Renormalization of critical circle maps

<u>Def'n</u>. $f : \mathbb{T} = \mathbb{R}/\mathbb{Z} \circlearrowleft$ is a critical circle map if :

- f is a homeomorphism$^+$
- $f \in C^3$
- f has a single critical point at $x=0$, of cubic type

$$f(x) - f(0) = ax^3 + o(x^3)$$

<u>Example:</u> Arnold's family:

$$A_\theta(x) = x + \theta - \frac{1}{2\pi} \sin 2\pi x$$

$A_\theta : \mathbb{R} \nearrow \mathbb{R}$, cubic critical point at integers.

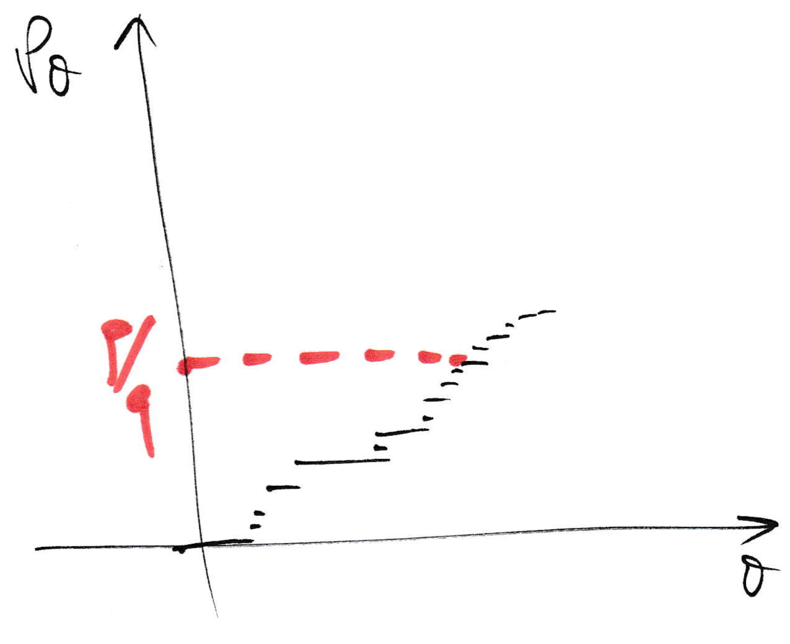
$A_\theta(x+1) = A_\theta(x) + 1 \implies A_\theta$ projects to $f_\theta : \mathbb{T} \circlearrowleft$

Note: let

$\rho_\theta \equiv \rho(f_\theta)$ be the rotation number

then $\theta \mapsto \rho_\theta$ is monotone,

the graph is a "devil's staircase": segments at rational heights, points at irrational heights.



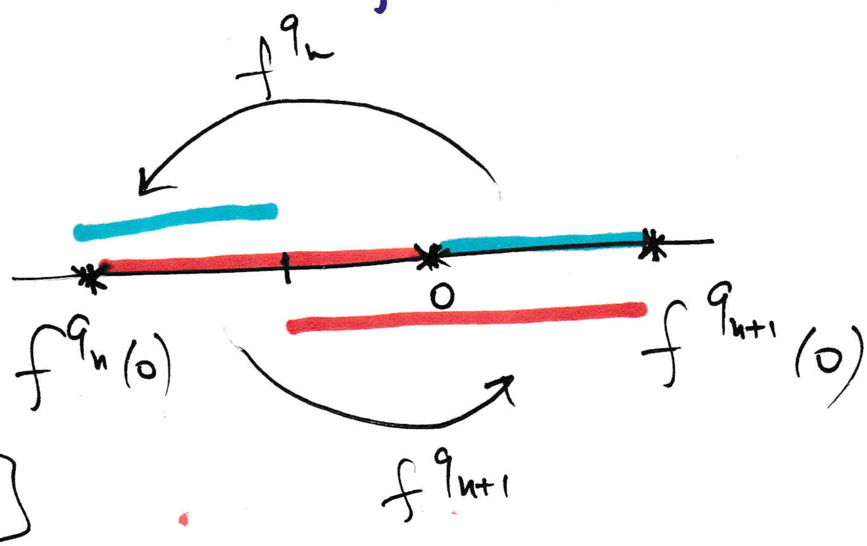
Renormalization = a rescaled first return map

