On a Quartic Polynomial Family

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We study the dynamical plane and the parameter space of complex quartic polynomials $P_{ab}(z) = P_a(P_b(z))$, where $P_a(z) = az + z^2$.

In the real case, this family was introduced by Kot-Schaffer (1984). They were motivated by the problem of getting some insight about the growth of a population with two differentiated seasons of reproduction. They supposed that each season the population grew according to a logistic model. In this way, the annual population is given by the composition of two quadratic maps.

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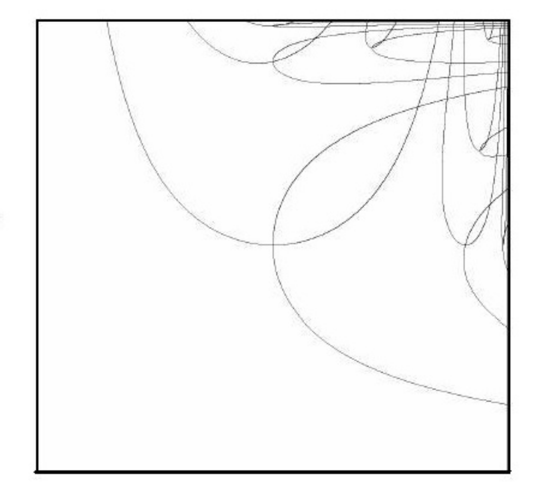
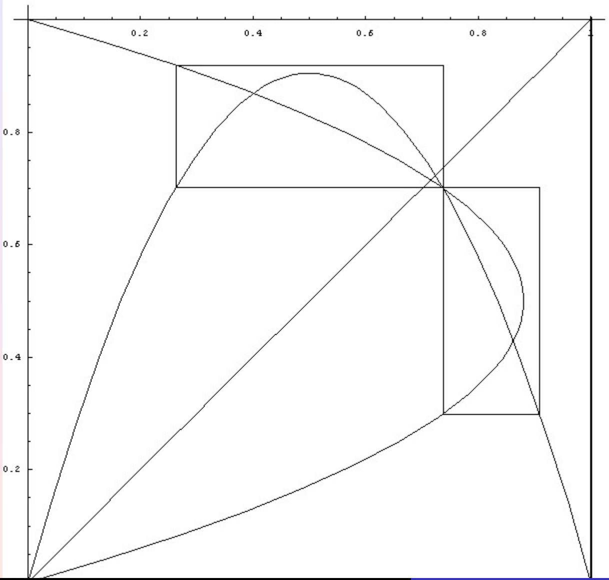
Bones in a Quartic Family



Renormalization in a Quartic Family



Complex Quartic Family

Let $Pol_d(\mathbb{C})$ be the set of polynomial maps of degree $d \geq 2$. The group $\mathcal{G}(\mathbb{C})$ of affine transformation acts on $Pol_d(\mathbb{C})$ by conjugation: $g \in \mathcal{G}(\mathbb{C})$ and $P \in Pol_d(\mathbb{C})$ yield $g \circ P \circ g^{-1} \in Pol_d(\mathbb{C})$.

Two polynomials maps of $Pol_d(\mathbb{C})$ are said to be holomorphically conjugate if they belong to the same orbit. The quotient space of $Pol_d(\mathbb{C})$ under this action is denoted by

$$M_d(\mathbb{C}) = Pol_d(\mathbb{C})/\mathcal{G}.$$

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Symmetry Locus

Definition

An automorphism of a polynomial $P \in Pol_d(\mathbb{C})$ is an affine map $\psi \in \mathcal{G}(\mathbb{C})$ such that $\psi \circ P \circ \psi^{-1} = P$.

The collection $Aut(P)$ of all automorphisms of P forms a finite group which measures how much the action of $\mathcal{G}$ on $\mathcal{P}_d$ fails to be free at P.

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The set

$$\mathcal{S}_d = \{P \in Pol_d(\mathbb{C}) : Aut(P) \text{ is non trivial } \}.$$

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Monic and Centered Polynomials

Any polynomial map $P \in Pol_d(\mathbb{C})$ is affinely conjugate to one which is monic and centered,

$$P(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_1 z + a_0.$$

This normal form is unique up to conjugation by a $(d-1)$-th root of unity.

We denote by $\mathcal{P}_d$, the set of all monic centered polynomials. It forms a complex $(d-1)$ dimensional affine space $\mathcal{A}_{d-1}$ with coordinates $(c_0, c_1, \cdots, c_{d-2})$.

We can use $\mathcal{A}_d$, as coordinate space for $M_d(\mathbb{C})$, although there remains the ambiguity up to $(d-1)$-th root of unity. The map from $\mathcal{A}_{d-1}$ to $\mathcal{P}_d$ is a $(d-1)$-fold covering of $\mathcal{P}_d$ ramified along the symmetry locus.

Class under conjugacy

In $\mathcal{P}_4$, there are three polynomials in the same class under conjugacy, and they are conjugate under the affine map $\phi(z) = \omega z$, where ω is a cubic root of unity.

The polynomial

$$P(z) = z^4 + a_2 z^2 + a_1 z + a_0$$

is conjugate by $\phi(z)$ to

$$Q(z) = z^4 + a_2 \omega^2 z^2 + a_1 z + a_0 \omega.$$

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Each $P \in Pol_4(\mathbb{C})$ has four fixed point $z_1, z_2, z_3, z_4 \in \mathbb{C}$. We denoted by μ_1, μ_2, μ_3 and μ_4 their respective multipliers. The **holomorphic index** of a rational map R at a fixed point $z_0 \in \mathbb{C}$ is defined as

$$\iota(R, z_0) = \frac{1}{2\pi i} \oint \frac{dz}{z - f(z)},$$

where we integrate in a small loop in the positive direction around z_0.

