Infinitesimal Computations in Arithmetic Dynamics

Adam Epstein

University of Warwick

Parameter Spaces

- The parameter space $\mathbf{Rat}_D$ of all degree D rational maps
$$f : \mathbb{P}^1 \to \mathbb{P}^1$$
is a smooth affine algebraic variety of dimension $2D + 1$.
- The group **Aut** of all projective transformations acts on $\mathbf{Rat}_D$ by conjugation. For $D > 1$ the quotient moduli space $\mathbf{rat}_D$ is an affine algebraic variety of dimension $2D - 2$.

We consider various infinitesimal questions concerning such spaces.

Questions

- To what extent is the quotient projection

$$\mathbf{Rat}_D \to \mathbf{rat}_D$$

  a submersion ?
- When does iteration

$$\begin{array}{ccc} \mathbf{Rat}_D & \to & \mathbf{Rat}_{D^n} \\ f & \mapsto & f^n \end{array}$$

  induce an immersion ?
- These spaces have various dynamically significant subspaces, determined by such conditions as the existence of :
  - cycles with specified period and multiplier,
  - parabolic cycle with specified degeneracy and index,
  - critical orbit relations with specified combinatorics.

  When are these smooth ? When do they intersect transversely ?

Overview

We must address certain basic issues :

- We need to describe the tangent and cotangent spaces of $\mathbf{Rat}_D$ intrinsically. This is not merely a matter of aesthetics : we are interested in results valid over fields for which the standard machinery of complex analytic geometry is unavailable.
- We need a calculus for the intrinsic computation of derivatives and coderivatives.

We begin by reviewing notions and formalism from algebraic geometry.

Fields

Let $\mathbb{K}$ be an algebraically closed field.

- Up to isomorphism, there is a unique minimal algebraically closed field of any given characteristic :
  - $\overline{\mathbb{Q}}$ for characteristic 0,
  - $\overline{\mathbb{F}}_p$ for characteristic p.
- Various properties of algebraically closed fields, and all **first order** properties, depend only on the characteristic.
- **Lefschetz Principle** : There is just one algebraic geometry in any given characteristic.
- In particular, $\overline{\mathbb{Q}}$ and $\mathbb{C}$ yield the same algebraic geometry.

Varieties

A *variety* over $\mathbb{K}$ is a **locally ringed space** which is everywhere locally isomorphic to the maximal ideal spectrum of a finitely generated **reduced** K-algebra.

- The specification of a variety X consists of an underlying set equipped with a **Zariski topology**, together with a **structure sheaf** $\mathcal{O}_X$ organizing the data of which $\mathbb{K}$-valued functions are defined and holomorphic on which open subsets.
- Fundamental examples : Affine spaces $\mathbb{A}^n$, projective spaces $\mathbb{P}^n$, affine and (quasi-)projective varieties cut out by radical ideals in the corresponding polynomial rings.
- The formalism provides algebraic definitions of such notions as smoothness, tangent and cotangent bundles, and other machinery of differential geometry. These specialize to the standard notions for varieties over $\mathbb{C}$.
- **Reduced** means only trivial nilpotent elements : $h^m = 0 \Rightarrow h = 0$.

Morphisms

A *morphism* of $\mathbb{K}$-varieties

$$f : X \to Y$$

is specified by a continuous map between the underlying topological spaces which is suitably compatible with the structure sheaves :

$$h \in \mathcal{O}_Y(W) \;\Rightarrow\; h \circ f \in \mathcal{O}_X(f^{-1}(W))$$

for any open $W \subseteq Y$. A morphism between smooth varieties induces $\mathbb{K}$-linear maps between tangent spaces

$$D_x f : T_x X \to T_{f(x)} Y$$

and $\mathbb{K}$-linear maps between cotangent spaces

$$D_x^* f : T_{f(x)}^* Y \to T_x^* X.$$

Ramification

A morphism of smooth $\mathbb{K}$-varieties

$$f : X \to Y$$

is *inseparable* if $D_x f = 0$ at every $x \in X$, and *separable* otherwise.

- If $\operatorname{char} \mathbb{K} = 0$ then f is inseparable if and only if it is constant.
- If $\operatorname{char} \mathbb{K} = p$ then $f : \mathbb{P}^1 \to \mathbb{P}^1$ given by $z \mapsto z^p$ is inseparable.

A separable morphism of smooth algebraic curves has local degree 1 at all but finitely many points. Such a map has *wild* ramification at $x \in X$ if the local degree is a multiple of the field characteristic

$$\operatorname{char} \mathbb{K} | \deg_x f$$

and *tame* ramification otherwise. A morphism is *tame* if it is everywhere tamely ramified. If $\operatorname{char} \mathbb{K} = 0$ then every nonconstant morphism is tame.

A version of the Riemann-Hurwitz Theorem applies to separable morphisms of curves. Such a morphism $f : X \to Y$ has a ramification divisor $\mathbf{\Gamma}_f$ whose order at x is the vanishing order of $D_x f$: this quantity is always at least $\deg_x f$, with equality if and only if f is tamely ramified at x.

