Cubic polynomials with one periodic critical point: irreducibility

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Celebrating John Milnor's 85th. birthday
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Spaces

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(f, c_0, c_1) and (g, ω_0, ω_1) are in the same conjugacy class if there exists $A : \mathbb{C} \to \mathbb{C}$ affine such that:

- $A \circ f = g \circ A$,
- for $i = 0, 1$ we have $\omega_i = A(c_i)$.

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$$P_{a,v}(z) = -P_{-a.-v}(-z).$$

Periodic critical point

The curve $\mathcal{S}_n$ of period n is formed by all conjugacy classes $[f, c_0, c_1] \in \mathrm{Poly}_3^{cm}$ such that:

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Compare with Epstein.

Question

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Theorem (Arfeux and K.)

$\mathcal{S}_n$ is connected.

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$\mathcal{S}_4$ is connected (Bonifant-Milnor) of genus 6 and 14 punctures.

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What is the Euler characteristic of the smooth compactification of $\mathcal{S}_n$?

Requires to compute the number N_p of punctures.
(Algorithms by De Marco-Schiff (2010) based on De Marco-Pilgrim (2010 approx).)

Dichotomy

The connectedness locus

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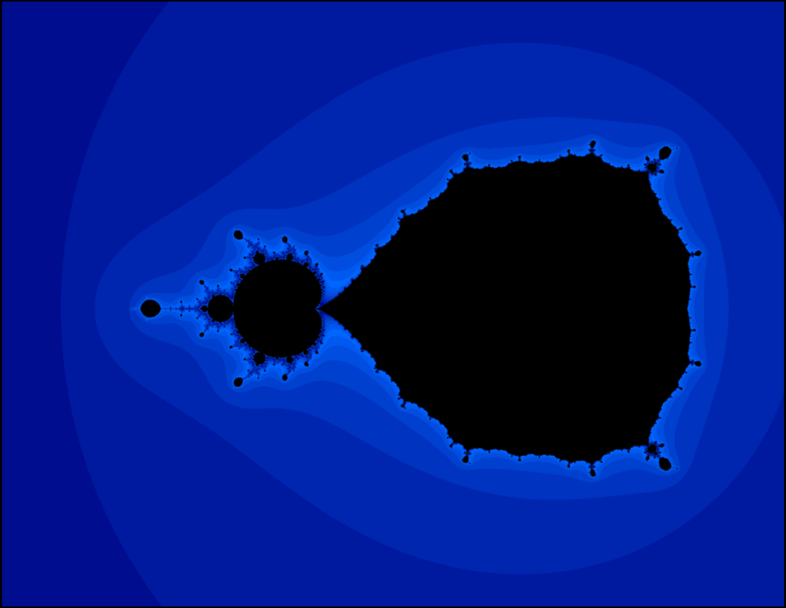
The escape locus

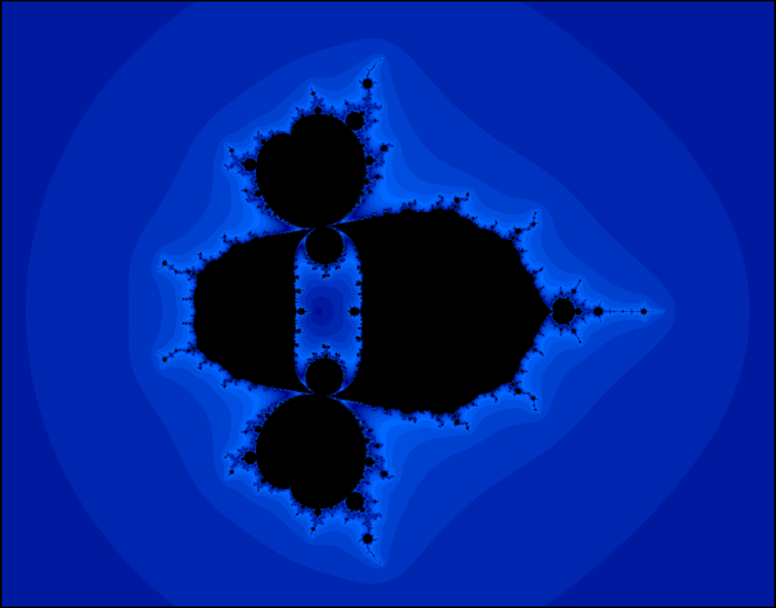
$$\mathcal{E}(\mathcal{S}_n) = \left\{[f, c_0, c_1] \in \mathcal{S}_n \mid f^k(c_1) \to \infty\right\}$$

is open and every connected component is unbounded.

