A disconnected deformation space of rational maps

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Preliminaries

S^2

$$A, B \subseteq S^2 \text{ finite}$$

$$3 \leqslant |A|, |B|$$

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S^2

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$3 \leqslant |A|, |B|$

$$\begin{tikzcd} (S^2, A) \arrow[d, "f"] \\ (S^2, B) \end{tikzcd}$$

Teichmüller Space

$\mathcal{T}_A :=$ classes of $\phi : S^2 \to \mathbb{P}^1$ where $\phi_1 \sim \phi_2$ if
there is $\mu \in \mathrm{Aut}(\mathbb{P}^1)$ so that $\phi_1 = \mu \circ \phi_2$ on A, and
ϕ_1 is isotopic to $\mu \circ \phi_2$ rel A.

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$$\mathcal{T}_B \xrightarrow{\sigma_f} \mathcal{T}_A$$

$f : (S^2, A) \to (S^2, B)$

B contains critical values of f

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$$\begin{tikzcd} \mathcal{T}_B \arrow[r, bend left, "\sigma_f"] \arrow[r, bend right, "\sigma_{A,B}"'] & \mathcal{T}_A \end{tikzcd}$$

$$\mathrm{Def}_A^B(f) := \{\tau \in \mathcal{T}_B \mid \sigma_f(\tau) = \sigma_{A,B}(\tau)\}$$

The forgetful map

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$$\begin{tikzcd}
(S^2, A) \arrow[d, "f"] & \\
(S^2, B) \arrow[r, "\phi"] & (\mathbb{P}^1, \phi(B))
\end{tikzcd}$$

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$$\sigma_f : [\phi] \mapsto [\psi]$$

$$\begin{tikzcd} \mathrm{Def}_A^B(f) \arrow[r, hook] & \mathcal{T}_B \arrow[r, bend left, "\sigma_f"] \arrow[r, bend right, "\sigma_{A,B}"'] & \mathcal{T}_A \end{tikzcd}$$

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$\phi|_A = \psi|_A$ and

ϕ is isotopic to ψ rel A.

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$$\mathrm{Def}_A^B(f) \neq \emptyset$$

$$\Updownarrow$$

$f : (S^2, A) \to (S^2, B)$ is combinatorially equivalent to a rational map

The dynamical case

If $A = B$, Thurston proved:

Theorem. The space $\mathrm{Def}_B^B(f) \neq \emptyset$ if and only if the map $f : (S^2, B) \to (S^2, B)$ admits no obstructing multi-curves.

Theorem. The space $\mathrm{Def}_B^B(f)$ is connected.

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Local structure

The following result is due to Epstein.

Theorem. Let $f : (\mathbb{P}^1, A) \to (\mathbb{P}^1, B)$ be a rational map which is not of Lattès type. Then $\mathrm{Def}_A^B(f)$ is a complex analytic submanifold of $\mathcal{T}_B$ of dimension $|B - A|$.

A disconnected deformation space

$$\mathrm{Per}_4(0) \subseteq \mathrm{M}_2$$

$$\begin{tikzcd} 0 \arrow[r, "2"] & \infty \arrow[r] & 1 \arrow[r] & a \arrow[lll, bend left] \end{tikzcd} \qquad * \xrightarrow{2} b$$

$$A = \{0, 1, \infty, a\} \quad B = A \cup \{b\} \quad b \notin A$$

$$\dim(\mathrm{Def}_A^B(f)) = 1 \quad \dim(\mathcal{T}_B) = 2 \quad \dim(\mathcal{T}_A) = 1$$

Theorem. (H–K)
For $\langle f \rangle \in \mathrm{Per}_4(0)^*$, $\mathrm{Def}_A^B(f)$ has infinitely many connected components.

Tanya Firsova, Jeremy Kahn, and Nikita Selinger
proved a related result

Work of Mary Rees

Moduli space

$$\mathcal{M}_A := \{\text{injective } \varphi : A \hookrightarrow \mathbb{P}^1 \text{ up to postcomposition with Möbius transformations}\}$$

$\mathcal{M}_A$ is a complex manifold, isomorphic to $\mathbb{C}^{|A|-3} - \{\text{finitely many hyperplanes}\}$

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$$\begin{tikzcd} \mathcal{T}_A \arrow[d] & {[\phi]} \arrow[d, mapsto] \\ \mathcal{M}_A & {[\phi|_A]} \end{tikzcd}$$

Modular group Mod_A isomorphic to pure mapping class group of (S^2, A)

The induced homomorphism

$$\sigma : \mathcal{T}_B \to \mathcal{T}_A$$

$$\mathrm{G}_\sigma := \{g \in \mathrm{Mod}_B \mid \text{there exists } g' \in \mathrm{Mod}_A \text{ so that}$$
$$\text{for all } \tau \in \mathcal{T}_B,\ \sigma(g \cdot \tau) = g' \cdot \sigma(\tau)\}$$

