Rotation Sets and Polynomial Dynamics

Jackfest, Cancún
May 30, 2016

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For every rational number p/q there is a unique periodic orbit in $\mathbb{R}/\mathbb{Z}$ under the doubling map $t \mapsto 2t \pmod{\mathbb{Z}}$ whose rotation number is p/q:

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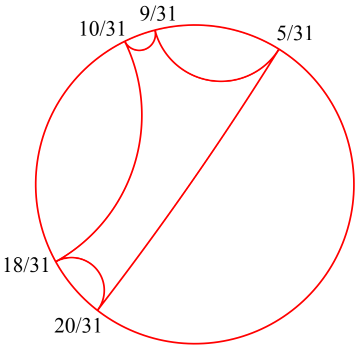

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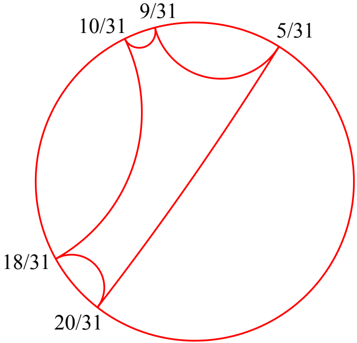

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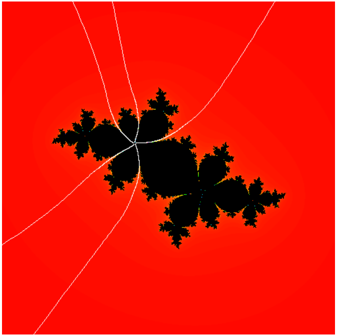

$$z \mapsto e^{2\pi i(2/5)}z + z^2$$

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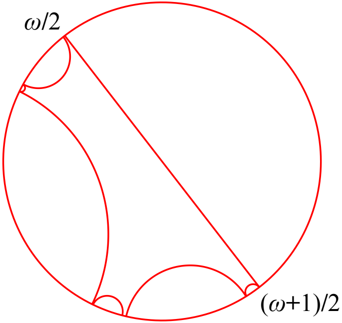

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$$\omega = 0.7098034428\cdots$$

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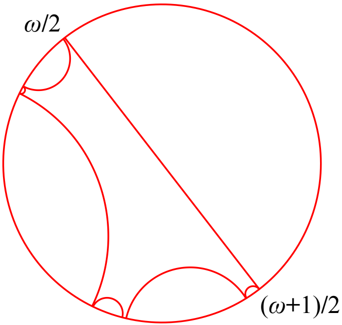

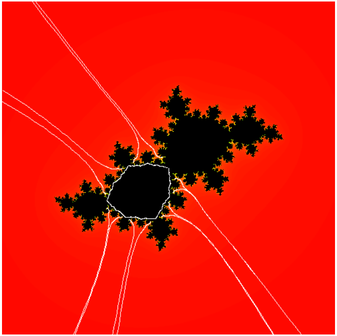

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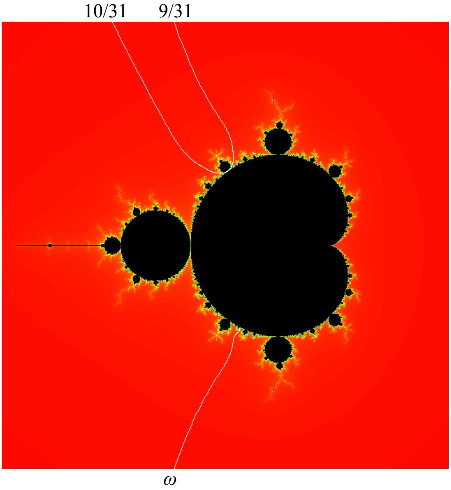
These “rotation sets” describe angles of the external rays that land on the boundary of the main cardioid of the Mandelbrot set:



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- Abstract part: Understanding the structure of rotation sets under multiplication by $d \geq 2$.

- Concrete part: Realizing rotation sets in suitable spaces of degree d polynomials.

2. Earlier work

(1993) Goldberg and Milnor: Rational rotation sets, fixed point portraits of polynomials

(1994) Bullett and Sentenac: Rotation sets under doubling

(1996) Goldberg and Tresser: Irrational rotation sets via Farey trees

(2006) Blokh, Malaugh, Mayer, Oversteegen, and Parris: Rotation sets under multiplication by d

(2015) Bonifant, Buff, and Milnor: Rotation sets under tripling, antipode preserving cubic maps

3. Rotation sets

Fix an integer $d \geq 2$.

