Surfaces in the space of surfaces

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Planes in compact hyperbolic manifolds

$f : \mathbb{H}^2 \to M^n = \mathbb{H}^n/\Gamma$

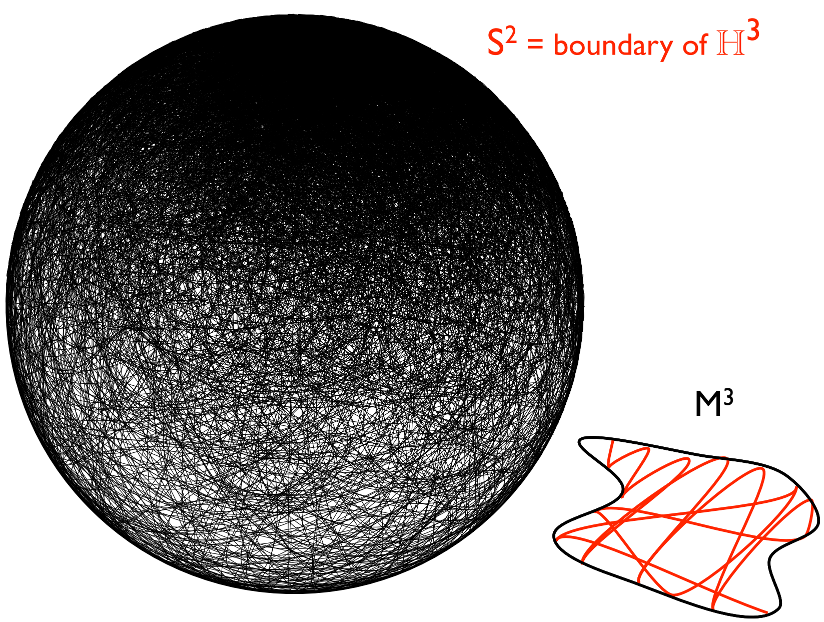
Theorem (Shah, Ratner)

The closure of $f(\mathbb{H}^2)$ is a compact, immersed, totally geodesic submanifold N^k inside M^n.

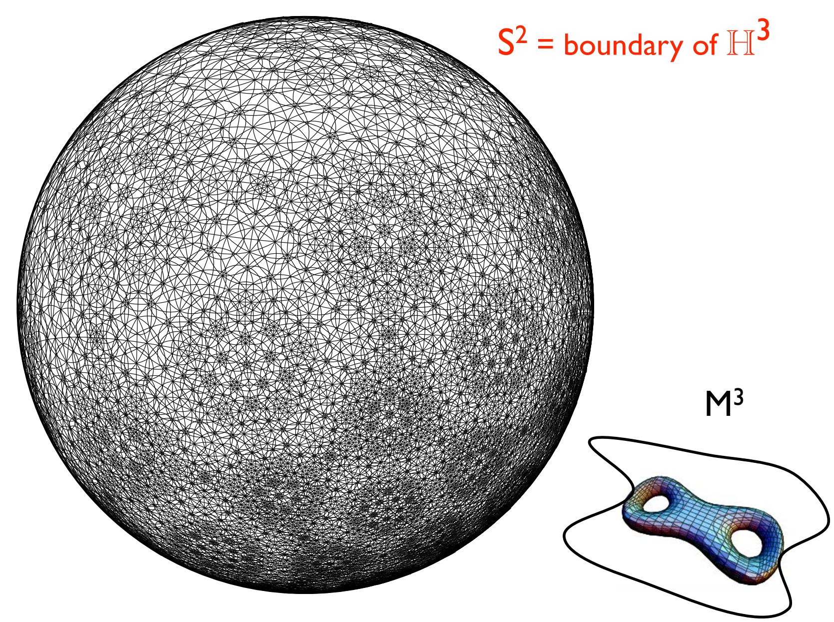
Ex: $f : \mathbb{H}^2 \to M^3$

Im(f) = a closed surface, or
Im(f) is dense in M^3.

