Bott-Samelson-Demazure-Hansen Varieties for Projective Homogeneous Varieties with Nonreduced Stabilizers

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Abstract of the Dissertation

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Over a field of positive characteristic, a reductive algebraic group G may have some nonreduced parabolic subgroup P. In this thesis, we study the Schubert and Bott-Samelson-Demazure-Hansen (BSDH) varieties of the associated exotic flag varieties G/P, with P nonreduced. It is shown that in general the Schubert and BSDH varieties of such a G/P are not normal, and the projection of the BSDH variety onto the Schubert variety has nonreduced fibers at closed points. When the base field is finite, the convolution morphisms between BSDH varieties are also studied. It is shown that the decomposition theorem holds for such morphisms, and the pushforward of intersection complexes by such morphisms are Frobenius semisimple.

Contents

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Chapter 1

Introduction

This thesis stems from a special phenomenon in algebraic geometry in positive characteristics: the existence of non-reduced parabolic subgroup schemes of a reductive group scheme.

Given a reductive group scheme G over a field k, a subgroup scheme P of G is said to be parabolic if P contains a Borel subgroup of G. Given such a pair (G, P), it is natural to study three kinds of varieties associated with it:

1. The flag variety G/P;

2. The Schubert varieties on the flag variety;

3. The Bott-Samelson-Demazure-Hansen (BSDH) varieties associated to a Schubert variety.

When P is reduced, we call the corresponding three kinds of varieties classical. When P is non-reduced, we call them exotic.

Here is a quick example for a first impression: when $G = GL_2$, a non-reduced parabolic P can be made of matrices of the following form:

$$\begin{bmatrix} * & * \\ \epsilon & * \end{bmatrix}, \quad \epsilon^p = 0.$$

The resulting exotic flag variety G/P parametrizes Frobenius twists of one dimensional vector subsapces of a fixed two dimensional vector space, and is isomorphic to the projective line.

1.1 Background and History

BSDH varieties and convolution morphisms

The study of the classical flag and Schubert varieties dates back at least to the 1800s [Sch89]. These varieties enjoy many nice properties, which have a myriad of applications to combinatorics, representation theory, and geometry. In the study of the geometry of the classical Schubert varieties, the classical BSDH varieties naturally arise as resolutions of singulartities, see the textbook accounts [Jan03, §II.14], [BL00, §9.1], and [BK07, §2.1] in the context of representation theory, singularity of Schubert varieties, and Frobenius splittings of Schubert varieties, respectively.

The classical BSDH varieties are given by certain iterated fibrations p_i's with fibers isomorphic to some Schubert varieties. The prototype of them is first constructed by Bott and Samelson in [BS58, p.970] in the context of compact real Lie groups. They are used there to study the cohomology of the loop spaces of Lie groups. In algebraic geometry, Hansen introduced them in [Han73, §3] in the case where G is a complex reductive group, P is a Borel subgroup of G, and the Schubert varieties that occur as fibers of p_i's are all isomorphic to the projective line over the base. Around the same time, Demazure introduced them in [Dem74, §3.7] with G a split reductive group over a base ring, and same assumptions on P and p_i's as Hansen's. The name "BSDH varieties" is thus settled, but the development of such varieties went on: the classical BSDH varities in our sense, where P can be any reduced parabolic and no restrictions on the fibers of p_i's are imposed, are introduced in later works [GM82, §2.11], [Zel83, p.143], and [SV94, §2].

There are natural convolution morphisms between classical BSDH varieties, generalizing the desingularization morphism from BSDH varieties to Schubert varieties. The topology of such natural convolution morphisms are important in areas in representation theory, such as the Kazhan-Lusztig Conjecture [Spr81], the Koszul duality patterns [BGS96], Soergel bimodules [EMTW20], and parity sheaves [JMW14].

Furthermore, the definitions of the classical BSDH varieties and the convolution morphisms among them are extended to the Kac-Moody setting in [Kum02, §7.1.3], and the affine flag variety setting in [MV00, §2], [Hai06, §2.1], and [dCHL18, §4.1]. The convolution morphisms between classical BSDH varieties in this thesis are based on the definition of the "generalized convolution morphisms" in *loc. cit.* The aforementioned applications to representation theory also extend to these settings, see, e.g. [AR13] and [BY11].

Exotic varieties

In contrast to the nice properties enjoyed by the well-studied classical flag, Schubert, and BSDH varieties, the exotic ones are sources of various curious phenomena, and they appeared relatively late in the literature:

Let us first list some of the curious phenomena demonstrated by the exotic flag varieties: Haboush and Lauritzen found examples of exotic flag varieties which

1. violate the Kodaira and Kempf vanishing theorems [Lau93, Ex. 6.3.1-2];

2. admit ample line bundles with negative Euler characteristic [Lau93, Ex. 6.3.3];

3. do not admit a flat lift to $\mathbb{Z}$ [HL93, 56];

4. are not Frobenius split [Lau93, Theorem 5.2];

5. are not $\mathcal{D}$-affine; [Lau97, §4.4], i.e., admit a $\mathcal{D}$-module, which is quasi-coherent as an $\mathcal{O}$-module, with nonvanishing higher cohomology.

All of the five phenomena above cannot happen for the classical flag varieties.

Moreover, in [Tot19, Cor. 2.2], Totaro shows that the affine cone over certain exotic flag variety has an isolated terminal singularity that is not Cohen-Macaulay. This is the first discovery of such singularity.

Let us recount the (short) history of the development of the exotic flag, Schubert, and BSDH varieties: The first systematic studies of exotic flag varieties are done by Wenzel in [Wen93], and Haboush and Lauritzen in [HL93]. The first systematic study of exotic Schubert varieties is done by Lauritzen in [Lau97]. The first systematic study of exotic BSDH varieties is done by the author in [Zha22]. The main content of this thesis is based on *loc. cit.*

1.2 Plan of Thesis

In this thesis, I define and study the exotic BSDH varieties and the convolution morphisms among them.

In Chapter 2, I start by recalling and establishing some basic structural results of the non-reduced parabolics, the exotic flag varieties, and their corresponding exotic Schubert varieties and Richardson varieties.

In Chapter 3, the definitions of the exotic BSDH varieties and functors are given. I also establish some basic properties of them, and explain why some other naive definitions cannot work.

Chapters 4 and 5 contain the main results of mine on the exotic BSDH varieties and convolution morphisms. A slogan of my main results is that "geometry is wild, and topology is nice".

Geometry is Wild (Chapter 4):

1. While the classical Schubert and BSDH varieties are normal, we show that the exotic ones can be non-normal.

   Namely, we construct a non-normal exotic BSDH variety in Example 4.2.7.

   Moreover, in Example 4.2.8, for each $n \geq 2$, we construct an n-dimensional non-normal exotic Schubert variety.

2. While the classical BSDH varieties are given by iterated fibrations p_i's with fibers isomorphic to some Schubert varieties, we show that the corresponding

morphisms p_i's for the exotic ones are in general not such fibrations.

Namely, in Examples 4.2.1 and 4.2.6, we construct exotic BSDH varieties whose morphisms p_1 have non-reduced fibers, not isomorphic to any Schubert varieties.

Moreover, in Exmample 4.2.2, we construct an exotic BSDH variety whose morphism p_1 has fibers with different degrees of non-reducedness.

3. While a classical BSDH variety admits a birational morphism res to a Schubert variety, which usually can be used as a desingularization of the target, we construct an exotic BSDH variety in Example 4.2.12 whose corresponding morphism res is not birational.

We obtain these examples by explicit calculations using (1) the Bruhat and Deodhar decompositions in terms of the Schubert and Richardson cells, (2) modular interpretation of the exotic BSDH varieties, and (3) the explicit equations given by various incidence varieties.

Topology is nice (Chapter 5):

1. In Theorem 5.2.3, we show that over a finite or algebraically closed base field, the intersection complexes of the exotic BSDH varieties, and the derived pushforwards of them by convolution morphisms, are semisimple, Frobenius semisimple, very pure of weight zero, even and Tate.
2. The BBDG Decomposition Theorem in general only holds for schemes over algebraically closed fields. In Theorem 5.2.2, we show that, over any finite field, the Decomposition Theorem package still holds for any convolution morphism.

These topological results were shown for the classical BSDH varieties and some infinite dimensional versions of them in [BGS96], [BY11], [AR13], and [dCHL18]. Here we take the finite dimensional part of their results and use the fact that the exotic BSDH varieties are universally homeomorphic to the classical ones.

In the Appendix, Chapter 6, I clarify the relation between the notion of k-functors and k-spaces used in works [Jan03], [DG70], and the notion of algebraic space as in [Ols16]. I also explicate the relation among different definitions of images of a scheme morphism using k-functor, k-spaces, monomorphisms of schemes, and closed immersions of schemes.

Let us end this Introduction with three remarks for future research in this (exotic) field:

1. This thesis is only the beginning step in the study of the exotic BSDH varieties in the algebro-geometric perspective. The Kac-Moody and affine versions of them, as well as their implications in representation theory, are still completely open. However, our “topology is nice” results seem to imply that their Kazhdan-Lusztig theory should be the same as the classical ones. This is because the intersection cohomologies of the exotic Schubert varieties are the same as the classical ones, due to the fact that they are universally homeomorphic.

2. We still lack a concentrator-like description of the non-reduced parabolics: The reduced parabolics can be described as the concentrator subschemes of G under the conjugation actions of cocharacters of G. Leveraging on this fact, it is shown in [Hei18, Cor. 1.1.6] that, given any G-bundle E on a curve, the followings are equivalent:

   (a) E is Θ-(semi)stable;

   (b) E is (semi)stable.

   The classical work [Ram75, Lemma 2.1] shows that item (b) above is also equivalent to:

   (c) given any reduction of structure group E_P to a reduced parabolic subgroup scheme P of G, and any dominant character χ of P, the associated line bundle $E_P \times^P \chi$ has nonpositive degree.

   Turning to non-reduced parabolics, in [Sun99], it is shown that the followings are equivalent:

   (b′) E is Frobenius (semi)stable;

   (c′) Same as item (c) above, but we consider reduction of structure group E_P to *any* parabolic P, reduced or not.

   It is then natural to ask if Frobenius (semi)stability can also be described as some kind of Θ-semistability. To answer this question in a similar fashion as in [Hei18], we need the desired concentrator-like description of non-reduced parabolics.

3. The non-reduced parabolics should give rise to new (global) Springer theories as in [Yun11], and they have not been studied yet. Namely, in analogy to the

exposition in [Yun15, §4.5], given a non-reduced parabolic P of a reductive group G, the moduli stack $\mathcal{M}^{par}$ in the exotic global Springer theory should parametriz the quadruples (E, x, ϕ, E_x^P), where (E, ϕ) is a G-Higgs bundle on a fixed curve C, x is a point of C, and E_x^P is a reduction of the fiber E_x to a P-torsor E_x^P compatible with ϕ. What can we say about the topology and symmetry of the resulting Hitchin morphism?

Chapter 2

Basic Structure of Exotic Flag Varieties

In this chapter, we first fix notations in §2.1. In §2.2, we recall and establish some basic structural restuls for the nonreduced parabolics in Prop. 2.2.1; the Chow rings of the exotic flag varieties in Prop. 2.2.5; and the exotic Schubert and Richardson varieties in Prop. 2.2.6.

2.1 Setup

Let k be a perfect field with characteristic $p > 0$. All the schemes in this thesis are k-schemes of finite presentation.

Group-Theoretic Preliminaries

Let G be a connected split reductive linear algebraic group over k. Fix a split maximal torus T in G and B a Borel subgroup containing T. Let W be the Weyl group.

We say that a subgroup scheme P of G is a parabolic if P contains a Borel subgroup of G. Using the argument in [Spr98, §6.2], we have that

Lemma 2.1.1. *A subgroup scheme P of G is parabolic if and only if the quotient G/P exists as a complete variety over k.*

Throughout this thesis, we only consider the parabolics P that contain the Borel B as a subscheme. The maximal reduced subscheme of P is denoted by P_{red}. Since k is perfect, we have that P_{red} is also a group subscheme by [Mil17, Cor. 1.39].

The set of roots R is inside the character lattice $X(T)$ of T. For each root $\alpha \in R$, $-\alpha \in R \subset X(T)$ is the opposite root, and there is a root homomorphism $x_\alpha : \mathbb{G}_a \to G$. Let the subgroup $U(\alpha)$ be the image of x_α.

Let $\Delta \subset R$ be the subset of simple roots.

The positive roots are those roots which are also roots of B.

For every subset $I \subset R$, define $U(I)$ to be the subscheme of G that is the scheme-theoretic image of the product morphism $(\prod_{\alpha\in I} x_\alpha)(\mathbb{G}_a^{\#I}) \subset G$. Note that $U(I)$ may not be a subgroup of G.

The unipotent radical $R_u(Q)$ of a reduced parabolic Q is $U(I)$ for a unique subset $I \subset R$. In this case we also write $Q = P_I$. The opposite unipotent $R_u^-(Q)$ is $U(-I)$.

Note: our notation P_I differs from the standard textbooks, e.g. [Spr98], [Jan03], and [Mil17] : in those books our P_I is their $P_{\Delta\setminus\pm I}$. We opt for the lighter notation because we will be using the symbols P_I and $U(I)$ a lot.

Let $\mathbb{G}_{a,n} := \operatorname{Spec}(k[\epsilon]/(\epsilon^{p^n}))$ be the additive infinitesimal group schemes, and $U(\alpha, n) := Image(x_\alpha(\mathbb{G}_{a,n}))$.

Define $\mathbb{G}_{a,\infty} := \mathbb{G}_a$ and $U(\alpha, \infty) := U(\alpha)$.

Let $L_I = T \cdot U(R \setminus \pm I)$ be the Levi factor of P_I. Let $W^I \subset W$ be the Weyl group for (L_I, T). Let $W_I \subset W$ be the sub-poset consisting of the longest representatives for the classes in the double coset $W^I \backslash W / W^I$.

For a subset $I \subset R$ and a function $\boldsymbol{n} : I \to \mathbb{N}_{>0} \cup \{\infty\}$, $\boldsymbol{n}(\alpha) = n_\alpha$, define the subscheme of G

$$U(I, \boldsymbol{n}) := \prod_{\alpha \in I} x_\alpha(\mathbb{G}_{a,n_\alpha}) = \prod_{\alpha \in I} U(\alpha, n_\alpha).$$

For $w \in W_I$, define the subscheme of G

$$U(w(I, \boldsymbol{n})) := \prod_{\alpha \in I} x_{w(\alpha)}(\mathbb{G}_{a,n_\alpha}) = \prod_{\alpha \in I} U(w(\alpha), n_\alpha).$$

Bruhat Decompositions

Recall a reduced scheme X is *paved by affine spaces* if there exists a sequence of closed subschemes $\emptyset \subset X_0 \subset X_1 ... \subset X_n = X$ so that each $X_i \setminus X_{i-1}$ is isomorphic to a disjoint union of affine spaces $\mathbb{A}^{n_i}$ for some $n_i \in \mathbb{N}$.