- For any separable $f : \mathbb{P}^1 \to \mathbb{P}^1$ of degree D, the ramification divisor $\mathbf{\Gamma}_f$ has degree $2D - 2$.
- If $\mathrm{char}\,\mathbb{K} = p$ the morphism given by $z \mapsto z + z^p$ is wild, and it is ramified only at ∞.

Sheaves

We work mainly with the sheaves associated to line bundles on $\mathbb{P}^1$:

- Θ the sheaf of germs of holomorphic vector fields,
- Ω the sheaf of germs of holomorphic differentials,
- $\mathcal{Q}$ the sheaf of germs of holomorphic quadratic differentials.

Given such a sheaf $\mathcal{L}$ and a divisor $\mathbf{D}$ on $\mathbb{P}^1$, we denote by $\mathcal{L}_{\mathbf{D}}$ the sheaf of germs of meromorphic sections s with

$$\mathrm{ord}_x s + \mathrm{ord}_x \mathbf{D} \geq 0$$

of the original line bundle. The quotient $\mathcal{L}_{\mathbf{D}}/\mathcal{L}$ is a *skyscraper sheaf*.

Cohomology

A short exact sequence of sheaves

$$0 \to \mathcal{A} \to \mathcal{B} \to \mathcal{C} \to 0$$

induces a llong exact sequence of cohomology groups

$$0 \to H^0(\mathcal{A}) \to H^0(\mathcal{B}) \to H^0(\mathcal{C}) \to H^1(\mathcal{A}) \to H^1(\mathcal{B}) \to H^1(\mathcal{C}) \to \cdots .$$

The functor H^0 delivers tthe space of global sections. For sheaves given by line bundles on smooth algebraic curves, the functor H^1 may be computed via répartitions (adeles).

Tangent Spaces

Proposition

For any $f \in \mathbf{Rat}_D$,

$$T_f\mathbf{Rat}_D \overset{\text{can}}{\cong} H^0(f^\star\Theta)$$
$$\overset{\text{can}}{\cong} H^0(\Theta_{\Gamma_f}) \quad \textit{if } f \textit{ is separable.}$$

Proof : For a curve $\lambda \mapsto f_\lambda$ in $\mathbf{Rat}_D$ with $f_0 = f$, the tangent is given by the function which sends $x \in \mathbb{P}^1$ to the vector $\dot{f}(x) \in T_f\mathbb{P}^1$ tangent to the curve $x \mapsto f_\lambda(x)$ in $\mathbb{P}^1$. If f is separable then $D_X f$ is invertible at all but finitely many $x \in \mathbb{P}^1$ whence

$$(D_x f)^{-1}\dot{f}(x)$$

is a meromorphic vector field on $\mathbb{P}^1$. □

Composition

Composition induces morphisms

$$\begin{array}{ccc} \mathbf{Rat}_{D_1} \times \mathbf{Rat}_{D_2} & \to & \mathbf{Rat}_{D_2 D_1} \\ (f_1, f_2) & \mapsto & f_2 \circ f_1 \end{array} .$$

Proposition

At separable f_1, f_2 the derivative of composition is given by

$$\begin{array}{ccc} H^0(\Theta_{\Gamma_{f_1}}) \oplus H^0(\Theta_{\Gamma_{f_2}}) & \to & H^0(\Theta_{\Gamma_{f_2 \circ f_1}}) \\ (\mathfrak{v}_1, \mathfrak{v}_2) & \mapsto & \mathfrak{v}_1 + f_1^* \mathfrak{v}_2 \end{array} .$$

Proof : Chain Rule. □

Orbits

Conjugation induces morphisms

$$\begin{array}{ccc} \mathbf{Aut} & \to & \mathbf{Rat}_D \\ \alpha & \mapsto & \alpha^{-1} \circ f \circ \alpha \end{array} .$$

Corollary

For separable f the derivative of the orbit at the identity is given by

$$\begin{array}{ccc} H^0(\Theta) & \to & H^0(\Theta_{\Gamma_f}) \\ \mathfrak{v} & \mapsto & \mathfrak{v} - f^*\mathfrak{v} \end{array} .$$

Iteration

Iteration induces endomorphisms

$$\begin{array}{ccc} \mathbf{Rat}_D & \to & \mathbf{Rat}_{D^n} \\ f & \mapsto & f^n \end{array} .$$

Corollary

At separable f the derivative of iteration is given by

$$\begin{array}{ccc} H^0(\Theta_{\Gamma_f}) & \to & H^0(\Theta_{\Gamma_{f^n}}) \\ \mathfrak{v} & \mapsto & \sum_{k=0}^{n-1} (f^k)^* \mathfrak{v} \end{array} .$$

Invariant Vector Fields

Theorem

Let $f \in \mathbf{Rat}_D$ be separable, let $\mathfrak{v} \neq 0$ a meromorphic vector field on $\mathbb{P}^1$, and suppose $f^*\mathfrak{v} = \lambda\mathfrak{v}$ for some $\lambda \in \mathbb{K}$.