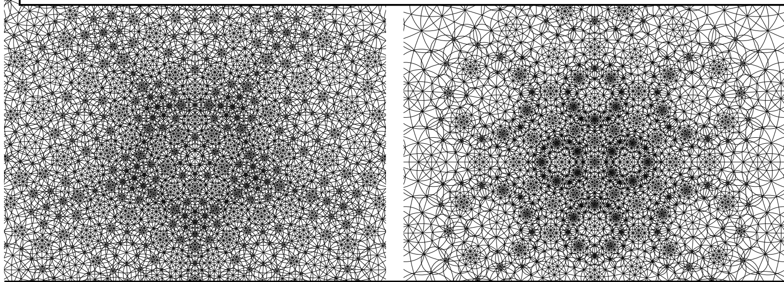
Dense plane in M^3



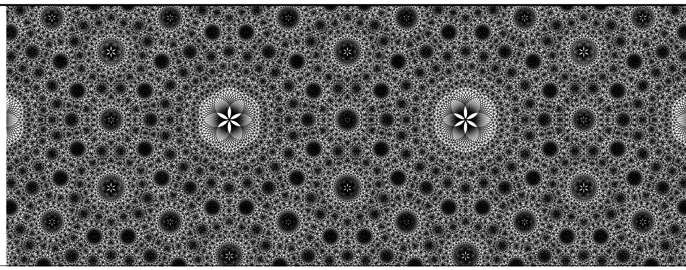
Closed, totally geodesic surface in M^3



Arithmetic tetrahedra

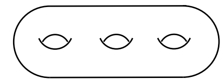


Open problem: Do ∞ many closed geodesic surfaces $\Rightarrow$ M is arithmetic?



Moduli space

$\mathcal{M}_g$ = moduli space of Riemann surfaces X of genus g

{  }

-- a complex variety, dimension 3g-3

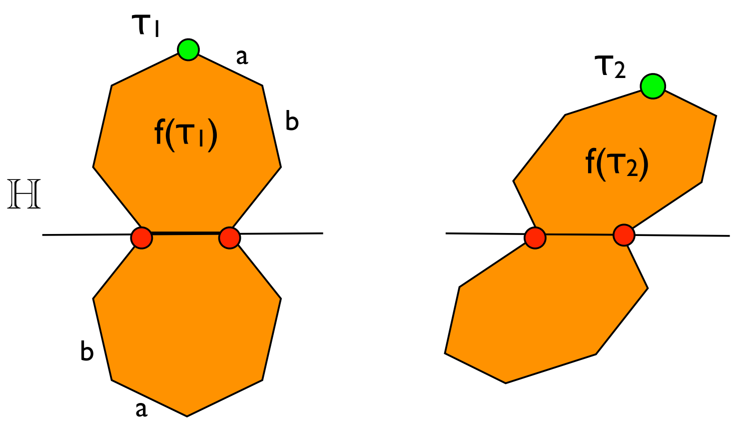
Teichmüller metric

There exists a holomorphic, isometrically immersed *complex geodesic*

$$f : \mathbb{H}^2 \to \mathcal{M}_g$$

through every point in every possible direction.

Example of a complex geodesic $f : \mathbb{H}^2 \to \mathcal{M}_3$



f(τ) = Polygon(τ)/gluing = genus 3 X(τ)

Planes in $\mathcal{M}_g$

$$f : \mathbb{H}^2 \to \mathcal{M}_g = \mathcal{T}_g/\mathrm{Mod}_g$$

Theorem (M, Eskin-Mirzakhani-Mohammadi, Filip)

The closure of f($\mathbb{H}^2$) is an algebraic subvariety of moduli space.

2002, 2014

Example: For g=2, the closure of f($\mathbb{H}^2$) can be
a Teichmüller curve,
a Hilbert modular surface,
or the whole space.

The Hilbert modular surface is ruled, but not totally geodesic.

Totally geodesic subvarieties

$\mathcal{M}_g \subset \mathbb{P}^N$ is a projective variety

Almost all subvarieties $V \subset \mathcal{M}_g$ are *contracted*.

PROBLEM

What are the *totally geodesic*[*] subvarieties $V \subset \mathcal{M}_g$?

(*Every complex geodesic *tangent* to V is contained in V.)

Known geodesic subvarieties in $\mathcal{M}_g$

I. Covering constructions

$$\begin{tikzcd} Y \arrow[d, "d"] & \mathcal{M}_g & \widetilde{\mathcal{M}}_h \arrow[l, "f"'] \arrow[dl] \\ X & \mathcal{M}_h & \end{tikzcd}$$

Im(f) = a totally geodesic subvariety

Example: $\widetilde{\mathcal{M}}_{1,2} \to \mathcal{M}_{1,3}$

II. Teichmüller curves

$$\begin{tikzcd} \mathbb{H} \arrow[rr, "f"] \arrow[dr] & & \mathcal{M}_g \\ & V=\mathbb{H}/\Gamma \arrow[ur] & \end{tikzcd}$$