Given any parabolic P of G, an affine paving of G/P is given by the Bruhat decomposition and the Bruhat order:

$$G/P = \coprod_{w \in W/W^I} BwP/P. \tag{2.1}$$

We call ${}^B X'_P(w) = BwP/P$ (resp. ${}^B X_P(w) = \overline{BwP/P}$) the Schubert cell (resp. Schubert varieties) of G/P corresponding to w.

There is another affine paving of G/P:

$$G/P = \coprod_{w \in W_I} P_{red} wP/P. \tag{2.2}$$

We denote $X'_P(w) = P_{red} wP/P$ and $X_P(w) = \overline{P_{red} wP/P}$.

In this thesis, we consider both decompositions above.

Name of the Exotic Flag Varieties

About the name of G/P with P nonreduced: G/P has been originally called the varieties of unseparated flag (vufs) by Haboush, Lauritzen, and Wenzel, e.g. in

[HL93], and the projective pseudo-homogeneous spaces in [Sri17]. However, G/P is a separated variety and the action of G on G/P is transitive. To avoid confusion, we do not use either of the two names. In [Tot19] and [Zha22], it is called homogeneous varieties with non-reduced stabilizer groups. In this thesis, we keep this name, and also use the name exotic flag varieties, as it is less wordy.

2.2 Basic Structures

Proposition 2.2.1. *With the settings in 2.1,*

1. *For every parabolic P, reduced or not, in G, P_{red} is a parabolic subgroup of G, the unipotent radical of P_{red} is $R_u(P_{red}) = U(I)$ for a unique subset $I \subset R$, i.e., $P_{red} = P_I$. Furthermore, $P = P_I \cdot (P \cap U(-I))$, i.e., P is the image of the multiplication morphism over k, $P_I \times (P \cap U(-I)) \to G$, where $P \cap U(-I)$ is the scheme-theoretic intersection.*

2. $$P \cap U(-I) = U(J, \boldsymbol{n}) \cong \prod_{\beta \in J} Spec(k[T]/(T^{p^{n_\beta}})),$$

   for a uniquely determined subset $J \subset -I$ and uniquely determined $n_\beta < \infty$ for each $\beta \in J$.

Proof. When k is algebraically closed and $p > 3$, both items above were proved in [Wen93, Th. 4, Th. 10(i)]. Our proof is similar in spirit.

Item (2) follows from item (1) the same way as in the argument from Lemma 5 to Theorem 10 (i) in [Wen93]. In fact, the argument there over algebraically closed fields holds verbatim over any field. Therefore it is enough to show item (1).

Because k is perfect, P_{red} is a reduced parabolic subgroup scheme of G [Mil17, Cor. 1.39], therefore $P_{red} = P_I$ for a unique subset $I \subset R$ [Mil17, Th. 21.91].

Clearly, P contains $P_I \cdot (P \cap U(-I))$ as a subscheme. If P is a subscheme of $P_I \cdot U(-I)$, then P is a subscheme of $P_I \cdot (P \cap U(-I))$. Hence suffices to show P is a subscheme of $P_I \cdot U(-I)$.

By [Jan03, 162], $P_I \cdot U(-I)$ is an open and dense subscheme of G, with a complement closed subscheme $Z \subset G$.

If P is not a subscheme of $P_I.U(-I)$, then the scheme $P \cap Z$ is not empty, thus $(P_I \cap Z_{red})_{red} = (P \cap Z)_{red}$ [Gro60, Sec. I-5.1.8] is nonempty. Therefore $P_I \cap Z \neq \emptyset$, which contradicts the definition of Z.

□

Remark 2.2.2. Let us keep the notation from Prop. 2.2.1. When k has characteristic $p > 3$ or when G is simply laced, it is proved in [Wen93, Th. 10.(iii)] that

$$n_\beta = \min\{n_\delta |\ \delta \in \Delta \cap J,\ \langle \beta^\vee, \delta \rangle \neq 0\}, \tag{2.3}$$

where $\beta^\vee \in X^\vee(T)$ is the dual root of β in the cocharacter lattice. Let

$$\mathcal{S} := \{\boldsymbol{n} : \Delta^- \to \mathbb{N}_{>0} \cup \{\infty\} |\ \boldsymbol{n} \text{ is not constant } \},$$

where Δ^- is the set of negative simple roots. The equality (2.3) means that for a fixed Borel subgroup, there is a bijection $f : \mathcal{S} \xrightarrow{\sim} \mathcal{P}$, where $\mathcal{P}$ is the set of nonreduced parabolics containing B.
When $p \leq 3$ it is shown in [Wen93, Rmk. 15] that in general f is only an injection.

Remark 2.2.3. When k is imperfect, there exists an algebraic subgroup scheme $H \subset \mathbb{G}_a^n$ over k whose reduced part H_{red} is not a group [Gab70, Sec. VI_A.1.3.2]. This leads to a natural question of which we do not know the answer: does there exist a parabolic subgroup P of a linear algebraic group G so that P_{red} is not a group anymore?

From the general properties of quotients by subgroup schemes of a smooth algebraic group [Bri17, Thm. 2.7.2.(2)], we see that G/P is a smooth scheme and a homogeneous space. The natural map $\pi : G/P_{red} \to G/P$ is purely inseparable and the fiber of π over the identity flag $F_{id} \in G/P$ is the infinitesimal scheme P/P_{red} [Mil17, Rmk. 7.16.(a)].

The following proposition establishes the Bruhat decomposition for G/P when P is nonreduced. This can be seen as a manifestation of the slogan "topology is nice" that will be discussed in detail in Section 5.

We call the scheme-theoretic images under π of the Schubert cells (resp. varieties) of G/P_{red} as the Schubert cells (resp. varieties) of G/P, and denote them as

$${}^B X'_P(w) := \pi({}^B X'_{P_{red}}(w)),\ \ {}^B X_P(w) := \pi({}^B X_{P_{red}}(w)).$$

Proposition 2.2.4. *Let P be a nonreduced parabolic of G and let $\pi : G/P_{red} \to G/P$ be the natural quotient map. Then*

1. *The Schubert cells form an affine paving of G/P;*
2. *The integral Chow groups $Ch(G/P)$ and $Ch(G/P_{red})$ are isomorphic as abstract abelian groups.*

Proof. Both items are proved in [Lau97, Sec. 2.3] in the context that k is algebraically closed and $p > 3$. Over perfect fields, both items are proved in [Sri17, Lem. 5.1-2] following Lauritzen's idea. $\square$

The next proposition compares the ring structures of $Ch(G/P)$ and $Ch(G/P_{red})$. It is stated and proved in [Lau97, 6] over an algebraically closed field k. The same proof applies also in the case when k is perfect.

Let P, I and J be as in Proposition 2.2.1, we define $d_w := \sum_{\alpha \in w^{-1}(I) \cap J} n_\alpha$. Let $w_0 \in W/W^I$ be the longest element.

Proposition 2.2.5. *Let $P \subset G$ be a nonreduced parabolic. The pullback $\pi^* : Ch(G/P) \to Ch(G/P_{red})$ between the integral Chow rings is a ring embedding and the cokernel of π^* is a p^N-torsion abelian group. Moreover,*

$$\pi_*[{}^B X_{P_I}(w)] = p^{d_w}[{}^B X_P(w)], \quad \pi^*[{}^B X_P(w)] = p^{d_{w_0} - d_w}[{}^B X_{P_I}(w)].$$

For any $w \in W/W^I$, let $Y'_P(w)$ be the $U(-I)$-orbit of the T-fixed point wP/P. Let $Y_P(w)$ be the closure of $Y'_P(w)$. When P is reduced, $Y'_P(w)$ (resp. $Y_P(w)$) is called the opposite Schubert cell (resp. variety). It is easy to see that $Y_P(w)$ is the scheme theoretic image of $Y_{P_{red}}(w)$ under the natural morphism $G/P_{red} \to G/P$. The following Proposition 2.2.6 can be proved in the same way as [KL79, Lemma A4.(b)].

Proposition 2.2.6. *Let $v \in W/W^I$ be such that $w > v$. We then have*

$${}^B X_P(w) \cap v Y'_P(id) \cong {}^B X'_P(v) \times (Y'_P(v) \cap {}^B X_P(w)).$$

Proposition 2.2.6 reflects another piece of the basic structure of a flag variety: Assume now that P is reduced. The open Richardson varieties in G/P are defined as

$$Z'_P(w, v) := {}^B X'_P(w) \cap Y'_P(v).$$

For each T-fixed point $vP/P \in G/P$, the big cell around it is $vU(-I)P/P = vY'_P(id)$. Intersecting with a Schubert variety, we have that the big cell of a Schubert variety ${}^B X_P(w) \cap v Y'_P(id)$ at a T-fixed point vP/P is isomorphic to the direct product of a Schubert cell and a slice through it, which lies between an open Richardson variety and its closure [Bri05, Prop. 1.3.5]. Proposition 2.2.6 says that we have a similar decomposition for G/P with P nonreduced.

Chapter 3

Exotic BSDH Varieties: Definitions

We assume the set up in 2.1 and 2.2.1. In particular, the base field k is perfect.

Let $w_\bullet$ be a sequence of elements $w_1, ..., w_r \in W$. In this chapter, we introduce the protagonist of this paper: the Bott-Samelson-Demazure-Hansen (BSDH) variety $X_P(w_\bullet)$ when the parabolic P is nonreduced.

3.1 Classical BSDH varieties

When P is reduced, the BSDH varieties are well known and thoroughly studied in the literature. Here we give a quick review of two equivalent definitions of them, following the expositions in [Jan03, §II.13] and [dCHL18, §4.1].

There are two kinds of classical BSDH varieties, corresponding to the two affine pavings (2.1) and (2.2) of the flag varieties.

B-BSDH varieties

The first one, which we call as the B-BSDH variety ${}^BX_P(w_\bullet)$, is defined as the closed subscheme of $(G/B)^{r-1} \times (G/P)$ whose functor of points are of the form $(g_1B/B, g_2B/B, ..., g_rP/P)$ such that $g_0 = id$, the point $g_{i-1}^{-1}g_i$ lies in $\overline{Bw_iB}$ for $i = 1, ..., r-1$, and $g_{r-1}^{-1}g_r$ lies in $\overline{Bw_rP}$.

The first projection $p_1 : {}^BX_P(w_\bullet) \to P_1/B$ is a Zariski-locally trivial fibration with fiber ${}^BX_P(w_2, ..., w_r)$. Using the first projection iteratively, we can realize ${}^BX_P(w_\bullet)$ as an iterated fibration with fibers isomorphic to $P_2/B, ..., {}^BX_P(w_r)$.

In particular, we have that ${}^BX_P(w_\bullet)$ is an integral scheme. When $r = 1$, we have that ${}^BX_P(w_1)$ is the same as the subvariety ${}^BX_P(w_1) = \overline{Bw_1P/P}$ of G/P as in (2.2).

Given a a reduced expression $w = s_1...s_r$ in terms of the simple transpotisions $s_1, ..., s_r \in W$, we can define ${}^BX_P(s_\star)$ equivalently as the quotient:

$$
{}^BX_P(s_\bullet) := P_1 \times^B P_2 \times^B \cdots \times^B {}^BX_P(s_r), \tag{3.1}
$$

where $P_i = \overline{Bs_iB}$ is the parabolic group corresponding to s_i. The assignment $(g_1, ..., g_rP/P) \mapsto (g_1B/B, g_1g_2B/B, ..., g_1...g_rP/P)$ gives the equivalence of the two definitions of ${}^BX_P(w_\bullet)$. In this case, we have that ${}^BX_P(s_\bullet)$ is an interated $\mathbb{P}^1$-fibrations, thus a smooth variety. Moreover, in this case, the last projection ${}^BX_P(s_\bullet) \to G/P$ is birational onto the Schubert variety ${}^BX_P(w)$, thus giving a resolution of singularity of the latter.

P-BSDH varieties, which are more important in this thesis

The second one, which we call as the P-BSDH variety $X_P(w_\bullet)$, is defined as the closed subscheme of $(G/P)^r$ whose functor of points are of the form $(g_1P/P, ..., g_rP/P)$ such that $g_0 = id$, the point $g_{i-1}^{-1}g_i \in \overline{Pw_iP}$ for $i = 1, ..., r$.

Using the first projections iteratively, again we see that $X_P(w_\bullet)$ is an interated Zariski-locally trivial fibration with fibers isomorphic to $X_P(w_i)$'s.

We can still consider the last projection $p_r : X_P(w_\bullet) \to (G/P)$. Its scheme-theoretic image is a Schubert variety $X_P(w_\star)$, where $W_I \ni w_\star := w_1 \star ... \star w_r$ is the Demazure product as defined in [dCHL18, §4.2]. The morphism $p_r : X_P(w_\bullet) \to X_P(w_\star)$ is an example of a convolution morphism, to be studied in more detail in Chapter 5.

Caution: In this thesis, we emphasize on the classical and exotic P-BSDH varieties. In particular, all the results in Chapters 4 and 5 are on the exotic P-BSDH varieties. We do this partly because we are following the presentation of the BSDH varieties in [dCHL18, §4]. However, we note that very little is lost after this restriction: In Chapter 4, we mainly provide some wild examples of exotic P-BSDH varieties, but these examples are all special cases of B-BSDH varieties because in these examples, we always have $P_{red} = B$. In Chapter 5, we prove some topological results for the convolution morphisms between P-BSDH varieties. Such results have been shown in [dCHL18] for classical P-BSDH varieties and their infinite-dimensional analogues. In both of our work and [dCHL18], convolution morphisms between B-BSDH varieties are not defined. However, it is not hard to define them provided what in our works. Moreover, the results for the convolution morphisms between B-BSDH varieties should be just as expected, so we omit them from our exposition.

3.2 Exotic BSDH varieties

When P is nonreduced, as we have seen in Sec. 2.2, the Schubert varieties of G/P are defined as the scheme theoretic image of the Schubert varieties of G/P_{red} under the natural universal homeomorphism $\pi : G/P_{red} \to G/P$. We define the BSDH varieties similarly:

Definition 3.2.1 (BSDH Varieties)**.** Let P be a nonreduced parabolic of G. For $w_\bullet = (w_1, ..., w_r) \in W_I^r$, we define the P-BSDH variety $X_P(w_\bullet)$ to be the scheme theoretic image of the classical BSDH variety $X_{P_{red}}(w_\bullet) \subset (G/P_{red})^r$ under the natural morphism $\pi^r : (G/P_{red})^r \to (G/P)^r$.

