Quasiconformal surgery and applications to transcendental dynamics

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Lecture Plan

Gentle introduction to qc surgery

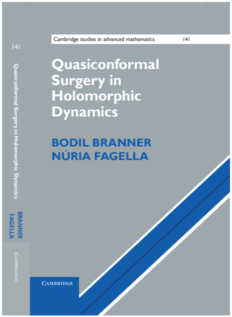
- Quasiconformal and quasiregular mappings
- Almost complex structures and the MRMT
- Surgery: classical examples and general principles

Applications to transcendental dynamics

- Moving asymtptotic values (linearization of neutral fixed points)
- Bishop's qc folding construction (existence of oscillating wandering domains)

For details

Branner, F. *Quasiconformal Surgery in Holomorphic Dynamics*
Cambridge University Press, 2014.



Contributions by Buff – Henriksen, Bullet, Epstein – Yampolsky, Haïsinsky, Petersen and Pilgrim – Tan Lei.

1. Quasiconformal maps

- Quasiconformality is a degree of regularity : the right one to study (e.g.) structural stability of holomorphic maps.
  - Topological conjugacies between holomorphic maps can be upgraded to qc conjugacies.

- $QC \Rightarrow \mathcal{C}^0$

- $QC \not\Rightarrow \mathcal{C}^1$ and $\mathcal{C}^1 \not\Rightarrow QC$

- They are very flexible (as opposed to holomorphic maps) — good to construct models.

Quasiconformal maps

- If F is conformal at z_0, it preserves angles between curves crossing at z_0, because

$$DF(z_0) : \mathbb{C} \to \mathbb{C} \text{ is a complex linear map } z \mapsto F'(z_0)\, z.$$

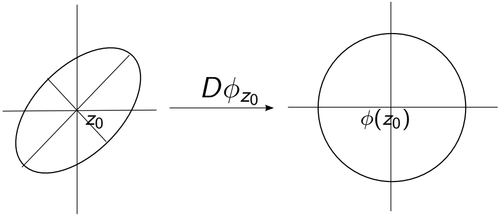
- In general , if ϕ is differentiable at z_0:

$$D\phi(z_0) : \mathbb{C} \to \mathbb{C} \text{ is a linear map } z \mapsto a\,z + b\,\overline{z}.$$

with $a = \partial_z \phi(z_0)$ and $b = \partial_{\overline{z}} \phi(z_0)$

- It defines an ellipse in the tangent space at z_0

$$E_{z_0} = \left(D\phi_{z_0}\right)^{-1}\left(\mathbb{S}^1\right)$$



Quasiconformal maps

If F is orientation preserving, the dilatation of the ellipse E_{z_0} is given by

$$K_\phi(z_0) = \frac{|\text{major axis of } E_{z_0}|}{|\text{minor axis of } E_{z_0}|} = \frac{|a| + |b|}{|a| - |b|} \in [1, \infty).$$

The Beltrami coefficient at z_0 is the quantity

$$\mu(z_0) = \frac{b}{a} = \frac{\partial_{\overline{z}}\phi}{\partial_z \phi}(z_0) \in \mathbb{D}$$

which codes the information of the ellipse, up to scaling.

A map which is OP and differentiable almost everywhere defines a field of ellipses σ (up to scaling) a.e. in the tangent bundle, The dilatation of the field of ellipses, or the dilatation of the map is

$$K_\phi = \operatorname*{ess\,sup}_{z \in U} K_\phi(z).$$

Quasiconformal maps

Formal definition and examples

Definition (Analytic definition of qc maps)

A map $\phi : U \to V$, $U, V \subset \mathbb{C}$ is $K-$quasiconformal if :

- *ϕ is an orientation preserving homeomorphism,*
- *ϕ is absolutely continuous on lines ($\Rightarrow$ differentiable a.e.)*
- *$K_\phi(z) \leq K < \infty$ a.e. where defined.*

Examples:

- ϕ conformal $\Leftrightarrow$ ϕ is $1-$quasiconformal $\Leftrightarrow$ ϕ is qc and $\partial_{\overline{z}} = 0$ a.e (Weyl's lemma).
- Every o.p. $\mathbb{R}^2-$linear map is qc ($K_\phi(z)$ ctant)
- $\mathcal{C}^1$ o.p. diffeos are quasiconformal on any compact set.

Quasiconformal maps can also be defined in terms of what they do to the moduli of annuli.

Definition (Geometric definition of qc maps)

A map $\phi : U \to V$, $U, V \subset \mathbb{C}$ is $K-$quasiconformal if and only if ϕ is an orientation preserving homeomoprhism and for every annulus A with $\overline{A} \subset U$

$$\frac{1}{K} \operatorname{mod}(A) \; \leq \; \operatorname{mod}(\phi(A)) \; \leq \; K \operatorname{mod}(A).$$

Conformal maps preserve the moduli of annuli.

Quasiconformal/Quasiregular maps

Properties

(1) Pasting quasiconformal maps along "reasonable" curves preserves quasiconformality.
(This gives great flexibility!)

(2) If ϕ is K_1-qc and φ is K_2-qc $\quad \Rightarrow \quad \phi \circ \varphi$ is $(K_1 K_2)-$qc

(3) If ϕ is $K-$qc then ϕ^{-1} is also $K-$qc.

Definition

$f : U \to V$ is a $K-$quasiregular map if f is locally $K-$quasiconformal except at a discrete set of points.

conformal	$\longleftrightarrow$	holomorphic
quasiconformal	$\longleftrightarrow$	quasiregular

2. Almost complex structures

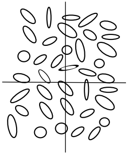
An almost complex structure (or conformal structure) σ on U is a measurable field of ellipses $(E_z)_z$ defined almost everywhere, up to scaling, with bounded dilatation

$$K = \operatorname*{ess\,sup}_{z \in U} K(E_z) < \infty.$$

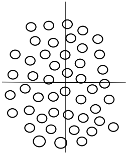
Any measurable function $\mu : U \to \mathbb{D}$ defines an almost complex structure with dilatation

$$K := K(\sigma) = \frac{1+k}{1-k} > 1 \text{ where } k = \|\mu\|_\infty < 1.$$

The standard complex structure σ_0 is defined by circles at every point, or by $\mu_0 \equiv 0$.



σ



σ_0

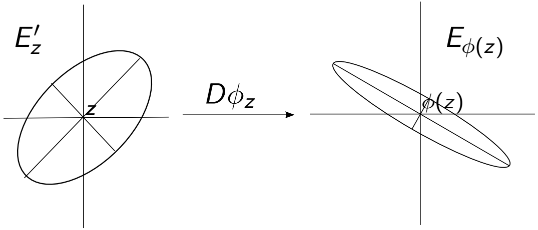
Pullbacks

Quasiconformal (or quasiregular) maps $\phi : U \to V$ can be used to pull back almost complex structures in V to almost complex structures in U.

Given σ and a.c.s. on V, we may define a new a.c.s. $\sigma' = \phi^* \sigma$ on U, by the field of ellipses

$$E'_z = (D\phi_z)^{-1} E_{\phi(z)}$$

defined almost everywhere.



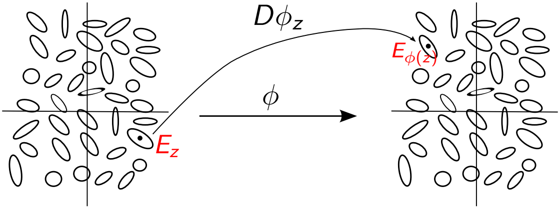
Pullbacks

We say that ϕ transports σ to σ', and write it as a pullback

$$\sigma' = \phi^*(\sigma) \quad \text{or} \quad \mu' = \phi^*\mu \quad \text{or} \quad (U, \sigma') \xrightarrow{\phi} (V, \sigma).$$

Finally, σ is $\phi-$invariant if $\phi : U \to U$ transports σ to itself, i.e.

$$\phi^*\sigma = \sigma.$$



Remark

- *If ϕ is holomorphic, then $K(\phi^*(\sigma)) = K(\sigma)$.*
- *If ϕ is $\widetilde{K}-$ q.c. then $K(\phi^*(\sigma)) = \widetilde{K} \cdot K(\sigma)$.*

3. Main tool for surgery: MRMT

We saw that every quasiconformal map $\phi : U \to V$ induces and almost complex structure $\sigma = \phi^*(\sigma_0)$ with bounded dilatation.

Question: Is the converse true?

Integrability Theorem (Morrey, Ahlfors, Bers, Bojarski)

Let $U \simeq \mathbb{D}$ (resp. $\mathbb{C}$ or $\widehat{\mathbb{C}}$ via charts) and σ be an almost complex structure on U with dilatation $K_\sigma < \infty$. Then, there exists a quasiconformal homeomorphism $\phi : U \to \mathbb{D}$ (resp. $\mathbb{C}$ or $\widehat{\mathbb{C}}$) such that

$$\sigma = \phi^* \sigma_0.$$

Moreover,

- *ϕ is unique up to post-composition with conformal self maps of $\mathbb{D}$ (resp. $\mathbb{C}$ or $\widehat{\mathbb{C}}$).*
- *If σ depends continuosly on a parameter, so does ϕ. (If $U \simeq \mathbb{C}$ or $\widehat{\mathbb{C}}$, we also have holomorphic dependence).*

4. Surgery: constructing dynamical models

Surgery is about constructing maps with prescribed dynamics.
Quasiregular maps often are used as dynamical models of holomorphic maps.
We say that a qr map f is a dynamical model of F holomorphic if there exists ϕ qc such that

$$\phi \circ f = F \circ \phi$$

or equivalently, the following diagram conmutes

$$\begin{array}{ccc} U & \xrightarrow{f(qr)} & V \\ \phi(qc)\big\downarrow & & \big\downarrow\phi(qc) \\ U & \xrightarrow{F(hol)} & V \end{array}$$

We write $f \underset{qc}{\sim} F$

Surgery: constructing dynamical models

Question: Given a quasiregular map f,

- when is it a dynamical model for some holomorphic map F?
- equivalently, when does there exist F holomorphic such that $f \underset{qc}{\sim} F$?

The answer comes from the MRMT.

Theorem (Key lemma)

If $\exists\, \sigma$ such that $f^*\sigma = \sigma \quad \Longrightarrow \quad \exists F$ holomorphic such that $f \underset{qc}{\sim} F$

Apply the MRMT to σ to obtain ϕ the integrating map. Then define F as:

$$\begin{array}{ccc} (\widehat{\mathbb{C}}, \sigma) & \xrightarrow{\quad f \quad} & (\widehat{\mathbb{C}}, \sigma) \\ \phi \big\downarrow & & \big\downarrow \phi \\ (\widehat{\mathbb{C}}, \sigma_0) & \xrightarrow{F := \phi f \phi^{-1}} & (\widehat{\mathbb{C}}, \sigma_0) \end{array}$$

Since F is qc and $F^*\sigma_0 = \sigma_0$ we conclude (Weyl's lemma) that F is holomorphic.

