Modular groups over the quaternions

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Since the time of Carl Friedrich Gauss one of the most fascinating and important objects in mathematics is the modular group and its action on the upper half-plane of complex numbers.

$$SL(2,\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}$$

$$z \mapsto \frac{az+b}{cz+d}$$

$$PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\{I, -I\}$$

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Complex Modular Group

The modular group can be shown to be generated by the two transformations

$$T : z \mapsto -1/z$$
$$S : z \mapsto z + 1$$

so that every element in the complex modular group can be represented (in a non-unique way) by the composition of powers of T and of S.

Geometrically, T represents inversion in the unit circle followed by reflection with respect to the origin, while S represents a unit translation to the right.

The generators T and S obey the relations $T^2 = I$ and $(TS)^3 = I$ and it can be shown that these are a complete set of relations, so the modular group has the following presentation:

$$\{T, S \mid T^2 = I,\ (TS)^3 = I\}$$

$$PSL(2, \mathbb{Z}) \cong \mathbb{Z}_2 \star \mathbb{Z}_3$$

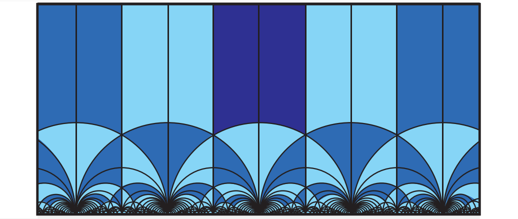

Figure 14. A fundamental domain for the action of the modular group $PSL(2, \mathbb{Z})$ on the hyperbolic plane $\mathbf{H}^2_{\mathbb{R}}$ and the corresponding tessellation .

Let

$$\mathbb{H} = \{x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \,:\, x_n \in \mathbb{R},\ n = 0, 1, 2, 3\}$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = -\mathbf{ji} = \mathbf{k}.$$

be the division algebra of the quaternions.
Then $\mathbb{H}$ is can be identified in the natural way with $\mathbb{R}^4$.

We define the hyperbolic half-space $\mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \,:\, \Re(\mathbf{q}) > 0\}$.

Definition

Let $\mathbf{H}^1_{\mathbb{H}}$ be the half-space model of the *one-dimensional quaternionic hyperbolic space*

$$\mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \ : \ \Re(\mathbf{q}) > 0\}.$$

It can be identified with the hyperbolic space of dimension four $\mathbf{H}^4_{\mathbb{R}}$:

$$\mathbf{H}^1_{\mathbb{H}} \cong \mathbf{H}^4_{\mathbb{R}} = \{(x_0, x_1, x_2, x_3) \in \mathbb{R}^4 : \ x_0 > 0\}$$

with the element of hyperbolic metric given by $(ds)^2 = \frac{(dx_0)^2+(dx_1)^2+(dx_2)^2+(dx_3)^2}{x_0^2}$ where s measures length along a parametrized curve

Definition

For any $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{H})$, the associated real analytic function

$$F_A : \mathbb{H} \cup \{\infty\} \to \mathbb{H} \cup \{\infty\}$$

defined by

$$F_A(\mathbf{q}) = (a\mathbf{q} + b) \cdot (c\mathbf{q} + d)^{-1} \tag{1}$$

is called the linear fractional transformation associated to A. We set $F_A(\infty) = \infty$ if $c = 0$, $F_A(\infty) = ac^{-1}$ if $c \neq 0$ and $F_A(-c^{-1}d) = \infty$.

Let $\mathbb{F} := \{F_A : A \in GL(2,\mathbb{H})\}$ the set of linear fractional transformations.
Since $\mathbb{H} \times \mathbb{H} = \mathbb{H}^2 := \left\{(\mathbf{q}_0, \mathbf{q}_1) \ : \ \mathbf{q}_0,\, \mathbf{q}_1 \in \mathbb{H}\right\}$ as a real vector space is $\mathbb{R}^8$, the group $GL(2,\mathbb{H})$ can be thought as a subgroup of $GL(8,\mathbb{R})$. Using this identification we define:

Definition

Let $SL(2,\mathbb{H}) := SL(8,\mathbb{R}) \cap GL(2,\mathbb{H})$ be the *special linear group* and

$$PSL(2,\mathbb{H}) := SL(2,\mathbb{H})/\{\pm\mathcal{I}\},$$

where $\mathcal{I}$ denotes the identity matrix, the *projective special linear group*.

Theorem

The set $\mathbb{F}$ is a group with respect to the composition operation and the map

$$\Phi : GL(2, \mathbb{H}) \to \mathbb{F}$$

defined as $\Phi(A) = F_A$ is a surjective group antihomomorphism with $\ker(\Phi) = \{t\mathcal{I} \ : \ t \in \mathbb{R} \setminus \{0\}\}$. Furthermore, the restriction of Φ to the special linear group $SL(2, \mathbb{H})$ is still surjective and has kernel $\{\pm\mathcal{I}\}$.

Let **B** denote the open unit ball in $\mathbb{H}$ and $\mathcal{M}_{\mathbf{B}}$ the set of linear transformations that leave invariant **B** or *Möbius transformations*. This is the set

$$\mathcal{M}_{\mathbf{B}} := \{F \in \mathbb{F} \ : F(\mathbf{B}) = \mathbf{B}\}.$$

$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and

$$Sp(1,1) := \{A \in GL(2, \mathbb{H}) \quad : \quad {}^t\bar{A}HA = H\}.$$

There is an interesting characterization of these Möbius transformations:

Theorem

Given $A \in GL(2, \mathbb{H})$, then the linear fractional transformation $F_A \in \mathcal{M}_{\mathbf{B}}$ if and only if there exist $u, v \in \partial\mathbf{B}$, $q_0 \in \mathbf{B}$ (i.e., $|u| = |v| = 1$ and $|q_0| < 1$) such that

$$F_A(\mathbf{q}) = v(\mathbf{q} - q_0)(1 - \overline{q_0}\mathbf{q})^{-1}u \tag{2}$$

for $\mathbf{q} \in \mathbf{B}$. In particular, the antihomomorphism Φ can be restricted to a surjective group antihomomorphism $\Phi : Sp(1,1) \to \mathcal{M}_{\mathbf{B}}$ whose kernel is $\{\pm\mathcal{I}\}$.

Proposition

The Poincaré distance $d_{\mathbf{B}}$ given by:

$$\frac{4|d\mathbf{q}|^2}{(1-|\mathbf{q}|^2)^2}$$

in $\mathbf{B}$ is invariant under the action of the group $\mathcal{M}_{\mathbf{B}}$ of Möbius transformations.
In other words: $\mathcal{M}_{\mathbf{B}} = Isom^{+}_{d_{\mathbf{B}}}(\mathbf{B})$.

The compactification $\widehat{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$ of $\mathbb{H}$ can be identified with $\mathbb{S}^4$ via the stereographic projection. The elements of $\mathcal{M}_{\mathbf{B}}$ act conformally on the 4-sphere with respect to the standard metric and they also preserve orientation and preserve the unit ball. Therefore we conclude that

$$\mathcal{M}_{\mathbf{B}} \subset Conf_+(\mathbb{S}^4),$$

where $Conf_+(\mathbb{S}^4)$ is the group of conformal and orientation-preserving diffeomorphisms of the 4-sphere $\mathbb{S}^4$. As a differentiable manifold, $Conf_+(\mathbb{S}^4)$ is diffeomorphic to $SO(5) \times \mathbf{H}^5_{\mathbb{R}}$ with $\mathbf{H}^5_{\mathbb{R}} = \{(x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5 \ : \ x_1 > 0\}$, so $Conf_+(\mathbb{S}^4)$ has real dimension 15.

Let us give a different description of this group. We recall that $\mathbb{S}^4$ can be thought of as being the projective quaternionic line $\mathbf{P}^1_{\mathbb{H}} \cong \mathbb{S}^4$. This is the space of *right quaternionic lines* in $\mathbb{H}^2$, *i.e.*, subspaces of the form

$$\mathcal{L}_{\mathbf{q}} := \{(\mathbf{q}_1\lambda, \mathbf{q}_2\lambda,) \, : \, \lambda \in \mathbb{H}\} \; , \; (\mathbf{q}_1, \mathbf{q}_2) \, \in \, \mathbb{H}^2 \setminus \{(0,0)\}.$$

We recall that $\mathbb{H}^2$ is a right module over $\mathbb{H}$ and the action of $GL(2,\mathbb{H})$ on $\mathbb{H}^2$ commutes with multiplication on the right, i.e. for every $\lambda \in \mathbb{H}$ and $A \in GL(2,\mathbb{H})$ one has,

$$AR_\lambda = R_\lambda A$$

where R_λ is the multiplication on the right by $\lambda \in \mathbb{H}$. Thus $GL(2,\mathbb{H})$ carries right quaternionic lines into right quaternionic lines, and in this way an action of $GL(2,\mathbb{H})$ on $\mathbf{P}^1_{\mathbb{H}} \cong \mathbb{S}^4$ is defined.

Any $F_A \in \mathbb{F}$ lifts canonically to an automorphism $\widetilde{F_A}$ of $\mathbf{P}^3_{\mathbb{C}}$, the complex projective 3-space and the map $\Psi : F_A \mapsto \widetilde{F_A}$ injects $\mathbb{F}$ into the complex projective group $PSL(4, \mathbb{C}) := SL(4, \mathbb{C})/\{\pm\mathcal{I}\}$.

$$\mathcal{M}_{\mathbf{B}} \subset PSL(2, \mathbb{H}) := SL(2, \mathbb{H})/\{t\mathcal{I},\ t \neq 0\} \simeq Conf_+(\mathbb{S}^4).$$

Poincaré extension to the fifth dimension.