We define the B-BSDH varieties ${}^B X_{P_{red}}(w_\bullet)$ to be the scheme theoretic image of the classical B-BSDH variety ${}^B X_{P_{red}}(w_\bullet) \subset (G/B)^{r-1} \times (G/P_{red})$ under the natural morphism $\mathrm{id}^{r-1} \times \pi : (G/B)^{r-1} \times (G/P_{red}) \to (G/B)^{r-1} \times (G/P)$.

We immediately have the integrality of $X_P(w_\bullet)$ and the characterization of its closed points as in the classical case:

Proposition 3.2.2. *Let P be a not necessarily reduced parabolic.*

1. *The P-BSDH variety $X_P(w_\bullet)$ is an integral closed subscheme of $(G/P)^r$ that is universally homeomorphic to $X_{P_{red}}(w_\bullet)$;*

2. *A closed point $(a_1, ..., a_r)$ of $(G/P)^r$, with residue field k', is in $X_P(w_\bullet)$ if and only if there exists k'-points $g_1, ..., g_r$ of G so that $a_i = g_iP/P$ and $g_{i-1}^{-1}g_i \in \overline{P_{red}w_iP_{red}}(k') = \overline{Pw_iP}(k')$, where $i = 1, ..., r$ and $g_0 := id$ is the identity element of G.*

3. *The B-BSDH variety ${}^B X_P(w_\bullet)$ is an integral closed subscheme of $(G/B)^{r-1} \times (G/P)$ that is universally homeomorphic to ${}^B X_{P_{red}}(w_\bullet)$;*

4. *A closed point $(a_1, ..., a_r)$ of $(G/B)^{r-1} \times (G/P)$, with residue field k', is in ${}^B X_P(w_\bullet)$ if and only if there exists k'-points $g_1, ..., g_r$ of G so that $a_{i<r} = g_iB/B$, $a_r = g_rP/P$, $g_{i-1}^{-1}g_i \in \overline{Bw_iB}(k')$ for $i < r$, and $g_{r-1}^{-1}g_r \in \overline{P_{red}w_iP_{red}}(k') = \overline{Pw_iP}(k')$, where $g_0 := id$ is the identity element of G.*

Proof. (1) As π^r is proper, the underlying space of $X_P(w_\bullet)$ is the image of the underlying space of $X_{P_{red}}(w_\bullet)$ under π^r. Therefore $X_{P_{red}}(w_\bullet)$ is universally homeomorphic to $X_P(w_\bullet)$ via π^r. Since $X_{P_{red}}(w_\bullet)$ is irreducible, we have that $X_P(w_\bullet)$ is also irreducible. By [Sta23, Lem. 29.6.7], the scheme theoretic image of a reduced scheme is reduced. Hence $X_P(w_\bullet)$ is also reduced, as $X_{P_{red}}(w_\bullet)$ is.

(2) Let $(a_1, ..., a_r)$ be a closed point of $X_P(w_\bullet)$ with residue field k'. Let $(b_1, ..., b_r)$ be the closed point of $X_{P_{red}}(w_\bullet)$ over $(a_1, ..., a_r)$. As π^r is finite and purely inseparable, the residue field k'' of $(b_1, ..., b_r)$ is a finite purely inseparable extension of k'. Since k is perfect, we have that k' is also perfect. Thus $k'' = k'$. By the equality $(\overline{Pw_iP})_{red} = \overline{P_{red}w_iP_{red}}$ of reduced closed subschemes of G, we have that $\overline{Pw_iP}$ and $\overline{P_{red}w_iP_{red}}$ have the same K-points, for any field extension $K \supset k$. The rest follows from the characterization of closed points of $X_{P_{red}}(w_\bullet)$.

(3) and (4) follows similarly. ∎

The Prop. 3.2.4 below gives us another way to define the BSDH varieties, no matter whether P is reduced or not.

Definition 3.2.3 (P-BSDH Functor)**.** Let P be a not necessarily reduced parabolic. We define a k-functor $\mathcal{X}_P(w_\bullet)$ to be the subfunctor of $(G/P)^r$ that sends every k-algebra A to the subset $\mathcal{X}'_P(w_\bullet)(A)$ of $(G/P)^r(A) = ((G/P)(A))^r$ that consists of the points $(a_1, ..., a_r)$ such that there exist $g_0 \in P_{red}(A)$, $g_1, ..., g_r \in G(A)$ satisfying that $g_iP/P = a_i$ and $g_{i-1}^{-1}g_i \in \overline{(P_{red}w_iP_{red})}(A)$ for $i = 1, ..., r$.

Proposition 3.2.4. *Let P be a not necessarily reduced parabolic. The BSDH variety $X_P(w_\bullet)$ is the smallest closed subscheme of $(G/P)^r$ that contains $\mathcal{X}_P(w_\bullet)$ as a subfunctor, i.e., the inclusion of k-functors $\mathcal{X}_P(w_\bullet) \hookrightarrow (G/P)^r$ factors as $\mathcal{X}_P(w_\bullet) \xrightarrow{i_1} X_P(w_\bullet) \xrightarrow{i_2} (G/P)^r$, where i_1 is an inclusion of k-functors and i_2 is a closed immersion of schemes.*

Proof. When P is reduced, this characterization of $X_P(w_\bullet)$ is indeed one of the standard definitions of the BSDH varieties. For example, let us refer to [Jan03, Sec. II.13]. There $X_{P_{red}}(w_\bullet)$ is defined to be the big fppf sheaf associated with the image functor, under the quotient morphism $G^r \to (G/P_{red})^r$, of the closed subscheme $V(w_\bullet)$ of G^r, whose functor of points sends a k-algebra A to the set

$$V(w_\bullet)(A) := \{(g_1, ..., g_r) \in G^r(A) |\ \ g_{i-1}^{-1}g_i \in \overline{P_{red}w_iP_{red}}(A)\},$$

where g_0 is the identity element. By definition, the image functor of $V(w_\bullet)$ is just $\mathcal{X}_{P_{red}}(w_\bullet)$. Since $V(w_\bullet)$ is P-invariant, the big fppf sheaf $X_{P_{red}}(w_\bullet)$ is indeed a closed subscheme of $(G/P_{red})^r$ by [Jan03, Sec. I.5.21].

When P is nonreduced, the scheme $X_P(w_\bullet)$ is defined to be the scheme theoretic image of $X_{P_{red}}(w_\bullet)$ under π^r, it hence suffices to show that $\mathcal{X}_P(w_\bullet)$ is the image functor, i.e., $\mathcal{X}_P(w_\bullet)(A) = \pi(A)\mathcal{X}_{P_{red}}(w_\bullet)(A)$. To see this, take any $(a_1, ..., a_r) \in \mathcal{X}_P(w_\bullet)(A)$, take the corresponding g_i in Def. 3.2.3, then the A-point

$$(g_1P_{red}/P_{red}, ..., g_rP_{red}/P_{red})$$

of $(G/P_{red})^r$ is in $\mathcal{X}_{P_{red}}(w_\bullet)$ because of the conditions satisfied by the g_i's. Since $\pi(A)(g_iP_{red}/P_{red}) = g_iP/P$, we have that $(a_1, ..., a_r) = \pi(A)(g_iP_{red}/P_{red})_i$. Therefore $\pi(A)(\mathcal{X}_{P_{red}}(w_\bullet)(A)) \supset \mathcal{X}_P(w_\bullet)(A)$. On the other hand, if $(b_1, ..., b_r) \in \mathcal{X}_{P_{red}}(w_\bullet)(A)$, then take the corresponding g_i's in Def. 3.2.3. Again because $\pi(A)(g_iP_{red}/P_{red}) = g_iP/P$, we have $\pi(A)(b_1, ..., b_r) \in \mathcal{X}_P(w_\bullet)(A)$. $\square$

Some basic information about k-functors and big fppf sheaves are included in the Appendix §6. In Example 6.2.6, we show that the big fppf-sheaf associated with an image k-functor in general is not a scheme.

Remark 3.2.5. In the proof above, we mentioned that the scheme $X_{P_{red}}(w_\bullet)$ is indeed the big fppf sheaf associated to the functor (presheaf) $\mathcal{X}_{P_{red}}(w_\bullet)$. However, Rmk. 4.2.9 below entails that, when P is nonreduced, in general the big fppf sheaf $\underline{\mathcal{X}_P}(w_\bullet)$ associated to the functor $\mathcal{X}_P(w_\bullet)$ is not a scheme.

Indeed, if $\underline{\mathcal{X}_P}(w_\bullet)$ is a scheme, then by the universal property of fppf sheafification and Prop.3.2.4, we see that the inclusion of functors $\underline{\mathcal{X}_P}(w_\bullet) \hookrightarrow X_P(w_\bullet)$ gives rise to an isomorphism between schemes. By [Jan03, Sec. I.5.4.4], we have that for every k-algebra A,

$$\underline{\mathcal{X}_P}(w_\bullet)(A) = (G/P)^r(A) \cap \bigcup_B \pi^r(B) X_{P_{red}}(B),$$

where B ranges over all fppf-A-algebras. However, from Rmk. 4.2.9 below, we see that the morphism $X_{P_{red}}(w_\bullet) \to X_P(w_\bullet)$ in general is not flat. Therefore the A-points of $X_P(w_\bullet)$ in general contain some points in $\pi^r(B)X_{P_{red}}(B)$ where B is not a flat A-algebra. Therefore in general there is a strict inclusion of sets $X_P(w_\bullet)(A) \supsetneq \underline{\mathcal{X}_P}(w_\bullet)(A)$.

Remark 3.2.6. We can reproduce results similar to Def. 3.2.3, Prop. 3.2.4, and Rmk. 3.2.5 for B-BSDH functors by replacing $(G/P)^r$ by $(G/B)^{r-1} \times (G/P)$.

When P is reduced, we can define the BSDH varieties using the relative positions between two flags. For the rest of the section, which is not used in other parts of this thesis, we define a natural generalization of relative positions when P is nonreduced, and show that they define in general nonreduced schemes. Therefore, we cannot define BSDH varieties using relative relations when P is nonreduced, at least not using the natural definition of them below:

Definition 3.2.7 (Relative Positions)**.** Let P be a not necessarily reduced parabolic. For any k-algebra A, and A points $a_1, ..., a_r \in G/P(A)$, we define the relation $a_1 \xrightarrow{w_2} a_2 \xrightarrow{w_3} ... \xrightarrow{w_r} a_r$ (resp. $a_1 \xrightarrow{\leq w_2} a_2 \xrightarrow{\leq w_3} ... \xrightarrow{\leq w_r} a_r$) if there are $g_1, ..., g_r \in G(A)$ so that $g_i P/P = a_i$ and that for $i = 2, ..., r$, $g_{i-1}^{-1} g_i \in P_{red} w_i P_{red}(A) \subset G(A)$ (resp. $g_{i-1}^{-1} g_i \in \overline{P_{red} w_i P_{red}}(A)$).

Note that the subscheme $X'_{P_{red}}(w) = P_{red} w P_{red}/P_{red}$ of G/P_{red} can be defined as all the flags that are of relative position w to the identity flag $P_{red}/P_{red} \in G/P_{red}$.

The following proposition shows that when P is nonreduced, all the flags that are of relative position w to the identity flag form a scheme that is in general nonreduced. Therefore relative position is not very useful to define Schubert or BSDH varieties when P is nonreduced.

Proposition 3.2.8. *For every k-algebra A, we have an equality of sets*

$$\{a \in G/P(A) |\ P/P \overset{w}{—} a\} = P(A) \cdot wP/P.$$

Let PwP/P be the scheme theoretic image of $P \times wP/P$ under the action morphism $G \times G/P \to G/P$. Then we have that PwP/P in general is nonreduced.

Proof. Every element in $P(A) \cdot wP/P$ has the form $\epsilon pwP/P$ where $\epsilon \in U(J, \boldsymbol{n})(A)$, $p \in P_I(A)$. Because $P/P = \epsilon P/P$, in the notation in Def. 3.2.7, we can choose $g_1 = \epsilon$, $g_2 = \epsilon pw$, so that $g_1P/P = P/P$, $g_2P/P = a$ and $g_0^{-1}g_1 \in P_IwP_I(A)$.

For an example of nonreduced PwP/P. Let G and P be as in Example 4.2.1. Let $w = s_\alpha$. Then PwP/P is isomorphic to $U(-\alpha-\beta, 1) \cdot U(-\alpha)P/P$, which is not reduced. ☐

Chapter 4

Wild Geometry

In this chapter, we give multiple examples to demonstrate that the geometry exotic BSDH varieties and their associated natural morphisms is wildly distinct than the geometry of the classical ones.

We keep the notation in 2.1 and 2.2.1. Recall for $w_1, ..., w_r \in W_I$, the geometric Demazure product $w_\star := w_I \star ... \star w_r \in W_I$ is defined so that the image of the last projection $pr_r : X_{P_I}(w_\bullet) \to G/P_I$ is $X_{P_I}(w_\star)$. For a detailed discussion of the $\star$ operation, let us refer to [dCHL18, Sec. 4.2-3].

In this section, by BSDH varieties, we always mean P-BSDH varieties, see the discussion at the end of §3.1.

When the parabolic P is reduced, two morphisms among classical BSDH varieties are especially useful: the first projection $p_1 : X_P(w_1, ..., w_r) \to X_P(w_1)$ and the last projection $p_r : X_P(w_1, ..., w_r) \to X_P(w_\star)$. By studying the first projection, we see that $X_P(w_1, ..., w_r)$ is a Zariski locally trivial fibration with fiber isomorphic to $X_P(w_2, ..., w_r)$. In particular, if $P = B$ and $s_\star = s_1 \cdot ... \cdot s_r$ is a reduced expression in terms of simple reflections, then we have that $X_B(s_\bullet)$ is an iterated $\mathbb{P}^1$-fibration, and the last projection gives a resolution of singularities of the normal Schubert variety $X_B(s_\star)$.

The picture is very different when P is nonreduced:

In Ex. 4.2.1, Ex. 4.2.2 Ex. 4.2.6 and Ex. 4.2.12, we show that the fibers of the first and last projections are in general nonreduced. In Ex. 4.2.1, all the fibers are non-reduced and are isomorphic to each other. In Ex. 4.2.2, the general fibers are reduced while some special fibers are not. In Ex. 4.2.6, we give defining equations for the fibers. In Ex. 4.2.12, the fibers of the last projection are not reduced.

In Ex. 4.2.8 and Ex. 4.2.7 below, we give examples of non-normal Schubert and BSDH varieties. Explicit equations are provided for both examples using incidence varieties.

To explain these examples, we first need some general results about the fibers, included in §4.1 below.

4.1 General Results

For a not necessarily reduced parabolic P, let $I \subset R$ so that $P_{red} = P_I$.

