Satellite copies of the Mandelbrot set

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Quadratic polynomials on $\widehat{\mathbb{C}}$

$P_c(z) = z^2 + c$, ∞ (super)attracting fixed point, with basin $\mathcal{A}_c(\infty)$

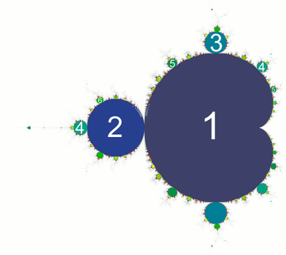
Filled Julia set $K_c = K_{P_c} = \widehat{\mathbb{C}} \setminus \mathcal{A}_c(\infty)$

Mandelbrot set: set of parameters for which K_c is connected

$\heartsuit$ (or H_1): P_c has an attracting fixed point (α f.p.),

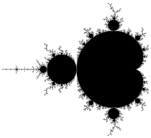
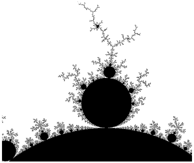
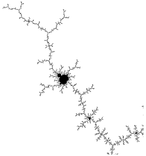
$H_{p/q}$: P_c has a period q attracting cycle,
α repelling of rotation number p/q,

$\partial\heartsuit \cap \partial H_{p/q} = c_{p/q}$, and $P'_{c_{p/q}}(\alpha) = e^{2\pi i p/q}$.
At $c_{p/q}$, α collides with a p/q-repelling cycle.
In $H_{p/q}$
the cycle becomes attracting and α repelling.



Little copies of M inside M

- Striking: apparent little copies of M in M.

- Their presence is explained by the theory of polynomial-like maps.

Polynomial-like mappings

- A (deg d) polynomial-like map is a triple (f, U', U), where $U' \subset\subset U$ and $f : U' \to U$ is a (deg d) proper and holomorphic map.
- $K_f = \{z \in U' \mid f^n(z) \in U', \forall n \geq 0\}$,
- **Straightening theorem (Douady-Hubbard, '85)** Every (deg d) polynomial-like map $f : U' \to U$ is hybrid equivalent to a (deg d) polynomial, a unique such member if K_f is connected.

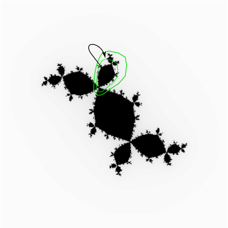
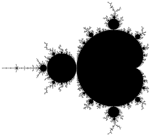
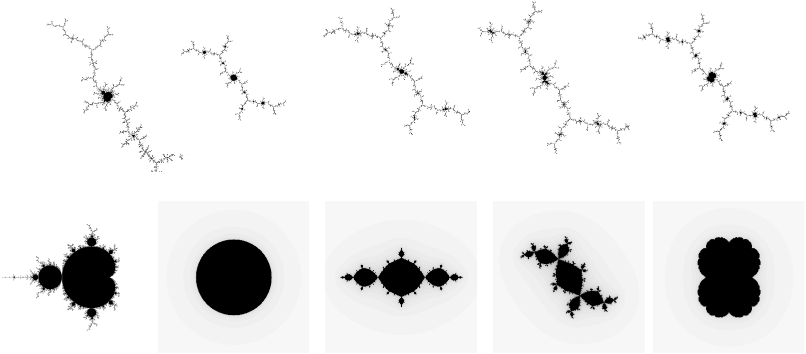


Figure : K_c, c center of the period 3 component

Figure : K_0, 0 center of the main component



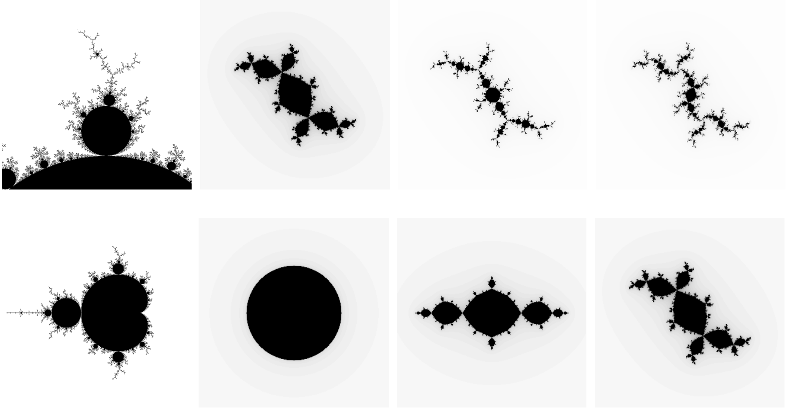
- **Theorem (D-H,'85)** (Under some conditions) there exists a homeomorphism χ between the connectedness locus of an analytic family of deg 2 polynomial-like maps and the Mandelbrot set M.



