Local Lagrangian and Fixed-Point Floer (Co)homologies

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Abstract of the Dissertation

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This work consists of two logically-independent parts. In the first, we define a broad class of local Lagrangian intersections coined quasi-minimally degenerate (QMD) before developing techniques for studying their local Floer homology. The main result is: if L_0, L_1 are two Lagrangian submanifolds whose intersection decomposes into QMD sets, there is a spectral sequence converging to their Floer homology $HF_*(L_0, L_1)$ whose E^1 page is obtained from local data given by the singular homologies of the QMD pieces. We then give some applications of these techniques towards studying affine varieties, reproducing some prior results using our more general framework. The second part studies the fixed-point Floer cohomology of monodromies of Milnor fibrations arising from algebraic isolated hypersurface singularities. The main result is a novel proof that families of such singularities with constant Milnor number also have constant multiplicity and log canonical threshold. This answers a conjecture of Zariski and recovers a theorem of Varchenko.

Contents

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1 Introduction

This dissertation is the combination of two separate but related projects. The first presents a general framework which aids in the computation of certain Lagrangian Floer-theoretic invariants using local data. It extends the work of Poźniak [Po9] to the so-called quasi-minimally degenerate (QMD) case and the packaging comes in the form of a spectral sequence as observed by Seidel [Sei99]. After presenting the general framework, we give four demonstrations of studying affine varieties, using the spectral sequence.

The second project focuses on a different type of Floer-invariant, one associated to a symplectomorphism. In particular, we study families of Milnor fibration of isolated hypersurface singularities and their monodromies which are symplectomorphisms. By combining a topological quantum field theory (TQFT) perspective with adjacency of singularities, we give a novel proof of a conjecture due to Zariski [Zar71]: if a family of isolated hypersurface singularities have constant Milnor number, it also has constant multiplicity. Note that the statement does not contain symplectic geometry content. Our proof also shows the log canonical threshold remains constant, recovering a result of Varchenko [Var82].

Though the two projects may appear unrelated, the inspiration for the definition of QMD intersections comes from singularities of algebraic varieties and some of our current work is aims to use our spectral sequence to compute the symplectic cohomology of Milnor fibers or more generally, smoothings of normal and numerically $\mathbb{Q}$-Gorenstein singularities. On the other hand, for the sake of organization, we will present some foundational concepts relevant to both projects at the start but further on, the main content is split into two parts that may be read independently.

2 Some Needed Symplectic and Contact Geometry

The purpose of this brief section is to establish some basic definitions, terminology, and conventions; more definitions will be later introduced as needed. A **symplectic manifold** is a smooth manifold M equipped with a symplectic form ω; i.e. a nondegenerate, closed 2-form. The existence of such a 2-form implies that $\dim_{\mathbb{R}} M = 2n$ and that M is orientable, with a preferred orientation coming from the volume form ω^n. In the case that M is a closed manifold; i.e. compact and without boundary, we also have that $H^{2k}(M, \mathbb{R}) \neq 0$ for $0 \leq k \leq n$. However, the symplectic manifolds of interest in this paper will be open symplectic manifolds with additional conditions. Here are some definitions for objects that we can associate to a symplectic manifold.

Definition 2.1. *Let $H : M \to \mathbb{R}$ be any smooth function on a symplectic manifold (M, ω). Then, because of nondegeneracy, there exists a vector field X_H which is defined by $\iota_{X_H}\omega = dH$; we call X_H the **Hamiltonian vector field** of H and it is the ω-dual of dH. The function H is called a **Hamiltonian function**.*

Such a X_H has the property that the time t flow ψ_t defined by the vector field satisfies $\psi_t^*\omega = \omega$ for every t. In general, any diffeomorphism $\psi : M \to M$ which satisfies $\psi^*\omega = \omega$ is called a **symplectomorphism** and those that arise from a function H are called **Hamiltonian diffeomorphisms.** In fact, if $H_t : M \to \mathbb{R}$ is a time-dependent smooth family of functions, we can still define vector fields X_{H_t} and ψ_t which satisfy the flow equation and we still call these Hamiltonian diffeomorphisms. We also mention subobjects that are abundantly used in symplectic and contact geometry.

Definition 2.2. *Let $V \subset (W, \omega)$ be a subspace of a dimension $2n$ symplectic vector space and define $V^\omega := \{w \in W : \omega(w, v) = 0 \, \forall v \in V\}$. A subspace V is called **isotropic** if $V \subset V^\omega$ and **coisotropic** $V^\omega \subset V$. If $V = V^\omega$, then V is called **Lagrangian**.*

An easy fact is that Lagrangian subspaces are n dimensional (half the dimension). If we have a smooth submanifold $S \subset (M, \omega)$ in a symplectic manifold, then we say that it is **isotropic (resp. coisotropic or Lagrangian)** if for all $p \in S$, $T_pS \subset T_pM$ is isotropic (resp. coisotropic or Lagrangian).

3 Preliminary Remarks on Minimal Degeneracy

We now begin a discussion of the first project mentioned above which will continue until Section 10. The main inspiration for the later definition of quasi-minimally degenerate intersection begins with considering a large class of minimally degenerate functions, a term coined in Kirwan's thesis [Kir84]. Here, we give a definition that is somewhat more general than the one Kirwan originally gave and later on in Section 2, will further generalize the definition. However, in more recent work with Penington, Kirwan has considered very general functions as well [KP20].

Definition 3.1. *Let $f : M \to \mathbb{R}$ be a smooth function on a manifold M. A set C is called **minimally degenerate** if the following conditions hold.*

1. *C is a compact set contained in the set of critical points for f and f is constant on C. C has an isolating open neighborhood V which means that inside of $V \setminus C$, f does not have any critical points. Such a C is called a critical subset of f.*
2. *There is a submanifold S containing C such that $f|_S$ takes C as its minimum set.*
3. *At every point $x \in C$, the tangent space T_xS is maximal among all subspaces of T_xM on which the Hessian $Hess_x f$ is positive semi-definite (synonymously, nonnegative).*

If the critical points of f is a disjoint union $\bigsqcup C$ where each C is minimally degenerate, then we say that f is minimally degenerate.

In effect, minimal degeneracy means that critical sets can be as degenerate as minima but no worse. This large class of functions includes many interesting examples such as Morse-Bott functions or functions on varieties which have subvarieties with singularities as critical sets (see Section 4. In symplectic geometry, the importance of this definition arises when considering the norm square of a moment map $|\mu|^2$. In general, $|\mu|^2$ is not Morse-Bott but may be minimally degenerate and hence, can still be studied via Morse theoretic techniques. For this reason, Definition 3.1 is sometimes referred to as a Morse-type definition. Kirwan applied such tools to $|\mu|^2$ which are a major element of her proof of Kirwan surjectivity. This result is celebrated for its importance towards studying symplectic quotients and geometric invariant theory.

3.1 Organization

In Section 4, we first summarize some of the properties of minimal degenerate functions before expanding on Definition 3.1 by introducing the definitions of **flattened degeneracy** and **quasi-minimal degeneracy**. As is often the case in mathematics, defining something isn't difficult but defining something *useful* can be. We hope to demonstrate the usefulness of these definitions by proving a series of results. The main result is the existence of a C^1-small perturbation which enlarges a flattened degenerate critical set into a submanifold with boundary without changing the homotopy type of the critical set.

In Section 5, we define flattened and quasi-minimal degeneracy for a subset C of the intersection of a pair of Lagrangians. Part of the definition involves a submanifold S, much like in Definition 3.1 and the definitions of Section 4. In fact, although the Lagrangian definition of quasi-minimal degeneracy is fairly general, in some cases, one can think of C as locally modeled on minimally degenerate functions. Indeed, later on in Section 5, we prove a result relating the "Morse" and Lagrangian definitions of quasi-minimal degeneracy in the case that one of the Lagrangians is the graph of an exact 1-form.

Before proving the "Morse" implies Lagrangian result however, we establish the existence of a C^1-small perturbation of the Lagrangians locally around an isolated subset C of the intersection. Like the "Morse" case, the perturbation yields a codim 0 submanifold with boundary Σ of S. The process by which we do this can be intuitively thought of as "thickening" the intersection and we shall refer to these Σ as **thickenings** of C. This is the content of Theorem 5.2 and is the key technical result which we use to extend a result of Poźniak in Section 6. One primary motivation behind constructing such a specific perturbation is this: in Floer theory, genericity is a double-edged sword. For example, a small generic perturbation of a Hamiltonian function results in gaining the favorable property of nondegeneracy yet the perturbed function can hardly be studied precisely because it is generic. Therefore, it is often more helpful to perturb in a controlled way at the expense of having some amount degeneracy as a result.

The purpose of Section 6 is to give an exposition of the results from Poźniak's thesis [Po9]. The main result that we will use is Theorem 6.4. It roughly says: if two Lagrangians intersect cleanly in an isolated neighborhood, then the local Floer homology is determined by the singular homology of the clean intersection. The necessary definitions for understanding this theorem are provided in the section.

As stated above, in Section 7, we use Theorem 5.2 to extend Poźniak's main result to give a stronger Theorem 7.2: if two Lagrangians have a quasi-minimally degenerate intersection in an isolated neighborhood, then the local Floer homology is determined by the singular homology of the intersection.

In Section 8, we extract a spectral sequence from Theorem 7.2, much in the same way that Seidel extracted a spectral sequence from Theorem 6.4 in [Sei99]. The local data obtained from the isolated neighborhoods form the E^1 page of the spectral sequence and converges to the (global) Lagrangian Floer homology. For the sake of simplifying the exposition, we shall ignore orientations and work with $\mathbb{Z}_2$ coefficients.

Theorem 3.2. *Suppose $L_0 \cap L_1$ decomposes into $\bigsqcup C_p$ where each C_p is quasi-minimally degenerate. Let $\Sigma_p := \Sigma_{C_p}$ be the thickening for C_p. Then there is a spectral sequence which converges to $HF_*(L_0, L_1)$ and whose E^1-term is*

$$E^1_{pq} = \begin{cases} H_{p+q-\iota(\Sigma_p)}(C_p; \mathbb{Z}/2), & 1 \leq p \leq r; \\ 0, & \textit{otherwise.} \end{cases}$$

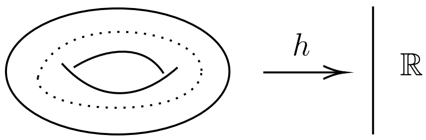
As applications, we perform four brief demonstrations in Section 9: we compute the Hamiltonian Floer homology of a particular affine variety, give an alternative method for studying certain manifolds with corners, study the E^1 page for a particular log Calabi-Yau, and show how the spectral sequence may be applied to situations beyond that of log Calabi-Yau.

The first and fourth examples may have been computed before but the author does not know where they may appear in the literature. The second and third examples have been previously computed but relied heavily on structure which would not be available in more general settings. For example, Ganatra and Pomerleano in [GP20] computed local Hamiltonian Floer cohomology of certain types of minimally degenerate families of orbits which appear as manifolds with corners. In this paper, our result is able to compute local Hamiltonian Floer cohomology for all such families of minimally degenerate orbits. In a different vein, Pascaleff, in [Pas14], computed wrapped Lagrangian Floer cohomology of certain Lagrangian sections in a log Calabi-Yau surface. In this paper, we indicate how the spectral sequence aids in computing wrapped Floer cohomology for many other Lagrangians inside smooth affine surfaces beyond the log Calabi-Yau case.

The final section is less mathematical and more conjectural. We speculate about other applications and research directions of minimal degeneracy.

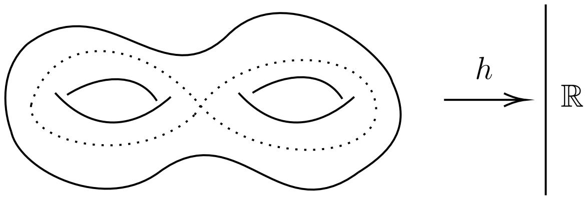
4 Definitions and Basic Results Regarding Minimal Degeneracy

In the preliminary remarks, Definition 3.1 tells us what it takes for a set to be minimally degenerate and also what it means for functions to be minimally degenerate. Here are two concrete examples to compare. One Morse-Bott example is that of the height function h on a torus, "laid on its side." The critical submanifold for the height function is a disjoint union of two circles; the function takes its maximum on one circle and minimum on the other. Call C_M the circle on which it is a maximum. The definition of a minimally degenerate function requires that C_M is contained in a submanifold S such that when restricting h to S, $h|_S$ takes its minimum on C_M. Here, it is convenient to simply let $S := C_M$ so that $h|_S$ is constant and thus, C_M is both the maximum and minimum set of $h|_S$. Here is a cartoon of the situation where we depict only one of the circles.



The height function on a torus

Example 4.1. A minimally degenerate example to have in mind begins with a compact genus-2 surface M embedded in $\mathbb{R}^3$ (we'll suppress notation ordinarily used to denote embeddings). The embedding is such the height function $h(x, y, z) = z$ has critical points which form two figure 8's—a subvariety, call them E_1 and E_2. Observe that the dimension of $\ker \operatorname{Hess} h$ is not constant along the connected components of the critical points. For E_1, the minimum figure 8, M itself serves as the needed submanifold containing this minimal set. Here is a picture.



The height function on a genus 2 surface

However, the maximum figure 8 E_2 does not have a submanifold S containing it such that $h|_S$ takes a minimum on this figure 8. So as it stands, h is not minimally degenerate though E_1 is minimally degenerate. If we perturb h locally around E_2 so that its new maximum is achieved only at a single point p, then the new function is minimally degenerate.

4.1 Comparing Kirwan's Original Definition to Definition 3.1

Having seen some examples, it is worth pointing out that though Definition 3.1 is similar to one found in Kirwan's thesis [Kir84] (p. 65), there are a few important differences.

Firstly, we focus on individual critical sets C because we wish to later consider isolated sets $C \subset \Lambda \cap L$ that are contained in the intersection of Lagrangian submanifolds. Such a C has no intrinsic reference to a smooth function but nonetheless, may display minimal degeneracy type properties such as admitting a submanifold S with some nice properties. This will be made precise later.

Moreover, we make no assumptions about the normal bundle of S and the relevant restrictions of the Hessian are positive semi-definite instead of positive definite. The first relaxing of the definition is simply because we don't need the assumptions but the second condition is quite crucial and will be explained in due time. There is also a third difference in definition: we don't require the critical set to be a finite union but instead, we require the compact sets to have isolating neighborhoods.

This third difference is made for two reasons. The first is that we wish to avoid certain pathological compact sets such as $A = \{1/n : n \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ or the "Hawaiian earring"

$$\mathcal{H} = \bigcup_{n=1}^{\infty} \left\{ (x, y) \in \mathbb{R}^2 \mid \left(x - \frac{1}{n}\right)^2 + y^2 = \left(\frac{1}{n}\right)^2 \right\}.$$

Indeed, if A were to arise as the minimum set of some smooth function, then somewhere between each $\frac{1}{n}$ and $\frac{1}{n+1}$ would be a maximum. These maxima would converge towards 0 and hence, A is not isolated. A similar argument also shows that H is not isolated. In point of fact, Kirwan's definition also prohibits such closed sets.

However, finite unions are not general enough in Floer theory; one often encounters infinite unions of Reeb orbits. So in order to continue to prohibit pathological closed sets but also expand the definition to allow for infinite unions, we've opted to use isolated closed sets in our definition.

4.2 Generalization of Minimal Degeneracy

Since Lagrangian intersections are our main motivation, consider the following example.

Example 4.2. Let L_0 be the zero section of $T^*\mathbb{R} \cong \mathbb{R}^2$ with standard symplectic form and L_1 be the graph of $df = 2x$. The linear symplectomorphisms on $\mathbb{R}^2$ can be thought of as elements of $SL(2,\mathbb{R}) \cong Sp(2,\mathbb{R})$. One such example is the shearing map represented by

$$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix},$$

which sends L_1 to the graph of $df = -2x$ while fixing L_0. Hence, before applying the linear symplectomorphism, L_1 is described by the Morse function $f(x) = x^2$ and afterwards, described by $-f$.

This example illustrates that even in the case of transverse intersections, a choice of Weinstein neighborhood affects whether the intersection behaves like a minimum or a maximum. Indeed, one may construct examples of Lagrangians L_0 and L_1 intersecting transversally at a point p and then choose a Weinstein neighborhood so that L_1 is the graph of df where f is a Morse function with a critical point at p of arbitrary index. Hence, in the Lagrangian setting, any attempt to define $C \subset L_0 \cap L_1$ to be minimally degenerate should not rely on a Weinstein neighborhood since, depending on the neighborhood, C may or may not be minimal. Thus, this motivates us to give a few definitions that generalize Definition 3.1.

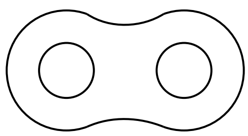
Definition 4.3. *Let $f : M \to \mathbb{R}$ be a smooth function and let C be an isolated family of critical points of f. Let $S \subset M$ be a submanifold containing C. We say that f is* ***flattened degenerate along*** (C, S) *if:*

1. $f|_S$ *is minimal along* C.
2. $\ker Hess_x f = T_x S$ *for all* $x \in C$.

If the critical points of f form a disjoint union $\bigsqcup C$ where each C has a submanifold S_C such that f is flattened degenerate along (C, S_C), we will say that f is flattened degenerate.

Observe that if we take a smaller submanifold $S' \subset S$ that still contains C, then f is also flattened degenerate along (C, S'). Now, for an example:

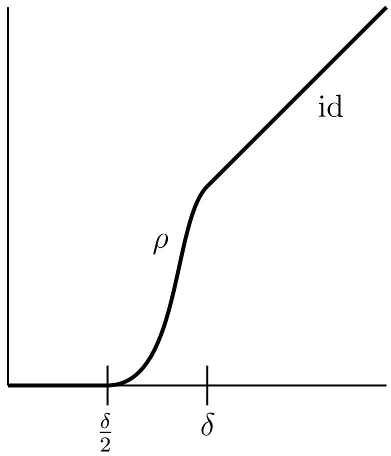
Example 4.4. Consider the genus 2 surface M from Example 4.1, embedded into $\mathbb{R}^3$. We may deform the embedding by "flattening" the bottom of the surface so that the height function has a set of minima C that looks like a "mask" (see figure below) and is a codim 0 submanifold-with-boundary. If we pick local coordinates, along C, h is constant and hence its 2nd derivatives and hence, Hessian, is trivial along C. Thus, the height function is flattened degenerate along (C, M). It's obvious here but worth pointing out that the submanifold S we chose is M itself).



As this example illustrates, one way to obtain flattened degenerate functions is to perform this flattening process. The following lemma demonstrates the usefulness of this definition and also that the flattening procedure can always be applied to flattened degenerate functions along (C, S) so that we get a new submanifold-with-boundary Σ. Moreover, C will be homotopy equivalent to Σ. In the example above, there is no need to undergo this procedure since C itself is already a submanifold-with-boundary.

Lemma 4.5. *If f is flattened degenerate along (C, S), then there is a codim 0 submanifold $\Sigma \subset S$ with boundary which is an isolated critical set of $\check{f}$ containing C, a function that is C^1 close to f. Moreover, $C \hookrightarrow \Sigma$ is a homotopy equivalence.*

Proof. **Case 1:** Suppose $\dim S = \dim M$. In this case, f is minimal on C and $\mathrm{Hess}_x f = 0$ for all $x \in C$. Let $\delta > 0$ (a parameter we may adjust) and $\rho : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $\rho(x) = 0$ for $x \leq \delta/2$, $\rho(x) = x$ for $x \geq \delta$, and $0 < \rho'(x) < 3$ for $x \in (\delta/2, \delta)$.



Then, let $\check{f} = \rho \circ f$. Note that $\check{f}^{-1}(0) = f^{-1}([0, \delta/2])$ and by Sard's theorem, for generic δ, $f^{-1}(\delta)$ is a submanifold. Hence, $\Sigma := \check{f}^{-1}(0)$ is a submanifold with boundary and clearly a critical set of $\check{f}$. Since C is isolated by some open set U, we may choose δ small enough such that $\Sigma \subset U$. We may also choose δ small enough so that the vector field $-\nabla \check{f}$ has complete flow. This gives the desired homotopy inverse to $C \hookrightarrow \Sigma$ since the only points on Σ that are stationary are points of C and in the limit, the flow of the gradient of any point goes to a point in C . The bounds on the 1st derivative plus the fact that $\mathrm{Hess}_x f = 0$ for $x \in C$ makes $\check{f}$ C^1-close to f.

Case 2: $\dim S < \dim M$. We work in a tubular neighborhood of S which is diffeomorphic to the normal bundle of S: $\nu : NS \to S$. Then, of course S is a codim 0 submanifold of S and $f|_S$ has C as a minimum and also satisfies the Hessian condition. We may apply Case 1 to this and obtain a codim 0 submanifold $\Sigma \subset S$ which is the critical set of $\rho \circ f|_S$.

Next, we wish to pullback $\rho \circ f|_S$ to M in some way. There are two possibilities: we may take $\nu^*(\rho \circ f)$ which is constant on the fibers. Hence, the critical set of this function is $NS|_\Sigma$, not merely Σ. However, $NS|_\Sigma$ is homotopy equivalent to Σ due to the contractible fibers. We may then extend this function from the tubular neighborhood to all of M via bump functions.

Alternatively, we fix a complete metric on NS and let $r(x, v) = |v|$ be the radial function with respect to this metric. Then $(1 + r^4)(\rho \circ f|_S)$ is a function on NS which has Σ as an isolated critical set. We can then extend this function to all of M once again, using bump functions. ∎

We think of the process in which Σ is obtained as a sort of "flattening" process because ρ is constant on $[0, \delta/2]$. We will often refer to a Σ obtained in this way as a **thickening** of C. We now give another definition.

Definition 4.6. *We say that f is **quasi-minimally degenerate** (QMD) along C if there exists a smooth function $\tau \geq 0$ and a submanifold S so that:*

1. $\tau^{-1}(0) = C$.

