

# A disconnected deformation space of rational maps

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# Preliminaries

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$(S^2, A)$

$\downarrow f$

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# Teichmüller Space

$\mathcal{T}_A :=$  classes of  $\phi : S^2 \rightarrow \mathbb{P}^1$  where  $\phi_1 \sim \phi_2$  if there is  $\mu \in \text{Aut}(\mathbb{P}^1)$  so that  $\phi_1 = \mu \circ \phi_2$  on  $A$ , and  $\phi_1$  is isotopic to  $\mu \circ \phi_2$  rel  $A$ .

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$$\begin{array}{ccc} & \xrightarrow{\sigma_f} & \\ \mathcal{T}_B & & \mathcal{T}_A \\ & \xleftarrow{\sigma_{A,B}} & \end{array}$$

$$\text{Def}_A^B(f) := \{\tau \in \mathcal{T}_B \mid \sigma_f(\tau) = \sigma_{A,B}(\tau)\}$$



# The forgetful map

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$$\sigma_f : [\phi] \mapsto [\psi]$$

$$\text{Def}_A^B(f) \hookrightarrow \mathcal{T}_B \begin{array}{c} \xrightarrow{\sigma_f} \\ \xrightarrow{\sigma_{A,B}} \end{array} \mathcal{T}_A$$

$$\begin{array}{l} \text{cv}(f) \subseteq B \\ A \subseteq B \end{array}$$

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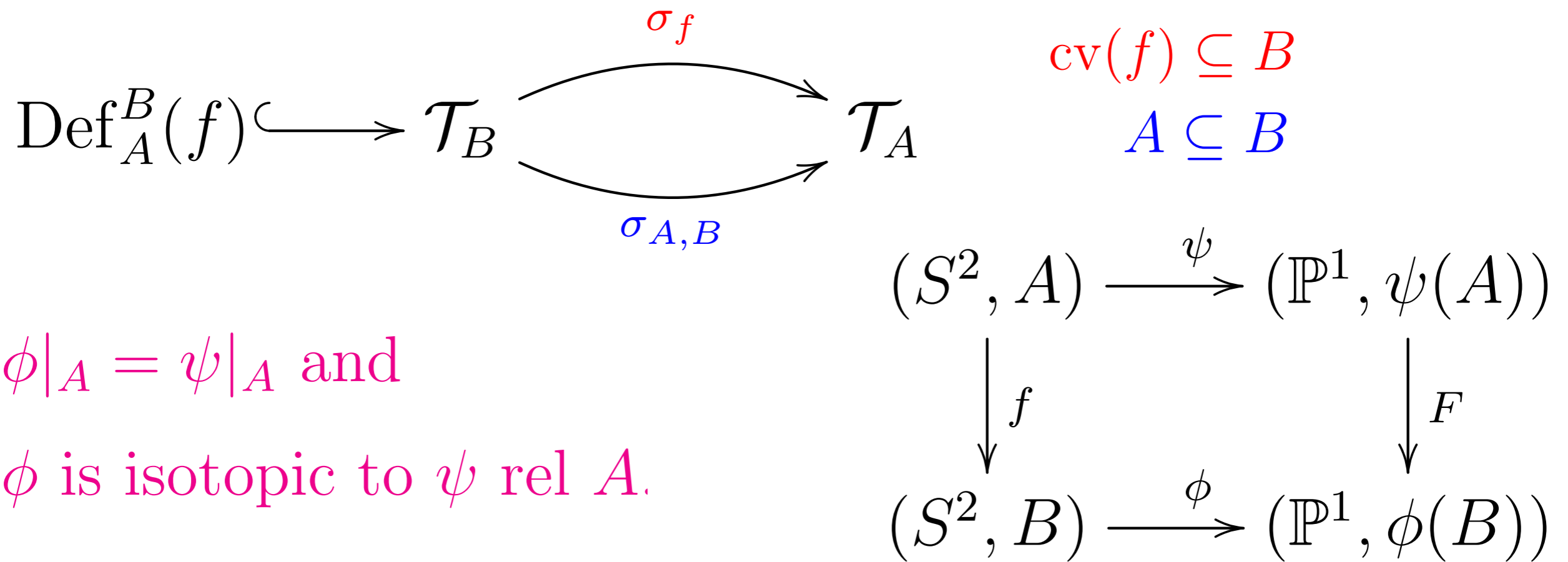
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$\phi|_A = \psi|_A$  and

$\phi$  is isotopic to  $\psi$  rel  $A$ .

$$\begin{array}{ccc} (S^2, A) & \xrightarrow{\psi} & (\mathbb{P}^1, \psi(A)) \\ \downarrow f & & \downarrow F \\ (S^2, B) & \xrightarrow{\phi} & (\mathbb{P}^1, \phi(B)) \end{array}$$





$$\text{Def}_A^B(f) \neq \emptyset$$



$f : (S^2, A) \rightarrow (S^2, B)$  is combinatorially equivalent to a rational map

# The dynamical case

If  $A = B$ , Thurston proved:

**Theorem.** The space  $\text{Def}_B^B(f) \neq \emptyset$  if and only if the map  $f : (S^2, B) \rightarrow (S^2, B)$  admits no obstructing multicurves.