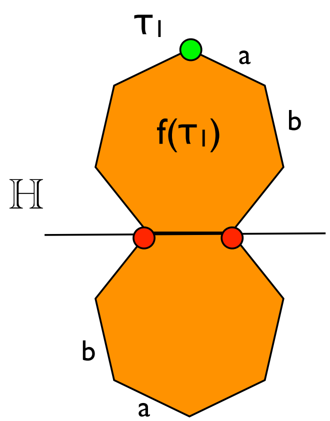
finite area

Im(f) = a totally geodesic curve

Example of a Teichmüller curve

$$\begin{tikzcd} \mathbb{H} \arrow[rr, "f"] \arrow[dr] & & \mathcal{M}_3 \\ & V=\mathbb{H}/\Gamma(2,7,\infty) \arrow[ur] & \end{tikzcd}$$

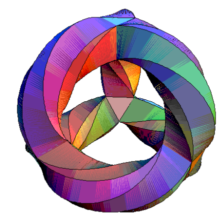
= the Klein quartic!

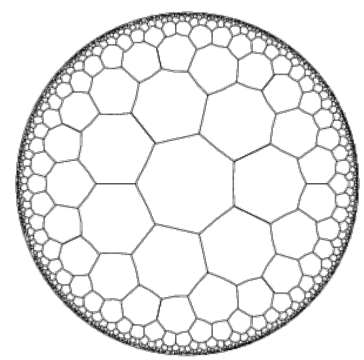


Klein quartic

Helaman Ferguson, 1993

Thurston, MSRI director, 1992-1997





$168 = 7\times 24 = |PSL_2(Z/7)|$

1st example of a Teichmüller surface

Theorem

There is a primitive, totally geodesic complex surface F (the flex locus) properly immersed into $\mathcal{M}_{1,3}$.

Complement

The universal cover of F is not isomorphic, as a complex manifold, to any $\mathcal{T}_{g,n}$.

A new Teichmüller space?

Proof that F does not exist

Let V be a totally geodesic hypersurface in $\mathcal{M}_g$.
Given [X] in V, let $q_0,\ldots,q_n$ be a basis for $Q(X) = T^*_X \mathcal{M}_g$.
Assume q_0 generates the normal bundle to V.

Then the *highly nonlinear condition*:

$$\int_X q_0 \frac{\sum \overline{a}_i \overline{q}_i}{|\sum a_i q_i|} = 0$$

Is equivalent to a linear condition on (a_i) of the form

$$\sum a_i b_i = 0.$$

Description of the Teichmüller surface

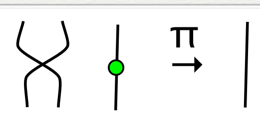
The flex locus $F \subset \mathcal{M}_{1,3}$
is the set of

(A,P) in $\mathcal{M}_{1,3}$:

$\exists$ degree 3 map $\pi: A \to \mathbb{P}^1$ such that

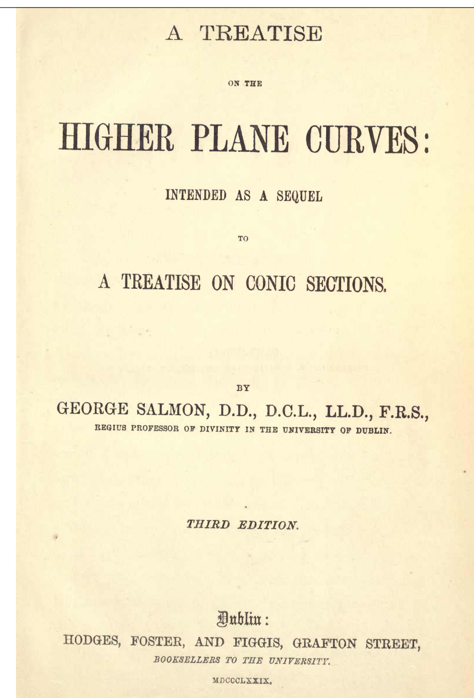
(i) $P \sim Z$ = any fiber of π; and

(ii) $P \subset$ cocritical points of π.



What is the dimension of F? Is F irreducible? ....

1879


A TREATISE

ON THE

HIGHER PLANE CURVES:

INTENDED AS A SEQUEL

TO

A TREATISE ON CONIC SECTIONS.

BY

GEORGE SALMON, D.D., D.C.L., LL.D., F.R.S.,
REGIUS PROFESSOR OF DIVINITY IN THE UNIVERSITY OF DUBLIN.

THIRD EDITION.

