:

$$f := L \circ J \;:\; \Sigma_j \to L_j$$

where L_j denotes the jth column of the matrix L. We will have a pseudo-automorphism if

$$L_j \mapsto f(L_j) \mapsto \cdots \mapsto f^{m_j}(L_j) = e_{\sigma_j}, \quad f^{\ell}(L_j) \notin \bigcup_i \Sigma_i$$

where σ is a permutation of $\{0, \dots, k\}$. Let $\pi : X \to \mathbb{P}^k$ blow up the orbits of the L_j. The induced map $f_X := \pi^{-1} \circ f \circ \pi$ will have no exceptional locus.

Theorem (B-Kyounghee Kim)

The $\delta_1(L \circ J) = \delta_1(m_1, \dots, m_k, \sigma)$ is given by an explicit formula involving the orbit lengths m_j and the permutation σ.

Since $\ell \mapsto \delta_\ell$ is concave in ℓ, it follows that $\delta_1 = 1$ if and only if $\delta_\ell = 1$ for any (equivalently, all) $1 \le \ell \le k$.

Pseudo-automorphisms with invariant curves

As a practical matter, however, the strategy on the previous page is not a feasible for giving existence because it involves solving equations of very high degree in many variables. The relevant computations are possible, however, if we assume that all the centers of blowup like in a curve (which must be invariant). The existence of automorphisms of blowups of $\mathbb{P}^2$ with invariant curves was studied by McMullen, Diller and Uehara.

Let us consider a parametrized curve $\psi : \mathbb{C} \to \mathcal{C} \subset \mathbb{P}^k$. We say that $\mathcal{C}$ *satisfies a group law* if the following holds: For each hyperplane $H \subset \mathbb{P}^k$, the solutions $t_1, \ldots, t_d$ of $\psi(t_i) \in H$ satisfy $\sum t_i = 0$. There are several cases of curves with group law; all the curves we work with have degree $k+1$. For instance:
$t \mapsto \psi(t) = [1 : t : t^2 : \cdots : t^{k-1} : t^{k+1}]$ cusp at ∞
$t \mapsto \psi(t) = [1 : t : \cdots : t^k] \cup [0 : \cdots : 0 : (-1)^{k-1} : t]$ union of rational normal curve and a tangent line.

The group law lets us find the points of blowup

If there is an invariant curve, then the points which will be blown up are of the form $\psi(t_j)$. Thus this becomes a problem of determining the points $\{t_i\} \subset \mathbb{C}$. Using the group law on the curve, we can obtain:

Theorem (B-Diller-Kyounghee Kim)

For most choices of orbit lengths $(m_0, \ldots, m_k)$ and permutations σ, there is a matrix L such that the space $\pi : X \to \mathbb{P}^k$ obtained by blowing up the orbits yields an induced pseudo-automorphism $f_X : X \dashrightarrow X$

See:

Pseudoautomorphisms with invariant elliptic curves, arXiv:1401.2386

Concrete existence of pseudo-automorphisms

The abstract existence of pseudo-automorphisms in the case where the orbit lengths are $(1, \ldots, 1, n)$, and the permutation is $\sigma = (0\,1\,2\,\ldots k)$ had been given earlier by **F. Perroni** and **D-Q Zhang**. The motivation for the Theorem on the previous slide was to replace “abstract” existence by “concrete” existence. We find, for instance $f := L \circ J$ in the case of the curve with a cusp, such maps are of the form:

$$L = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 & 1 \\ \beta_1 & 0 & 0 & \ldots & 0 & 1-\beta_1 \\ 0 & \beta_2 & 0 & \ldots & 0 & 1-\beta_2 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & \ldots & \beta_k & 1-\beta_k \end{pmatrix}$$

where $\beta_i = (\delta^i - 1)/(\delta(\delta^{k+1} - \delta^i))$, and δ is any root of the polynomial χ_n which gives the dynamical degree of $L \circ J$.

Different choices of invariant curve lead to different matrices L.

In dimension k: $\delta_0 = 1$, and δ_k is the topological degree. Thus for invertible maps, $\delta_k = 1$.

f is said to be *cohomologically hyperbolic* if there is a unique $1 \leq p \leq k-1$ such that $\delta_p(f)$ is maximal. In this case, the maximal growth occurs uniquely in bidegree (p, p) (codimension p).

Theorem (Dinh-Sibony)

If f is a cohomologically hyperbolic automorphism, then there are invariant currents $T^{s/u}$, and these may be used to form an invariant measure μ with interesting dynamical properties.

Problem: How many of the details of the heuristic picture for dimension 2 be carried over to higher dimension?

Let $f : X \dashrightarrow X$ be a meromorphic map. Suppose that there is a dominant, meromorphic map $\phi : X \dashrightarrow Y$ and a meromorphic map $g : Y \dashrightarrow Y$ such that $g \circ \phi = \phi \circ f$. Then the sets $\{\phi = \text{const}\}$ form an *invariant fibration.*
In the presence of an invariant fibration, there is a *dynamical degree on the fiber*, written $\delta_j(f|\phi)$, and it is related to the other dynamical degrees by:

Theorem (Dinh-Nguyen, Dinh-Nguyen-Truong)

Suppose that the map f has an invariant fibration as above. Then

$$\delta_p(f) = \max_{\max\{0,p-k+\ell\} \leq j \leq \min\{p,\ell\}} \delta_j(g)\delta_{p-j}(f|\phi)$$

Corollary

If X is a 3-fold, and $f : X \dashrightarrow X$ is a birational map with an invariant fibration, then $\delta_1 = \delta_2$. In this case, f is not cohomologically hyperbolic.