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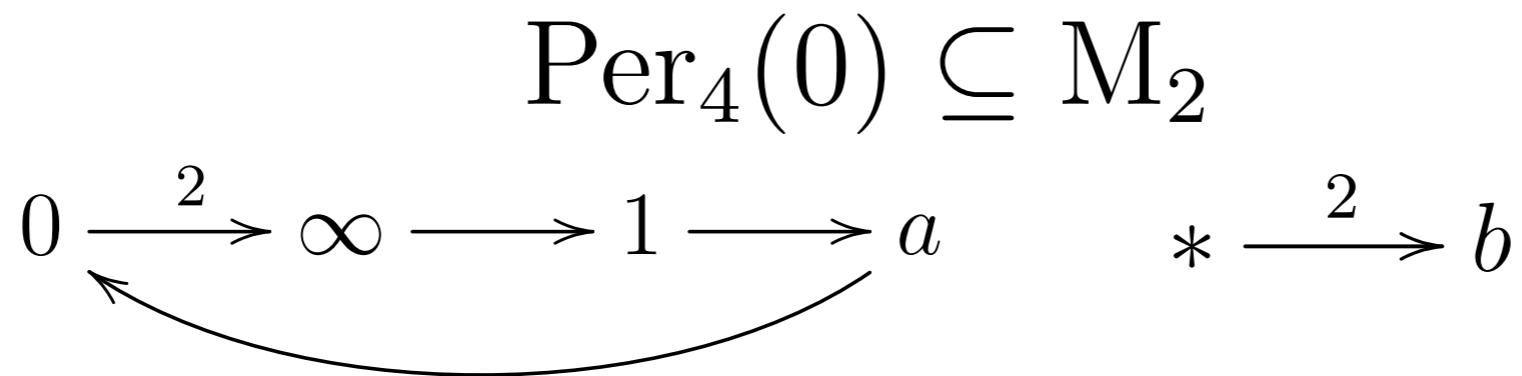
**Theorem.** The space  $\text{Def}_B^B(f)$  is connected.

## Local structure

The following result is due to Epstein.

**Theorem.** Let  $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$  be a rational map which is not of Lattès type. Then  $\text{Def}_A^B(f)$  is a complex analytic submanifold of  $\mathcal{T}_B$  of dimension  $|B - A|$ .

# A disconnected deformation space

$$\text{Per}_4(0) \subseteq M_2$$
$$0 \xrightarrow{2} \infty \longrightarrow 1 \longrightarrow a \quad * \xrightarrow{2} b$$


$$A = \{0, 1, \infty, a\} \quad B = A \cup \{b\} \quad b \notin A$$

$$\dim(\text{Def}_A^B(f)) = 1 \quad \dim(\mathcal{T}_B) = 2 \quad \dim(\mathcal{T}_A) = 1$$

**Theorem.** (H–K)

For  $\langle f \rangle \in \text{Per}_4(0)^*$ ,  $\text{Def}_A^B(f)$  has infinitely many connected components.

Tanya Firsova, Jeremy Kahn, and Nikita Selinger  
proved a related result

Work of Mary Rees

# Moduli space

$\mathcal{M}_A := \{\text{injective } \varphi : A \hookrightarrow \mathbb{P}^1 \text{ up to postcomposition}$   
 $\text{with Möbius transformations}\}$

$\mathcal{M}_A$  is a complex manifold, isomorphic to  
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$$\begin{array}{ccc} \mathcal{T}_A & [\phi] & \\ \downarrow & \downarrow & \\ \mathcal{M}_A & [\phi|_A] & \end{array}$$

Modular group  $\text{Mod}_A$   
isomorphic to pure mapping  
class group of  $(S^2, A)$

# The induced homomorphism

$$\sigma : \mathcal{T}_B \rightarrow \mathcal{T}_A$$

$$G_\sigma := \{g \in \text{Mod}_B \mid \text{there exists } g' \in \text{Mod}_A \text{ so that} \\ \text{for all } \tau \in \mathcal{T}_B, \sigma(g \cdot \tau) = g' \cdot \sigma(\tau)\}$$

# The induced homomorphism

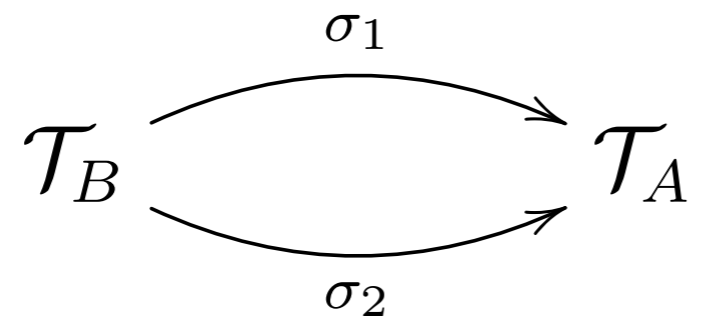
$$\sigma : \mathcal{T}_B \rightarrow \mathcal{T}_A$$

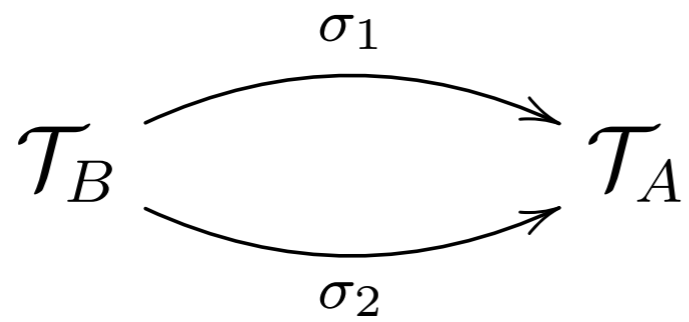
$$\mathsf{G}_\sigma := \{g \in \text{Mod}_B \mid \text{there exists } g' \in \text{Mod}_A \text{ so that} \\ \text{for all } \tau \in \mathcal{T}_B, \sigma(g \cdot \tau) = g' \cdot \sigma(\tau)\}$$

$$\Phi_\sigma : \mathsf{G}_\sigma \rightarrow \text{Mod}_A \quad \text{given by} \quad \Phi_\sigma : g \mapsto g'$$

$$\begin{array}{ccc} \mathcal{T}_B & \xrightarrow{\sigma} & \mathcal{T}_A \\ \downarrow & & \downarrow \\ \mathcal{T}_B / \mathsf{G}_\sigma & & \\ \downarrow & \searrow & \downarrow \\ \mathcal{M}_B & & \mathcal{M}_A \end{array}$$