- *In this situation, $\lambda \neq 0$ and $\mathfrak{v}$ is holomorphic.*
- *Furthermore, if f is tame then $\lambda = \pm\frac{1}{D}$ and, up to conjugacy, $f(z) = z^{\pm D}$ with $\mathfrak{v}$ a scalar multiple of $z\frac{\partial}{\partial z}$.*

Thus, if $\operatorname{char}\mathbb{K} = 0$ and $D > 1$ then $f^*\mathfrak{v} \neq \mathfrak{v}$. However, if $\operatorname{char}\mathbb{K} = p$,

- $f^*\mathfrak{v} = \mathfrak{v}$ for the tame $f(z) = z^{\pm D}$ and $\mathfrak{v} = z\frac{\partial}{\partial z}$ when $p|(D \pm 1)$.
- $f^*\mathfrak{v} = \lambda\mathfrak{v}$ for the wild $f(z) = \frac{1}{\lambda}z + z^p$ and $\mathfrak{v} = \frac{\partial}{\partial z}$.

Lemma

Let $f \in \mathbf{Rat}_D$ where $D > 1$. Then any finite backward invariant set contains at most two points. Moreover, any such point is periodic of period 1 or 2, and f has local degree D at any such point.

Proof : Since $\mathbb{K}$ is algebraically closed, if x has finite backward orbit then some point in the backward orbit is periodic, hence every point in the backward orbit of x is periodic, whence each is the unique preimage of its image. Thus, f has local degree D at every such point, so the period is at most 2. $\square$

Proof of Theorem : By the invariance of $\mathfrak{v}$, for any point x

$$\mathrm{ord}_x \mathfrak{v} - 1 = \mathrm{ord}_x f^* \mathfrak{v} - 1 \leq \deg_x f \cdot (\mathrm{ord}_{f(x)} \mathfrak{v} - 1)$$

with equality if and only if f is tamely ramified at x. Thus, $\lambda \neq 0$, since f is tamely ramiified at all but finitely many points. Moreover,

$$\mathrm{ord}_{f(x)} \mathfrak{v} < 0 \quad \Rightarrow \quad \mathrm{ord}_x \mathfrak{v} < 0$$

and if $f(x) = x$ then

$$\mathrm{ord}_x \mathfrak{v} < 0 \quad \Rightarrow \quad \deg_x f = 1.$$

Thus, the pole set of $\mathfrak{v}$ is a finite backward invariant set containing no fixed critical points. Furthermore, if f is tame then

$$\mathrm{ord}_{f(x)} \mathfrak{v} > 0 \quad \Rightarrow \quad \mathrm{ord}_x \mathfrak{v} > 0$$

so the zero set of $\mathfrak{v}$ is also a finite backward invariant set. The conclusions follow by the Lemma. □

Immersions

Proposition

If $\operatorname{char}\mathbb{K} = 0$ *and* $D > 1$*, or if* $\operatorname{char}\mathbb{K} \not| (D \pm 1)$ *and* f *is tame, then*

$$\begin{array}{ccc} \mathbf{Aut} & \to & \mathbf{Rat}_D \\ \alpha & \mapsto & \alpha^{-1} \circ f \circ \alpha \end{array}$$

is an immersion.

Proof : The derivative at α is the linear map

$$\begin{array}{ccc} H^0(\Theta) & \to & H^0(\Theta_{\Gamma_f}) \\ \mathfrak{v} & \mapsto & \mathfrak{v} - f^*\mathfrak{v} \end{array}$$

whose kernel consists of all $\mathfrak{v}$ such that $f^*\mathfrak{v} = \mathfrak{v}$. $\square$

Immersions

Corollary

In the above setting, if f has trivial automorphism group then $\mathbf{rat}_D$ *is smooth at the corresponding point, and the quotient projection*

$$\mathbf{Rat}_D \rightarrow \mathbf{rat}_D$$

is a submersion at f.

Proposition

If $\mathrm{char}\,\mathbb{K} = 0$ *and* $D > 1$*, or if* $D < \mathrm{char}\,\mathbb{K} \nmid ((\pm D)^n - 1)$*, then iteration*

$$\mathbf{Rat}_D \;\to\; \mathbf{Rat}_{D^n}$$

is an immersion.

Proof : If $\mathrm{char}\,\mathbb{K} = 0$ or $\mathrm{char}\,\mathbb{K} > D$ then every $f \in \mathbf{Rat}_D$ is tame, hence separable. At separable f, the derivative of immersion is the linear map

$$\begin{array}{ccc} H^0(\Theta_{\Gamma_f}) & \to & H^0(\Theta_{\Gamma_{f^n}}) \\ \mathfrak{v} & \mapsto & \sum_{k=0}^{n-1} (f^k)^* \mathfrak{v} \end{array}$$

whose kernel consists of all $\mathfrak{v}$ such that $f^*\mathfrak{v} = \lambda\mathfrak{v}$ for some $\lambda \in \mathbb{K}$ with $\sum_{k=0}^{n-1} \lambda^k = 0$, and $\lambda^n = 1$ for any such λ. □

Corollary

In the above setting, the iteration morphism

$$\mathbf{Rat}_D \;\rightarrow\; \mathbf{Rat}_{D^n}$$

has finite fibres.