As we have seen before the quaternionic projective line $\mathbf{P}^1_{\mathbb{H}}$ can be identified with the unit sphere $\mathbb{S}^4$ in $\mathbb{R}^5$ and therefore $\mathbb{S}^4$ is the boundary of the closed unit ball $\mathbf{D}^5 \subset \mathbb{R}^5$. As usual, we identify the interior of $\mathbf{D}^5$ with the real hyperbolic 5-space $\mathbf{H}^5_{\mathbb{R}}$. Since $PSL(2, \mathbb{H})$ acts conformally on $\mathbb{S}^4 \cong \mathbf{P}^1_{\mathbb{H}}$, by Poincaré Extension Theorem each element $\gamma \in PSL(2, \mathbb{H})$ extends canonically to a conformal diffeomorphism of $\mathbf{D}^5$ which restricted to $\mathbf{H}^5_{\mathbb{R}}$ is an orientation preserving isometry $\tilde{\gamma}$ of the open 5-disk $\mathbf{B}^5$ with the Poincaré's metric.

Reciprocally, any orientation preserving isometry of $\mathbf{H}^5_{\mathbb{R}}$ extends canonically to the ideal boundary $\mathbb{R}^4 \cup \{\infty\}$ as an element of $PSL(2,\mathbb{H})$. Thus the map $\gamma \mapsto \tilde{\gamma}$ is an isomorphism and $PSL(2,\mathbb{H}) = Isom_+\mathbf{H}^5_{\mathbb{R}}$.

This is the connection between twistor spaces, conformal geometry, hyperbolic geometry in 5-dimensions and complex Kleinian groups.

Isometries in the half-space $\mathbf{H}^1_{\mathbb{H}}$.

Consider $\mathbf{H}^1_{\mathbb{H}}$ be the half-space model of the *one-dimensional quaternionic hyperbolic space*

$$\mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \ : \ \Re(\mathbf{q}) > 0\}.$$

Via the Cayley transformation $\Psi : \mathbf{B} \to \mathbf{H}^1_{\mathbb{H}}$ defined as $\Psi(\mathbf{q}) = (1+\mathbf{q})(1-\mathbf{q})^{-1}$ one can show explicitely that the unit ball $\mathbf{B}$ of $\mathbb{H}$ is diffeomorphic to $\mathbf{H}^1_{\mathbb{H}}$ and introduce a Poincaré distance in $\mathbf{H}^1_{\mathbb{H}}$ in such a way that the Cayley transformation $\Psi : \mathbf{B} \to \mathbf{H}^1_{\mathbb{H}}$ is an isometry; moreover the Poincaré distance in $\mathbf{H}^1_{\mathbb{H}}$ is invariant under the action of the group $\Psi \mathbb{M}_{\mathbf{B}} \Psi^{-1}$ which is denoted by $\mathbb{M}_{\mathbf{H}^1_{\mathbb{H}}}$.

Since the unit ball **B** in $\mathbb{H}$ can be identified with the lower hemisphere of $\mathbb{S}^4$ and any transformation $F_A \in \mathcal{M}_{\mathbf{B}}$ is conformal and preserves orientation, we conclude that (see also [**?**, **?**])

$$\mathcal{M}_{\mathbf{B}} \simeq Conf_{+}(\mathbf{H}^1_{\mathbb{H}})$$

where $Conf_{+}(\mathbf{H}^1_{\mathbb{H}})$ represents the group of conformal diffeomorphisms orientation–preserving of the half-space model $\mathbf{H}^1_{\mathbb{H}}$.

Definition

Let $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ the subgroup of $PSL(2, \mathbb{H})$ whose elements are associated to invertible linear fractional transformations which preserve $\mathbf{H}^1_{\mathbb{H}}$.

According to Lars V. Ahlfors it was Karl Theodor Vahlen who iintroduces in 1901 the idea of using the Clifford numbers to define Möbius groups of 2 by 2 matrices with entries in the Clifford numbers (Clifford matrices) acting on hyperbolic spaces. In 1984 Ahlfors pushes forward the idea of Vahlen and considers groups of 2 by 2 Clifford matrices and gives the necessary and sufficient conditions to leave invariant a half-space and the corresponding hyperbolic metric. In particular in dimension 4 he gives conditions on 2 by 2 Clifford matrices with entries in the quaternions $\mathbb{H}$ to induce Möbius transformations which act as orientation-preserving isometries of a half-space of $\mathbb{H}$ with its Poincaré metric.

In 2009 Cinzia Bisi and Graziano Gentili reformulate the conditions of Ahlfors and reduce it to a *single relation* in the quaternionic setting:

$$\bar{A}^t K A = K$$

where A is a 2 by 2 matrix with entries in $\mathbb{H}$ and $K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $\bar{A}^t$ is the transpose of the quaternionic conjugate of A. We remark that the (BG) conditions are equivalent to Ahlfors' conditions but they are much simpler, very efficient and useful.

Proposition

[**Conditions** (**BG**)] *The subgroup $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ can be characterized as the group induced by matrices which satisfy one of the following (equivalent) conditions:*

$$\begin{cases} \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} a, b, c, d \in \mathbb{H} \ : \ \overline{A}^t K A = K \right\} \\ \text{with } K = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ \\ \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ \Re(a\overline{c}) = 0, \ \Re(b\overline{d}) = 0, \ \overline{b}c + \overline{d}a = 1 \right\} \\ \\ \left\{ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \ \Re(c\overline{d}) = 0, \ \Re(a\overline{b}) = 0, \ a\overline{d} + b\overline{c} = 1 \right\}. \\ a, b, c, d \in \mathbb{H} \end{cases}$$

Definition

The group of invertible linear transformations satisfying (BG) conditions consists of orientation preserving hyperbolic isometries of $\mathbf{H}^1_{\mathbb{H}}$, therefore it be denoted $Isom_+(\mathbf{H}^1_{\mathbb{H}})$; however, in analogy with the previous notations, we will also called it the *Möbius group of* $\mathbf{H}^1_{\mathbb{H}}$ *and we denote alternatively this group as* $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}} := \Psi \mathcal{M}_{\mathbf{B}} \Psi^{-1}$.
Moreover, we have $Isom(\mathbf{H}^1_{\mathbb{H}}) = \mathbb{M}_{\mathbf{H}^1_{\mathbb{H}}}$, *where* $Isom(\mathbf{H}^1_{\mathbb{H}})$ *is the full group of isometries of* $\mathbf{H}^1_{\mathbb{H}}$.

Remark

The group $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ acts by orientation-preserving conformal transformations on the sphere at infinity of the hyperbolic 4-space defined as follows

$$\mathbb{S}^3 = \partial \mathbf{H}^1_{\mathbb{H}} := \{\mathbf{q} \in \mathbb{H} \; : \; \Re(\mathbf{q}) = 0\} \cup \{\infty\}.$$

In other words $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}} \cong \mathit{Conf}_+(\mathbb{S}^3)$.

Groups which are generalizations of the Picard and modular groups have been considered before by C. Maclachlan, P.L. Waterman and N.J. Wielenberg and N. W. Johnson and A. I. Weiss in particular the group corresponding to the Lipschitz integers is considered and a fundamental domain of this group is described.
Ruth Kellerhalls using quaternions, describes the Margulis decomposition of non compact 5-dimensional hyperbolic manifolds of finite volume and gives estimates of the minimal length of closed geodesics.

The affine subgroup $\mathcal{A}(\mathbb{H})$ of the isometries of $\mathbf{H}^1_{\mathbb{H}}$

Consider now the *affine subgroup* $\mathcal{A}(\mathbb{H})$ of $PSL(2, \mathbb{H})$ consisting of transformations which are induced by matrices of the form

$$\begin{pmatrix} \lambda a & b \\ 0 & \lambda^{-1} a \end{pmatrix} \quad \textit{i.e.} \quad \mathbf{q} \mapsto ((\lambda a)\mathbf{q} + b)(\lambda^{-1} a)^{-1}$$

with $|a| = 1$, $\lambda > 0$ and $\Re(\bar{b}a) = 0$. Such matrices satisfy (BG) conditions of Proposition 1 and therefore are in $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$.

We enumerate some properties of $\mathcal{A}(\mathbb{H})$ in the next proposition.

Proposition

The group $\mathcal{A}(\mathbb{H})$ is

1. *the maximal subgroup of $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ which fixes the point at infinity.*
2. *a Lie group of real dimension 7 and*
3. *a conformal group. This is each matrix in $\mathcal{A}(\mathbb{H})$ acts as a conformal transformation on the hyperplane at infinity $\partial\mathbf{H}^1_{\mathbb{H}}$. Therefore $\mathcal{A}(\mathbb{H})$ is the group of conformal and orientation preserving transformation acting on the space of pure imaginary quaternions at infinity which can be identified with $\mathbb{R}^3$ so that this group is isomorphic to the conformal group $Conf_+(\mathbb{R}^3)$.*

Iwasawa decomposition of the isometries of $\mathbf{H}^1_{\mathbb{H}}$

In analogy with the complex and real case, we can state a generalization of Iwasawa decomposition for any element of $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ as follows

Proposition

Every element of $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$ i.e., elements in $PSL(2, \mathbb{H})$ which satisfies (BG) conditions and which is represented by the matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ can be written in a unique way as follows

$$M = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}, \tag{3}$$

with $\lambda > 0$, $\Re(\omega) = 0$, $|\alpha|^2 + |\beta|^2 = 1$ and $\Re(\alpha\overline{\beta}) = 0$.

Definition

A *Lipschitz quaternion* (or Lipschitz integer) is a quaternion whose components are all integers. The ring of all Lipschitz quaternions $\mathbb{H}(\mathbb{Z})$ is the subset of quaternions with integer coefficients:

$$\mathbb{H}(\mathbb{Z}) := \{a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \ : , a, b, c, d \in \mathbb{Z}\}$$

This is a subring of the ring of *Hurwitz quaternions*:

$$\mathbb{H}ur(\mathbb{Z}) :=$$

$$\left\{ a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \ : a, b, c, d \in \mathbb{Z} \ \text{or} \ a, b, c, d \in \mathbb{Z} + \frac{1}{2} \right\}.$$

Indeed it can be proven that $\mathbb{H}ur(\mathbb{Z})$ is closed under quaternion multiplication and addition, which makes it a subring of the ring of all quaternions $\mathbb{H}$.