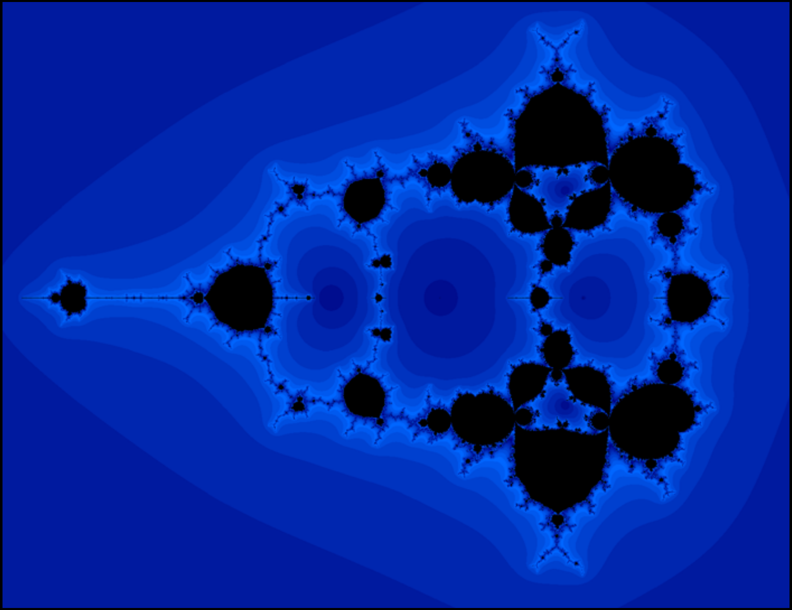
S_1



S_2



S_3



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n	1	2	3	4	5
$\nu_3(n)$	3	6	24	72	240

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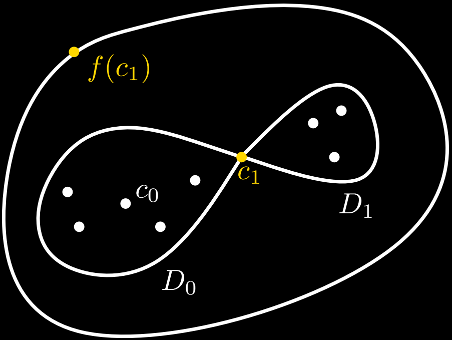
if $\mathcal{U}$ and $\mathcal{U}'$ are escape regions of $\mathcal{S}_n$ then,

there exists a path **contained in** $\mathcal{S}_n$ joining $\mathcal{U}$ and $\mathcal{U}'$.

Dynamics on escape regions: itinerary



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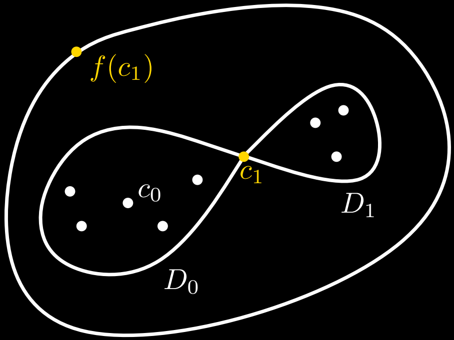

For $f^k(z) \not\to \infty$, define

$$\text{itin}(z) := (i_0, i_1, i_2, \dots) \in \{0, 1\}^{\mathbb{N}}.$$

where, for all $k \geq 0$

$$f^k(z) \in D_{i_k}.$$

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The first n symbols of the itinerary of $f(c_0)$ form the kneading word of $\mathcal{U}$:

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There exists one and only one escape region $\mathcal{U}$ with such that:

$$\kappa(\mathcal{U}) = 1^{n-1}0.$$

Strategy

Join all escape regions $\mathcal{U}$ to $\mathcal{U}_\star$ where $\kappa(\mathcal{U}_\star) = 1^{n-1}0$.

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$$\kappa(\mathcal{U}) \neq 1^{n-1}0,$$

then join $\mathcal{U}$ to $\mathcal{U}'$ such that:

$$\kappa(\mathcal{U}') \text{ has more 1's than } \kappa(\mathcal{U}).$$

Spaces of topological maps

Let B be the space of degree 3 **topological branched coverings**

$$F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$$

with marked branched points ∞, c_0 and c_1 such that:

$F(\infty) = \infty$ and F is locally 3-to-1 around ∞,

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Let $\mathcal{B}$ be the space of affine conjugacy classes of (F, c_0, c_1).

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If $(F_t)_{t\in[0,1]}$ *is a path in* $\mathcal{B}$ *such that* F_0 *and* F_1 *belong to* $\mathcal{S}_n$,

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$$p_t \in \mathcal{S}_n \text{ for all } t \in [0,1].$$

Aim: change symbol 0 *to* 1

Given $f \in \mathcal{U}$ such that

$$\kappa(U) = i_1 \ldots i_{m-1} 0 i_{m+1} \ldots i_{n-1} 0$$

construct a path (F_t) from $F_0 = f$ to $F_1 \in \mathcal{U}'$ with

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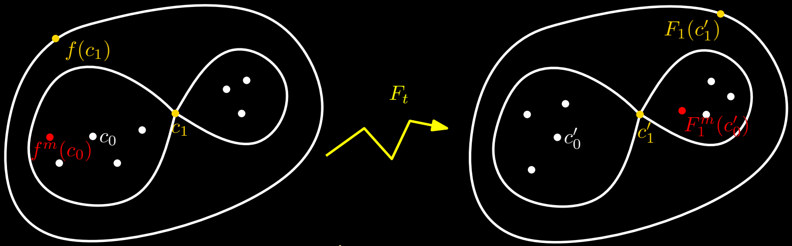