Lemma

For rational $f : \mathbb{P}^1 \to \mathbb{P}^1$ and any fixed point ζ,

- *The variation of the multiplier is given by $[q]_\zeta$ where $q = \frac{1+o(1)}{z^2}\,dz^2$ has invianriant polar part.*
- *The variation of the holomorphic index is given by $[\frac{f'(z)}{(z-f(z))^2}\,dz^2]_\zeta$.*

Infinitesimal Holomorphic Index Formula :

$$\sum_{f(\zeta)=\zeta} [\frac{f'(z)}{(z-f(z))^2}\,dz^2]_\zeta$$

is 0 in $H^1(\mathcal{Q}_{-\mathbf{\Gamma}_f})$.

Theorem

For any proper subset of the fixed point set, the corresponding indices yield independent local coordinates for $\mathbf{Rat}_D$ at any f.

Short Exact Sequences

Let A and B be finite subsets of $\mathbb{P}^1$ such that $\#A \geq 3$ and $B \supseteq A \cup f(A) \cup S(f)$ where $S(f)$ is the critical value set of f.

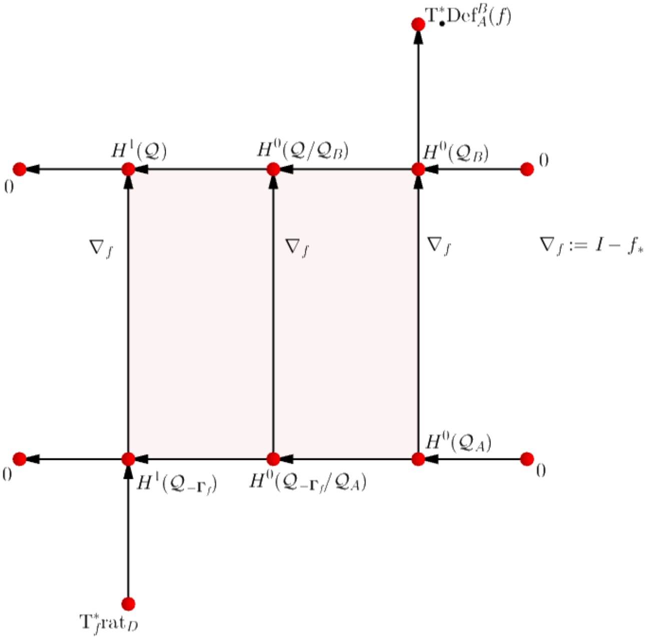
Consider the morphism of short exact sequences of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Theta_{-B} & \longrightarrow & \Theta & \longrightarrow & \Theta/\Theta_{-B} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Theta_{-A} & \longrightarrow & \Theta_{\mathbf{\Gamma}_f} & \longrightarrow & \Theta_{\mathbf{\Gamma}_f}/\Theta_{-A} & \longrightarrow & 0 \end{array}$$

where the vertical arrows are given by $I - f^*$.

There is an induced morphism of long exact sequences in cohomology.

Serre Duality yields the following diagram of $\mathbb{K}$-linear maps :


$T^*_\bullet \mathrm{Def}_A^B(f)$
$H^1(\mathcal{Q})$
$H^0(\mathcal{Q}/\mathcal{Q}_B)$
$H^0(\mathcal{Q}_B)$
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$\nabla_f := I - f_*$
$H^0(\mathcal{Q}_A)$
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$H^1(\mathcal{Q}_{-\Gamma_f})$
$H^0(\mathcal{Q}_{-\Gamma_f}/\mathcal{Q}_A)$
$T^*_f \mathrm{rat}_D$

Now let **A** and **B** be positive divisors with support A and B.

Consider the morphism of short exact sequences of sheaves

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Theta_{-\mathbf{B}} & \longrightarrow & \Theta_{-B} & \longrightarrow & \Theta_{-B}/\Theta_{-\mathbf{B}} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \Theta_{-\mathbf{A}} & \longrightarrow & \Theta_{-A} & \longrightarrow & \Theta_{-A}/\Theta_{-\mathbf{A}} & \longrightarrow & 0 \end{array}$$

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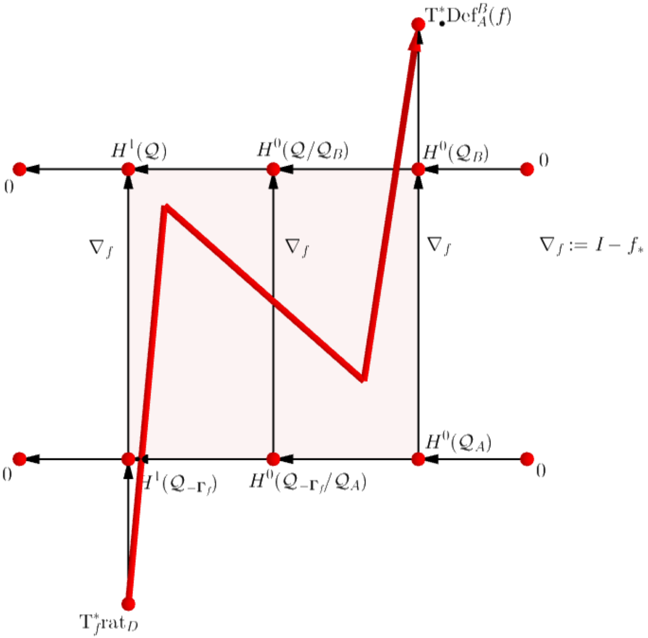