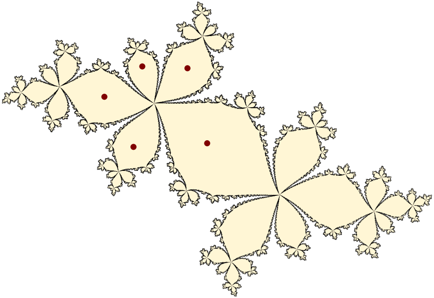
Soft Surgery

Deforming holomorphic maps

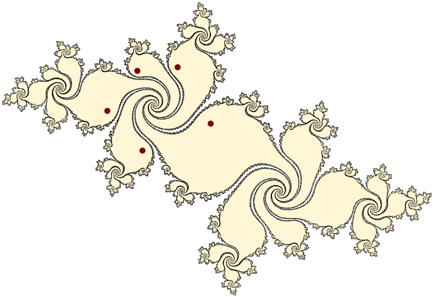
- If f is holomorphic to start with, we may still find an almost complex structure $\sigma \neq \sigma_0$ which is $f-$invariant.
- After conjugating by the integrating map ϕ we obtain $F := \phi f \phi^{-1}$ holomorphic and $f \underset{qc}{\sim} F$. We then say that F is a quasiconformal deformation of f.
- In some cases, all quasiconformal deformations of f result int he same map we started with. Then we say that f is (quasiconformally) rigid.
- **Example:** Centers of hyperbolic components are rigid. All other maps in the component are deformations of each other.

Soft Surgery

Deforming holomorphic maps



$z \mapsto z^2 + c_1$

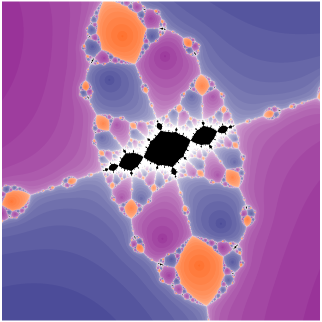


$z \mapsto z^2 + c_2$

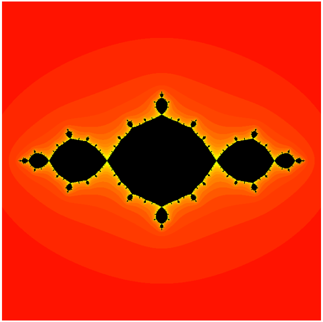
c_1, c_2 belong to the same hyperbolic component (2/5) of the Mandelbrot set. The two maps are quasiconformally conjugate.
(Sullivan, Dounady and Hubbard, 1984)

5. Application: the straightening theorem

Question: Why do we see polynomial Julia sets in the dynamical plane of non-polynomial mappings?



Newton's method of a cubic pol.



$z \mapsto z^2 - 1$

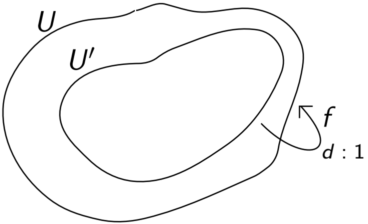
The straightening theorem

The answer is given by the Straightening theorem, one of the first surgery applications.

Theorem (Douady and Hubbard, 1985)

Let $U', U \subset \mathbb{C}$ *be topological discs such that* $\overline{U'} \subset U$. *Suppose* $f : U' \to U$ *is a proper holomorphic map of degree* $d \geq 2$. *Then, there exists a polynomial* $P(z)$ *of degree* d *such that*

$$f \underset{qc}{\sim} P \quad \text{on } U'.$$



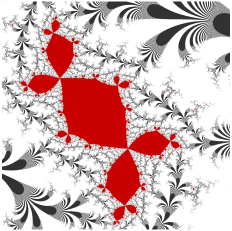
The straightening theorem

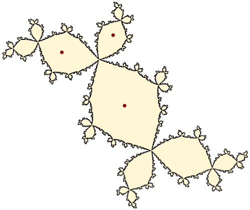
In particular, there is a (qc) homeo between the small filled Julia set of f

$$\mathcal{K}_f = \{z \in U' \mid f^n(z) \in U' \text{ for all } n \geq 0\}.$$

and the filled Julia set of P

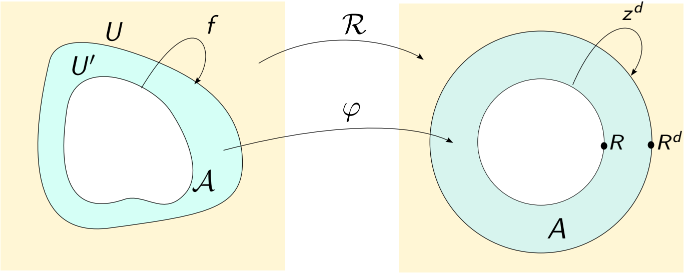
$$\mathcal{K}_P = \{z \in \mathbb{C} \mid f^n(z) \nrightarrow \infty\}.$$





The straightening theorem

(1) Let $\mathcal{R} : \widehat{\mathbb{C}} \setminus \overline{U} \to \widehat{\mathbb{C}} \setminus \mathbb{D}_{R^d}$ be a conformal map. It extends to $\varphi : \partial U \to \mathbb{S}_{R^d}$. (w.l.o.g. we assume ∂U and $\partial U'$ are at least $\mathcal{C}^2$).



(2) Define $\varphi : \partial U' \to \partial \mathbb{D}_R$ so that

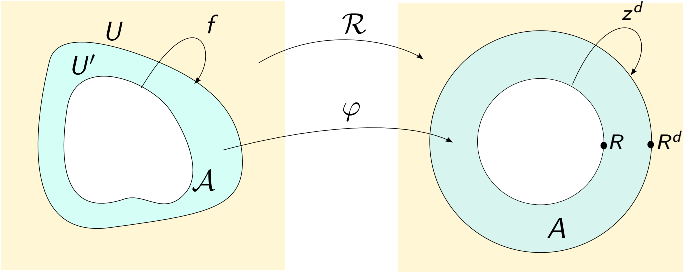
$$\varphi^{-1}(\varphi(z))^d = f(z).$$

Extend to $\varphi : \overline{\mathcal{A}} \to \overline{A}$ $K-$quasiconformally (Extension lemma).

The straightening theorem

(3) Define a new map (the model)

$$g = \begin{cases} f & \text{on} \quad U' \\ \mathcal{R}^{-1}(\varphi(z))^d & \text{on} \quad \overline{\mathcal{A}} \\ \mathcal{R}^{-1}(\mathcal{R}(z))^d & \text{on} \quad \mathbb{C} \setminus U \end{cases}$$



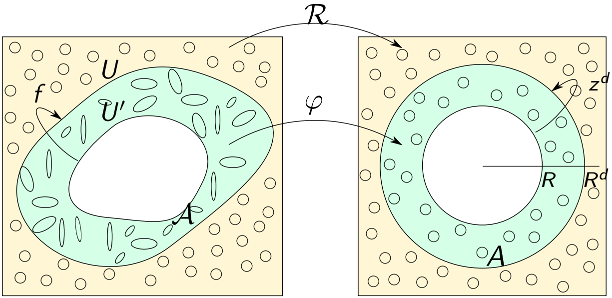
The straightening theorem

(4) g is quasiregular: it is continuous, locally quasiconformal in $\mathcal{A}$ and holomorphic in $\mathbb{C} \setminus \mathcal{A}$.

(5) We construct a $g-$invariant complex structure: let $\mathcal{A}_n := f^{-n}\mathcal{A}$

(spreading by the dynamics)

$$\sigma = \begin{cases} \varphi^*(\sigma_0) & \text{on } \mathcal{A} \\ (f^n)^*\sigma & \text{on } \mathcal{A}_n \\ \sigma_0 & \text{elsewhere} \end{cases}$$



The straightening theorem

(6) Apply the MRMT to σ to obtain ϕ the integrating map. Normalize so that $\phi(\infty) = \infty$. Then define P as:

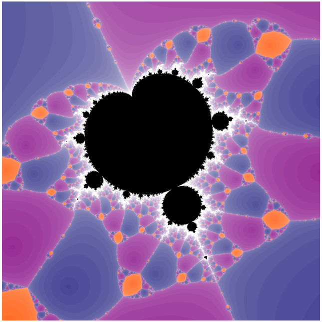
$$\begin{array}{ccc} (\widehat{\mathbb{C}}, \sigma) & \xrightarrow{\quad g \quad} & (\widehat{\mathbb{C}}, \sigma) \\ \phi \Big\downarrow & & \Big\downarrow \phi \\ (\widehat{\mathbb{C}}, \sigma_0) & \xrightarrow{P := \phi g \phi^{-1}} & (\widehat{\mathbb{C}}, \sigma_0) \end{array}$$

(7) P is holomorphic (Weyl's lemma). In fact P is entire and ∞ is a superattracting fixed point . Hence P is a polynomial of degree d, and it is qc-conjugate (by ϕ) to f inside U.

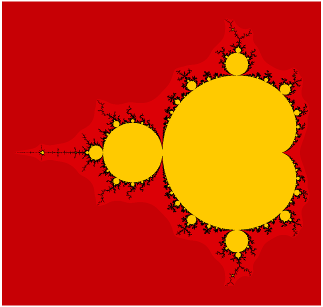
(8) $\sigma = \sigma_0$ on $\mathcal{K}(f) \Longrightarrow \partial_{\overline{z}}\phi = 0$ on $\mathcal{K}_f$.
Hence, P and f are conformally conjugate in the interior of $\mathcal{K}_f$ (if any).

Parameter version

In parameter space, we see that bifurcations occur with the same patterns.



Newton parameter space



Bifurcations of $z \mapsto z^2 + c$

To prove the existence of copies of M involves dealing with the continuity of integrating maps, and of the resulting polynomials with respect to the parameters.

6. Surgery: general principles

There exist some criteria which are useful in many occasions.

Shishikura's Principle 1

Suppose

- *f is holomorphic in $\mathbb{C} \setminus X$ and $K-$quasiregular in X*
- *$f^j(X) \cap X = \emptyset$ for all $j \geq N$ (orbits pass through X at most N times).*

Then, there exists F holomorphic such that $f \underset{qc}{\sim} F$.

Proof:

We construct an $f-$invariant almost complex structure by defining $\sigma = \sigma_0$ on $U := \cup_{j \geq N} f^j(X)$ and spreading σ by the dynamics.

Observe that U is invariant and f is holomorphic in U, so $f^* \sigma = \sigma$.

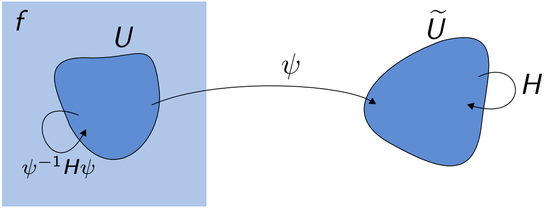
Surgery: general principles

Shishikura's Principle 2

Let $U \subset \widehat{\mathbb{C}}$ *and suppose*

- f *is holomorphic in* $\widehat{\mathbb{C}} \setminus U$*;*
- $f\,|_U = \psi^{-1} \circ H \circ \psi$ *where* $\psi : U \to \widetilde{U}$ *is quasiconformal (the gluing map) and* $H : \widetilde{U} \to \widetilde{U}$ *is holomorphic.*

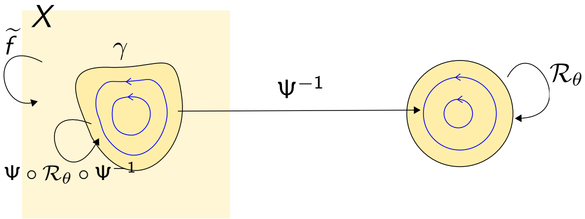
Then, there exists F *holomorphic such that* $f \underset{qc}{\sim} F$*.*

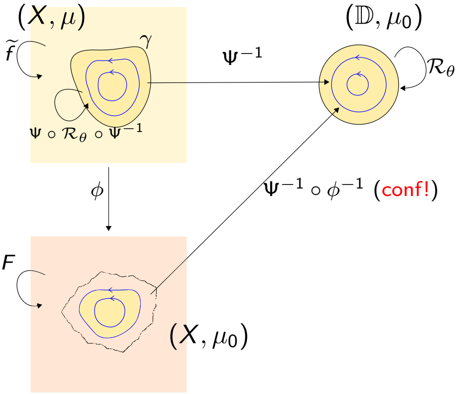


Application: Turning Herman rings into Siegel disks

- Suppose f has a Herman ring and γ is an invariant curve. Then $\exists\, \psi : \mathbb{S}^1 \to \gamma$ analytic conjugation to $\mathcal{R}_\theta(z) = e^{i\theta} z$.
- We can "glue" a Siegel disk in the interior of γ (or also in the exterior).
- Ahlfors-Berling or Douady-Earle: $\psi : \mathbb{S}^1 \to \gamma$ extends to $\Psi : \overline{\mathbb{D}} \to \overline{D_\gamma}$, which is quasiconformal in $\mathbb{D}$. This map is the "gluing" map.
- Define

$$\widetilde{f}(z) = \begin{cases} (\Psi \circ \mathcal{R}_\theta \circ \Psi^{-1})(z) & \text{if } z \in D_\gamma, \\ f(z) & \text{if } z \in X \setminus D_\gamma \end{cases}$$





- $\widetilde{f}$ is quasiregular
- Pull back σ_0 on $\mathbb{D}$ and spread by the dynamics.
- (MRMT) Obtain ϕ qc and $F := \phi f \phi^{-1}$ is holomorphic.
- $\phi(D_\gamma)$ is contained in a Siegel disk of F with the same rotation number.