$m_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the ***multiplication by*** d map defined by

$$m_d(t) = d\,t \pmod{\mathbb{Z}}$$

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Definition

A non-empty compact set $X \subset \mathbb{R}/\mathbb{Z}$ is a ***rotation set*** for m_d if

- $m_d(X) = X$, and
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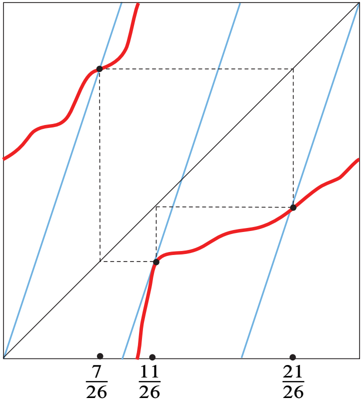
The ***rotation number*** $\rho(X) \in [0, 1)$ is defined as the rotation number of any degree 1 monotone extension of $m_d|_X$.

3. Rotation sets

Example:

$$X : \frac{7}{26} \overset{m_3}{\longmapsto} \frac{21}{26} \overset{m_3}{\longmapsto} \frac{11}{26} \qquad \rho(X) = \frac{2}{3}$$



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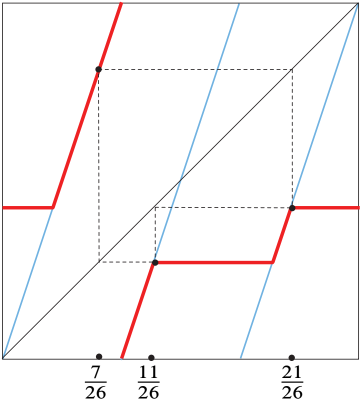

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Every rotation set is nowhere dense, whereas a randomly chosen point on the circle has a dense orbit under m_d.

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Theorem

The union of all rotation sets for m_d has Lebesgue measure zero.

4. Gaps

Let X be a rotation set for m_d.

Definition

- Each connected component I of $(\mathbb{R}/\mathbb{Z}) \smallsetminus X$ is called a ***gap*** of X.
- I is ***minor*** if $|I| < 1/d$, and ***major*** otherwise.
- I is ***taut*** if $|I| = n/d$ for some integer n, and ***loose*** otherwise.
- The ***multiplicity*** of I is the integer part of $d|I|$.

4. Gaps

Assume $\rho(X) \neq 0$. Define the ***standard monotone map*** g as follows:

On a minor gap, set $g = m_d$.

On a major gap $(a, a + \ell)$ of multiplicity n, set

$$g(t) = \begin{cases} m_d(a) & t \in (a, a + n/d] \\ m_d(t) & t \in (a + n/d, a + \ell). \end{cases}$$

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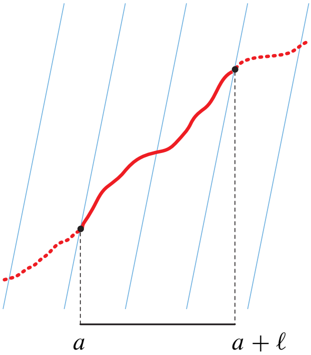

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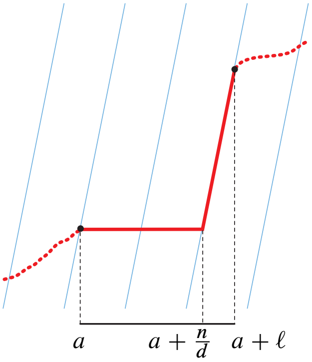

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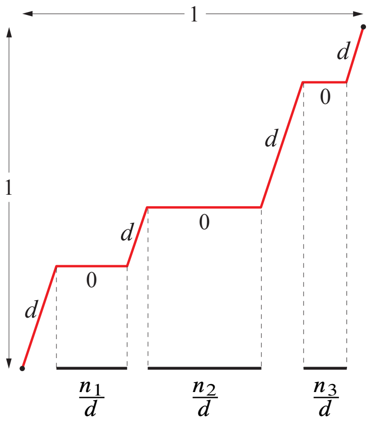