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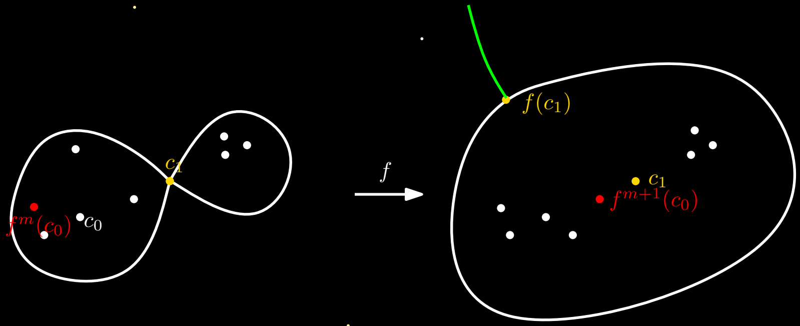
Extended Green line and twisting loop



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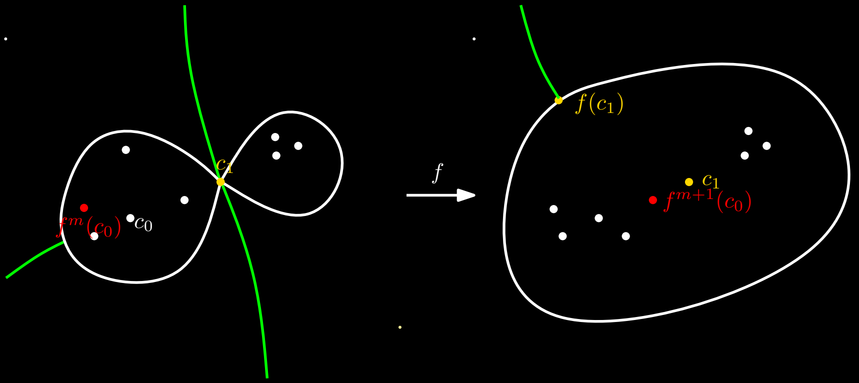

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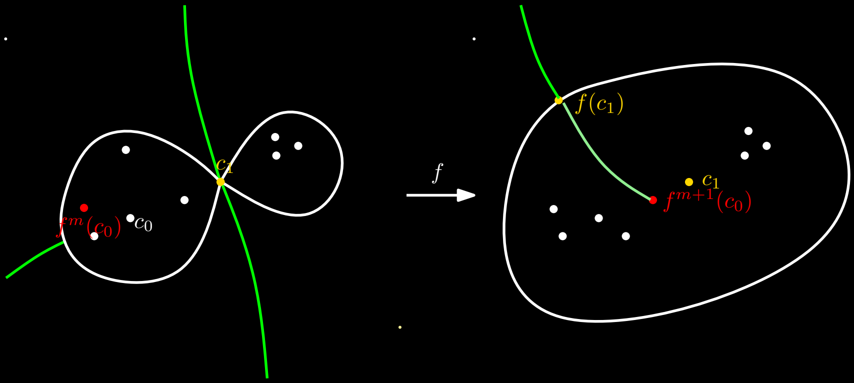

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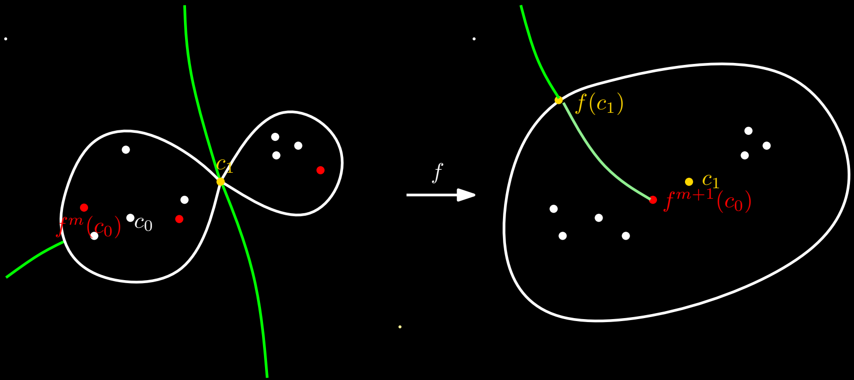

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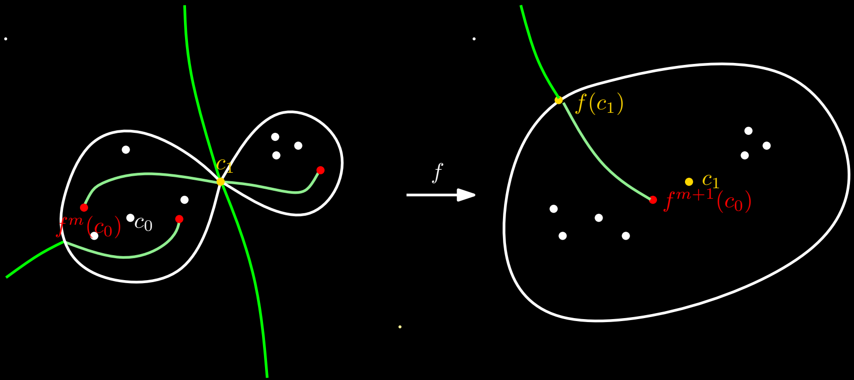