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$\nabla_f := I - f_*$
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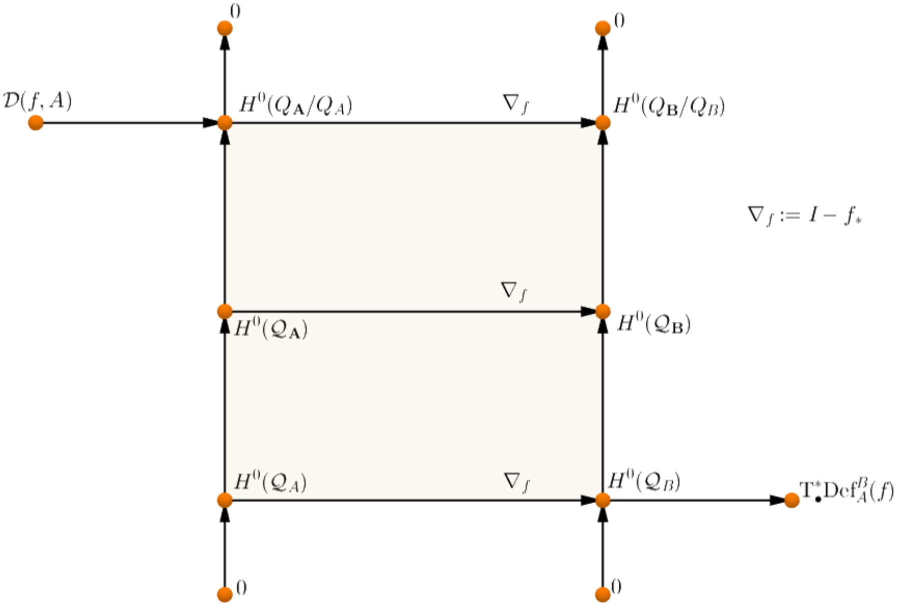
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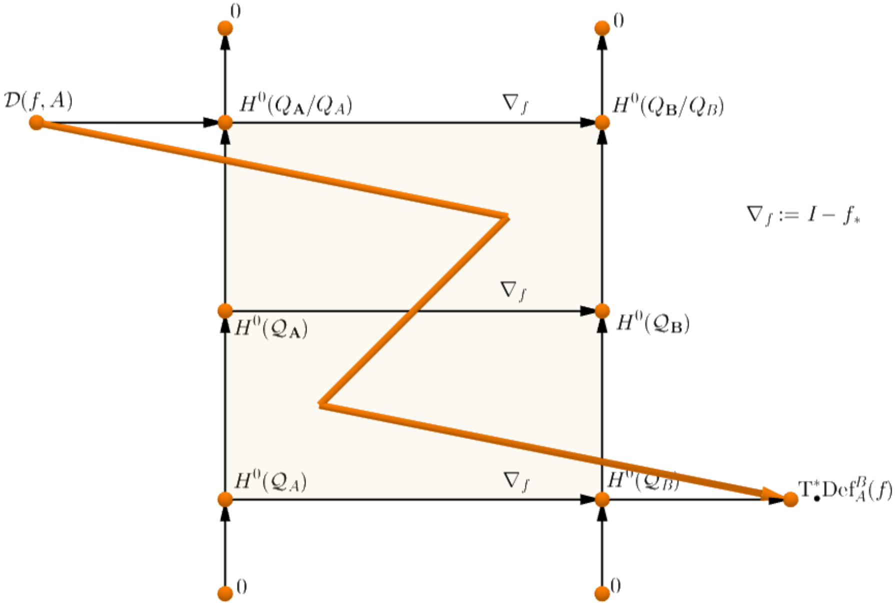
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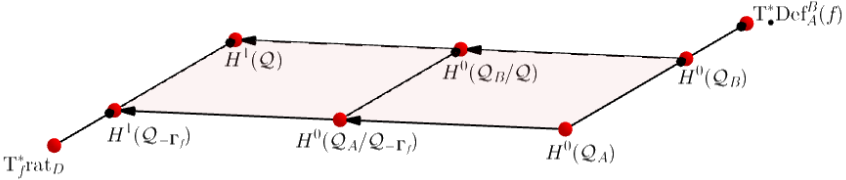
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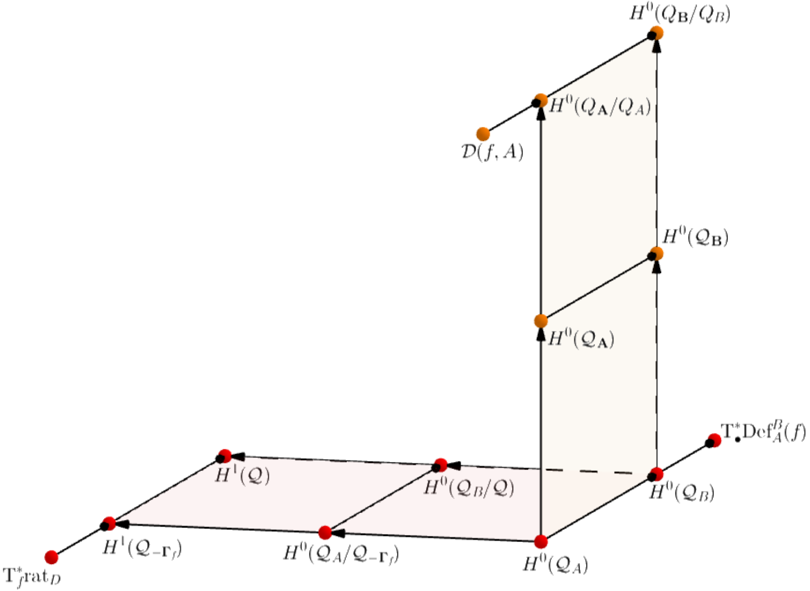