2. $\ker \mathit{Hess}_x \tau$ *is transverse to* $T_x S$ *for each* $x \in C$.

3. $f - \tau$ *is flattened degenerate along* (C, S).

If the critical points of f *form a disjoint union* $\bigsqcup C$ *such that* f *is QMD along each* C*, then we say that* f *is quasi-minimally degenerate.*

Remark: Here are a few immediate and important observations.

- The height function in Example 4.1 is QMD along the set C which is a minimal figure 8. This is because we can find a $\tau \geq 0$ with $\tau^{-1}(0)$ (this is always doable for any closed set C using partition of unity). Then, using $S = M$, the transversality condition is automatically satisfied. Lastly, we can arrange τ to have these properties and in addition, be such that $f - \tau$ is flattened degenerate. Intuitively, the "flattening" procedure of Example 4.4 uses such a τ.

- Since $\tau \geq 0$ and C is a set of minima for τ, then $(d\tau)|_C = 0$.

- Note that if we take a smaller submanifold S' inside of S which still contains C, there's no issue since all the properties of τ still hold on S'. In this way, we may elect to "shrink" S while maintaining the relevant properties. Put another way, S should not be viewed as part of the data but rather the germ of submanifolds containing C is what's essential. Here is an immediate consequence. Note that since C is a minimum for τ, then for each $x \in C$, $\mathrm{Hess}_x \tau \geq 0$ on $T_x S$ and the same holds for points near C. Hence, by choosing a small enough S, we can assume that for any $x \in S$, $\mathrm{Hess}_x \tau \geq 0$ on $T_x S$ rather than only those $x \in C$.

- Similarly, for small enough S, points $x \in S$ will also be such that $\ker \mathrm{Hess}_x \tau$ is transverse to $T_x S$. The advantage to defining quasi-minimal degeneracy in this way is that we don't need to invoke a metric since $d\tau$ vanishes on C. Off of C, the Hessian requires a choice of metric.

- Note that $\frac{d}{ds}(d(f - s\tau)_x) = d\tau_x$ for each x. If $x \in C$, then $d\tau_x = 0$ and hence, for $x \in C$, $\frac{d}{ds}(d(f - s\tau)_x) = 0$; i.e. $d(f - s\tau)_x$ is independent of s. In particular, set $s = 0$ and thus, $d(f - s\tau)_x = df_x$. If $x \notin C$ but is close to C, then $d\tau_x \neq 0$ because $\tau^{-1}(0) = C$. So $\frac{d}{ds}(d(f - s\tau)_x) \neq 0$. By being very near C, we can assume this means that $d(f - s\tau)_x \neq 0$ as well for any $s \in [0, 1]$. Hence, we have a isotopy between a QMD function f and a flattened degenerate function $f - \tau$ where during the isotopy, no new critical points are introduced near C nor are any critical points lost.

A less immediate observation is that, by adding one more condition, we have a function that is minimally degenerate in the spirit of Kirwan's definition. More precisely:

Lemma 4.7. *If, in addition to the properties listed in 4.6,* $\mathit{Hess}_x(f - \tau)$ *has no positive eigenvalues, then* C *is minimally degenerate in the sense of Definition 3.1. Conversely, if* C *is a minimally degenerate set, then* f *is QMD along* C *and* $\mathit{Hess}_x(f - \tau)$ *has no positive eigenvalues.*

Proof. ($\Longrightarrow$) $f - \tau$ is flattened degenerate which means $(f-\tau)|_S$ has C as minimum and $\ker \operatorname{Hess}_x(f-\tau) = T_xS$ for $x \in C$. This means that along C, the 1st order derivatives of $f - \tau$ restricted to the directions tangent to S are not varying. In other words, $f - \tau$ has S as a critical set: $d(f-\tau)_x = 0$ for $x \in S$ (if necessary, we shrink S, treating it as a germ). Hence, $df_x = d\tau_x$ for $x \in S$. Since τ has C as minimum, $f|_S$ has C as minimum.

Let $x \in C$ and consider $\operatorname{Hess}_x f$. Let $V \subset \ker \operatorname{Hess}_x \tau$ be a subspace transverse to T_xS satisfying $V \cap T_xS = 0$ (i.e. it is of complementary dimension). Now, $\operatorname{Hess}_x(f-\tau) = \operatorname{Hess}_x f - \operatorname{Hess}_x \tau$ has no positive eigenvalues (the additional property mentioned above). And restricting $\operatorname{Hess}_x \tau$ to V (which is in its own kernel) gives a trivial quadratic form. Hence, $\operatorname{Hess}_x f$ restricted to V is negative definite.

Also, since $\ker \operatorname{Hess}_x(f-\tau) = T_xS$, we have that $\operatorname{Hess}_x f$ and $\operatorname{Hess}_x \tau$ agree when restricted to T_xS. $\tau \geq 0$ so $\operatorname{Hess}_x \tau$ is non-negative definite on T_xS and hence, so is $\operatorname{Hess}_x f$. Together, these two facts show that T_xS is the maximal subspace for which $\operatorname{Hess}_x f$ is non-negative definite.

($\Longleftarrow$) Conversely suppose that C is a connected, isolated minimally degenerate critical locus of f in the sense of Kirwan's thesis (generalized slightly in this paper) and let S be the corresponding "minimizing submanifold." We construct the auxiliary function τ as follows: By using a complete metric on M, we can identify a neighborhood of S with a tubular neighborhood $U \subset NS$ of its normal bundle NS. Let $r : U \to [0,\infty)$ be the radial coordinate for this tubular neighborhood; i.e. $(x,v) \in U$ is mapped to the norm $|v| \in [0,\infty)$. Let $\pi : U \to S$ be the projection map. We define $\tau = r^4 + \pi^*(f|_S)$ on U and then use a bump function to extend τ to the whole of M.

Note that $\tau^{-1}(0) \subset S$ because if we have $(x,v) \in U$ where $v \neq 0$, then $|v|^4 > 0$. Since $f|_S$ has C as its minimum (we can assume it takes values 0), we conclude that $C = \tau^{-1}(0)$. This tells us that $d\tau|_C = 0$, thanks to C being a set of critical points of f. Also, when restricted to S, the r^4 part vanishes and π is trivial. So then, $d(f-\tau)|_S = 0$ because $f - \tau$ vanishes along S. Moreover, S is the minimum for $f - \tau$ and so $f - \tau$ is negative definite along NS. This means that $\ker \operatorname{Hess}(f-\tau) = T_xS$.

Lastly, when restricted to S, r^4 vanishes. So for a point $x \in S$, we only need to consider the $\pi^*(f|_S)$ piece of τ. But along a fiber of U, this is constant and hence the Hessian of $\pi^*(f|_S)$ vanishes on NS. This means that $NS \subset \ker \operatorname{Hess} \tau$ which implies that this kernel is transverse to T_xS. □

This lemma shows us that a function being minimally degenerate is equivalent to it having a non-negative function τ with some properties, the essential one being that $f - \tau$ is flattened degenerate. We will shortly see the usefulness of this notion when studying Lagrangian intersections.

5 Lagrangian Quasi-Minimal Degeneracy

Similar to above, we will give two definitions concerning the intersection of any pair of Lagrangians.

Definition 5.1. *Let $\Lambda, L \subset M$ be two Lagrangian submanifolds of a symplectic manifold (M,ω) and $C \subset \Lambda \cap L$. We say that C is* ***flattened degenerate along a submanifold*** *$S \subset \Lambda$ with respect to Λ, L if:*

1. *$T_xS = T_x\Lambda \cap T_xL$ for each $x \in C$.*

2. *There exists a time dependent Hamiltonian H_t whose derivative and Hessian vanish along C and satisfies $\frac{d}{dt}(H_t) \geq 0$.*

3. $\phi_1^H(S) \subset L$.

Remark: Here are some important points.

- As promised, this definition is intrinsic in the sense that it does not depend on a choice of Weinstein neighborhood.
- The time-dependence of H is natural in symplectic geometry and gives a more flexible definition than requiring an autonomous Hamiltonian.
- Much in the case of functions, the submanifold S is best thought of as a germ since we may "shrink" the submanifold to some S' and use the same H without modification since it has all the same properties on the subset $S' \subset S$. Therefore, some of the remarks we made for functions also applies here.

Theorem 5.2. *Suppose that C is flattened degenerate along S with respect to Λ, L. There exists a C^1-close family of Lagrangians Q_s with $Q_0 = L$ realized by a Hamiltonian isotopy such that:*

1. *There exists a fixed open neighborhood V of C such that $Q_s \cap \Lambda \cap V$ is a compact,connected subset inside V for every $s \in [0,1]$; i.e. there is a fixed isolating neighborhood.*

2. *The intersection $Q_s \cap \Lambda$ near C is homotopy equivalent to a codim 0 submanifold-with-boundary $\Sigma \subset S$ which contains C. Moreover, the inclusion $C \hookrightarrow \Sigma$ is a homotopy equivalence.*

Proof. We will break this into two cases: $\dim S = n$ where $\dim M = 2n$ and $\dim S < n$. In reality, the may be treated as one case, as we will indicate. But hopefully, this presentation is more digestible.

We begin by choosing a Weinstein neighborhood $\mathcal{U}$ of $C \subset \Lambda$. Hence, we may view $\mathcal{U}$ as being symplectomorphic to a neighborhood of $C \subset \Lambda$ inside of $T^*\Lambda$. Let $\pi : T^*\Lambda \to \Lambda$ be the bundle map. Even if we did not have the condition above that $S \subset \Lambda$, viewing S as a germ of a submanifold, we may pick a codim 0 submanifold $S' \subset S$ and project it, via π, to land in Λ. Our choice of S' is made so that $\pi(S')$ is a submanifold (without boundary) of Λ. This shows that we do not really lose any generality by assuming $S \subset \Lambda$.

Case 1: When $\dim S = n$, then for $x \in C$, $T_xS = T_x\Lambda \cap T_xL$ implies that $T_x\Lambda = T_xL$; i.e. Λ and L are tangent along C. This means that along C, L is transverse to the fibers of $T^*\Lambda$ and thus, is a graph of some section. Since L is Lagrangian, the section is a closed 1-form and hence, a locally exact 1-form. This means that near C, L is the graph of some df_1. In fact, each Lagrangian in the family $\Lambda_t = \phi_t^H(\Lambda)$ is a graph of some df_t. Now, condition (2) of the definition of flattened degenerate tells us that $dH_t|_C = 0$ and for each $x \in C$, $\mathrm{Hess}_x H_t = 0$, and $\frac{d}{dt}H_t \geq 0$. This implies that $C \subset \Lambda_t \cap \Lambda$ for each t.

Lemma 5.3. *Let $\Lambda_t = \phi_t^H(\Lambda)$ be a family of Lagrangians in $T^*\Lambda$ where H has the properties above. Then, Λ_t is the graph of some df_t. There exists a time-dependent vector field $\widehat{Z}_t$ on $T^*\Lambda$ such that $\widehat{Z}_t$ vanishes on C. Additionally, if ψ_t is the flow of $-\widehat{Z}_t$, then $\frac{d}{dt}f_t \circ \psi_t \geq 0$. In particular, $f_t \geq 0$.*

Proof. Since Λ_t is a graph, at each point of Λ_t, the tangent space of $T^*\Lambda$ splits into a vertical direction and a "horizontal" direction (tangent direction to Λ_t). We may then split X_{H_t} into two components as well. More precisely, first restrict X_{H_t} to $TT^*\Lambda|_{\Lambda_t}$. Then, it equals $Y_t + Z_t$ where Y_t is tangent to the fibers and Z_t is tangent to Λ_t.

This horizontal component Z_t may behave in such a way that the functions f_t used to define Λ_t are decreasing. However, we may project Z_t to Λ where it generates a diffeomorphism on Λ. If we pull back this projection, we get a vector field on the cotangent bundle (call it $\widehat{Z}_t$) which generates a symplectomorphism. Observe that $Z_t - \widehat{Z}_t$ is a vector field in the fiber direction and also that Z_t and $\widehat{Z}_t$ both vanish on C. Let ψ_t be the flow generated by $-\widehat{Z}_t$. The flow counteracts the horizontal movement that comes from the original Z_t and hence, the piece of X_{H_t} that matters when flowing $\psi_t(\Lambda_t)$ is the vertical Y_t.

Because $\frac{d}{dt}H_t \geq 0$, the Y_t will only flow the $\psi_t(\Lambda_t)$ in such a way that the defining functions f_t increase as well. That is, $\frac{d}{dt}f_t \circ \psi_t \geq 0$.

Since $L_0 = \Lambda$, we may assume $f_0 \equiv 0$. Pick $x \in T^*\Lambda$ and $t_0 \in [0,1]$. Then there is a path on the interval $[0, t_0]$ given by $\gamma(t) = \psi_t(\psi_{t_0}^{-1}(x))$ which starts at $\psi_{t_0}^{-1}(x)$ and ends at x. Along this path, $\frac{d}{dt}f_t(\gamma(t)) \geq 0$ and $f_0(\gamma(t)) = 0$. Hence, $f_{t_0}(x) \geq 0$. □

Let $\delta > 0$ be a parameter and $\rho : \mathbb{R} \to \mathbb{R}$ be the function we saw earlier. Let $\rho_s(x) = (1-s)x + s\rho(x)$ be a linear interpolation of ρ. Observe that for $x \in \rho^{-1}(0)$, unless $s = 1$, $\rho_s(x) > 0$.

Let $\check{f}_s = \rho_s \circ f_1$. We adjust δ such that $\check{f}_1^{-1}(0) = (\rho \circ f_1)^{-1}(0) = f_1^{-1}([0, \delta/2])$ is a submanifold with boundary also being a smooth manifold. This is possible since regular values are dense by Sard's theorem. Moreover, we choose δ to be small enough that so that $\check{f}_1^{-1}(0)$ is contained within a neighborhood U of C in which f_1 has no critical points other than those in C.

We can say something similar about the $\check{f}_s$. We have that $d\check{f}_s = d\rho_s \circ df_1$. Since $\rho_s(x) = (1-s)x + s\rho(x)$, then $d\rho_s = (1-s)\,\mathrm{id} + sd\rho$. For $x \in [0, \delta/2]$, $d\rho_s$ acts by scalar multiplication via $1 - s$. For $x \in [\delta, \infty)$, $d\rho_s = \mathrm{id}$. Hence, in a neighborhood of C and when $s < 1$, the critical points of $\check{f}_s$ are precisely C. It is only when $s = 1$ do we have $\Sigma := f_1^{-1}([0, \delta/2])$ as a set of critical points.

Setting Q_s to be the graph of $d\check{f}_s$, we have a family of Lagrangians where $Q_0 = L$ and another description of Σ as the intersection $Q_1 \cap \Lambda$. Again, Σ is a codim 0 submanifold-with-boundary of S containing C. If we pick a complete metric, then $-\nabla f_1$ does not vanish anywhere in $U \setminus C$ and this vector field gives a deformation retract of Σ onto C. We will postpone showing that the family Q_s form a Hamiltonian isotopy until we introduce Lemma 5.4 below.

Case 2: As before, we work in a Weinstein neighborhood and assume that $S \subset \Lambda$. $\pi : T^*\Lambda \to \Lambda$ is the cotangent bundle. When $\dim S = k < n$, $\pi^{-1}(S) = T^*\Lambda|_S$ is a coisotropic submanifold of $T^*\Lambda$. Coisotropic submanifolds admit foliations and in this case, the foliation is actually a fibration over the leaf space which is symplectomorphic to the symplectic reduction T^*S.

In more detail, let $\eta \in T_x^*S$. Then a fiber over $(x, \eta) \in T^*S$ is $F = \{\varphi \in T_x^*\Lambda : \varphi|_{T_x^*S} = \eta\}$. If we use a metric to get an orthogonal decomposition: $T^*\Lambda = T^*S \oplus T^*S^\perp$, all such φ decompose as $\varphi = \eta + \alpha$. And the set of α form a vector space. So this fibration is a vector bundle.

So let $p : T^*\Lambda|_S \to T^*S$ be the fibration. As in the first case, we have a family of

Lagrangians $\Lambda_t = \phi_t^H(\Lambda) \subset T^*\Lambda$. Then let $\widetilde{\Lambda}_t := p(\Lambda_t \cap T^*\Lambda|_S) \subset T^*S$; it is a Lagrangian. See [MS17], p. 221.

Claim: $\widetilde{\Lambda}_t$ is the graph of some df_t where $f_t : S \to \mathbb{R}$ are functions satisfying $f_0 \equiv 0$ and $f_t \geq 0$.

Proof. One way to show that $\widetilde{\Lambda}_t$ is a graph near C is to show that for $x \in C$, $T_xS = T_x\widetilde{\Lambda}_t$. Now, we have the "clean intersection" condition that $T_xS = T_x\Lambda \cap T_xL$ for $x \in C$. Also, $dH_t|_C = 0$ and $\mathrm{Hess}_x H_t = 0$ for $x \in C$. This means that the flow does not move C at all and that 1st order derivatives of H_t do not change in any directions at $x \in C$ because the Hessians vanish. Hence, for $x \in C$ $T_xS = T_x\Lambda \cap T_x\Lambda_t$ for each t.

Armed with that fact, then projecting to the leaf space, we have for $x \in C$, $T_xS = T_xS \cap T_x\widetilde{\Lambda}_t$ which just means $T_x\widetilde{\Lambda}_t = T_xS$. Arguing as in Case 1, this means that near C, each $\widetilde{\Lambda}_t$ is the graph of some section. Being Lagrangians, the sections are closed 1-forms which are locally exact. Hence, near C, $\widetilde{\Lambda}_t$ is the graph of df_t where f_t is a smooth family of functions $f_t : S \to \mathbb{R}$ and $f_0 \equiv 0$.

As for showing that $f_t \geq 0$, since $\frac{d}{dt}H_t \geq 0$, the restriction of the H_t to T^*S also satisfies the same property. Hence, the result follows from Lemma 3.3. □

We are now in a position to apply the same sort of argument towards "flattening" the intersection by using a family of functions $\rho_s : \mathbb{R} \to \mathbb{R}$ with the same properties as before. Hence, we obtain a family of Lagrangians $\widetilde{Q}_s$ with $\widetilde{Q}_0 = \widetilde{\Lambda}_1$. Since $\phi_1^H(S) \subset L \cap \phi_1^H(\Lambda)$ and dH_t and the Hessians of H_t vanish on C, then along C, $\phi_1^H(S)$ is tangent to S. Hence, near C, $p(\phi_1^H(S) \cap T^*\Lambda|_S)$ is a Lagrangian submanifold of dimension k, coinciding with $\widetilde{L} = p(L \cap T^*\Lambda|_S)$. Hence, we'll also denote $\widetilde{Q}_0$ by $\widetilde{L}$. We also denote $\widetilde{\Sigma} := \widetilde{Q}_1 \cap S$ which is a submanifold with boundary that deformation retracts onto C.

Now, recall that for each $x \in C$, $T_x\widetilde{\Lambda}_t = T_xS$ due to C being a minimum set for f_t. In particular, $T_x\widetilde{L} = T_x\widetilde{\Lambda}_1 = T_xS$ for $x \in C$. Now, the Hessian of $\rho_s \circ f_1$ at a critical point will have 1st order terms of ρ_s paired with 2nd order terms of f_1 and 2nd order terms of ρ_s paired with 1st order terms of f_1. Since the 1st order terms of f_1 vanish near C and the 1st order terms of ρ_s vanish near $x = 0$, we may conclude that near C, both the derivative and Hessian of $\rho_s \circ f_1$ vanish and hence, for each s, $T_x\widetilde{Q}_s = T_x\widetilde{L}$ for $x \in C$.

What we need to do now is show that these Lagrangians lift to our original setting; i.e.

Claim (restatement of what is to be proved): There exists a C^1-close Lagrangian family Q_s realized by a Hamiltonian isotopy X_{K_s} such that $Q_0 = L$, $T_xQ_s = T_xL$ for all $x \in C$, and $p(Q_s \cap T^*\Lambda|_S)$ is the graph of $d(\rho_s \circ f_1)$. Moreover, the Hamiltonian K_s has vanishing derivative and Hessian on C.

Proof. The family $\widetilde{Q}_s$ is given by a isotopy $i_s : \widetilde{L} \to (T^*S, d\lambda)$.

Following [Oh15], in more generality, let $i : [0,1] \times L \to (M, \omega)$ be an isotopy and L_s be the image of $\{s\} \times L$ under i. The pullback $i^*\omega$ can be written as $i^*\omega = ds \wedge \alpha + \beta$ where α and β both vanish when contracted with ∂_s, a vector field tangent to the interval $[0,1]_s$. Let $i_s : L \to [0,1] \times L$ be the natural inclusion.

Next, define $\alpha_s := \iota_{i_*\partial_s}\omega|_{L_s}$ where $i_*\partial_s$ is the pushforward of ∂_s. Note that $i^*\alpha_s(V) = \omega(i_*\partial_s, i_*(V)) = \iota_{\partial_s} i^*\omega = \alpha$. Now, suppose that $i_s^*\alpha$ is closed. This implies that $i_s^* i^* d\alpha_s = 0$. But also, α_s is defined on L_s and the map $i \circ i_s : L \to L_s$ is a diffeomorphism. This implies that $d\alpha_s = 0$. Hence, the ω-dual X_{α_s} is a symplectic vector field defined on L_s.

Next, by definition, $\alpha_s - \iota_{i_*\partial_s}\omega|_{L_s} = 0$. The ω-dual of this vector field is $V_s := X_{\alpha_s} - i_*\partial_s$. This vector field V is tangent to L_s. To show this, suppose that for any Lagrangian L and vector field V, $\iota_V\omega|_L = 0$. This means that V is in TL^ω, the ω-orthogonal space. But L is Lagrangian and so $TL = TL^\omega$. Applying this to our situation, $X_{\alpha_s} - i_*\partial_s \in TL_s$ for each s. This means that the vector field $i_*\partial_s$ which realizes the isotopy, can always be upgraded to a symplectic vector field X_{α_s} simply by reparametrizing the domain L_s. This was achieved using only the assumption that $i_s^*\alpha$ is closed for each s.