Proposition 4.1.1 (Fibers of the First Projection p_1)**.** *Given any closed point gP/P in $X_P(w_1)$, the first projection $p_1 : X_P(w_1, ..., w_r) \to X_P(w_1)$ has fibers isomorphic* *to*

$$(gP \cap \overline{P_{red}w_1P_{red}}) \cdot X_P(w_2, ..., w_r),$$

i.e., the scheme theoretic image of $(gP \cap \overline{P_{red}w_1P_{red}}) \times X_P(w_2, ..., w_r)$ *under the morphism* $G \times (G/P)^{r-1} \to (G/P)^{r-1} : (g, a_2, ..., a_r) \mapsto (ga_2, ..., ga_r)$.

In particular, the largest reduced subscheme of a fiber of p_1 *is always isomorphic to* $X_P(w_2, ..., w_r)$.

Proof. Let k' be the residue field of the closed point gP/P. Let $\mathcal{X}_P(w_\bullet)$ be the image k-functor of $X_{P_{red}}(w_\bullet)$ under the morphism $\pi^r : (G/P_{red})^r \to (G/P)^r$ as defined in Def. 3.2.3. By Prop. 3.2.4, we have that $X_P(w_\bullet)$ is the smallest closed subscheme of $(G/P)^r$ such that the inclusion of k-functors $\mathcal{X}_P(w_\bullet) \hookrightarrow (G/P)^r$ factors as

$$\mathcal{X}_P(w_\bullet) \hookrightarrow X_P(w_\bullet) \hookrightarrow (G/P)^r,$$

where the first arrow is an inclusion of functors and the second arrow is the closed immersion of schemes. Similarly, let

$$\mathcal{I} := (gP \cap \overline{P_{red}w_1P_{red}}) \cdot \mathcal{X}_P(w_2, ..., w_r)$$

be the image k'-functor under the multiplication. We have that

$$(gP \cap \overline{P_{red}w_1P_{red}}) \cdot X_P(w_2, ..., w_r)$$

is the smallest closed subscheme of $(G/P)^{r-1}$ that contains $\mathcal{I}$ as a subfunctor. Therefore, it suffices to show that for any k'-algebra A, the preimage of $gP/P \in \mathcal{X}_P(w_1)(A)$ under $p_1(A) : \mathcal{X}_P(w_1, ..., w_r)(A) \to \mathcal{X}_P(w_1)(A)$ is

$$gP/P \times \mathcal{I}(A) \subset \mathcal{X}_P(w_1, ..., w_r)(A).$$

Below we only consider A-points of functors.

By Def. 3.2.3, the fiber consists of the points $(a_1, ..., a_r)$ such that there exists $g_1, ..., g_r \in G(A)$ with $g_1P/P = gP/P$, $g_1 \in \overline{P_{red}w_1P_{red}}(A)$, and that $g_{i-1}^{-1}g_i \in \overline{P_{red}w_iP_{red}}(A)$ for $i = 2, ..., r$. The first two conditions give that an A-point $g_1 \in G(A)$ can be a representative of the first factor of the fiber of p_1 over gP/P if and only if $g_1 \in gP \cap \overline{P_{red}w_1P_{red}}$. The third condition gives that, for example, an element $g_2 \in G(A)$ can be a representative of the second factor of $p_1^{-1}(gP/P)$ if and only if

$$g_2 \in (gP \cap \overline{P_{red}w_1P_{red}}) \cdot X_P(w_2),$$

and the representatives of the i-th factor for $i \geq 3$ are determined so iteratively. Therefore we have the first statement.

The second statement follows from the fact that the scheme

$$gP \cdot X_P(w_2, ..., w_r) = gU(J, \boldsymbol{n}) \cdot X_P(w_2, ..., w_r)$$

is an infinitesimal thickening of $X_P(w_2, ..., w_r)$. $\square$

As $X_P(w_2, ..., w_r)$ is invariant under the left multiplication by P_{red}, Prop. 4.1.1 tells us that to understand the fiber of the first projection p_1 over gP/P is the same as to understand $gP/P_{red} \cap X_{P_{red}}(w_1)$, which in turn is the fiber of $\pi : X_{P_{red}}(w_1) \to X_P(w_1)$ over gP/P. In Propositions 4.1.2 and 4.1.3 below, we give finer descriptions of the fibers of p_1 in terms of roots and Weyl groups.

Recall that we have the decomposition

$$X_P(w_1) = \coprod_{v \leq w, v \in W_I} X'_P(v).$$

In Proposition 4.1.2, we describe the fibers of p_1 over a point in the largest part $X'_P(w_1)$. In Proposition 4.1.3, we describe the fiber of p_1 over the smallest part $X_P(id) = P/P$ when $P_{red} = B$.

Proposition 4.1.2 (Finer Description of $p_1^{-1}(gP/P)$, I)**.** *If gP/P is in $X'_P(w_1) = P_{red}w_1P/P$, then the fiber of the first projection $p_1 : X_P(w_\bullet) \to X_P(w_1)$ at gP/P is isomorphic to*

$$(U(J, \boldsymbol{n}) \cap w_1^{-1}P_{red}w_1) \cdot X_P(w_2, ..., w_r).$$

In Prop. 4.1.3 below, let $P_{red} = B$, and $w_1 = s_{\alpha_1}...s_{\alpha_n}$ be a reduced expression of w_1 in terms of reflections s_{α_i} that exchanges the simple root α_i with its negative $-\alpha_i$. Let $\mathcal{U}$ be the scheme theoretic image of the multiplication $U(-\alpha_1)\times...\times U(-\alpha_n) \to G$ (the order of $U(-\alpha_i)$'s are fixed).

Proposition 4.1.3 (Finer Description of $p_1^{-1}(gP/P)$, II)**.** *The fiber of $p_1 : X_P(w_\bullet) \to X_P(w_1)$ over the identity flag $P/P \in X_P(w_1)$ is isomorphic to the scheme*

$$(U(J, \boldsymbol{n}) \cap \mathcal{U}) \cdot X_P(w_2, ..., w_r).$$

Proof of Proposition 4.1.2. The closed points of $X'_P(w_1)$ are identified with the closed points of $X'_{P_{red}}(w_1)$ via the universal homeomorphism $\pi : G/P_{red} \to G/P$. Therefore the closed point gP/P of $X'_P(w_1)$ has the form $u'wP/P$ with $u' \in P_{red}(k')$.

Since $X_P(w_2, ..., w_r)$ is invariant under the left multiplication by P_{red}, combined with Prop. 4.1.1, it suffices to show that we have the following identity of closed subschemes of G/P_{red}:

$$u'w_1P/P_{red} \cap \overline{P_{red}w_1P_{red}/P_{red}} = u' \cdot (U(w_1(J, \boldsymbol{n})) \cap P_{red}) \cdot w_1P_{red}/P_{red}. \quad (4.1)$$

Since $u'w_1P/P_{red}$ is an infinitesimal thickening of the closed point $u'w_1P_{red}/P_{red}$, which is in the interior $P_{red}w_1P_{red}/P_{red}$, we have that

$$u'w_1P/P_{red} \cap \overline{P_{red}w_1P_{red}/P_{red}} = u'w_1P/P_{red} \cap P_{red}w_1P_{red}/P_{red}. \quad (4.2)$$

It is then easy to see that the right hand sides of (4.1) and (4.2) agree. ☐

Proof of Proposition 4.1.3. From above we see that it suffices to show that

$$P/B \cap \overline{Bw_1B/B} = (U(J, \boldsymbol{n}) \cap \mathcal{U})B/B. \quad (4.3)$$

Since P/B is an infinitesimal thickening of the identity flag B/B, which is in the interior of the open and dense opposite Schubert cell $Y'_B(id) = U(-R^+)B/B$ as discussed in Prop. 2.2.6, we have

$$P/B \cap \overline{Bw_1B/B} = P/B \cap (U(-R^+)B/B \cap \overline{Bw_1B/B}). \quad (4.4)$$

The latter intersection is the closure of the open Richardson variety

$$Z_B(w_1, v) = X'_B(w_1) \cap Y'_B(id)$$

inside $Y'_B(id)$. We use the Deodhar decomposition to obtain a parametrization of the open Richardson variety. Let us refer to [MR04, (4.6)& Prop. 5.2] for a detailed discussion of Deodhar decomposition. What is useful for us is that [MR04, Prop 5.2] gives us that

$$Z_B(w_1, v) = x_{-\alpha_1}(k^*) \cdot ... \cdot x_{-\alpha_n}(k^*)B/B,$$

where $w_1 = s_{\alpha_1}...s_{\alpha_n}$ is a reduced expression of w_1 in terms of simple reflections, and k^* is the units in the base field k. Taking the closure in $Y'_B(-id)$, we have the (4.3) as desired. ☐

From the proof of Prop. 4.1.3 above, we can see that, by embedding the infinitesimal scheme gP/P_{red} into an open subscheme of G/P_{red} containing the point

gP_{red}/P_{red}, the problem of understanding the nonreducedness of the fibers of p_1 is related to the structure of Richardson varieties, as defined after Prop. 2.2.6, in G/P_{red}.

Below we use Propositions 4.1.2 and 4.1.3 to give examples of exotic phenomena related to the morphisms among BSDH varieties with nonreduced P.

4.2 Concrete Examples

In Ex. 4.2.1 below, we give an example where all the fibers of p_1 over k-points are isomorphic and nonreduced.

Example 4.2.1 (Nonreduced Fibers of p_1)**.** Let $G = SL_5$. Let $P_{red} = B$ be a fixed Borel subgroup. Let α, β, γ, and δ be the four positive simple roots labeling the four nodes in the Dynkin diagram A_4 from left to right respectively. Let $P = U(-\beta, 1) \cdot B$, i.e., $U(J, \boldsymbol{n}) = U(-\beta, 1)$. We consider the first projection

$$p_1 : X_P(s_\alpha s_\beta,\ s_\delta) \to X_P(s_\alpha s_\beta).$$

By Prop. 4.1.2, over a general point in the Schubert cell $Bs_\alpha s_\beta P/P$, the fiber of p_1 is isomorphic to

$$(U(-\beta, 1) \cap U(s_\beta s_\alpha(R^+))) \cdot X_P(s_\delta) = U(-\beta, 1) \cdot X_P(s_\delta).$$

As $U(-\beta, 1)$ is not contained in the stabilizer of points of $X_P(s_\delta)$, we have that $U(-\beta, 1) \cdot X_P(s_\delta)$ is nonreduced.

Prop. 4.1.2 also entails that the fiber of p_1 over the identity flag P/P is isomorphic to

$$(U(-\beta, 1) \cap U(\{-\alpha, -\beta\})) \cdot X_P(s_\delta) = U(-\beta, 1) \cdot X_P(s_\delta),$$

which is isomorphic to the general fiber.

We now consider the fibers over points in the Schubert cells $Bs_\alpha P/P$ and $Bs_\beta P/P$. Over $Bs_\alpha P$, it suffices to determine

$$s_\alpha P/B \cap X_B(s_\alpha s_\beta) = s_\alpha P/B \cap s_\alpha U(-R^+)B/B \cap X_B(s_\alpha s_\beta).$$

By Prop. 2.2.6 or [Bri05, Prop. 1.3.5], we have that

$$s_\alpha U(-R^+)B/B \cap X_B(s_\alpha s_\beta) = Bs_\alpha B/B \times (U(-R^+)s_\alpha B/B \cap X_B(s_\alpha s_\beta)).$$

By Deodhar decomposition [MR04, (4.6)& Prop. 5.2], the second factor is

$$U(-R^+)s_\alpha B/B \cap X_B(s_\alpha s_\beta) = s_\alpha U(-\beta)B/B.$$

Therefore we have that

$$s_\alpha P/B \cap X_B(s_\alpha s_\beta) = s_\alpha U(-\beta,1)B/B \cap s_\alpha U(\{-\alpha,-\beta\})B/B = s_\alpha U(-\beta,1)B/B.$$

Therefore we have that the fiber over a closed point in $Bs_\alpha P/P$ is isomorphic to $U(-\beta,1)\cdot X_P(\delta)$.

Over the Schubert cell $Bs_\beta P/P$, we use Prop. 2.2.6 and Deodhar decomposition again to obtain that

$$s_\beta P/B \cap X_B(s_\alpha s_\beta) = s_\beta U(-\beta,1)B/B \cap s_\beta U(\{-\beta,-\alpha-\beta\})B/B = s_\beta U(-\beta,1)B/B.$$

In conclusion, we see that the fibers of p_1 over all the k-points of $X_P(w_1)$ are isomorphic to an infinitesimal thickening of $X_P(\delta)$, $U(-\beta,1)\cdot X_P(\delta)$.

The calculation in Ex. 4.2.1 shows that, in order to determine the fiber of $p_1 : X_P(w_\bullet) \to X_P(w_1)$ over a point in the Schubert cell BvP/P with $v < w$, we can always first use the identity

$$vP/P_{red} \cap X_{P_{red}}(w_1) = vP/P_{red} \cap (vU(-R^+)P_{red}/P_{red} \cap X_{P_{red}}(w_1)).$$

The latter intersection is then isomorphic to $X'_{P_{red}}(v) \times Z$, where Z is the closure of the open Richardson variety (defined after Prop. 2.2.6) $Z'_{P_{red}}(w_1, v)$ inside $Y'_{P_{red}}(v)$. The part $X'_{P_{red}}(v)$, combined with the first half of the proof of Prop. 4.1.2, entails that the fiber is an infinitesimal thickening of

$$(U(J,\boldsymbol{n}) \cap U(v^{-1}(I)))\cdot X_P(w_2,...,w_r),$$

which may already be nonreduced. The study of the fiber of p_1 is then reduced to the study of closures of open Richardson varieties in opposite Schubert cells. From this procedure, we see that the fibers of $p_1 : X_P(w_\bullet) \to X_P(w_1)$ (or $\pi : X_{P_{red}}(w_1) \to X_P(w_1)$) have different descriptions over different parts of $X_P(w_1)$.

Example 4.2.2 below is an example where the fibers of $p_1 : X_P(w_\bullet) \to X_P(w_1)$ are not isomorphic to each other:

Example 4.2.2 (Non-isomorphic fibers of p_1)**.** Let G, P_{red}, and the roots be as in

Example 4.2.1. Let $P = U(-\alpha, 1) \cdot B$. We consider the first projection

$$p_1 : X_P(s_\alpha s_\beta, s_\delta) \to X_P(s_\alpha s_\beta).$$

Running the calculation as in Example 4.2.1 again, we have that the fibers of p_1 over the fixed points of $X_P(s_\alpha s_\beta)$ are:

$$p_1^{-1}(s_\alpha s_\beta P/P) = X_P(s_\delta); \qquad p_1^{-1}(s_\alpha P/P) = U(-\alpha, 1) \cdot X_P(s_\delta);$$
$$p_1^{-1}(s_\beta P/P) = X_P(s_\delta); \qquad p_1^{-1}(P/P) = U(-\alpha, 1) \cdot X_P(s_\delta).$$

In the Example 4.2.2, we have reduced general fibers and nonreduced special fibers (over $X_P(s_\alpha)$). We then have the natural question: Can the situation be reversed? The following general lemma gives a negative answer. We thank Andres Fernandez Herrero for pointing out a mistake in the previous version and suggesting a formulation and proof for the current version.