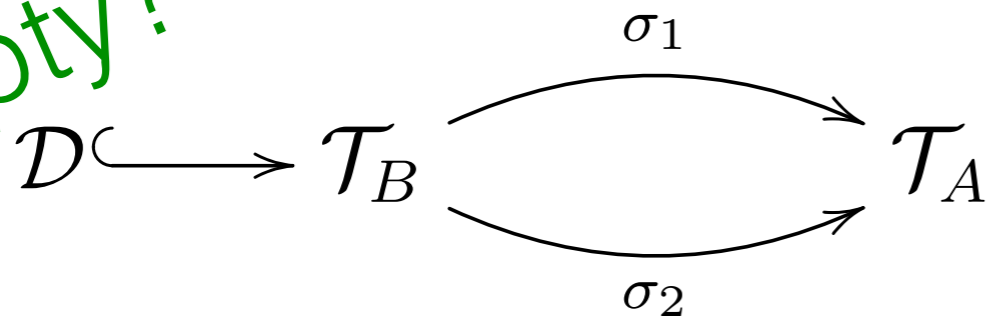




$$\Phi_1 : \mathbb{G}_1 \rightarrow \text{Mod}_A$$

$$\Phi_2 : \mathbb{G}_2 \rightarrow \text{Mod}_A$$

empty?



$$\Phi_1 : G_1 \rightarrow \text{Mod}_A$$

$$\Phi_2 : G_2 \rightarrow \text{Mod}_A$$

$$\mathcal{D} := \{\tau \in \mathcal{T}_B \mid \sigma_1(\tau) = \sigma_2(\tau)\}$$

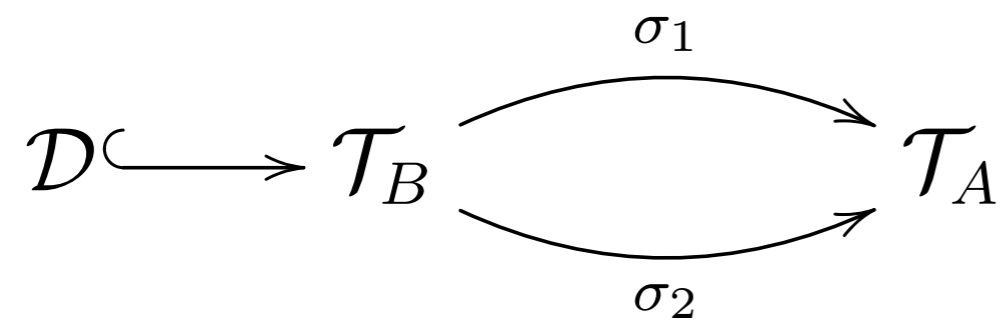
$$S := \{g \in G_1 \cap G_2 \mid \Phi_1(g) = \Phi_2(g)\}$$

Let  $g \in G_1 \cap G_2$ . TFAE:

(1)  $g \in S$

(2)  $g \cdot \mathcal{D} \cap \mathcal{D} \neq \emptyset$

(3)  $g \cdot \mathcal{D} = \mathcal{D}$

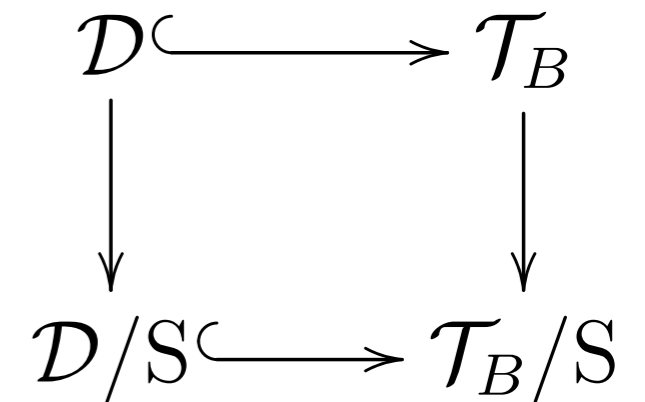


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$$\begin{array}{ccc}
\mathcal{D} & \hookrightarrow & \mathcal{T}_B \\
\downarrow & & \downarrow \\
\mathcal{D}/S & \hookrightarrow & \mathcal{T}_B/S
\end{array}$$

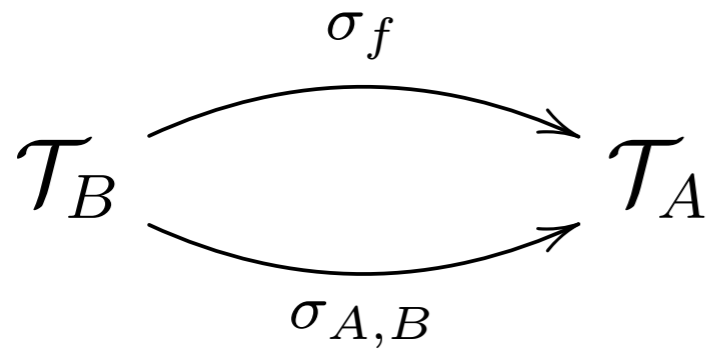
Let  $\ast \in \mathcal{D}$

$\mathcal{D}_0 :=$  the component of  $\mathcal{D}$  containing  $\ast$

$$E := \{g \in S \mid g \cdot \mathcal{D}_0 = \mathcal{D}_0\}$$

$\mathcal{D}$  is connected if and only if  $E = S$

**Proposition.** If  $\mathcal{D}/S$  is connected, there is a bijection between the connected components of  $\mathcal{D}$  and the cosets of  $E$  in  $S$ .