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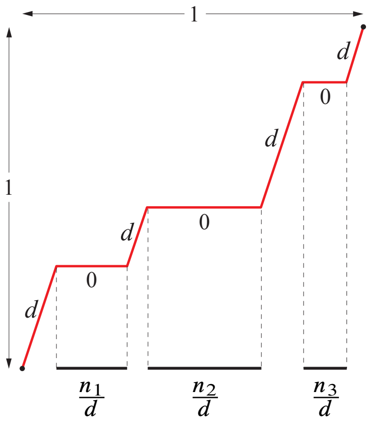

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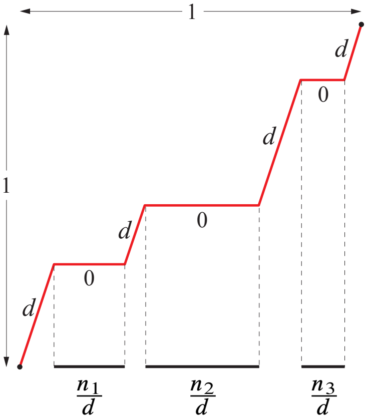

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$$\sum \frac{n_i}{d} = 1 - \frac{1}{d}$$

$$\Longrightarrow \sum n_i = d - 1.$$

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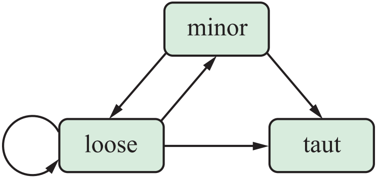

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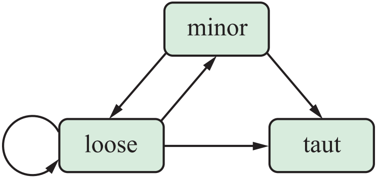

Corollary

If $\rho(X)$ is irrational, every gap of X eventually maps to a taut gap. In particular, at least one major gap of X is taut.

5. Minimal rotation sets

• Let X be a ***minimal*** rotation set with $\rho(X) = \theta$. Then X is a q-cycle if $\theta = p/q$ and is a Cantor set if θ is irrational.

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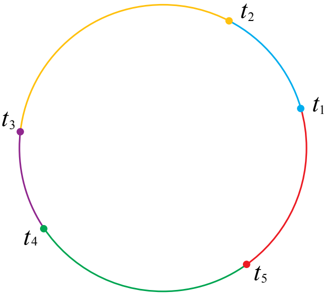
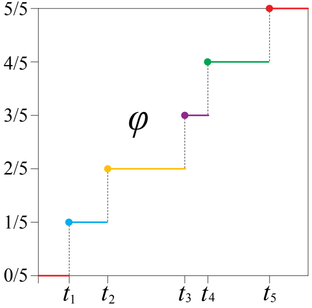
• There is a degree 1 monotone map $\varphi : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, normalized by $\varphi(0) = 0$, which satisfies

$$\varphi \circ m_d = R_\theta \circ \varphi \qquad \text{on } X$$

and is constant on every gap of X. We call this φ the ***semiconjugacy*** associated with X.

5. Minimal rotation sets





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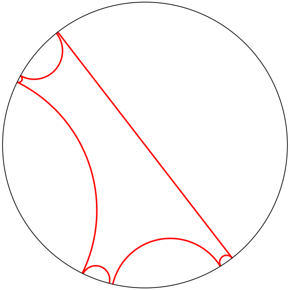

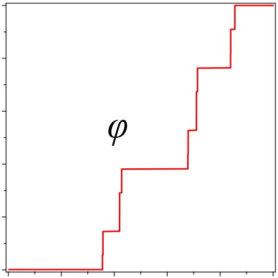

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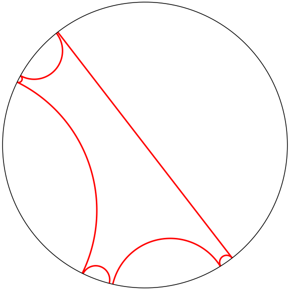