Milnor shows that the index has the following properties:

1. If z_0 is a fixed point with multiplier $\mu \neq 1$, then

$$\iota(R, z_0) = \frac{1}{1-\mu}.$$

2. For any polynomial P which is not the identity map,

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Elementary Symmetric Functions

Let $\sigma_1, \sigma_2, \sigma_3$ and σ_4 be the elementary symmetric functions of these multipliers.

$$\begin{aligned}
\sigma_1 &= \mu_1 + \mu_2 + \mu_3 + \mu_4, \\
\sigma_2 &= \mu_1\mu_2 + \mu_1\mu_3 + \mu_1\mu_4 + \mu_2\mu_3 + \mu_2\mu_4 + \mu_3\mu_4, \\
\sigma_3 &= \mu_1\mu_2\mu_3 + \mu_1\mu_2\mu_4 + \mu_1\mu_3\mu_4 + \mu_2\mu_3\mu_4, \\
\sigma_4 &= \mu_1\mu_2\mu_3\mu_4.
\end{aligned}$$

$$4 - 3\sigma_1 + 2\sigma_2 - \sigma_3 = 0.$$

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Let

$$P(z) = z^4 + a_2 z^2 + a_1 z + a_0,$$

be a monic centered polynomial in $\mathcal{P}_d$.

$$\begin{aligned}
\sigma_1 &= 12 - 8a_1 \\
\sigma_2 &= 4a_2^3 - 16a_0 a_2 + 18a_1^2 - 60a_1 + 48 \\
\sigma_4 &= 16a_0 a_2^4 + 4a_1 a_2^3(2 - a_1) + 16a_0 a_2(9a_1^2 - 18a_1 + 8) + \\
&\quad -27a_1^4 + 108a_1^3 - 144a_1^2 + 64a_1 + 256a_0^3.
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Parameters and Elementary Symmetric Functions

Solving the parameters a_i's in term of σ_is

$$
\begin{aligned}
a_2 &= r\,, \\
a_1 &= \frac{3}{2} - \frac{\sigma_1}{8}\,, \\
a_0 &= \frac{128r^3 + 24\sigma_1 + 9\sigma_1^2 - 32\sigma_2 - 48}{512r}\,,
\end{aligned}
$$

where r is a root of the quadratic polynomial (Fujimura-Nishizawa-2005):

$$P_2(z) = A_2(\sigma_1, \sigma_2, \sigma_4)z^2 + A_1(\sigma_1, \sigma_2, \sigma_4)z + A_0(\sigma_1, \sigma_2, \sigma_4), \quad (1)$$

and

$$\begin{aligned} A_2 &= 262144(\sigma_1 - 4)^2, \\ A_1 &= 1024\sigma_1 \left(1280 - 576\sigma_1 + 27\sigma_1^3 - 144\sigma_1\sigma_2 + 384\sigma_2\right) \\ &\quad +1024 \left(128\sigma_2^2 - 256\sigma_2 - 512\sigma_4 - 768\right), \\ A_0 &= \left(24\sigma_1 + 9\sigma_1^2 - 32\sigma_2 - 48\right)^3. \end{aligned}$$

There is a natural projection $\Psi_4 : M_4(\mathbb{C}) \to \mathbb{C}^3$, defined by

$$\Psi_4\left(p_{a_2,a_1,a_0}(z)\right) = (\sigma_1, \sigma_2, \sigma_4)$$

From the quadratic poynomial P_2, we have that

1. $A_2 = 0$ if and only if $\sigma_1 = 4$,
2. $A_2 = A_1 = 0$ if and only if $\sigma_1 = 4$ and $\sigma_4 = \dfrac{(\sigma_2 - 4)^2}{4}$.
3. if $\sigma_1 = 4$ and $\sigma_4 = \dfrac{(\sigma_2 - 4)^2}{4}$, then $A_0 = (192 - 32\sigma_2)^3$.

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Exceptional Set

The set $\mathcal{E}_4 = \{(4, z, \frac{(z-4)^2}{4}) \in \mathbb{C}^3 \,:\, z \in \mathbb{C}\}$ is called the exceptional set for the quartic polynomial family. It is a complex curve in $\mathbb{C}^3$.

Proposition

$\Psi_4(M_4(\mathbb{C})) = \mathbb{C}^3 \setminus \mathcal{E}_4$.

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A $P \in Pol_4(\mathbb{C})$ is in the symmetry locus $\mathcal{S}_4$ in $\mathbb{C}^3$ if and only if it is conjugate to a polynomial map in the normal form

$$\tilde{P}_{a_1}(z) = z^4 + a_1 z, \ \text{with } a_1 \in \mathbb{C}.$$

Moreover,

$$Aut(\tilde{P}_{a_1}) = \{\psi(z) = \omega z \ : \ \omega \text{ is a cubic root of unity}\}$$

Proposition

The symmetry locus $\mathcal{S}_4$ in the parameter space $\mathbb{C}^3$ is given by the complex curve $\gamma : \mathbb{C} \to \mathbb{C}^3$, defined as

$$\gamma(s) = \left(c, \frac{3(3s-4)(s+4)}{32}, \frac{-(3s-4)^3(s-12)}{4096}\right).$$

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In the quadratic polynomials $Pol_2(\mathbb{C})$, we have the family of monic centered polynomials $P_c(z) = z^2 + c$, $c \in \mathbb{C}$, which fixes the critical point, and the family $P_\lambda(z) = \lambda z + z^2$, $\lambda \in \mathbb{C}$, which has a fixed point in zero. For any parameter $c \in \mathbb{C}$, there are two parameters λ_1 and λ_2, such that P_{λ_1}, P_{λ_2} and P_c are affinely conjugated. We denote by $P_{c_2 c_1} = P_{c_2} \circ P_{c_1}$

Quartic Polynomials

1. The quartic polynomial

$$P_{\lambda_2\lambda_1} = \lambda_1\lambda_2 z + \lambda_2 z^2 + \lambda_1^2 z^2 + 2\lambda_1 z^3 + z^4$$

is conjugated to the monic centered polynomial

$$Q(z) = z^4 + (\lambda_2 - \frac{\lambda_1^2}{2})z^2 + \frac{\lambda_1(\lambda_1^3 - 4\lambda_1\lambda_2 + 8)}{16},$$

by the affine map $\psi(z) = z - \dfrac{\lambda_1}{2}$.