Let $\rho(f) \notin \mathbb{Q}$, $\rho = \cfrac{1}{a_0 + \cfrac{1}{a_1 + \dots}} = [a_0, a_1, \dots]$

$a_i \in \mathbb{N}$

$p_n / q_n = [a_0, \dots, a_{n-1}]$ ⟵ closest return times of a point in $\mathbb{T}$

First return map of $[f^{q_n}(0), f^{q_{n+1}}(0)]$



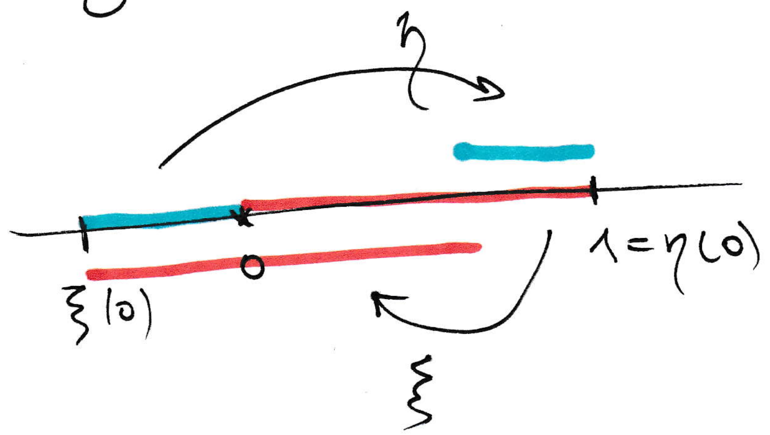
<u>Difficulty with rescaling</u>: how do we glue the ends of the interval into $\mathbb{T} = \mathbb{R}/\mathbb{Z}$?

A non-affine gluing is easy:

$$[f^{q_n}(0), f^{q_{n+1}}(0)] \big/ {x \sim f^{q_n}(x)} \overset{C^3}{\simeq} \text{circle}$$

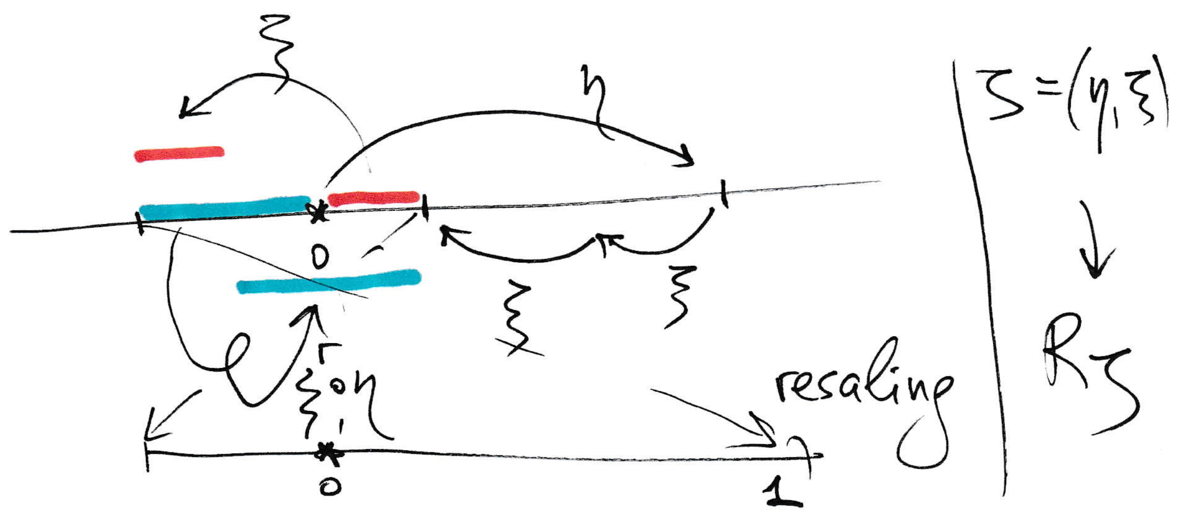
A "classical" solution: do not glue.