$F \underset{\text{conf}}{\sim} f$ on any domain whose forward orbit is disjoint from D_γ.

This is a local procedure, but depending the map we start with, we end up with a different result.

- Starting with the degree 3 Blaschke family

$$B_{a,t} := e^{2\pi it} z^2 \frac{z-a}{1-az}, \quad \text{with } a > 3 \text{ (so } B_{a,t} \text{ is a diffeo on } \mathbb{S}^1.$$

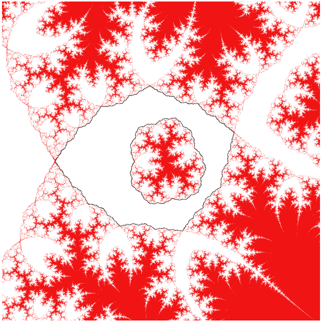
  after the surgery we would obtain a quadratic polynomial.

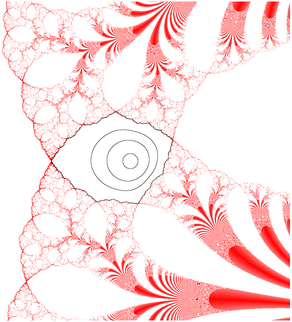
- Starting in the complexified Arnold family

$$A_{t,a}(z) = e^{2\pi it} z e^{\frac{a}{2}(z-1/z)} \quad \text{with } 0 < a, t < 1,$$

  after the surgery we would land in the semistandard family $E_\theta(z) = e^{2\pi i\theta} z e^z$.
  This is used to transport features from certain families to others.

Application: Turning Herman rings into Siegel disks





Some results proven with this tool

The gluing can in fact be done in much more general settings, and it can also be reversed [Shishikura], converting maps with Siegel disks into Herman rings. Both have been used to prove many results in holomorphic dynamics.

- Existence of Herman rings of any Brjuno rotation number.
- Existence of critical points on the boundaries of rotation domains of certain maps.
- Existence of Siegel disks (or Herman rings) with Jordan curve boundaries with no critical points.
- etc etc

Surgery: general principles

The most powerful of all principles is due to D. Sullivan (1981).

Sullivan's Principle

$f : U \to \mathbb{C}$ is quasiconformally conjugate to some holomorphic $F : U \to \mathbb{C}$

if and only if

$\exists K < \infty$ such that for all $n \geq 1$, the iterates $\{f^n\}$ are (uniformly) $K-$quasiregular.

Moreover, F is unique up to conformal conjugacies.

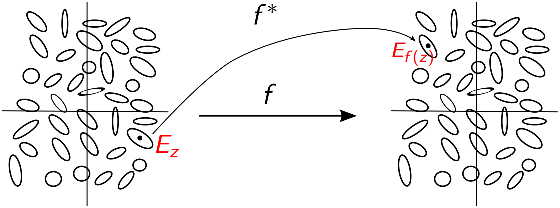
- One implication is obvious: If $f = \phi \circ F \circ \phi^{-1}$ with ϕ $K-$qc, then $f^n = \phi \circ F^n \circ \phi^{-1}$ are uniformly K^2-qc.
- Now suppose f^n is $K-$qc for all $n \geq 0$, and let

$$\mu_n(z) := (f^n)^*(\mu_0(fn(z))).$$

We may interpret these as a (uniformly) bounded set of points in $\mathbb{D}$, with the Poincaré metric.

- Special case: Using that $f^* : \mathcal{M}(f(z)) \to \mathcal{M}(z)$ is an isometry in the hyperbolic metric plus other ingredients one can prove (in some special cases) that $\mu_n(z)$ converges pointwise and $\mu(z) := \lim_{n\to\infty} \mu_n(z)$ is a measurable function with $||\mu||_\infty < 1$.

- Moreover
$$\mu_{n+1}(z) = f^*(\mu_n(f(z)))$$
and therefore
$$\mu(z) = f^*\mu(f(z)).$$
which means that μ is f^*-invariant or $f^*\mu = \mu$.



- (MRMT) There exists ϕ qc such that $F = \phi \circ f \circ \phi^{-1}$ is holomorphic.

- General case: In general, $\mu_n(z)$ does not converge.
- We then define $\mu(z)$ as the circumcenter of the set $\{\mu_n(z)\}$ (the center of the unique minimal hyperbolic disc containing the set).
- Again using that $f^* : \mathcal{M}(f(z)) \to \mathcal{M}(z)$ is an isometry, and the uniqueness of the circumcenter, we have that $f^*\mu = \mu$.

(Tukia'86)

7. Extension theorems

- The actual construction of the models (quasiregular maps f) is often done by pasting different maps in several pieces of the plane.
- Often (as in the examples) we need to interpolate in between to maps or to fill in certain regions with given boundary values. There are many Extension theorems which deal with these situations.
- These theorems have nohthing to do with dynamics – they are examples of how complex analysis plays an important role in complex dynamics.

Extension theorems

Here are two examples which can be proved by simple interpolation.

Theorem (Extension of boundary maps to an annulus)

Let A_1 and A_2 be standard annuli. Suppose $\psi : \partial A_1 \to \partial A_2$ is an o.p. $\mathcal{C}^1$ covering of degree $d \geq 1$. Then, there exists a $\mathcal{C}^1$ covering $\Psi : \overline{A_1} \to \overline{A_2}$ of degree d, such that $\Psi_{\partial A_1} = \psi$.
Moreover, Ψ can be chosen to be conformal on any annulus $\overline{A} \subset A_1$ as long as $\mathrm{mod}(A) < \mathrm{mod}(A_2)$.

Theorem (Extension of boundary maps to Jordan domains)

Let $\psi : \mathbb{S}^1 \to \mathbb{S}^1$ be a $\mathcal{C}^1$ diffeomorphism. Then there exists a piecewise $\mathcal{C}^1$ map $\Psi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ such that $\Psi = \psi$ on $\mathbb{S}^1$.
Moreover, Ψ can be chosen to be conformal on any smaller disc D compactly contained in $\mathbb{D}$.

Conditions can be relaxed to a great extend (quasicircles, quasisymmetry, etc)

8. Other applications

Surgery techniques have been used to prove plenty of results of very different nature.

- Nonexistence of wandering domains for rational maps (Sullivan 82)
- Construction of examples of and counterexamples to dozens of conjectures.
  - There exist Siegel discs and Herman rings of any Brjuno rotation number.
  - For rotation numbers of bounded type, Siegel disks have Jordan boundaries with a critical point (rational maps).
- Connectivity of the Julia set under certain hypothesis (e.g. Newton's methods).
- Bound the number of non-repelling cycles for a given system.
- Parametrization of structurally stable components in parameter spaces.
- Constructing homeomorphisms between parameter spaces of different families.
- and a very very large ETC.

9. Transcendental dynamics - Generalities

If $f : \mathbb{C} \to \mathbb{C}$ (or to $\widehat{\mathbb{C}}$) has an essential singularity at infinit we say that f is transcendental.

The set $S(f)$ of singularities of f^{-1} consists of critical values and asymtpotic values (and the closure of such).
$f : \mathbb{C} \setminus f^{-1}(S(f)) \longrightarrow \mathbb{C} \setminus S(f)$ is a covering map of infinite degree.

Definition

A point $a \in \widehat{\mathbb{C}}$ is an asymptotic value if there exists a curve $\gamma(t) \to \infty$ such that $f(\gamma(t)) \to a$. (Morally, a has infinitely many preimages collapsed at infinity).

Example: $z = 0$ for $f(z) = \lambda e^z$.

Special classes of transcendental maps are:

- The Speisser class or finite type maps:

$$\mathcal{S} = \{f \text{ ETF such that } S(f) \text{ is finite}\}$$

- The Eremenko-Lyubich class

$$\mathcal{B} = \{f \text{ ETF such that } S(f) \text{ is bounded}\}$$

  Example: $\lambda \frac{z}{\sin(z)}$.

Transcendental maps allow for two extra types of Fatou components: wandering domains an Baker domains. However,

- maps in $\mathcal{S}$ cannot have either of them;
- maps in $\mathcal{B}$ cannot have Baker domains nor wandering domains whisch escape to infinty uniformly.

10. Surgery: Moving asymptotic values

Theorem (Geyer '01)

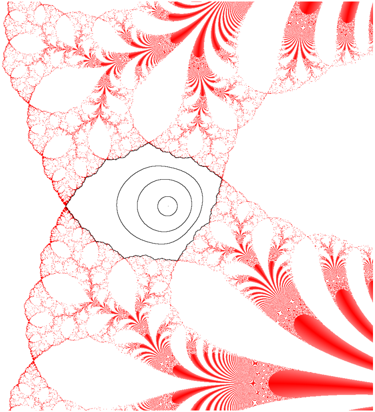
Let $E(z) = \lambda z e^z$, with $\lambda = e^{2\pi i\theta}$ and $\theta \in \mathbb{R} \setminus \mathbb{Q}$. Then, E is linearizable at $z = 0$ iff $\theta \in \mathcal{B}$ (set of Brjuno numbers).

One implication holds generally for any analytic function. To prove the other one we shall do some surgery and use the following proposition.

Proposition (Yoccoz, Pérez-Marco '93)

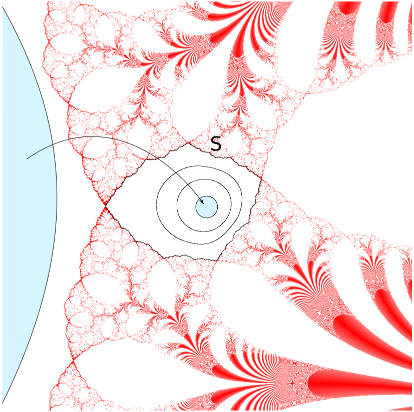
If $f_a(z) = f(z) + c(a)z^2 g(z)$ is uniformly linearizable for $|a| \leq \varepsilon$, where f, g and c are analytic, $f(0) = c(0) = 0$, $f'(0) = \lambda = e^{2\pi i\alpha}$, $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, $g(0) \neq 0$ and c is non-constant, then the quadratic polynomial $P_\lambda(z) = \lambda z + z^2$ is linearizable and $\alpha \in \mathcal{B}$.

Assume $E(z) = \lambda z e^z$ is linearizable.