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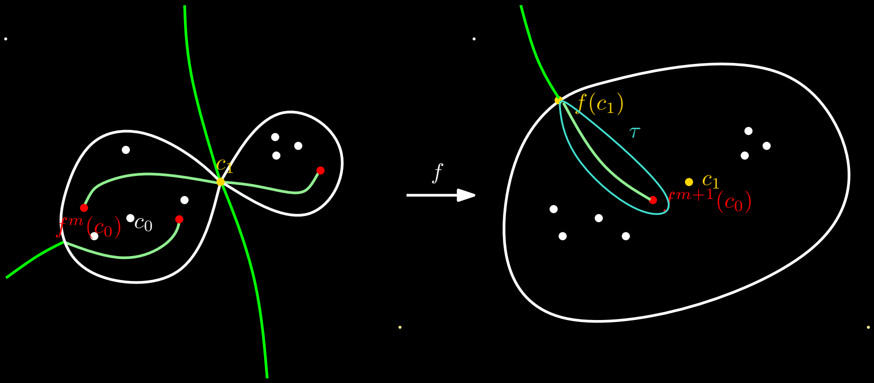

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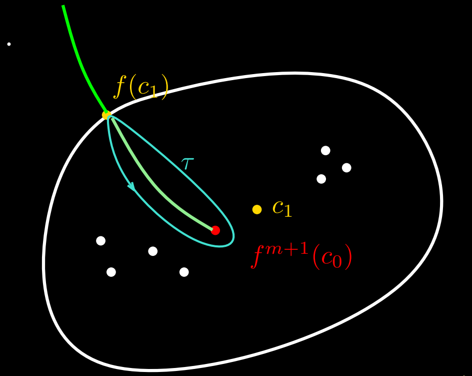

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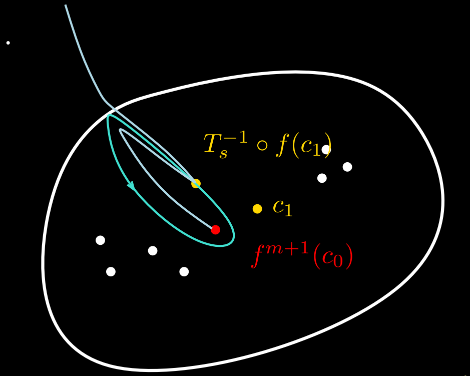

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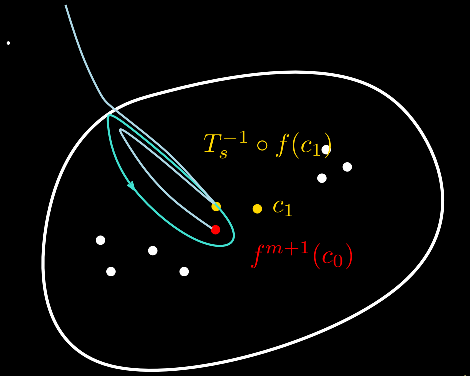

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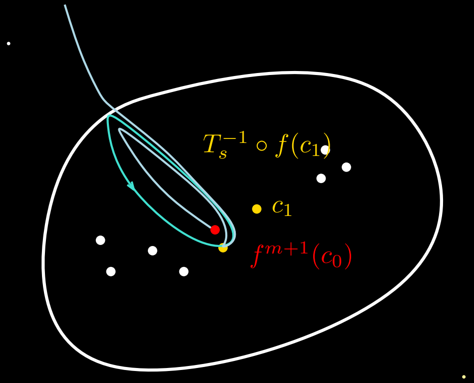

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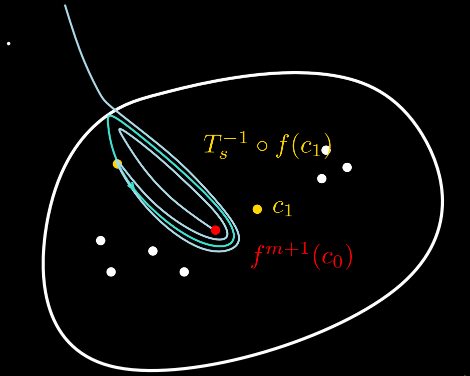

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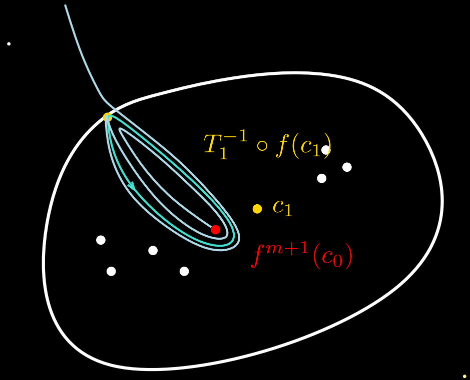

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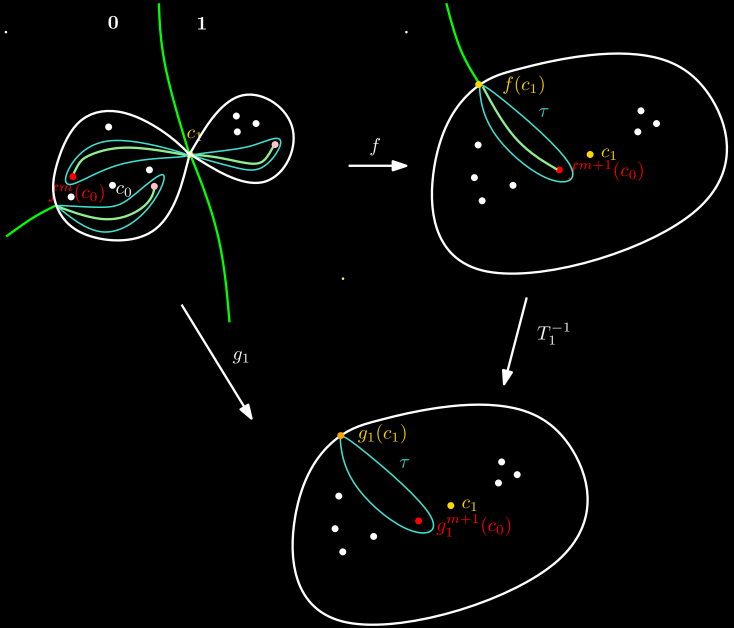