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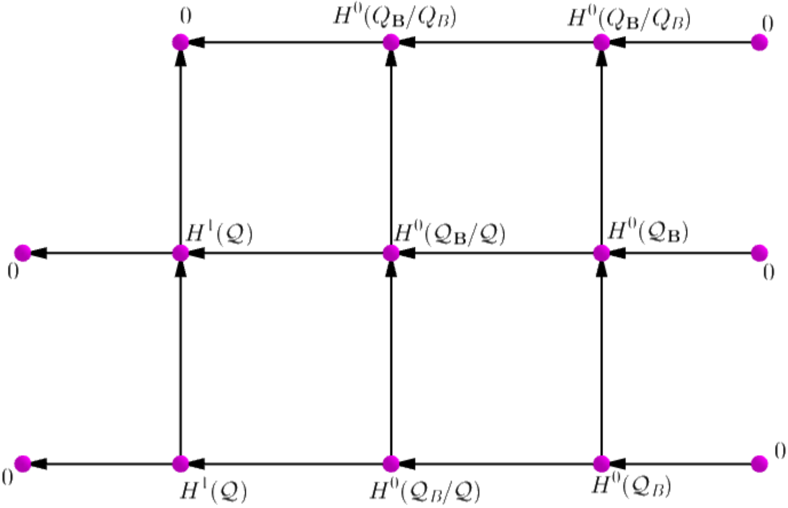
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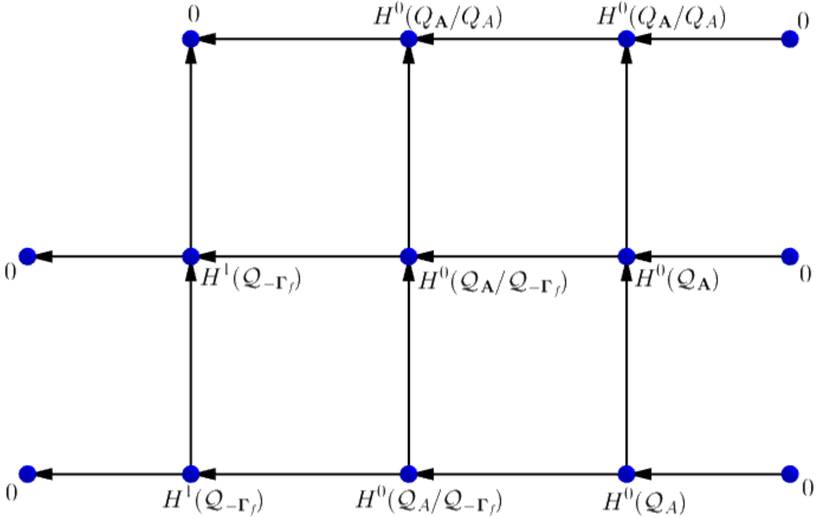
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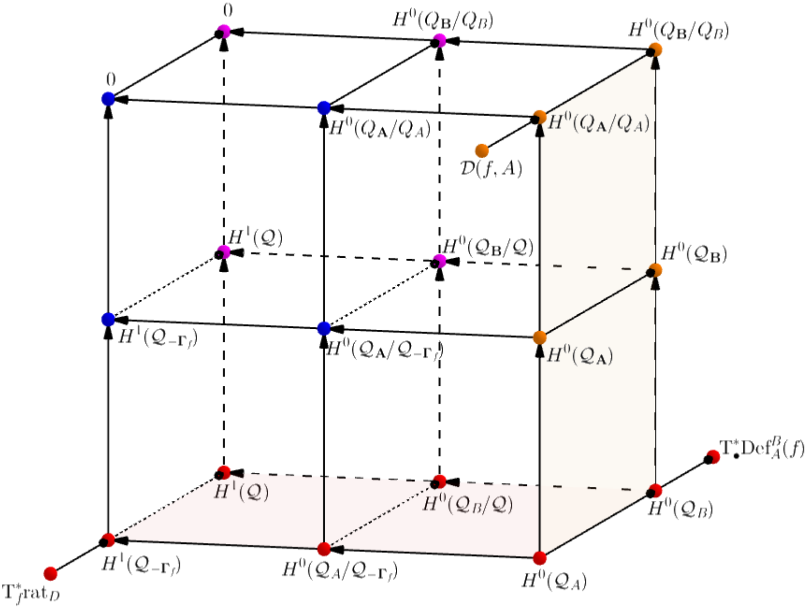
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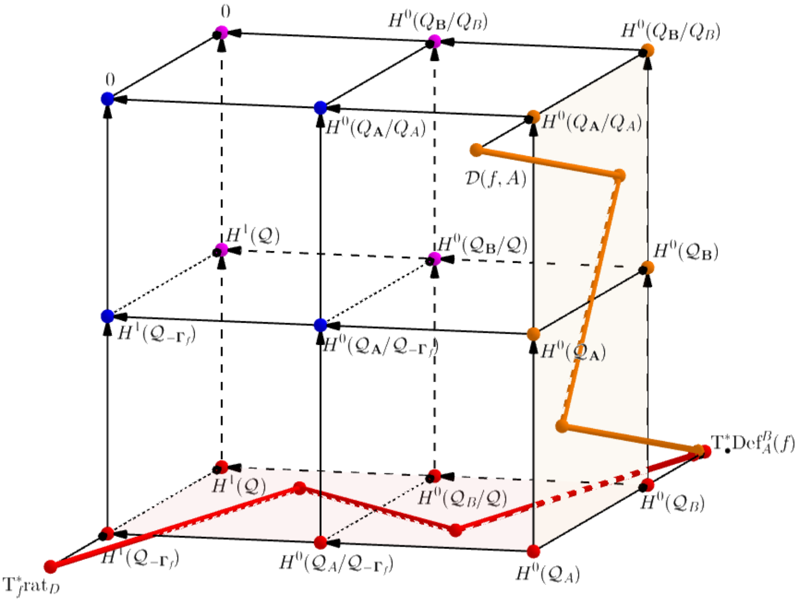