- $z = -1$ is the only critical point.
- $z = 0$ is an asymptotic value and a fixed point. The surgery will split them up.
- For all U small nbd of 0, $E^{-1}(U)$ has one unbounded component U' and $E : U' \to U$ is a universal covering.

Assume $E(z) = \lambda z e^z$ is linearizable.



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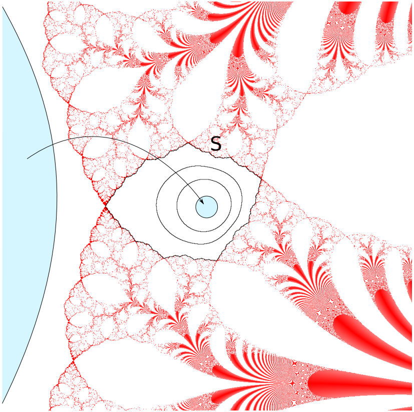

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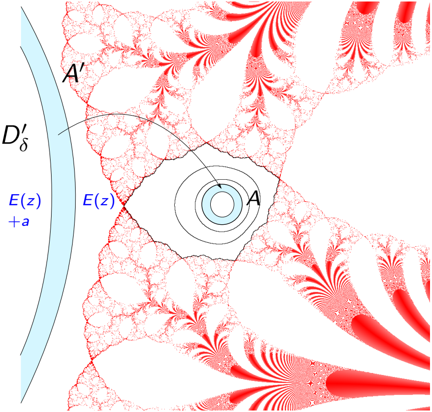
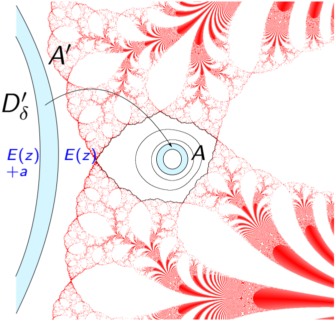


- Choose δ very small. Let $A = D_{2\delta}(0) \setminus D_\delta(0)$ and $A' =$ (unique c.c of) $E^{-1}(A)$.
- Define a new map:

$$E_a(z) = \begin{cases} E(z) & \text{if } z \notin D'_{2\delta} \\ E(z) + a & \text{if } z \in D'_\delta \\ \varphi(z) & \text{if } z \in A' \end{cases}$$

where φ is a qc interpolating map between $E(z)$ and $E(z) + a$. Formula



- E_a is quasiregular (holomorphic off A', quasiconformal on A').
- In fact $\left|\mu_{E_a(z)}\right| \leq k_a \longrightarrow 0$ as $a \to 0$.
- Orbits pass through A' at most once.
- Hence $\exists \phi_a : \mathbb{C} \to \mathbb{C}$ qc, such that $F_a := \phi_a \circ E_a \circ \phi_a^{-1}$ is holomorphic.
- ϕ_a is conformal on S.
- Normalize setting $\phi_a(0) = 0$ and $\phi_a(-1) = -1$. Then $a \mapsto \phi_a(z)$ is holomorphic for all fixed z.

$$\begin{array}{ccc} \mathbb{D}_r & \xrightarrow{\lambda z} & \mathbb{D}_r \\ \varphi \downarrow & & \downarrow \varphi \\ S & \xrightarrow{E} & S \\ \phi_a \downarrow & & \downarrow \phi_a \\ \tilde{S}_a & \xrightarrow{F_a} & \tilde{S}_a \end{array}$$

- $\tilde{S}_a := \phi_a(S)$ is still a Siegel disk, with multiplier λ .
- $\phi_a \circ \varphi$ is a linearizing map of the Siegel disk.
- $z = 0$ is NOT an asymptotic value any more. The new asymp. value is $v_a = \phi_a(a) \in \tilde{S}$. The map $a \mapsto v_a$ is analytic.

- Since $\phi_a'(0) \to 1$ as $a \to 0$, for a small enough F_a has a Siegel disk of uniform size.

- Imposing the conditions on F_a and rescaling one can prove that F_a is conjugate to

$$G_a(z) = \lambda(c_a + e^z(z + c_a(z-1))) = \lambda z e^z + c_a z^2 \left(\frac{\lambda}{2} + \mathcal{O}(z)\right)$$

  with c_a analytic, non-constant and $c_0 = 0$.

- This satisfies the conditions of Pérez-Marco's proposition and therefore we conclude that θ is a Brjuno number.

- Observation: We have embedded $E(z)$ into a one parameter family of maps G_a. This works for small values of a but the family is interesting in itself and it has a persistent Siegel disk and two singular values which can interchange their roles (Berenguel, F '10).

11. Constructing entire functions

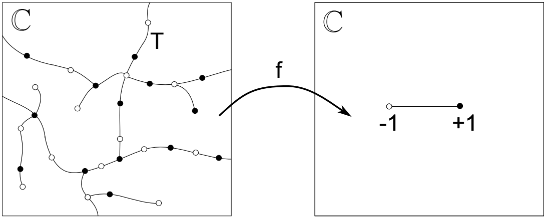
Bishop's qc-folding construction

Let $f : \mathbb{C} \to \mathbb{C}$ be a transcendental entire function with

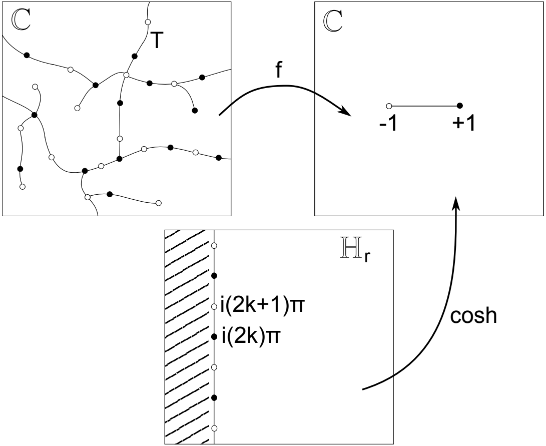
- exactly two critical values, say -1 and $+1$
- no finite asymptotic values

Question: What does f look like ?

$T = f^{-1}([-1, +1])$ is an infinite bipartite tree.

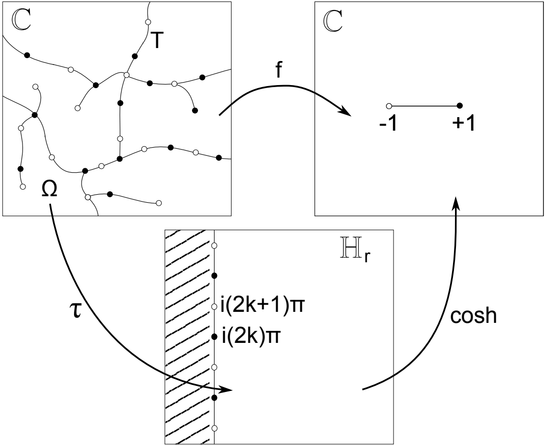


$T = f^{-1}([-1,+1])$ is an infinite bipartite tree.



cosh : $\mathbb{H}_r \to \mathbb{C}\setminus[-1,+1]$ is a universal cover.

$T = f^{-1}([-1,+1])$ is an infinite bipartite tree.



$\forall \Omega$ c.c. of $\mathbb{C} \setminus T$, $\tau_{|\Omega} = (\cosh^{-1} \circ f_{|\Omega}) : \Omega \to \mathbb{H}_r$ is conformal.

Conversely: How to construct f from (T, τ) ?

More precisely: Given

- an infinite bipartite tree $T \subset \mathbb{C}$ with "good enough" geometry
- a map τ such that $\tau_{|\Omega} : \Omega \to \mathbb{H}_r$ is conformal, $\forall \Omega$ c.c. of $\mathbb{C} \setminus T$

Question: Does there exist an entire function $f : \mathbb{C} \to \mathbb{C}$ such that $f = \cosh \circ \tau$?

Main problem: In general, $\cosh \circ \tau$ is not continuous across T.

Strategy:

Step 1: Modify (T, τ) in a small neighborhood $T(r_0)$ of T.

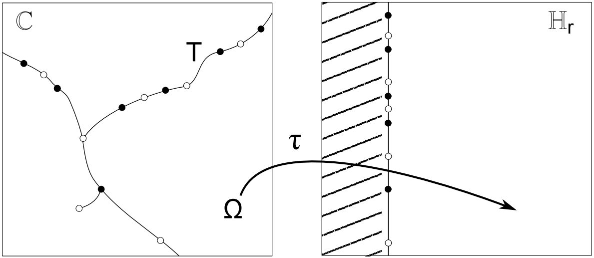
More precisely, replace (T, τ) by (T', η) such that

- $T \subset T' \subset T(r_0)$
- $\eta = \tau$ off $T(r_0)$
- $\eta_{|\Omega'} : \Omega' \to \mathbb{H}_r$ is K-quasiconformal, $\forall \Omega'$ c.c. of $\mathbb{C} \setminus T'$
- $\cosh \circ \eta$ continuous across T' (quasiregular map)

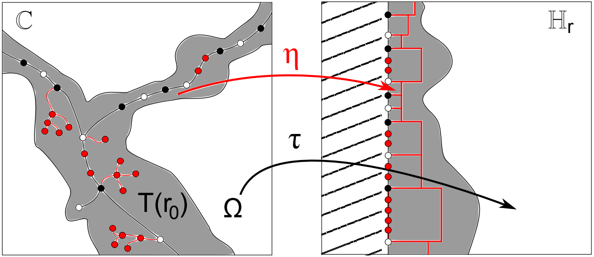
Step 2: Apply MRMT .

Obtain a qc map ϕ (the integrating map of $\mu_{\cosh \circ \eta}$) so that $f := \cosh \circ \eta \circ \phi^{-1}$ is entire. In particular

$$f \circ \phi = \cosh \circ \tau \text{ off } T(r_0)$$


$\mathbb{C}$
T
τ
Ω
$\mathbb{H}_r$

Bishop's qc-folding construction



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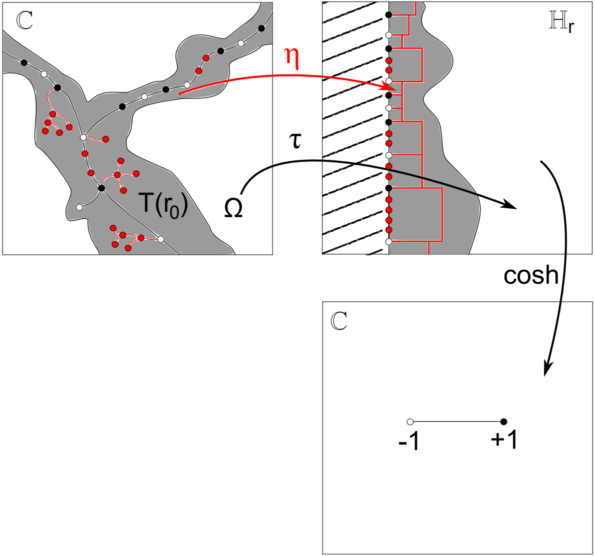