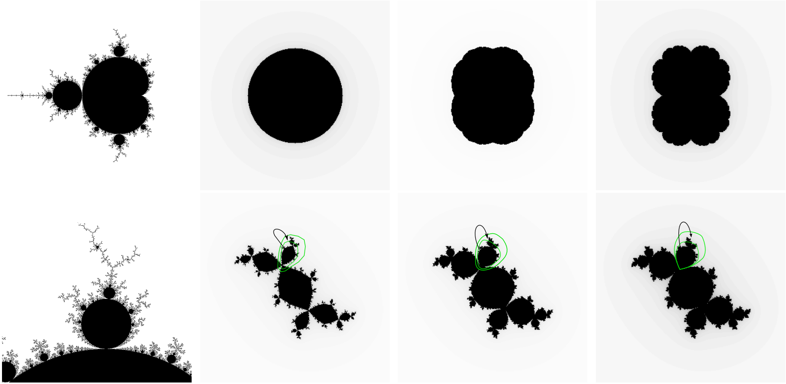
Copies of M inside M

There are 2 kinds of copies:

1. Primitive copies of M, when the the little Julia sets are disjoint (like in the previous slide)
2. Satellite copies of M, when the little Julia sets touch at their β-fixed point (like in this slide).

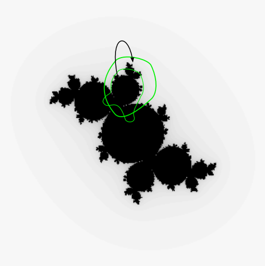


- Primitive copies of M: χ homeo (DH,'85), qc (Lyubich, '99),
- Satellite copies of M: χ homeo *except at the root* (DH,'85), qc *outside a neighborhood of the root* (Ly, '99).



M and its little copies

- Haissinsky ('00): χ homeomorphism at the root in the satellite case.



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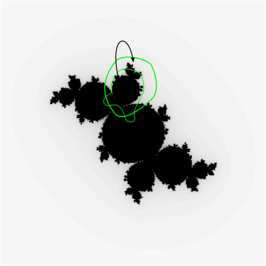
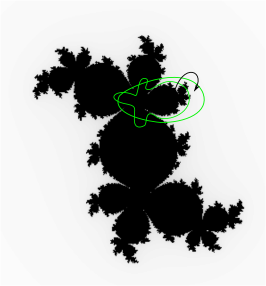
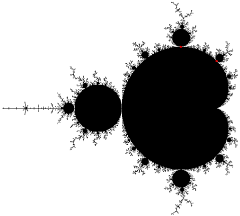






M and its little copies

- We can take a parabolic-like restriction of the roots
- (parabolic-like map: 'polyn.-like map with parab. external map'
- model family: $P_A(z) = z + 1/z + A, \ A \in \mathbb{C}$)

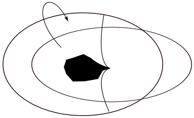
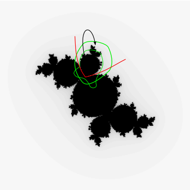
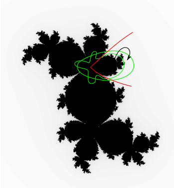


Figure : Parabolic-like map.





- L. ('14): roots of any 2 satel. copies have restrictions qc conj.

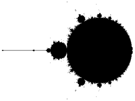


Figure : M_1.

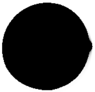


Figure : Filled Julia set of $P_0(z) = z^2 + 1/z$.

M and its little copies

Are the satellite copies mutually qc homeomorphic?

- Consider $\xi_{\frac{p}{q},\frac{P}{Q}} := \chi_{P/Q}^{-1} \circ \chi_{p/q} : M_{p/q} \to M_{P/Q}$:
  1. $\xi_{\frac{p}{q},\frac{P}{Q}}$ qc away from nbh of root,
  2. roots hybrid conjugate on corresponding ears (but not nbh)

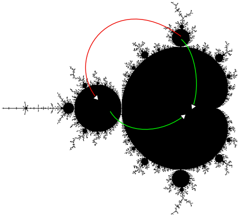


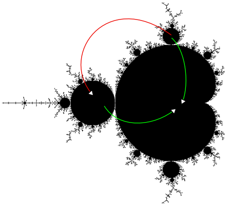
Figure : The map $\xi_{\frac{1}{3},\frac{1}{2}} = \chi_{1/2}^{-1} \circ \chi_{1/3} : M_{1/3} \to M_{1/2}$.