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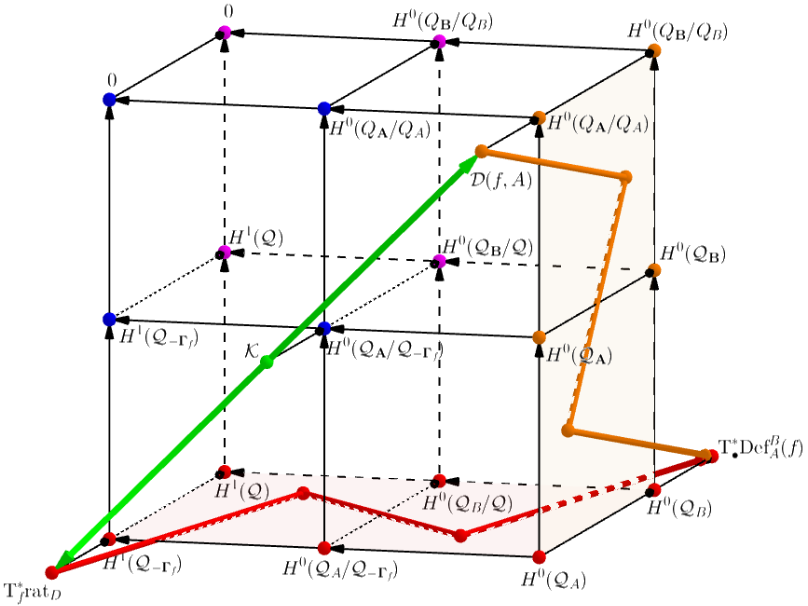
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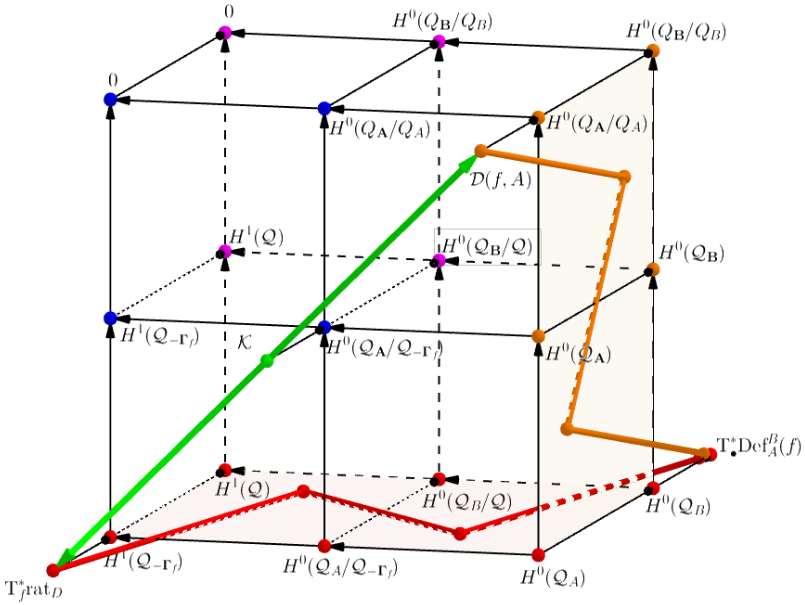
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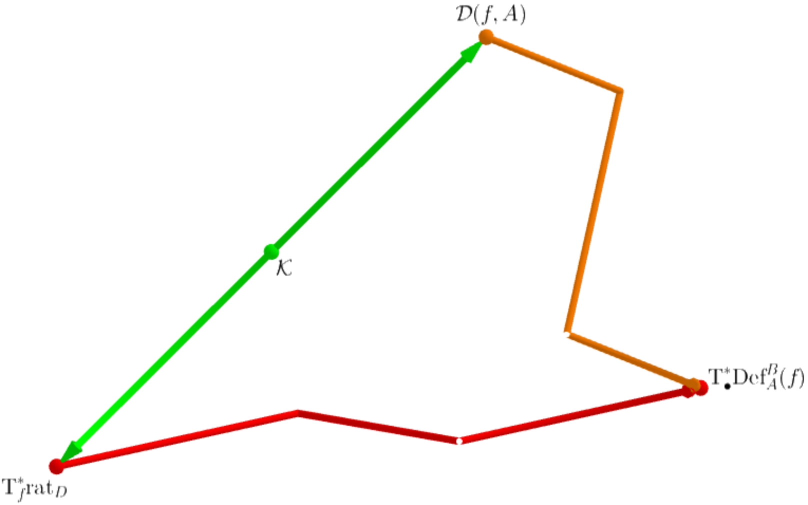
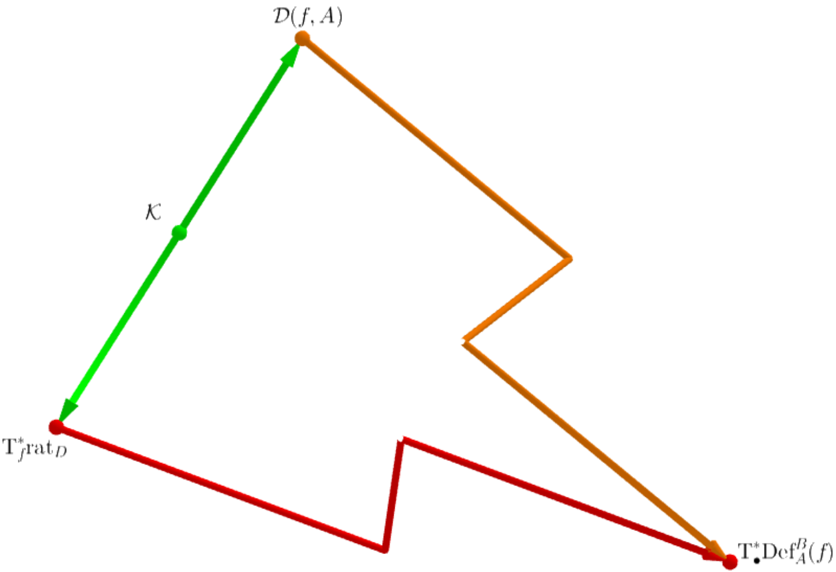
$H^0(\mathcal{Q}_B/\mathcal{Q})$
$H^0(\mathcal{Q}_B)$
$H^1(\mathcal{Q}_{-\mathbf{r}_f})$
$H^0(\mathcal{Q}_A/\mathcal{Q}_{-\mathbf{r}_f})$
$H^0(\mathcal{Q}_A)$
$\mathrm{T}^*_f\mathrm{rat}_D$

n


$\mathcal{D}(f, A)$
$\mathcal{K}$
$\mathrm{T}^*_\bullet\mathrm{Def}^B_A(f)$
$\mathrm{T}^*_f\mathrm{rat}_D$


$\mathcal{D}(f, A)$
$\mathcal{K}$
$\mathrm{T}_f^*\mathrm{rat}_D$
$\mathrm{T}_\bullet^*\mathrm{Def}_A^B(f)$

Thus,

$$\mathcal{K} \cong T^*_{[f]}\mathbf{rat}_D \oplus \mathcal{D}(f, A)$$

canonically, and the maps

$$\mathcal{K} \to T^*_\bullet Def^B_A(f)$$

sum to 0. It follows that if $\mathbf{q}$ is a system of invariant polar parts of quadratic differentials, and if q is any meromorphic quadratic differential on $\mathbb{P}^1$ with the corresponding invariant divergences and with all poles in A, then

$$\langle \blacktriangle_f \varpi, [\mathbf{q}] \rangle = -\langle \varpi, \nabla_f q \rangle$$

where

$$\blacktriangle_f : T_\bullet Def(f) \to T_{[f]}\mathbf{rat}_D$$

is the connecting homomorphism.

Happy Birthday, Jack !