The next question is, when is α_s exact? In our situation, the vector fields X_{α_s} vanish on the critical set C and hence, α_s vanishes on C. Being closed, we may conclude that $\alpha_s = df_s$ If we take a neighborhood U of C that deformation retracts to C, then we may pull α_s back by the retraction, which is a homotopy equivalence, thereby extending the α_s to a neighborhood of C. Since de Rham cohomology is a homotopy invariant, this means that the extension is also exact. Hence, the ω-dual of the extension is a Hamiltonian vector field which we'll continue to call X_{α_s}. Let us summarize the last few paragraphs as a general lemma.

Lemma 5.4. *Let $i : [0,1]_s \times L \to (M, \omega)$ be an isotopy of embeddings and $i_s : L \to [0,1] \times L$ the natural map sending L to $\{s\} \times L$. Then we may write $i^*\omega = ds \wedge \alpha + \beta$ where $\iota_{\partial_s}\alpha = \iota_{\partial_s}\beta = 0$. If $i_s^*\alpha$ is closed, then the vector field $i_*\partial_s$ may be modified to a family of symplectic vector fields X_s defined on L_s, the image of $i \circ i_s$. In particular, if L_0 is Lagrangian, then each L_s is Lagrangian.*

Furthermore, if each of the X_s vanish along a set $C \subset M$, then in a neighborhood of C, these X_s may be taken to be Hamiltonian vector fields.

Returning to the proof of the claim, by this lemma, there are time-dependent Hamiltonian vector fields which generate the flow to realize the isotopy i_s. Denote the corresponding Hamiltonian by $\widetilde{K}_s$. Because of the tangency of the $\widetilde{Q}_s$ all along C, the 1st order derivatives of the Hamiltonian vector fields has to vanish; i.e. $\operatorname{Hess}_x \widetilde{K}_s = 0$ for $x \in C$.

The $\widetilde{Q}_s$ differ from each other only in the region of $\kappa := f_1^{-1}([0,\delta])$ because $\rho_s = \mathrm{id}$ on $[\delta, \infty)$. Hence, $\widetilde{K}_s$ is such that the Hamiltonian vector fields vanish on $\widetilde{Q}_s \setminus \kappa$. Let us take $\widetilde{K}_s$ to be constant ouside of this region.

Next, we want to pullback the $\widetilde{Q}_s$ and $\widetilde{L}$ somehow to $T^*\Lambda$ so that the resulting Lagrangians satisfy the tangency condition. We cannot simply pullback by p and then onto some tubular neighborhood of $T^*\Lambda|_S$. Instead, we begin by choosing a metric g on Λ; such a metric defines a section ψ of $p : T^*\Lambda|_S \to T^*S$ in the following way. For a point $(x, \phi) \in T^*S$, $\psi(x, \phi) = (x, \Phi)$ where Φ is the unique covector such that $\Phi|_S = \phi$ and it vanishes on the subspace g-orthogonal to T_xS.

Next, with $k = \dim S$, if we pick a tubular neighborhood $\nu : N \to T^*\Lambda|_S$, then by construction $Q_s := \nu^{-1}(\psi(\widetilde{Q}_s))$ is an n-dim Lagrangian of $T^*\Lambda$ where at each $x \in C$, its tangent space splits as $T_xQ_s = T_xS \oplus T_xF$. Here, F is the fiber of ν and is $n-k$-dimensional.

So we want to pick the tubular neighborhood in such a way that for each $x \in C$, the tangent space $T_xL = T_xS \oplus T_xF$. We already know that L descends to $\widetilde{L}$ and at points $x \in C$, $T_xS = T_x\widetilde{L}$ so we simply need $T_xF \subset T_xL$.

Now, for a point $x \in C \subset S \subset \Lambda$, $T_{(x,0)}T^*\Lambda \cong T_xS \oplus T_xS^\perp \oplus T_x^*\Lambda$. Here, $T_xS \oplus T_xS^\perp \cong T_x\Lambda$ (recall, we chose a metric on Λ). On the otherhand, $T_{(x,0)}T^*\Lambda|_S \cong T_xS \oplus T_x^*\Lambda$ and hence, the fibers of the normal bundle of $T^*\Lambda|_S$ at x can be identified with $T_xS^\perp$. Because $T_xS^\perp \subset T_x\Lambda \subset T_{(x,0)}T^*\Lambda$ and $T_x\Lambda$ is a Lagrangian subspace, then $T_xS^\perp$ is isotropic. So we simply choose a tubular neighborhood of $T^*\Lambda|_S$ with fibers F such that $T_xF = T_xS^\perp$. This is possible since $T_xS^\perp$ is complementary to $T_{(x,0)}T^*\Lambda|_S$. Therefore, by construction, Q_s and L are tangent along C. By Lemma 5.4, we may conclude that near C in $T^*\Lambda$, we also have a Hamiltonian family of vector fields X_{K_s} generating the isotopy. Moreover, similar to in the symplectic reduction, because of the tangency of all the Q_s along C, we may conclude that $\text{Hess}_x K_s = 0$ for $x \in C$. We had postponed showing this Hamiltonian isotopy result for Case 1 but the argument is exactly the same. In some sense, there aren't really two cases. Case 1 is simply the scenario where the symplectic reduction is trivial.

Lastly, the family is C^1-small simply because we have bounds on the first derivative of ρ and hence on the first derivative of ρ_s. On the other hand, df_1 vanishes on C and hence, is small near C. □

Let Σ be the lift of $\widetilde{\Sigma}$. It is the total space of some bundle over $\widetilde{\Sigma}$, produced by lifting twice. The first time by p, we would have the total space of a fibration over $\widetilde{\Sigma}$ with fiber F. And then we lift a second time by ν. Each time, the fibers are contractible so Σ deformation retracts to $\widetilde{\Sigma}$ which itself deformation retracts to C.

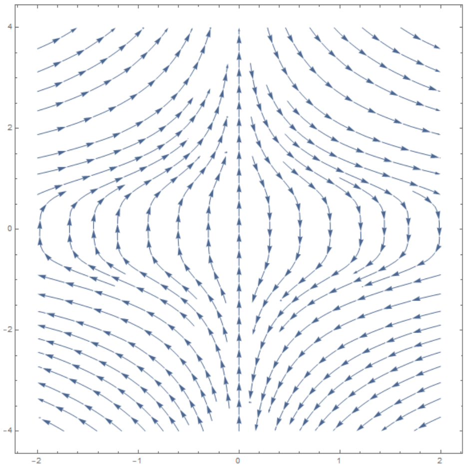
In both Case 1 and Case 2, for $s < 1$, $Q_s \cap \Lambda = C$ and $Q_1 \cap \Lambda = \Sigma$. So we may fix an open neighborhood V containing Σ and that will suffice as an isolating neighborhood. □

Remarks:

- Similar to the "Morse" case, we call Σ a **thickening**. We also have an analog of quasi-minimal degeneracy for Lagrangians.

- It is important to note that in Case 2, L is not the graph of an exact 1-form. It is only when we pass to the symplectic reduction that we have a graph of an exact 1-form in the vicinity of C.

- For the last step of the proof, we also have the option of taking a similar approach to the alternative outlined in the proof of Lemma 4.5. This would give us $\widetilde{\Sigma}$ itself as an isolated critical set rather than its lift. However, one technical but resolvable issue is that if we multiply the pullback function $\nu^*p^*\widetilde{K}_s$ by $(1 + r^4)$ where r is the radial function, the generated flow of the resulting Hamiltonian will not be tangent to the fibers. To briefly illustrate this, consider the simple example $K : \mathbb{R} \to \mathbb{R}, x \mapsto x^2$. If y is the fiber coordinate of $T^*\mathbb{R}$, then $\check{K} := (1 + r^4)\pi^*K = (1 + y^4)x^2$ and $X_{\check{K}} = 4x^2y^3\,\partial_x - 2x(1 + y^4)\,\partial_y$. When $x \neq 0$ and $y \neq 0$, the vector field is not tangent to the fibers. See figure (when $x = 0$, the vector field vanishes; this is not depicted in the image).



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However, in general, one can modify the Hamiltonian so that it coincides with the $\nu^* p^* \widetilde{K}_s$ when restricted to Q_s and hence, the flow is tangent to the fibers along the family of Lagrangians.

Having defined flattened degeneracy and proven a thickening result, we now give a definition for quasi-minimal degeneracy.

Definition 5.5. *We say that $C \subset \Lambda \cap L$ is* ***quasi-minimally degenerate*** *(QMD) if there is a time-dependent Hamiltonian K so that C is an isolated family of intersection points of Λ and $\phi_t^K(L)$ for each $t \in [0,1]$ and moreover, C is flattened degenerate along S with respect to $\Lambda, \phi_1^K(L)$.*

Much like the "Morse" situation, we also have an isotopy which keeps C as an isolated set in the intersections and at time 1, we require flattened degeneracy. Hopefully, context makes it clear whether we mean QMD in the "Morse" sense or in the Lagrangian sense. In the next section, we will show that there is a good reason for using the same name.

5.1 Relating the "Morse" and Lagrangian Definitions

In this section, we prove a result which relates the "Morse" definition of quasi-minimally degenerate to the Lagrangian definition of quasi-minimally degenerate.

Proposition 5.6. *If L is the graph of df in T^*M where f, as a function, is quasi-minimally degenerate in the sense of Definition 4.6 along a critical locus $C \subset M$, then C is also quasi-minimally degenerate in the Lagrangian sense of Definition 5.5.*

To prove this proposition, we first make the following general observation. If $\pi : T^*M \to M$ is the cotangent bundle of M and $\tau : M \to \mathbb{R}$ is any function, then $\pi^*\tau$ is a function on T^*M. Observe that since $d(\pi^*\tau) = d\tau \circ d\pi$, if $Y \in TF$ where F is a fiber of the cotangent bundle (so Y is in the vertical directions of TT^*M), then $0 = d(\pi^*\tau)(Y) = \omega(X_{\pi^*\tau}, Y)$ where $X_{\pi^*\tau}$ is the Hamiltonian vector field associated to $\pi^*\tau$. This holds for every Y in the vertical direction. Hence, $X_{\pi^*\tau}$ is ω-orthogonal to the fiber F. But the fiber is Lagrangian and so $TF = TF^\omega$. This means $X_{\pi^*\tau} \in TF$ as well. The main point is that the flow of $\pi^*\tau$ is tangent to the fibers of the bundle.

Next, let $U \subset M$ be an open subset diffeomorphic to an open subset of $\mathbb{R}^n$ such that $T^*M|_U \cong U \times \mathbb{R}^n$; i.e. the cotangent bundle is trivialized on this set, and $U \times \mathbb{R}^n$ is a

Darboux chart. This means that if we let $(x_1, ..., x_n)$ be coordinates of U and $(y_1, ..., y_n)$ be fiber coordinates, then the symplectic structure of $U \times \mathbb{R}^n$ is symplectomorphic to $\omega_{st} = \sum_i^n dx_i \wedge dy_i$.

Next, let $\tau : U \to \mathbb{R}$ be a smooth function. Then, $X_{\pi^*\tau}$ can be computed. Indeed, as is well known in classical mechanics:

$$X_{\pi^*\tau} = \sum_i^n \frac{\partial(\pi^*\tau)}{\partial y_i}\partial_{x_i} - \frac{\partial(\pi^*\tau)}{\partial x_i}\partial_{y_i}.$$

We know $X_{\pi^*\tau}$ is tangent in the fiber directions; this is because $\pi^*\tau$ is constant on the fibers and hence the $\frac{\partial(\pi^*\tau)}{\partial y_i}$ vanish. On the other hand, the coefficients of the ∂_{y_i} are simple $\frac{\partial\tau}{\partial x_i}$. This means that the flow is translation in the fibers by $-t\, d\tau$ where t is the time. This shows us that for a function $\tau : M \to \mathbb{R}$, the image of $d\tau$ viewed as a section of $\pi : T^*M \to M$ is a Lagrangian which is mapped to the zero section by the time 1 flow of $\pi^*\tau$. We summarize this in a lemma.

Lemma 5.7. *Let $\tau : M \to \mathbb{R}$ be a smooth function and $\pi : T^*M \to M$ be the cotangent bundle of M. Then the autonomous Hamiltonian vector field $X_{\pi^*\tau}$ is tangent to the fibers and the time t flow $\phi_t^{\pi^*\tau}$ acts by translation on the fibers via $-t\, d\tau$.*

Proof of Prop. 5.6 In our situation, we assume that f is quasi-minimally degenerate along C with corresponding submanifold S and that L is the graph of df inside of T^*M. By definition, there exists a $\tau : M \to [0, \infty)$ with several properties including $\tau^{-1}(0) = C$, $(f - \tau)|_S$ is minimal on C, and $\ker \mathrm{Hess}_x(f - \tau) = T_xS$ for all $x \in C$. Though it's not necessary, we write this as two cases for the purpose of illustrating why we need that $\ker \mathrm{Hess}_x\tau$ is transverse to T_xS for $x \in C$.

Case 1: Suppose $\dim S = \dim M$. In this case, if $\pi : T^*S \to S$ is the contangent bundle, then the flow of $K := \pi^*\tau$ fixes the points $x \in C$. Moreover, $(f - \tau)|_S$ has C as a minimum; WLOG, suppose the minimum value is 0. Because $\tau \geq 0$, for x near C, $\tau(x) > 0$. In order for C to be a minimum of $(f-\tau)|_S$, we need $f(x) > \tau(x)$. This implies that in a small neighborhood of C, f increases more rapidly than C in all directions of S (and hence of M). In particular, for x near but not in C, $df_x - t\, d\tau_x \neq 0$ for $t \in [0, 1]$. This translation, as Lemma 5.7 tells us, is the flow of $\pi^*\tau$. Hence, C remains an isolated set of the intersection $M \cap \phi_t^K(L)$ for each t.

Next, since S is codim 0, $\mathrm{Hess}_x(f - \tau)$ vanishes completely on T_xM for $x \in C$. Conceptually, this means that the 1st order derivatives of $f - \tau$ are not deviating at all from zero along C and hence $\phi_1^K(L)$ is tangent to M (and hence S) at points in C; i.e. $T_xM = T_xS = T_xM \cap T_x\phi_1^K(L)$ which means $T_xM = T_x\phi_1^K(L)$ for $x \in C$.

Lastly, to show that C is flattened degenerate with respect to $\phi_1^K(L), M$, we need a Hamiltonian H such that $\phi_1^H(S) \subset \phi_1^K(L)$ along with the other properties. Since $\phi_t^K(L)$ can be described as the graph of $d(f - t\tau)$, we may simply let $H = \pi^*(\tau - f)$. Then, $dH|_C = 0$ and $\mathrm{Hess}_xH = 0$ for $x \in C$. And since it is an autonomous Hamiltonian, $\frac{d}{dt}H = 0$. The flow of $-H$ brings $\phi_1^K(L)$ to the zero section; hence, the flow of H brings the zero section to $\phi_1^K(L)$, including $\phi_1^H(S)$.

Case 2: Suppose $\dim S < \dim M$. In this case, one concern is that though $(f - \tau)|_S$ is minimal on C and hence, C is isolated in S, it may be that $K = \pi^*\tau$ will behave badly in the normal directions to S. This is why it is crucial that we have another condition on τ:

ker Hess$_x\tau$ is transverse to T_xS for $x \in C$. If we pick a metric, we can then consider the Hessian of τ at a point $y \in S$, near C. Since it is near C and transversality is an open condition, ker Hess$_y\tau$ is transverse to T_yS. In other words, τ doesn't do anything in the directions normal to S, such as introduce new critical points.

Thus, we can still use $K = \pi^*\tau$ as the Hamiltonian and C remains an isolated subset of $\phi_t^K(L) \cap M$ for all $t \in [0,1]$. Once again, $\phi_1^K(L)$ is the graph of $d(f-\tau)$. Moreover, since $\ker \mathrm{Hess}_x(f-\tau) = T_xS$ for $x \in C$, we have that $T_xS = T_xM \cap T_x\phi_1^K(L)$. And lastly, the same H still has all the correct properties. □

The following corollary is immediate.

Corollary 5.8. *Let $\Lambda, L \subset (M,\omega)$ be Lagrangians and $C \subset \Lambda \cap L$. Suppose there exists a Weinstein neighborhood U around C and a symplectomorphism $\varphi : U \to V$ where V is a neighborhood of the zero section of $T^*\Lambda$, $\Lambda \cap U$ is mapped to the zero section, and $L \cap U$ is mapped to a Lagrangian in V which is the graph of df where f is QMD. Then, C is QMD in the Lagrangian sense.*

5.2 Local Lagrangian Floer Theory

In the definitions above, we took the effort to ensure that if C is an isolated subset of a Lagrangian intersection, then whenever we perturbed the Lagrangians by Hamiltonian isotopies, C remained isolated or at least its homotopy type does not change. The reason for this is because we want to study Lagrangian intersections using Lagrangian Floer theory. Let's begin with a rough intuitive description of the usual Lagrangian Floer homology before discussing a local homology theory.

In nice cases, Lagrangian Floer homology associates to a pair (L_0, L_1) of Lagrangians a group $HF^*(L_0, L_1)$ which has a few properties.

1. $HF^*(L_0, L_1)$ "categorifies" intersection numbers in the sense that $\chi(HF^*(L_0, L_1)) = L_0 \cdot L_1$ (intersection number of the smooth topology).

2. $HF^*(L_0, L_1)$ is Hamiltonian isotopy invariant. So if ϕ^{H_0}, ϕ^{H_1} are Hamiltonian diffeomorphisms, then $HF^*(\phi^{H_0}(L_0), \phi^{H_1}(L_1)) \cong HF^*(L_0, L_1)$.

3. If $L = L_0 = L_1$, then $HF^*(L, L) \cong H^*(L)$, the singular homology of L.

4. If L_0 and L_1 intersect transversally, then $HF^*(L_0, L_1)$ is the homology of a chain complex generated by the intersection points. This implies that the rank of HF^* gives a (refined) lower bound for Lagrangian intersections: $\#(\phi_1^H(L_0) \cap L_1) \geq \mathrm{rk}\, HF^*(L_0, L_1) \geq L_0 \cdot L_1$.

We are deliberately vague about what "nice cases" means but these properties cannot always hold. Indeed, a compact Lagrangian L in $\mathbb{C}^n$ can be displaced by a Hamiltonian ϕ so that $L \cap \phi(L) = \varnothing$. Hence, $0 \geq \mathrm{rk}\, HF^*(L, \phi(L)) = \mathrm{rk}\, HF^*(L, L) = \mathrm{rk}\, H^*(L)$; this obviously cannot happen. The example can be modified so that L is displaced by a compactly supported Hamiltonian isotopy and hence, the compact manifold $\mathbb{CP}^n$ also serves as a counterexample.

However, moving forward, we will not be too concerned with these issues as they have been addressed in many other texts. We shall simply proceed to the local situation and add some rigor to the description. For full details, consult section 3 of [Po9]. Let

$P(L_0, L_1)$ be the space of paths starting on L_0 and ending on L_1. Let $\mathcal{U} \subset P(L_0, L_1)$ be a closed subset and let $ev : \mathcal{U} \times I \to M$ be the map sending $(\gamma, t) \mapsto \gamma(t)$. We say that $\mathcal{U}$ is bounded if the image of ev is precompact.

Next, we would like to define an action functional. In general, the action functional is defined only on the universal cover $\widetilde{P}(L_0, L_1)$. Choose a base point $\gamma_0 \in P(L_0, L_1)$. Let $u : I \times I \to M$ represent an element $\tilde{\gamma} \in \widetilde{P}$; i.e. $u(0,t) = \gamma_0(t), u(1,t) = \gamma(t)$, and $u(s,i) \in L_i$. We also introduce a Hamiltonian $H : M \times [0,1] \to \mathbb{R}$. Then, the action functional is

$$A_H(\gamma) := \int u^*\omega + \int_0^1 H_t(\gamma(t))\, dt.$$

The critical points are precisely the paths $\gamma(t) \in P(L_0, L_1)$ satisfying $\gamma'(t) = X_t(\gamma(t))$. Here, X_t is the time-dependent Hamiltonian vector field. To make this a local theory, we simply consider the critical points of A_H inside of $\mathcal{U}$.

We also consider a moduli space $\mathcal{M}_{J,H}(L_0, L_1, \mathcal{U})$ of J-holomorphic strips $u : \mathbb{R}_s \times I_t \to M$; these strips satisfy a twisted Cauchy-Riemann equation $\overline{\partial} u + \nabla_u H = 0$. Additionally, we require the elements of this moduli space to satisfy $u(s, \cdot) \in \mathcal{U}$ for all $s \in \mathbb{R}$. The **maximal invariant subset** $\mathcal{S}_{J,H}(\mathcal{U})$ is defined to be the image of $\mathcal{M}_{J,H}(L_0, L_1, \mathcal{U})$ under the evaluation map $ev : \mathbb{R} \times \mathcal{M}_{J,H}(L_0, L_1) \to P(L_0, L_1)$, $ev(s,u)(t) = u(s,t)$.

We also say that $\mathcal{S}_{J,H}(\mathcal{U})$ is isolated if its closure under the compact-open topology is contained in the interior of $\mathcal{U}$. If it is isolated, then whenever a sequence $u_n \in \mathcal{M}_{J,H}(L_0, L_1, \mathcal{U})$ converges to $u \in \mathcal{M}_{J,H}(L_0, L_1)$, in fact, $u \in \mathcal{M}_{J,H}(L_0, L_1, \mathcal{U})$.