Ideas

(f_λ) an. family of pol-like with connectedness locus $M_{p/q} \setminus root$,
(g_ν) an. family of pol-like with connectedness locus $M_{P/Q} \setminus root$.

Assume $\exists$ a uniform external equivalence between corresponding pol-like: a family of uniformly quasiconformal maps $\Psi_\lambda : A_\lambda \to A_\nu$ between fundamental annuli of f_λ and g_ν respectively. Then

- by Rickmann lemma, for each $\lambda \in M_{p/q} \setminus root$, $f_\lambda \sim_{qc} g_\nu$ uniformly.
- hope to construct some holomorphic motion between $M_{p/q} \setminus root$ and $M_{p/q} \setminus root$, and so prove that ξ is qc.



Problem

When $q \neq Q$, *setting* $\Lambda_\beta = \mathrm{Log}(f'_\lambda(\beta))$ *and* $N_\beta = \mathrm{Log}(g'_\nu(\beta))$,

$$d_{\mathbb{H}}(\Lambda_\beta, N_\beta) \to \infty$$

approaching the root, so the corresponding pol-like f_λ, g_ν *are not uniformly hybrid equivalent approaching the root.*

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Eureka!

- If corresponding families uniformly qc equivalent
- then Teichmüller distance between corresponding pol-like is uniformly bounded
- so in particular $d_{\mathbb{H}}(\Lambda_\beta, N_\beta) < C$

Satellite copies, result

- $M_{p/q}$ satellite copy attached to ♡ at c, where P_c has a fixed point with multiplier $\lambda = e^{2\pi i p/q}$ (so $\chi_{p/q}^{-1}(\heartsuit) = H_{p/q}$)
- **Theorem (L-Petersen, 2015)**: For p/q and P/Q irreducible rationals with $q \neq Q$,

$$\xi_{\frac{p}{q},\frac{P}{Q}} := \chi_{P/Q}^{-1} \circ \chi_{p/q} : M_{p/q} \to M_{P/Q}$$

is not quasi-conformal, *i.e.* it does not admit a quasi-conformal extension to any neighborhood of the root.

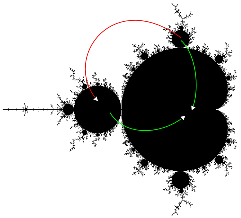


Figure : The map $\xi_{\frac{1}{3},\frac{1}{2}} = \chi_{1/2}^{-1} \circ \chi_{1/3} : M_{1/3} \to M_{1/2}$.

Setting

- Parametrize: $P_\lambda = \lambda z + z^2$ (rel: $\lambda \to c(\lambda) = \frac{\lambda}{2} - \frac{\lambda^2}{4}$), so λ is multiplier of α-fixed point 0
- $\lambda \in M_{p/q}$, $(f_\lambda, U'_\lambda, U_\lambda)$ polynomial-like restriction of P^q_λ, then 0 is β-fixed point for f_λ with multiplier λ^q,
- $\nu = \xi(\lambda) \in M_{P/Q}$, (g_ν, V'_ν, V_ν) polynomial-like res. of P^Q_ν, then 0 is β-fixed point for g_ν with multiplier ν^Q,
- $\Lambda = \mathrm{Log}(\lambda^q)$, $N(\nu) = \mathrm{Log}(\nu^Q)$, and lift $\xi : \lambda \to \nu$ to $\hat{\xi} : \Lambda \to N$.

Strategy:

1. Find a lower bound for K_ϕ for any ϕ: $f_\lambda \sim_{qc} g_\nu$
2. Translate it to a lower bound for K_ξ
   (*Plough in the dynamical plane, and harvest in parameter space*)
3. Send the lower bound to infinity

1-Lower bound for qc conjugacy, dynamical plane

- **Proposition:** $\lambda \in M_{p/q}$, $\nu = \xi(\lambda) \in M_{P/Q}$, $\Lambda = \mathrm{Log}(\lambda^q)$, $N(\nu) = \mathrm{Log}(\nu^Q)$. Any quasi-conformal conjugacy ϕ between f_λ and g_ν has:

$$\lim_{r \to 0} \mathrm{Log}\, ||K_\phi||_{\infty, \mathbb{D}(r)} \geq d_{\mathbb{H}_+}(\Lambda, N),$$