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$$\Phi_\sigma : \mathrm{G}_\sigma \to \mathrm{Mod}_A \quad \text{given by} \quad \Phi_\sigma : g \mapsto g'$$

$$\begin{tikzcd}
\mathcal{T}_B \arrow[r, "\sigma"] \arrow[d] & \mathcal{T}_A \arrow[dd] \\
\mathcal{T}_B/\mathrm{G}_\sigma \arrow[d] \arrow[dr] & \\
\mathcal{M}_B & \mathcal{M}_A
\end{tikzcd}$$

$$\begin{tikzcd} \mathcal{T}_B \arrow[r, bend left, "\sigma_1"] \arrow[r, bend right, "\sigma_2"'] & \mathcal{T}_A \end{tikzcd}$$

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$$\Phi_1 : \mathrm{G}_1 \to \mathrm{Mod}_A$$

$$\Phi_2 : \mathrm{G}_2 \to \mathrm{Mod}_A$$

empty?

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] & \mathcal{T}_B \arrow[r, bend left, "\sigma_1"] \arrow[r, bend right, "\sigma_2"'] & \mathcal{T}_A \end{tikzcd}$$

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$$\mathcal{D} := \{\tau \in \mathcal{T}_B \mid \sigma_1(\tau) = \sigma_2(\tau)\}$$

$$\mathrm{S} := \{g \in \mathrm{G}_1 \cap \mathrm{G}_2 \mid \Phi_1(g) = \Phi_2(g)\}$$

Let $g \in \mathrm{G}_1 \cap \mathrm{G}_2$. TFAE:

(1) $g \in \mathrm{S}$

(2) $g \cdot \mathcal{D} \cap \mathcal{D} \neq \emptyset$

(3) $g \cdot \mathcal{D} = \mathcal{D}$

$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] & \mathcal{T}_B \arrow[r, bend left, "\sigma_1"] \arrow[r, bend right, "\sigma_2"'] & \mathcal{T}_A \end{tikzcd}$$

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$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] \arrow[d] & \mathcal{T}_B \arrow[d] \\ \mathcal{D}/\mathrm{S} \arrow[r, hook] & \mathcal{T}_B/\mathrm{S} \end{tikzcd}$$

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$$\begin{tikzcd} \mathcal{D} \arrow[r, hook] \arrow[d] & \mathcal{T}_B \arrow[d] \\ \mathcal{D}/\mathrm{S} \arrow[r, hook] & \mathcal{T}_B/\mathrm{S} \end{tikzcd}$$

Let $\circledast \in \mathcal{D}$

$\mathcal{D}_0 :=$ the component of $\mathcal{D}$ containing $\circledast$

$\mathrm{E} := \{g \in \mathrm{S} \mid g \cdot \mathcal{D}_0 = \mathcal{D}_0\}$

$\mathcal{D}$ is connected if and only if $\mathrm{E} = \mathrm{S}$

Proposition. If $\mathcal{D}/\mathrm{S}$ is connected, there is a bijection between the connected components of $\mathcal{D}$ and the cosets of E in S.

$$
\begin{tikzcd}
\mathcal{T}_B \arrow[r, bend left, "\sigma_f"] \arrow[r, bend right, "\sigma_{A,B}"'] & \mathcal{T}_A
\end{tikzcd}
$$

$$\Phi_f : \mathrm{G}_f \to \mathrm{Mod}_A$$

G_f contains the *liftable mapping classes*

$$\Phi_{A,B} : \mathrm{G}_{A,B} \to \mathrm{Mod}_A$$

$$\mathrm{G}_{A,B} = \mathrm{Mod}_B$$

$$\mathcal{D} = \mathrm{Def}_A^B(f)$$

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\mathcal{T}_B \arrow[r, bend left, "\sigma_f"] \arrow[r, bend right, "\sigma_{A,B}"'] & \mathcal{T}_A
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$$
\begin{tikzcd}
\mathcal{D} \arrow[r, hook] \arrow[d] & \mathcal{T}_B \arrow[r, bend left, red, "\sigma_f"] \arrow[r, bend right, blue, "\sigma_{A,B}"'] \arrow[d] & \mathcal{T}_A \arrow[ddd] \\
\mathcal{D}/\mathrm{S} \arrow[r, hook] \arrow[dr, "\text{injective}"'] & \mathcal{T}_B/\mathrm{S} \arrow[d] & \\
 & \mathcal{T}_B/\mathrm{G}_f \arrow[d, blue] \arrow[ddr, red] & \\
 & \mathcal{M}_B \arrow[r, blue] & \mathcal{M}_A
\end{tikzcd}
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$\mathcal{D}/\mathrm{S}$ maps to