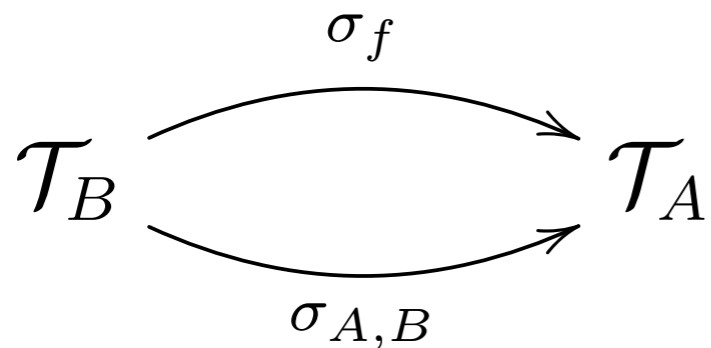
$$\Phi_f : \mathbb{G}_f \rightarrow \text{Mod}_A$$

$\mathbb{G}_f$  contains the *liftable mapping classes*

$$\Phi_{A,B} : \mathbb{G}_{A,B} \rightarrow \text{Mod}_A$$

$$\mathcal{D} = \text{Def}_A^B(f)$$

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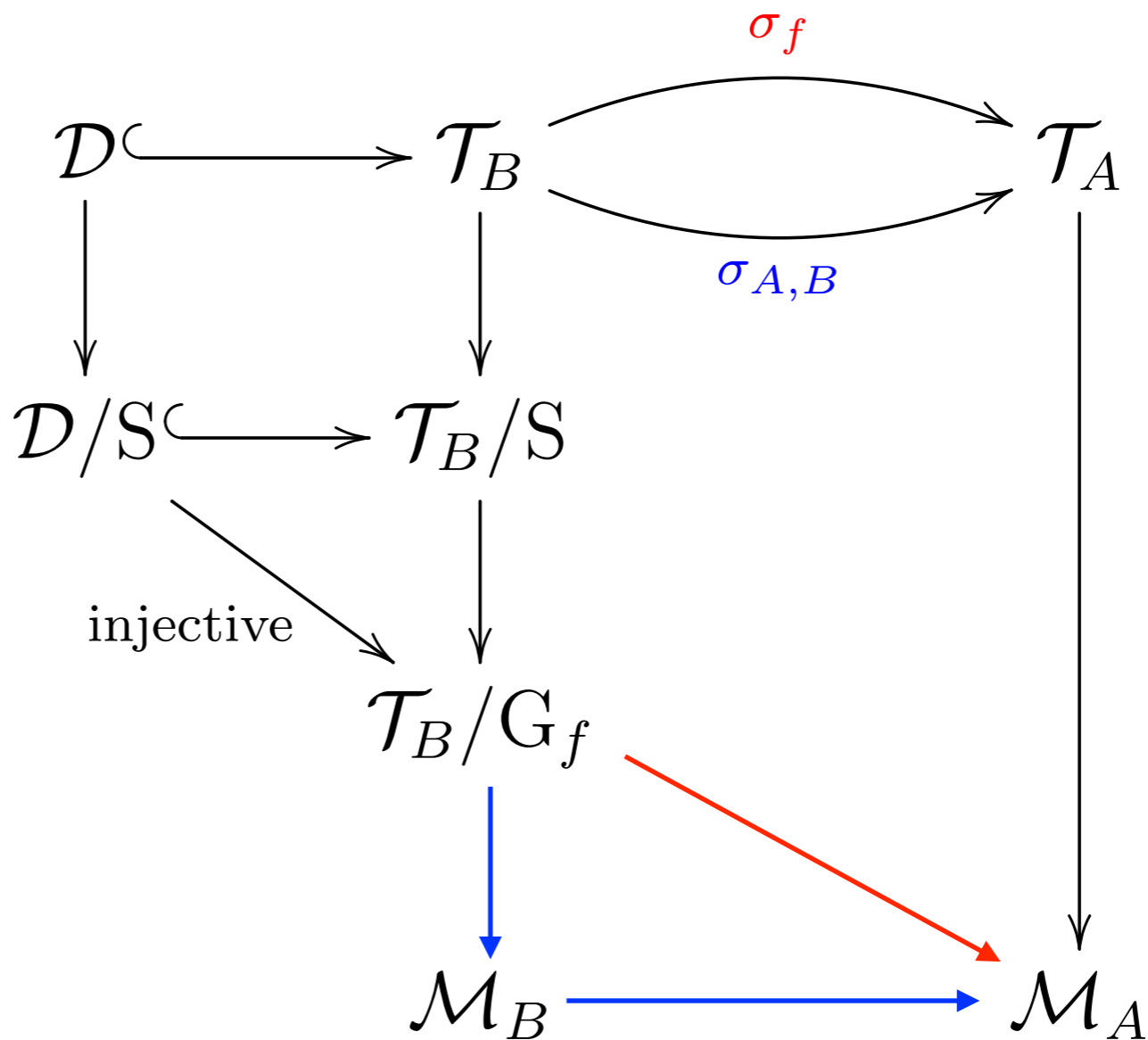