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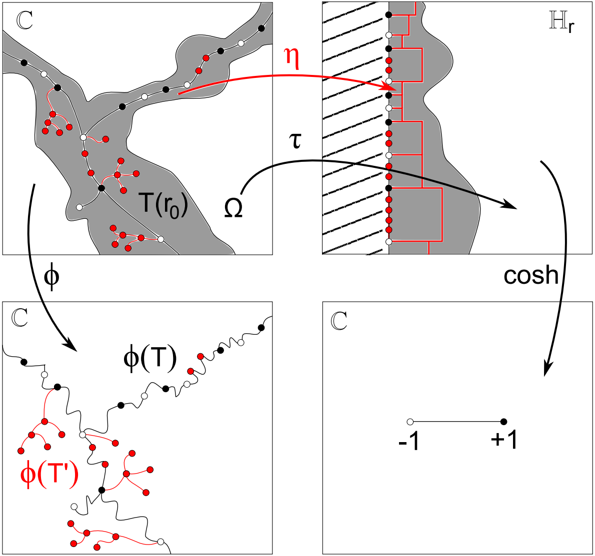
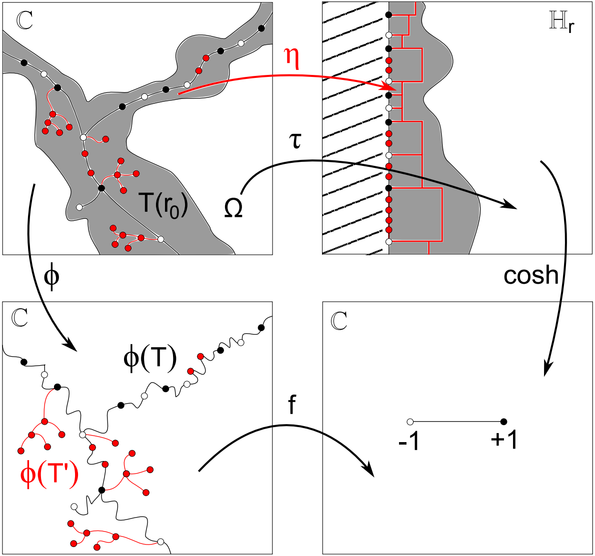


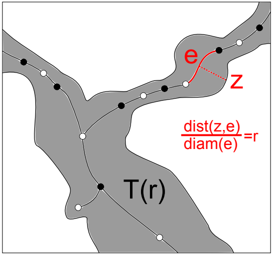
Bishop's qc-folding construction



Bounded geometry

Definition: We say that T has bounded geometry if

- edges of T are $\mathcal{C}^2$ with uniform bounds
- angles between adjacent edges are uniformly bounded away from 0
- $\forall e, f$ adjacent edges, $\frac{1}{M} \leqslant \frac{\operatorname{diam}(e)}{\operatorname{diam}(f)} \leqslant M$
- $\forall e, f$ non-adjacent edges, $\frac{\operatorname{diam}(e)}{\operatorname{dist}(e,f)} \leqslant M$

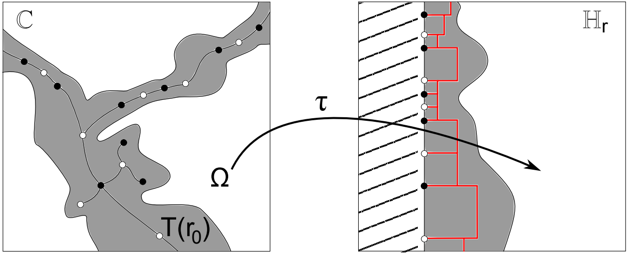


$$T(r) = \bigcup_{e \text{ edge of } T} \left\{ z \in \mathbb{C} \;/\; \operatorname{dist}(z,e) < r \operatorname{diam}(e) \right\}$$

Lemma

If T has bounded geometry, then $\exists r_0 > 0$ such that

$$\forall \Omega \text{ c.c. of } \mathbb{C} \setminus T,\ \forall \text{ square } Q \subset \mathbb{H}_r \text{ that has a } \tau_{|\Omega}\text{-edge as one side,}$$
$$Q \subset \tau_{|\Omega}\Big(T(r_0) \cap \Omega\Big)$$



Every edge has two τ-sizes!!!

Bishop's Theorem

Theorem (Bishop'12)

If (T, τ) *satisfies the following conditions*

1. T *has bounded geometry*
2. *every edge has* τ*-size* $\geqslant \pi$

then $\exists$ *an entire function* f *and a quasiconformal map* ϕ *such that*

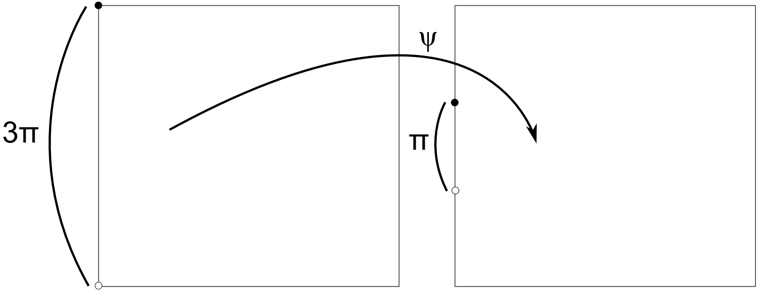
$$f \circ \phi = \cosh \circ \tau \text{ off } T(r_0)$$

Moreover

- f *has exactly two critical values,* -1 *and* $+1$
- f *has no finite asymptotic values*
- $\phi(T) \subset f^{-1}([-1, +1]) \quad (= \phi(T'))$
- $\forall c$ *critical point of* f, $\deg_{\text{loc}}(c, f) = \deg(c, \phi(T'))$

The main technical difficulty is to find a quasiconformal map ψ from a square to itself such that

$$\begin{cases} \psi \text{ maps the left side to an edge of length } \pi \\ \psi \text{ is the identity on the right side} \end{cases}$$



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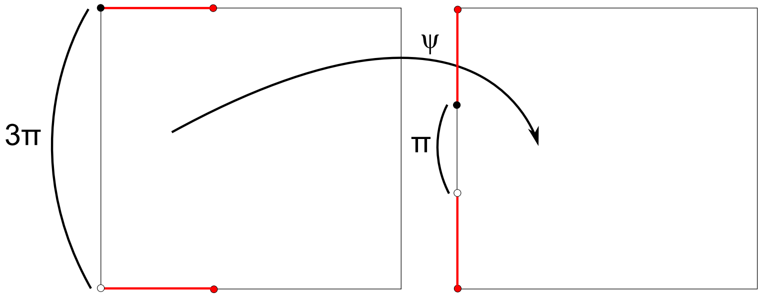
Solution: Add some extra edges and "unfold".



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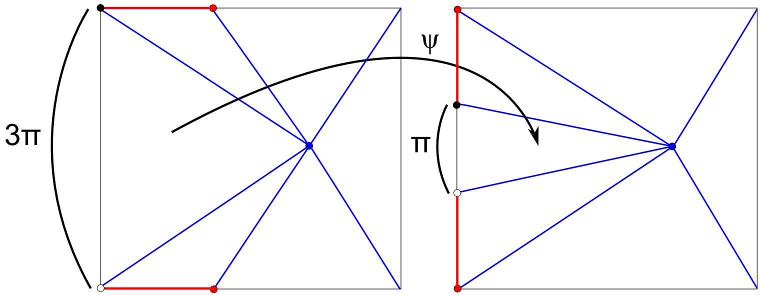

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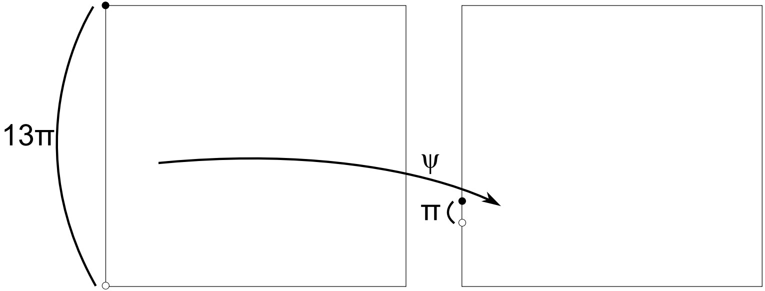

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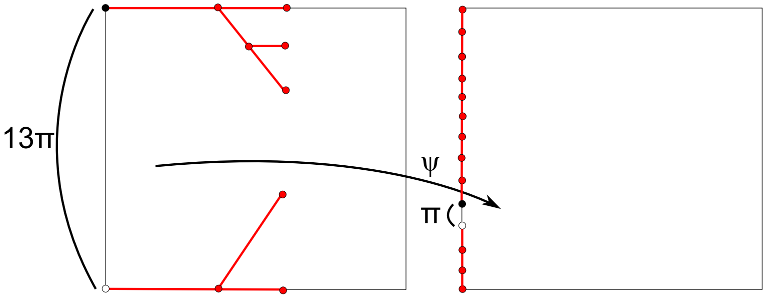
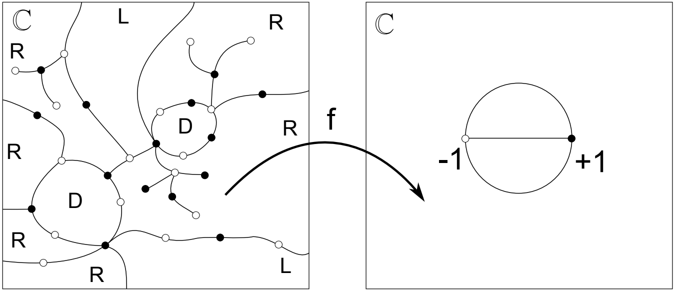


ψ^{-1} is called a **quasiconformal folding**.

Generalization: We may also construct f with

- more critical values than only -1 and $+1$
- some finite asymptotic values
- arbitrary high degree critical points

Let T be an infinite bipartite graph.



The c.c. of $\mathbb{C}\setminus T$ are sorted into three different types:

R-components: $\tau_{|\Omega} : \Omega \to \mathbb{H}_r$ conformally
D-components: $\tau_{|\Omega} : \Omega \to \mathbb{D}$ conformally
L-components: $\tau_{|\Omega} : \Omega \to \mathbb{H}_\ell$ conformally

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R	$\Omega \xrightarrow{\tau_{\vert\Omega}} \mathbb{H}_r$
D	$\Omega \xrightarrow{\tau_{\vert\Omega}} \mathbb{D}$
L	$\Omega \xrightarrow{\tau_{\vert\Omega}} \mathbb{H}_\ell$

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L-components: $\tau_{|\Omega} : \Omega \to \mathbb{H}_\ell$ conformally

R	$\Omega \xrightarrow{\tau_{\vert\Omega}} \mathbb{H}_r \xrightarrow{\cosh}$
D	$\Omega \xrightarrow{\tau_{\vert\Omega}} \mathbb{D}$
L	$\Omega \xrightarrow{\tau_{\vert\Omega}} \mathbb{H}_\ell$

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D-components: $\tau_{|\Omega} : \Omega \to \mathbb{D}$ conformally
L-components: $\tau_{|\Omega} : \Omega \to \mathbb{H}_\ell$ conformally

R	Ω	$\xrightarrow{\tau_{\vert\Omega}}$	$\mathbb{H}_r$	$\xrightarrow{\cosh}$	$\mathbb{C}\setminus[-1,+1]$
D	$(\Omega,\star)$	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{D},0)$		
L	Ω	$\xrightarrow{\tau_{\vert\Omega}}$	$\mathbb{H}_\ell$		

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R	Ω	$\xrightarrow{\tau_{\vert\Omega}}$	$\mathbb{H}_r$	$\xrightarrow{\cosh}$	$\mathbb{C}\setminus[-1,+1]$
D	$(\Omega,\star)$	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{D},0)$	$\xrightarrow{z\mapsto z^{d_\Omega}}$	$(\mathbb{D},0)$
L	Ω	$\xrightarrow{\tau_{\vert\Omega}}$	$\mathbb{H}_\ell$		

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R	Ω	$\xrightarrow{\tau_{\vert\Omega}}$	$\mathbb{H}_r$	$\xrightarrow{\cosh}$	$\mathbb{C}\setminus[-1,+1]$		
D	$(\Omega,\star)$	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{D},0)$	$\xrightarrow{z\mapsto z^{d_\Omega}}$	$(\mathbb{D},0)$	$\xrightarrow{\rho_\Omega}$	$(\mathbb{D},w_\Omega)$
L	(Ω,∞)	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{H}_\ell,-\infty)$				

where $\rho_\Omega : \mathbb{D} \to \mathbb{D}$ is quasiconformal with $\rho_\Omega(z) = z$, $\forall z \in \partial\mathbb{D}$.