2. If $c_1 = \dfrac{\lambda_2}{2} - \dfrac{\lambda_1^2}{4}$ and $c_2 = \dfrac{\lambda_1}{2} - \dfrac{\lambda_2^2}{4}$, then $P_{c_2 c_1}(z) = Q(z)$.

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Quartic Polynomials

$P_{c_2c_1}(z)$ and $P_{\lambda_2\lambda_1}(z)$ are conjugated to

$$P(z) = z^4 + A_2 z^2 + A_0.$$

We denote this family by $\mathcal{P}$.

Ψ_4 sends this family to the subspace $\{(12, \sigma_2, \sigma_4) \in \mathbb{C}^3\}$.

We denote by $Per_k(\mu)$ the set of $P \in \mathcal{P}$ such that P has a periodic orbit of period k and multiplier μ.

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Remark

If $P \in \mathcal{P}$, then it has three critical points, $\tilde{c}_0 = 0$, $\tilde{c}_1 = \sqrt{-\frac{A_2}{2}}$ and $\tilde{c}_2 = -\sqrt{-\frac{A_2}{2}}$. But, P has at most two different dynamics, since $P(\tilde{c}_1) = P(\tilde{c}_2)$ and these two critical points define a same dynamic for P.

The $Per_1(0)$ is defined in three cases, each one of them is determined by the polynomials fixing $\tilde{c}_j$, for $j = 0, 1, 2$.
If $\tilde{c}_0 = 0$ is fixed, we have the following quartic family

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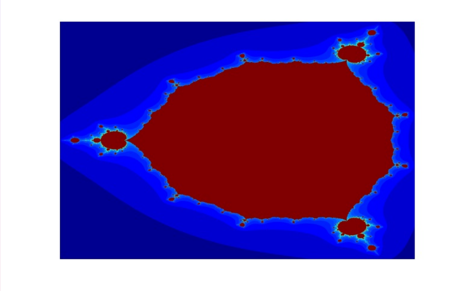
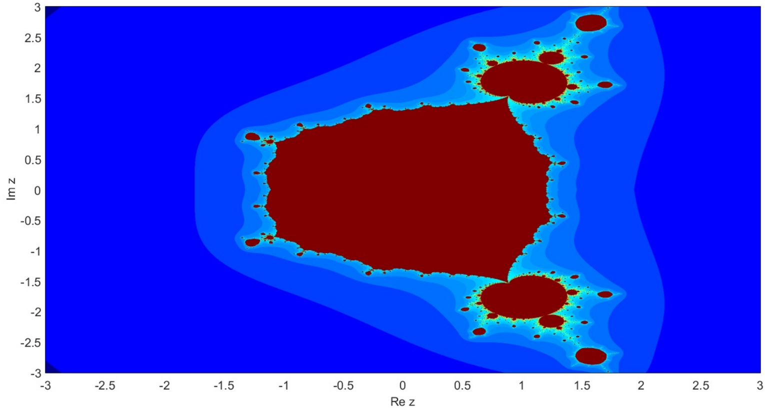
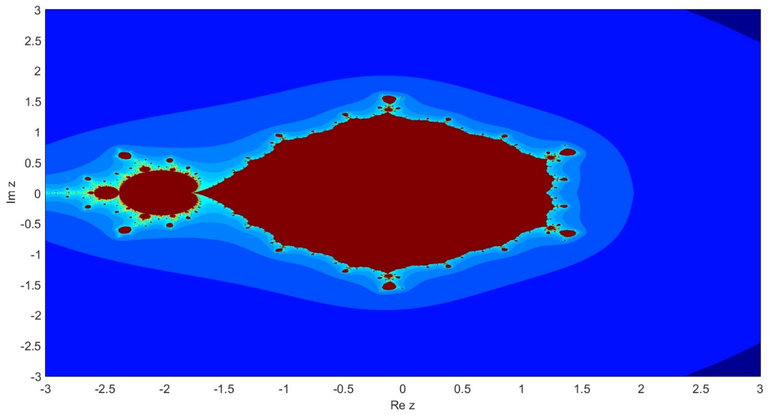


Connectedness Locus for $P_a(z) = z^4 + az^2 + \left(\frac{a}{2}\right)^2 + \sqrt{\frac{-a}{2}}$



Connectedness Locus for $P_a(z) = z^4 + az^2 + \left(\frac{a}{2}\right)^2 - \sqrt{\frac{-a}{2}}$



Julia Sets

We denote by $c_a = \sqrt{\frac{-a}{2}}$ the free critical point and $v_a = P_a(c_a) = \frac{-a^2}{4}$.
The *filled Julia set* K_a consists of the non escaping points, that is,

$$K_a = \{z \in \mathbb{C} \, : \, \{P_a^n(z)\} \text{ is bounded } \}.$$

And the *Julia set* J_a is its boundary.
Let $\mathcal{C}$ be the **connectedness locus** of this family, i.e.

$$\mathcal{C} = \{a \in \mathbb{C} \, : \, K_a \text{ is connected}\}$$

We can partition the plane in two loci: $\mathcal{C}$ and its complement $\mathcal{C}_\infty$ which consist of the parameters a such that the critical point c_a is attracted by infinity. Moreover, the connectedness locus $\mathcal{C}$ can be partitioned in hyperbolic components which are given by the hyperbolic parameters.

$$\mathcal{W} = \left\{ a \in \mathbb{C} \, : \, c_a \in \tilde{B}_a \right\},$$

where $\tilde{B}_a$ is the basin of attraction of zero.