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HODGES, FOSTER, AND FIGGIS, GRAFTON STREET,
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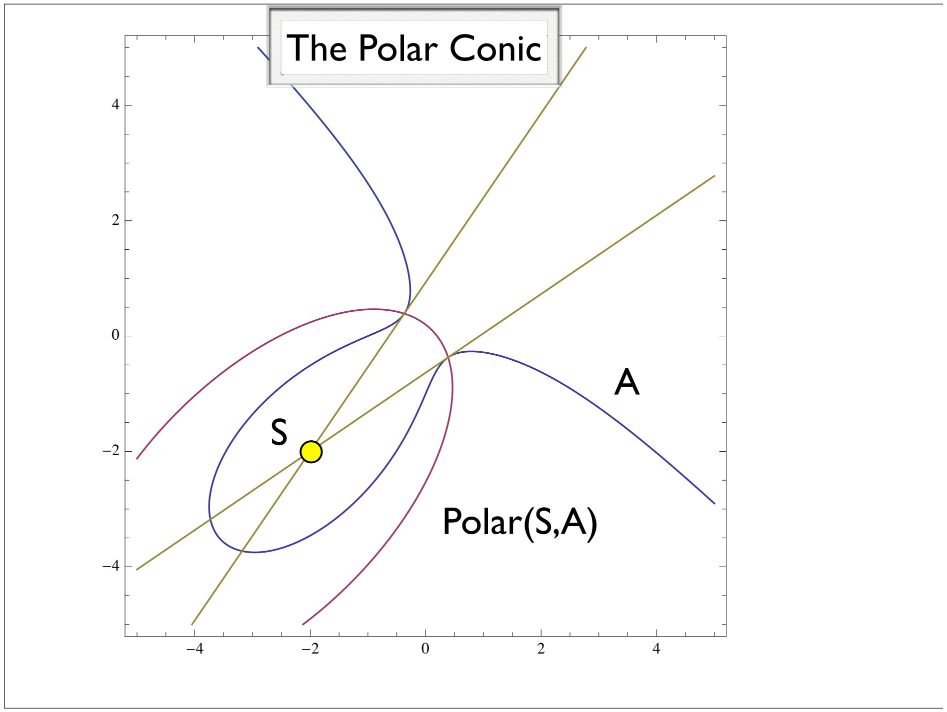
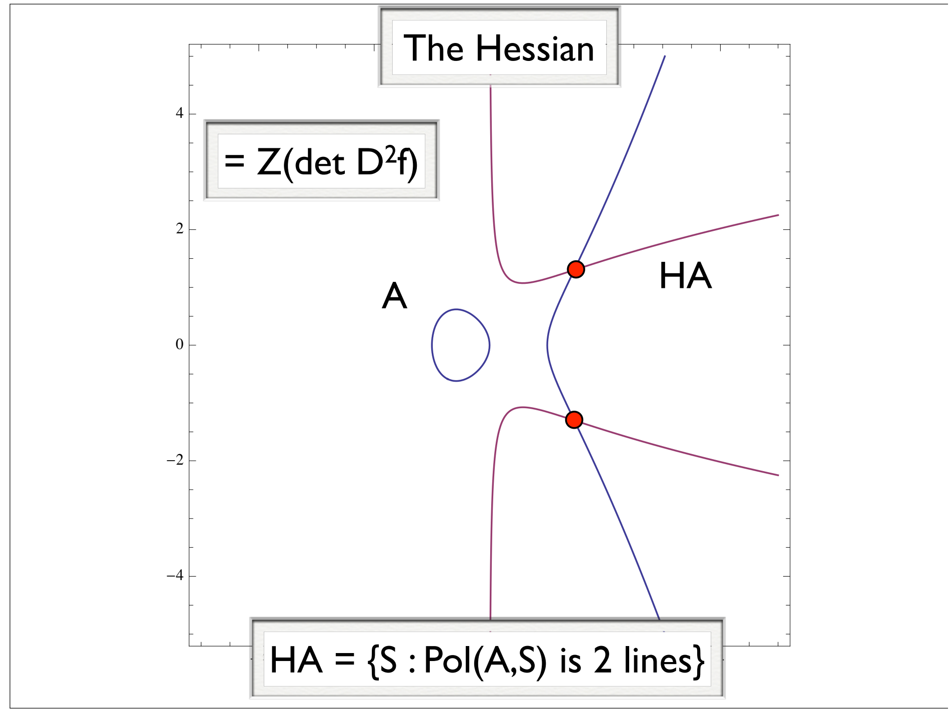
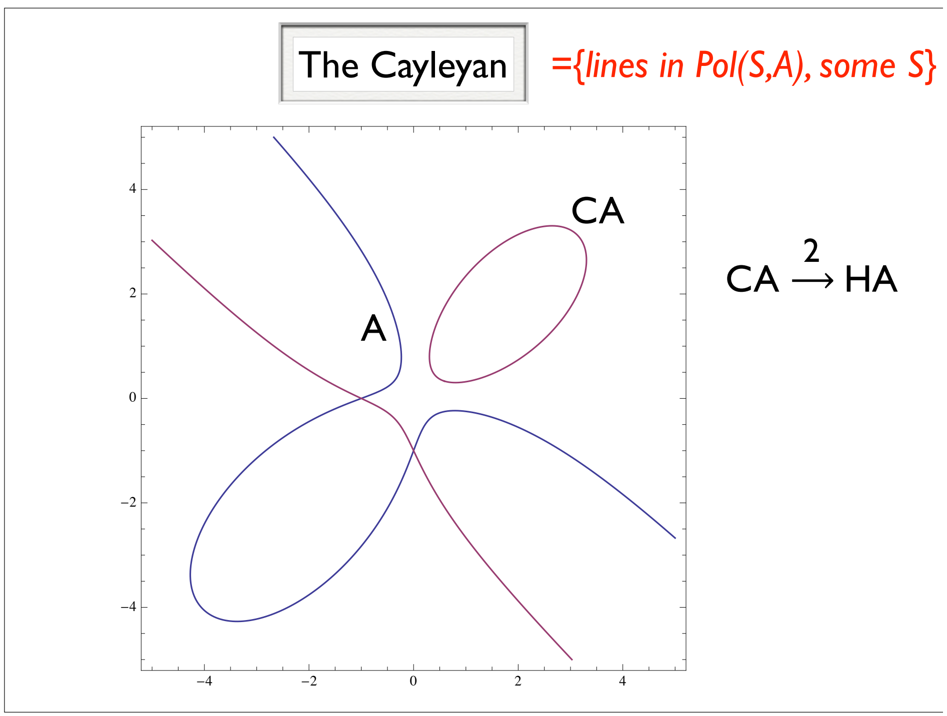
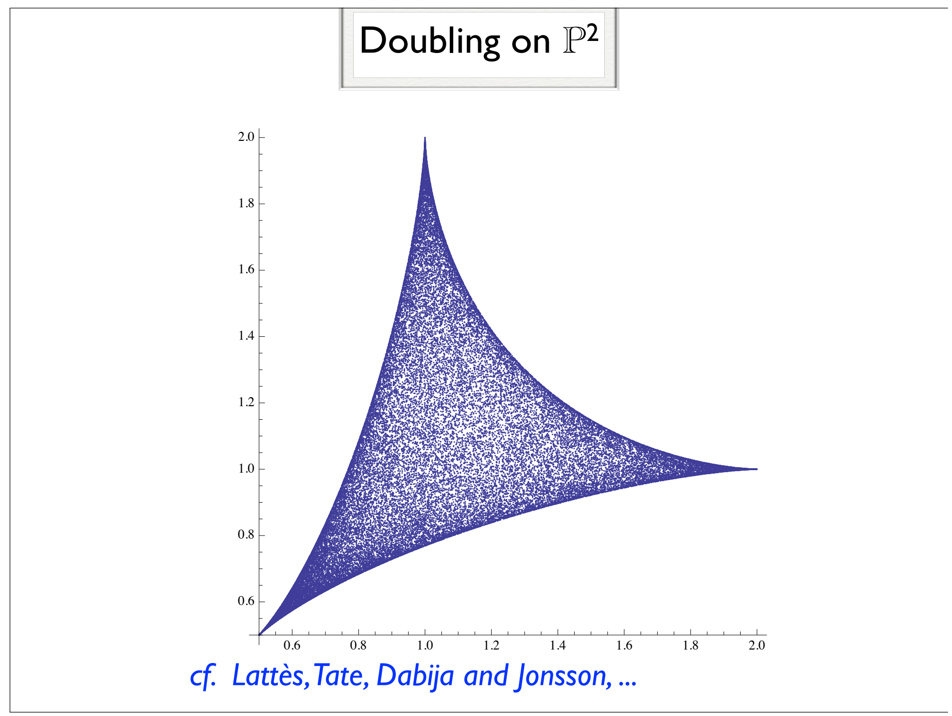
MDCCCLXXIX.