As a group, $\mathbb{H}ur(\mathbb{Z})$ is free abelian with generators $1/2(1 + \mathbf{i} + \mathbf{j} + \mathbf{k}), \mathbf{i}, \mathbf{j}, \mathbf{k}$. Therefore $\mathbb{H}ur(\mathbb{Z})$ forms a lattice in $\mathbb{R}^4$. This lattice is known as the $\mathcal{F}_4$ lattice since it is the root lattice of the semisimple Lie algebra $\mathcal{F}_4$. The Lipschitz quaternions $\mathbb{H}(\mathbb{Z})$ form an index 2 sublattice of $\mathbb{H}ur(\mathbb{Z})$ and it is a subring of the ring of quaternions.

In this section we investigate a class of linear transformations which will play a crucial role in the definition of the quaternionic modular groups.
A translation $\tau_\omega : \mathbf{H}^1_{\mathbb{H}} \to \mathbf{H}^1_{\mathbb{H}}$ is the transformation defined as $\mathbf{q} \mapsto \mathbf{q} + \omega$ which is a hyperbolic isometry in $\mathbf{H}^1_{\mathbb{H}}$. Then it is a transformation associated to the matrix $\begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \in \mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}}$, i.e. $\Re(\omega) = 0$.
In what follows we consider translations where ω is the imaginary part of a Lipschitz or Hurwitz integer. We remark that the imaginary part of a Lipschitz integer is still a Lipschitz integer but the imaginary part of a Hurwitz integer is not necessarily a Hurwitz integer.

Definition

An *imaginary Lipschitz quaternion* (or imaginary Lipschitz integer) is the imaginary part of a Lipschitz quaternion, a quaternion whose real part is 0 and the others components are all integers. The set of all imaginary Lipschitz quaternions is

$$\Im\mathbb{H}(\mathbb{Z}) = \{b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \in \mathbb{H} \;:\; b, c, d \in \mathbb{Z}\}\,.$$

Definition

Let $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ be the abelian group of translations by the imaginary Lipschitz group $\Im\mathbb{H}(\mathbb{Z})$, i.e. such that $\mathbf{q} \mapsto \mathbf{q} + \omega$, $\omega = n_2\mathbf{i} + n_3\mathbf{j} + n_4\mathbf{k}$ where the n's are all integers; equivalently $\mathbf{q} \mapsto \mathbf{q} + \omega$ belongs to $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ if and only if $\omega \in \Im\mathbb{H}(\mathbb{Z})$. This is

$$\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})} := \left\{ \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \in GL(2, \mathbb{H}) : \Re(\omega) = 0 \right\}$$

The group $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ acts freely on $\mathbf{H}^1_{\mathbb{H}}$ as a representation of the abelian group with 3 free generators $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. A fundamental domain is the following set $\{\mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbf{H}^1_{\mathbb{H}} \ : \ |x_n| \leq 1/2,\ n = 1, \cdots, 3\}$. This set is referred as the *chimney*. It has two ends, one of finite volume which is asymptotic at the point at infinity. The other end has infinite volume, it is called a *hyperbolic trumpet*.

Inversion
Let
$$T(\mathbf{q}) = \mathbf{q}^{-1} = \frac{\overline{\mathbf{q}}}{|\mathbf{q}|^2}.$$
Clearly T is a linear fractional transformation of $\mathbf{H}^1_{\mathbb{H}}$ and its representative matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

The only fixed point of T in $\mathbf{H}^1_{\mathbb{H}}$ is 1 since the other fixed point of T in $\mathbb{H}$ is -1 which is not in $\mathbf{H}^1_{\mathbb{H}}$. Furthermore T is an isometric involution * of $\mathbf{H}^1_{\mathbb{H}}$ because it satisfies (BG) conditions of Proposition 1. In particular T is an inversion on $\mathbb{S}^3$ which becomes the antipodal map on any copy of $\mathbb{S}^2$ obtained as intersection of $\mathbb{S}^3$ with a plane perpendicular to the real axis. Finally, this isometry T leaves invariant the hemisphere (which is a hyperbolic 3-dimensional hyperplane) $\Pi := \{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} \quad : \quad |\mathbf{q}| = 1\}$. The only fixed point of T in $\mathbf{H}^1_{\mathbb{H}}$ is 1 and all other points in $\mathbf{H}^1_{\mathbb{H}}$ are periodic of period 2.

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*In the following sense; T sends every point of a hyperbolic geodesic parametrized by arc length $\gamma(s)$, passing through 1 at time 0 (i.e. such that $\gamma(0) = 1$), to its opposite $\gamma(-s)$. In other words, T is a hyperbolic symmetry around 1.

Definition

Let

$$\mathcal{C} =$$

$$\{\mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} \in \mathbf{H}^1_{\mathbb{H}} \ : |\mathbf{q}| = 1,\ |x_n| \leq 1/2,\ n = 1, \cdots, 3\}.$$

Then, $\mathcal{C}$ is a regular hyperbolic cube in Π. The 8 points of the form $\frac{1}{2} \pm \frac{1}{2}\mathbf{i} \pm \frac{1}{2}\mathbf{j} \pm \frac{1}{2}\mathbf{k}$, are the vertices of $\mathcal{C}$ and in particular they are periodic of period 2 for T. These eight points are Hurwitz units (but not Lipschitz units).

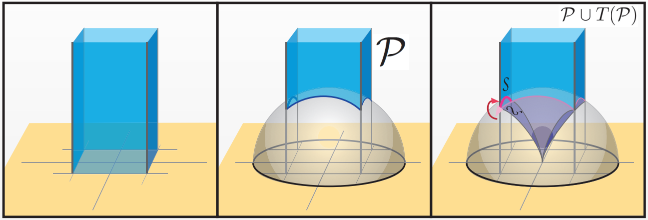


Figure 1. Schematic picture of the chimney which is the fundamental domain of the parabolic group $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ (generated by the translations $\tau_{\mathbf{i}}$, $\tau_{\mathbf{j}}$ and $\tau_{\mathbf{k}}$), the polytope $\mathcal{P}$ and the polytope $\mathcal{P}$ and its inversion $T(\mathcal{P})$. The horizontal plane represents the purely imaginary quaternions that forms the ideal boundary $\partial\mathbf{H}^1_{\mathbb{H}}$ and above it the open half-space of quaternions with positive real part $\mathbf{H}^1_{\mathbb{H}}$.

Composition of translations and inversion

We observe that if $\tau_\omega(\mathbf{q}) := \mathbf{q} + \omega$, $\omega \in \mathbb{H}$, then $L_\omega := \tau_\omega T$ has as corresponding matrix

$$\begin{pmatrix} \omega & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

similarly $R_\omega := T\tau_\omega$ has as corresponding matrix

$$\begin{pmatrix} 0 & 1 \\ 1 & \omega \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & \omega \\ 0 & 1 \end{pmatrix}.$$

Therefore R_ω is represented by interchanging the elements on the diagonal of the matrix which represents L_ω. In the following table, we list the matrices associated to iterates of L_ω with suitable choices of ω.

Table

$\omega = \pm\mathbf{i}$ or $\omega = \pm\mathbf{j}$ or $\omega = \pm\mathbf{k}$ $\omega^2 = -1$	$\omega = \pm\mathbf{i} \pm \mathbf{j}$ or $\omega = \pm\mathbf{i} \pm \mathbf{k}$ or $\omega = \pm\mathbf{j} \pm \mathbf{k}$ $\omega^2 = -2$	$\omega = \pm\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}$ $\omega^2 = -3$
$L_\omega^2 = \begin{pmatrix} 0 & \omega \\ \omega & 1 \end{pmatrix}$	$L_\omega^2 = \begin{pmatrix} -1 & \omega \\ \omega & 1 \end{pmatrix}$	$L_\omega^2 = \begin{pmatrix} -2 & \omega \\ \omega & 1 \end{pmatrix}$
$L_\omega^3 = \begin{pmatrix} \omega & 0 \\ 0 & \omega \end{pmatrix}$	$L_\omega^3 = \begin{pmatrix} 0 & -1 \\ -1 & \omega \end{pmatrix}$	$L_\omega^3 = \begin{pmatrix} -\omega & -2 \\ -2 & \omega \end{pmatrix}$
$L_\omega^4 = \begin{pmatrix} -1 & \omega \\ \omega & 0 \end{pmatrix}$	$L_\omega^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$	$L_\omega^4 = \begin{pmatrix} 1 & -\omega \\ -\omega & -2 \end{pmatrix}$
$L_\omega^5 = \begin{pmatrix} 0 & -1 \\ -1 & \omega \end{pmatrix}$		$L_\omega^5 = \begin{pmatrix} 0 & 1 \\ 1 & -\omega \end{pmatrix}$
$L_\omega^6 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$		$L_\omega^6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

We can see that the order of L_ω depends on ω; in particular, each of the six transformations L_ω with $\omega = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, has order 6 but when restricted to the plane $\mathbf{H}^1_{\mathbb{C}}$ or more generally $S_\omega := \{\mathbf{q} = x_1 + x_{\mathbf{i}}\mathbf{i} + x_{\mathbf{j}}\mathbf{j} + x_{\mathbf{k}}\mathbf{k} \in \mathbf{H}^1_{\mathbb{H}} \ : \ x_\alpha = 0 \text{ if } \alpha \neq \omega, 0\}$, with $\omega = \mathbf{i}, \mathbf{j}, \mathbf{k}$ has order 3. Furthermore $\mathbf{q}_0 \in \mathbb{H}$ is a fixed point for $L_\omega = \tau_\omega T$ with $\omega = 0, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, if and only if $\mathbf{q}_0$ is a root of $\mathbf{q}^2 - \omega\mathbf{q} - 1 = 0$. In the same way $\mathbf{q}_0$ is a fixed point for $R_\omega = T\tau_\omega$ with $\omega = 0, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, if and only if $\mathbf{q}_0$ is a root of $\mathbf{q}^2 + \mathbf{q}\omega - 1 = 0$. Briefly, if $\omega = \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}$, the only fixed point of L_ω in $\mathbf{H}^1_{\mathbb{H}}$ is $\frac{\sqrt{3}}{2} + \frac{\omega}{2}$ and the only fixed point of R_ω is $\frac{\sqrt{3}}{2} - \frac{\omega}{2}$.

The Lipschitz Quaternionic modular group

We are now in the position of introducing the following:

Definition

The *Lipschitz quaternionic modular group* is the group generated by the inversion T and the translations $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$. It will be denoted by $PSL(2, \mathfrak{L})$.