It was shown in Pozniak's thesis [Po9] that:

Proposition 5.9. *Assume that $\mathcal{U}$ is bounded, $\mathcal{S}_{J,H}(\mathcal{U})$ is isolated and the symplectic action $\mathcal{A}_H$ is defined on $\mathcal{U}$. There is an $\epsilon > 0$ such that if $\|J' - J\|_{C^1} < \epsilon$ and $\|H' - H\|_{C^1} < \epsilon$, then $\mathcal{S}_{J',H'}(\mathcal{U})$ is also isolated. Moreover, if both pairs $(J,H), (J',H')$ are regular, then*

$$H_*(C_*(L_0, L_1, \mathcal{U}, J', H')) \cong H_*(C_*(L_0, L_1, \mathcal{U}, J, H)).$$

In this context we say $\mathcal{S}_{J',H'}(\mathcal{U})$ is a continuation of $\mathcal{S}_{J,H}(\mathcal{U})$. When the context is clear and we have the given Lagrangians, H, and J, we may sometimes simplify notation and just write $HF(\mathcal{U})$ for the local Floer homology.

5.3 Hamiltonian Floer Theory

It should be stated that we may define flattened and quasi-minimal degeneracy for Hamiltonian Floer theory as well since we can recover Hamiltonian Floer theory from Lagrangian Floer theory. If $K : M \times [0,1] \to \mathbb{R}$ is a time dependent Hamiltonian and ϕ_1^K is its time one flow, then we study the fixed points of ϕ_1^K. Let $\Gamma \subset M \times M$ be the graph of ϕ_1^K and $\Delta \subset M \times M$ the diagonal. The fixed points of ϕ_1^K are in one-to-one correspondence with the intersection points $\Gamma \cap \Delta$ (for details of how to relate the differentials of the two complexes, see [Hut10]). In the symplectic manifold $(M \times M, \omega \oplus (-\omega))$, Γ and Δ are Lagrangian submanifolds. Hence, we say that a set C of fixed points of ϕ_1^K is flattened degenerate (QMD resp.) if and only if $C \times C$ is flattened degenerate (QMD resp.) in $\Gamma \cap \Delta$. We conjecture that there is an equivalent definition that is more natural or at least easier to work with for the Hamiltonian setting but we do not yet have good candidates. As a suggestion, if K is the time-dependent Hamiltonian from above, then the definition should involve studying $\ker(\phi_1^K - \mathrm{id})$.

6 Clean Intersections and Pozniak's Results

From now on, we shall prefer the Lagrangian viewpoint but everything translates over to the Hamiltonian viewpoint as outlined above. In Pozniak's thesis [Po9], he gives us a way to compute local Floer homology for Lagrangians that cleanly intersect along a submanifold.

Definition 6.1. *Let L_0, L_1 be Lagrangians and N a submanifold. We say that L_0 and L_1 have a clean intersection along N if $N \subset L_0 \cap L_1$ and for every $x \in N$, $T_xN = T_xL_0 \cap T_xL_1$.*

In general, the intersection may be wild but if some part of the intersection is clean along N and there are no other intersection points in a small neighborhood of N, there is a way to define local Floer homology in a neighborhood of N. If $\partial N \neq \varnothing$, then we do not have a clean intersection but it is straightforward to adapt Pozniak's arguments to intersections along manifolds with boundary; we will do that in section 5. We first record Pozniak's original results.

6.1 A Standard Model

The first of Pozniak's results shows that cleanly intersecting Lagrangians have a standard model.

Theorem 6.2. *Let (M, ω) be a symplectic manifold and L_0, L_1 two Lagrangian submanifolds of M which intersect cleanly along a compact manifold N. There exist a vector bundle $\tau : L \to N$, a neighborhood V_0 of N in T^*L, a neighborhood U_0 of N in M, and a symplectomorphism $\phi : U_0 \to V_0$ such that*

$$\phi(L_0 \cap U_0) = L \cap V_0 \text{ and } \phi(L_1 \cap U_0) = TN^{ann} \cap V_0.$$

Before sketching Pozniak's proof, here is a short outline. First, he proved that cleanly intersecting Lagrangians may be put into a standard form. Then, by way of a Moser-type argument, Pozniak showed that there exists a vector bundle $\tau : L \to N$ and also neighborhoods U of N in M and V of N in T^*L, and a symplectomorphism $\phi : U \to V$ which satisfies: $\phi(L_0 \cap U) = L \cap V$ and $\phi(L_1 \cap U) = TN^{ann} \cap V$. Here, $TN^{ann} = \{\alpha \in T^*L_N : \alpha|_{TN} = 0\}$ (the annihilator). The L he chose is $L = TN^{\perp} \subset TL_0$ for a chosen metric on L_0 and the exponential map gives the desired tubular neighborhood. The proof does not actually rely on compactness of N and can be adapted to open manifolds.

Sketch of Pozniak's proof:

1. Use the Weinstein neighborhood theorem to view a neighborhood of L_0 as symplectomorphic to a neighborhood of the zero section of T^*L_0. Choose a metric on L_0 and let $L = TN^{\perp} \subset TL_0$. The exponential map gives a diffeomorphisms of neighborhoods of N in L_0 and L which induces a symplectomorphism. Therefore, without loss of generality, treat $L_0 = L$ and $M = T^*L$.

2. Let $L_2 = TN^{ann}$. The goal now is to show there exists a symplectomorphism $\chi_1 : U_1 \subset T^*L \to V_1 \subset T^*L_2$ where both the domain and range are neighborhoods of N such that

   (a) $\chi_1|_{L_2 \cap U_1} = \text{id}$

(b) $\chi_1(L \cap U_1) \subset T^*L_2|_N$

(c) $\chi_1(L_1 \cap U_1) = \Gamma_\alpha$, the graph of a 1-form α on L_2.

3. Assuming χ_1 exists, note that $N \subset \Gamma_\alpha$ which implies that $\alpha_N = 0$. Then for $x \in L_2 \cap U_1$, the map

$$\psi_\alpha : T_x^*L_2 \to T_x^*L_2, \beta \mapsto \beta - \alpha(x)$$

is a symplectomorphism. We may choose a sufficiently small neighborhood V_2 of N in T^*L_2 so that $\psi_\alpha(V_2) \subset \chi_1(U_1)$.

4. Let $\phi : U_0 \to V_0$ be defined by $\phi(x) = \chi_1^{-1} \circ \psi_\alpha \circ \chi_1(x)$. Letting $U_0 = \chi_1^{-1}(V_2)$, we can check that ϕ satisfies each of the properties we want.

So now, we need to show that χ_1 exists.

1. Note that we need only show that for a defined map χ_1, $\chi_1(L_1)$ should be transverse to the fibers of T^*L_2 in order for the image to be a graph.

2. Let $E = \ker d\tau$ be the vertical subbundle of TL (recall that $L = TN^\perp$). For $x \in N$,

$$T_x(T^*L) = T_xL \oplus T_x^*L = E_x \oplus T_xN \oplus T_xN^{ann} \oplus E_x^{ann}.$$

3. It is straightforward to show that $T_xL_1 \cap (E_x \oplus E_x^{ann}) = 0$. Then, L_1 is transverse to E_x^{ann}. If show $\chi_1(E^{ann} \cap U_1) \subset T_x^*L_2$, then we'll have shown that $\chi_1(L_1)$ is transverse to the fibers of T^*L_2.

4. We need two lemmas:

   (a) There exists a vector bundle $\sigma : T^*L \to TN^{ann}$ with fibers as Lagrangian submanifolds of T^*L. In particular, $\sigma^{-1}(x) = E_x^{ann}$ for all $x \in N$.

   The proof mainly involves checking that a proposed σ does have a vector bundle structure.

   (b) Let $\sigma : V \to L$ be a vector bundle such that (V, ω_0) is a symplectic manifold and the fibers V_x are Lagrangian. Then for every compact set $K \subset L$, there is a fiber preserving symplectomorphism χ defined in a neighborhood of U of K in V; $\chi : (U, \omega_0) \to (T^*L, \omega)$ where ω is the standard symplectic form on T^*L. Moreover, $\pi_L \circ \chi = \sigma|_U$ and $\chi|_L = \text{id}$.

   The proof involves a Moser-type argument to show the fiber preserving property.

5. The two lemmas immediately show the existence of χ_1. $\chi|_L = \text{id}$ gives (a), $\pi_L \circ \chi = \sigma|_U$ gives (b), and the fiber preserving property gives (c). □

6.2 Morse and Floer Data Coincide

The second result of Pozniak's results shows that a C^1 small Morse function allows us to identify Morse and Floer critical points and flow lines.

Theorem 6.3. *Let (N, g_N) be a compact, Riemannian manifold, $\tau : L \to N$ a vector bundle over N and $f : N \to \mathbb{R}$ a C^2 function on N. Let $\pi : T^*L \to L$, $f_L = f \circ \tau$, and $H = f \circ \tau \circ \pi$. We can construct a metric g on L by lifting g_N (see p. 81-82 for details). Let $J = J_g$ be the associated almost complex structure defined using $d\lambda$ and g (λ is the canonical 1-form on T^*L). We also suppose there is a neighborhood U of N in L such that $\|\nabla^g df_L(x)\| \leq 1$ for all $x \in U$. Then the following holds:*

1. *All critical points and gradient lines with respect to J for the action functional $\mathcal{A}_H$ in $\Omega(\pi^{-1}(U), U, TN^{ann})$ are t-independent and so they are in 1-1 correspondence with the critical points and the gradient lines of f with respect to g_N.*

2. *The critical points of $\mathcal{A}_H$ are nondegenerate if f is a Morse function. In this case, if $x^{\pm} \in Crit(f)$ and $u : \mathbb{R} \to N$ is a t-independent element of $\mathcal{P}(x^-, x^+)$, then the linearized operator $D_{J,H}(u)$ is onto if and only if the operator $D_f(u) : W^{1,p}(u^*TN) \to L^p(u^*TN)$, $D_f(u)\xi = \nabla_s \xi + \nabla_\xi \nabla f(u)$ is onto and the assignment $\xi \mapsto \xi'(s,t) = \xi(s)$ gives the isomorphism $\ker D_f(u) \cong \ker D_{J,H}(u)$.*

Remark: Note that once a metric and function are fixed on N, Pozniak gives a *specific* metric and almost complex structure on T^*L, rather than take generic pairs. Despite the non-genericity, the second part of the result asserts that we still have smooth moduli spaces.

Sketch of Pozniak's proof:

1. There exists local coordinates $x = (q, q', p, p')$ on T^*L such that $\frac{\partial H}{\partial p} = \frac{\partial H}{\partial p'} = \frac{\partial H}{\partial q'} = 0$ and $X_H(0, 0, df(q), 0)$. So the Hamiltonian flow is $\phi_t(q, q', p, p') = (q, q', p + tdf(q), p')$.

   Now consider paths γ with boundary conditions $\gamma(0) \in L$ and $\gamma(1) \in TN^{ann}$. When $x \in L$, $p = p' = 0$ and when $x \in TN^{ann}$, $q' = p + df(q) = 0$. Hence, the only Hamiltonian paths are constant: $x(t) = (q, 0, 0, 0)$ with q being a critical point of f.

2. Suppose that $D^2 f(q) := \text{Hess}_q f$ is nondegenerate. Then in these coordinates,

$$D\phi_1(x) = \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ D^2 f(q) & 0 & I & 0 \\ 0 & 0 & 0 & I \end{pmatrix}$$

   Let $v = (Q, Q', 0, 0)$ be a vector tangent to L. Then, $D\phi_1(x)v = (Q, Q', D^2 f(q)Q, 0)$ is tangent to TN^{ann} if and only if $Q' = D^2 f(q)Q = 0$. In this case, $Q = 0$ as well as $D^2 f(q)$ is nondegenerate. So $Q = Q' = 0$. Hence, $D\phi_1(x)(T_x L) \cap T_x TN^{ann} = \{0\}$. Therefore, x is nondegenerate as a critical point of the action functional $\mathcal{A}_H$.

3. Let g^D be the Kaluza-Klein metric on T^*L which is a "diagonal" lift of g. The important feature of g^D is that it is compatible with the canonical symplectic structure and J that we've defined. Let $V = \ker d\pi$ be the vertical subbundle of $T(T^*L)$. Then $dH = df_L \circ d\pi$ vanishes on V which means ∇H with respect to g^D is in the horizontal subspace: $\nabla H(\xi) \in H_\xi$.

Moreover, $d\pi|_{H_\xi}$ is an isometry so $d\pi(\nabla H(\xi)) = \nabla^g f_L(\pi(\xi))$ which tells us that ∇H is a horizontal lift of $\nabla^g f_L$. Thus, if $x \in N$, then $\nabla H(x) = \nabla^(g_N) f(x) \in TN$. This means that if $u : \mathbb{R} \to N$ is a gradient line of f, then $v(s,t) := u(s)$ is a gradient line of $\mathcal{A}_H$ satisfying our boundary conditions.

4. Suppose now that $v : \mathbb{R}_s \times I_t \to U$ satisfies the Floer equation. We want to show that v is t-independent. Let $x(s,t)$ and $y(s,t)$ be the horizontal and vertical components of $v(s,t)$. Then $\frac{\partial v}{\partial s}$ and $\frac{\partial v}{\partial t}$ also decompose into horizontal and vertical components, giving a new form of the Floer equation:

$$\frac{\partial x^*}{\partial s} - \nabla_t y + dH_1(x) = 0, \qquad \nabla_s y + \frac{\partial x^*}{\partial t} = 0.$$

Here, $*$ means the dual using the metric g. Of course, there are also boundary conditions for these equations. We wish to show that $y \equiv 0$ and hence, $\frac{\partial x}{\partial t}$ which implies that $v(s,t) = x(s) : \mathbb{R} \to N$ is a gradient line of f.

5. Define

$$\gamma(s) := \frac{1}{2}\int_0^1 |y(s,t)|^2 \, dt.$$

Note that $\lim_{s\to\pm\infty} \gamma(s) = 0$ and also $\gamma \geq 0$. So it attains a maximum on $\mathbb{R}$. However, Pozniak showed the following lemma: if $\|\nabla^g f_L\|_{L^\infty} < 1$, then $\gamma''(s) \geq 0$ for all $s \in \mathbb{R}$. When the hypothesis holds, this means that γ is both concave up everywhere but also achieves a maximum. This implies that γ must be constant and in fact, $\gamma \equiv 0$ because it limits to 0. Hence $y \equiv 0$.

The proof of this lemma requires the crucial fact: y is a solution to the elliptic equation $\Delta y - \langle \nabla df_L, \nabla_s y\rangle = 0$.

6. To show $\ker D_f(u) \cong \ker D_{J,H}(u)$, we similarly decompose a vector field $\xi = (\zeta, \eta) \in \Gamma(u^* T(T^*L))$ into horizontal and vertical components and obtain a way of writing the linearized Floer equation in these horizontal and vertical components.

We may show that $\zeta \in \ker D_f(u)$ is a solution to the linearized Floer equation and hence in $D_{J,H}(u)$. Conversely, let $\xi = (\zeta,\eta) \in D_{J,H}(u)$. If we similarly define

$$\gamma_1(s) := \frac{1}{2}\int_0^1 |\eta(s,t)|^2 \, dt,$$

we may prove as Pozniak did that when $\|\nabla^g df_L\|_{L^\infty} < 1$, then $\gamma_1''(s) \geq 0$ for all s. Hence, $\eta \equiv 0$ and $\xi = (\zeta, 0)$ is in $\ker D_f(u)$.

7. Lastly, similar arguments show that $\ker D_f^*(u) \cong \ker D_{J,H}^*(u)$ and so $D_f(u)$ is onto if and only if $D_{J,H}(u)$ is onto. □

6.3 Pozniak's Main Theorem

Finally, we state the main result from Pozniak's thesis about the local Floer homology of clean intersections.

Theorem 6.4. *Let (M, ω) be a symplectic manifold and L_0, L_1 two Lagrangian submanifolds that intersect cleanly along a compact, connected submanifold N. Fix base point $x_0 \in N$. If U is any relatively compact neighborhood of N such that*

1. *Aside from those in N, there are no other critical points of $\mathcal{A}$ in the connected component $P(U, L_0, L_1, x_0)$ of the constant path x_0 in the path space $P(U, L_0, L_1)$.*

2. *The action function of ω is well-defined in $P(U, L_0, L_1, x_0)$, meaning, we do not need to lift to the universal cover.*

Then $\mathcal{U} = P(U, L_0, L_1, x_0)$ is an isolating neighborhood and $\mathcal{S}_{J,0}(\mathcal{U}) = N$ for any almost complex structure J. There exists an almost complex structure J_0 and a Hamiltonian $H_0 : M \to \mathbb{R}$ such that

1. *$\mathcal{S}_{J_0,H_0}(\mathcal{U})$ is a continuation of N.*

2. *(J_0, H_0) is a regular pair and if $g_N = g_J|_N$, $f = H_0|_N$, then (g_N, f) is Morse-Smale.*

3. *The Floer complex $CF_*(\mathcal{U}, J_0, H_0)$ coincides with the Morse complex $CM_*(N, g_N, f)$ and thus*
$$HF_*(\mathcal{U}, \mathbb{Z}_2) \cong H_*^{sing}(N, \mathbb{Z}_2).$$

For the main theorem, Theorem 6.4, Pozniak first assumes we're in the setting of the standard form for a clean intersection. His theorem 6.3 shows that for a C^1 small Morse function f, under a canonical metric g, almost complex structure J, and Hamiltonian H on T^*L defined from f, the Floer and Morse critical points and flow line all live within a small enough neighborhood U of N and they all coincide.

Hence, the proof of the main theorem is mainly showing that for any chosen isolated neighborhood U of N which sits inside the neighborhood of our standard form for the clean intersection, there exists ϵ such that when $\|H\|_{C^1} < \epsilon$, the Floer critical points and trajectories are contained within U.

Sketch of Pozniak's proof: Given an isolated neighborhood U, suppose that there is no such ϵ; that is, there is a positive sequence $\epsilon_n \to 0$ and Hamiltonians H_n satisfying $|H_n|_{C^1} < \epsilon_n$ such that for each n, there is some Floer trajectories u_n that leaves the neighborhood U.

As a reminder, u_n satisfies:

$$\frac{\partial u_n}{\partial s} + J_n(u_n)\frac{\partial u_n}{\partial t} + \nabla H_n(u_n) = 0$$

plus some boundary and limiting conditions. Then we may write the energy as

$$E(u_n) = \mathcal{A}_0(x_n^-) - \mathcal{A}_0(x_n^+) + \int_0^1 H_n(x_n^+) - H_n(x_n^-)\, dt$$

Pozniak proves a useful lemma 3.4.5: $|\mathcal{A}_0(\gamma) - \mathcal{A}_0(x)| \leq \|\dot{\gamma}\|_{L^2}^2$. With the lemma, he showed that $\|\dot{x}_n^+\| \leq \|H_n\|_{C^1} \leq \epsilon_n$ which bounds the first term. Similarly, there is a bound for the second term. The terms within the integral are bounded above by the C^0 norms of H_n which are in turn, also bounded by ϵ_n. Therefore, $E(u_n) \to 0$. This shows that $u_n \to u_0 \equiv$ *const*. Moreover, u_0 must be in $N \subset U$. This contradicts the fact that the u_n all leave the neighborhood U at some point. □

7 Adapting Pozniak's Results

For manifolds with boundary, it is customary to define the tangent space in such a way that even on the boundary, the tangent spaces are the same dimension as the interior. Thus, we'll keep the same definition:

Definition 7.1. *Two Lagrangians L_0, L_1 intersect cleanly along a submanifold with or without boundary Σ if for all $x \in \Sigma$, $T_x\Sigma = T_xL_0 \cap T_xL_1$.*

We now state a generalization of Theorem 6.2.

Theorem 7.2. *Let (M, ω) be a symplectic manifold and L_0, L_1 two Lagrangian submanifolds. Suppose that $C \subset L_0 \cap L_1$ is a Lagrangian quasi-minimally degenerate set. There exists a perturbation for C resulting in a submanifold Σ with boundary which deformation retracts to C. Fix base point $x_0 \in \Sigma$. If U is any relatively compact neighborhood of Σ such that*

1. *There are no critical points of $\mathcal{A}$ other than those in Σ in the connected component $P(U, L_0, L_1, x_0)$ of x_0 in the path space $P(U, L_0, L_1)$.*
2. *The action function of ω is well-defined in $P(U, L_0, L_1, x_0)$.*

Then $\mathcal{U} = P(U, L_0, L_1, x_0)$ is an isolating neighborhood and $\mathcal{S}_{J,0}(\mathcal{U}) = \Sigma$ for any almost complex structure J. There exists an almost complex structure J_0 and a Hamiltonian $H_0 : M \to \mathbb{R}$ such that

1. *$\mathcal{S}_{J_0,H_0}(\mathcal{U})$ is a continuation of Σ.*
2. *(J_0, H_0) is a regular pair and if $g_\Sigma = g_J|_\Sigma$, $f = H_0|_\Sigma$, then (g_Σ, f) is a Morse-Smale.*
3. *The Floer complex $CF_*(\mathcal{U}, J_0, H_0)$ coincides with the Morse complex $CM_*(\Sigma, g_\Sigma, f)$ and thus*
$$HF_*(\mathcal{U}, \mathbb{Z}_2) \cong H_*^{sing}(\Sigma, \mathbb{Z}_2).$$

Remark: Since Pozniak shows $CF_*(\mathcal{U}, \mathbb{Z}_2)$ coincides with the Morse complex $CM_*(\Sigma, g_\Sigma, f)$ and since Morse theory works also for open submanifolds, we find that $HF_*(\mathcal{U}, \mathbb{Z}_2) \cong H_*^{sing}(\Sigma, \mathbb{Z}_2)$ and in particular, $HF_k(\mathcal{U}, \mathbb{Z}_2) = 0$ where $k = \dim \Sigma$.

Proof. The brunt of the work falls to Theorem 5.2 which gives the perturbation yielding a thickening Σ that deformation retracts onto C. So we just need to work with Σ. In the proof of Theorem 6.2, N is boundaryless and Pozniak puts a metric on L_0 and lets $L = TN^\perp$. We cannot directly follow suit because our Σ has boundary and therefore, $L := T\Sigma^\perp$ would also have boundary. As a result, there cannot be a boundaryless neighborhood of Σ in T^*L symplectomorphic to a neighborhood of Σ in M.