the equalizer

of the two maps

$\mathcal{T}_B/\mathrm{G}_f \to \mathcal{M}_A$

$$\begin{tikzcd}
\mathcal{D} \arrow[r, hook] \arrow[d] & \mathcal{T}_B \arrow[r, bend left, "\sigma_f"] \arrow[r, bend right, "\sigma_{A,B}"'] \arrow[d] & \mathcal{T}_A \arrow[ddd] \\
\mathcal{D}/\mathrm{S} \arrow[r, hook] \arrow[dr, "\text{injective}"'] & \mathcal{T}_B/\mathrm{S} \arrow[d] & \\
 & \mathcal{T}_B/\mathrm{G}_f \arrow[d] \arrow[ddr] & \\
 & \mathcal{M}_B \arrow[r] & \mathcal{M}_A
\end{tikzcd}$$

$$
\begin{tikzcd}
\mathcal{D} \arrow[r, hook] \arrow[d] & \mathcal{T}_B \arrow[r, bend left, "\sigma_f"] \arrow[r, bend right, "\sigma_{A,B}"'] \arrow[d] & \mathcal{T}_A \arrow[ddd] \\
\mathcal{D}/\mathrm{S} \arrow[r, hook] \arrow[dr, "\text{injective}"'] & \mathcal{T}_B/\mathrm{S} \arrow[d] & \\
& \mathcal{W} \arrow[d] \arrow[dr] & \\
& \mathcal{M}_B \arrow[r] & \mathcal{M}_A
\end{tikzcd}
$$

For our $f : (\mathbb{P}^1, A) \to (\mathbb{P}^1, B)$,

$$\circledast = [\mathrm{id}] \in \mathcal{D}$$

$\mathcal{D}/\mathrm{S}$ is connected

$$
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$\mathcal{V} :=$ image of $\mathcal{D}/\mathrm{S}$ in $\mathcal{W}$

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\mathcal{D}/\mathrm{S} \arrow[r, hook] \arrow[dr, "\text{injective}"'] & \mathcal{T}_B/\mathrm{S} \arrow[d] & \\
& \mathcal{W} \arrow[d, "\mu_B"'] \arrow[ddr, phantom, ""] \arrow[dr, "\mu_A"] & \\
& \mathcal{M}_B \arrow[r, "\mu_{A,B}"'] & \mathcal{M}_A
\end{tikzcd}
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$$
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$$\mathrm{S} = \mathrm{Equalizer}((\mu_A)_*, (\mu_{A,B} \circ \mu_B)_*)$$

$$
\begin{aligned}
\mathrm{Mod}_B &= \pi_1(\mathcal{M}_B, \circledast_B) \\
\mathrm{Mod}_A &= \pi_1(\mathcal{M}_A, \circledast_A) \\
\mathrm{G}_f &= \pi_1(\mathcal{W}, \circledast_{\mathcal{V}}) \\
\mathrm{E} &= \text{image of } \pi_1(\mathcal{V}, \circledast_{\mathcal{V}}) \text{ in } \pi_1(\mathcal{W}, \circledast_{\mathcal{V}})
\end{aligned}
$$

slogan?

Our example

$$f : (\mathbb{P}^1, A) \to (\mathbb{P}^1, B)$$

$$f : z \mapsto \frac{(4z-3)(z+2)}{4z^2}$$

$$\begin{tikzcd} 0 \arrow[r, "2"] & \infty \arrow[r] & 1 \arrow[r] & \frac{3}{4} \arrow[lll, bend left] \end{tikzcd} \qquad * \xrightarrow{2} \frac{121}{96}$$

$$A = \{0, 1, \infty, 3/4\} \quad B = A \cup \{121/96\}$$

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$$\mathcal{W}$$

$$(x, y, z, F) \leftrightarrow (x, y)$$

$$\begin{tikzcd}
0 \arrow[d, "2"] & \infty \arrow[d] & 1 \arrow[d] & x \arrow[d] & * \arrow[d, "2"] \\
\infty & 1 & y & 0 & z
\end{tikzcd}$$

$$F(t) = \frac{(x-t)(-tx+y+t+x-1)}{(x-1)t^2}$$

$$z = \frac{(-x^2+y+2x-1)^2}{4x(y-1+x)(1-x)}$$

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$$(x, y, z, F) \leftrightarrow (x, y)$$

$$\begin{tikzcd} (x,y) \arrow[d] \arrow[rr] & & x \\ (y,z) \arrow[rr] & & y \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[d, "2"] & \infty \arrow[d] & 1 \arrow[d] & x \arrow[d] & * \arrow[d, "2"] \\ \infty & 1 & y & 0 & z \end{tikzcd}$$

$$F(t) = \frac{(x-t)(-tx+y+t+x-1)}{(x-1)t^2}$$

$$z = \frac{(-x^2+y+2x-1)^2}{4x(y-1+x)(1-x)}$$