Remark

The group $PSL(2, \mathfrak{L})$ is a discrete subgroup of $\mathcal{M}_{\mathbf{H}^1_{\mathbb{H}}} \subset PSL(2, \mathbb{H})$. It is important to emphasize that the Lipschitz quaternionic modular group $PSL(2, \mathfrak{L})$ is a proper subgroup of $PSL(2, \mathbb{H}(\mathbb{Z}))$; indeed, the subgroup generated by (proper) translations and by the inversion T in $\mathbf{H}^1_{\mathbb{H}}$ has elements which are represented by matrices with Lipschitz integers as entries, but in general an arbitrary element in $PSL(2, \mathbb{H}(\mathbb{Z}))$ does not satisfy (BG) conditions and therefore it does not preserve $\mathbf{H}^1_{\mathbb{H}}$.

Lipschitz unitary and affine subgroups of $PSL(2,\mathfrak{L})$.

Let $\mathfrak{L}_u$ the group (of order 8) of *Lipschitz units*

$$\mathfrak{L}_u := \{\pm 1,\ \pm\mathbf{i},\ \pm\mathbf{j}, \pm\mathbf{k} :\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1\}.$$

This group is the *quaternion group* which is a non-abelian group of order eight. Moreover, its elements are the 8 vertices of a 16-cell in the 3-sphere $\mathbb{S}^3$ and the 8 barycentres of the faces of its dual polytope which is a hypercube also called 8-cell.

Definition

The subgroup $\mathcal{U}(\mathfrak{L})$ of $PSL(2,\mathfrak{L})$ whose elements are the 4 diagonal matrices

$$D_{\mathbf{u}} := \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u} \end{pmatrix}$$

with $\mathbf{u}$ a Lipschitz unit is called *Lipschitz unitary group*.

The Lipschitz unitary group $\mathcal{U}(\mathfrak{L})$ is isomorphic to the so called *Klein group* of order 4 which is $\mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, since $\mathbf{ij} = \mathbf{k}$. Moreover, we observe that the action on $\mathbf{H}^1_{\mathbb{H}}$ of the transformation associated to $D_{\mathbf{u}}$, where $\mathbf{u} = \mathbf{i}, \mathbf{j}$ or $\mathbf{k}$ is for conjugation and sends a quaternion $\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}}$ to $\mathbf{uqu}^{-1}$. It acts as a rotation of angle π with axis the hyperbolic 2-plane completely geodesic

$$S_{\mathbf{u}} = \{x + y\mathbf{u} \ : \ x, y \in \mathbb{R}, x > 0\}.$$

Definition

The Lipschitz affine subgroup (or the Lipschitz parabolic subgroup) $\mathcal{A}(\mathfrak{L})$ is the group generated by the unitary group $\mathcal{U}(\mathfrak{L})$ and the group of translations $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$. This is

$$\mathcal{A}(\mathfrak{L}) := \langle \mathcal{U}(\mathfrak{L}), \mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})} \rangle.$$

Equivalently

$$\mathcal{A}(\mathfrak{L}) = \left\{ \begin{pmatrix} \mathbf{u} & \mathbf{u}b \\ 0 & \mathbf{u} \end{pmatrix} : \ \mathbf{u} \in \mathfrak{L}_u, \ \Re(b) = 0 \right\} \tag{4a}$$

$$= \left\{ \begin{pmatrix} \mathbf{u} & b\mathbf{u} \\ 0 & \mathbf{u} \end{pmatrix} : \ \mathbf{u} \in \mathfrak{L}_u, \ \Re(b) = 0 \right\}. \tag{4b}$$

Remark

The Lipschitz affine subgroup $\mathcal{A}(\mathfrak{L})$ is the maximal Lipschitz parabolic subgroup of $PSL(2,\mathfrak{L})$. Moreover $\mathcal{A}(\mathfrak{L}) \subset PSL(2,\mathfrak{L}) \cap \mathcal{A}(\mathbb{H})$. Furthermore, for every $x_0 > 0$ this subgroup leaves invariant the horospheres $\{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} : \Re(\mathbf{q}) = x_0\}$ and also the horoballs $\{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} : \Re(\mathbf{q}) > x_0\}$.

Since $\mathbf{ij} = \mathbf{k}$, it is generated by hyperbolic isometries associated to the three matrices

$$\begin{pmatrix} \mathbf{i} & 0 \\ 0 & \mathbf{i} \end{pmatrix}, \quad \begin{pmatrix} \mathbf{j} & 0 \\ 0 & \mathbf{j} \end{pmatrix}, \quad \begin{pmatrix} 1 & \mathbf{u} \\ 0 & 1 \end{pmatrix},$$

where $\mathbf{u} = \mathbf{i}, \mathbf{j}$ and $\mathbf{k}$.

Remark

The transformation represented by the matrix $\begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u} \end{pmatrix}$ is a rotation of angle π which keeps fixed each point of the plane $S_{\mathbf{u}}$ (the "axis of rotation"). The composition of such rotation and the inversion T leads to a transformation represented by the matrix $\begin{pmatrix} 0 & \mathbf{u} \\ \mathbf{u} & 0 \end{pmatrix}$ with $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$. For these transformations the plane $S_{\mathbf{u}}$, with $\mathbf{u} = \mathbf{i}, \mathbf{j}, \mathbf{k}$ is invariant. Both rotations and inversion composed with a rotation of the plane leave invariant the hyperbolic hyperplane Π and have 1 as a fixed point.

The Hurwitz modular group and its unitary and affine subgroups.

In analogy with the introduction of the unitary, affine and modular groups in the Lipschitz integers setting we give the following generalization

Definition

Let $\mathfrak{H}_u$ be the group of *Hurwitz units*

$$\mathfrak{H}_u :=$$

$$\{\pm 1,\ \pm \mathbf{i},\ \pm \mathbf{j}, \pm \mathbf{k},\ \frac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k}) :\ \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1, \mathbf{ij} = \mathbf{k}\}$$

where in $\frac{1}{2}(\pm 1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$ all 16 possible combinations of signs are allowed.

This group is of order 24 and it is known as the *binary tetrahedral* group. Its elements can be seen as the vertices of the 24-cell. We recall that the 24-cell is a convex regular 4-polytope, whose boundary is composed of 24 octahedral cells with six meeting at each vertex, and three at each edge. Together they have 96 triangular faces, 96 edges, and 24 vertices. It is possible to give an (ideal) model of the 24-cell by considering the convex hull (of the images) of the 24 unitary Hurwitz numbers via the Cayley transformation $\Psi(\mathbf{q}) = (1+\mathbf{q})(1-\mathbf{q})^{-1}$.

Definition

The subgroup $\mathcal{U}(\mathfrak{H})$ of $PSL(2,\mathbb{H})$ given by the 12 diagonal matrices

$$D_{\mathbf{u}} := \begin{pmatrix} \mathbf{u} & 0 \\ 0 & \mathbf{u} \end{pmatrix}$$

with $\mathbf{u}$ a Hurwitz unit is called *Hurwitz unitary group*.

The group $\mathcal{U}(\mathfrak{H})$ is of order 12 and in fact it is isomorphic to the group of orientation preserving isometries of the regular tetrahedron. It clearly contains $\mathcal{U}(\mathfrak{L})$ as a subgroup but is not contained in the Lipschitz modular group $PSL(2,\mathfrak{L})$.

Definition

The Hurwitz modular group is the group generated by the inversion T, by the translations $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ and by the Hurwitz unitary group $\mathcal{U}(\mathfrak{H})$. It will be denoted by $PSL(2,\mathfrak{H})$.

Proposition

The group $PSL(2,\mathfrak{L})$ is a subgroup of index three of the group $PSL(2,\mathfrak{H})$.

Proof.

This is so since the order of the group $\mathcal{U}(\mathfrak{L})$ of transformations induced by the diagonal matrices with entries in the Lipschitz units is of index three in the group $\mathcal{U}(\mathfrak{H})$ of transformations induced by diagonal matrices with entries in the Hurwitz units. ■

Definition

The Hurwitz affine subgroup (or the Hurwitz parabolic subgroup) $\mathcal{A}(\mathfrak{H})$ is the group generated by the unitary Hurwitz group $\mathcal{U}(\mathfrak{H})$ and the group of translations $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$. Thus,

$$\mathcal{A}(\mathfrak{H}) = \left\{ \begin{pmatrix} \mathbf{u} & \mathbf{u}b \\ 0 & \mathbf{u} \end{pmatrix} : \ \mathbf{u} \in \mathfrak{H}_u, \ \Re(b) = 0 \right\} \tag{5a}$$

$$= \left\{ \begin{pmatrix} \mathbf{u} & b\mathbf{u} \\ 0 & \mathbf{u} \end{pmatrix} : \ \mathbf{u} \in \mathfrak{H}_u, \ \Re(b) = 0 \right\}. \tag{5b}$$

It follows from the definition that $PSL(2, \mathfrak{L}) \subset PSL(2, \mathfrak{H})$. It is worth observing here that using the Cayley transformations $\Psi(\mathbf{q}) = (1 + \mathbf{q})(1 - \mathbf{q})^{-1}$ one can represent the actions (in terms of mutiplication/rotations) of the Hurwitz units on the unitary sphere $\mathbb{S}^3$ as transformations of $\mathbf{H}^1_{\mathbb{H}}$. Indeed, to any such a transformation it is possible to associate one of the following 24 matrices

$$P_{\mathbf{u}} := \frac{1}{2}\begin{pmatrix} \mathbf{u}+1 & \mathbf{u}-1 \\ \mathbf{u}-1 & \mathbf{u}+1 \end{pmatrix} \in PSL(2, \mathfrak{H})$$

(with $\mathbf{u}$ a Hurwitz unit); each of these matrices represents a rotation around 1 given by the formula

$$\mathbf{q} \mapsto ((\mathbf{u}+1)\mathbf{q} + \mathbf{u} - 1)((\mathbf{u}-1)\mathbf{q} + \mathbf{u} + 1)^{-1}. \qquad (6)$$

This way of representing (the group of) Hurwitz units in terms of matrices can be considered as a way to generalize Pauli matrices. Let $P(\mathfrak{H}) \subset PSL(2, \mathbb{H})$ be the group of order 24 of rotations as in (6). This group is obviously isomorphic to $\mathfrak{H}_u$. The orbit of 0 under the action of $\mathfrak{H}_u$ on the boundary $\partial \mathbf{H}^1_{\mathbb{H}} \cup \{\infty\}$ are the vertices of the 24-cell and are the images under the Cayley transformation of the Hurwitz units. Therefore $P(\mathfrak{H})$ is a subgroup of the group of symmetries of the 24-cell.