The c.c. of $\mathbb{C} \setminus T$ are sorted into three different types:

R-components: $\tau_{|\Omega} : \Omega \to \mathbb{H}_r$ conformally
D-components: $\tau_{|\Omega} : \Omega \to \mathbb{D}$ conformally
L-components: $\tau_{|\Omega} : \Omega \to \mathbb{H}_\ell$ conformally

R	Ω	$\xrightarrow{\tau_{\|\Omega}}$	$\mathbb{H}_r$	$\xrightarrow{\cosh}$	$\mathbb{C} \setminus [-1, +1]$
D	$(\Omega, \star)$	$\xrightarrow{\tau_{\|\Omega}}$	$(\mathbb{D}, 0)$	$\xrightarrow{z \mapsto z^{d_\Omega}}$	$(\mathbb{D}, 0)$
L	(Ω, ∞)	$\xrightarrow{\tau_{\|\Omega}}$	$(\mathbb{H}_\ell, -\infty)$	$\xrightarrow{\exp}$	$(\mathbb{D}, 0)$

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R	Ω	$\xrightarrow{\tau_{\vert\Omega}}$	$\mathbb{H}_r$	$\xrightarrow{\cosh}$	$\mathbb{C}\setminus[-1,+1]$		
D	$(\Omega, \star)$	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{D}, 0)$	$\xrightarrow{z \mapsto z^{d_\Omega}}$	$(\mathbb{D}, 0)$		
L	(Ω, ∞)	$\xrightarrow{\tau_{\vert\Omega}}$	$(\mathbb{H}_\ell, -\infty)$	$\xrightarrow{\exp}$	$(\mathbb{D}, 0)$	$\xrightarrow{\rho_\Omega}$	$(\mathbb{D}, v_\Omega)$

Recall that functions in class $\mathcal{B}$ have no "escaping" WD.

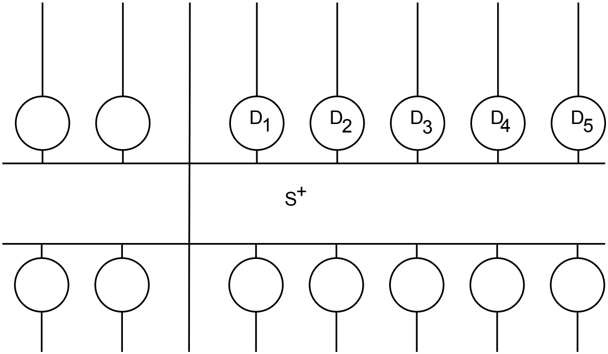
But the question remained of whether functions in class $\mathcal{B}$ could have WD, which would need to be oscillating or bounded (in the sense of limit functions).

Theorem (Bishop'12)

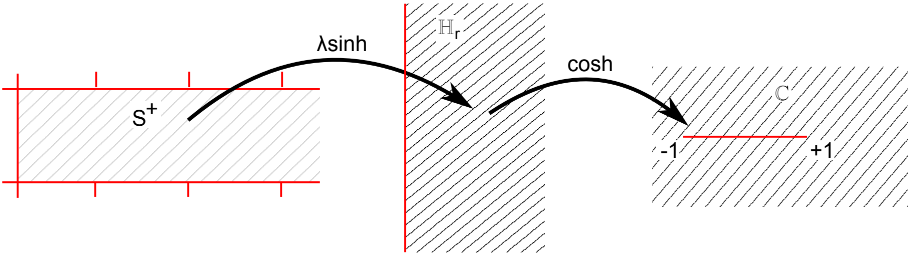
There exists $f \in \mathcal{B}$ which has an oscillating wandering domain.

$$f = F \circ \phi \quad \text{with} \quad \begin{cases} F : \mathbb{C} \to \mathbb{C} \text{ quasiregular (transcendental)} \\ \phi : \mathbb{C} \to \mathbb{C} \text{ quasiconformal so that } \mu_{\phi^{-1}} = F^*(\mu_0) \end{cases}$$

F is constructed using an infinite graph.



$$F: \quad S^+ \xrightarrow{\lambda \sinh} \mathbb{H}_r \xrightarrow{\cosh} \mathbb{C}\setminus[-1,+1]$$
$$z \longmapsto \cosh(\lambda \sinh(z))$$



$\boxed{\lambda > 0}$ is fixed so that $f^n\left(\frac{1}{2}\right) \xrightarrow[n\to+\infty]{} +\infty$ very fast.

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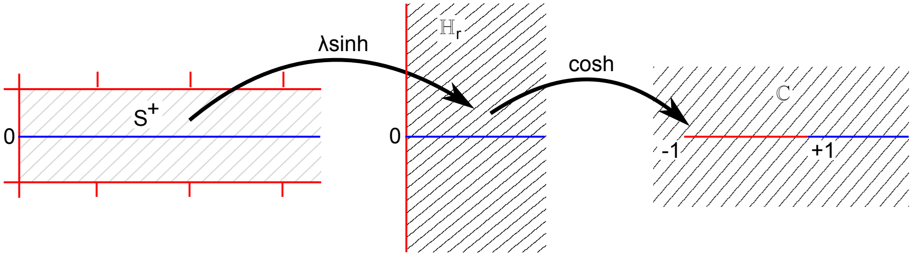

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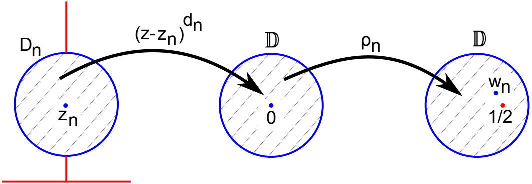
For every $n \geqslant 1$,

$$F: \quad (D_n, z_n) \xrightarrow{z \mapsto (z-z_n)^{d_n}} (\mathbb{D}, 0) \xrightarrow{\rho_n} (\mathbb{D}, w_n)$$
$$z \longmapsto \rho_n\left((z-z_n)^{d_n}\right)$$

with $\begin{cases} \rho_n : \mathbb{D} \to \mathbb{D} \text{ quasiconformal} \\ \rho_n(0) = w_n \end{cases}$



for some parameters $\boxed{d_n \xrightarrow[n\to+\infty]{} +\infty}$ and $\boxed{w_n \xrightarrow[n\to\infty]{} \frac{1}{2}}$.

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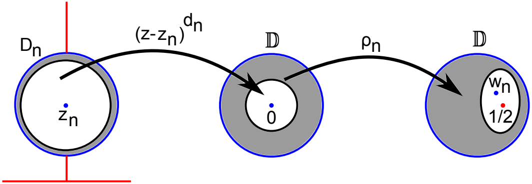
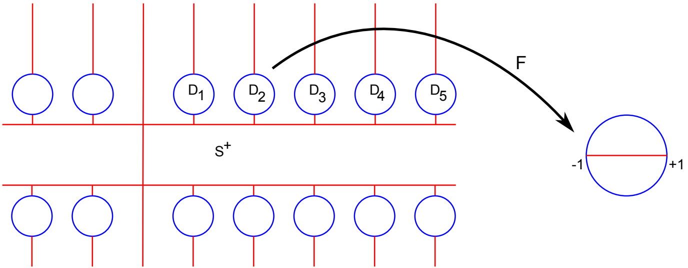
with $\begin{cases} \rho_n : \mathbb{D} \to \mathbb{D} \text{ quasiconformal} \\ \rho_n(0) = w_n \\ \mathrm{supp}(\mu_{\rho_n}) \subset \left\{\frac{1}{2} \leqslant |z| \leqslant 1\right\} \end{cases}$



for some parameters $\boxed{d_n \xrightarrow[n\to+\infty]{} +\infty}$ and $\boxed{w_n \xrightarrow[n\to\infty]{} \frac{1}{2}}$.

Using Bishop's construction F may be extended to a quasiregular map $F : \mathbb{C} \to \mathbb{C}$ such that:

- $\forall z \in \mathbb{C},\ F(-z) = F(z)$ and $F(\overline{z}) = \overline{F(z)}$
- $\mathrm{Crit}(F) = \{-1, +1\} \cup \{w_n,\ n \geqslant 1\} \cup \left\{\frac{1}{2}\right\} \subset \overline{\mathbb{D}}$ with $w_n \xrightarrow[n\to+\infty]{} \frac{1}{2}$
- $\mathrm{Asym}(F) = \emptyset$



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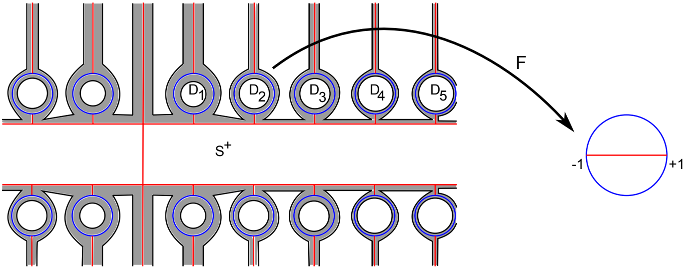
- $\mathrm{Asym}(F) = \emptyset$
- $\mathrm{supp}\,(F^*(\mu_0))$ is small enough



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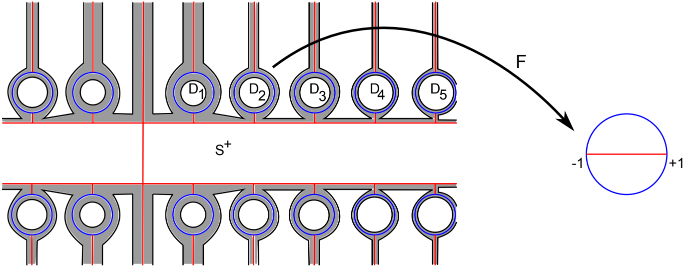
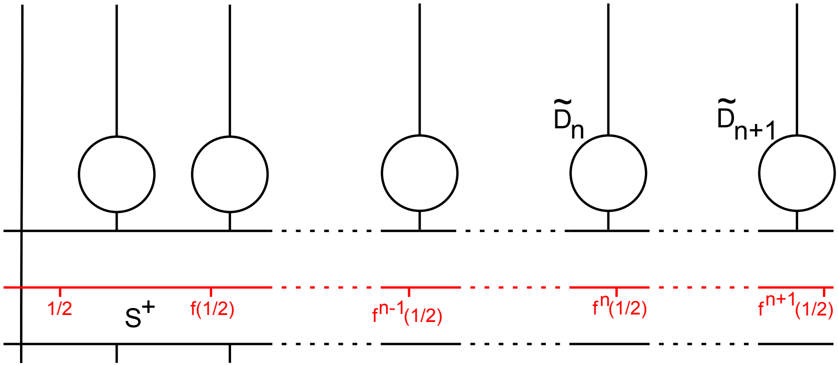
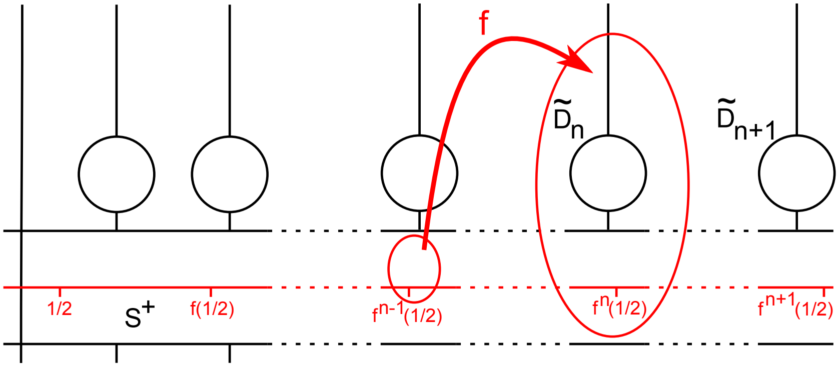
- $\mathrm{Asym}(F) = \emptyset$
- $\mathrm{supp}\,(F^*(\mu_0))$ is small enough in order that $\phi|_{S^+} \approx \mathrm{Id}_{S^+}$