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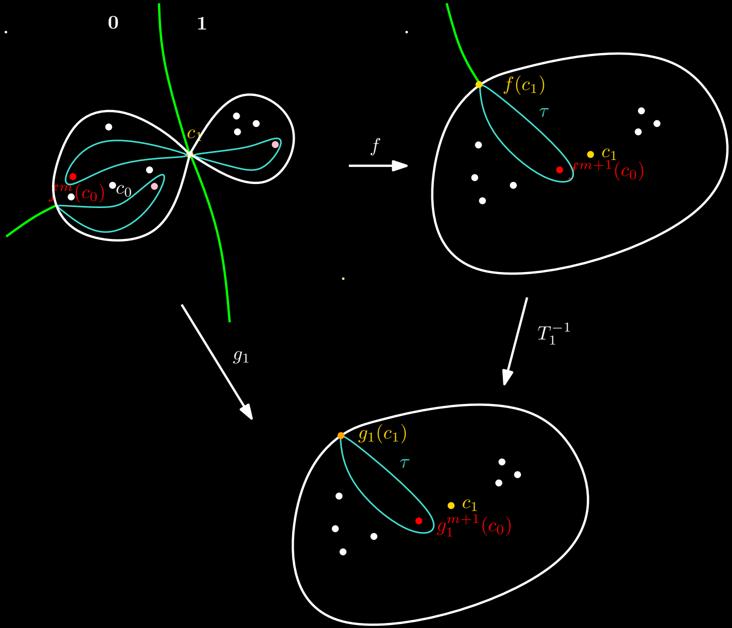

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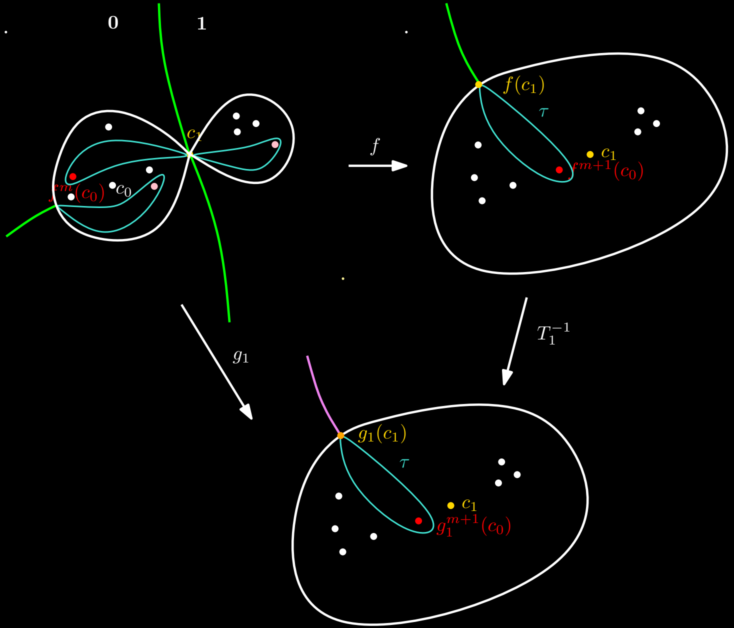

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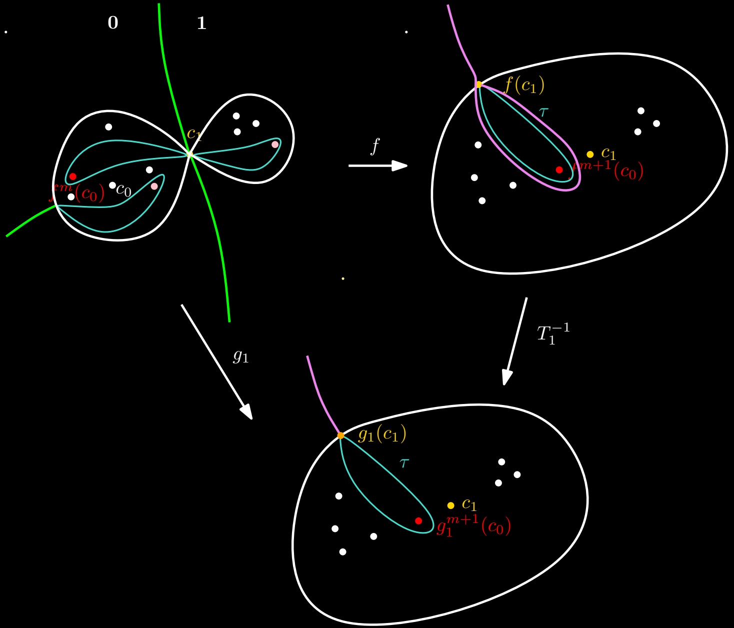