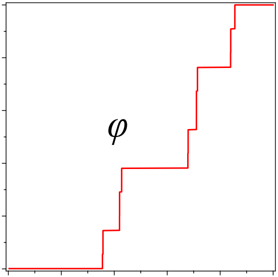

- X supports a unique m_d-invariant probability measure μ, which satisfies

$$\varphi(t) = \int_0^t d\mu.$$

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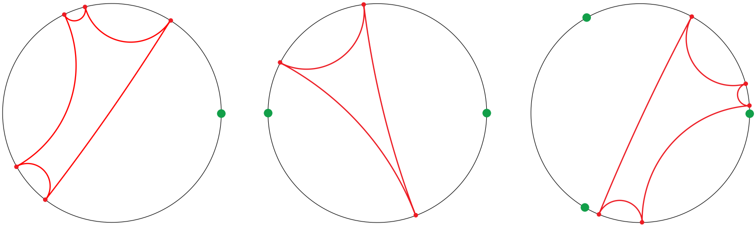

$d = 2$ $\quad$ $d = 3$ $\quad$ $d = 4$

5. Minimal rotation sets

Definition

The ***deployment vector*** of X is

$$\delta(X) = (\delta_1, \ldots, \delta_{d-1}) \in \Delta^{d-2} \subset \mathbb{R}^{d-1},$$

where

$$\delta_i = \mu[z_{i-1}, z_i).$$

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Note that when $\theta = p/q$ in lowest terms, $q\delta(X) \in \mathbb{Z}^{d-1}$.

5. Minimal rotation sets

Theorem (Goldberg-Tresser)

Given an "admissible" pair $(\theta, \delta) \in (\mathbb{R}/\mathbb{Z}) \times \Delta^{d-2}$ *there is a unique minimal rotation set* $X = X_{\theta,\delta}$ *with* $\rho(X) = \theta$ *and* $\delta(X) = \delta$.

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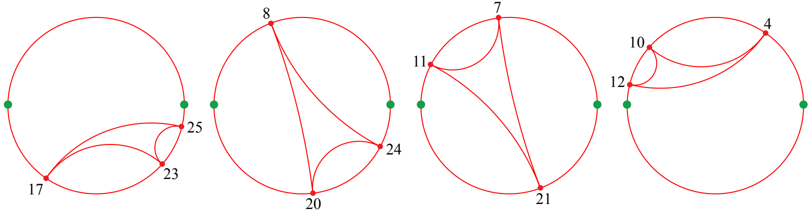
Thus, the space of all minimal rotation sets for m_d of a given rotation number θ is

- finite with $\binom{q+d-2}{q}$ elements if $\theta = p/q$.
- isomorphic to the simplex Δ^{d-2} if θ is irrational.

6. The cubic case

Example: Under the tripling map m_3, there are four 3-cycles with rotation number $\theta = 2/3$:



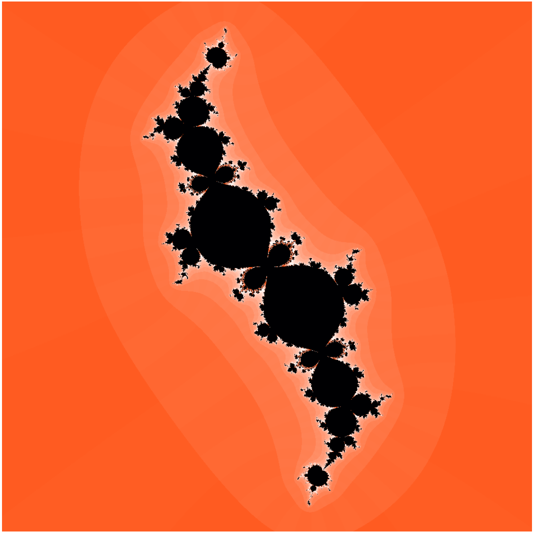
A	B	C	D
$\delta = (0, 1)$	$\delta = (\frac{1}{3}, \frac{2}{3})$	$\delta = (\frac{2}{3}, \frac{1}{3})$	$\delta = (1, 0)$

6. The cubic case

Connectedness locus of the cubic family

$$f_a(z) = e^{2\pi i\theta} z + az^2 + z^3 \qquad \text{with} \qquad a \in \mathbb{C}$$