Instead, we first take a slightly smaller perturbation of the QMD set to obtain a submanifold with boundary Σ' inside of Σ. We have that $\partial\Sigma' \cap \partial\Sigma = \varnothing$. Furthermore, when the perturbation is sufficiently small, the local Floer homology is unaffected.

Next, we take the interior of Σ' to obtain an open manifold $\widehat{\Sigma}$ and extend the vector bundle structure to $\widehat{\Sigma}$. We then take $L = T\widehat{\Sigma}^\perp$. A similar construction gives us a desired $L_2 = T\widehat{\Sigma}^{ann}$.

Therefore, we have a version of the standard form of clean intersections for an open manifold. Moreover, Pozniak's results do not really rely on the whether Σ' is compact so we may use his arguments. For instance, one can do Morse theory on open manifolds. If there's any concern about the Floer data being pathological near the boundary of Σ', we can simply take an even smaller perturbation before taking the interior. This never changes the homotopy type and hence, the Morse theoretic data is unchanged.

Thus, to broaden Pozniak's main theorem to the case of minimally degenerate intersections, one can follow his arguments almost verbatim. The Morse and Floer data will coincide and the theorem follows. □

8 A Spectral Sequence

From the results of Pozniak, onef can draw out a spectral sequence analogous to the Morse-Bott spectral sequence. Indeed, in a paper by Paul Seidel [Sei99], he formulates Pozniak's result in spectral sequence terms.

Suppose that L_0, L_1 are two Lagrangians with intersection decomposed into $\bigsqcup C_p$ where $C_1, ..., C_r$ are the connected components and are submanifolds. By Pozniak's results, there exist disjoint neighborhoods U_p of the C_p for a Hamiltonian H that is sufficiently C^1 small. We may patch together an almost complex structure J from the local data. Once done, each neighborhood U_p has the property that all the relevant Floer theoretic data for (H, J) stay within the U_p.

Therefore, there exists a filtration on $CF_*(H)$ induced by the action functional. The filtration is preserved by the differential ∂. In more detail, let $x_p \in C_p$ and $a_p = \mathcal{A}(x_p)$ where we now view x_p as a constant path. The C_p are ordered so that $a_1 \leq a_2 \leq ... \leq a_r$. Then, there is a filtration on CF by CF^p which is generated by the critical points inside of $U_1 \cup ... \cup U_p$. Clearly, $CF^p/CF^{p-1} = CF(U_p)$ and the homology there is the local Floer homology $HF(L_0, L_1, U_p)$.

This provides a summary for how the usual Floer homology is related to the local Floer homology. One obtains a spectral sequence using this action filtration which converges to the usual Floer homology $HF_*(L_0, L_1)$ with the E^1-page being $E^1_{p,q} = HF_{p+q}(L_0, L_1, U_p)$. The local Floer homology is related to singular homology, albeit with some shift in grading.

8.1 Gradings

To discuss the shift in the gradings, we'll first recall some facts about the Maslov index. For more detail, one may consult Section 4 of Seidel's paper [Sei99] or the Robbin-Salamon papers that Seidel references [RS93],[RS95]. We will mostly stick to Seidel's notation.

Let $\mathcal{L}(n)$ denote the Lagrangian Grassmannian for $(\mathbb{R}^{2n}, \omega_0)$ and consider paths $\gamma, \gamma' : I = [0, 1] \to \mathcal{L}(n)$. The Maslov index $\mu(\gamma, \gamma')$ assigns values in $\frac{1}{2}\mathbb{Z}$ to these two paths and has some basic properties. Of primary importance to us are:

1. $\mu(\gamma, \gamma')$ depends on γ, γ' only up to homotopy with fixed endpoints.
2. μ is unchanged if one conjugates both γ and γ' by a path $\Psi : [0, 1] \to Sp(2n, \mathbb{R})$.
3. μ is additive under concatenation of paths.
4. $\mu(\gamma, \gamma')$ vanishes if the dimension of $\gamma(t) \cap \gamma'(t)$ is constant.
5. $\mu(\gamma, \gamma') \equiv \frac{1}{2} \dim(\gamma(0) \cap \gamma'(0)) - \frac{1}{2} \dim(\gamma(1) \cap \gamma'(1)) \pmod 1$.

Letting γ_x denote the constant path at a point x, choose a path $I \to P(L_0, L_1)$ which begins at γ_{x_-} and ends at γ_{x_+}. Here, $x_\pm \in L_0 \cap L_1$. Then, this path is described by a map $u : I \times I \to M$ with some obvious Lagrangian boundary conditions. We can assign an index to u in the following way.

Let $E = u^*TM$ and choose a Lagrangian subbundle $F \subset E$ such that $F|_{(s,0)} = T_{u(s,0)}L_0$ and $F|_{(s,1)} = T_{u(s,1)}L_1$ for all s. After picking a trivialization, we can view the paths $u(s, 0)$ and $u(s, 1)$ being paths $I \to \mathcal{L}(n)$; call these γ_0, γ_1. Then, the index of u can be defined as $I(u) := \mu(\gamma_0, \gamma_1)$. It is a result of the first two properties above that $I(u)$ depends only on homotopies of u which keep the end points $\gamma_{x_\pm}$ fixed. So the choice of trivialization does not matter.

Moreover, if u, u' are two paths in $P(L_0, L_1)$ with the same endpoints, then $I(u) - I(u') = \chi(v)$. This $\chi \in H^1(P(L_0, L_1), \mathbb{Z})$ is some class determined by the Maslov index for loops and v is the loop obtained by concatenating u and u'. We are interested in cases where this class $\chi = 0$; this happens, for example, when $c_1(M) = 0$ and $H^1(L_0) = H^1(L_1) = 0$. The latter condition ensures that the Maslov class, which obstructs the existence of gradings on Lagrangians, vanishes. For some details on this, see chapter 12 of [Sei08b].

When $\chi = 0$, we can find numbers $i(\gamma_x) \in \frac{1}{2}\mathbb{Z}$ for each $x \in L_0 \cap L_1$. Then, if u is a path between γ_{x_+} and γ_{x_-}, then $I(u) = i(\gamma_{x_-}) - i(\gamma_{x_+})$. Because of property (5), it can be arranged that

$$i(\gamma_x) \equiv \frac{1}{2}\dim(T_xL_0 \cap T_xL_1) \pmod 1 \tag{8.1}$$

Seidel calls numbers $i(\gamma_x)$ satisfying these properties *coherent choices of indices.*

We can also extend the above discussion to incorporate Hamiltonians. Choose two Hamiltonians H_-, H_+ and suppose that $\gamma_\pm$ are critical points of the actional functionals $\mathcal{A}_{H_\pm}$. Then if $u : I \to P(L_0, L_1)$ is a path between γ_+ and γ_-, there we can assign u an index: $I_{H_-,H_+}(u) \in \frac{1}{2}\mathbb{Z}$. As before, if $\chi = 0$, then there is a coherent choice of indices $i_H(\gamma) \in \frac{1}{2}\mathbb{Z}$ for any choice of Hamiltonian H and critical point γ of $\mathcal{A}_H$ so that

$$I_{H_-,H_+}(u) = i_{H_-}(\gamma_{x_-}) - i_{H_+}(\gamma_{x_+}).$$

Furthermore, it can be arranged that

$$i_H(\gamma) \equiv \frac{1}{2}\dim(D\phi_1^H(T_{\gamma(0)}L_0) \cap T_{\gamma(1)}L_1) \pmod 1.$$

Also, if we begin with a coherent choice of indices $i(\gamma_x)$, we can choose $i_H(\gamma)$ such that $i(\gamma_x) = i_H(\gamma_x)$ when $H \equiv 0$.

When L_0, L_1 are compact and the associated action functional is well-defined on $P(L_0, L_1)$ and also $\chi = 0$, then we have coherent choices of indices $i_H(\gamma)$. If (H, J) are a regular pair, then we have a grading on the Floer homology groups $HF(L_0, L_1, H, J)$ which also induces gradings on local Floer homology.

As was discussed at the beginning of Section 6, when $L_0 \cap L_1$ is a finite union of components C_p with isolating neighborhoods U_p, then we have a filtration for the Floer chains and hence, a spectral sequence converging to $HF_*(L_0, L_1)$ with the E^1 page given by local Floer homology: $E^1_{p,q} = HF_{p+q}(L_0, L_1, U_p)$.

If L_0 and L_1 have clean intersection, then property (4) above implies that for any coherent choice of indices the function $x \mapsto i(\gamma_x)$ is locally constant on $L_0 \cap L_1$. Let $i(C_p)$ be the value of this function on C_p and $i'(C_p) = i(C_p) - \frac{1}{2}\dim C_p$. This is an integer because of Equation 8.1. Hence, $HF_*(L_0, L_1; U) \cong H_{*-i'(C)}(C, \mathbb{Z}_2)$ and the spectral sequence which converges to $HF_*(L_0, L_1)$ has E^1 page described completely by the topology of the C_p.

8.2 An Index Lemma

We wish to prove that $HF_*(L_0, L_1; U) \cong H_{*-\iota(C)}(C, \mathbb{Z}_2)$ in the case that C is a minimally degenerate set and $\iota(C)$ is some type of Maslov index. From there, we will immediately have a spectral sequence as before. The results of Section 5 prove most of this result. We also know that there is a coherent choice of index $i(\gamma_x)$ for $x \in C$ that, because of property (4), is locally constant. What remains to be shown is that the index of C and the index of a thickening Σ_C are the same.

Lemma 8.1. *The index $i(C) = i(\Sigma_C)$.*

Proof. Since minimally degenerate intersections are modeled locally on minimally degenerate functions, let $f : L \to \mathbb{R}$ be a minimally degenerate function and C a minimally degenerate isolated critical locus with S_C as the associated submanifold. Let $x \in S_C$. Then, the $H_x f$ is positive semi-definite along $T_x S_C$ but negative definite along some subspace V transverse to $T_x S_C$ in $T_x L$. Perturbing f gives a new function $\tilde{f}$ whose critical locus is a thickening Σ_C of C. As a reminder, $\Sigma_C \subset S_C$ is a codimension 0 submanifold. The thickening is such that $H_x \tilde{f}$ is zero along $T_x S_C$ and negative definite along V. In particular this perturbation gets rid of all the positive eigenvalues of the Hessian of f at x but the dimension of the negative eigenspace stays the same. As a result, the index of f and $\tilde{f}$ at x are the same and equal $\dim V$. Here, index simply means the dimension of the negative eigenspace. So the perturbation does not change the "Morse-type" index.

Next, we choose a regular C^2-small Hamiltonian H and obtain a coherent choice of index $i_H(\gamma)$ for γ being a critical point of $\mathcal{A}_H$. Because H is C^2-small, Theorem 6.3 says that for $x \in L_0 \cap L_1$, γ_x is a critical point of $\mathcal{A}_H$. We may further conclude that $i(\gamma_x) = i_H(\gamma_x)$ as the Morse and Floer theoretic data coincide.

Therefore, the Maslov index will equal this Morse-type index up to a shift. Since the Morse-type index does not change, neither does the Maslov index as the Hamiltonian is C^2 small. □

With this lemma, we have the following result which is almost verbatim the statement by Seidel:

Theorem 8.2. *Suppose $L_0 \cap L_1$ decomposes into $\bigsqcup C_p$ where each C_p is quasi-minimally degenerate. Let $\Sigma_p := \Sigma_{C_p}$ be the thickening for C_p. Then there is a spectral sequence which converges to $HF_*(L_0, L_1)$ and whose E^1-term is*

$$E^1_{pq} = \begin{cases} H_{p+q-\iota(\Sigma_p)}(C_p; \mathbb{Z}/2), & 1 \le p \le r; \\ 0, & \textit{otherwise.} \end{cases}$$

This is a homology spectral sequence, i.e. the k-th differential ($k \ge 1$) has degree $(-k, k-1)$.

Remark: We've presented the theorem in this way because that was the setting of Poźniak but as indicated in the proofs of Section 7, we can drop the compactness condition. Thus, if there are infinitely many QMD disjoint sets C_p that form $L_0 \cap L_1$, so long as they are isolated, we can use the action filtration to build a finite E^1 page with action less than A and form a directed system of finite E^1 pages. Then, since direct limits commute with homology, the limit E^1 page indeed is the E^1 page of a spectral sequence. In applications where we introduce a Hamiltonian, say, in Hamiltonian Floer theory (we can switch back and forth between Hamiltonian and Lagrangian settings by Section), for each cutoff A, there are only finitely many isolated families C_p. This is because we're working not on the manifold but rather the loop (or path) space and thus, if two loops have different action, we can find isolating neighborhoods.

9 Some Applications

In this section, we present some applications of this method. Several of these examples have already been studied but often with rather *ad hoc* tools. Our purpose in presenting them is to demonstrate that our result can be used as a general and systematic tool. Before we do that, we'll first sketch why manifolds-with-corners are examples of QMD sets because these appear in the first two examples.

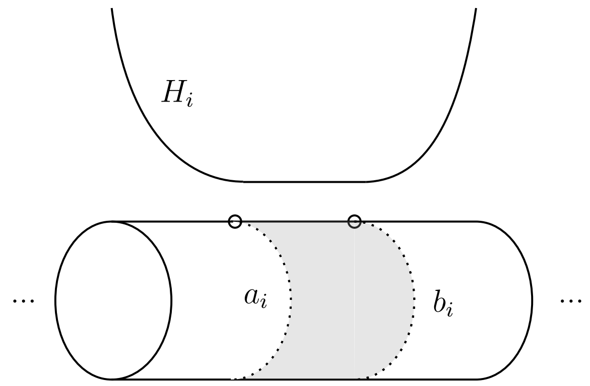
Recall that a manifold-with-corners (or more succinctly, cornered manifolds) is a topological space $M = \mathring{M} \sqcup \partial M$ which decomposes into two pieces. The first piece is the interior $\mathring{M}$ which is an open manifold of, say, dimension n. The other piece, ∂M, also decomposes into pieces which are all manifolds themselves. The pieces have positive codimension and every point $p \in \partial M$ admits a neighborhood homeomorphic to $[0,1)^k \times (0,1)^{n-k}$ for some $0 < k \leq n$.

Now, working in this local model, let $\mathbb{R}^k$ have coordinates $x_1, ..., x_k$ and $f(x_1, ..., x_k) = \prod_{i=1}^k x_i$. Then $f^{-1}(0)$ is a union of hyperplanes and removing this union divides up $\mathbb{R}^k$ into 2^k connected components with one of them being distinguished as the **positive orthant** where all the $x_i \geq 0$; it is a cornered manifold. So then, the set $f^{-1}((\epsilon, \infty))$ intersected with the positive orthant gives us a manifold-with-boundary (more succinctly, a bordered manifold) which is a smoothing of the cornered manifold and the function gives us a deformation retracts onto the positive orthant. Moreover, note that f does not have any critical points outside of $f^{-1}(0)$, the union of hyperplanes.

With some modifications to this scenario (which we do not write in detail), we can make a function $\tilde{f} \geq 0$ so that the positive orthant is $\tilde{f}^{-1}(0)$ is the set of minima and the thickening is diffeomorphic to the smoothing (by an isotopy). Essentially, it is the smoothing but we work in such a way that the thickening includes the positive orthant. This is all doable once we leave this model behind and work with more general manifolds because a cornered submanifold C lives in an ambient manifold and locally, we may as well think of the ambient manifold as $\mathbb{R}^k \times \mathbb{R}^{n-k}$. We'll now turn to our examples.

Example 9.1. Consider $M = \mathbb{C}^*$ which is $\mathbb{CP}^1$ minus two points. Then $\mathbb{C}^* \times \mathbb{C}^*$ can be thought of as $\mathbb{P}^1 \times \mathbb{P}^1$ minus four projective lines which is, of course, an affine variety.

Now, view $\mathbb{C}^* \cong \mathbb{R}_s \times S^1_t$ as a cylinder and pick real constants $a_1, b_1 \in \mathbb{R}$. We define a Hamiltonian $H_1(s,t) = H_1(s)$ which is 0 on $[a_1, b_1]$ and $\frac{d}{ds}H_1(s) < 0$ when $s < a_1$ and $\frac{d}{ds}H_1(s) > 0$ when $s > b_1$. Then the critical set of H_1 is $A_1 = [a,b] \times S^1$, an annulus. We may do something similar on the other copy of $\mathbb{C}^*$ to produce a Hamiltonian H_2 with the critical set being an annulus, A_2.



Hamiltonian H_i on a cylinder; the shaded region is the critical set

Then letting $H = H_1 + H_2$ on M, the critical set is $A_1 \times A_2$ which is a manifold-with-corners. Since the graph of the time-1 flow of H, when intersected with the diagonal $\Delta \subset M \times M$, is precisely $A_1 \times A_2$, we can move everything over to the Lagrangian Floer setting as briefly outlined in Section 5.3. Since the critical set is a manifold-with-corners, it is not Morse-Bott. But it is QMD in the Lagrangian sense. This is because away from the boundary and corners, the critical set is Morse-Bott and so we only need to focus on the codim 1 stratum. We can always find a Hamiltonian with which to perturb the critical set into a manifold-with-boundary (so we smooth out the corners) which puts us in the Lagrangian flattened degenerate situation. Lastly, being flattened degenerate automatically implies QMD.

If we prefer, we can proceed with computing the Lagrangian Floer homology of the graph intersecting the diagonal and recover Hamiltonian Floer homology that way. But the present situation is simple enough here that we can just pick a Morse function f and small constant $\epsilon > 0$. Perturbing H by ϵf breaks the critical set into isolated points. Then,

$$HF_*(A_1 \times A_2) \cong HM_*(A_1 \times A_2) \cong H_*^{sing}(A_1) \otimes H_*^{sing}(A_2).$$

Here, HM_* is Morse homology and of course, we have a Künneth formula.

Example 9.2. Let X be a smooth complex affine variety; Hironaka's theorem gives a compactification of X in the sense that $X \cong M \setminus D$ where M is a smooth projective variety and D is a simple normal crossings divisor which supports an ample line bundle. In a paper by Ganatra and Pomerleano [GP20], their main result produces a spectral sequence which converges to the symplectic cohomology $SH^*(X)$ and its E_1 page is ring isomorphic to the logarithmic cohomology of (M, D):

$$H^*_{log}(M, D) \cong \bigoplus_{p,q} E_1^{p,q}$$

The $E_1^{p,q}$ are formed from local Floer homology groups. In order to obtain the isomorphism on the group level, they conduct a study of families of Hamiltonian orbits which are manifolds with corners via a variant of Morse-Bott analysis. An alternative approach to their study is to choose a neighborhood on which to perturb a manifold with corners, thereby smoothing out the corners to become a manifold with boundary (and thus, placing us in the QMD setting like the previous example). One then computes the local Hamiltonian Floer groups by translating to the Lagrangian setting. Ganatra and Pomerleano were aware of this alternative approach which they allude to in Remark 4.17 on

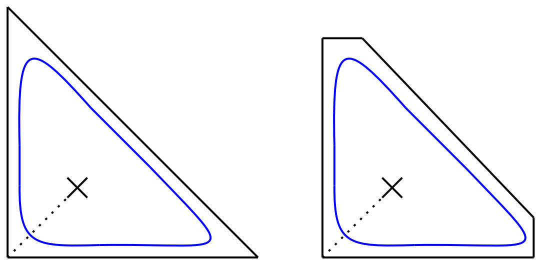
p. 72 (arxiv version) in their paper. Since local Floer homology is invariant under such perturbations, we may freely perturb in this manner.

Remark: In order, to produce the multiplicative structure, Ganatra and Pomerleano produce a novel log PSS map which they developed in a prior paper [GP21].

Example 9.3. This next example was studied by Pascaleff in his thesis [Pas14] and we will spend considerably more time on it. He computed the wrapped Floer homology of certain Lagrangians, including the ring structure that comes from counting Floer triangles. Here, we will give a weaker result as a technical demonstration of the principles from above. The purpose of choosing this example is to compare these methods to known results and to also fill in some details of Pascaleff's work. For some excellent pictures, consult [Pas14], [Pas19].

Consider a line L and conic C in $\mathbb{CP}^2$ which intersect transversally. Letting $D = L \cup C$, this is an anticanonical divisor of $\mathbb{CP}^2$ and has the properties needed to view the pair $(\mathbb{CP}^2, D)$ as a log Calab-Yau. For instance, D is a normal crossings divisor. Further details of the definition are found in Pascaleff's thesis.

Next, we opt to blowup the two points of intersection since any blowup along D will not affect the $\mathbb{CP}^2 \setminus D$. What we obtain then is a new divisor $\widetilde{D}$ inside of the twice blowup, call it X, which is the union of the proper transforms $\widetilde{L}$ and $\widetilde{C}$ as well as two exceptional divisors, E and F. Below is the toric picture where we take $\mathbb{CP}^2$, represented by its moment polytope Δ. The preimate $\mu^{-1}(\partial\Delta)$ of the boundary under the moment map is a union of three lines. We can smooth one of the corners, the bottom left one at the cost of introducing a nodal fiber, marked with an x.



Toric Picture with $\mathbb{CP}^2$ and $Bl_2\mathbb{CP}^2$

The reason for this blowup is because it gives us control over neighborhoods of the intersections of these four curves. In particular, we may choose holomorphic coordinates (z, w) such that the a neighborhood of an intersection appears as $\mathbb{C}^2$ with one divisor locally appearing as $\{z = 0\}$ and the other as $\{w = 0\}$. The blowup parameters allow us to obtain a symplectic form which sees the two divisors as symplectically orthogonal. This is a consequence of the $U(2)$ invariance of the blowup form. In fact, we only need $U(1) \times U(1)$ invariance. The argument here is essentially what is outlined in [Sei08a], Theorem 4.5.