Lemma 4.2.3. Let $f : X \to Y$ be a proper morphism with finite presentation whose geometric fibers are of the same dimension. Assume that all the geometric fibers of f are irreducible. Then the set

$$\{y \in Y | \; X_y \text{ is geometrically reduced}\} \tag{4.5}$$

is open in Y.

Proof. We first reduce to the case where Y is the spectrum of a discrete valuation ring as in the proof of [Sta23, Tag 0C0E]: We may assume that Y is affine and write it as a cofiltered limit of affine schemes of fintie type over $\mathbb{Z}$. Therefore we have that f is the base change of a morphism $f_0 : X_0 \to Y_0$ where Y_0 is affine of finite type over $\mathbb{Z}$ and f_0 is proper, of finite presentation and has equidimensional geometric fibers, see [Sta23, Tag 01ZM,Tag 081F, Tag 0EY2]. Given any geometric point in $y_0 \in Y_0(k)$, there is a field extension k'/k such that the induced k' point of Y_0 factors through $Y \to Y_0$. Therefore, by [Sta23, Tag 054P, Tag 038I], we have that the geometric fibers of f_0 are also irreducible. By [Sta23, Tag 0567], forming the set (4.5) commutes with base change. Therefore, we can replace f and f_0. In particular, we can assume that Y is Noetherian. Since the set (4.5) is constructible [Sta23, Tag 0579], it suffices to show that (4.5) is stable under generalization [Sta23, Tag 0542]. By [Sta23, Tag 054F], we can replace Y with the spectrum of a discrete valuation ring R.

Let s be the special point of Y and η be the generic point of Y. We know that X_s is geometrically reduced and we want to show that X_η is geometrically reduced. Let $\overline{X_\eta}$ be the scheme theoretic closure of X_η in X. We now show that $\overline{X_\eta}$ is flat over Y. By [Har13, Prop. III.9.7], we are reduced to show that every associated point of $\overline{X_\eta}$ lies in X_η. We are then reduced to the case where X is the spectrum of an R-algebra A. Let π be the uniformizer of R. The ideal of $\overline{X_\eta}$ is $K := ker(A \to A[\pi^{-1}])$. The inclusion $X_\eta \to \overline{X_\eta}$ is given by the injective ring map $A/K \hookrightarrow A[\pi^{-1}]$. Let $\mathfrak{p}$ be an associated prime of A/K. We see that $\mathfrak{p}A[\pi^{-1}]$ is an associated prime of $A[\pi^{-1}]$. Furthermore, $(\mathfrak{p}A[\pi^{-1}]) \cap (A/K)$ consists of elements in $\mathfrak{p}A[\pi^{-1}]$ with nonnegative valuation, thus recovers $\mathfrak{p}$. We have thus shown that $\overline{X_\eta}$ is flat over Y.

By upper-semicontinuity of fiber dimension [Sta23, Tag 0D4I] and our equidimensional assumption, we have that $dim((\overline{X_\eta})_s) = dim(X_\eta) = dim(X_s)$. Therefore $(\overline{X_\eta})_s$ is a closed subscheme of the integral scheme X_s with maximal dimension. Thus we have an equality $(\overline{X_\eta})_s = X_s$. In particular, we have that $(\overline{X_\eta})_s$ is geometrically reduced. Using the flatness from the last paragraph and [Sta23, 0C0E], we have that X_η is geometrically reduced. $\square$

Remark 4.2.4. In Lemma 4.2.3, the condition that all geometric fibers of f are irreducible is necessary: Consider the example where $Y = Spec(k[x])$ and $X \subset Y \times_k \mathbb{P}^2_{y:z:w}$ is defined by the ideal

$$(y - w) \cdot (z, x(xw - y^2)^2).$$

The projection $X \to Y$ is visibly proper with finite presentation. When $x \neq 0$, the fiber consists of two nonreduced points on the line $z = 0$ together with the line $y = w$. When $x = 0$, the fiber consists of the line $z = 0$ and the line $y = w$. Therefore, all the fibers of f has dimension 1, but its generic fiber contains two nonreduced points while the special fiber is reduced.

Remark 4.2.5. We also provide a "minimal" example to demonstrate the necessaity of the irreducible fiber condition: Let $Y = Spec(k[t])$ and $X \subset Y \times_k \mathbb{P}^2_{x:y:z}$ be defined by the ideal

$$(tx^2, xy).$$

Over the locus where t is invertible, the fiber in X is a line with an embedded point, thus nonreduced. When $t = 0$, the fiber is just two lines, thus reduced.

In the examples above, no explicit equations are used. In Ex. 4.2.6 below, we give another example of the nonreduced fibers of $p_1 : X_P(w_\bullet) \to X_P(w_1)$ using defining equations.

Example 4.2.6 (Equations for Nonreduced Fibers)**.** Let $G = SL_3$, and let $P_{red} = B$, the fixed Borel subgroup. Let α and β be the two simple positive roots. Let $P = U(-\alpha, 1) \cdot P_{red}$. By [LM97, 231], we have an embedding

$$X_B(s_\alpha, s_\beta) \hookrightarrow Gr(1,3) \times Gr(2,3) \cong \mathbb{P}^2 \times \check{\mathbb{P}}^2$$

given by the map $(g_1B, g_2B) \mapsto (g_1\langle e_1\rangle, g_2\langle e_1, e_2\rangle)$, where each e_i is a standard coordinate of $\mathbb{A}^3$. The image satisfies the relation

$$\langle e_1, e_2\rangle \supset g_1\langle e_1\rangle \subset g_2\langle e_1, e_2\rangle.$$

Let the homogeneous coordinates be $(x : y : z;\ a : b : c)$, then the defining equations for the image of i are $ax + by + cz = 0$ and $z = 0$.

Similarly, we have an embedding

$$X_P(s_\alpha, s_\beta) \hookrightarrow Frob(Gr(1,3)) \times Gr(2,3) \cong \mathbb{P}^2 \times \check{\mathbb{P}}^2$$

given by the map

$$(g_1P, g_2P) \mapsto (Frob(g_1\langle e_1\rangle), g_2\langle e_1, e_2\rangle).$$

The image satisfies the relation

$$\langle e_1, e_2\rangle \supset g_1\langle e_1\rangle \subset Frob(g_2\langle e_1, e_2\rangle).$$

The defining equations are $z = 0$ and $a^px + b^py + c^pz = 0$.

Restricting the embedding above to the first factor, we have an embedding $X_P(s_\alpha) \hookrightarrow Frob(Gr(1,3))$ with image isomorphic $\mathbb{P}^1$, defined by $z = 0$. The first projection

$$p_1 : X_P(s_\alpha, s_\beta) \to X_P(s_\alpha)$$

is the first projection $\mathbb{P}^2 \times \check{\mathbb{P}}^2 \to \mathbb{P}^2$ restricted to $X_P(s_\alpha, s_\beta)$. For every point with coordinate $(x_0 : y_0 : 0) \in X_P(s_\alpha)$, the fiber of p_1 is the subscheme in $\check{\mathbb{P}}^2$ defined by $x_0a^p + y_0b^p + 0c^p = 0$. Since the field k is perfect and $char(k) = p$, we have that the fiber is defined by

$$(x_0^{1/p}a + y_0^{1/p}b)^p = 0,$$

hence nonreduced.

The example above is also an example of non-normal BSDH varieties when P is nonreduced:

Example 4.2.7 (Non-normal BSDH Variety)**.** Take the $X_P(s_\alpha, s_\beta)$ as in Ex. 4.2.6. It is the subvariety of $\mathbb{P}^2 \times \check{\mathbb{P}}^2$ defined by the homogeneous ideal $\langle a^p x + b^p y, z\rangle$. On the chart $x \neq 0$ $c \neq 0$, the variety is defined by the ideal $\langle a^p + b^p y, z\rangle$. By computing the Jacobian, we see that the singular locus consists of the points with coordinate $(a : 0 : c;\ x : y : 0)$. Therefore the singular locus is of codimension 1.

Example 4.2.8 (Non-normal Schubert Varieties)**.** For each $n \geq 2$, we can find a non-normal Schubert variety of dimension n:

We use the twisted incidence varieties, which are called unseparated incidence varieties in [Lau96, Sec. 2.2] (Schubert varieties are not discussed there). Take $G = SL_{n+1}$. The parabolic P is set up so that G/P_{red} is the incidence variety $\sum_{i=1}^{n+1} x_i y_i = 0$ in $\mathbb{P}^n \times \check{\mathbb{P}}^n$, and G/P is the unseparated incidence variety

$$\sum_{i=1}^{n+1} z_i w_i^p = 0.$$

The quotient $\pi : G/P_{red} \to G/P$ is given by the ring map $z_i \mapsto x_i^p$ and $w_i \mapsto y_i$. For example, when $n = 3$, for any k-algebra A, the A-points $P(A)$ consists of the matrices of the form

$$\begin{bmatrix} * & * & * & * \\ \epsilon & * & * & * \\ \epsilon & * & * & * \\ 0 & 0 & 0 & * \end{bmatrix}, \quad \epsilon^p = 0.$$

By [Bri05, Ex. 1.2.3.5], the Schubert varieties of G/P_{red} are of the form $I'_{i,j}$ with $1 \leq i, j \leq n+1$, $i \neq j$, where each $I'_{i,j}$ is defined by the homogeneous ideal

$$x_{i+1} = ... = x_{n+1} = y_1 = ... = y_{j-1} = 0.$$

Therefore the Schubert varieties of G/P are of the form $I_{i,j}$ where each $I_{i,j}$ is defined by the homegenous ideal

$$z_{i+1} = ... = z_{n+1} = w_1 = ... = w_{j-1} = 0.$$

When $n = 2$, we take the Schubert variety $I_{2,1}$. The homogeneous ideal is

$\langle z_1 w_1^p + z_2 w_2^p, z_3 \rangle$, comparing the equations, we see that $I_{2,1}$ is isomorphic to the BSDH variety $X_P(s_\alpha, s_\beta)$ discussed in Ex. 4.2.7, hence non-normal.

When $n \geq 3$, we take the Schubert variety $I_{3,1}$. This is an n-dimensional Schubert variety. In the affine chart $z_1 \neq 0, w_{n+1} \neq 0$, the defining ideal is

$$\langle z_2 w_2^p + z_3 w_3^p, w_1, z_4, ..., z_{n+1} \rangle.$$

By computing the Jacobian and noticing that $char(k) = p$, we have that the singular locus is defined by

$$w_1 = w_2^p = w_3^p = z_4 = ... = z_{n+1} = 0,$$

which has codimension 1 in $I_{3,1}$. Therefore $I_{3,1}$ is non-normal.

M. Brion points out that the Ex. 4.2.8 above shows that in general the natural morphism ${}^B X_{P_{red}}(w) \to {}^B X_P(w)$ is not flat.

Remark 4.2.9 (Nonflat Morphisms ${}^B X_{P_{red}}(w) \to {}^B X_P(w)$)**.** We show that the natural morphisms $\pi : I'_{2,1} \to I_{2,1}$ and $\pi : I'_{3,1} \to I_{3,1}$, as in the Ex. 4.2.8 above, are not flat.

Suppose $\pi : I'_{2,1} \to I_{2,1}$ is flat, then by [Gro65, Cor. 6.5.2.(i)], we see that if x is a regular point of $I'_{2,1}$, then $\pi(x)$ is a regular point of $I_{2,1}$. However, the singular locus of $I_{2,1}$ is of codimension 1, while $I'_{2,1}$ is normal. Hence $\pi : I'_{2,1} \to I_{2,1}$ is not flat. The same reasoning shows that $\pi : I'_{3,1} \to I_{3,1}$ is not flat either.

In contrast to Ex. 4.2.8, we have the following proposition:

Proposition 4.2.10. *No matter whether the parabolic P is reduced or not, one dimensional Schubert varieties of G/P are always isomorphic to $\mathbb{P}^1$.*

Proof. The argument in Ex. [Bri05, Ex. 1.3.4.(2)] still works here. Namely, any one dimensional Schubert variety in G/P has the form ${}^B X_P(s)$ with s a simple reflection, we have that ${}^B X_P(s) \cap U(-I)P/P$, being T-invariant, is the affine line $\mathbb{A}^1$ in the direction s. Therefore it remains to check that ${}^B X_P(s)$ is smooth at the other T-fixed point sP/P, which follows from the smoothness of the Schubert cell BsP/P. □

Proposition 4.1.1 implies that the first projection $X_P(w_1, ..., w_r) \to X_P(w_1)$ in general cannot be a Zariski locally trivial fibration as the domain is reduced but, in general, the fibers over closed points are not. This failure to be a Zariski locally trivial fibration can be explained from another perspective as shown in the following Remark 4.2.11:

Remark 4.2.11. Fix a k-algebra A and we only consider A-points in this remark.

Let us first recall why $X_{P_{red}}(w_\bullet)$ is a Zariski locally trivial fibration in the classical case: Let $g_0 = id \in G$.

$$V'(w_\bullet) := \{(g_1, ..., g_r) \in G^r |\ g_{i-1}^{-1} g_i \in P_{red} w_i P_{red}\}.$$

The multiplication map gives an isomorphism

$$m : V'(w_1) \times V'(w_2) \xrightarrow{\sim} V'(w_1, w_2),$$

which induces an isomorphism

$$X'_{P_{red}}(w_1) \times X'_{P_{red}}(w_2) \xrightarrow{\sim} X'_{P_{red}}(w_1, w_2).$$

Now we consider $X_P(w_\bullet)$ instead of $X_{P_{red}}(w_\bullet)$. We have that

$$X'_P(w_1, w_2) \cong V'(w_1, w_2)/\sim_1,$$

where $\sim_1$ is an equivalence relation defined as $(g_1, g_2) \sim_1 (g_1 p_1, g_2 p_2)$ for some $p_i \in P$. Note this is not a quotient by group action, as P does not act on $V'(w_1, w_2)$.

Pulling back $\sim_1$ along m and taking the quotient by P on the second factor, we have that

$$X'_P(w_1, w_2) \cong (V'(w_1) \times X'_P(w_2))/\sim_2,$$

where $\sim_2$ is an equivalence relation defined as $(g_1, b) \sim_2 (g_1 p, p^{-1} b)$ for some $p \in P$.