Definition

Let $\hat{\mathcal{U}}(\mathfrak{L})$ and $\hat{\mathcal{U}}(\mathfrak{H})$ be the isotropy subgroups of 1 in $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$, respectively. These subgroups are the maximal subgroups which also preserve the cube $\mathcal{C}$ and the hyperplane Π.

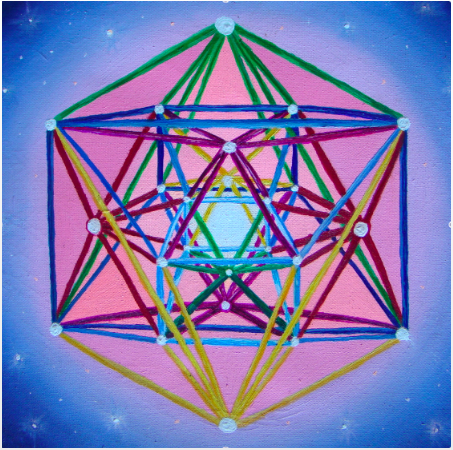


Figure 13. The 24-cell $\{3, 4, 3\}$.

Proposition

The groups $\hat{\mathcal{U}}(\mathfrak{L})$ and $\hat{\mathcal{U}}(\mathfrak{H})$ are the subgroups generated by T and $\mathcal{U}(\mathfrak{L})$ and T and $\mathcal{U}(\mathfrak{H})$, respectively. We write

$$\hat{\mathcal{U}}(\mathfrak{L}) = \langle T, \mathcal{U}(\mathfrak{L}) \rangle \qquad \text{and} \qquad \hat{\mathcal{U}}(\mathfrak{H}) = \langle \mathrm{T}, \mathcal{U}(\mathfrak{H}) \rangle.$$

Since $T^2 = \mathcal{I}$ and T commutes with all of the elements of $\mathcal{U}(\mathfrak{L})$ and $\mathcal{U}(\mathfrak{L})$ we have:

$$\hat{\mathcal{U}}(\mathfrak{L}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathcal{U}(\mathfrak{L}) \qquad \text{and} \qquad \hat{\mathcal{U}}(\mathfrak{H}) = \mathbb{Z}/2\mathbb{Z} \oplus \mathcal{U}(\mathfrak{H}).$$

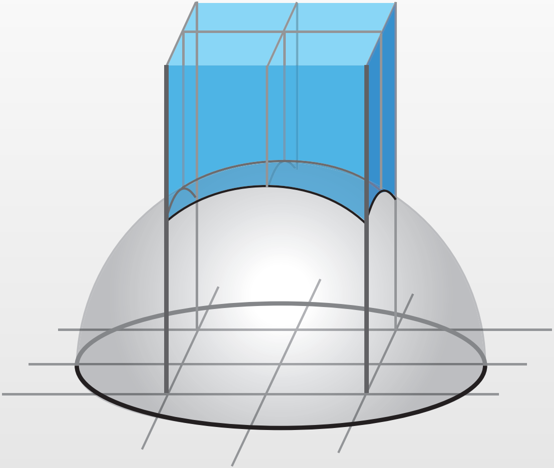
Remark

The groups $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$ are discrete and preserve the half-space $\mathbf{H}^1_{\mathbb{H}}$ and the hyperbolic metric ds so they are *4-dimensional hyperbolic Kleinian groups* in the sense of Henri Poincaré.

A quaternionic kaleidoscope.

We begin with the ideal convex hyperbolic polytope $\mathcal{P}$ with one vertex at infinity which is the intersection of the half-spaces which contain 2 and which are determined by the following set of hyperbolic hyperplanes

$$\text{Hyperplanes of the faces of } \mathcal{P} = \begin{cases} \Pi := \{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} \ : \ |\mathbf{q}| = 1\} \\ \Pi_{-\frac{\mathbf{i}}{2}} := \{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} \ : \ \mathbf{q} = x_0 - \frac{1}{2}\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},\ x_0 > 0,\ x_2, x_3 \in \mathbb{R}\} \\ \Pi_{\frac{\mathbf{i}}{2}} := \{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} \ : \ \mathbf{q} = x_0 + \frac{1}{2}\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k},\ x_0 > 0,\ x_2, x_3 \in \mathbb{R}\} \\ \Pi_{-\frac{\mathbf{j}}{2}} := \{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} \ : \ \mathbf{q} = x_0 + x_1\mathbf{i} - \frac{1}{2}\mathbf{j} + x_3\mathbf{k},\ x_0 > 0,\ x_1, x_3 \in \mathbb{R}\} \\ \Pi_{\frac{\mathbf{j}}{2}} := \{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} \ : \ \mathbf{q} = x_0 + x_1\mathbf{i} + \frac{1}{2}\mathbf{j} + x_3\mathbf{k},\ x_0 > 0,\ x_1, x_3 \in \mathbb{R}\} \\ \Pi_{-\frac{\mathbf{k}}{2}} := \{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} \ : \ \mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} - \frac{1}{2}\mathbf{k},\ x_0 > 0,\ x_1, x_2 \in \mathbb{R}\} \\ \Pi_{\frac{\mathbf{k}}{2}} := \{\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}} \ : \ \mathbf{q} = x_0 + x_1\mathbf{i} + x_2\mathbf{j} + \frac{1}{2}\mathbf{k},\ x_0 > 0,\ x_1, x_2 \in \mathbb{R}\} \end{cases}$$



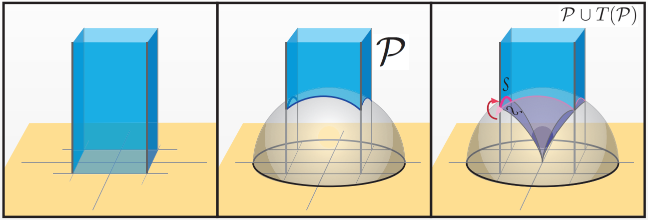


Figure 1. Schematic picture of the chimney which is the fundamental domain of the parabolic group $\mathcal{T}_{\Im\mathbb{H}(\mathbb{Z})}$ (generated by the translations $\tau_{\mathbf{i}}$, $\tau_{\mathbf{j}}$ and $\tau_{\mathbf{k}}$), the polytope $\mathcal{P}$ and the polytope $\mathcal{P}$ and its inversion $T(\mathcal{P})$. The horizontal plane represents the purely imaginary quaternions that forms the ideal boundary $\partial\mathbf{H}^1_{\mathbb{H}}$ and above it the open half-space of quaternions with positive real part $\mathbf{H}^1_{\mathbb{H}}$.

The polytope $\mathcal{P}$ is bounded by the hemisphere Π and the six hyperplanes Π_n ($n = \frac{\mathbf{i}}{2}, -\frac{\mathbf{i}}{2}, \frac{\mathbf{j}}{2}, -\frac{\mathbf{j}}{2}, \frac{\mathbf{k}}{2}, -\frac{\mathbf{k}}{2}$) that are orthogonal to the ideal boundary and pass through the point at infinity that is denoted by ∞.
The only ideal vertex of $\mathcal{P}$ is the point at infinity. The (non ideal) vertices of $\mathcal{P}$ are the eight points $\frac{1}{2}(1 \pm \mathbf{i} \pm \mathbf{j} \pm \mathbf{k})$ which are the vertices of the cube $\mathcal{C} \subset \Pi$ which was defined in subsection 3.2.
The politope $\mathcal{P}$ has seven 3-dimensional faces: one compact cube $\mathcal{C}$ and six pyramids with one ideal vertex at ∞ as their common apex and the six squares of the cube $\mathcal{C}$ as their bases.
Moreover $\mathcal{P}$ has 20 2-dimensional faces (6 compact squares and 12 triangles with one ideal vertex which is the point at infinity) and 20 edges (12 compact and 8 which are asymptotic to the point at infinity).

The convex polytope $\mathcal{P}$ satisfies the conditions of the Poincaré's polyhedron theorem, therefore the group generated by reflections on the faces of $\mathcal{P}$ is a discrete subgroup of hyperbolic isometries of $\mathbf{H}^1_{\mathbb{H}}$. We denote this subgroup by $G(3)$. The index-two subgroup generated by composition of an even number of reflections has as fundamental domain the convex polytope $\mathcal{P} \cup T(\mathcal{P})$. This subgroup of $PSL(2, \mathbb{H}(\mathbb{Z}))$ which consists of orientation-preserving isometries will denoted by $G(3)_+$. The polytope $\mathcal{P}$ can be tessellated by four copies of the fundamental domain of the action of $PSL(2, \mathfrak{L})$ and by twelve copies of the fundamental domain of the action of $PSL(2, \mathfrak{H})$ on $\mathbf{H}^1_{\mathbb{H}}$. The quotient space $\mathbf{H}^1_{\mathbb{H}}/G(3)$ is a quaternionic kaleidoscope which is a good non-orientable orbifold.

The orientable orbifold $\mathbf{H}^1_{\mathbb{H}}/G(3)_+$ is obtained from the double pyramid $\mathcal{P} \cup T(\mathcal{P})$ by identifying in pairs the faces with an ideal vertex at infinity with corresponding faces with an ideal vertex at zero. These 3-dimensional faces meet at the square faces of the cube $\mathcal{C}$ in Π and they are identifying in pairs by a rotation of angle $2\pi/3$ around the hyperbolic plane that contains the square faces. The underlying space is $\mathbb{R}^4$ and the singular locus of $\mathcal{O}_{G(3)_+}$ is a cube. This group is generated by the six rotations of angle $2\pi/3$ around the hyperbolic planes that contain the square faces of the cube $\mathcal{C}$.