By a relative Moser argument, we are able to extend the symplectic form to a neighborhood U of $\widetilde{D}$ without disrupting the symplectic orthogonality of the curves. Thus, we have another log Calabi-Yau $(X, \widetilde{D})$ and we shall study $X \setminus \widetilde{D}$ or more precisely, $X \setminus U$. In order to do this, let us give a more refined view of U. The main issue is to consider neighborhoods of the nodes as there are concerns about smoothness. We take polar coordinates for $\mathbb{C}^2$, $(r_1, \theta_1, r_2, \theta_2)$ and consider the real hypersurface $\{r_1 r_2 = \delta\}$ for some small $\delta > 0$. Then in this locale, U may be thought of as $\{r_1 r_2 < \delta\}$. When we extend the hypersurface, it gives a 3-manifold M which is, in fact, a T^2 bundle over S^1. This is clear from the toric picture where the blue curve represents the S^1 over which M is T^2-fibered. It is then also clear that it doesn't matter whether we use $\mathbb{CP}^2$ or its blowup at two points. This was presented in Section 7 of Pascaleff's thesis.

M can be classified by an element of $SL(2,\mathbb{Z})$ which gives the monodromy. The normal bundles of the curves in X are $\mathcal{O}(-1) \to \mathbb{CP}^1$ for the exceptional divisors and the proper transform $\widetilde{L}$. For $\widetilde{C}$, it is $\mathcal{O}(2) \to \mathbb{CP}^1$. To construct M, we need to "plumb" the circle bundles of these normal bundles together. The map to plumb these bundles at the nodes corresponds to

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

since we are basically interchanging circles fibers in an orientation-preserving way. On the other hand, by choosing meromorphic sections with single poles for $\mathcal{O}(-1)$ and a holomorphic section with two zeros for $\mathcal{O}(2)$, we obtain some contributions to the monodromy map. Let

$$T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then, the sections above correspond to T^{-1} and T^2, respectively. Thus, the monodromy map (in this basis) is given by multiplication of a sequence of these matrices:

$$\mu = JT^{-1}JT^{-1}JT^{-1}JT^2 = \begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}.$$

The vector field $V = r_1\partial_{\theta_1} + r_2\partial_{\theta_2}$ gives a characteristic foliation of M and is tangent to the fibers. Pascaleff wrote down a contact form α which realizes V as its Reeb vector field. We may also write down a Liouville vector field in order to produce a Liouville domain which appears as the affine variety $X \setminus U$ with contact boundary given by M.

In the sequel, our goal is to study the wrapped Floer homology of the Lagrangian defined by removing the neighborhood of the divisor from the real part of X. For the definition of wrapped Floer homology, one may consult [McL20a], [Pas14]. One can describe the real part as the fixed point set of an antisymplectic involution which coincides with complex conjugation away from the blowup points and so, topologically, it will be $\mathbb{RP}^2 \setminus D$ where D is the real part of the conic plus line; of course, this is the same thing as removing a conic from $\mathbb{R}^2$. Call this Lagrangian Λ.

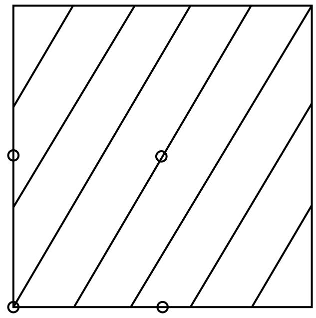
Λ is cylindrical at infinity which means it is, at infinity, the product of a Legendrian submanifold of the contact boundary, product with $\mathbb{R}$. This parameter $\mathbb{R}$ can be thought of as changing δ. The submanifold is a 1-manifold and is disconnected. Indeed, if we look at the real picture of Λ, the real part of the divisor separates Λ into two or three components, depending on the real conic. So Λ has two or three cylindrical ends. Each boundary component is a Legendrian knot and may be viewed as a section of the T^2 bundle over S^1.

Near the nodes, $V = r_1\partial_{\theta_1} + r_2\partial_{\theta_2}$. Λ intersects each torus fiber at four points since the real part requires $\theta_1, \theta_2 \in \{0, \pi\}$. If we view the torus as $\mathbb{R}^2/(2\pi\mathbb{Z})^2 = [0,1]^2/\sim$, then the points are $(0,0), (\frac{1}{2}, 0), (0, \frac{1}{2}), (\frac{1}{2}, \frac{1}{2})$. We have Reeb orbits whenever $r_2/r_1 \in \mathbb{Q}$. However, we also have Reeb chords between the distinct points depending on r_1 and r_2. We'll let $q = r_1, p = r_2$. It can be easily checked that

- $(0,0)$ connects to $(0, \frac{1}{2})$ if and only if q is even.

- Whether $(0,0)$ connects to $(\frac{1}{2},0)$ is symmetric to the above. We have solutions exactly when p is even.
- $(0,0)$ connects to $(\frac{1}{2},\frac{1}{2})$ exactly when both p and q are odd.
- Whether $(\frac{1}{2},0)$ connects to $(0,\frac{1}{2})$ is equivalent to the previous; think of $(\frac{1}{2},0)$ as $(\frac{1}{2},1)$, then translate the plane down by a half.
- Whether $(\frac{1}{2},0)$ connects to $(\frac{1}{2},\frac{1}{2})$ or whether $(0,\frac{1}{2})$ connects to $(\frac{1}{2},\frac{1}{2})$ is the same as some of the above situations; just translate the plane.

We also note that the lengths of the orbits/chords depends both on the numerator and denominator in r_2/r_1; this means that we're able to isolated the families via length of orbit. Here is a picture where p,q are both odd.



Example of Reeb chords with slope $p/q = 5/3$

If we pick an admissible Hamiltonian H, this shows that there are plenty of generators for the wrapped Floer chain complex $CW_*(\Lambda,\Lambda,H)$. Away from the nodes, the situation is tamer and the total space M admits a Boothby-Wang structure (we can ensure the symplectic form is integral). The circle action gives Morse-Bott submanifolds similar to the first example above of $\mathbb{C}^* \times \mathbb{C}^*$.

Pascaleff gives reasons for why the generators are all concentrated in degree zero and thus, why the differential of $CW_*(\Lambda,\Lambda,H)$ is trivial; he then computes that $CW_0(\Lambda,\Lambda,H) \cong HW_*(\Lambda,\Lambda) \cong \mathbb{K}[x,y][(xy-1)^{-1}]$. We can supply some evidence for this from a local Floer theoretic perspective. If we restrict our attention to low energy Floer strips, they must connect Reeb chords of some "type" to Reeb chords of the same "type" because of the boundary conditions. To be more precise, consider an interval $I \subset S^1$ and $T^2 \times I$. Then since Λ intersects T^2 at four points, one can imagine four line segments in $T^2 \times I$ which represent the intersection with Λ. The boundary conditions imposed on Floer strips makes it so that these low energy strips must connect a Reeb chord to a translation of the Reeb chord along I. Thus, regardless of what the degree of the Reeb chord is, the Floer strip is not connecting Reeb chords of differing index. As such, in the local Floer complex where we take only low energy strips, there is no differential in the low-energy regime. This fact shows that we may obtain the E^1 page of the spectral sequence from the local Floer data. Moreover, the rank of the underlying vector space of the E^1 page is countably infinite which corroborates Pascaleffs calculation and so the differentials in the spectral sequence must have large kernels and not too large of images. The countable infinity is not a problem for us; this was addressed in the remark following Theorem 8.2.

To summarize, Pascaleff was able to compute the wrapped Floer homology without this spectral sequence and gave a description of the triangle product to determine the ring structure. To do this, he relies on the example being a log Calabi-Yau pair which allowed him to use some mirror symmetry techniques. Here, we've given a weaker understanding of the example by using our spectral sequence but without reliance on the log Calabi-Yau condition nor mirror symmetry. Notably, the spectral sequence can be applied to non-compact Lagrangians.

Example 9.4. The spectral sequence we construct from the local Floer data applies to examples beyond log Calabi-Yau. For instance, if we have any complex algebraic surface, so long as we have an antisymplectic involution that has fixed points, the fixed point set is a Lagrangian. It is convenient to take the one which fixes the real locus but is certainly not the only option.

Now, let's take four generic lines in $\mathbb{CP}^2$ fixed by an antisymplectic involution. Then the union of them gives a divisor that is not anticanonical. Any given line will intersect all the other three lines, giving a total of six intersections. Blowing up at those points, we'll obtain six exceptional lines for a total of 12 nodes. We apply our study of the nodes from the previous example to these 12. Away from the nodes, the topology differs from the previous example. Before, the Morse-Bott manifolds were all annuli. This time, the proper transforms of the original lines give thrice-punctured spheres while the exceptional divisors continue to contribute annuli. So the local Floer data which feeds into the spectral sequence is different from before but still tractable.

There are certainly many other line arrangements which produce a multitude of affine varieties and Lagrangians to which we may also apply these techniques.

10 Concluding Remarks to Minimal Degeneracy

The above examples are not exhaustive. There may be situations in which these techniques would be helpful for computing triangle products or other A^∞ products on local Lagrangian Floer homology. Minimal degeneracy may also appear naturally in low dimensional topology. For example, minimal degeneracy techniques may be introduced into Heegaard Floer homology where one studies Lagrangian intersections of a Heegaard surface. In practice, actual computations tend to be of combinatorial flavor because Lagrangian intersections can be rather complicated. The results of this paper may alleviate some of those complications. Various versions of Heegaard Floer theory are known to be isomorphic to other homology theories, such as embedded contact homology and monopole Floer homology. It would be interesting to see whether QMD has some analogous meaning in these theories and whether anything new can be proven. Moreover, in [Fuk21], Fukaya suggested some possible relationships to Atiyah-Floer conjectures. Namely, the instanton homology of M, an S^1 bundle over a Riemann surface has a Chern-Simons functional that looks similar to those appearing in Kirwan's study of moment maps. If we take a Heegaard decomposition of M, the moduli of flat connections on M is a certain Lagrangian intersection and the local properties of the Chern-Simons functional can be related to the mildness of the intersection. It is not Morse-Bott in general but may be QMD.

Another direction could be that of contact homology. Bourgeois studied Morse-Bott techniques in his PhD thesis [Bou03]. Perhaps one could find minimally degenerate or QMD contact forms and do computations once these notion have been properly defined. We are currently considering QMD families of Reeb orbits related to certain algebraic singularities with good smoothings and using our spectral sequence to compute the symplectic cohomology of these smoothings.

Lastly, there may also be interesting questions about minimally degeneracy itself to explore. Holm and Karshon show in [HK16] that Kirwan's definition of minimal degeneracy is local. That is, if $f : M \to \mathbb{R}$ is a smooth function and is minimally degenerate near each critical point, then f is minimally degenerate. It may be worth exploring whether the same can be said of quasi-minimally functions in both the "Morse" and Lagrangian setting. It seems plausible for there to be obstructions for a Lagrangian intersection to be locally QMD at each point but not globally QMD. Such obstructions are likely to be completely topological.

11 Preliminary Remarks on Isolated Hypersurface Singularities and Milnor Fibrations

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial such that $f(0) = 0$ and 0 is an isolated singularity. This means that all the first partial derivatives of f with respect to the standard complex coordinates vanish at 0 and there exists a neighborhood of 0 in $V := V(f) = f^{-1}(0)$ such that there are no singularities in the neighborhood other than 0. We will often refer to f as the singularity though to be precise, we ought to consider the germ of 0 in $f^{-1}(0)$.

In any case, in [Mil68], Milnor proved that there exists a fibration structure that we can associate to f in the following way. The norm squared function $r : V \to [0, \infty)$ sending $z \mapsto \|z\|^2$ is a real algebraic. Then its set of critical values is a real algebraic set and hence is Zariski closed set in $\mathbb{R}$; i.e. is finite. Hence there exists $\epsilon_0 > 0$ such that r does not have any positive critical values smaller than ϵ_0^2. Take any $\epsilon < \epsilon_0$ and consider the intersection of V with a small sphere S^{2n+1}_ϵ; since we've avoided critical points with our small ϵ, V intersects the sphere transversally and hence, the intersection is a manifold L, called the link. Milnor showed that $f/|f| : S^{2n+1}_\epsilon \setminus L \to S^1$ is a fibration which means that, in particular, all its fibers are diffeomorphic as smooth manifolds by Ehresman's theorem. In fact, this result does not require f to have an isolated singularity at 0 but in that case, L might not be a manifold; e.g. $f(x, y, z) = xyz$. When the singularity *is* isolated, the fibration has further desirable properties such as the fiber having the homotopy type of a bouquet of n-spheres.

It is well known that S^{2n+1} has a tight contact structure $\xi = TS^{2n+1} \cap JTS^{2n+1}$ where J is the standard complex structure on $\mathbb{C}^n$ and the link is a contact submanifold. A generalization of a theorem by Eliashberg-Gromov [EG91] says that contact manifolds admitting symplectic fillings are tight. This applies to the standard contact spheres as they admit the ball as a symplectic filling. Since the link L of a singularity is contained in S^{2n+1}, it inherits contact structure and the Milnor fibration gives $S^{2n+1} \setminus L$ an open book decomposition which supports this contact structure. The fibers M_f are Stein domains with $\partial M_f = L$ so we see the inherited contact structure on L is also tight. It should noted that the complex structures of the fibers can vary but the exact symplectic structures are all deformation equivalent.

There is also a different fibration which we can associate to a polynomial f with isolated singularity called the **mapping torus**. This is the fibration $f : f^{-1}(S^1_\delta) \cap B_\epsilon \to S^1_\delta$. Basically, we take the preimage of a small circle and intersect that with a small ball; we take $\delta \ll \epsilon$. So the main difference is that we do not divide by the norm. However, Milnor showed that the interior of a fiber of the mapping torus is diffeomorphic to the interior of a fiber of the Milnor fibration. Hence, they are essentially equivalent fibrations. The main difference is that each fiber of the mapping torus has boundary being a manifold contactomorphic to the link L_f whereas each fiber of the Milnor fibration share the exact same boundary which is L_f. Since the fibers are the same, we'll call them Milnor fibers and denote them by M_f.

Both fibrations come equipped with monodromy maps; since the fibers are symplectomorphic, we treat them as the same monodromy map $\phi_f : M_f \to M_f$. The monodromy can be used to recover the two fibrations since it gives the gluing data for how to build each fibration. For example, the mapping torus for f is $T(\phi_f) := M_f \times [0,1]/\sim$ where $(x,0) \sim (\phi_f(x), 1)$. Moreover, this monodromy map is compactly supported (or can be made so without altering anything significant) and hence, the mapping torus near the boundary has a symplectic collar neighborhood. Being compactly supported, we also have a Floer invariant $HF^*(\phi)$ that can be associated to ϕ (and any compactly supported symplectomorphism at that). Its definition involves forming a chain complex generated by fixed points of ϕ and whose differential involves counts of Floer trajectories.

One of the insights in this paper is that HF^* is a TQFT (topological quantum field theory) in that a symplectic cobordism between two mapping tori will induce a map on the level of the algebraic invariant HF^*. Another insight is that when two singularities are adjacent, there is a standard cobordism. Therefore, one may use this to compare families of monodromy maps that arise from families of isolated hypersurface singularities. In the case of μ-constant families, Zariski conjectured that the multiplicity remains constant. This was proven by de Bobadilla-Pełka [dBP22] using some algebro-geometric machinery combined with the main result of McLean [McL19]. However, by leveraging the TQFT structure of fixed-point Floer cohomology, an alternative (and shorter) proof which remains more in the realm of Floer theory is achievable and additionally, recovers a theorem of Varchenko [Var82] that the log canonical threshold is also constant. So we have:

Theorem 11.1. *Let $f_t : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ define a family of isolated hypersurface singularities with constant Milnor number. Then the multiplicity and log canonical threshold is also constant in the family.*

11.1 Organization

In Section 13, we introduce the relevant algebraic invariants which fixed-point Floer cohomology HF^* will recover. Section 14 is where the definition of HF^* is outlined. Secion 16 introduces notion of adjacent singularities and constructs cobordisms between the mapping tori of monodromies arising from an adjacent pair. The cobordism induces a map which is the subject of Section 17. Sections 18 and 19 give the needed notions for showing why the induced map is well-defined. As is usual in Floer theory, we appeal to a version of Gromov compactness. Section 20 gives further language of TQFTs and the main result there is to show that our induced map is dependent only on the link of the singularities and the canonical framing. Section 21 gives the new proof and the later sections are more colloquial, posing questions and examples of various phenomena.

12 Liouville Manifolds

We now present some definitions for more specialized classes of symplectic manifolds.

Definition 12.1. *An **exact symplectic manifold** (M, λ) is a smooth manifold M equipped with a choice of primitive 1-form λ so that $\omega := d\lambda$ is symplectic.*

By Stokes theorem, such manifolds either have boundary or are noncompact.

Definition 12.2. *A* ***Liouville structure*** *on an exact symplectic manifold* (M, λ) *is the datum of a vector field* Z *which satisfies* $\iota_Z d\lambda = \lambda$*; i.e. it is* ω*-dual to* λ*. If*

- M *is compact with boundary, then we additionally require that* Z *is outward pointing and transverse to* ∂M*. Then the triple* (M, λ, Z) *is called a* ***Liouville domain****.*
- M *is noncompact, then we require that* Z *is a complete vector field and that* M *can be exhausted by Liouville domains. We call the triple* (M, λ, Z) *a* ***Liouville manifold****.*

Often, λ *is called the* ***Liouville 1-form****.*

This definition has several immediate consequences. Firstly, the condition that $\iota_Z\omega = \lambda$ is equivalent to the following Lie derivative condition because $\omega = d\lambda$ is closed: $\mathcal{L}_Z\omega = \omega$ and hence, the forward flow of Z exponentially expands volume. Secondly, in the case that M is compact with boundary, the 1-form $\alpha := \lambda|_{\partial M}$ is a **contact form** on ∂M which means that $\alpha \wedge (d\alpha)^{n-1} > 0$ is a volume form and gives a preferred orientation. By Frobenius theorem, this means that $\xi := \ker \alpha$ is a maximally nonintegrable rank $(2n-2)$ distribution. We then see that $d\alpha$ is a symplectic form on the vector bundle $\xi \to \partial M$ and that $d\alpha$ has a 1-dim kernel on $T(\partial M)$. This means that the structure group of $T(\partial M)$ is $Sp(2n-2) \times 1$ and in fact, $T(\partial M) \cong \xi \oplus \mathbb{R}\langle X_R\rangle$. This X_R is the so-called **Reeb vector field** which satisfies $\alpha(X_R) \equiv 1$ and $\iota_{X_R} d\alpha = 0$. Observe that it coorients ξ and that $TM|_{\partial M} = \xi \oplus \mathbb{R}\langle X_R\rangle \oplus \mathbb{R}\langle Z\rangle$. All told, $(\partial M, \alpha)$ is a **contact manifold** with a specified contact form α. The definition for a contact manifold is exactly what's been written here except that we needn't begin with the odd-dimensional manifold being the boundary of anything.

It is often desirable to take a contact manifold (Y, α) and consider its **symplectization** $(Y \times \mathbb{R}, d(e^t\alpha))$. Here, t is the coordinate on $\mathbb{R}$ and one can see that $d(e^t\alpha)$ defines a symplectic form on this cylinder. Since Liouville domains have contact-type boundary, we're able to glue on the symplecization of the boundary and obtain the so-called **completion**. This completion is an example of a **finite-type** Liouville manifold. There are also infinite-type Liouville manifolds but we won't study them in this paper. Suffice to say, finite-type Liouville manifolds are such that the limiting set of the backwards flow of Z is compact.

Example 12.3. Let Q be any closed smooth manifold and consider its cotangent bundle $\pi : T^*Q \to Q$. On the total space, there is a 1-form λ defined as follows. Let $q \in Q$, and $p : T_qQ \to \mathbb{R}$ be a covector. Also, let $V \in T_{(q,p)}(T^*Q)$. Then $\lambda_{(q,p)}(V) = p(d\pi(V))$; i.e. we push V down to T_qQ and evaluate that vector with p. This λ is often called the tautological 1-form and in coordinates, $\lambda = \sum p\, dq$ and $d\lambda = \sum dp \wedge dq$ is a symplectic form. The Liouville vector field is $\hat{Z} = \sum p\, \partial_p$.

This is enough to conclude that T^*Q is a Liouville manifold and it's easy to see that the restriction of λ to the zero section Q vanishes. Hence, the restriction of ω to Q also vanishes, showing that Q is Lagrangian. If we pick a Riemannian metric on Q, we get inner products on each tangent space T_qQ and dually, on T_q^*Q. Then, the function $\hat{\chi}(q, p) = \frac{1}{2}|p|^2$ has the zero section as its set of critical points and any other level set admits contact structure.

If we also pick a small Morse function on $f : Q \to \mathbb{R}$ and let $F(q, p) = \langle p, \nabla f(q)\rangle$ where the gradient is defined using our chosen Riemannian metric, then the Hamiltonian vector field X_F coincides with ∇f along the zero section of T^*Q. We can modify our Liouville vector field to be $Z = \sum p\,\partial_p + X_F$ and it will be gradient-like for the Morse function $\chi(q,p) = \hat{\chi}(q,p) + f(q)$.

In the example with cotangent bundles, the advantage of the modification is that the Liouville vector field can be viewed as arising from a Morse function which then allows one to perform surgery techniques and handlebody decompositions in a way that's compatible with the symplectic structure. Without giving a precise definition, we remark that this is the gist of a class of Liouville manifolds called **Weinstein manifolds**. Through amazing work [CE12], a Weinstein manifold also admits a Stein structure which we will discuss in the next section. But first, we give a definition for saying when two Liouville manifolds are isomorphic.

Definition 12.4. *If (M_i, λ_i) with $i = 0, 1$ are two Liouville manifolds (we've suppressed the vector fields in the notation), then a* ***Liouville isomorphism*** *is a diffeomorphism $\psi : M_0 \to M_1$ such that $\psi^*\lambda_1 = \lambda_0 + df$. An isomorphism for Liouville domains is similarly defined by first completing the Liouville domains.*

Note that such a ψ is automatically a symplectomorphism; some authors will call such a ψ an **exact symplectomorphism**. If we're dealing with finite type Liouville manifolds, we can even ask that f in the definition be compactly supported. As we will see later in Lemma 16.9, when we're dealing with finite type Liouville manifolds, there's not really any reason to distinguish between symplectomorphisms and Liouville isomorphisms.