On the other hand, we have that

$$X_P(w_1) \times X_P(w_2) \cong (X_P(w_1) \times P_I) \times^{P_I} X_P(w_2).$$

The map $V'(w_1) \to X_P(w_1) \times P_I$, defined by $u w_1 p \mapsto (u w_1 P, p)$ with $u \in U(I \cap w_1(-I))$, is the quotient by the relation $u w_1 p \sim u \epsilon w_1 p$ for $\epsilon \in U(I \cap w(J, \boldsymbol{n}))$. Therefore we see that

$$X'_P(w_1) \times X'_P(w_2) \cong (V'(w_1) \times X'_P(w_2))/\sim_3,$$

where $\sim_3$ is an equivalence relation defined as follows: Every $g \in V'(w_1)$ can be written as $g = u(g) w_1 p(g)$ uniquely for $u(g) \in U(I \cap w(-I))$ and $p(g) \in P_I$. The equivalence relation $\sim_3$ is defined as

$$(g,b) \sim_3 (gp, p^{-1}b) \sim_3 (u(g)\epsilon w_1 p(g), b),$$

for $p \in P_I$ and $\epsilon \in U(I) \cap U(w(J, \boldsymbol{n}))$.

When $P \neq P_{red}$, the first part of $\sim_3$ is included in $\sim_2$ but the second part of $\sim_3$ is not, so we see that $\sim_2$ and $\sim_3$ are in general not comparable, hence the multiplication m above in general does not induce a morphism between $X'_P(w_1) \times X'_P(w_2)$ and $X'_P(w_1, w_2)$.

Below we give an example of nonreduced fibers of the last projection, where $p > 0$ can be any prime number. This example 4.2.12 is also curious because the target of the last projection is the whole variety G/P.

Example 4.2.12 (Nonreduced Fibers of p_3)**.** We use the setup in Ex.4.2.6, i.e., $G = SL_3$ and $P = U(-\alpha, 1) \cdot B$. Consider the last projection

$$p_3 : X_P(s_\beta, s_\alpha, s_\beta) \to X_P(s_\beta s_\alpha s_\beta) = G/P.$$

A point in the big Schubert cell $X'_P(s_\beta s_\alpha s_\beta)$ has the form $us_\beta s_\alpha s_\beta P/P$ for some $u \in R_u(B) = U(\alpha, \beta, \alpha + \beta)$. We have that

$$us_\beta s_\alpha s_\beta P/P = us_\beta s_\alpha s_\beta U(-\alpha, 1)P/P = uU(\beta, 1)s_\beta s_\alpha s_\beta P/P.$$

Therefore the fiber $p_3^{-1}(us_\beta s_\alpha s_\beta P/P)$ is isomorphic to the direct product

$$uU(\beta,1)s_\beta P/P \times uU(\beta,1)s_\beta s_\alpha P/P = us_\beta U(-\beta,1)P/P \times us_\beta s_\alpha U(-\alpha-\beta,1)P/P.$$

Since $-\beta, -\beta-\alpha \notin J$, we have that $p_3^{-1}(us_\beta s_\alpha s_\beta P/P)$ is nonreduced. Therefore the last projection p_3 is not birational.

By Lemma 4.2.3, we have that all the fibers of p_3 are thus non-reduced.

Chapter 5

Nice Topology

Although the geometry of $X_P(w_\bullet)$ differs greatly from that of classical BSDH varieties, the topology of the newly constructed exotic BSDH varieties remains the same as for the corresponding classical BSDH varieties, due to the fact that $\pi : G/P_{red} \to G/P$ is purely inseparable, finite, and surjective, hence a universal homeomorphism [Gro65, Prop. 2.4.4].

Over a finite or algebraically closed field, we can then generalize some results in [dCHL18] to the case involving $X_P(w_\bullet)$ for a nonreduced parabolic P. Namely, given a generalized convolution morphism $f : X_P(w_\bullet) \to X_Q(w''_{\theta,\bullet})$ (defined in Def. 5.2.1), a special case of which is the last projection $p : X_P(w_\bullet) \to X_P(w_\star)$, we prove that the decomposition theorem holds for f, and that the push forward of the intersection complex $f_*\mathcal{IC}_{X_P(w_\bullet)}$ is *good*, which is a notion defined in [dCHL18, Def. 1.2.1] and recalled below in Def. 5.1.1.

5.1 Derived Categories and Galois Actions

We first recall some notation. Let l be a prime number so that $l \neq char(k) = p$. When the field k is finite, let $D^b_m(X, \overline{\mathbb{Q}_l})$ be the bounded mixed constructible derived category with the middle perversity t-structure as in [BBD82]. We denote the derived direct image Rf_* just as f_*.

Let k be a finite field. For X a separated scheme of finite type over k, the triangulated category $D^b_m(X, \overline{\mathbb{Q}}_l)$ is the mixed, bounded, and constructible "derived" category with self dual perversity as in [BBD82, § 2.2.10-19]. The truncation functors for the standard t-structure are denoted $\tau_{\leq i}$ and $\tau_{\geq i}$ for $i \in \mathbb{Z}$. Let $\bar{x} \in X(\bar{k})$ be a geometric point, and let x be the closed point that is the image of $\bar{x} : \mathrm{Spec}(\bar{k}) \to X$. For every $F \in D^b_m(X, \overline{\mathbb{Q}}_l)$, the stalk of the i-th cohomology sheaf $\mathcal{H}^i(F)_{\bar{x}}$ is a $Gal(k(x)_s/k(x))$-module, where $k(x)_s$ is the separable closure of $k(x)$. In the rest of the paper, when we consider the stalk of $F \in D^b_m(X, \overline{Q}_l)$ at some $\bar{x} \in X(\bar{k})$ as a Galois module, we always mean the structure of a $Gal(k(x)_s/k(x))$-module.

Now we can define the notion of *good*:

Definition 5.1.1. We say that $F \in D^b_m(X, \overline{\mathbb{Q}}_l)$ is good if F is

1. Semisimple, i.e., it is isomorphic to a direct sum of shifted simple perverse sheaves, which, by [BBD82, Sec. 4.3], are of the form $j_{!*}(L[dimV])$, where $j : V \hookrightarrow X$ is an inclusion of irreducible smooth subvariety, and L is an irreducible lisse $\overline{\mathbb{Q}}_l$-sheaf over V;

2. Frobenius semisimple, i.e., for every $\bar{x} \in X(\bar{k})$, the stalks of the cohomology sheaves $\mathcal{H}^i(F)_{\bar{x}}$ are semisimple as graded Galois modules;

3. Even, i.e., $\mathcal{H}^i(F)_{\bar{x}}$ is trivial for i odd;

4. Very pure with weight zero, i.e., let $F^\vee$ be the Verdier dual of F, then for each degree $i \in \mathbb{Z}$ and $\bar{x} \in X(\bar{k})$, both stalks $\mathcal{H}^i(F)_{\bar{x}}$ and $\mathcal{H}^i(F^\vee)_{\bar{x}}$ have weight i;

5. Tate, i.e., for every $\bar{x} \in X(\bar{k})$, the stalk $\mathcal{H}^i(F)_{\bar{x}}$ is isomorphic to a direct sum of Tate modules $\overline{\mathbb{Q}}_l(-k)$ of weight $2k$ for possibly varying $k \in \mathbb{Z}$.

The following lemma is probably well known, but we cannot find an explicit reference:

Lemma 5.1.2. Given any finite purely inseparable field extension $k \hookrightarrow k'$, we have a commutative diagram of field extensions:

$$\begin{tikzcd} \bar{k} & k'_s \arrow[l, hook'] & k_s \arrow[l, hook'] \\ & k' \arrow[u, hook] & k \arrow[l, hook'] \arrow[u, hook] \end{tikzcd}, \tag{5.1}$$

where k_s and k'_s are separable closures of k and k', and $\bar{k}$ is an algebraic closure of both k and k'. By restriction, we have a group isomorphism $\psi : Gal(k'_s/k') \to Gal(k_s/k)$.

Proof. Firstly, we can take k'_s to be $k' \otimes_k k_s$ by the primitive element theorem from basic algebra.

The extension $\bar{k}/k_s$ is purely inseparable hence any intermediate extension of it is again purely inseparable. Therefore $k_s \hookrightarrow k'_s$ is purely inseparable.

We then have that $Aut(k'_s/k_s) = 1$, i.e., a field automorphism of k'_s that fixes k_s must be trivial, so ψ is injective.

On the other hand, any $\sigma \in Aut(k_s)$ can be extended to an automorphism $\sigma' \in Aut(k'_s)$. If σ fixes k, then for any element $a \in k'$, we must have

$$0 = \sigma'(a^m) - a^m = \sigma'(a)^m - a^m = (\sigma'(a) - a)^m,$$

where m is a power of p so that $a^m \in k$. Therefore σ' fixes k', and $\sigma' = \psi^{-1}(\sigma)$, hence ψ is bijective. ☐

Proposition 5.1.3. *Let k be a finite field. Let $f : X \to Y$ be a finite, surjective, and purely inseparable morphism of schemes over k. Let $F \in D^b_m(X, \overline{\mathbb{Q}}_l)$ be Frobenius semisimple as defined in Def. 5.1.1.(2). Then $f_*F \in D^b_m(Y, \overline{\mathbb{Q}}_l)$ is again Frobenius semisimple. Moreover, the weights of f_*F are the same as the weights of F.*

Proof. We first show that f_*F is Frobenius semisimple. By defintion, we need to show that for every geometric point $\bar{y} \in Y(\bar{k})$, where y is the image closed point of Y, we have that the stalk $\mathcal{H}^i(f_*F)_{\bar{y}}$ is a semisimple Galois module.

We first reduce to the case where F can be identified with a $\overline{\mathbb{Q}}_l$-sheaf.

Since f is finite, the functor f_* equals the functor Rf_* and is thus exact. Therefore, we have

$$\mathcal{H}^i(f_*F) \cong \mathcal{H}^i(f_*\tau_{\geq i}\tau_{\leq i}F) \cong f_*\tau_{\geq i}\tau_{\leq i}F[i] \cong f_*\mathcal{H}^i(F),$$

isomorphisms of $\overline{\mathbb{Q}}_l$-sheaves placed at degree 0.

By assumption, F is Frobenius semisimple, so $\mathcal{H}^i(F)_{\bar{x}}$ is a semisimple Galois module for every geometric point $\bar{x} \in X(\bar{k})$.

Therefore we are reduced to show that if F is a $\overline{\mathbb{Q}}_l$-sheaf on X so that $F_{\bar{x}}$ is Galois semisimple for every geometric point $\bar{x} \in X(\bar{k})$, then $f_*G_{\bar{y}}$ is also Galois semisimple for every geometric point $\bar{y} \in Y(\bar{k})$.

For every closed point y of Y, let x be the closed point of X so that $f(x) = y$ (recall that $f : X \to Y$ is purely inseparable). Let x' be the scheme theoretic fiber of f over y. We have the following commutative diagram,

$$\begin{tikzcd} x \arrow[r, "i_1"] \arrow[dr, "\phi_1"] & x' \arrow[r, "i_2"] \arrow[d, "\phi"] & X \arrow[d, "f"] \\ & y \arrow[r, "i_3"] & Y \end{tikzcd}$$

By proper base change, we then have the isomorphisms of $Gal(k(y)_s/k(y))$-modules

$$(f_*F)_{\bar{y}} \cong (i_3^*f_*F)_{\bar{y}} \cong (\phi_*i_2^*F)_{\bar{y}}.$$

Note that $i_1 : x \hookrightarrow x'$ is a closed immersion which is also a universal homeomorphism. By [Sta23, 03SI], we have that the etale topologies of x and x' are the same, and the equivalences are given by i_1^* and i_{1*}. Therefore, we have isomorphisms of Galois modules

$$(f_*F)_{\bar{y}} \cong (\phi_*i_2^*F)_{\bar{y}} \cong (\phi_*i_{1*}i_1^*i_2^*F)_{\bar{y}} \cong (\phi_{1*}(i_2 \circ i_1)^*F)_{\bar{y}}.$$

Since $(i_2 \circ i_1)^*F$ is a $\overline{\mathbb{Q}}_l$-sheaf on x, and for every $\bar{x} \in x(\bar{k})$, we have that $((i_2 \circ i_1)^*F)_{\bar{x}} = F_{\bar{x}}$, the pull back $(i_2 \circ i_1)^*F$ is Galois semisimple.

Therefore we are reduced to the case where $X = x = \mathrm{Spec}(k(x))$ and $Y = y = \mathrm{Spec}(k(y))$ are two points over k.

Let $n \in \mathbb{N}$. Let L be a lisse sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules over x, it is identified as a $\mathbb{Z}/l^n\mathbb{Z}$-module, which we still denote as L, provided with a continuous representation

$$\xi_L : Gal(k(x)_s/k(x)) \to Aut_{\mathbb{Z}/l^n\mathbb{Z}\text{-mod}}(L).$$

Recall that, by Lemma 5.1.2, we have a group isomorphism

$$\psi : Gal(k(x)_s/k(x)) \xrightarrow{\sim} Gal(k(y)_s/k(y)) =: G.$$

By [Mil80, §II.3.1.(e)], the $\mathbb{Z}/l^n\mathbb{Z}$-module f_*L, when viewed as a continuous representation of G, is the coinduction of the $Gal(k(x)_s/k(x))$-module L, i.e., we have an isomorphism of G-modules:

$$f_*L \xrightarrow{\sim}_{G\text{-mod}} Hom_{\mathbb{Z}[G]}(\mathbb{Z}[Im(\psi)],\ L^{ker(\psi)}),$$

where $L^{ker(\psi)}$ denotes the invariant part of the $\mathbb{Z}/l^n\mathbb{Z}$-module under the action of $ker(\psi) \subset Gal(k(x)_s/k(x))$.

As ψ is a group isomorphism, we see that $f_*L \cong L$ as $\mathbb{Z}/l^n\mathbb{Z}$-modules and the representation

$$Gal(k(y)_s/k(y)) \to Aut_{\mathbb{Z}/l^n\mathbb{Z}\text{-mod}}(f_*L) \cong Aut_{\mathbb{Z}/l^n\mathbb{Z}\text{-mod}}(L)$$

factors as

$$Gal(k(y)_s/k(y)) \xrightarrow{\psi^{-1}} Gal(k(x)_s/k(x)) \xrightarrow{\xi_L} Aut_{\mathbb{Z}/l^n\mathbb{Z}\text{-mod}}(L).$$

As $\xi \circ \psi^{-1}$ and ξ has the same image, we see that the proposition "if L is a semisimple $Gal(k(x)_s/k(x))$-module, then f_*L is a semisimple $Gal(k(y)_s/k(y))$-module" is true for every L, a lisse sheaf $\mathbb{Z}/l^n\mathbb{Z}$-modules on x, and every $n \in \mathbb{Z}_{>1}$. Taking the projective limit with respect to $n \in \mathbb{Z}_{>1}$, we see that the same proposition is true if L is a lisse $\mathbb{Z}_l$-sheaf, as such a sheaf corresponds to the projective limit of the representations of the Galois group on $\mathbb{Z}/l^n\mathbb{Z}$ for $n \in \mathbb{Z}_{>1}$. Tensoring with finite fields extensions of $\mathbb{Q}_l$ and take the direct limit, we see that the same proposition is true if L is a lisse $\overline{\mathbb{Q}}_l$-sheaf. Finally, we can take L to be G above and the proof that $f_*F \in D^b_m(Y, \overline{\mathbb{Q}}_l)$ is Frobenius semisimple is finished.