The Polar Conic
S
A
Polar(S,A)


The Hessian
= Z(det $D^2 f$)
A
HA
HA = {S : Pol(A,S) is 2 lines}


The Cayleyan
={lines in Pol(S,A), some S}
CA
A
$CA \xrightarrow{2} HA$


Doubling on $\mathbb{P}^2$
cf. Lattès, Tate, Dabija and Jonsson, ...

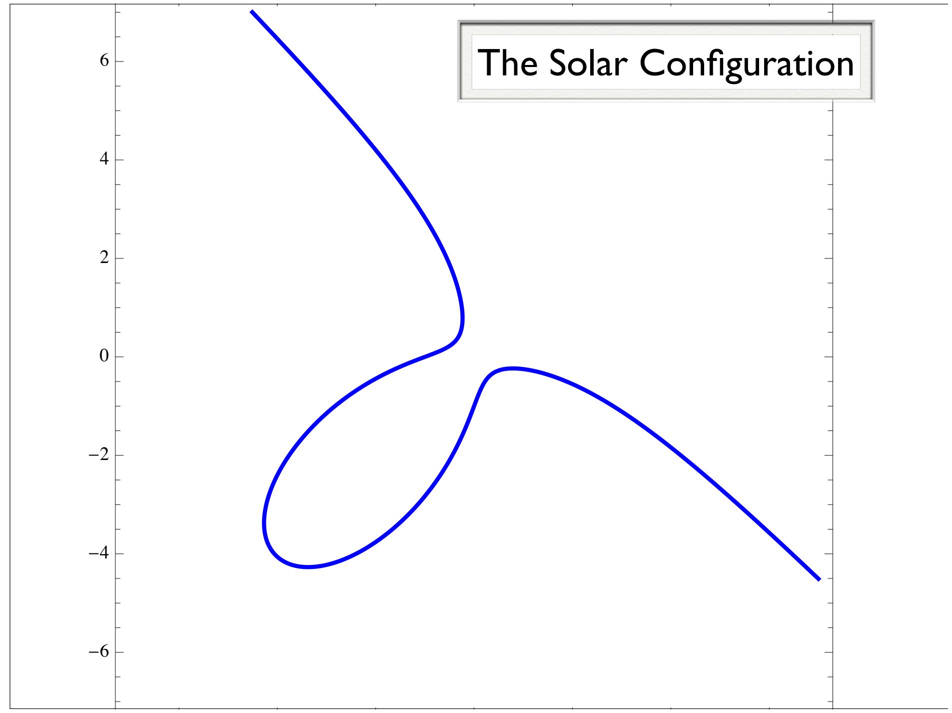
The flex locus $F \subset \mathcal{M}_{1,3}$

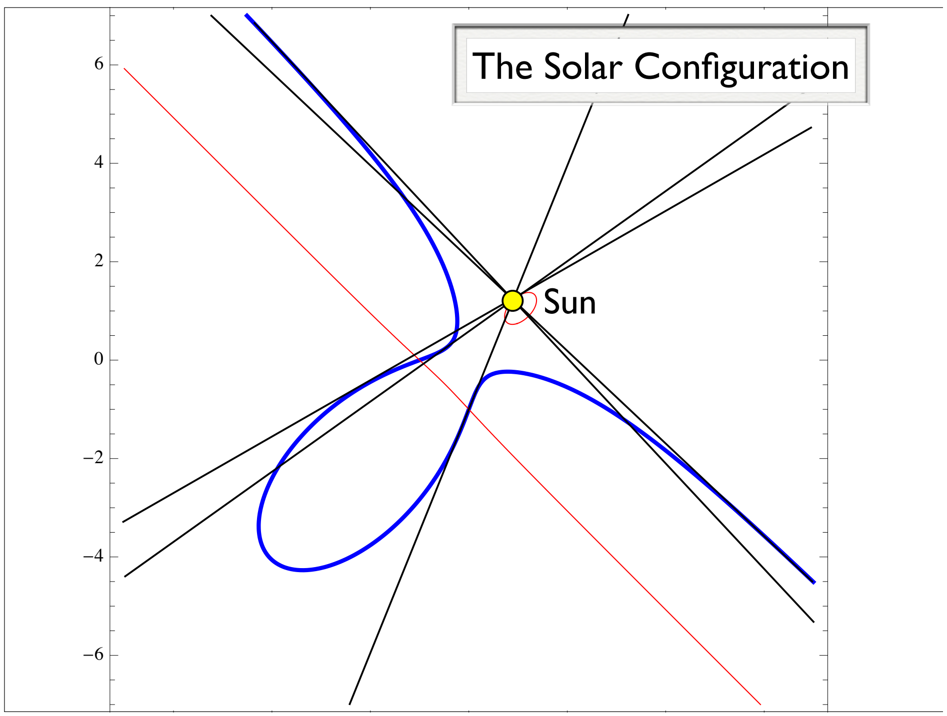
is:

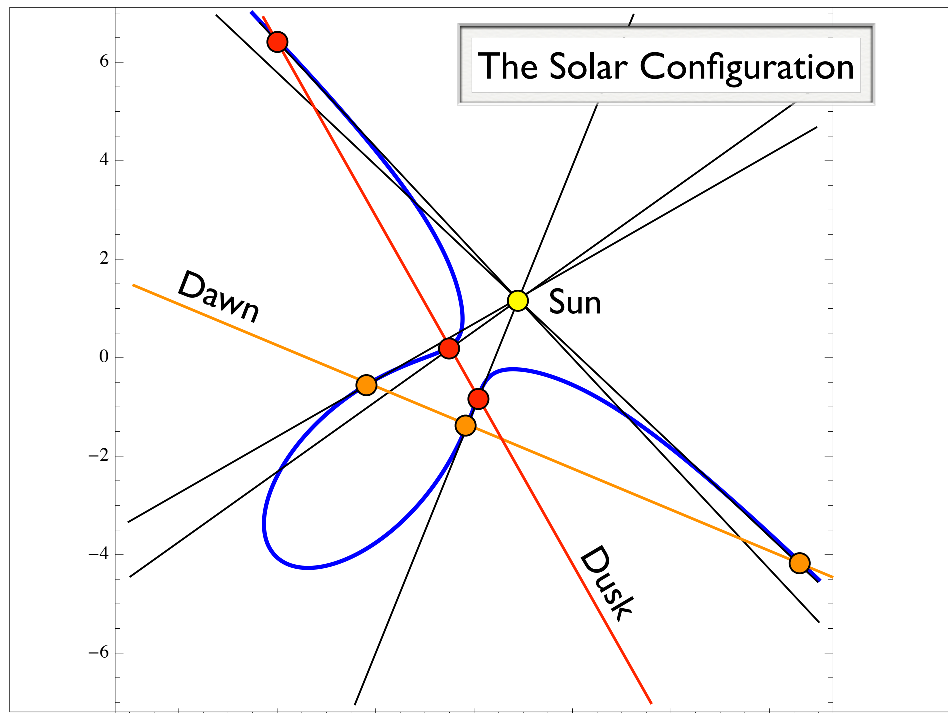
(A,P) in $\mathcal{M}_{1,3}$:
P = double(L) ∩ A, for some L in CA

$CA \to F \to \mathcal{M}_1$

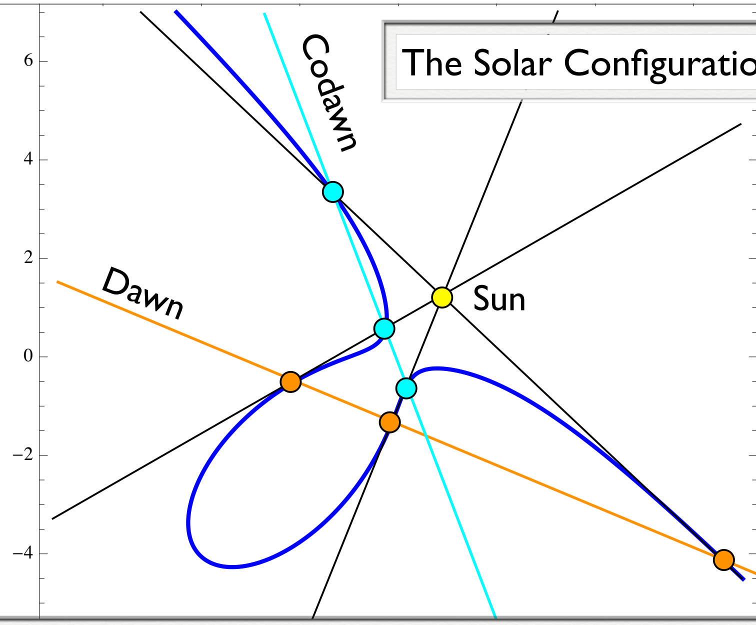
Corollary: F is 2 dimensional.







The Solar Configuration



F is the set of codawn configurations {(A,P)}

The gothic locus $\Omega G \subset \Omega\mathcal{M}_4$

ΩG = {(X,ω) in $\Omega\mathcal{M}_4(2,2,2)$:

(i) ∃ J with A=X/J of genus 1;

(ii) ω is odd for J;

(iii) ∃ odd cubic map p: X → B, genus 1;

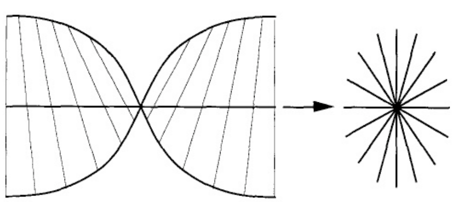
(iv) p(Z(ω)) = one point.