12.1 Stein Manifolds and Convexity

In this subsection, we mainly follow the exposition of [CE12]. The quickest definition of a Stein manifold is:

Definition 12.5 (Affine Definition)**.** *A* ***Stein manifold*** *is a complex manifold (V, J) which admits a proper holomorphic embedding into some $\mathbb{C}^N$.*

This is a very straightforward definition; a proper map $V \to \mathbb{C}^N$ is just one where the preimage of compact sets is compact. An immediate consequence of this definition is that V cannot contain closed holomorphically embedded complex submanifolds because they would violate the maximum principle. However, this definition is also rather misleading because historically, Stein manifolds were defined from a more complex analytic viewpoint rather than this affine viewpoint. The classical definition is:

Definition 12.6 (Classical Definition)**.** *A complex manifold (V, J) is Stein if it has the following properties:*

1. *V is holomorphically convex; i.e. for compact $K \subset V$, the* ***holomorphic hull*** *$\hat{K}_V := \{x \in V : |f(x)| \le \sup_K |f| \,\forall f \in \mathcal{O}(V)\}$ is also compact.*
2. *for every $x \in V$, there exist holomorphic functions $f_1, ..., f_n : V \to \mathbb{C}$ which form a holomorphic coordinate system at x;*
3. *for any $x \neq y \in V$, there exists a holomorphic function $f : V \to \mathbb{C}$ with $f(x) \neq f(y)$; i.e. a separating holomorphic function.*

A complex analyst of several variables would likely read this definition and recall that domains $V \subset \mathbb{C}^N$ are Stein if and only if they are, in some sense, the largest domains on which holomorphic functions can be extended. Still, this definition doesn't make it clear why Stein manifolds are a subclass of Liouville manifolds. To arrive at that point, we first need to define another concept. Let (V, J) be an almost complex manifold and let $\phi : V \to \mathbb{R}$ be a smooth function. Then $d^c f$ is the 1-form defined by $d^c f(X) = df(JX)$. We say that ϕ is ***J*-convex** if the 2-form $\omega_\phi := -dd^c f$ satisfies $\omega_\phi(X, JX) > 0$ for all tangent vectors $X \neq 0$. It should be noted that the definition here doesn't require J to be integrable but an integrable J guarantees that ω_ϕ is J-invariant; i.e. $\omega_\phi(JX, JY) = \omega_\phi(X, Y)$.

From this discussion and choice of notation, the reader may guess that a J-convex function ϕ on a complex manifold (V, J) gives us a symplectic form ω_ϕ. Indeed this is the case and so we're led to give a third definition:

Definition 12.7 (J-Convex Definition)**.** *A complex manifold (V, J) is Stein if it admits an exhausting J-convex function $\phi : V \to \mathbb{R}$. Recall that exhausting means ϕ is bounded below and proper. Hence, it has the property that sets $\{x \in V : \phi(x) \leq r\}$ are compact.*

With this third definition, we also see that if we fix a J-convex ϕ, then $\lambda_\phi := -d^c\phi$ gives us a primitive 1-form and ω_ϕ is our exact symplectic form. Moreover, we have a J-invariant Riemannian metric $g_\phi(X, Y) := \omega_\phi(X, JY)$ and the gradient of ϕ with respect to this metric gives us our Liouville vector field Z.

Before giving some examples, it would be amiss if we did not mention why these three definitions are equivalent. Bishop and Narasimhan proved that the classical definition implies the affine definition. Grauert proved that the J-convex definition implies the affine definition. It is clear that the affine definition implies the other two.

Example 12.8. The simplest example of a Stein manifold is $\mathbb{C}^n$ itself where $\phi(z) = |z|^2$ is the J-convex function and ω_ϕ is in fact, the usual Kähler form. Note that the regular level sets of ϕ are spheres S^{2n-1} and they have a contact structure coming from λ_ϕ restricted to their tangent bundles. One can check that the same contact structure $\xi = TS^{2n-1} \cap J(TS^{2n-1})$. Though this example is very simple, it will be important in the sequel when we discuss Milnor fibers and their links which sit inside such spheres.

Example 12.9. Let $f(z_0, ..., z_n) = \sum_{k=0}^{n} z_k^2$ be a polynomial on $\mathbb{C}^{n+1}$. Observe that it has only one critical point at 0 and hence $V = f^{-1}(1)$ is a smooth manifold. Since it is defined by a complex algebraic equation, it is properly holomorphically embedded just by inclusion.

If we express $z_k = x_k + iy_k$ with real and imaginary parts, then $z_k^2 = x_k^2 - y_k^2 + 2ix_ky_k$. Hence, the set $\{f(z) = 1\}$ can also be described as pairs of vectors $(x, y) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ such that $|x|^2 - |y|^2 = 1$ and $\langle x, y\rangle = 0$; we're using the Euclidean inner product here. When $y = 0$, the points $(x, 0)$ form the unit sphere $S^n \subset \mathbb{R}^{n+1}$. It turns out that the restriction of the Kähler form to this Stein manifold gives a Liouville structure that is symplectomorphic to (T^*S^n, λ) that we saw in the previous section and the S^n mentioned here can be viewed as the zero section; it's easy to see that it is Lagrangian.

Example 12.10. Suppose we have a complete intersection of affine varieties with an isolated singularity in $\mathbb{C}^N$ locally defined by polynomials $f_1, ..., f_k$. Then let $F : \mathbb{C}^n \to \mathbb{C}^k$ be defined by $F = (f_1, ..., f_k)$ and a generic regular value of F will be a smoothing of the singularity. When $k = 1$, we have a Milnor fiber. By using the affine definition for Stein manifold, we clearly see that such a smoothing is Stein.

13 Some Algebraic Invariants and Open Book Decompositions

Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a polynomial with isolated singularity and let $\mathbb{C}[\![z_1, ..., z_{n+1}]\!]$ be the ring of formal power series in n variables. For convenience, we often assume $f(0) = 0$. The maximal ideal $\mathfrak{m}$ consists of all power series without a constant term.

13.1 Algebraic Invariants of Singularities

Here are three algebraic invariants one can define for the germ of the singularity $(f^{-1}(0), 0)$. The first involves integration but in fact, can be rendered in a completely algebro-geometric way.

Definition 13.1. *Let $\phi_{c,f} : \mathbb{C}^n \setminus \{0\} \to \mathbb{C}$ be $1/|f|^{2c}$. The* ***log canonical threshold*** *of $f^{-1}(0)$ is $lct(f^{-1}(0)) = \sup\{c > 0 : \phi_{c,f}$ is locally integrable$\}$. Other common notation is $lct(f)$.*

The next invariant involves differentiation instead of integration.

Definition 13.2. *The* ***multiplicity*** *of f, denoted $\nu(f)$ is the largest positive integer such that $f \in \mathfrak{m}^{\nu(f)}$.*

Definition 13.3. *The* ***Milnor number*** *of f, denoted $\mu(f)$, is the complex dimension of the algebra $\mathbb{C}[z_1, ..., z_{n+1}]/Jac(f)$ where $Jac(f)$ is the Jacobian ideal of f. This ideal is generated by the first partial derivatives of f.*

Though the Milnor number defined in this way is completely algebraic, there is another notion which Milnor defined in [Mil68]. In fact, there are several ways other ways to think of $\mu(f)$. For sufficiently small $\epsilon \gg \delta > 0$, the **Milnor fiber** is $M_f := f^{-1}(\delta) \cap B_\epsilon(0)$ where $B_\epsilon(0) \subset \mathbb{C}^{n+1}$ is a radius ϵ ball centered at 0. We now have the following theorem:

Theorem 13.4. *[Mil68] Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial with isolated singularity at 0 and $\mu(f)$ be the Milnor number defined above. Then:*

1. *The Poincaré-Hopf index of the vector field ∇f at 0 (under the standard metric of $\mathbb{C}^{n+1}$) is $\mu(f)$.*
2. *The homotopy type of the Milnor fiber M_f is a bouquet of spheres $\bigvee^{\mu(f)} S^n$. In particular, the Betti number $b_n(M_f) = \mu(f)$.*
3. *If $\tilde{f}$ is a Morsification of f, then the number of critical points of $\tilde{f}$ is $\mu(f)$.*

Here, a Morsification of a function f is a C^∞-small perturbation so that the resulting $\tilde{f}$ has only Morse critical points. This theorem shows that the algebraically defined invariant $\mu(f)$ is also a topological invariant since the Poincaré-Hopf index, middle Betti number, and Morsifications are all topological. In the introduction, we also described the result by Milnor concerning the fibration structure. We'll restate the result here:

Theorem 13.5. *[Mil68] Let $f : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ be a polynomial with isolated singularity at 0. For a sufficiently small $\epsilon > 0$, let $L := S^{2n+1}_\epsilon \cap f^{-1}(0)$. Then the map $\frac{f}{|f|} : S^{2n+1}_\epsilon \setminus L \to S^1$ is a fibration whose fibers are diffeomorphic to the interior of M_f, previously defined.*

In fact, much more is true. We observe that M_f, by virtue of being holomorphically embedded in $\mathbb{C}^{n+1}$ is also exact symplectically embedded. The parallel transport map along the base circle actually gives a symplectomorphism $\phi : M_f \to M_f$ supported away from $\partial M_f \cong L$ which is called the **monodromy map**. Hence, we form the mapping torus from the interior of M_f: $T_\phi := \mathring{M}_f \times [0,1]/\sim$ where $(x, 0) \sim (\phi(x), 1)$. If we only consider the smooth structures, we recover Milnor's fibration. This leads us to a definition:

Definition 13.6. *An* ***open book decomposition*** (M, π, B) *is the data of:*

- *a binding $B \subset M$ which is a codim 2 smooth submanifold;*
- *a fibration $\pi : M \setminus B \to S^1$.*

Hence, this mapping torus fibration is an example of an **open book**. But we see that the essential data towards defining it was an open exact symplectic manifold $\mathring{M}$ with a compactly supported symplectomorphism $\phi : \mathring{M} \to \mathring{M}$ from which we can construct the mapping torus T_ϕ. And the binding is ∂M. We could just use this data to define a mapping torus with ∂M as the binding. Hence, the data (M, ϕ) is called an **abstract contact open book**. To be more precise, we don't actually need any symplectic structure; it makes sense to define abstract open books with just diffeomorphisms and smooth pages.

13.2 Supporting Open Book Decompositions of Contact Manifolds

However, it turns out that when we take into account the symplectic structures, the total space of the mapping torus defined from an abstract open book admits a contact structure $\xi := \ker \alpha$ (for some contact 1-form α) via the Thurston-Winkelnkemper construction that is compatible with the open book structure. What this means is that B is transverse to xi and the Reeb field of α is always transverse to the pages. This perspective begins with a symplectic discussion but we could also go the other way and begin with the contact form.

Definition 13.7. *A* ***supporting open book decomposition*** *of a contact manifold* (Y, ξ) *is*

- *$B \subset Y$, a codim 2 submanifold that is transverse to ξ;*
- *a fibration $Y \setminus B \to S^1$*
- *there is a Reeb vector field that is always transverse to the fibers (pages).*

From this point of view which does not begin with symplectic structures, the idea is that, as much as possible, we want the pages to be tangent to the contact structure which means we get a symplectic form on the pages.

Example 13.8. Let $f = x^2 + y^3$ and $g = x^2 + y^2$. Then $\mathfrak{m}^2$ contains elements of degree at least 2: $x^2, y^2, xy, x^3, \ldots$ and $\mathfrak{m}^3$ contains elements of degree at least 3. So the multiplicities are $\nu(f) = \nu(g) = 2$. It is clear that $\mu(f) = 2$ and $\mu(g) = 1$. Using some additional results from [Mil68] for counting the number of boundary components, since the Milnor fibers are complex curves, we're able to deduce that M_g is a genus 0 curve with two boundary components whereas M_f is a genus 1 curve with one boundary component.

Having introduced some objects from symplectic and contact geometry, we'll now introduce the main result of [McL19] which relates these with some of the algebraic invariants we saw earlier:

Theorem 13.9. *Let $f, g : \mathbb{C}^{n+1} \to \mathbb{C}$ be polynomials with isolated singularities at 0. Then, if the links of f and g are (graded) contactomorphic, the multiplicity and log canonical threshold of f and g are equal.*

The key technical tool is a spectral sequence relating the exceptional divisors of a log resolution and the fixed-point Floer cohomology of the monodromy which we reproduce here (with slight rephrasing) though we will not explain the technical aspects of it unless needed. See the paper for details.

Theorem 13.10 (Theorem 1.2 of [McL19])**.** *Suppose that $\pi : Y \to \mathbb{C}^{n+1}$ is a multiplicity m separating resolution for some $m \in \mathbb{Z}_+$ and $\check{S}$ an indexing set of the exceptional divisors. Let $(w_i)_{i\in\check{S}}$ be positive integers so that $-\sum_{i\in\check{S}} w_i E_i$ is ample. Let a_i be the discrepancy of E_i, $m_i = ord_f(E_i)$, and $k_i := m/m_i$ for all $i \in S_m$ which is the set of i where m_i divides m. Then there is a cohomological spectral sequence converging to the fixed-point Floer cohomology $HF^*(\phi^m, +)$ of the mth iterate of the monodromy ϕ with E_1 page*

$$E_1^{p,q} = \bigoplus_{i\in S_m;\, k_i w_i = -p} H_{n-(p+q)-2k_i(a_i+1)}(\widetilde{E}_i^o; \mathbb{Z})$$

where $\widetilde{E}_i^o$ is a particular m_i-fold cover of $E_i^o := E_i \setminus \bigcup_{j\neq i} E_j$.

The spectral sequence can be viewed as a categorification of a result of [A'C75] who proved the the classical Lefschetz number $\Lambda(\phi^m) = \sum_{i\in S_m} m_i \chi(E_i^0)$ for all $m > 0$. That is, $\Lambda(\phi^m)$ can be computed from the data of *any* log resolution of the singularity. McLean's result is a categorification in the sense that this numerical statement is lifted to the level of homological algebra (when we view a spectral sequence as implied by a filtered complex) and recovered when we take the Euler characteristic. This is because the Euler characteristic of a spectral sequence equals the Euler characteristic of what it converges to and $\chi(HF^*(\phi^m, +) = \Lambda(\phi^m)$. Given that this Floer invariant gives us a way to break into the algebro-geometric setting, we now give a quick recap of what the invariant is and its properties.

14 Review of Fixed-Point Floer Cohomology

We follow Mclean [McL19] but the original definition can also be found in [Sei01]. Let (M, θ) be a Liouville domain. An almost complex structure J on M is **cylindrical** near ∂M if it is compatible with the symplectic form $d\theta$ and if $dr \circ J = -\alpha$ near ∂M inside the standard collar neighborhood $(0, 1] \times \partial M$ where $\alpha = \theta|_{\partial M}$.

We will consider compactly supported exact symplectomorphisms $\phi : M \to M$ because then fixed point Floer cohomology will have finite rank. Such a ϕ is **non-degenerate** if for every fixed point p of ϕ the linearization of ϕ at p does not have 1 as an eigenvalue. It has **small positive slope** if it is equal to the time-1 Hamiltonian flow of δr near ∂M where $\delta > 0$ is smaller than the period of the smallest periodic Reeb orbit of α. This means that it corresponds to the time δ Reeb flow near ∂M. If ϕ is an exact symplectomorphism, then a small positive slope perturbation $\check{\phi}$ of ϕ is an exact symplectomorphism $\check{\phi}$ equal to the composition of ϕ with a C^∞-small Hamiltonian symplectomorphism of small positive slope. The **action** of a fixed point p is $-F_\phi(p)$ where F_ϕ is a function satisfying $\phi^*\theta = \theta + dF_\phi$. The action depends on a choice of F_ϕ which has to be fixed when ϕ is defined although usually F_ϕ is chosen so that it is zero near ∂M (if possible). All of the symplectomorphisms coming from isolated hypersurface singularities will have such a unique F_ϕ. An isolated family of fixed points is a path connected compact subset $B \subset M$ consisting of fixed points of ϕ of the same action and for which there is a neighborhood $N \supset B$ where $N \setminus B$ has no fixed points. Such an isolated family of fixed points is called a **codimension 0 family of fixed points** if in addition there is an autonomous Hamiltonian $H : N \to (-\infty, 0]$ so that $H^{-1}(0) = B$ is a connected codimension 0 submanifold with boundary and corners, the time t flow of X_H is well-defined for all $t \in \mathbb{R}$ and $\phi|_N : N \to N$ is equal to the time 1 flow of H. The action of an isolated family of fixed points $B \subset M$ is the action of any point $p \in B$.

Let (M, θ, ϕ) be an abstract contact open book. Let $(J_t)_{t\in[0,1]}$ be a smooth family of almost complex structures with the property that $\phi^*J_0 = J_1$. A Floer trajectory of $(\phi, J_t)_{t\in[0,1]})$ joining $p_-, p_+ \in M$ is a smooth map $u : \mathbb{R}\times[0,1] \to M$ so that $\partial_s u + J_t\partial_t u = 0$ where (s,t) parameterizes $\mathbb{R}\times\mathbb{R}/\mathbb{Z}$, $u(s,0) = \phi(u(s,1))$ and so that $\lim_{s\to\pm\infty} u(s,t) = p_\pm$ for all $t \in [0,1]$. We write $\mathcal{M}(\phi, J_t, p_-, p_+)$ for the set of such Floer trajectories and define $\overline{\mathcal{M}}(\phi, J_t, p_-, p_+) := \mathcal{M}(\phi, J_t, p_-, p_+)/\mathbb{R}$, where $\mathbb{R}$ acts by translation in the s coordinate.

In order to properly define the fixed point Floer cohomology of an exact symplectomorphism, we need to discuss gradings. However, we move that discussion to the appendix.

Let (M, θ, ϕ) be a graded abstract contact open book and let $\check{\phi}$ be a small positive slope perturbation of ϕ. This can be done so that $\check{\phi}$ is C^∞-close to ϕ and so that the fixed points of $\check{\phi}$ are non-degenerate. We can also ensure that $\check{\phi}$ is a graded symplectomorphism due to the fact that it is isotopic to ϕ through symplectomorphisms. We now choose a C^∞-generic family of cylindrical almost complex structures $(J_t)_{t\in[0,1]}$ satisfying $\phi^*J_0 = J_1$. The genericity property then tells us that $\overline{\mathcal{M}}(\phi, J_t, p_-, p_+)$ is a compact oriented manifold of dimension 0 for all fixed points p_-, p_+ of ϕ satisfying $CZ(p_-) - CZ(p_+) = +1$.

We define $\#^\pm\overline{\mathcal{M}}(\phi, J_t, p_-, p_+)$ to be the signed count of elements of $\overline{\mathcal{M}}(\phi, J_t, p_-, p_+)$. Let $CF^*(\check{\phi})$ be the free abelian group generated by fixed points of ϕ and graded by the Conley-Zehnder index taken with **negative** sign. The differential ∂ on $CF^*(\check{\phi})$ is a $\mathbb{Z}$-linear map satisfying $\partial(p_+) = \sum_{p_-} \#^\pm\overline{\mathcal{M}}(\phi, J_t, p_-, p_+) \cdot p_-$ for all fixed points p_+ of $\check{\phi}$ where the sum is over all fixed points p_- satisfying $(-CZ(p_-)) - (-CZ(p_+)) = +1$. Since we use $-F_\phi(p)$ for the action, the differential takes fixed points of lower action to fixed points of higher action.

Because $(J_t)_{t\in[0,1]}$ is C^∞-generic, one can show that $\partial^2 = 0$ and we define the resulting homology group to be $HF^*(\phi, +) := HF^*(\check{\phi}, (J_t)_{t\in[0,1]})$. The notation is justified because this does not depend on the choice of perturbation $\check{\phi}$ nor on the choice of almost complex structure (J_t). These conventions then tell us that if $\phi : M \to M$ is the identity map with the trivial grading and $\dim M = n$ then $HF^k(\phi, +) = H^{n+k}(M, \mathbb{Z})$.

Having defined fixed-point Floer cohomology, we list some of their properties which can be found in [McL19].

1. For a graded abstract contact open book (M, λ, ϕ), the Lefschetz number $\Lambda(\phi)$ is equal to the Euler characteristic of $HF^*(\phi, +)$ multiplied by $(-1)^n$ where $n = \frac{1}{2}\dim M$.

2. If (M_i, λ_i, ϕ_i), $i = 0, 1$ are graded abstract contact open books so that the graded contact pairs associated to them are graded contactomorphic, then $HF^*(\phi_0, +) \cong$

$HF^*(\phi_1, +)$

3. Let (M, λ, ϕ) be a graded abstract contact open book where $\dim M = 2n$. Suppose that the set of fixed points of a small positive slope perturbation $\check{\phi}$ of ϕ is a disjoint union of codimension 0 families of fixed points $B_1, ..., B_\ell$ and $\iota : \{1, ..., \ell\} \to \mathbb{N}$ is a function where $\iota(i) = \iota(j)$ if and only if the actions of B_i and B_j equal and $\iota(i) < \iota(j)$ if the action of B_i is less than that of B_j. Then there is ac ohomological spectra sequence converging to $HF^*(\phi, +)$ whose E_1 page is equal to

$$E_1^{p,q} = \bigoplus_{\iota(i)=p} H_{n-(p+q)-CZ(\phi,B_i)}(B_p, \mathbb{Z}).$$

The last property is not one that we'll use directly but it's important for McLean's result. Here are some brief comments. We can make ι easily using the actions on the B_i and this gives a filtration on a chain complex built out of the fixed points. It's a general fact that whenever we have a filtered complex, we can obtain from it a spectral sequence.