Fundamental domains of the quaternionic modular groups $PSL(2,\mathfrak{L})$ **and** $PSL(2,\mathfrak{H})$**.**

We start from the following important lemma:

Lemma

Let $\gamma \in PSL(2,\mathbb{H})$ *satisfy (BG) conditions. If* $\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}}$*, then*

$$\Re(\gamma(\mathbf{q})) = \frac{\Re(\mathbf{q})}{|\mathbf{q}c + d|^2} \tag{7}$$

Proof.

We recall that if $\mathbf{q} \in \mathbf{H}^1_{\mathbb{H}}$ the action of γ in $\mathbf{H}^1_{\mathbb{H}}$ is given by the rule

$$\begin{aligned}
\gamma(\mathbf{q}) &= (a\mathbf{q} + b)(c\mathbf{q} + d)^{-1} \\
&= (a\mathbf{q} + b)(\overline{\mathbf{q}c} + \overline{d})\left(\frac{1}{|\mathbf{q}c + d|^2}\right).
\end{aligned}$$

Then:

$$\begin{aligned}
\Re(\gamma(\mathbf{q})) &= \frac{\Re(a\mathbf{q} + b)(\overline{\mathbf{q}}\,\overline{c} + \overline{d}) + (c\mathbf{q} + d)(\overline{\mathbf{q}}\,\overline{a} + \overline{b})}{2|\mathbf{q}c + d|^2} \\
&= \frac{|\mathbf{q}|^2 a\overline{c} + a\mathbf{q}\overline{d} + b\overline{\mathbf{q}c} + b\overline{d} + |\mathbf{q}|^2 c\overline{a} + c\mathbf{q}\overline{b} + d\overline{\mathbf{q}a} + d\overline{b}}{2|\mathbf{q}c + d|^2} \\
&= \frac{\Re(b\overline{\mathbf{q}}\,\overline{c} + a\mathbf{q}\,\overline{d})}{|\mathbf{q}c + d|^2}.
\end{aligned}$$

∎

Proof.

Let $\mathbf{q} = x + yI$, where $x > 0$, $y \in \mathbb{R}$ and $I^2 = -1$. Then $\overline{\mathbf{q}} = x - yI$ and

$$
\begin{aligned}
\Re(\gamma(\mathbf{q})) &= \frac{\Re(b(x - yI)\overline{c} + a(x + yI)\overline{d})}{|\mathbf{q}c + d|^2} \\
&= \frac{\Re(xb\overline{c} - ybI\overline{c} + xa\overline{d} + yaI\overline{d})}{|\mathbf{q}c + d|^2} \\
&= \frac{x + \Re(-ybI\overline{c} + yaI\overline{d})}{|\mathbf{q}c + d|^2} \\
&= \frac{x - ybI\overline{c} + ycI\overline{b} + yaI\overline{d} - ydI\overline{a})}{|\mathbf{q}c + d|^2} \\
&= \frac{x + y(-bI\overline{c} + cI\overline{b} + aI\overline{d} - dI\overline{a})}{|\mathbf{q}c + d|^2}
\end{aligned}
$$

∎

Proof.

On the other hand, since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL(2, \mathbb{H})$ and $\begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} \in PSL(2, \mathbb{H})$ and both satisfy (BG) conditions, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & al+b \\ c & cl+d \end{pmatrix} \in PSL(2, \mathbb{H})$$

and satisfies (BG) conditions. Therefore (BG) conditions imply:

$$(al+b)(-l\overline{c}+\overline{d}) + (cl+d)(-l\overline{a}+\overline{b}) = 0,$$

Then,

$$-al^2\overline{c} + al\overline{d} - bl\overline{c} + b\overline{d} - cl^2\overline{a} + cl\overline{b} - dl\overline{a} + d\overline{b} = 0,$$

$$al\overline{d} - dl\overline{a} + cl\overline{b} - bl\overline{c} = 0.$$

Finally, we have

$$\Re(\gamma(\mathbf{q})) = \frac{x}{|\mathbf{q}c+d|^2} = \frac{\Re(\mathbf{q})}{|\mathbf{q}c+d|^2}.$$

∎

We notice that if one restricts the entries of the matrices to the set $\mathbb{H}(\mathbb{Z})$ or $\mathbb{H}ur$, then there are only a finite number of possibilities for c and d in such a way that $|\mathbf{q}c + d|$ is less than a given number; therefore we obtain the following important

Corollary

For every $\mathbf{q} \in \mathbb{H}$ *one has*

$$\sup_{\gamma \in PSL(2,\mathfrak{L})} \Re(\gamma(\mathbf{q})) < \infty \qquad \textit{and} \qquad \sup_{\gamma \in PSL(2,\mathfrak{H})} \Re(\gamma(\mathbf{q})) < \infty.$$

This corollary is the key reason why the orbifolds $\mathbf{H}^1_{\mathbb{H}}/PSL(2,\mathfrak{L})$ and $\mathbf{H}^1_{\mathbb{H}}/PSL(2,\mathfrak{H})$ are all of finite volume. See section 9. The orbifolds $\mathbf{H}^1_{\mathbb{H}}/PSL(2,\mathbb{H}(\mathbb{Z}))$ and $\mathbf{H}^1_{\mathbb{H}}/PSL(2,\mathbb{H}ur(\mathbb{Z}))$ of the actions of $PSL(2,\mathbb{H}(\mathbb{Z}))$ and $PSL(2,\mathbb{H}ur(\mathbb{Z}))$ on $\mathbf{H}^5_{\mathbb{R}}$ are of finite volume by a similar inequality.

Orbifolds and fundamental domains for translations and for the inversion.

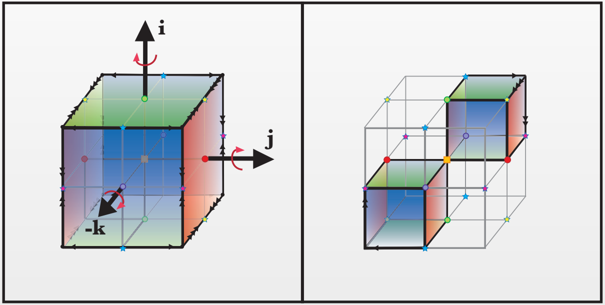


Figure 3. Left: The action of $U(\mathfrak{L})$ on the cube $\mathcal{C}$.
ÊRight: The two hyperbolic cubes $\mathcal{C}_1$ and $\mathcal{C}_2$ in $\mathcal{C}$ which are the bases of a fundamental domain $\mathcal{P}_{\mathfrak{L}}$ of $PSL(2, \mathfrak{L})$.

From the above considerations and taking into account the actions of the generators and the affine group one can describe the fundamental domains of $PSL(2,\mathfrak{L})$ and $PSL(2,\mathfrak{H})$

Fundamental domain of $PSL(2, \mathfrak{H})$

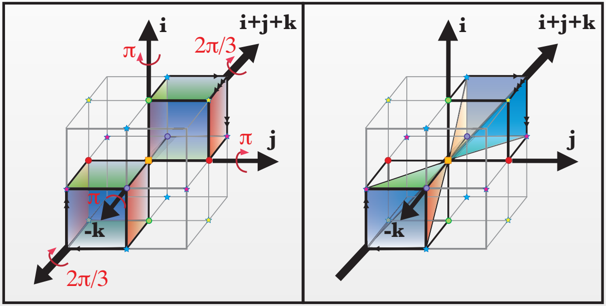


Figure 4. Left: The action of $U(\mathfrak{H})$ in the cube $\mathcal{C}$.
Right: the bases of the fundamental domain of $PSL(2, \mathfrak{H})$. The two hyperbolic pyramids $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathcal{C}$ which are the bases of a fundamental domain $\mathcal{P}_{\mathfrak{H}}$ of $PSL(2, \mathfrak{H})$.

A fundamental domain of $PSL(2,\mathfrak{L})$ as an ideal cone over a rhombic hyperbolic dodecahedron We can describe another fundamental domain of the modular group which is convex using "cut and paste" techniques

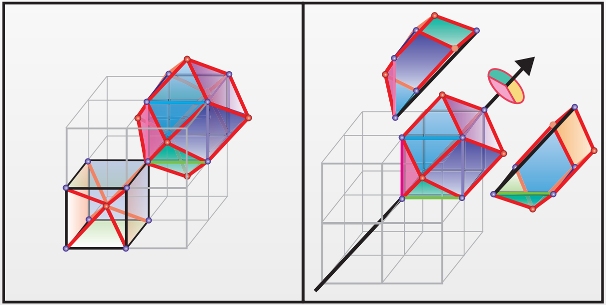


Figure 6. A fundamental domain $\mathcal{R}_{\mathfrak{L}}$ for $PSL(2,\mathfrak{L})$ can be taken to be the pyramid over the rhombic dodecahedron $\mathcal{R}$ with apex the point at infinity. A fundamental domain $\mathcal{R}_{\mathfrak{H}}$ for $PSL(2,\mathfrak{H})$ can be taken to be the pyramid over a third part of the rhombic dodecahedron $\mathcal{R}$.

Geometric characterization of the quaternionic modular groups

The following fundamental theorem gives the description of the Lipschitz quaternionic modular group $PSL(2, \mathfrak{L})$ as the group of quaternionic Möbius transformations whose entries are Lipschitz integers and which also satisfy (BG) conditions.

Theorem

Any element in $PSL(2, \mathbb{H}(\mathbb{Z}))$ which satisfies (BG) conditions belongs to the quaternionic modular group $PSL(2, \mathfrak{L})$.

Proof.

Let $A \in PSL(2, \mathbb{H}(\mathbb{Z}))$ satisfy (BG) conditions. Let $q = A(1)$ and $S \in PSL(2, \mathfrak{L})$ be such that $p := S(q) \in \mathcal{P}$. Then $(SA)(1) = p$ and it follows that $SA \in \mathcal{A}(\mathfrak{L})$. Hence $A \in \mathcal{A}(\mathfrak{L}) \subset PSL(2, \mathfrak{L})$. ∎

This theorem completely characterizes the group of Möbius transformations with entries in the Lipschitz integers which preserve the hyperbolic half-space $\mathbf{H}^1_{\mathbb{H}}$.
This proof can be adapted *verbatim* to prove the following Theorem which characterizes $PSL(2, \mathfrak{H})$:

Theorem

Any element in $PSL(2, \mathbb{H}ur)$ which satisfies (BG) conditions belongs to the quaternionic modular group $PSL(2, \mathfrak{H})$.