Commuting pairs of interval maps:



- $\eta, \xi \in C^3$ homeos of slightly larger intervals
- 0 is a cubic critical point of η, ξ
- $\eta \circ \xi = \xi \circ \eta$ where defined

Renormalization:

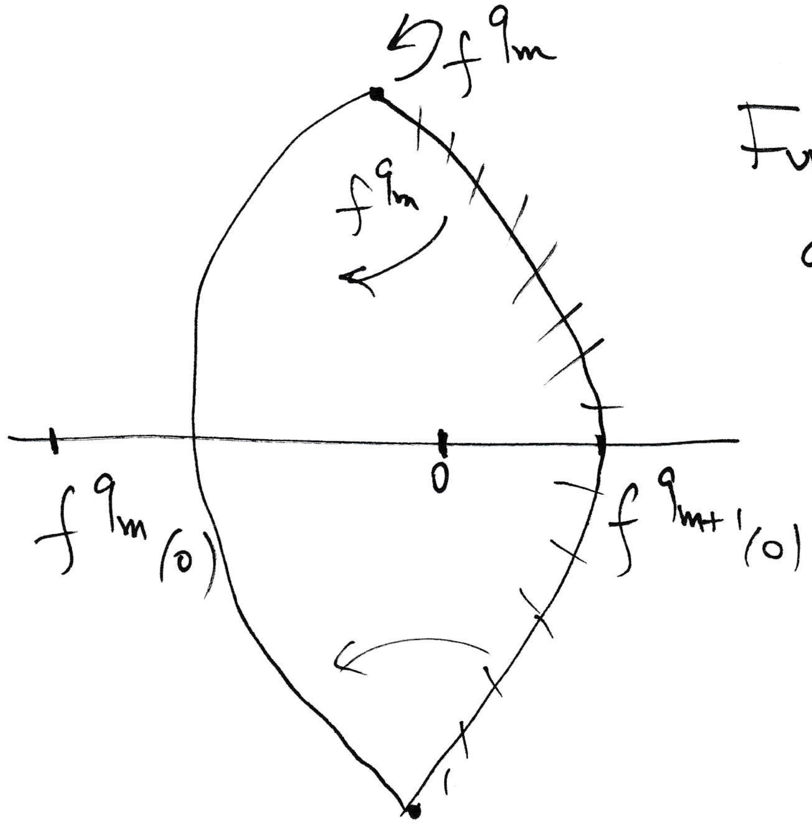


A conceptual difficulty: how to define a Banach manifold in which R is a smooth operator.

* the space of C^k-commuting pairs, $k \geqslant 3$ is preserved by R. However, composition is <u>not</u> smooth from $C^k \times C^k \to C^k$ (it is smooth from $C^k \times C^k \to C^{k-1}$)

* In the space $C^\omega \times C^\omega$, the commutation condition does not define a Banach manifold

Cylinder renormalization ([Y, ~1999])



$$\text{Fundamental domain}\Big/ f^{q_m} \simeq \mathbb{C}/\mathbb{Z}$$

$$\mathbb{C}/\mathbb{Z} \supset \mathbb{R}/\mathbb{Z}$$

$$\circlearrowleft$$

$$g \equiv R_{cyl} f$$

analytic operator in a Banach mfld.

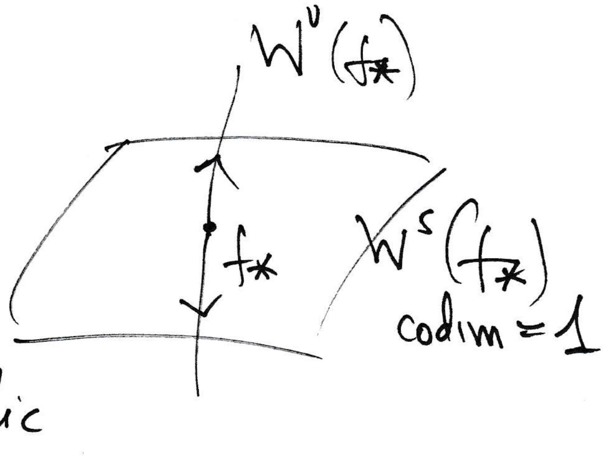
Hyperbolicity of renormalization [Y, 1999, 2000]