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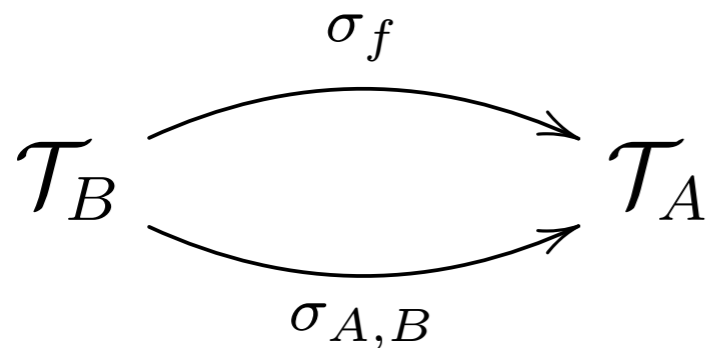
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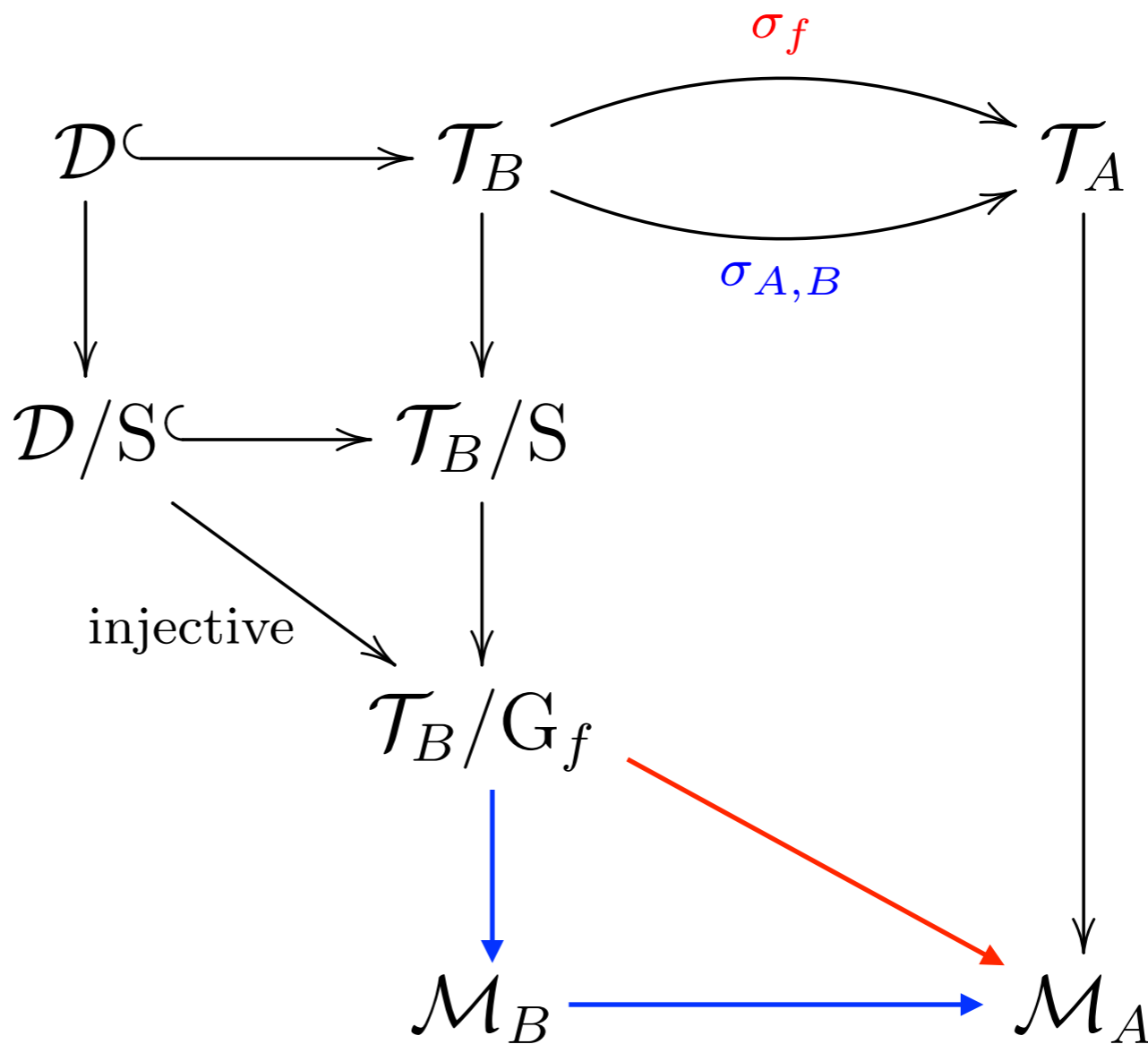
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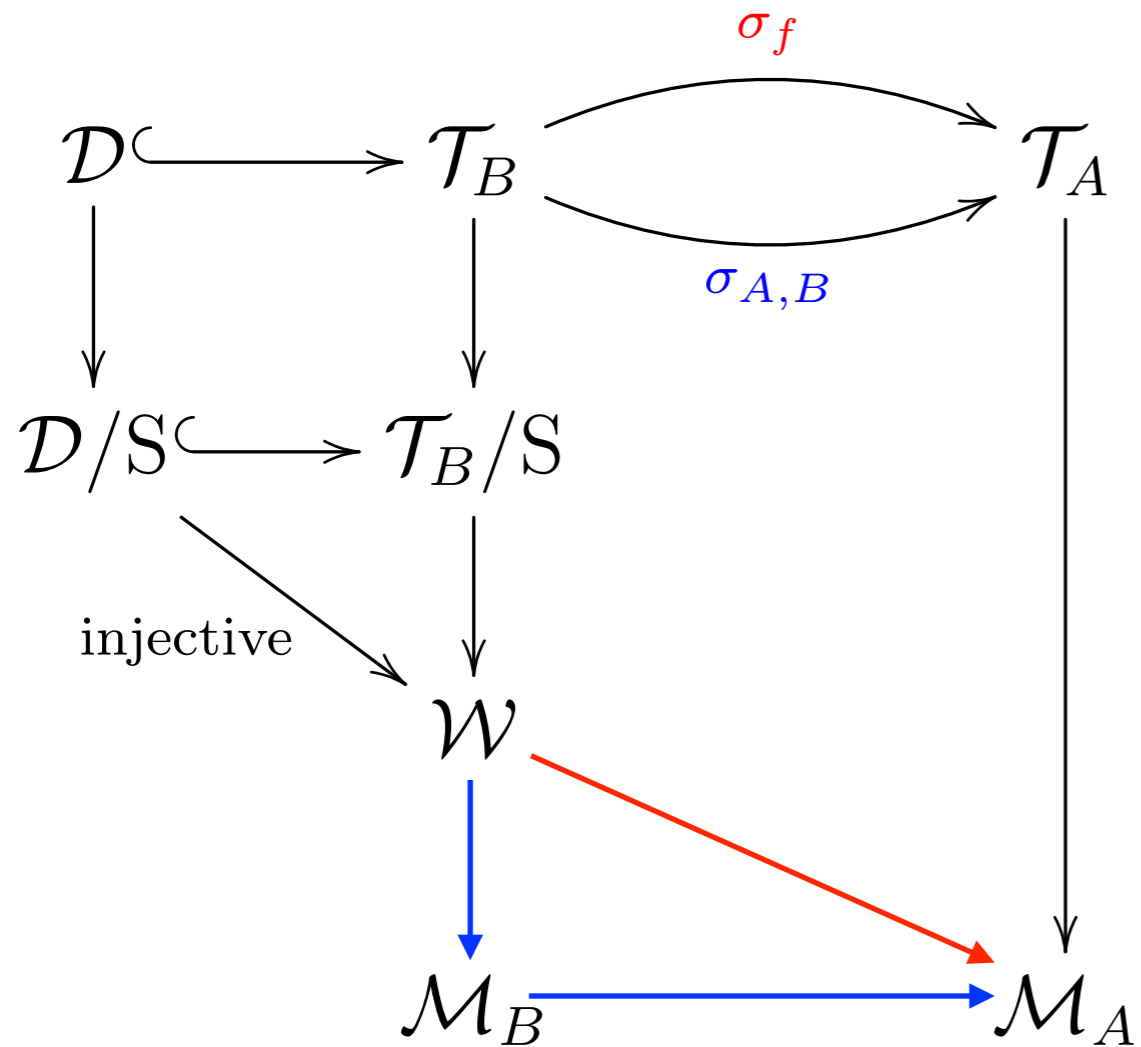
$$\mathcal{D} = \text{Def}_A^B(f)$$