Coxeter decomposition of the fundamental domains.

The polytope $\mathcal{P}$ is a *Coxeter polytope* i.e. the angles between its faces called *dihedral angles* are submultiples of π. The geometry of the hyperbolic tessellation of $\mathbf{H}^1_{\mathbb{H}}$ that is generated by reflections on the sides of $\mathcal{P}$ is codified by these angles. We denote this tessellation of $\mathbf{H}^1_{\mathbb{H}}$ by $\mathbf{Y}$. In order to understand it we will consider another tessellation of $\mathbf{H}^1_{\mathbb{H}}$, which is a refinement based on a barycentric decomposition of $\mathbf{Y}$ and whose cells all are isometric to a fixed hyperbolic 4-simplex which we denote by $\Delta_{\mathfrak{L}}$. This model simplex $\Delta_{\mathfrak{L}}$ is a Coxeter simplex with one ideal vertex.

It is important for us to describe the groups $PSL(2, \mathfrak{L})$ and $PSL(2, \mathfrak{H})$ as Coxeter subgroups of rotations of the symmetries of the tessellation generated by hyperbolic reflections of $\Delta_{\mathfrak{L}}$.

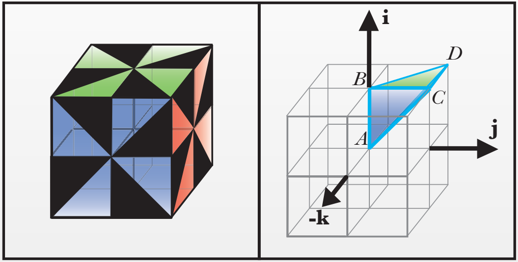


Figure 8. The Coxeter decomposition into 48 tetrahedra of a cube in the Euclidean 3-space.

The Euclidean tetrahedron $\Delta^e_{\mathfrak{L}}$ is the standard Coxeter's 3-simplex $\triangle(4,3,4)$. The Euclidean tessellation whose cells are isometric copies of the tetrahedron with Coxeter symbol $\triangle(4,3,4)$ is the refinament obtained by means of the barycentric subdivision of the classic tessellation by cubes of the Euclidean 3-space whose Schläfli symbol is $\{4,3,4\}$. Each cube is divided into 48 tetrahedra of type $\triangle(4,3,4)$. The Schläfli symbol of a cube is $\{4,3\}$. This symbol means that the faces of a cube are squares with Schläfli symbol $\{4\}$ and that the link of each vertex is an equilateral triangle with Schläfli symbol $\{3\}$. The symbol of the tessellation $\{4,3,4\}$ of the Euclidean 3-space means that the 3-dimensional cells are cubes with Schläfli symbol $\{4,3\}$ and that the link or verticial figure of each vertex in the tessellation $\{4,3,4\}$ is an octahedron with Schläfli symbol $\{3,4\}$.

We can compute the dihedral angles of $\triangle(4,3,4)$ by means of $\{4,3,4\}$.
In short, we list the 10 dihedral angles of $\Delta_{\mathfrak{L}}$:

$$\begin{array}{lll} \angle BCD = \pi/3, & \angle AC\infty = \pi/2, & \angle ABD = \pi/2, \\ \angle AB\infty = \pi/4, & \angle BD\infty = \pi/2, & \angle ABC = \pi/2, \\ \angle AD\infty = \pi/3, & \angle BC\infty = \pi/2, & \angle ACD = \pi/2. \\ \angle CD\infty = \pi/4, & & \end{array}$$

Definition

Let $\Gamma_{\{3,4,3,4\}}$ be the hyperbolic Coxeter group generated by reflections on the sides of $\Delta_{\mathfrak{L}}$. This group is a hyperbolic Kleinian group.

The polytope $\mathcal{P}$ is the union of the $6 \times 8 = 48$ simplexes asymptotic at ∞ and isometric to $\Delta_{\mathfrak{L}}$ which have bases contained in the cube $\mathcal{C}$. The Lipschitz fundamental domain $\mathcal{P}_{\mathfrak{L}}$ is obtained as the union of $6 \times 2 = 12$ simplexes asymptotic at ∞ and isometric to $\Delta_{\mathfrak{L}}$ with bases in the two cubes $\mathcal{C}_1$ and $\mathcal{C}_2$. The Hurwitz modular domain $\mathcal{P}_{\mathfrak{H}}$ is obtained as the union of 4 simplexes asymptotic at ∞ and isometric to $\Delta_{\mathfrak{L}}$ since $PSL(2,\mathfrak{L})$ is a subgroup of index 3 of $PSL(2,\mathfrak{H})$.

Finally, applying $24 \times 48 = 1152$ elements of the group $\Gamma_{\{3,4,3,4\}}$ to $\Delta_{\mathfrak{L}}$ we obtain an union of isometric copies of $\mathcal{P}$ that forms a right-angled 24-cell which is a cell of the regular hyperbolic honeycomb $\{3, 4, 3, 4\}$. See figure 13. There are 24 octahedra in the boundary of the 24-cell and there are 48 simplexes congruent to $\Delta_{\mathfrak{L}}$ over each one of the octahedrons. The non-compact, right-angled hyperbolic 24-cell has finite volume equal to $4\pi^2/3$.

We can summarize these previous results in the following theorem

Theorem

The Coxeter group $\Gamma_{\{3,4,3,4\}}$ *of finite covolume contains as subgroups the quaternionic modular groups* $PSL(2,\mathfrak{L})$ *and* $PSL(2,\mathfrak{H})$*. We have* $PSL(2,\mathfrak{L}) \subset PSL(2,\mathfrak{H}) \subset \Gamma_{\{3,4,3,4\}}$*.*
We have the following indices:

- $[\Gamma_{\{3,4,3,4\}} : PSL(2,\mathfrak{H})] = 4$*,*
- $[\Gamma_{\{3,4,3,4\}} : PSL(2,\mathfrak{L})] = 12$ *and*
- $[PSL(2,\mathfrak{H}) : PSL(2,\mathfrak{L})] = 3$*.*

Volumes of the fundamental domains

The Coxeter decomposition of the 24-cell implies that the group of symmetries (both orientation-preserving and orientation-reversing) of the 24-cell is of order $24 \times 48 = 1152$. With the action of these 1152 symmetries the 24-cell can be divided into 1152 congruent simplexes where each of them is congruent to $\Delta_{\mathfrak{L}}$. One knows from [**?**] that the volume of the hyperbolic right-angled 24-cell is $4\pi^2/3$, therefore the volume of $\Delta_{\mathfrak{L}}$ is $(4\pi^2/3)$ divided by 1152, this is $(\pi^2/864)$. Then, we have the following proposition

Proposition

The volume of $\mathcal{P}_{\mathfrak{L}}$ *is* $12(\pi^2/864) = \pi^2/72$ *and the volume of* $\mathcal{P}_{\mathfrak{H}}$ *is* $4(\pi^2/864) = \pi^2/216$*.*

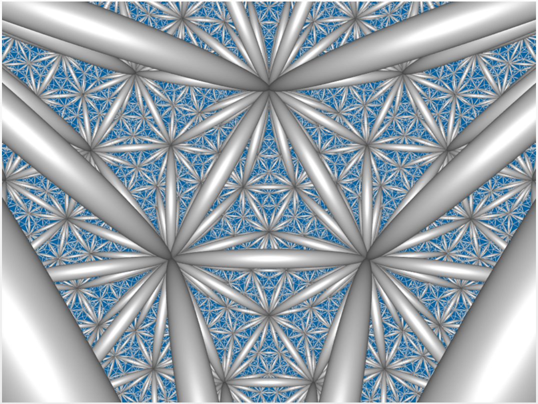


Figure 9. Schematic 3-dimensional version: the hyperbolic tessellation $\{3,4,4\}$. This figure is courtesy of Roice Nelson

Algebraic presentation of the Lipschitz modular group $PSL(2,\mathfrak{L})$

Theorem

The group $PSL(2,\mathfrak{L})$ has the following finite presentation:

$$PSL(2,\mathfrak{L}) = \left\langle T, \tau_{\mathbf{i}}, \tau_{\mathbf{j}}, \tau_{\mathbf{k}} \,|\, \mathfrak{R}_{\mathfrak{L}} \right\rangle,$$

where $\mathfrak{R}_{\mathfrak{L}}$ is the set of relations:

Theorem

$$\mathfrak{R}_{\mathfrak{L}} := \begin{cases} T^2,\ [\tau_{\mathbf{i}} : \tau_{\mathbf{j}}],\ [\tau_{\mathbf{i}} : \tau_{\mathbf{k}}],\ [\tau_{\mathbf{k}} : \tau_{\mathbf{j}}], \\ (\tau_{\mathbf{i}} T)^6,\ (\tau_{\mathbf{j}} T)^6,\ (\tau_{\mathbf{k}} T)^6, \\ (\tau_{\mathbf{i}}\tau_{\mathbf{j}} T)^4,\ (\tau_{\mathbf{i}}\tau_{\mathbf{k}} T)^4,\ (\tau_{\mathbf{j}}\tau_{\mathbf{k}} T)^4, \\ (\tau_{\mathbf{i}}\tau_{\mathbf{j}}\tau_{\mathbf{k}} T)^6, \\ (\tau_{\mathbf{i}} T)^3\,(\tau_{\mathbf{j}} T)^3\,(\tau_{\mathbf{k}} T)^3,\ (\tau_{\mathbf{i}} T)^3\,(\tau_{\mathbf{k}} T)^3\,(\tau_{\mathbf{j}} T)^3, \\ [(\tau_{\mathbf{i}} T)^3 : T], [(\tau_{\mathbf{j}} T)^3 : T], [(\tau_{\mathbf{k}} T)^3 : T], \\ [(\tau_{\mathbf{i}} T)^3 : \tau_{\mathbf{i}}], [(\tau_{\mathbf{j}} T)^3 : \tau_{\mathbf{j}}], [(\tau_{\mathbf{k}} T)^3 : \tau_{\mathbf{k}}], \\ (\tau_{\mathbf{u}} T)^3 \tau_{\mathbf{w}} (\tau_{\mathbf{u}} T)^3 \tau_{\mathbf{w}},\ \textit{where } \mathbf{u} \neq \mathbf{w} \textit{ are units in the set } \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}. \end{cases}$$