Böttcher's Coordinate

Remark

$Int(K_a) \neq \emptyset$, for all $a \in \mathbb{C}$ because $\tilde{B}_a \subset K_a$.

We denote by B_a the immediate basin of 0. In particular, if $\tilde{c}_1 \notin B_a$ then the map $P_a|_{B_a}$ is conjugated to z^2 on $\mathbb{D}$, else $B_a = \tilde{B}_a$.

By Böttcher's Theorem, there are conformal isomorphisms $\phi_a^\infty : U_a^\infty \to V_a^\infty$, $\phi_a^0 : U_a^0 \to V_a^0$, such that,

$$\phi_a^\infty \circ P_a = (\phi_a^\infty)^4 \quad \text{on } U_a^\infty \quad \text{and } \phi_a^0 \circ P_a = \left(\phi_a^0\right)^2, \text{ on } U_a^0,$$

with ϕ_a^∞ tangent to identity near to infinity. And ϕ_a^0 tangent to az near to 0.

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Symmetries

Let $\tau(z) = e^{\frac{2\pi i}{3}} z$ and $\sigma(z) = \overline{z}$. In the family P_a, the rotation τ is a conformal conjugation between two polynomials P_a and $P_{a'}$. Explicitly we have that $P_a(\tau z) = \tau P_{\tau a}(z)$. Moreover, P_a is conjugated to $P_{\overline{a}}$ by σ. Hence, we have that a "fundamental domain" for the family P_a is

$$\mathcal{D} = \left\{ a \in \mathbb{C} \,:\, 0 \leq arg(a) \leq \frac{1}{6} \right\}.$$

Remark

The sets $\mathcal{C}$, $\mathcal{W}$ and $\mathcal{H}_\infty$, admit the maps σ and τ as symmetries.

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Symmetries

Proposition

The set $\mathcal{H}_\infty$ is a connected component of hyperbolic parameters and $\mathcal{W}_0$ is exactly

$$\mathcal{W}_0 = \{a \in \mathbb{C} \,:\, c_a \in B_a\}\,.$$

$$\mathcal{W}_k = \{a \in \mathbb{C} \,:\, P_a^k(v_a) \in B_a \text{ and } P_a^{k-1}(v_a) \notin B_a\}.$$

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Hyperbolic Components

Proposition

The component $\mathcal{H}_\infty$ and any connected component of $\mathcal{W}$ are simply connected.

We denote by $\Psi_0 : \mathcal{W}_0 \to \mathbb{D}$ and $\Psi_\infty : \mathcal{W}_\infty \to \mathbb{C} \setminus \overline{\mathbb{D}}$, the conformal representation tangent to the identity at 0 and ∞, respectively.

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$\Psi_w(\sigma a) = \sigma \Psi_w(a)$ and $\Psi_w(\tau a) = \tau \Psi_w(a)$, for $w = 0$ or ∞.

If $\rho = e^{\frac{\pi i}{3}}$, then the line $\mathbb{R}^+\rho$ cut $\mathcal{C}$ in a connected set. Consequently, $\tau^k(\mathbb{R}^+\rho) \cap \mathcal{C}$, is connected for $k = 1, 2$.

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Proposition

If a and τa are in $\mathbb{C} \setminus \mathbb{R}^-$, then the Böttcher maps satisfy the following relation:

$$\sigma\left(\phi^{w}_{\sigma(a)}(\sigma(z))\right) = \phi^{w}_{a}(z) = \kappa_w(a)\phi^{w}_{\tau a}(\frac{z}{\tau}),$$

with $w \in \{0, \infty\}$, $\kappa_\infty(a) = \tau$ and $\kappa_0(a) = \dfrac{\lambda(a)}{\tau\lambda(\tau a)}$. Moreover the rays at parameter $a, \tau a$ and $\sigma(a)$ satisfy the relations:

$$R^{w}_{\sigma(a)}(t) = \sigma(R^{w}_{a}(-t)) \text{ and } R^{w}_{\tau a}(t) = \tau R^{w}_{a}(t + t_w(a)),$$

where $t_w(a) = arg(\kappa_w(a))$.

Hyperbolic Components

Theorem

The map $\Phi_\infty : \mathcal{W}_\infty \to \mathbb{C} \setminus \overline{D}$ defined as

$$\Phi_\infty(a) = \phi_a^\infty(v_a)$$

is a holomorphic covering map.

Proposition

The map

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Theorem

Let $\mathcal{U}$ be a connected component of $\mathcal{W}_k$ with $k > 0$ included in $\mathbb{C} \setminus \mathbb{R}^-$. The map $\Phi_{\mathcal{U}} : \mathcal{U} \to \mathbb{D}$ defined as

$$\Phi_{\mathcal{U}}(a) = \phi_a^0\left(P_a^k\left(v_a\right)\right)$$

is a conformal isomorphism.

Green's Function

The *Green function* is defined on U_a^∞ as

$$G_a^\infty(z) = log|\phi_a^\infty(z)|.$$

The *equipotential* of level $r > 0$, E_a^w, for $w = 0, \infty$ is the curve $(G_a^w)^{-1}(r)$. A *ray* $R_a^w(t)$, of angle $t \in \mathbb{R}/\mathbb{Z}$ is $(\phi_a^w)^{-1}(\mathbb{R}^+ e^{2\pi i t})$.

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External Rays

Proposition

Let $a_0 \in \mathbb{C}$, $w = 0$, or ∞, and $t \in \mathbb{Q}/\mathbb{Z}$. If the ray $R^w_{a_0}$ lands, then it lands at an eventually periodic point which is repelling or parabolic.