$\mathcal{D}/S$  maps to  
the equalizer  
of the two maps

$$\mathcal{T}_B/\mathbb{G}_f \rightarrow \mathcal{M}_A$$



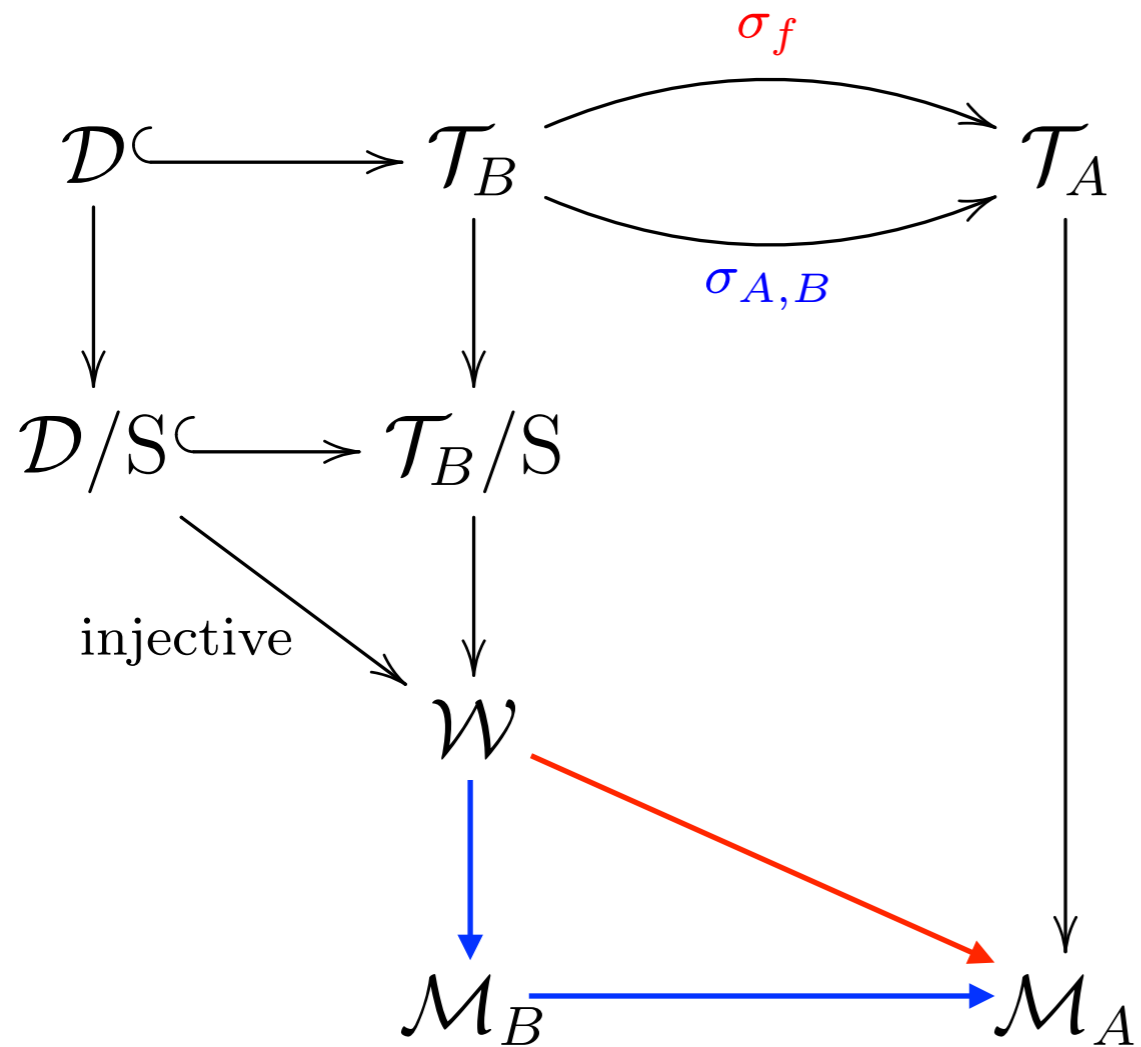




For our  $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ ,

$$\circledast = [\text{id}] \in \mathcal{D}$$