Theorem

The group $PSL(2, \mathfrak{H})$ has the following finite presentation:

$$PSL(2,\mathfrak{H}) = \left\langle T, \tau_{\mathbf{i}}, \tau_{\mathbf{j}}, \tau_{\mathbf{k}}, D_{\omega_1}, D_{\omega_{\mathbf{i}}}, D_{\omega_{\mathbf{j}}}, D_{\omega_{\mathbf{k}}} \,|\, \mathfrak{R}_{\mathfrak{H}} \right\rangle,$$

where $\mathfrak{R}_{\mathfrak{H}}$ is the set of relations:

Theorem

$$\mathfrak{R}_{\mathfrak{H}} := \begin{cases} T^2,\ [\tau_{\mathbf{i}} : \tau_{\mathbf{j}}],\ [\tau_{\mathbf{i}} : \tau_{\mathbf{k}}],\ [\tau_{\mathbf{k}} : \tau_{\mathbf{j}}], \\ (\tau_{\mathbf{i}} T)^6,\ (\tau_{\mathbf{j}} T)^6,\ (\tau_{\mathbf{k}} T)^6, \\ (\tau_{\mathbf{i}}\tau_{\mathbf{j}} T)^4,\ (\tau_{\mathbf{i}}\tau_{\mathbf{k}} T)^4,\ (\tau_{\mathbf{j}}\tau_{\mathbf{k}} T)^4, \\ (\tau_{\mathbf{i}}\tau_{\mathbf{j}}\tau_{\mathbf{k}} T)^6, \\ (\tau_{\mathbf{i}} T)^3\,(\tau_{\mathbf{j}} T)^3\,(\tau_{\mathbf{k}} T)^3,\ (\tau_{\mathbf{i}} T)^3\,(\tau_{\mathbf{k}} T)^3\,(\tau_{\mathbf{j}} T)^3, \\ [(\tau_{\mathbf{i}} T)^3 : T], [(\tau_{\mathbf{j}} T)^3 : T], [(\tau_{\mathbf{k}} T)^3 : T], \\ [(\tau_{\mathbf{i}} T)^3 : \tau_{\mathbf{i}}], [(\tau_{\mathbf{j}} T)^3 : \tau_{\mathbf{j}}], [(\tau_{\mathbf{k}} T)^3 : \tau_{\mathbf{k}}], \\ D_{\mathbf{u}}\tau_{\mathbf{w}}D_{\mathbf{u}}\tau_{\mathbf{w}},\ \text{where } \mathbf{u} \neq \mathbf{w} \text{ are units in the set } \{\mathbf{i},\mathbf{j},\mathbf{k}\} \\ [D_{\omega_1} : T], [D_{\omega_{\mathbf{i}}} : T], [D_{\omega_{\mathbf{j}}} : T], [D_{\omega_{\mathbf{k}}} : T], \\ (D_{\omega_1})^3, (D_{\omega_{\mathbf{i}}})^3, (D_{\omega_{\mathbf{j}}})^3, (D_{\omega_{\mathbf{k}}})^3, \\ D_{\omega_1} D_{\mathbf{i}} D_{\omega_{\mathbf{i}}} D_{\mathbf{j}} D_{\omega_{\mathbf{k}}} D_{\mathbf{k}}, \\ D_{\mathbf{k}} D_{\omega_{\mathbf{k}}} D_{\mathbf{i}} D_{\omega_{\mathbf{j}}}^{-1} D_{\mathbf{j}} D_{\omega_1}, \\ D_{\mathbf{i}} D_{\omega_{\mathbf{k}}} D_{\mathbf{j}} D_{\omega_{\mathbf{i}}} D_{\mathbf{k}} D_{\omega_{\mathbf{j}}}, \\ D_{\mathbf{j}} D_{\omega_{\mathbf{j}}}^{-1} D_{\mathbf{k}} D_{\omega_{\mathbf{i}}} D_{\mathbf{i}} D_{\omega_1}. \end{cases}$$

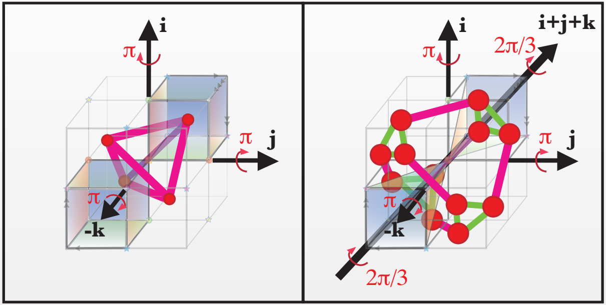


Figure 11. The unitary groups in the Cayley graphs of the quaternionic modular groups $PSL(2,\mathfrak{L})$ and $PSL(2,\mathfrak{H})$. The edges in red correspond to elements of order two and the edges in green correspond to elements of order three.

The volume of an orbifold is the same as the volume of its fundamental domain. Then we had computed $\mathrm{Vol}(\mathcal{O}^4_{\mathfrak{L}}) = 3\mathrm{Vol}(\mathcal{O}^4_{\mathfrak{H}})$ in the section 7.1. This is related to the Gauss-Bonnet-Euler theorem for orbifolds.

Selberg's covers and examples of hyperbolic 4-manifolds

By a *Selberg cover* we mean a covering space which is a manifold which corresponds to a torsion-free and finite-index subgroup. We have already remarked that the group $PSL(2, \mathfrak{L})$ is a subgroup of the symmetries of the honeycomb $\{3, 4, 3, 4\}$. This is a corollary of the previousresults. Then the fundamental domain $\mathcal{P}_{\mathfrak{L}}$ of the group $PSL(2, \mathfrak{L})$ is commensurable with a hyperbolic regular right–angled convex cell $\{3, 4, 3\}$ of the honeycomb $\{3, 4, 3, 4\}$). In other words, there is a finite subdivision of $\mathcal{P}_{\mathfrak{L}}$ and $\{3, 4, 3\}$ by congruent polyhedrons. The 24 vertices of this $\{3, 4, 3\}$ are:

$$0, \infty, \pm\mathbf{i}, \pm\mathbf{j}, \pm\mathbf{k}, \pm\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}, \frac{1}{2}(\pm\mathbf{i} \pm \mathbf{j} \pm \mathbf{k}). \tag{8}$$

The Selberg's theorem says that there exist smooth cover hyperbolic 4-manifolds of the orbifolds $\mathcal{O}^4_{\mathfrak{L}}$ and $\mathcal{O}^4_{\mathfrak{H}}$.

Proposition

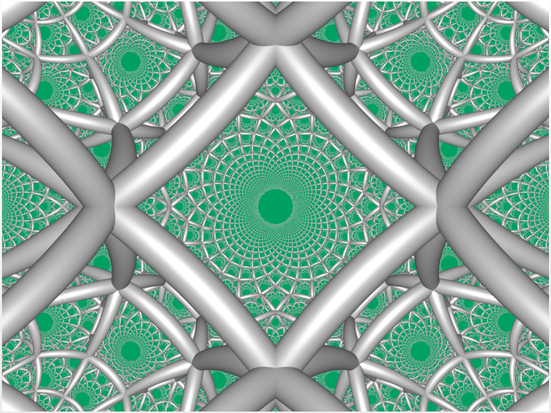
The minimal orders of Selberg covers of $O^4_{\mathfrak{L}}$ and $O^4_{\mathfrak{H}}$ are of orders 96 and 288, respectively.

In 1999, J. Ratcliffe and T. Tschantz found 1171 noncompact hyperbolic 4-manifolds which have Euler characteristic 1 by side-pairings in a fundamental region $\{3, 4, 3\}$ of the honeycomb $\{3, 4, 3, 4\}$. In 2004, D. Ivansić showed that the nonorientable 4-manifold numbered 1011 in the Ratcliffe and Tschantz's list, this is, the 4-manifold M_{1011} with the biggest order of their symmetry groups, is the complement of five Euclidean 2-torus in a closed 4-manifold with fundamental group isomorphic to $\mathbb{Z}/2\mathbb{Z}$. Moreover, the orientable double cover $\tilde{M}_{1011}$ is a complement of five 2-torus in the 4-sphere. In 2008 Ivansić showed that this 4-sphere has the same topology of the standard differentiable 4-sphere and not of an exotic 4-sphere. In his doctoral thesis J.P. Díaz provides diagrams of this link to give an explicit model of the isotopy class of the link.

We now recall the beautiful construction of a complete, nonorientable hyperbolic 4-manifold of finite volume with six cusps whose cross sections are $\mathbb{S}^1 \times \mathbf{K}^2$, where $\mathbf{K}^2$ is the Klein bottle. Let us consider the open unit ball $\mathbf{B}^4$ in $\mathbb{H}$ with the Poincaré metric. Let $\mathbf{C}_{24}$ denote the 24-cell whose vertices are the Hurwitz unit as seen before. There are 24 faces which are regular ideal hyperbolic octahedrons. Given a face F there is an opposite face $-F$ which is the face diametrically opposite to F (the image under multiplication by -1). One identifies F with $-F$ by a composition which consists of a reflection with respect to the hyperplane which contains F followed by multiplication by -1. This composition is an orientation-reversing hyperbolic isometry which sends $\mathbf{C}_{24}$ onto a contiguous cell of the honeycomb determined by the 24-cell. This pairing of each face with its opposite has the effect of creating a nonsingular, nonorientable, hyperbolic manifold with 6 cusps. We can take the orientable double covering.

In 2005, Ratcliffe, Tschantz and Ivansić showed that there is a dozen of examples of non-orientable hyperbolic 4-manifolds from this list whose orientable double covers are complements of five or six Euclidean surfaces (tori and Klein bottles) in the 4-sphere.

Ratcliffe showed that three of these dozen complements can be used to construct aspheric 4-manifolds that are homology spheres by means of Dehn's fillings.



HAPPY 85$^{\text{th}}$ BIRTHDAY PROFESSOR JOHN MILNOR!