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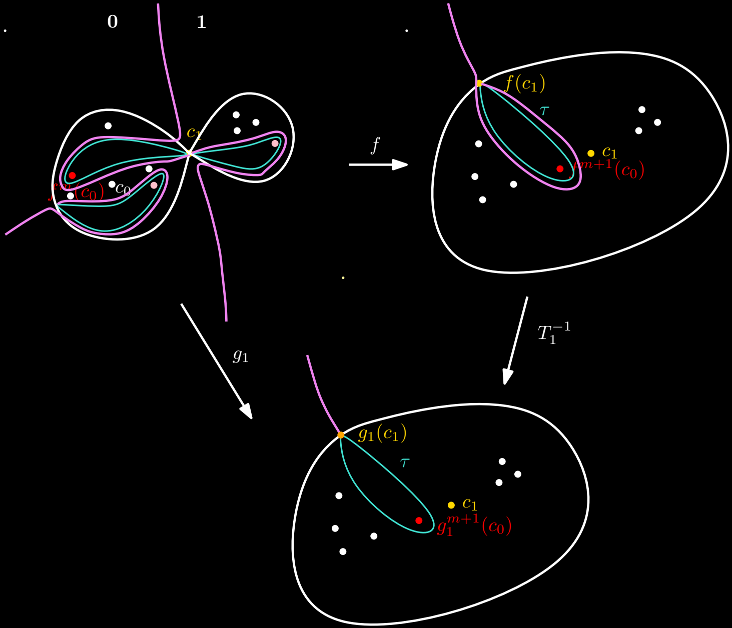
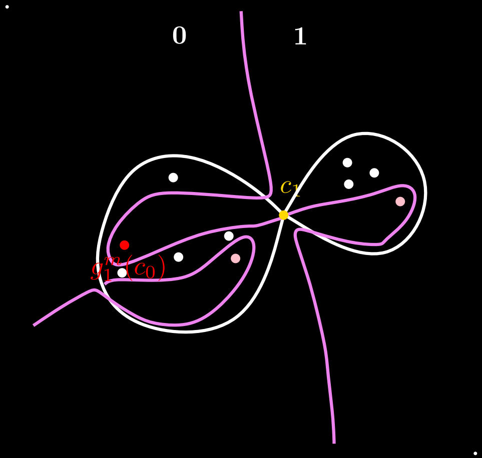


“*Green lines of g_1*”



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Example:

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Cui-Tan equivalence

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Theorem (Cui and Tan)

Let $F : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a semi-rational map.

F is c-equivalent to a rational map R

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In this case, R is unique up to Möbius conjugacy.

No Thurston obstruction

Lemma (à la Levy)

If g_1 is a semi-rational map such that:

$(g_1, c_0, c_1) \in B$,

$g_1^n(c_1) \to \infty$,

then g_1 has no Thurston obstructions.

Cui-Tan equivalence path: from g_1 *to* F_1

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(2) Conjugacy by isotopy of ψ with $\mathrm{id}_{\mathbb{C}}$:

$$\psi^{-1} \circ F_1 \circ \psi \to F_1.$$

Kneading of F_1

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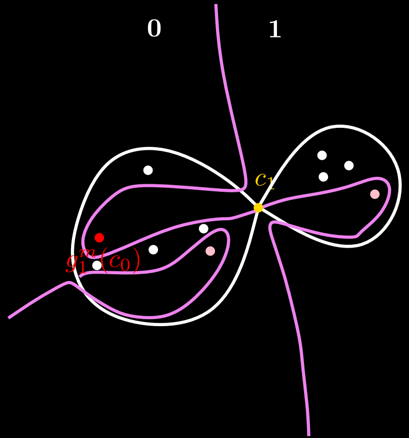
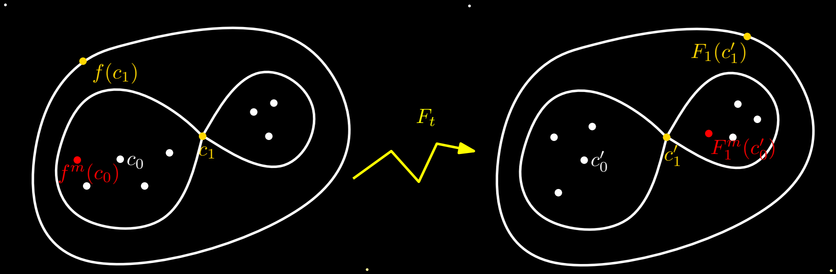


Summary

We found a path F_t such that:



Remarks

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Dynatomic curves.