Theorem

Let $a \in \mathbb{C}$ be a parameter such that J_a is connected. For every eventually periodic point of P_a that is repelling or parabolic, there exists a rational angle t such that R^∞_a lands at this point.

Theorem

The boundary of every connected component of $\mathcal{W}$ is a Jordan curve.

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Julia Sets

The quartic polynomial

$$P_{\lambda_2\lambda_1} = \lambda_1\lambda_2 z + \lambda_2 z^2 + \lambda_1^2 z^2 + 2\lambda_1 z^3 + z^4$$

is conjugated to the monic centered polynomial

$$Q(z) = z^4 + (\lambda_2 - \frac{\lambda_1^2}{2})z^2 + \frac{\lambda_1(\lambda_1^3 - 4\lambda_1\lambda_2 + 8)}{16}.$$

We have at least three different pairs $(\lambda_1^j, \lambda_2^j)$, such that $P_{\lambda_2^j\lambda_1^j} = P_{\lambda_2^k\lambda_1^k}$, for $j, k = 1, 2, 3$.

If $\lambda_j = e^{2\pi\theta_j}$ and $\theta_j \in \mathbb{R}/\mathbb{Z}$, for $j = 1, 2$, then zero is an indifferent point for P_{λ_1}, P_{λ_2} and $P_{\lambda_2\lambda_1}$.

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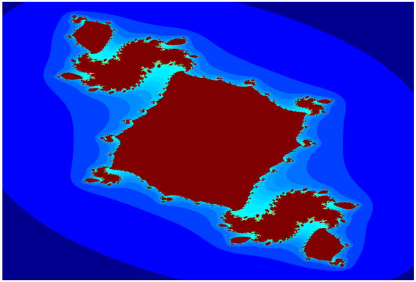
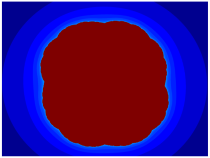
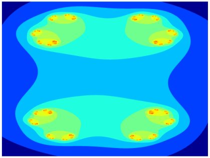


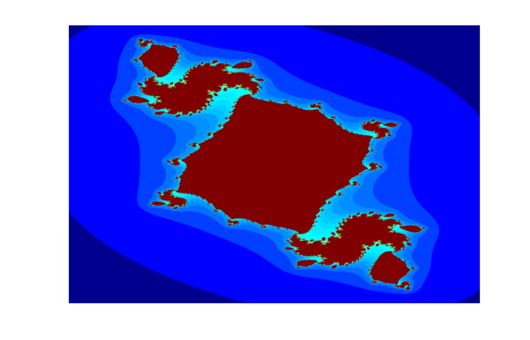
(a) $P(z) = z^4 + (\sqrt{-2}+1)z^2 + \frac{1}{7}$



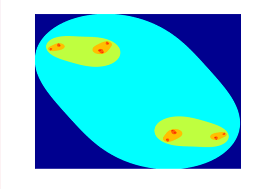
(b) $0.29252+0.08038i$



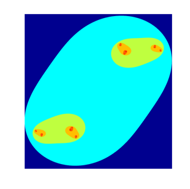
(c) $1.03955+1.4377i$



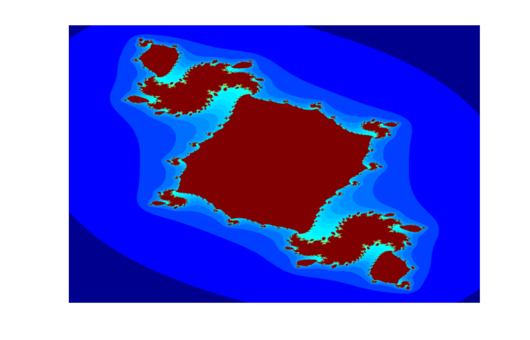
(d) $P(z) = z^4 + (\sqrt{-2} + 1)z^2 + \frac{1}{7}$



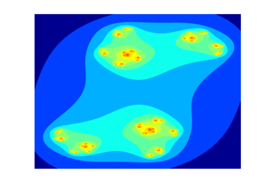
(e) -1.49644+2.71284i



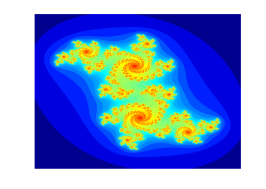
(f) -1.56008-2.64540i



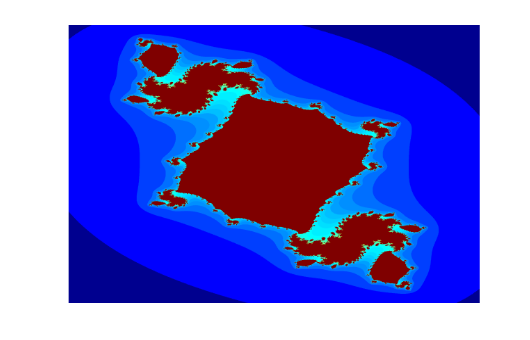
(g) $P(z) = z^4 + (\sqrt{-2}+1)z^2 + \frac{1}{7}$



(h) 0.26051-1.48545i



(i) -0.06934+1.02723i



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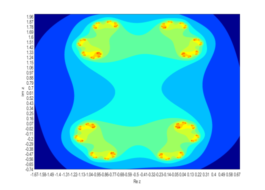
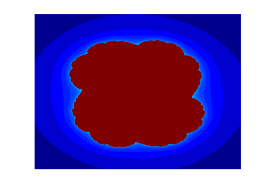
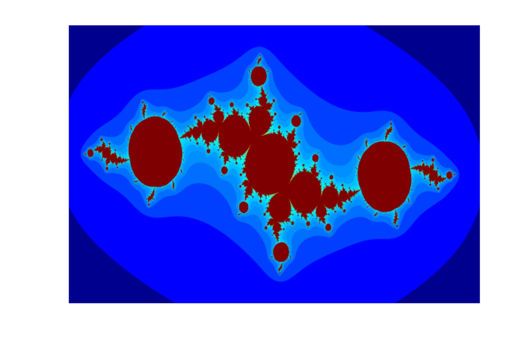