$\mathcal{D}/S$  is connected



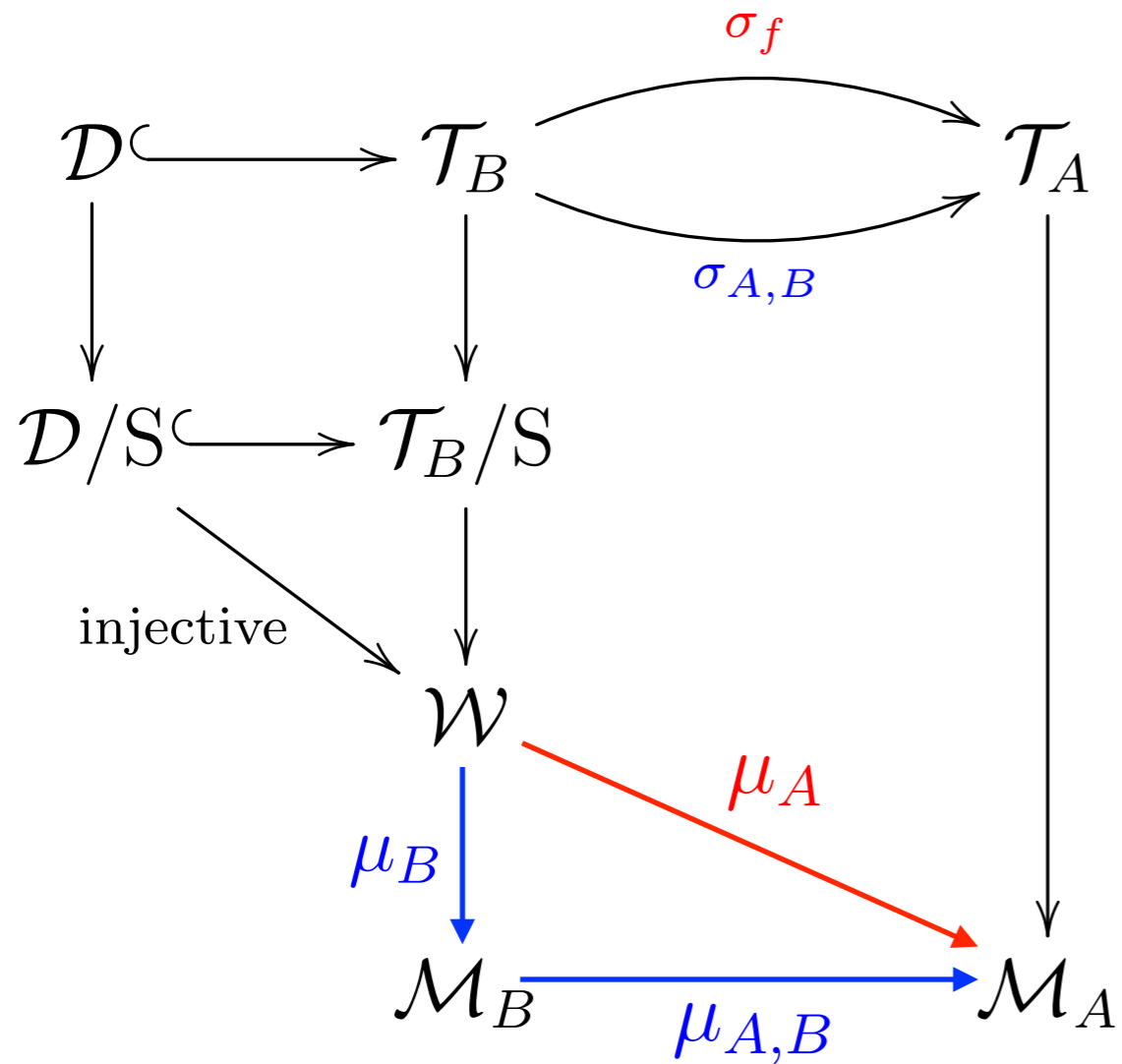
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$$\mathcal{W} := \mathcal{T}_B/\mathcal{G}_f$$

$\mathcal{V} := \text{image of } \mathcal{D}/\mathcal{S} \text{ in } \mathcal{W}$