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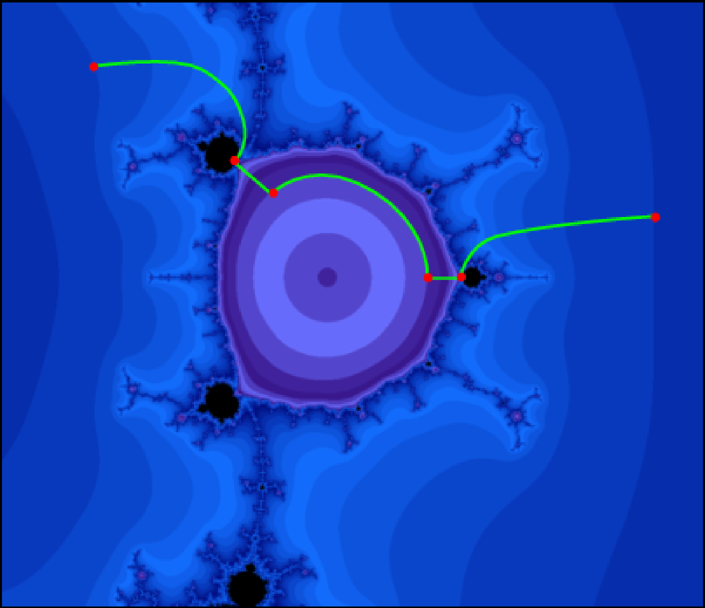
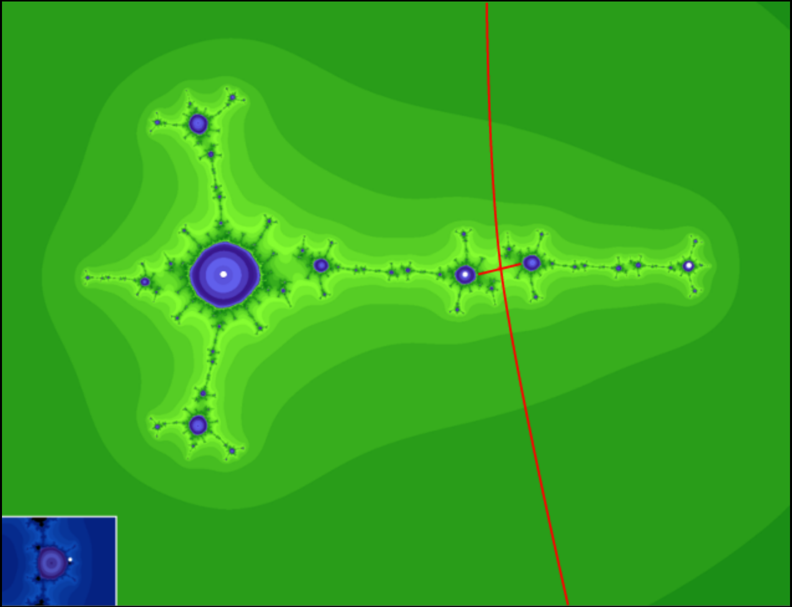
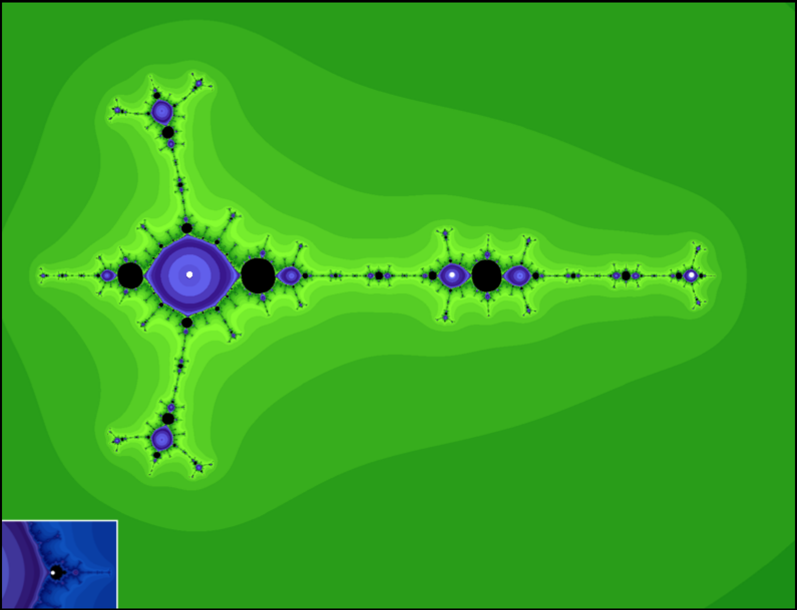
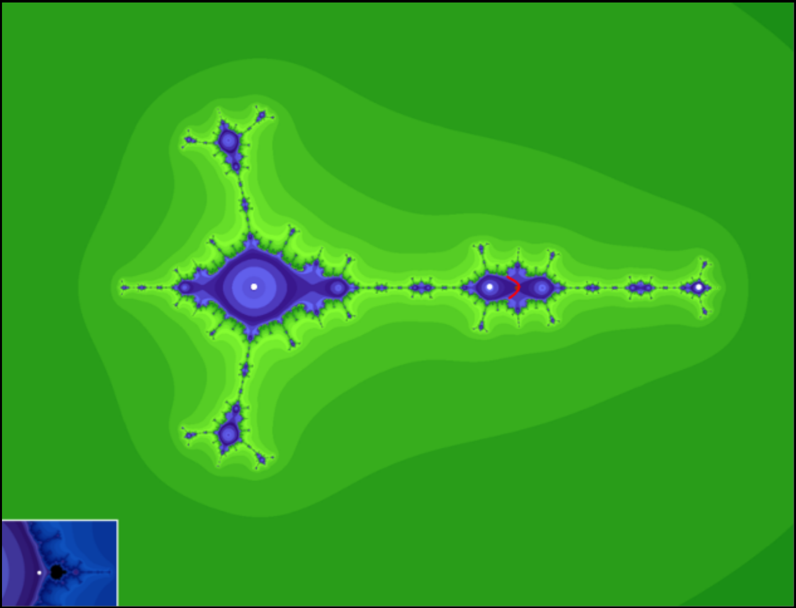
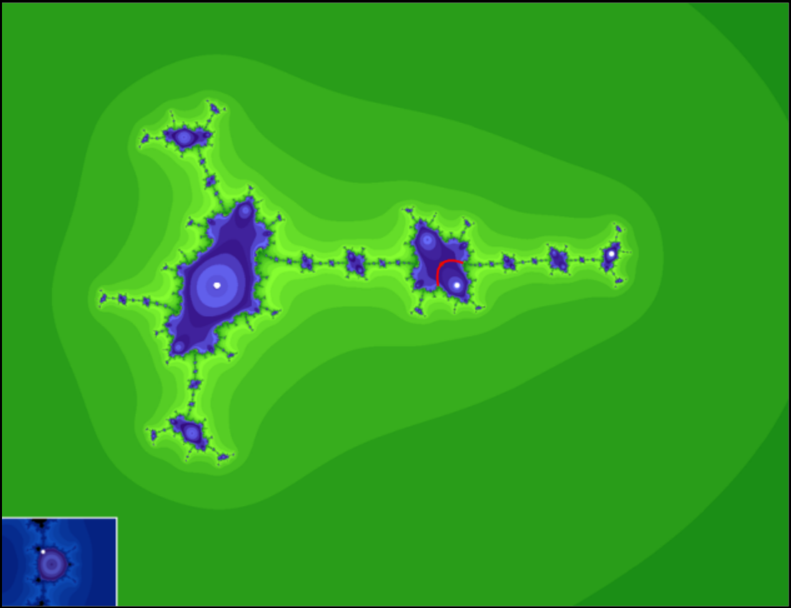
Explicit paths in $\mathcal{S}_n$.