6. Cubic polynomials



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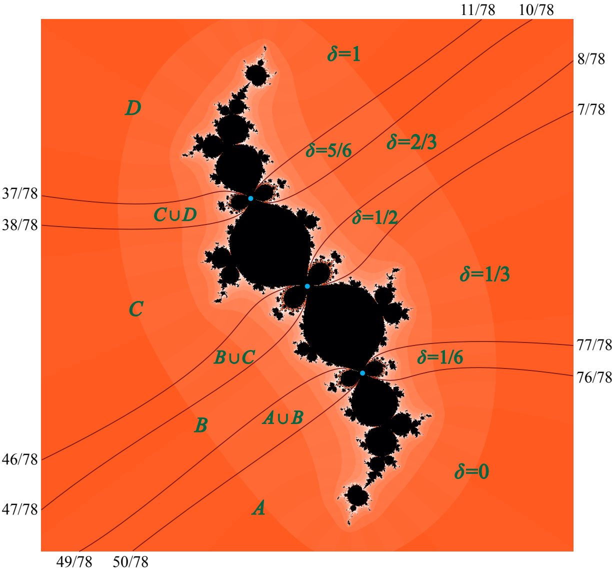

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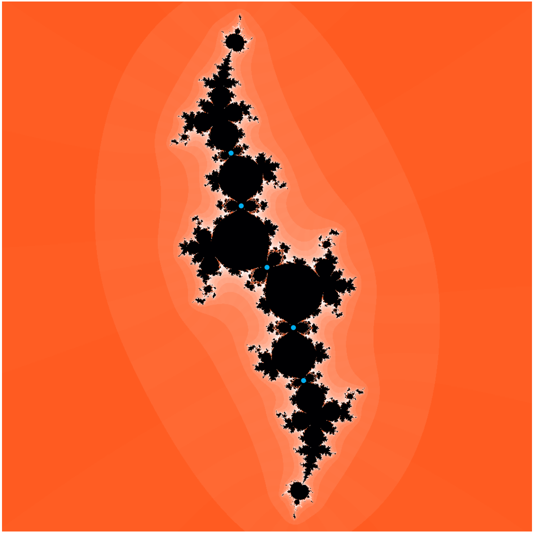

$\theta = 3/5$

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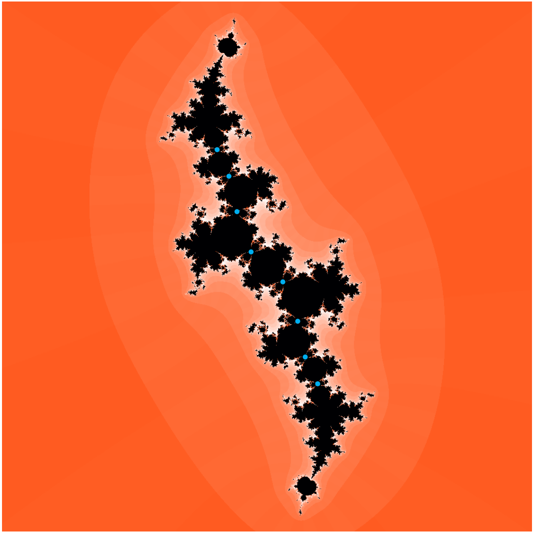

$$\theta = 5/8$$

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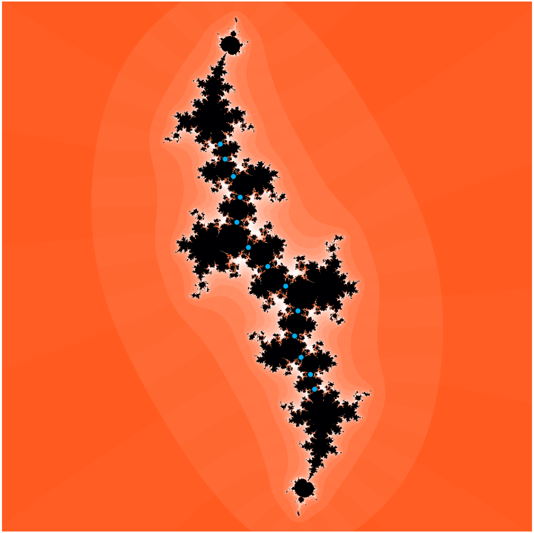

$\theta = 8/13$

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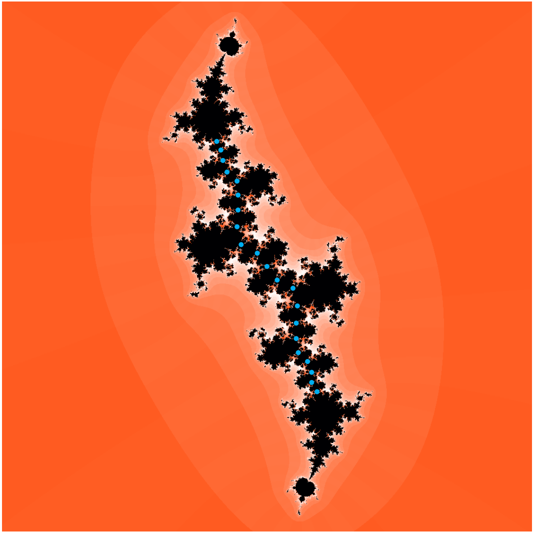