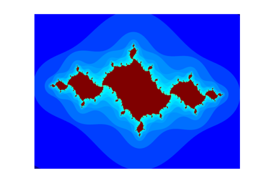
(k) 0.94341-1.30777i



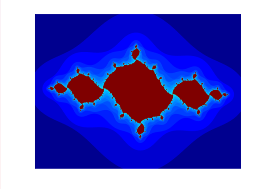
(l) 0.58987+0.18044i



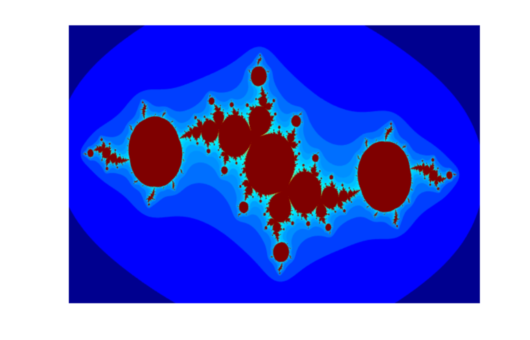
(a) $P(z) = z^4 + (-2.108893535 + .3570353803i)z^2 + 0.09068800125 - .2909712453i$



(b) 3.27449-.13815i



(c) 3.24272-0.09533i



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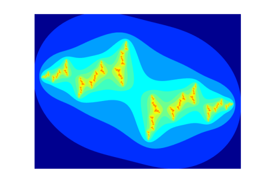
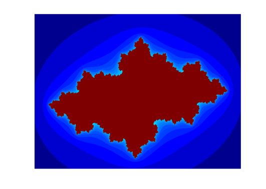
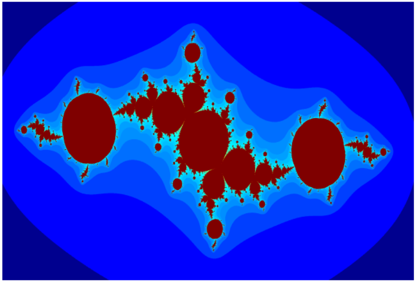


(e) -1.59397+0.34655i



(f) -.89857-.19536i



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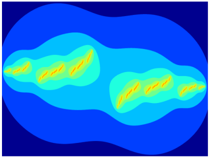
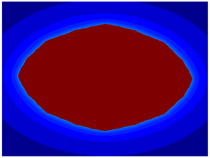
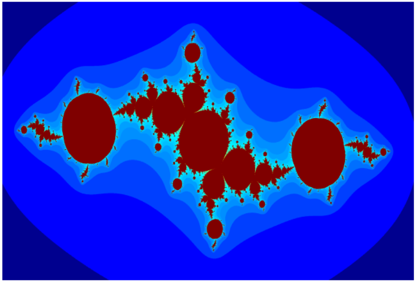


(h) -1.95833+0.16343i



(i) -.20471+0.03696i



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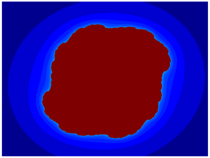
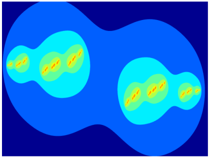
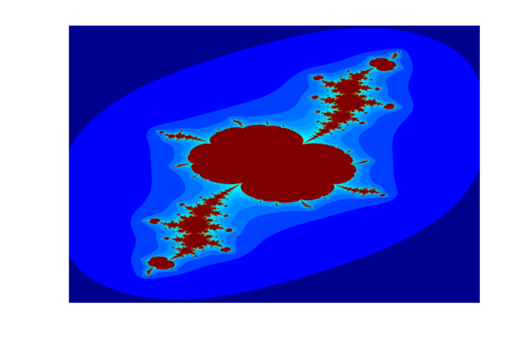


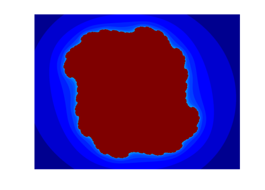
(k) 0.27780-.37184i



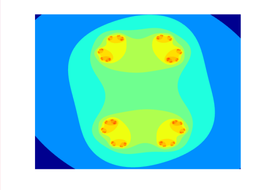
(l) -2.13943+0.25373i



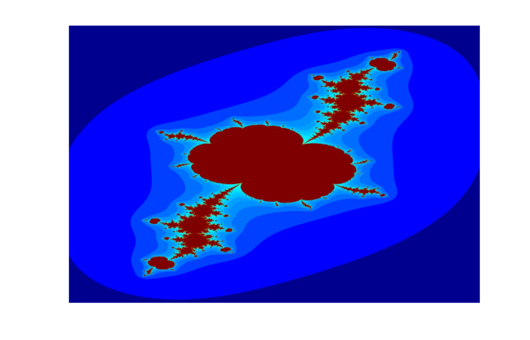
(a) $P(z) = z^4 + (1.133545861 - 1.631833500i)z^2 + 0.7465710888e - 1 + .1052770533i$



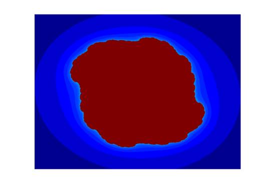
(b) 0.35423+0.48025i



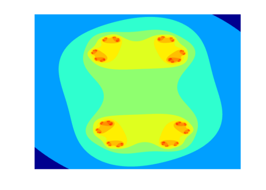
(c) 1.08096-1.46171i



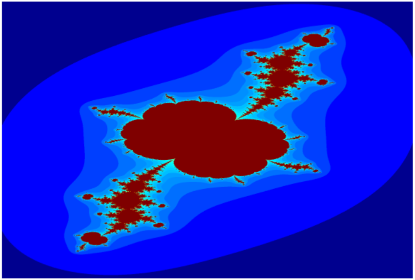
(d) $P(z) = z^4 + (1.133545861 - 1.631833500i)z^2 + 0.7465710888e - 1 + .1052770533i$