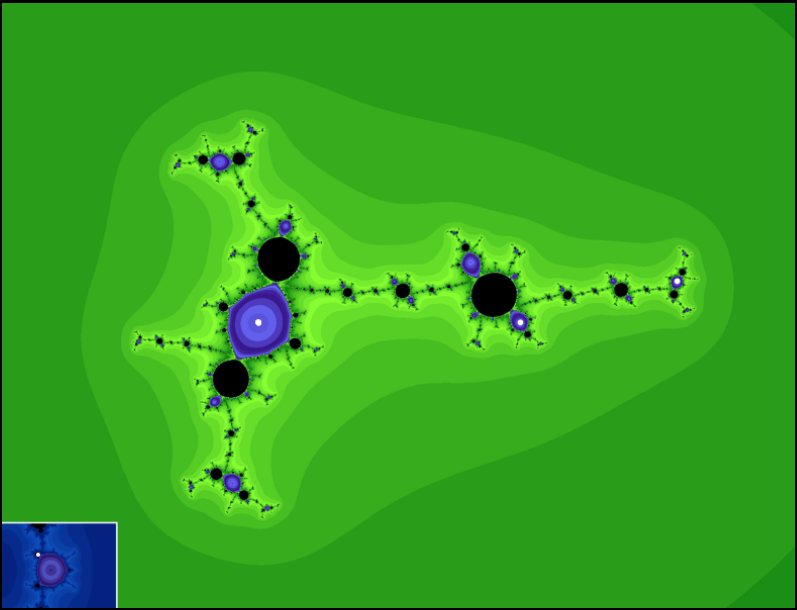


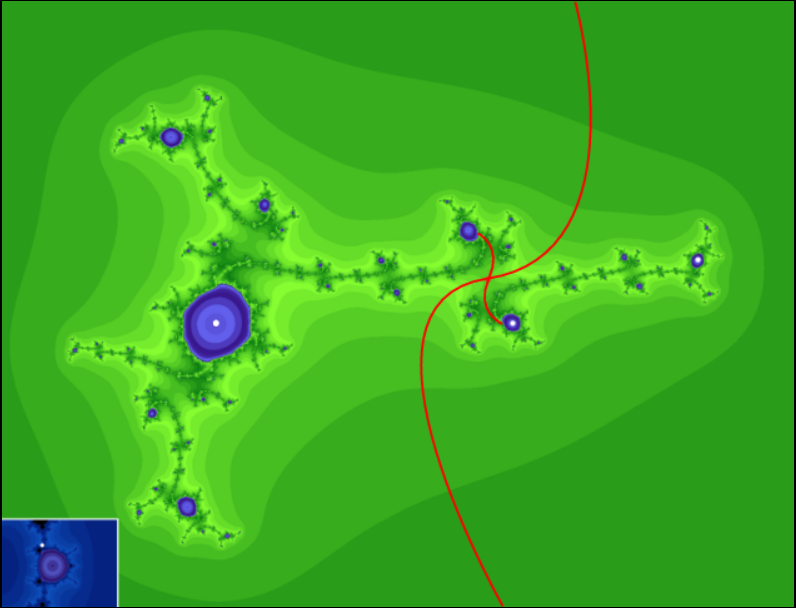












Thank you!

Happy Birthday Jack!