We now show that f_*F has the same weights as F:

As $k(x)$ and $k(y)$ are both perfect, the purely inseparable morphism $f : x \to y$ is induced by a field automorphism $f^\# : k(y) \xrightarrow{\sim} k(x)$. As Frobenius commutes with field automorphisms, we see that

$$\psi^{-1} : Gal(k(y)_s/k(y)) \to Gal(k(x)_s/k(x))$$

maps the Frobenius to Frobenius.

Alternatively, the fact that they have the same weights also follows from the general theory that Rg_* preserves weights if g is a proper morphism [BBD82, Sec. 5.1.14]. □

5.2 Topology of Convolution Morphisms

In this section, by BSDH varieties, we always mean P-BSDH varieties, see the discussion at the end of §3.1.

Our final goal is to prove the decomposition theorem and the goodness for $f_*\mathcal{IC}_{X_P(w_\bullet)}$, where $\mathcal{IC}_{X_P(w_\bullet)}$ is the shifted intersection complex on a BSDH variety $X_P(w_\bullet)$, and f is a generalized convolution morphism as defined in [dCHL18, Sec. 4.5].

We recall the definition of generalized convolution morphisms below. For the readers who want to skip the definition, it is useful to know two families of examples of generalized convolutions morphisms. The first family of examples is the i-th projection for $i \le r$, $p_i : X_P(w_1, ..., w_r) \to X_P(w_1 \star ... \star w_i)$. The second family of examples is the restriction of the natural morphism $(G/P)^r \to (G/Q)^r$, for some parabolics $P \subset Q$, to $X_P(w_\bullet)$.

To define the generalized convolution morphisms, we need some more notation.

Let $P \subset Q$ be two nonreduced parabolics of G containing a common Borel subgroup B. Let $H \subset I \subset R^+$ be such that

$$R_u(P_{red}) = U(I) \supset U(J) = R_u(Q_{red}).$$

The Weyl group of the Levi factor of P_{red}, W^I, is a subgroup of that of Q_{red}, W^H.

As W_I is defined as the set of longest representatives of the elements in $W^I \backslash W / W^I$, we have a natural map induced by the double quotient $w \mapsto w'' : W_I \to W_H$.

Let $\gamma : G/P_{red} \to G/Q_{red}$, $w \mapsto w''$ is defined so that for $w \in W_I$, we have an equality of schemes

$$Q_{red} \cdot \gamma(P_{red} w P_{red}/P_{red}) = Q_{red} w'' Q_{red}/Q_{red}.$$

Let $\Upsilon : G/P \to G/Q$ be the natural morphism. Then we have the equality

$$Q_{red} \cdot \Upsilon(X_P(w)) = X_Q(w'').$$

Let θ be the data of numbers $i_1, ..., i_m = r'$ with $1 \leq i_1 < ... < i_m = r' \leq r$. Let $w_1, ..., w_r \in W_I$ and $w''_1, ..., w''_r \in W_H$. Let $i_0 = 0$. Define

$$w_{\theta,k} := w_{i_{k-1}+1} \star ... \star w_{i_k}, \qquad w''_{\theta,k} := w''_{i_{k-1}+1} \star ... \star w''_{i_k}.$$

Definition 5.2.1. The generalized convolution morphism $f : X_P(w_\bullet) \to X_Q(w''_{\theta,\bullet})$ is defined as $f := \prod_{j=1}^{m} \Upsilon \circ p_{i_j}$, where p_{i_j} is the i_j-th projection of $(G/P)^r \to G/P$ restricted to $X_P(w_\bullet)$. Equivalently, for any k-algebra A and $(a_1, ..., a_r) \in X_P(w_\bullet)(A)$, we define

$$f(a_1, ..., a_r) := (\Upsilon(a_{i_1}), ..., \Upsilon(a_{i_m})).$$

We have the following commutative diagram, where the vertical morphisms are purely inseparable and f equals to the composition of two of the bottom sides:

$$\begin{tikzcd}
X_{P_{red}}(w_\bullet) \arrow[rr] \arrow[dr] \arrow[dd] & & X_{P_{red}}(w_{\theta,\bullet}) \arrow[dr] \arrow[dd] & \\
& X_{Q_{red}}(w''_\bullet) \arrow[rr, crossing over] & & X_{Q_{red}}(w''_{\theta,\bullet}) \arrow[dd] \\
X_P(w_\bullet) \arrow[rr] \arrow[dr] & & X_P(w_{\theta,\bullet}) \arrow[dr] & \\
& X_Q(w''_\bullet) \arrow[from=uu, crossing over] \arrow[rr] & & X_Q(w''_{\theta,\bullet})
\end{tikzcd} \tag{5.2}$$

We need one more piece of notation to state our final results. Let X be a variety over a finite field. By $\mathcal{IC}_X \in D^b_m(X, \overline{\mathbb{Q}}_l)$, we mean the intersection complex starting from degree zero, e.g., if X is smooth, then $\mathcal{IC}_X$ is the constant $\overline{\mathbb{Q}}_l$ placed at degree

0.

In the statement of the theorem below, the requirement $Q_{red} \cdot \overline{X_P(w_i)} = \overline{X_P(w_i)}$ is equivalent to say w_i is of Q-type as defined in [dCHL18, Def. 3.10.3]. It is equivalent to require $\overline{Q_{red} w_i P_{red}} = \overline{P_{red} w_i P_{red}}$. It guarantees that the generalized convolution morphism in the theorem below is surjective.

Theorem 5.2.2 (Decomposition Theorem)**.** *Let the base field k be algebraically closed or finite. Let $f : X_P(w_\bullet) \to X_Q(w''_{\theta,\bullet})$ be the generalized convolution morphism in Def. 5.2.1.*

If for each $i = 1, ..., r$, we have that $Q_{red} \cdot \overline{X_P(w_i)} = \overline{X_P(w_i)}$, then we have the decomposition theorem for f:

$$f_* \mathcal{IC}_{X_P(w_\bullet)} \cong \bigoplus_{\mathcal{O}} \mathcal{IC}_{\mathcal{O}} \otimes \Big(\bigoplus_{j=0}^{codim \mathcal{O}} \overline{\mathbb{Q}}_l^{m_{\mathcal{O},2j}} (-j)[-2j] \Big),$$

where $\mathcal{O}$ belongs to a finite collection of geometrically integral Q_{red}-invariant closed subvarieties.

Furthermore, Poincare-Verdier duality and Relative Hard Lefschetz theorem imply the following:

1. $m_{\mathcal{O},2j} = m_{\mathcal{O},2codim\mathcal{O}-2j}$;
2. $m_{\mathcal{O},2j} \leq m_{\mathcal{O},2j+2}$, *for* $2j < codim \mathcal{O}$.

Proof. This is true when P and Q are reduced by [dCHL18, Thm. 2.2.7].

Consider the northwest-southeast diagonal slice of the cube diagram (5.2):

$$\begin{tikzcd} X_{P_{red}}(w_\bullet) \arrow[r, "f"] \arrow[d, "\pi"'] & X_{Q_{red}}(w''_{\theta,\bullet}) \arrow[d, "\pi"] \\ X_P(w_\bullet) \arrow[r, "f"] & X_Q(w''_{\theta,\bullet}), \end{tikzcd} \tag{5.3}$$

where we abuse the language a bit for the arrows denoted f and π. Both vertical arrows denoted by π are finite, surjective, and purely inseparable.

Let $\mathcal{O}'$ be a stratum that appears in the decomposition theorem for $f : X_{P_{red}}(w_\bullet) \to X_{Q_{red}}(w''_{\theta,\bullet})$.

Let $\mathcal{O} := f(\mathcal{O}')$.

Let $j : U \hookrightarrow \mathcal{O}'$ be an open dense smooth subscheme so that $\mathcal{IC}_{\mathcal{O}'} = \mathcal{IC}_{\mathcal{O}'}((\overline{\mathbb{Q}}_l)_U)$.

Let j also denote the open embedding $j : \pi(U) \hookrightarrow \mathcal{O}$ by abuse of language. We then have that $\pi_* j_* = j_* \pi_*$. Moreover, as π is finite, the pushforward π_* is exact, and we have that π_* commutes with the truncation functors $\tau^{\leq i}$, $i \in \mathbb{Z}$, for the standard t-structure. Therefore, using the description of intermediate extension as iterated j_*'s and $\tau^{\leq i}$'s we have the isomorphism:

$$\pi_* \mathcal{IC}_{\mathcal{O}'} \cong \mathcal{IC}_{\mathcal{O}}(\pi_*(\overline{\mathbb{Q}}_l)_U).$$

Notice that this isomorphism is also given by (17) of [dCat15, Lem. 2.4.1]. As π is a universal homeomorphism, we have that π^* induces an equivalence of the category of étale covers of U and that of $\pi(U)$, hence $\pi_*(\overline{\mathbb{Q}}_l)_U = (\overline{\mathbb{Q}}_l)_{\pi(U)}$ and

$$\pi_* \mathcal{IC}_{\mathcal{O}'} \cong \mathcal{IC}_{\mathcal{O}}. \tag{5.4}$$

Therefore we have the decomposition

$$\begin{aligned} f_* \mathcal{IC}_{X_P(w_\bullet)} &\cong f_* \pi_* \mathcal{IC}_{X_{P_{red}}(w_\bullet)} \cong \pi_* f_* \mathcal{IC}_{X_{P_{red}}(w_\bullet)} \\ &\cong \bigoplus_{\mathcal{O}'} \pi_* \mathcal{IC}_{\mathcal{O}'} \otimes \Big(\bigoplus_{j=0}^{codim \mathcal{O}'} \overline{\mathbb{Q}}_l^{m_{\mathcal{O}',2j}}(-j)[-2j]\Big) \\ &\cong \bigoplus_{\mathcal{O}} \mathcal{IC}_{\mathcal{O}} \otimes \Big(\bigoplus_{j=0}^{codim \mathcal{O}} \overline{\mathbb{Q}}_l^{m_{\mathcal{O},2j}}(-j)[-2j]\Big). \end{aligned}$$

From this we also see that the (in-)equalities (1) and (2) follow from the corresponding (in-)equalities in the case where P and Q are reduced. $\square$

Theorem 5.2.3 (Goodness)**.** *Let the base field be finite. Let $f : X_P(w_\bullet) \to X_Q(w''_{\theta,\bullet})$ be a generalized convolution morphism as above. Then both $\mathcal{IC}_{X_P(w_\bullet)}$ and $f_* \mathcal{IC}_{X_P(w_\bullet)}$ are good as in Def. 5.1.1.*

Proof. In the proof of Thm. 5.2.2, it is shown that the pushforward of an intersection complex by π_* is still an intersection complex. Therefore, the pushforward of a semisimple complex by π_* is still semisimple.

It is easy to see that π_* preserves evenness. By Prop. 5.1.3, we have that π_* preserves Frobenius semisimplicity and weights. As π is a universal homeomorphism and preserves weights, we also have that π_* preserves Tateness.

Therefore the pushforward π_* preserves goodness.

By [dCHL18, Th. 2.2.1], we have that $\mathcal{IC}_{X_{P_{red}}(w_\bullet)}$ is good. Apply the equation (5.4) above to $\mathcal{O}' = X_{P_{red}}(w_\bullet)$, we have that

$$\pi_* \mathcal{IC}_{X_{P_{red}}}(w_\bullet) \cong \mathcal{IC}_{\pi(X_{P_{red}}(w_\bullet))} \cong \mathcal{IC}_{X_P(w_\bullet)}$$

is good.

By [dCHL18, Th. 2.2.2], we have that $f_*\mathcal{IC}_{X_{P_{red}}}(w_\bullet)$ is good. Therefore

$$f_*\mathcal{IC}_{X_P(w_\bullet)} \cong f_*\pi_*\mathcal{IC}_{X_{P_{red}}(w_\bullet)} = \pi_*f_*\mathcal{IC}_{X_{P_{red}}(w_\bullet)}$$

is good. $\square$

Chapter 6

Appendix: k-Functors, fppf-Sheaves, and Monomorphism of Schemes

In this Appendix, we clarify the relation between the language of k-functors and k-spaces used in [Jan03], [DG70], and the mainstream language of algebraic space as in [Ols16]. We also stress the relation among image k-functors, image k-spaces, and scheme-theoretic images.

6.1 k-spaces and k-algebraic spaces

Let k be a base ring. A k-functor is a functor from the category of k-algebras to the category of sets.

A k-functor X is called local if it is a sheaf for the Zariski topology on k-algebras, i.e., for every k-algebra A and every finite set $f_1, ..., f_n \in A$ so that $\sum_i f_i A = A$, the following sequence is exact:

$$X(A) \to \prod_i X(A_{f_i}) \rightrightarrows \prod_{i,j} X(A_{f_i f_j}).$$

In particular, k-schemes are all local k-functors.

Given a k-functor X and a k-subfunctor Y of X, we say that Y is an open subfunctor X if the inclusion morphism $Y \hookrightarrow X$ is representable by open immersions of schemes, i.e., given any scheme Z and morphism $f : Z \to X$, the base change $f^*Y \to Z$ is an open immersion of schemes [Jan03, Sec. I.1.7].

The Yoneda embedding gives an equivalence between the category of k-schemes and the category of local k-functors that admit open coverings of open subfunctors representable by affine schemes [Jan03, Sec. I.1.11].

A k-space X is a k-functor that is also an fppf sheaf [Jan03, Sec. I.5.2], i.e., for every k-algebra A and an fppf open covering $A_1, ..., A_n$ of A, the following sequence is exact:

$$X(A) \to \prod_i X(A_i) \rightrightarrows \prod_{i,j} X(A_i \otimes_A A_j).$$

We use the terms k-space and big fppf sheaf interchangeably. As schemes satisfy fppf descent, we have that schemes are also k-spaces [DG70, Sec. III.1.1.3].

Recall that a k-algebraic space X is a big étale sheaf on the category of k-schemes which admits a morphism $f : U \to X$ where U is a scheme, f is surjective, étale, and representable by schemes. Note that given any k-algebraic space X, the diagonal $\Delta : X \to X \times_k X$ is automatically representable by schemes, see Alper's argument in his book in progress [Alp23]. Therefore, our definition of k-algebraic space agrees with the one in [Ols16, Def. 5.1.10]. By [Sta23, Tag 076M], we have that k-algebraic spaces are also fppf-sheaves, thus k-spaces.