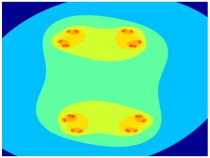
(e) 0.28484+0.39480i



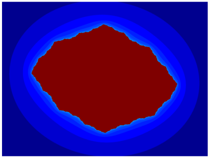
(f) 1.09617-1.51937i



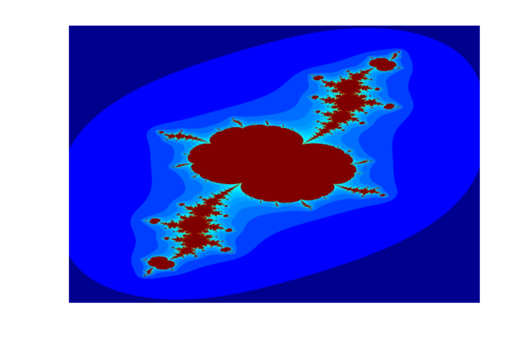
(g) $P(z) = z^4 + (1.133545861 - 1.631833500i)z^2 + 0.7465710888e - 1 + .1052770533i$



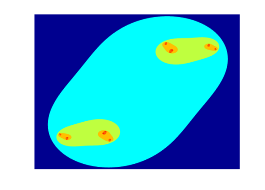
(h) $.9114 + 1.98533i$



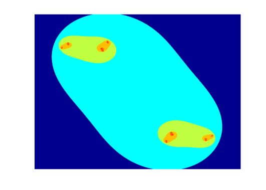
(i) $-.42184 + 0.17774i$



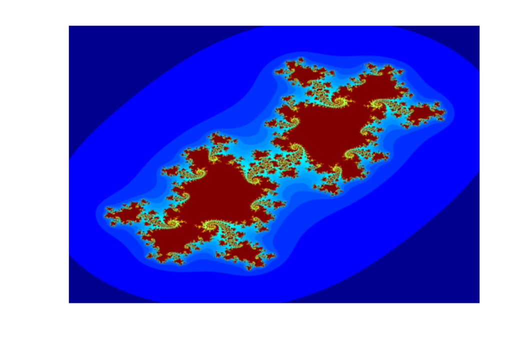
(j) $P(z) = z^4 + (1.133545861 - 1.631833500i)z^2 + 0.7465710888e - 1 + .1052770533i$



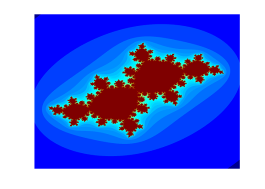
(k) -1.55054-2.86039i

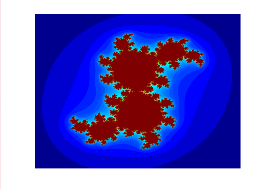


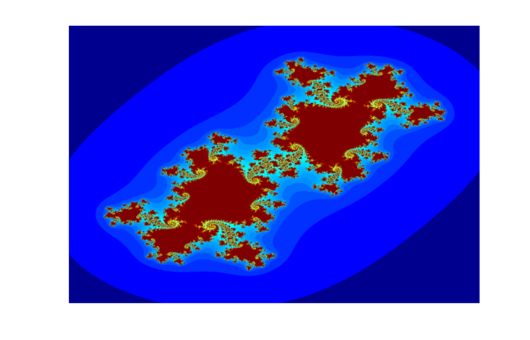
(l) -1.75529+2.80334i



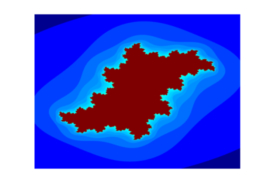
(a) $P(z) = z^4 + \left(e^{2i\pi\left(\frac{\sqrt{5}}{2}-\frac{1}{3}\right)} - \frac{1}{2}\left(e^{2i\pi\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)}\right)^2\right)z^2 + \frac{1}{16}e^{2i\pi\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)}\left(\left(e^{2i\pi\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)}\right)^3 - 4e^{2i\pi\left(\frac{\sqrt{5}}{2}-\frac{1}{2}\right)}e^{2i\pi\left(\frac{\sqrt{5}}{2}-\frac{1}{3}\right)} + 8\right)$

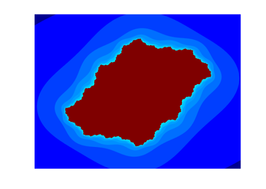


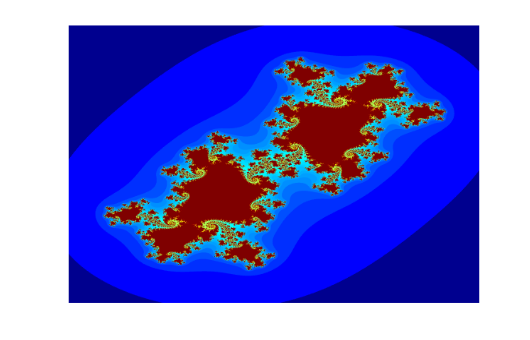




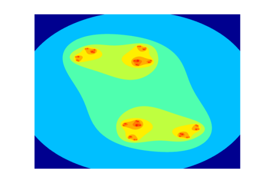
(d) $P(z) = z^4 + (e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{3})} - \frac{1}{2}(e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})})^2)z^2 + \frac{1}{16}e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})}((e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})})^3 - 4e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})}e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{3})}+8)$