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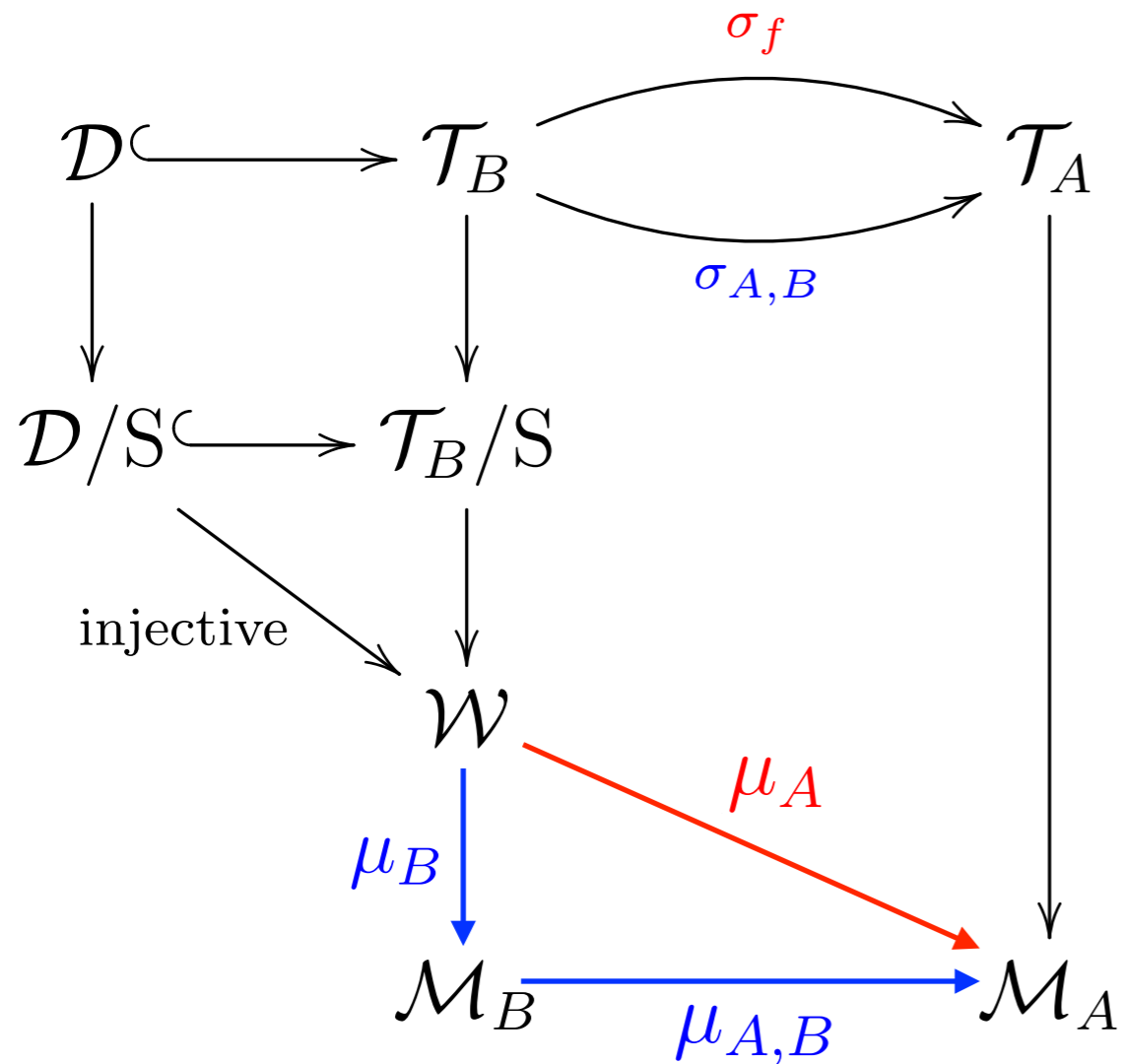
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For our  $f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$ ,

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$$\mathcal{S} = \text{Equalizer}((\mu_A)_*, (\mu_{A,B} \circ \mu_B)_*)$$

$$\text{Mod}_B = \pi_1(\mathcal{M}_B, *_B)$$

$$\text{Mod}_A = \pi_1(\mathcal{M}_A, *_A)$$

$$\mathcal{G}_f = \pi_1(\mathcal{W}, *_\mathcal{V})$$

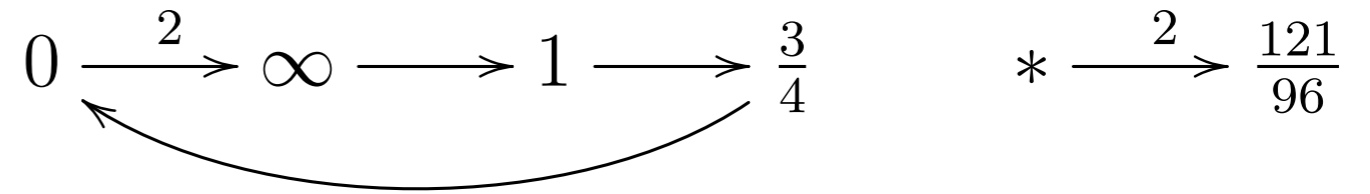
$$\mathcal{E} = \text{image of } \pi_1(\mathcal{V}, *_\mathcal{V}) \text{ in } \pi_1(\mathcal{W}, *_\mathcal{V})$$

**slogan?**

# Our example

$$f : (\mathbb{P}^1, A) \rightarrow (\mathbb{P}^1, B)$$

$$f : z \mapsto \frac{(4z - 3)(z + 2)}{4z^2}$$

$$0 \xrightarrow{2} \infty \longrightarrow 1 \longrightarrow \frac{3}{4} \quad * \xrightarrow{2} \frac{121}{96}$$


$$A = \{0, 1, \infty, 3/4\} \quad B = A \cup \{121/96\}$$

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$$(x, y, z, F) \leftrightarrow (x, y)$$

$$\begin{array}{ccccc}
 0 & \infty & 1 & x & * \\
 \downarrow 2 & \downarrow & \downarrow & \downarrow & \downarrow 2 \\
 \infty & 1 & y & 0 & z
 \end{array}$$

$$F(t) = \frac{(x - t)(-tx + y + t + x - 1)}{(x - 1)t^2}$$

$$z = \frac{(-x^2 + y + 2x - 1)^2}{4x(y - 1 + x)(1 - x)}$$

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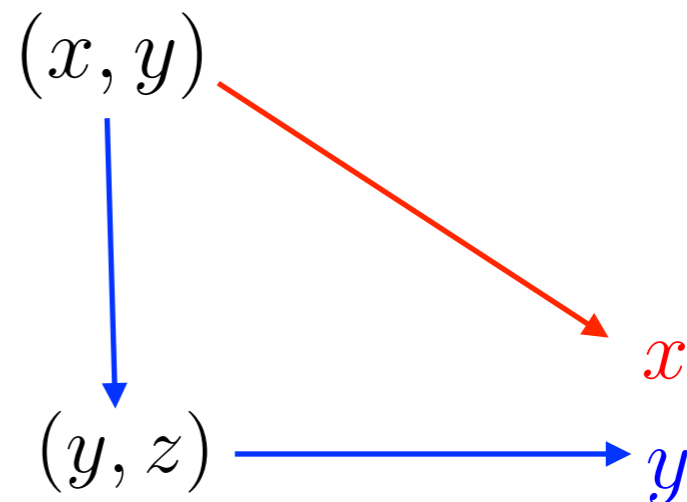
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