Theorem: ΩG is an $SL_2(\mathbb{R})$ invariant 4-manifold.

From ΩG to F

(X, ω) in ΩG → (A,q) = (X/J, ω^2/J)

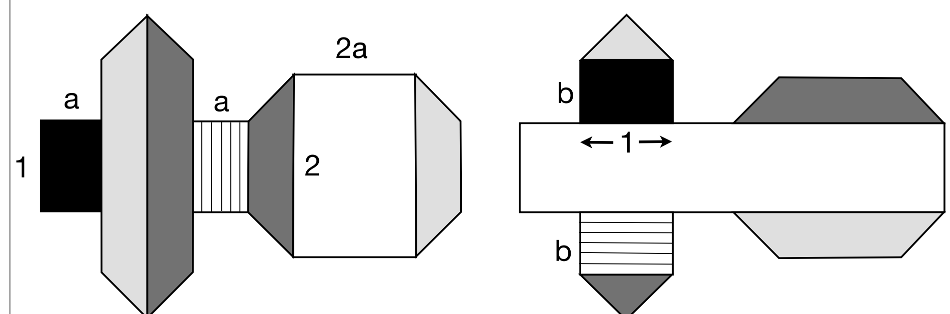
→ (A,P = poles(q)) in F

G  F

Corollary: F is totally geodesic

Cathedral polygons

Wright

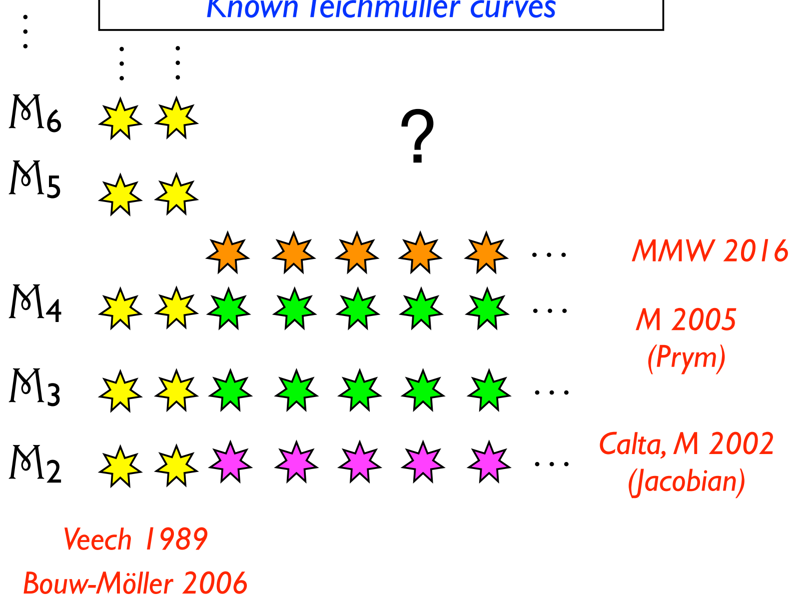


Theorem. Complement.

For every real quadratic field K = $\mathbb{Q}(\sqrt{d})$, there exists a,b in K such that P(a,b)/τ generates a Teichmüller curve V in $\mathcal{M}_4$.

We have a new infinite series of Teichmüller curves in (the gothic locus) $\mathcal{M}_4$.

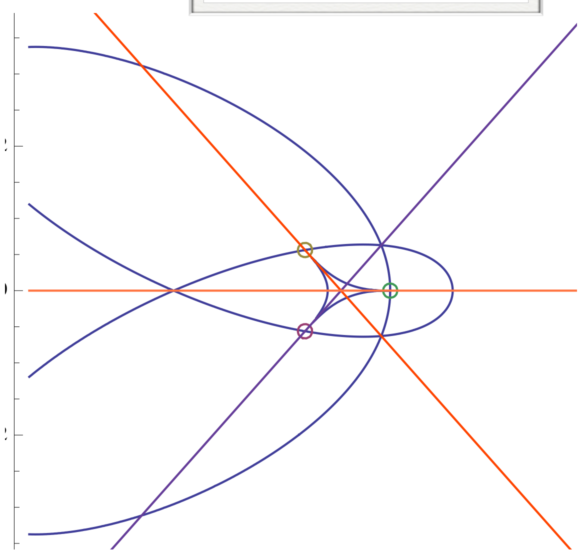
Known Teichmüller curves



Q. What *are* the totally geodesic surfaces in $\mathcal{M}_{1,3}$?

Why does F exist?

Satellite Cayleyan



degree 12,
27 nodes,
9 cusps