In summary, we have the following embeddings of categories:

$$\begin{tikzcd}
k\text{-schemes} \arrow[r, hook] \arrow[d, hook] & k\text{-spaces} \arrow[r, hook] & \text{local } k\text{-functors} \arrow[r, hook] & k\text{-functors} \\
k\text{-algebraic spaces} \arrow[ur, hook] & & &
\end{tikzcd}$$

6.2 Images and Monomorphisms

Every k-functor X has a unique fppf sheafification, which is called the associated k-space $\tilde{X}$. The functor $i : X \mapsto \tilde{X}$ is the left adjoint to the inclusion functor $\{k\text{-spaces}\} \hookrightarrow \{k-\text{functors}\}$ [Jan03, Sec. I.5.4].

Given a morphism $f : X \to Y$ between two k-spaces, the image k-functor $im(f)$ is given by $im(f)A := f(A)(X(A))$. The image k-space $Im(f)$ of f is the associated k-space of the image k-functor of f, i.e.,

$$Im(f) := \widetilde{im(f)}.$$

The image k-space has the universal property that it is the smallest subfunctor of Y, which is also a k-space, that f factors through. In [Jan03, Sec. I.5.4.4], it is shown that we have an isomorphism of sets

$$Im(f)(A) \cong \bigcup_B im(f)(B) \cap Y(A), \tag{6.1}$$

where B is taken over all fppf-A-algebras, and the intersection makes sense since we have the inclusions $im(f)(B) \subset Y(B) \supset Y(A)$. In particular, the canonical morphism $im(f) \to Im(f)$ induced by sheafification is already a monomorphism of k-functors.

Recall that given any morphism of schemes $f : X \to Y$, the scheme-theoretic image $Im^{sch}(f)$ is the smallest closed subscheme of Y that f factors through [Sta23, Tag 01R5]. As closed subschemes are naturally closed k-subfunctors and closed k-subspaces, we have canonical monomorphisms of k-functors:

$$im(f) \hookrightarrow Im(f) \hookrightarrow Im^{sch}(f) \hookrightarrow Y. \tag{6.2}$$

Lemma 6.2.1. Let $f : X \to Y$ be a flat morphism of finite presentation between schemes. If the set theoretic image of f is closed, then the image k-space $Im(f)$ is representable by the scheme $Im^{sch}(f)$

Proof. Since the set-theoretic image of f is closed, by [Sta23, Tag 01R8], we have that f surjects onto its scheme-theoretic image. By the description (6.1) above, we need to show that every A-point of $Im^{sch}(f)$ is in some $f(B)(X(B))$ for some fppf-A-algebra B. For this we can assume both X and Y are affine. Then the lemma follows from the properties of f that we impose: just take Spec(B) to be the fiber of f over Spec(A). $\square$

Given any morphism between schemes $f : X \to Y$, if there is a smallest monomorphism of schemes $Z \hookrightarrow Y$ that f factors through, then we denote Z by $Mono(f)$. As closed immersion is a monomorphism [Sta23, Tag 04XV], we have monomorphisms of schemes:

$$Mono(f) \hookrightarrow Im^{sch}(f) \hookrightarrow Y. \tag{6.3}$$

Lemma 6.2.2. A monomorphism of k-schemes is also a monomorphism of k-functors.

Proof. Given a morphism $f : X \to Y$ of k-functors, then f is a monomorphism if and only if the set function $f(A) : X(A) \subset Y(A)$ is injective for every k-algebra A. Suppose now that $f : X \to Y$ is a monomorphism of k-schemes. Given two A-points $x_1, x_2 \in X(A)$, suppose that $f(x_1) = f(x_2) \in Y(A)$, then $x_1 = x_2$ follows from the following commutative diagram of schemes:

$$\begin{tikzcd} \mathrm{Spec}(A) \arrow[r, shift left, "x_1"] \arrow[r, shift right, "x_2"'] & X \arrow[r, "f"] & Y. \end{tikzcd} \tag{6.4}$$

$\square$

Combining Lemma 6.2.2, (6.3), and (6.2), we obtain the following lemmas:

Lemma 6.2.3. Let $f : X \to Y$ be a morphism of schemes. Suppose that $Mono(f)$ exists. We have monomorphisms of k-functors

$$im(f) \hookrightarrow Im(f) \hookrightarrow Mono(f) \hookrightarrow Im^{sch}(f) \hookrightarrow Y. \tag{6.5}$$

If $im(f)$ (resp. $Im(f)$, resp. $Mono(f)$) is a k-space (resp. scheme, resp. closed subscheme of Y), then the first (resp. second, resp. thrid) monomorphism is an isomorphism.

Lemma 6.2.4. If $f : X \to Y$ is a monomorphism between schemes, then we have

$$im(f) = Im(f) = Mono(f) = X. \tag{6.6}$$

Combining Lemma 6.2.3 and Lemma 6.2.1, we obtain

Corollary 6.2.5. *Let $f : X \to Y$ be a flat morphism of finite presentation between k-schemes. Assume that the set theoretic image of f is closed in Y. Then we have identification of schemes:*

$$Im(f) = Mono(f) = Im^{sch}(f). \tag{6.7}$$

The following two examples explains that the scheme $Mono(f)$ in general does not exist.

Example 6.2.6. Let $f : X \to Y$ be a morphism of schemes.

1. The scheme $Mono(f)$ in general does not exist;

2. The k-space $Im(f)$ in general is not a scheme.

Proof. In general, there does not exist a smallest monomorphism that a morphism of schemes $f : X \to Y$. A counter example is the blow down map restricted to one chart $f : \mathbb{A}^2 \to \mathbb{A}^2$, $f(x, y) := (x, xy)$: see the math overflow post by R. van Dobben de Bruyn at mathoverflow.net/questions/19871/images-and-monomorphisms-of-schemes. This concludes item 1. Item 2 follows from item 1 and Lemma 6.2.3. □

Example 6.2.7. Let Y be a nodal cubic curve with the node o. Let $\tilde{f} : \tilde{X} \to Y$ be the normalization and let a_1 and a_2 be the two points lying above o. Let $X = \tilde{X} \setminus \{a_1\}$. Let $f : X \to Y$ be the restriction of $\tilde{f}$ to X. Then f is a monomorphism of schemes, and (6.5) becomes:

$$im(f) = Im(f) = Mono(f) = X \hookrightarrow Im^{sch}(f) = Y.$$

Bibliography

[AR13] P. Achar, S. Riche, *Koszul duality and semisimplicity of Frobenius*, in Annales de l'Institut Fourier **63.4** (2013), 1511–1612.

[Alp23] J. Alper, *Stacks and Moduli*, draft available at https://sites.math.washington.edu/ jarod/moduli.pdf (2023).

[BBD82] A. Beilinson, J. Bernstein, P. Deligne, *Faisceaux pervers*, in Analysis and Topology on Singular Spaces, I (Luminy, 1981), Astérisque, **100** (1982), 5–171.

[BGS96] A. Beilinson, V. Ginzburg, W. Soergel, *Koszul duality patterns in representation theory*, in Journal of the American Mathematical Society **9.2** (1996), 473–527.

[BY11] R. Bezrukavnikov, Z. Yun, *On Koszul duality for Kac-Moody groups*, in Representation Theory of the American Mathematical Society **17** (2011).

[Bri05] M. Brion, *Lectures on the geometry of flag varieties*, in Topics in Cohomological Studies of Algebraic Varieties: Impanga Lecture Notes, Birkhäuser Basel (2005), 33–85.

[Bri17] M. Brion, *Some structure theorems for algebraic groups*, in Proc. Symp. Pure Math. **94** (2017), 53–125.

[BK07] M. Brion, S. Kumar, *Frobenius splitting methods in geometry and representation theory*, **231**, Springer Science Business Media, (2007).

[BL00] S. Billey, V. Lakshmibai, *Singular loci of Schubert varieties*, **182**, Springer Science Business Media, (2000).

[BS58] R. Bott, H. Samelson, *Applications of the theory of Morse to symmetric spaces*, Amer. J . Maths., **80** (1958), 964-1029

[dCHL18] M. de Cataldo, T. Haines, L. Li, *Frobenius semisimplicity for convolution morphisms*, in Mathematische Zeitschrift **289.1** (2018), 119–169.

[dCat15] M. de Cataldo, *Proper toric maps over finite fields*, in International Mathematics Research Notices (2015), 13106–13121.

[Dem74] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Annales scientifiques de l'École Normale Supérieure. Vol. **7**, No. **1**, (1974), 53–88.

[DG70] M. Demazure, P. Gabriel, *Groupes algébriques. Tome I. Géométrie algébrique généralités. Groupes commutatifs*, North-Holland, 1970.

[EMTW20] B. Elias, S. Makisumi, U. Thiel, G. Williamson, *Introduction to Soergel bimodules*, Springer Nature, **Vol. 5**, (2020).

[Gab70] P. Gabriel, *Generalities sur les groupes algebriques*, in Propriétés Générales des Schémas en Groupes, Springer Berlin Heidelberg (1970), 287–317.

[GM82] S. Gelfand, R. MacPherson, *Verma modules and Schubert cells: a dictionary*, Séminaire d'Algébre Paul Dubreil et Marie-Paule Malliavin: Proceedings, Paris 1981 (34ème Année). Springer Berlin Heidelberg, (1982).

[Gro60] A. Grothendieck, *Éléments de géométrie algébrique : I. Le langage des schémas*, in Publications Mathématiques de l'IHÉS **4** (1960), 5–228.

[Gro65] A. Grothendieck, *Éléments de géométrie algébrique : IV. Étude locale des schémas et des morphismes de schémas, Seconde partie*, in Publications Mathématiques de l'IHÉS **24** (1965), 5–231.

[Gro67] A. Grothendieck, *Éléments de géométrie algébrique (rédigés avec la collaboration de Jean Dieudonné): IV. Étude locale des schémas et des morphismes de schémas, Quatrieme partie*, in Publications Mathématiques de l'IHÉS **32** (1967), 5–361.

[Hai06] T. Haines, *Equidimensionality of convolution morphisms and applications to saturation problems*, Advances in mathematics, **207.1** (2006), 297–327.

[Han73] H. C. Hansen, *On cycles in flag manifolds*, Mathematica Scandinavica **33.2** (1973), 269–274.

[Har13] R. Hartshorne, *Algebraic geometry,* Vol. 52. Springer Science Business Media, 2013.

[HL93] W. Haboush, N. Lauritzen, *Varieties of unseparated flags*, in Linear Algebraic Groups and Their Representations (Los Angeles, CA, 1992) **153**, Contemp. Math. Amer. Math. Soc., Providence, RI (1993), 35–57.

[Hei18] J. Heinloth, *Hilbert-Mumford stability on algebraic stacks and applications to $\mathcal{G}$-bundles on curves*, Épijournal de Géométrie Algébrique 1 (2018).

[Jan03] J. Jantzen, *Representations of Algebraic Groups, 2nd edition*, Mathematical Surveys and Monographs **107**, American Mathematical Society (2003).

[JMW14] D. Juteau, C. Mautner, G. Williamson, *Parity sheaves*, Journal of the American Mathematical Society, **27(4)**, (2014), 1169-1212.

[KL79] D. Kazhdan, G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, in Inventiones Mathematicae 53.2 (1979), 165-–184.

[Kum02] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, **204**, Springer Science Business Media, (2002).

[LM97] V. Lakshmibai, P. Magyar, *Standard monomial theory for Bott-Samelson varieties of GL(n)*, in Publications of the Research Institute for Mathematical Sciences **34** (1997).

[Lau93] N. Lauritzen, *Line bundles on projective homogeneous spaces*, PhD thesis, University of Illinois (1993).

[Lau96] N. Lauritzen, *Embeddings of homogeneous spaces in prime characteristics*, in American Journal of Mathematics **118.2** (1996), 377–387.

[Lau97] N. Lauritzen, *Schubert cycles, differential forms and D-modules on varieties of unseparated flags*, in Compositio Mathematica **109.1** (1997), 1–12.

[MR04] R. Marsh, K. Rietsch, *Parametrizations of flag varieties*, in Representation Theory of the American Mathematical Society **8.9** (2004), 212–242.

[Mil80] J. Milne, *Étale Cohomology (PMS-33)*, Princeton University Press (1980).

[Mil17] J. Milne, *Algebraic Groups: The Theory of Group Schemes of Finite Type over a Field*, Cambridge University Press (2017).

[MV00] I. Mirković, K. Vilonen, *Perverse sheaves on affine Grassmannians and Langlands duality*, Mathematical Research Letters **7.1** (2000) 13–24.

[Ols16] M. Olsson, *Algebraic spaces and stacks*, **62** American Mathematical Soc., (2016).

[Ram75] A. Ramanathan, Annamalai, *Stable principal bundles on a compact Riemann surface*, Mathematische Annalen 213 (1975), 129-152.

[Sch89] H. Schubert, *Lösung des Charakteristiken-Problems für lineare Räume beliebiger Dimension*, Mitt. Math. Gesellschaft Hamburg, **1** (1889) pp. 134–155

[Spr81] T. A. Springer, *Quelques applications de la cohomologie d' intersection*, S´eminaire Bourbaki, **24** (1981-1982), Expos´e No. 589.

[Spr98] T. A. Springer, *Linear Algebraic Groups, 2nd edition*, Springer Science & Business Media (1998).

[Sri17] S. Srinivasan, *Motivic decomposition of projective pseudo-homogeneous varieties* in Transformation Groups **22.4** (2017), 1125–1142.

[Sun99] X. Sun, *Remarks on semistability of G-bundles in positive characteristic*, Compositio Mathematica **119.1** (1999) 41–52.

[Sta23] The Stacks Project authors, *The Stacks project* (2023).

[SV94] P. Sankaran, P. Vanchinathan, *Small resolutions of Schubert varieties in symplectic and orthogonal Grassmannians*, Publications of the Research Institute for Mathematical Sciences **30.3** (1994): 443-458.

[Tot19] B. Totaro, *The failure of Kodaira vanishing for Fano varieties, and terminal singularities that are not Cohen-Macaulay*, in J. Algebraic Geom. **28** (2019), 751–771.

[Wen93] C. Wenzel, *Classification of All Parabolic Subgroup Schemes of a Semisimple Linear Algebraic Group over an Algebraically Closed Field of Positive Characteristic*, in Trans. Amer. Math. Soc. **337.1** (1993), 211–218.

[Yun11] Z. Yun, *Global springer theory*, Advances in Mathematics **228.1** (2011) 266–328.

[Yun15] Z. Yun, *Lectures on Springer Theories and Orbital Integrals*, Geometry of Moduli Spaces and Representation Theory. Ed. R. Bezrukavnikov, A. Braverman. Z. Yun. IAS/Park City Mathematics Series 24 (2017), 155–215.

[Zel83] A. V. Zelevinskii, *Small resolutions of singularities of Schubert varieties*, Functional Analysis and Its Applications **17.2** (1983), 142–144.

[Zha22] S. Zhang, *Bott-Samelson-Demazure-Hansen Varieties for Projective Homogeneous Varieties with Nonreduced Stabilizers*, in Transform. Groups. (2022), 1–26.