If $\dim(X) = 3$, and if $f : X \dashrightarrow X$ is birational, then the condition that f is not cohomologically hyperbolic is equivalent to the condition that $\delta_1(f) = \delta_2(f)$.

Guedj conjectured: *If $\delta_1(f) = \delta_2(f) > 1$, then f has an invariant fibration, or at least an invariant foliation.*

In dimension 2, any invertible map is cohomologically hyperbolic $\Leftrightarrow \delta > 1$. The Guedj Conjecture is true in this case because of:

Theorem (Diller-Favre)

If f is a bimeromorphic map of a compact, complex surface X, and if $\delta(f) > 1$ if and only if there is no invariant fibration.

Problem for dimension 3: *If f is a map which is not cohomologically hyperbolic, is there a heuristic picture for approaching the dynamics?*

An invariant fibration or foliation might let the dynamics be approached with lower-dimensional techniques.

Counterexample to Guedj's conjecture

For $n \geq 2$ we choose nonzero $a, c \in \mathbb{C}$ such that

$$na^2 + (n+1)ac + nc^2 = 0$$

and set

$$L = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & c \end{pmatrix}$$

and recall

$$J[x_0 : \cdots : x_3] = [x_0^{-1} : \cdots : x_3^{-1}]$$

Theorem (B-Cantat-Kyounghee Kim)

For $n \geq 2$, we set $f_{a,c} := L \circ J$. The dynamical degrees are $\delta_1(f) = \delta_2(f) > 1$. There is no (singular) foliation of dimension 1 or 2 which is invariant under $f_{a,c}$. In particular, there is no invariant (singular) fibration.

$$f(\Sigma_0) = e_1 := [0:1:0:0], \quad f(\Sigma_1) = e_2, \quad f(\Sigma_2) = e_3$$
$$f(\Sigma_3) = p := [1:a:0:c]$$

Theorem

Let Y denote $\mathbb{P}^3$ blown up at the points e_1 and e_3. Then the induced map f_Y is a dominant map of an invariant 4-cycle of surfaces:

$$\Sigma_0 \to E_1 \to \Sigma_2 \to E_3 \to \Sigma_0$$

The orbit of the exceptional image point q is inside this invariant set.

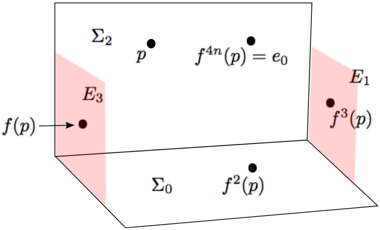


Figure: Construction of Y.

Theorem

If $na^2 + (n+1)ac + nc^2 = 0$, then the f_Y orbit of q lands on the point e_0. Let X denote the space obtained by blowing up the $4n+2$ points q, $f_Y(q)$, $\ldots$, $f_Y^{4n}q = e_0$, and e_3. Then the induced map f_X is a pseudo-automorphism.

One difference between these maps and the B-Diller-Kim maps is that that there are two "levels" of blowup, which is to say that it is not obtained by blowing distinct points in $\mathbb{P}^3$. Another difference is that these were not constructed using a group law. In fact:

Theorem

All of these surfaces are birationally equivalent to $\mathbb{P}^3$, but the maps (pairs) $(f_{a,c}, X_{a,c})$ are birationally inequivalent, and inequivalent to the maps obtained in the BDK construction

Theorem

Let $g := f^4|_{\Sigma_0}$. Then the dynamical degree of $\delta_1(g) > 1$, but it is not the dynamical degree is not a Salem number. Thus g is not birationally conjugate to a surface automorphism.

Using this and the fact that $\delta_1(f^4) > \delta_1(g) > 1$, we may show:

Theorem

f is not birationally conjugate to an automorphism of a 3-dimensional manifold.

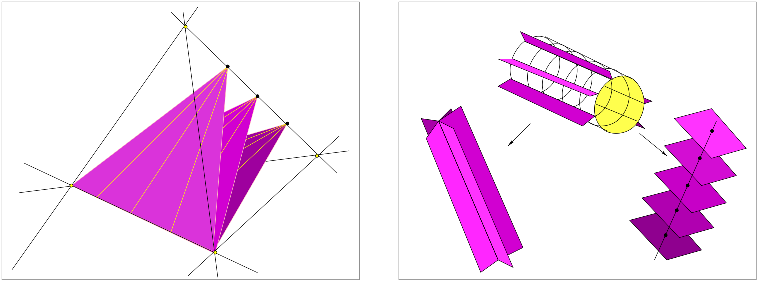
The divisor $\Sigma_0 + E_1 + \Sigma_2 + E_3$ represents the anti-canonical class. Furthermore, this spans the 1-eigenspace of f_X^* acting on $H^{1,1}(X)$. Every f_X-invariant divisor is a multiple of this. By studying g^*, we may show:

Theorem

There is a unique g-invariant curve on Σ_0.

One idea in the proof of the BCK Theorem

Suppose that $\mathcal{F}$ is an invariant foliation of codimension 1. Then the singular locus of $\mathcal{F}$ is an algebraic variety of dimension at most 1. If $\mathcal{F}$ "crosses" one of the indeterminate curves, then the image of that curve under J must be in the singular set of J.



The image of one of these singular curves must lie inside the invariant 4-cycle of surfaces. It is trapped inside this 4-cycle and has infinite orbit there. This contradicts the fact that it must lie inside the 1-dimensional singular locus of $\mathcal{F}$.