$\theta = 13/21$

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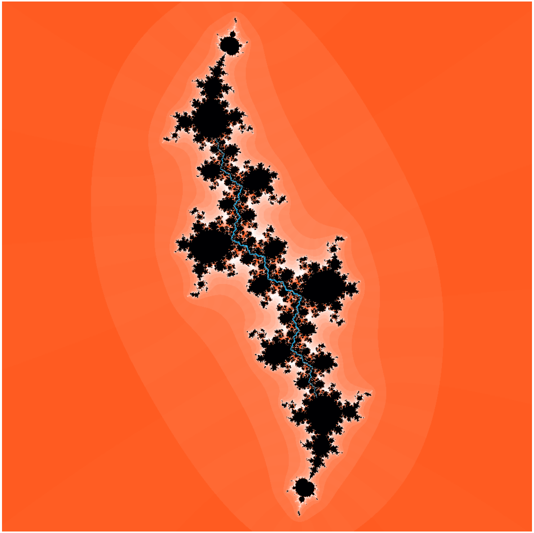

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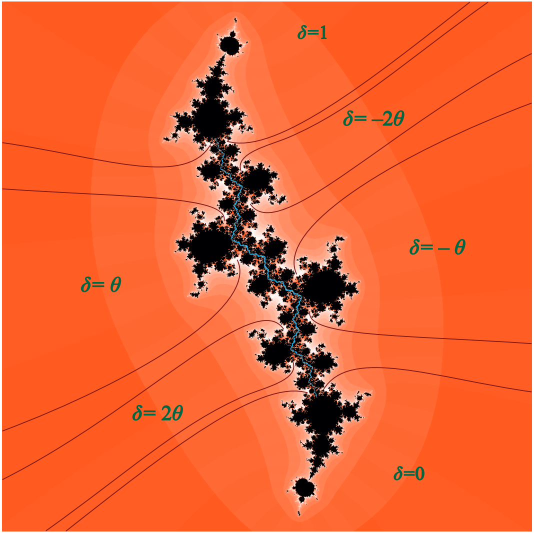

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- The semiconjugacy associated with X will be

$$\varphi(t) = \psi^{-1}(t+a)$$

for suitable a.

8. Some corollaries

In general, the assignment $(\theta, \delta) \mapsto X_{\theta,\delta}$ is only lower semicontinuous.

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Theorem

The following conditions are equivalent:

(i) $(\theta, \delta) \mapsto X_{\theta,\delta}$ *is continuous at* (θ_0, δ_0).

(ii) X_{θ_0,δ_0} *is maximal.*

(iii) X_{θ_0,δ_0} *is a Cantor set with* $d - 1$ *major gaps of length* $1/d$.

(iv) *The points* $\sigma_1, \ldots, \sigma_{d-1}$ *have disjoint orbits under* R_θ.

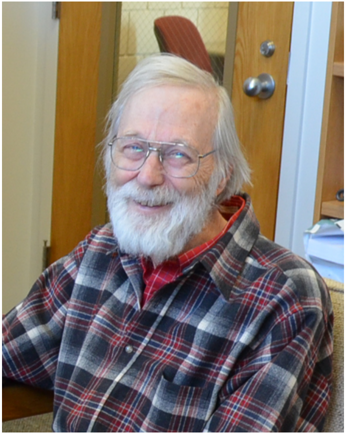
8. Some corollaries

Let ω denote the ***leading angle*** of $X_{\theta,\delta}$.

Theorem

$$\begin{aligned}\omega &= \frac{1}{d-1}\,\nu(0,\theta] + \frac{N_0}{d-1}\\ &= \frac{1}{d-1}\sum_{i=1}^{d-1}\sum_{0<\sigma_i-k\theta\le\theta}\frac{1}{d^{k+1}} + \frac{N_0}{d-1}\end{aligned}$$

where $N_0 \geq 0$ *is the length of the initial segment of* 0*'s in* δ.



HAPPY

BIRTHDAY

JACK!