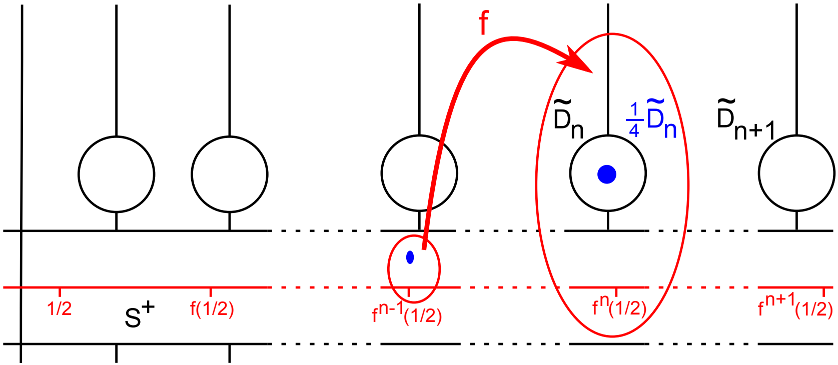


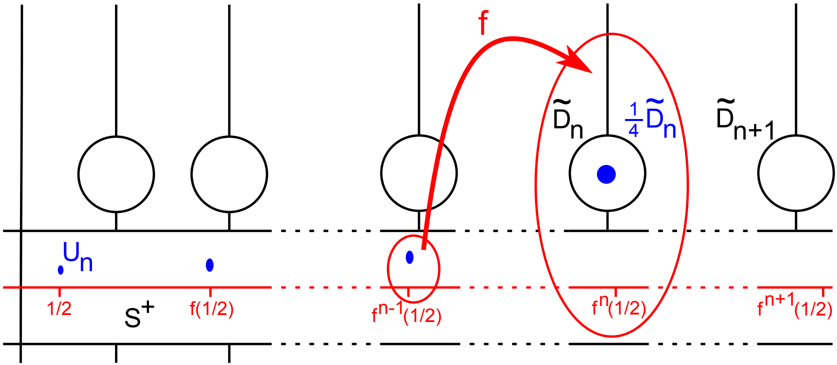
Let $f = F \circ \phi$ with $\phi : \mathbb{C} \to \mathbb{C}$ quasiconformal so that $\mu_{\phi^{-1}} = F^*(\mu_0)$.

Oscillating wandering domains in class $\mathcal{B}$

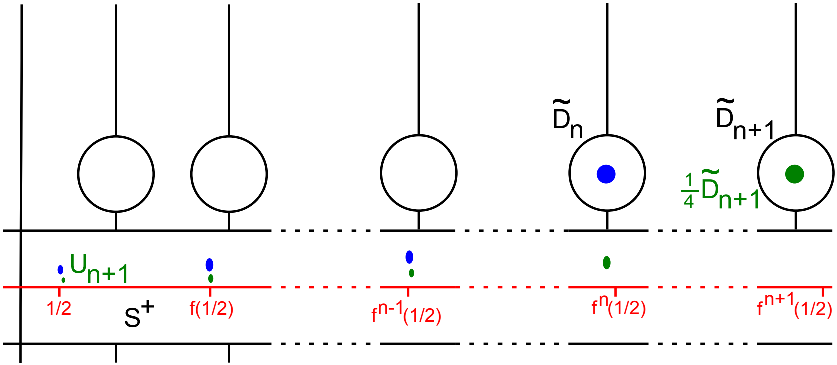



f
$\tilde{D}_n$
$\tilde{D}_{n+1}$
1/2
S^+
f(1/2)
$f^{n-1}(1/2)$
$f^n(1/2)$
$f^{n+1}(1/2)$


f
$\widetilde{D}_n$
$\frac{1}{4}\widetilde{D}_n$
$\widetilde{D}_{n+1}$
1/2
S^+
f(1/2)
$f^{n-1}(1/2)$
$f^n(1/2)$
$f^{n+1}(1/2)$

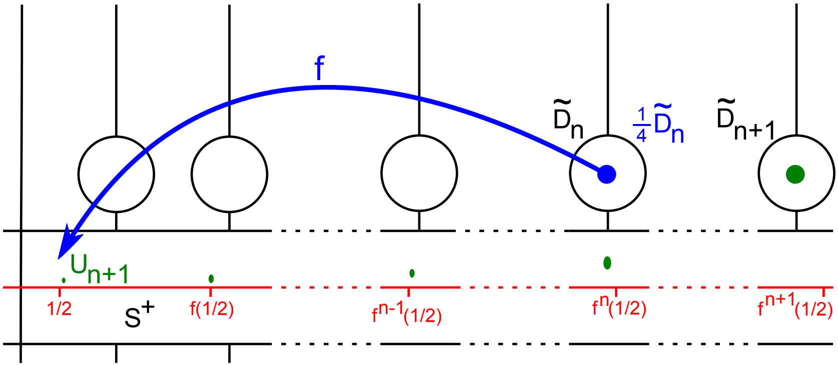


$f^n(U_n) = \frac{1}{4}\tilde{D}_n$ and $\mathrm{inradius}(U_n) \geqslant C.\left(\frac{df^n}{dx}\Big|_{x=\frac{1}{2}}\right)^{-1}$



$$f^n(U_n) = \tfrac{1}{4}\widetilde{D}_n \quad \text{and} \quad \text{inradius}(U_n) \geqslant C.\left(\frac{df^n}{dx}\Big|_{x=\frac{1}{2}}\right)^{-1}$$

$$f^{n+1}(U_{n+1}) = \tfrac{1}{4}\widetilde{D}_{n+1} \quad \text{and} \quad \text{inradius}(U_{n+1}) \geqslant C.\left(\frac{df^{n+1}}{dx}\Big|_{x=\frac{1}{2}}\right)^{-1}$$



$$f^{n+1}(U_{n+1}) = \frac{1}{4}\widetilde{D}_{n+1} \quad \text{and} \quad \mathrm{inradius}(U_{n+1}) \geqslant C.\left(\frac{df^{n+1}}{dx}\Big|_{x=\frac{1}{2}}\right)^{-1}$$

$$\widetilde{w}_n \in f\left(\frac{1}{4}\widetilde{D}_n\right) \quad \text{and} \quad \mathrm{diam}\left(f\left(\frac{1}{4}\widetilde{D}_n\right)\right) \leqslant C'.\left(\frac{1}{4}\right)^{\widetilde{d}_n}$$

Applications of Bishop's theorem

By considering appropriate trees and appropriate conformal maps, Bishop's theorem can be used to prove many different results (mostly in function theory but also in dynamics). We give two examples:

- (Bishop'12) There exists an ETF in class $\mathcal{S}$ for which the strong Eremenko conjecture fails to be true (i.e. not every point in the escaping set can be joined to ∞ by a curve in the escaping set)
- (Bishop'12) There exists an ETF with a spiriling tract that spirals arbitrarily fast.
- (F, Godillon and Jarque, '13) There exist $f, g \in \mathcal{B}$ having no wandering domains such that $f \circ g$ has a wandering domain. (This required to prove that Bishop's example has no "unexpected" wandering domains.)

12. Adding an essential singularity – Configurations of Herman rings

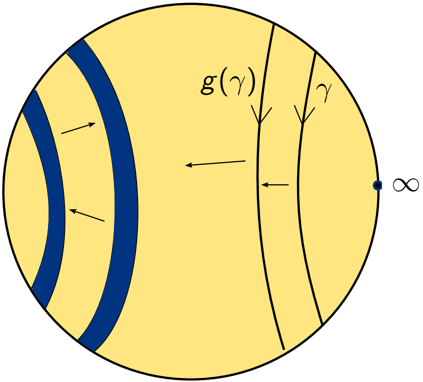
Shishikura in 1989 associated a tree to every configuration of Herman rings and showed that every tree satisfying certain conditions is realizable by a rational map. Later we it was shown that these configurations are also realizable by transcendental meromorphic maps.

Theorem (Peter, F. '12)

Let g be a rational function with Herman rings $A_1, \cdots, A_n$, realizing a certain configuration $\mathcal{C}$ in $\widehat{\mathbb{C}}$. Then, for every configuration $\mathcal{C}'$ in $\mathbb{C}$ "equivalent" to $\mathcal{C}$, there exists f meromorphic transcendental which realizes $\mathcal{C}'_{\mathbb{C}}$.

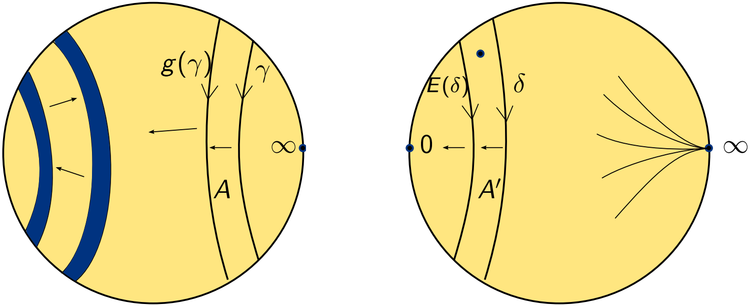
In other words, we can "glue" an essential singularity inside any component of $\widehat{\mathbb{C}} \setminus \cup_i A_i$, and produce a TMF realizing that configuration.

We shall sketch the surgery only in the simplest case, Shishikura's example of a nested two-cycle of Herman rings.