Example:

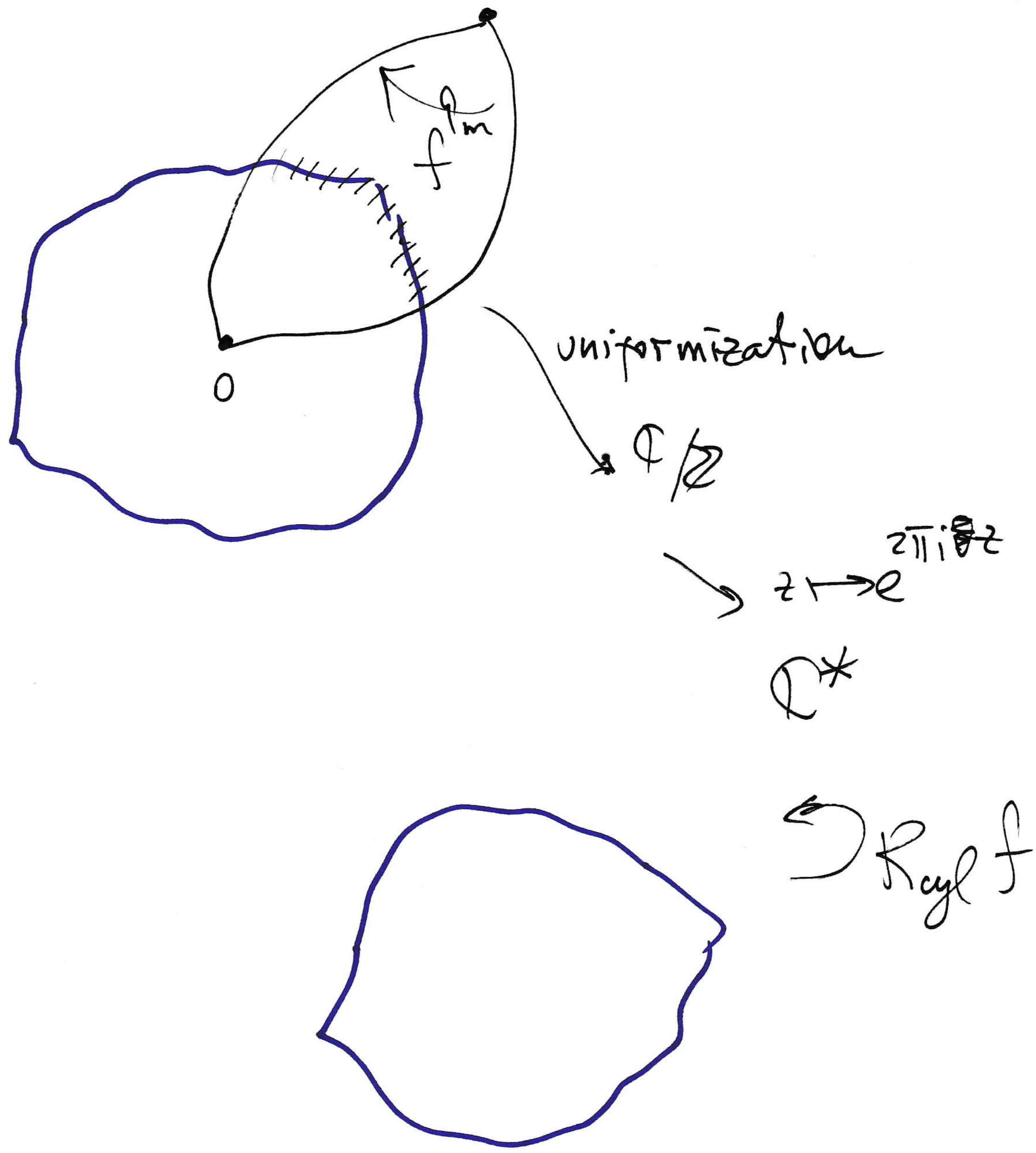
$$\rho_* = \frac{\sqrt{5}-1}{2} = [1,1,1,\ldots]$$

$$\exists!\, f_* : \quad R_{cyl} f_* = f_*$$

$$D R_{cyl}\big|_{f_*} \text{ is hyperbolic}$$



Bonus : works for Siegel disks:
[Y ~2005]