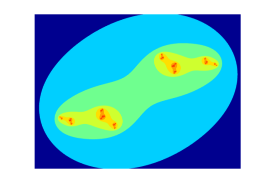




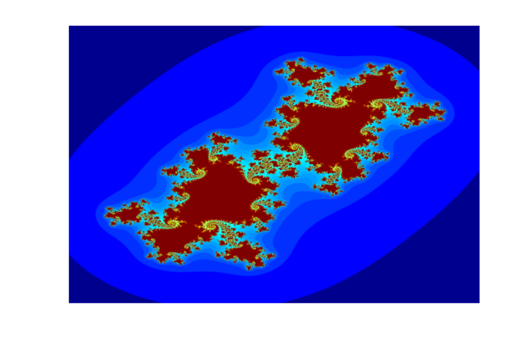
(g) $P(z) = z^4 + (e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{3})} - \frac{1}{2}(e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})})^2)z^2 + \frac{1}{16}e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})}((e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})})^3 - 4e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})}e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{3})}+8)$



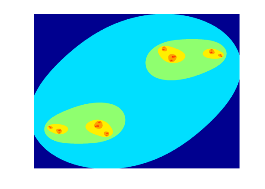
(h) 0.01563 + 1.00357*i*



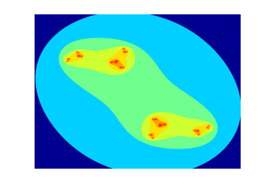
(i) 1.63007-1.44463*i*



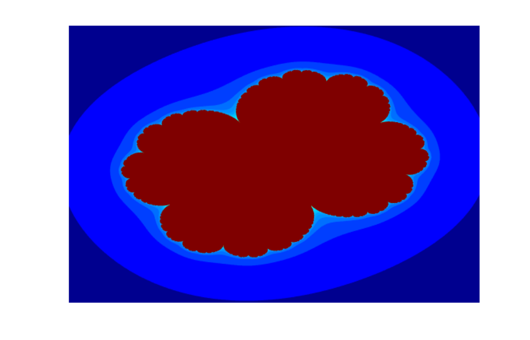
(j) $P(z) = z^4 + \left(e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{3})} - \frac{1}{2}\left(e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})}\right)^2\right)z^2 + \frac{1}{16}e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})}\left(\left(e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})}\right)^3 - 4e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{2})}e^{2i\pi(\frac{\sqrt{5}}{2}-\frac{1}{3})}+8\right)$



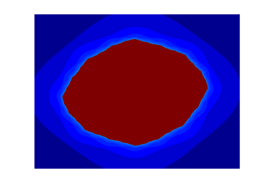
(k) 1.54784-2.13678i



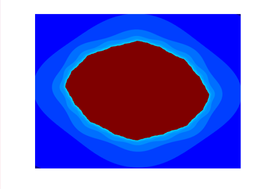
(l) .01242+1.83300i



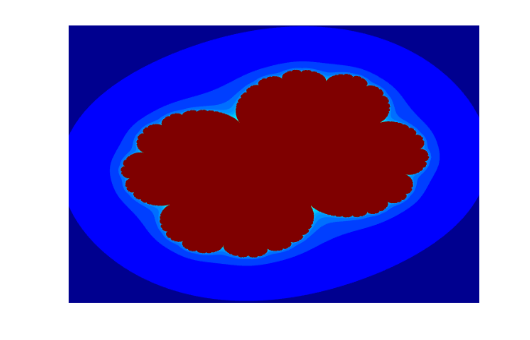
(a)



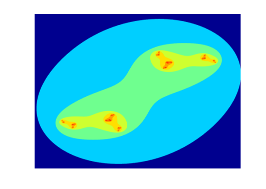
(b) $-\frac{i\sqrt{3}}{2} + \frac{1}{2} + i + \sqrt{3}$



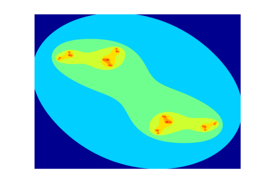
(c) $\frac{1}{2} - i + \frac{i\sqrt{3}}{2} + \sqrt{3}$



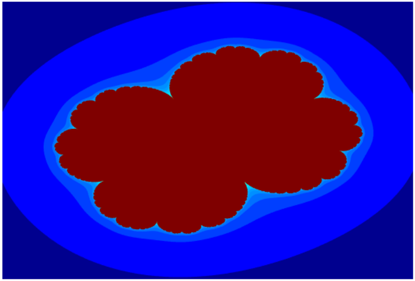
(d)



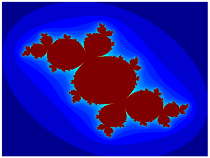
(e) $-\frac{i\sqrt{3}}{2} - \sqrt{3} + \frac{1}{2} - i$



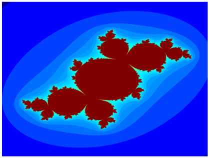
(f) $\frac{1}{2} + i + \frac{i\sqrt{3}}{2} - \sqrt{3}$



(g)



(h) $-\frac{1}{2}+\frac{i\sqrt{3}}{2}$



(i) $-\frac{1}{2}-\frac{i\sqrt{3}}{2}$

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2. B. Branner and J. H. Hubbard *The iteration of cubic polynomials, Part I : The global topology of parameter space*, Acta. Math. **160** (1988), 143-206.
3. A. Douady and J.H. Hubbard, *On the dynamics of polynomial-like mappings*, Ann. Sci. ENS. Paris, **18**, (1985), 287–343.
4. M. Fujimura and K. Nishizawa. *Moduli spaces and symmetry loci of polynomial maps.* In W. Kchlin, editor, Proceedings of the 1997 International Symposium on Symbolic and Algebraic Computation, (1997), 342-348.

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Thanks

Happy Birthday Jack