We want to "glue" an essential singularity at ∞ (or any point in $\mathrm{int}(\gamma)$). Consider $E(z) = \lambda(e^z - 1)$ with $|\lambda| < 1$ chosen so that

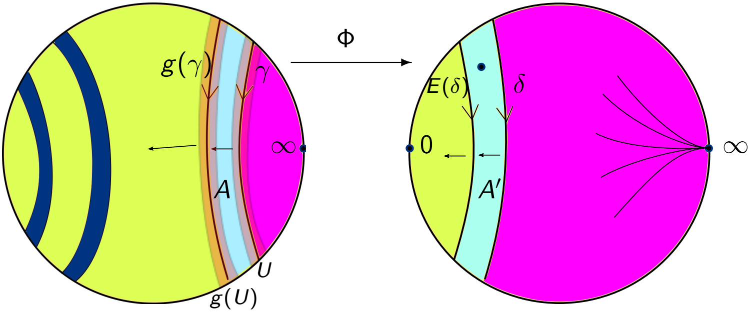
$$\text{mod }(A) = \quad \text{mod }(A_{|\lambda|,1}) = \quad \text{mod }(A').$$



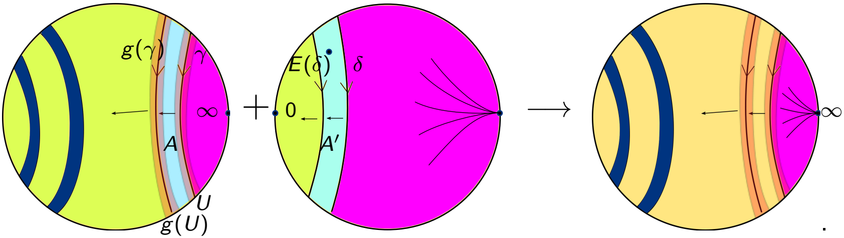
Construct a quasiconformal homeomorphism $\Phi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ (the gluing map) satisfying:

$$\Phi(\infty) = \infty; \quad \Phi(\gamma) = \delta; \quad \Phi(g(\gamma)) = E(\delta); \quad E \circ \Phi = \Phi \circ g \text{ on } \gamma$$

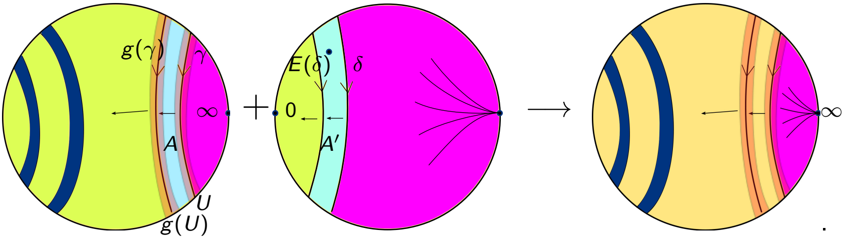
$$\Phi(\mathrm{int}(\gamma)) = \mathrm{int}(\delta); \quad \Phi(\mathrm{ext}(g(\gamma))) = \mathrm{ext}(E(\delta)); \quad \text{conformal off } U \cup g(U)$$



Adding an essential singularity



- Define a new map $F = \begin{cases} g & \text{on } \overline{\text{ext}(\gamma)} \\ \Phi^{-1} \circ E \circ \Phi & \text{on } \text{int}(\gamma). \end{cases}$.
- The iterates of F are uniformly $K-$quasiregular (since orbits pass through $U \cup g(U)$ at most once).
- Hence there exists $\varphi : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ qc fixing ∞ such that $f := \varphi^{-1} \circ F \circ \varphi$ is in $\mathcal{M}$ and realizes the "same" configuration of rings as g.



- Define a new map $$F = \begin{cases} g & \text{on } \overline{\text{ext}(\gamma)} \\ \Phi^{-1} \circ E \circ \Phi & \text{on } \text{int}(\gamma). \end{cases}$$
- Observe that we added no new poles: the poles of F are the poles of g and preimages under Φ of poles of E, which don't exist.
- Note that we added no new critical points (E has none) and one new asymptotic value at $\varphi(\Phi^{-1}(-\lambda)) \in \varphi(A)$.

13. Other applications

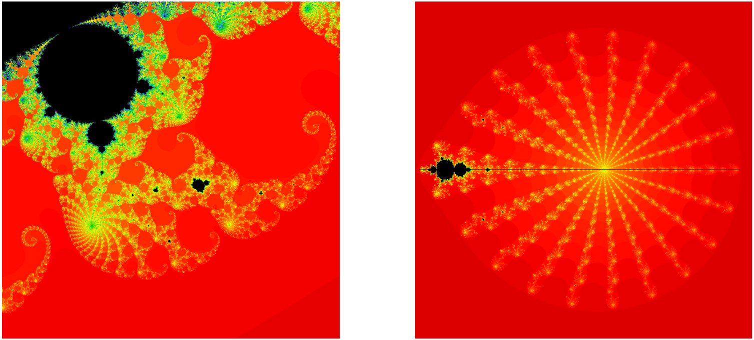
There are other surgery constructions with applications in transcendental dynamics that should have been here. One of them is due to Kisaka and Shishikura in 2005.

Theorem (Kisaka – Shisikura '05)

There exists a wandering domain W such that $f^n(W)$ has connectivity 2 for all $n \geq 0$.

- The regions where the model map g is quasiconformal are nested annuli A_m and g is $K_m - -$quasiconformal on A_m.
- Orbits visit every A_m at most once, but they may have the full orbit in the quasiconfromal reguion.
- It is essential that $\prod K_m = K_\infty < \infty$.
- This is an example where Sullivan's principle is useful.

Thank you for your attention!!!!



Homeomorphic bifurcation loci [B. Branner – N.F. 1999]

Interpolation on the infinite strip

We may explcitely define

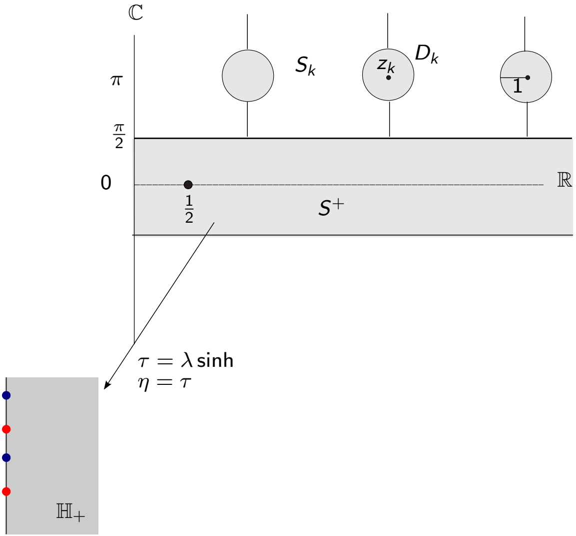
$$E_a(z) = E(z) + a\,\eta\left(\frac{|E(z)|}{\delta^2}\right)$$

where $\eta : \mathbb{R} \to \mathbb{R}$ is a $\mathcal{C}^\infty$ function equal to 1 in $(-\infty, 1)$ and equal to 0 in $(4, \infty)$.
Then one can actually compute the partial derivatives and prove that

$$|\mu_{E_a}| = \frac{\overline{\partial} E_a(z)}{\partial E_a(z)} \leq \frac{2M|a|}{\frac{\delta}{2} - 2M|a|} := k_a \xrightarrow[a \to 0]{} 0$$

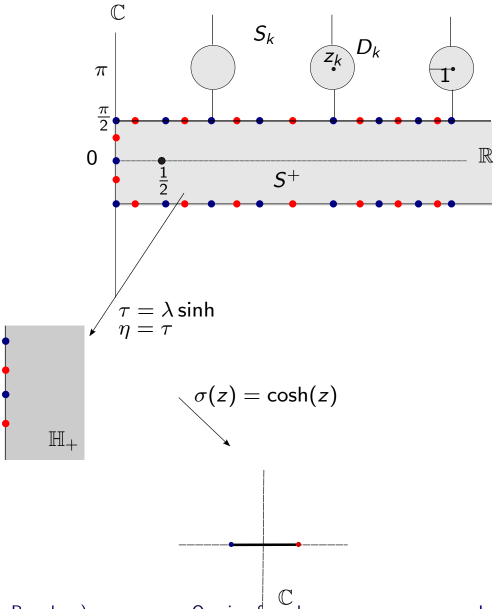
▸ Go back

Bishop's QC-folding for entire maps in class $\mathcal{B}$



Remark: The blue and red points correspond to $\pi i \mathbb{Z}$.

Bishop's QC-folding for entire maps in class $\mathcal{B}$



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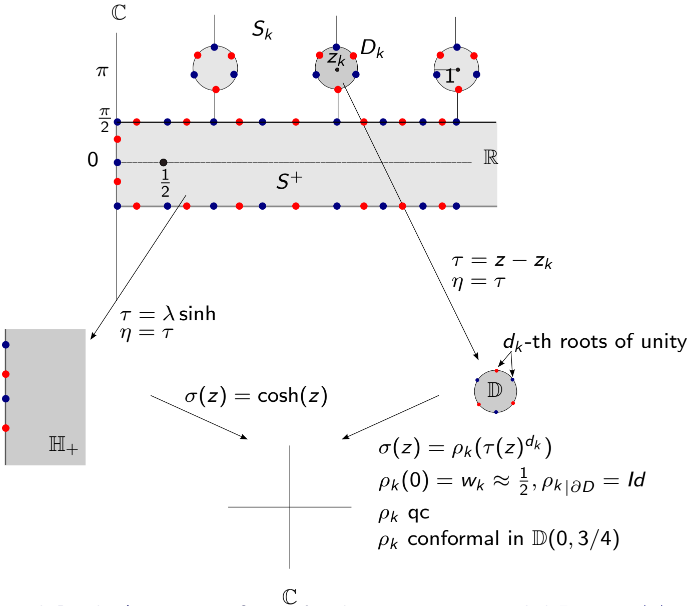

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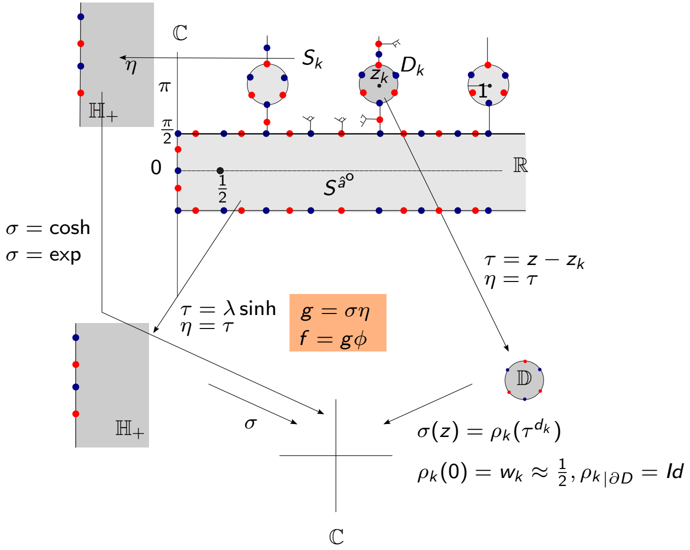


Bishop's QC-folding for entire maps in class $\mathcal{B}$

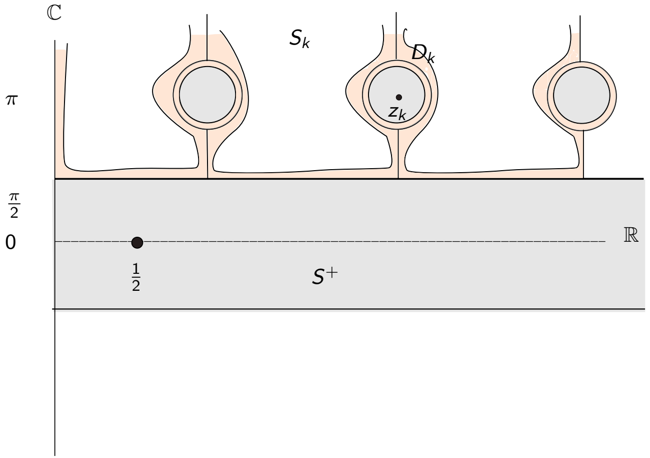
Important remarks about ϕ:

- ϕ is conformal on S^+ and $\frac{1}{4}D_k$'s.
- ϕ is uniformly K-quasiconformal for all the values of λ and d_k's.
- The dilatation of ϕ is supported inside $T(r_0)$ and this neighborhood decays exponentially in $|z|$.
- Moreover ϕ is symmetric (1-to-1 on $\mathbb{R}$), $\phi(0) = 0$, $\phi(\infty) = \infty$ and

$$\phi(z) = z + \frac{a}{z} + O\left(|z^{-2}|\right)$$

  for some $|z| > R$ (Dyn'kin's Theorem).
- ϕ' should be bounded by below from 0.
- Estimates get better when increasing the parameters.

Bishop's QC-folding for entire maps in class $\mathcal{B}$



Consequently: $f'(x) = \frac{d}{dx}\cosh\left(\lambda \sinh(\phi(x))\right) \geq 16x$ for λ large enough.
Hence we may choose λ so that $x_n = f^n\left(\frac{1}{2}\right)$ tends to infinity (exponential speed) as n does.