Hyperbolicity for high type:
$a_i > N_0 \gg 1$ (uses Inou-Shishikura)

[Y ~2005]

Problem: want to generalize renormalization to 2D maps.

• Motivating question:

Quadratic Hénon maps

$$H_{c,a}(x,y) = (x^2 + c + ay, ax), \quad a \neq 0$$

$$Jac = -a^2, \quad |a^2| \ll 1$$

Def'n a fixed point of $H_{c,a}$ is semi-Siegel if the eigenvalues are $\lambda = e^{2\pi i \theta}$, $\theta \in \mathbb{R}$, μ, $|\mu| < 1$ and $H_{c,a}$ is linearizable in the neighborhood

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Set $L(x,y) = (\lambda x, \mu y)$

• $\exists$ a maximal biholomorphic map $\varphi : \mathbb{D} \times \mathbb{C} \to \mathbb{C}^2$, s.t. $H_{c,a} \circ \varphi = \varphi \circ L$

• $S \equiv \varphi(\mathbb{D} \times \{0\})$ is the Siegel disk

Q: Is ∂S a topological circle?

Joint work w. Denis Gaidashev

Almost commuting pairs

$$\zeta = (\eta, \xi)$$

- (I) $\eta, \xi \in C^\omega$
- (II) ζ is a C^3-commuting pair

a Banach mfld in $C^\omega(\mathbb{R}) \times C^\omega(\mathbb{R})$

$\Updownarrow$

$$(\mathrm{II}')\ \eta \circ \xi - \xi \circ \eta(x) = o(x^3)$$

In the Siegel disk case:

$$(\mathrm{II}'):\quad \eta \circ \xi - \xi \circ \eta(x) = o(x^2)$$

Used before by Mestel, Stirnemann, Burbanks.

Theorem [Gaidashev - Y]

- Let $\rho_* = \frac{\sqrt{5}-1}{2}$. There exists a hyperbolic fixed point in the space of almost commuting pairs bot for crit. circle maps and for Siegel disks

- Extended $\mathcal{R}$ to 2D dissipative maps. Same fixed point, same spectrum

Theorem [Gaidashev, Radu, Y]

$\exists \delta > 0$ such that if $|\mu| < \delta$, then the Siegel disk of $H_{\lambda_*, \mu}$ is bounded by a topological circle.

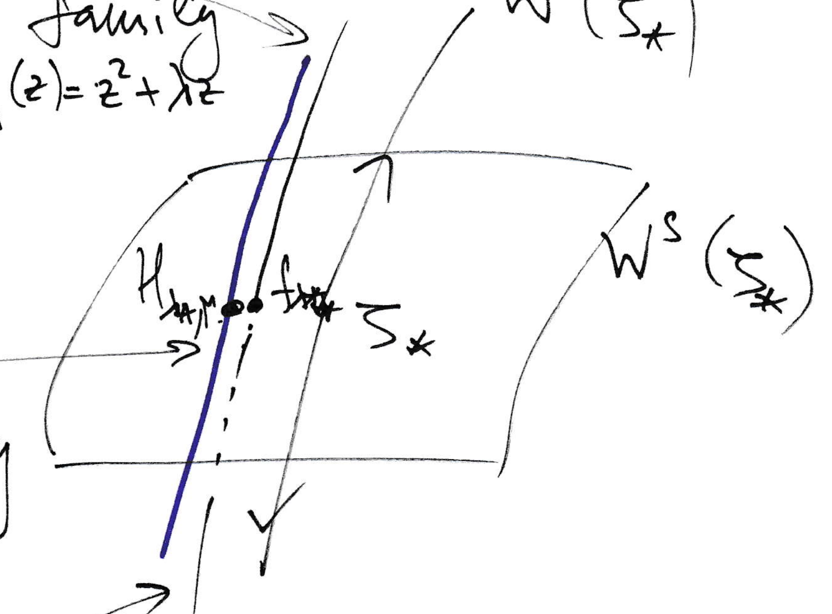
The conjugacy with rotation on ∂S is not C^1-smooth.