- Proof of the Proposition:
  1. ϕ induces a qc homeomorphism between the corresponding (marked) quotient tori

$$((D_1 \setminus \{0\})/f, \gamma_f) \text{ and } ((D_2 \setminus \{0\})/g, \gamma_g)$$

  2.

$$(T_\Lambda := \mathbb{C}/(\Lambda\mathbb{Z} - i2\pi\mathbb{Z}), \Pi_\Lambda([0, \Lambda]) \sim_T ((D_1 \setminus \{0\})/f, \gamma_f),$$
$$(T_N := \mathbb{C}/(N\mathbb{Z} - i2\pi\mathbb{Z}), \Pi_N([0, N]) \sim_T ((D_2 \setminus \{0\})/g, \gamma_g)$$

  3. $\lim_{r \to 0} \mathrm{Log}\, ||K_\phi||_{\infty, \mathbb{D}(r)} \geq inf_\varphi \mathrm{Log}\, K_\varphi =: d_T(T_\Lambda, T_N) = d_{\mathbb{H}_+}(\Lambda, N)$.

2-Lower bound for $K_{\hat{\xi}}$, parameter plane

1. Holomorphic motion argument
   (for passing from dynamical plane to parameter plane)
2. Generalization of the Teichmüller Thm for non-compact setting
   (we have map between grids, respecting homotopy type of 2 curves),

give:

Theorem: $\Lambda^* \in \Lambda(M_{p/q})$ Misiurewicz parameter s.t. the critical value is prefixed to β_f, $N^* = \hat{\xi}(\Lambda^*)$. Then

$$\lim_{r\to 0} \operatorname{Log} ||K_{\hat{\xi}}||_{\infty,\mathbb{D}(\Lambda^*,r)} \geq d_{\mathbb{H}_+}(\Lambda^*, N^*).$$

- By Yoccoz inequality, $\operatorname{diam}_{\mathbb{H}_+}((L^{p/q})_{\frac{n^2-1}{n^3}}) \stackrel{n\to\infty}{\longrightarrow} 0$
  so it's enough to prove $d_{\mathbb{H}_+}(\Lambda, N)$ unbounded on $\partial\heartsuit_{p/q}$!

3-About $d_{\mathbb{H}_+}(\Lambda^*, N^*)$

- For $\lambda \in M_{p/q}$, consider f_λ :
  - 0 is β-fixed point with multiplier λ^q,
  - $\alpha(\lambda)$ is α-fixed point with multiplier $\rho(\lambda)$ ($\rho : nbh(\heartsuit_{p/q}) \to nbh(\mathbb{D})$)
- Since ϕ_λ hybrid, if $\nu = \xi(\lambda)$, $\rho(\lambda) = \rho(\nu)$.
- Invert: take ρ as parameter! Computations show:

$$\Lambda(\rho) = -\frac{\mathrm{Log}(\rho)}{q} - \left(\frac{\mathrm{Log}(\rho)}{q}\right)^2 \cdot \mathrm{Resit}(f_{e^{2\pi i p/q}}, 0) + O\left(\left(\frac{\mathrm{Log}\rho}{q}\right)^3\right),$$

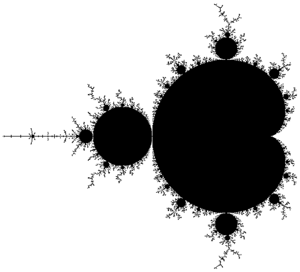
$$N(\rho) = -\frac{\mathrm{Log}(\rho)}{Q} - \left(\frac{\mathrm{Log}(\rho)}{Q}\right)^2 \cdot \mathrm{Resit}(g_{e^{2\pi i P/Q}}, 0) + O\left(\left(\frac{\mathrm{Log}\rho}{Q}\right)^3\right).$$

- So, for $q \neq Q$, and $\rho = e^{it} \in \mathbb{S}^1$, $d_{\mathbb{H}_+}(\Lambda(\rho), N(\rho)) \overset{\rho\to 1}{\longrightarrow} \infty$

End: for a sequence $\Lambda_n^* \in (L^{p/q})_{\frac{n^2-1}{n^3}} \cap M_{p/q}$,

$$\lim_{r\to 0} \mathrm{Log}\,||K_{\hat{\xi}}||_{\infty,\mathbb{D}(\Lambda^*,r)} \geq d_{\mathbb{H}_+}(\Lambda_n^*, N_n^*) \overset{n\to\infty}{\longrightarrow} \infty.$$

Thank you for your attention!



Happy birthday Jack!

happybirthday