Follows from:

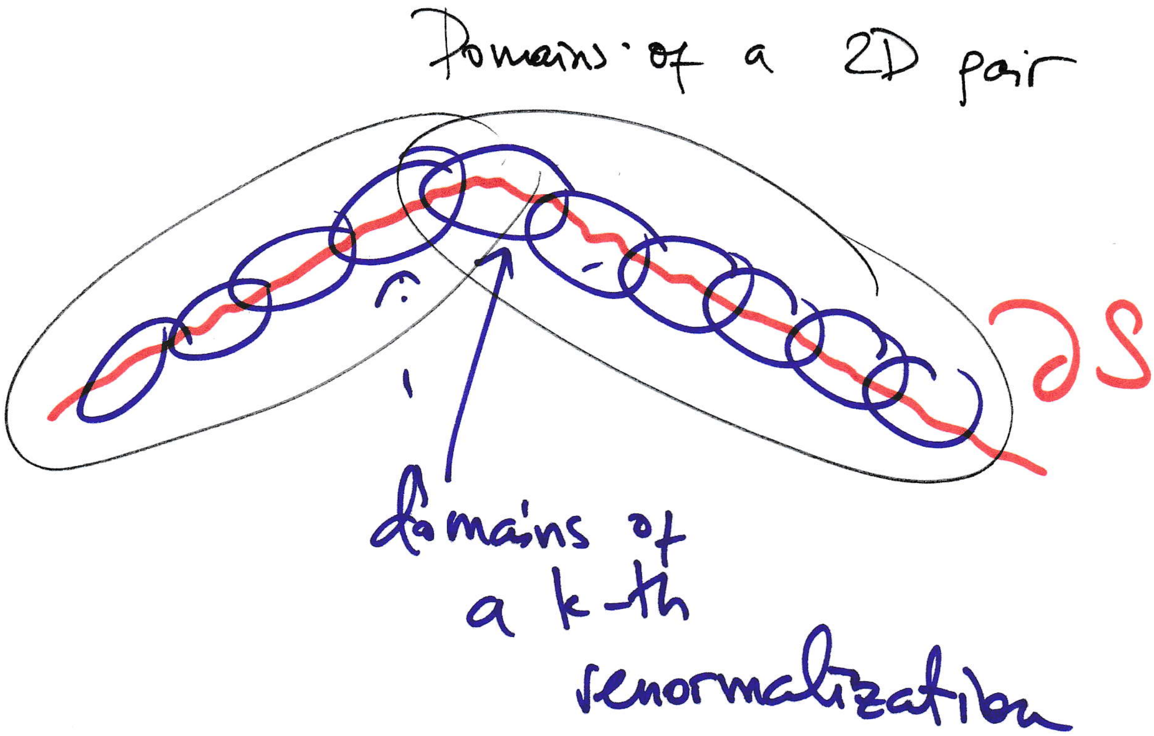
(1) Theorem [G, R, Y] If $H_{\lambda_*, \mu} \in$
$\in W^s$ (fixed point) $\Rightarrow \partial S$ is a top. circle

(2) Theorem [G, R, Y] $\exists \delta > 0$: If $|\mu| < \delta$, then $H_{\lambda_*, \mu} \in W^s$(f.p.)

The proof of 2



Proof of 1



Theorem [J. Yang - Y]
∂S is not a smooth curve

New application of Ruyl

(joint with I. Gorbovickis)

Critical circle maps with non-integer critical exponent α (locally can be written as $\phi\left(|x|^{\alpha-1}x\right)$

Theorem (Gorbovickis – Y)

Renormalization hyperbolicity for $|\alpha-(2n+1)|<\varepsilon$

Happy Birthday, Jack!!!