15 When $m = \nu$ is the Multiplicity

In the previous two sections, we've discussed algebraic invariants of singularities and also fixed-point Floer cohomology. Here, we briefly mention some relationships between them since our main result is to use this Floer invariant to prove a version of a conjecture by Zariski concerned with the multiplicity of singularities. For convenience in writing grading shifts in a moment, suppose $f : \mathbb{C}^{n+1} \to \mathbb{C}$. If the multiplicity is ν, then the McLean spectral sequence for ϕ^ν degenerates at the E^1 page; i.e. $E^1 = E^\infty$. This is because for any log resolution, there is one divisor whose order of vanishing is the multiplicity and the order of vanishing of all the other divisors are larger. Since those do not divide ν, they do not appear in the E^1 page and so there is only one entry in the E^1 page which means the spectral sequence degenerates immediately. Moreover, an m separating resolution will give us trivial input data for the E^1 page when $m < \nu$. In this way, McLean's spectral sequence gives us a way to determine the multiplicity ν from fixed-point Floer cohomology; just look for the first nontrivial $HF^*(\phi^m, +)$.

It is shown in [BdBLN22], Prop 1.6, that if we write $f = f_\nu + f_{\nu+1} + ...$ where each term f_d is a homogeneous polynomial of degree d, then $HF^*(\phi^\nu) \cong H_c^{*+2nd+n-1}(\chi_\nu(f)) \cong H_c^*(F)$. Here, $\chi_\nu(f)$ is the ν-th contact loci and F is the Milnor fiber of f_ν. This is also described on slide 14 of McLean's slides [McL20b].
Remark: The authors of [BdBLN22] prove the existence of an m-separating resolution because McLean's proof contains a mistake.

16 Adjacent Singularities and a Cobordism

Two isolated hypersurface singularities are **adjacent** if they come from a family of polynomials $f_t : \mathbb{C}^{n+1} \to \mathbb{C}$ where f_0 gives one of the singularities and f_t for $t \neq 0$ gives the other. Some shorthand notation for this is $[f_1] \to [f_0]$ and the relationship is not symmetric in the sense that there might not be a family where f_1 gives the central fiber and f_0 gives the rest. There is a more general definition for non-hypersurface singularities but this will suffice for us.

Example 16.1. Let $f = x^2 + y^3$ and $g = x^2 + y^2(1+y)$. The singularity type of g is A_1 and near 0, the singularity is given by $xy = 0$ and the Milnor fiber is isomorphic to $\mathbb{C}^*$ where the boundary is the Hopf link. It has Milnor number 1 and multiplicity 2. The singularity type of f is A_2 and the Milnor fiber is a once-punctured torus with boundary being the trefoil. It's Milnor number is 2 and its multiplicity is 2. Figure 1 shows the embedding of M_g into M_f.

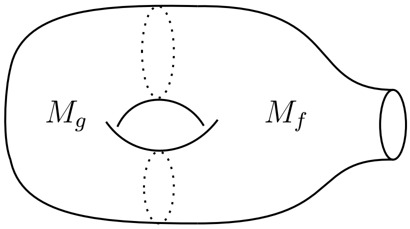
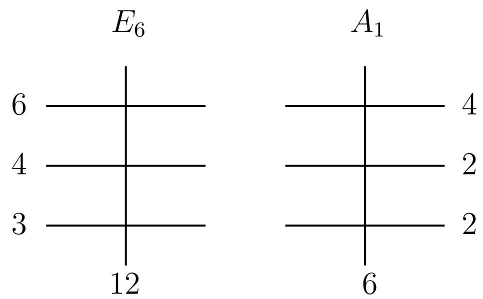


Figure 1: M_g embeds into M_f

Example 16.2. Let $f_t = x^2(x+t)+y^2(y^2+t)$. Observe that when $t = 0$, we have x^3+y^4 which defines a E_6 singularity. When $t \neq 0$, then the singularity at $(0,0)$ is of A_1 type. We have that $\mu(f_0) = 6, \nu(f_0) = 3$ and the Milnor fiber of f_0 has Euler characteristic -5 with boundary being the (connected) torus knot $T(4,3)$ (also called 8_{19}). Since $\chi = 2-2g-b$ where b is the number of boundary components, we find that the genus is 3.

We can perform four blowups to obtain a log resolution and in the process, look at the different chars to get the multiplicities. For example, in one of the charts at the end, the equation is $a^6b^2(a^6b + a^6b^2 + a^2t + t) = 0$. When $t = 0$, this simplifies to $a^{12}b^3(1+b)$. Below is a picture of the normal crossings divisor for the two singularities, labeled with their order of vanishing. The lowest order of vanishing is the multiplicity.



Remark 16.3. These two examples feature singularities of ADE type (also called simple singularities, Kleinian singularities, or du Val singularities). They can be obtained by taking a finite subgroup $G \subset SL(2,\mathbb{C})$ and quotienting to get $\mathbb{C}^2/G = \mathrm{Spec}\mathbb{C}[x,y]^G$. Here, $\mathbb{C}[x,y]^G$ are the polynomials invariant under G action. For instance, A_1 arises from $G = \mathbb{Z}/2 = \{\pm \mathrm{id}\}$ and we note that x^2, xy, y^2 are all invariant under this action. We also note that the $\mathbb{Z}/2$ action $(x,y) \mapsto (y,x)$ does not give a subgroup of $SL(2,\mathbb{C})$ but of $GL(2,\mathbb{C})$ and in this case, $\mathbb{C}^2/G \cong \mathbb{C}^2$ which shows that the analytic type does not remember information of the group G when the action is in $GL(2,\mathbb{C})$.

16.1 Examples: ADE Singularities

It was proved by Arnold that among singularities of ADE type, two are adjacent if and only if the Dynkin diagram of one embeds into the Dynkin diagram of the other. In the case of complex surface singularities, this result can also be recovered sympletically and was probably known to Arnold. We outline a different proof strategy in order to highlight a few important results in the literature. Firstly, Castelnuovo proved the uniqueness of minimal resolutions of complex surface singularities; i.e. resolutions where all (-1)-curves have been contracted. We then use the results of Brieskorn [Bri66] and Ohta-Ono [OO05]:

Theorem 16.4. *(Brieskorn) The minimal resolution of an isolated complex surface singularity and its Milnor fiber are diffeomorphic as a consequence of existence of simultaneous resolution.*

Theorem 16.5. *(Ohta-Ono) Let X be any minimal symplectic filling of the link of a simple surface singularity. Then, the diffeomorphism type of X is unique. Hence, it must be diffeomorphic to the Milnor fiber. Moreover, the symplectic deformation type of X is unique.*

Remark: Recall that two symplectic forms ω_0, ω_1 are **symplectic deformation equivalent** if there exists a diffeomorphism ϕ such that $\phi^*\omega_1$ and ω_0 are isotopic through a family of (not necessarily cohomologous) symplectic forms. In the proof of the theorem, Ohta-Ono attach a symplectic cap onto the fillings; these objects have many possible symplectic structures though they are all symplectic deformation equivalent. One can observe that the minimal resolution contains closed holomorphic curves and hence, cannot be a Stein filling. The Milnor filling, on the other hand, is Stein. One way to turn the minimal resolution into a Stein filling is to apply a hyperKähler rotation which gives some kind of symplectic deformation and might not be the one that brings us to the Milnor filling, but at least it would turn the holomorphic spheres into Lagrangian spheres. In [Ono23], Ono indicated that the result can be strengthened from one concerning symplectic deformation. The work of Lalonde-McDuff [LM96] can be used to pinpoint the cohomology class of the symplectic structure.

Corollary 16.6. *If two simple complex surface singularities are adjacent, then the Dynkin diagram of one embeds into the Dynkin diagram of the other.*

Proof sketch. Suppose that $f_0, f_1 : \mathbb{C}^3 \to \mathbb{C}$ are polynomials defining adjacent simple singularities; we will view f_0 as the "worse" singularity. The adjacency allows us to get a Stein cobordism between the link L_0 and L_1. Moreover, we can cap off the cobordism by using the unique minimal symplectic filling of L_1 to get a filling for L_0. Call these X_1 and X_0. This filling is also a minimal symplectic filling and hence, must be unique. By construction, the filling for L_1 embeds into the filling for L_0. Now, the Dynkin diagram for L_1 is given by studying the exceptional divisors and their intersections. This gives us a graph and the graph is precisely a Dynkin diagram of ADE type. Since we have an embedding $X_1 \hookrightarrow X_0$, we have an embedding $H_2(X_1, \mathbb{Z}) \hookrightarrow H_2(X_0, \mathbb{Z})$ which respects the intersection form. One needs to do some linear algebra since its possible that the basis for X_1 is not sent to the basis for X_0 but rather some linear combination. But this is doable and once done, this implies that the Dynkin diagram for X_1 embeds into the Dynkin diagram for X_0 □

Now, if two hypersurface singularities are adjacent, then there is a natural cobordism between their mapping tori and also a natural cobordism between their Milnor fibrations. We'll first construct a cobordism between their mapping tori. Suppose we have a 1-parameter family of polynomials f_t giving us the adjacency of f_0 to f_η for small η. We will symplectically modify the zero loci via bump functions so that they agree near the boundary of B_ϵ in the following way. Let $\beta : \mathbb{C}^{n+1} \to \mathbb{R}$ be a bump function supported in the region where the radius is $\epsilon' \leq r \leq \epsilon$; we can assume that $\beta = 1$ for $r > (\epsilon' + \epsilon)/2$ and is radially symmetric. We can later specify what ϵ' is but it will be very near ϵ and should be chosen such that the compact support of the monodromy of g does not intersect $S^{2n+1}_{\epsilon'}$. Then, letting $g = f_\eta$ for η extremely small, we define $\tilde{g} = g + (f - g)\beta$.

Lemma 16.7. *Let f, g be adjacent as above. Then the Milnor fiber of g smoothly embeds into the Milnor fiber of $\tilde{g}$ which smoothly embeds into the Milnor fiber of f.*

Proof. We take this opportunity to recall some basic facts. The Milnor fiber of a polynomial with isolated singularity is isomorphic to the zero locus of a generic smoothing of f. That is, if μ is the Milnor number which is the dimension of $\mathbb{C}[z_1, ..., z_{n+1}]/\mathrm{Jac}(f)$ where $\mathrm{Jac}(f)$ is the Jacobian ideal of f, then we have a miniversal deformation space $\mathbb{C}^\mu$. Take a small ball $B \subset \mathbb{C}^\mu$ centered at 0. A tuple $(\eta_1, ..., \eta_\mu) \in B$ gives us $\eta = \sum \eta_i p_i$ where p_i is a basis for $\mathbb{C}[z_1, ..., z_{n+1}]/\mathrm{Jac}(f)$. We construct a fibration over B where the fiber is the zero locus of $f + \eta$ intersected with a small ball in $\mathbb{C}^{n+1}$. Then, when $\eta = 0$, this is our singularity and for generic η (one not in the discriminant locus which has codim 1), the fiber is isomorphic to the Milnor fiber. So in this sense, a generic η gives a generic smoothing of f.

Because we have an adjacency, what we can do is first perturb f so that we get the singularity type of g and then perturb again to get a smoothing of g. Since the latter step is done by choosing a generic η, we see that this is also a generic smoothing of f. The way to get the embedding is to use different radii of balls as our cutoff. Let $\hat{\epsilon} \ll \epsilon$. For the Milnor fiber of g, it is the zero locus of a smoothing of g intersected with $B_{\hat{\epsilon}}$. This embeds into the zero locus of the same smoothing of g intersected with a larger ball B_ϵ. The latter is also a smoothing of f. Hence, the Milnor fiber of g embeds into the Milnor fiber of f.

Observe also that in the modification above to obtain $\tilde{g}$, since it happens very near the boundary of B_ϵ, these smoothings can be chosen to not affect f and $\tilde{g}$ near ∂B_ϵ. For example, $f^{-1}(0), \tilde{g}^{-1}(0)$ are transverse to ∂B_ϵ. As such, the Milnor fiber M_g also embeds into the Milnor fiber $M_{\tilde{g}}$ which embeds into the Milnor fiber M_f. Moreover, M_g and $M_{\tilde{g}}$ are diffeomorphic as one can construct a vector field whose flow maps $M_{\tilde{g}}$ onto M_g. Here is a picture.

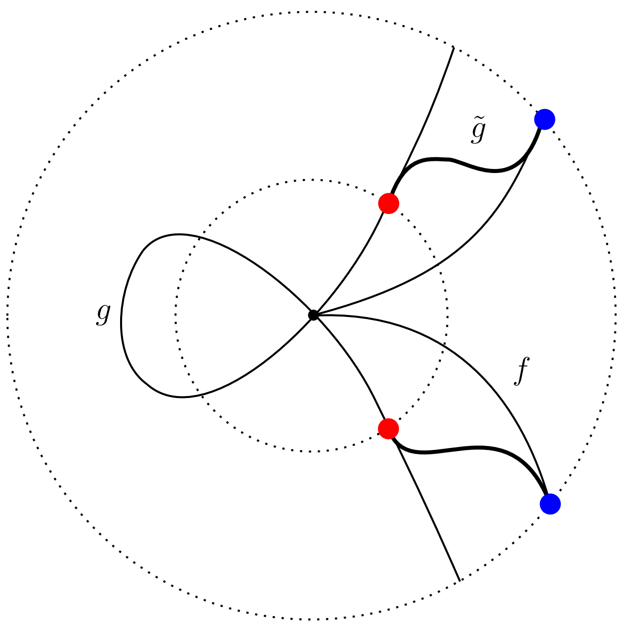


Figure 2: A modification

□

Next, we prove a symplectic result about the Milnor fiber of $\tilde{g}$.

Lemma 16.8. *Under the modification, $M_{\tilde{g}}$ still carries a symplectic structure and the symplectic type is independent of the choice of bump function β from Lemma 16.7. Moreover, $M_{\tilde{g}}$ and M_g are exact symplectomorphic.*

Proof. By virtue of being smoothly embedded in $\mathbb{C}^{n+1}$, $g^{-1}(0)$ and $\tilde{g}^{-1}(0)$ both have exact symplectic forms (away from the singularity) and the modification we've described. Observe that though β is not C^1-small, $f-g$ is C^1-small, as small as we would like by tuning the η from the proof of Lemma 16.7. Hence, $(f-g)\beta$ is C^1-small which means the polynomials $\tilde{g}$ and g are C^1-close.

If β_1, β_2 are two bump functions that are radially symmetric, then we may as well think of them as functions $\beta_1, \beta_2 : [\epsilon', \epsilon] \to [0,1]$ with non-negative first derivative. Then, if $t \in [0,1]$, $(1-t)\beta_1 + t\beta_2$ is also a bump function in that it equals 0 and 1 in the appropriate regimes and the first derivative is non-negative. This isotopy in t gives us a family of symplectic fibers. Since the symplectic forms are all exact, the Moser lemma shows that the fibers are all symplectomorphic (in the proof of Lemma 16.10, this is outlined). Lastly, we cite a result found in Cieliebak-Eliashberg's book; the proof is not difficult.

Lemma 16.9. *([CE12],Lemma 11.2) Any symplectomorphism $f : (V, \lambda) \to (V', \lambda')$ between finite type Liouville manifolds is diffeomorphic to an exact symplectomorphism.*

To apply this lemma, we point out that Milnor fibers are finite type Stein domains and we may complete them in the usual way to Stein (and hence, Liouville) manifolds. □

Lemma 16.10. *This modification does not change the Floer invariant HF^*.*

Proof. By tuning η to zero, what we actually get is a smooth family of mapping tori. We want HF^* to be invariant under this smooth family. At this point, it is more convenient to work with the classical Milnor fibration where the corresponding objects form a smooth family of Liouville domains where the variation takes place near the contact boundary. We may choose a smooth path in the disk and hence, obtain a smooth 1-parameter family of such Liouville domains between the Milnor fibers of f and g. By Gray's stability (see, for example, McDuff-Salamon [MS17], p. 136), when we have a 1-parameter family (in τ) of contact forms α_τ on a closed manifold, in this case, the link L considered as a smooth submanifold, there exists an isotopy ψ_τ and a family of functions f_τ such that $\alpha_\tau = \psi_\tau^*(f_\tau \alpha_0)$ where $\psi_0 = \mathrm{id}$ and $f_0 \equiv 1$. Hence, the ψ_τ are contactomorphisms since $d\psi_\tau$ maps $\ker \alpha_\tau$ to $\ker \alpha_0$; the scaling by f_τ doesn't affect the contact structure.

Similarly, we have a family of symplectic forms $\omega_\tau = d\alpha_\tau$ with exact derivative. Hence, we can use the Moser lemma to show that there is an isotopy Ψ_τ such that $\omega_\tau = \Psi_\tau^* \omega_0$ with $\Psi_0 = \mathrm{id}$ and $\Psi_\tau|_L = \psi_\tau$. Lastly, HF^* is a symplectic invariant and hence, for all τ, the corresponding HF^*'s of the $\phi_\tau : (F, \omega_\tau) \to (F, \omega_\tau)$ are all isomorphic. cf. property 2 in McLean's paper. □

Now, the links are contact submanifolds of the respective $(2n+1)$-spheres. Whenever we have a contact embedding $j : L \hookrightarrow (M, \alpha)$ of a submanifold into a contact manifold (M, α) where α is the contact form, some basic linear algebra shows us that the contact structure ξ_L embeds into ξ_M as a symplectic subbundle. Moreover, the Reeb field for L is the same as the Reeb field for M. Hence, $\xi_M = \xi_L \oplus \xi_L^\perp$ and $\xi_L^\perp$ is the normal bundle of L inside of S^{2n+1}. Let's now return to our situation where L is the link of an isolated singularity and $M = S^{2n+1}$.

Lemma 16.11. *Let $f : \mathbb{C}^{n+1} \to \mathbb{C}$ be a holomorphic function with isolated singularity at 0. The complex normal bundle of $F = f^{-1}(0) \setminus 0$ is trivialized by df and hence, the normal bundle of $L_f := f^{-1}(0) \cap S^{2n+1}_\epsilon$ also has a canonical trivialization induced by f.*

Proof. Since $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is a holomorphic map and 0 is the only point where $df = 0$, then $df : T\mathbb{C}^{n+1} \to T\mathbb{C}$ is surjective everywhere else. Since $F := f^{-1}(0) \setminus 0$ is a symplectic submanifold and part of the fiber, df maps TF to zero and the normal bundle of F maps surjectively onto $T\mathbb{C}$. In particular, since $L_f = F \cap S^{2n+1}_\epsilon$ is a transversal intersection of F with a regular level set of the norm squared function $|\cdot|^2 : \mathbb{C}^{n+1} \to \mathbb{R}$, the normal bundle of L_f in S^{2n+1}_ϵ is also trivialized. □

Let W be the portion of $\tilde{g}^{-1}(0)$ whose points have radius between ϵ' and ϵ. This means that the boundary of W is the union of the links L_g and L_f. In [Mil68], Milnor showed that Milnor fibers are smoothly parallelizable and that is the case here. In fact, a much stronger statement is true: the tangent bundle is algebraically trivial; we'll show why below. Moreover, for a generic fixed radius r, the level set $W_r = W \cap S^{2n+1}_r$ is a link with trivial normal bundle in S^{2n+1}_r. This shows that in fact, W itself has trivial normal bundle in B^{2n+2}_ϵ due to the adjacency of f and g. We call this the **canonical normal framing**.

Now, to prove that the tangent bundle of the Milnor fiber is algebraically trivial and hence, also trivial as both a complex and holomorphic bundle, we may use a powerful theorem of Suslin [Sus77]. This is much stronger than we need but we wish to advertise this result which may be less familiar to symplectic geometers.

Theorem 16.12. *Every stably trivial vector bundle of rank n on a smooth affine variety of dimension n over an algebraically closed field is algebraically trivial.*

The proof of Lemma 16.11 shows that any smooth complex affine hypersurface $F \subset \mathbb{C}^n$ has a trivial normal bundle ν and hence, has stably trivial tangent bundle since $TF \oplus \nu = \mathbb{C}^n$. Suslin's theorem then shows that TF is algebraically trivial.

16.2 A Cobordism for the Milnor Fibration

Since we showed above that W has trivial normal bundle, then the circle bundle of the normal bundle is $W \times S^1$. The cobordism that we construct will simply be the annulus $B^{2n+2}_\epsilon \setminus B^{2n+2}_{\epsilon'}$ with the normal bundle of W removed. This is a cobordism with corners between the Milnor fibrations; its boundary naturally decomposes into a horizontal and vertical part. $W \times S^1$ is the horizontal part and it is a contact hypersurface because it inherits its contact structure from the mapping torus. This structure is not $\xi = TW$ since that is integrable and stable Hamiltonian but not contact. Figure 2 depicts this; the red circles represent L_g and the blue circles represent L_f. The dotted lines denote the boundary and the bolded lines represent W. *Caveat lector*: Figure 2 portrays the real parts of a nodal and cuspidal singularity but in general, L_g, L_f, W are all connected. Also, the picture is drawn so that the zero loci appear to be far apart as we move further from the origin but that's for sake of having a picture that isn't too crowded.

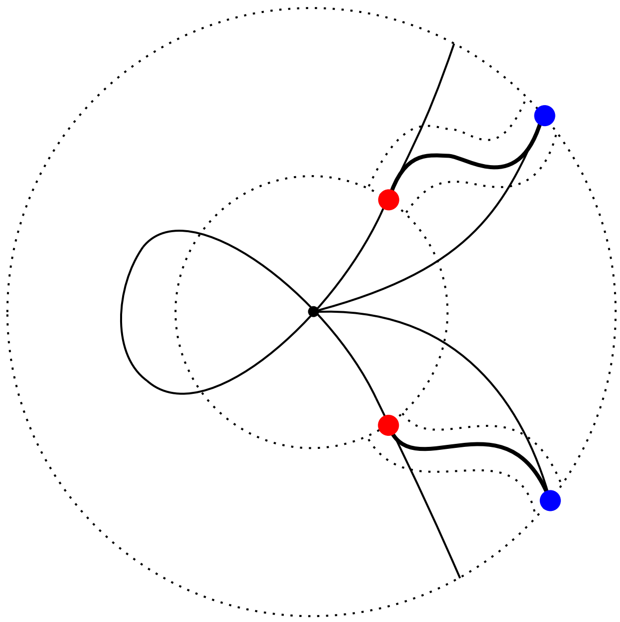


Figure 2: A cobordism between Milnor fibrations

16.3 A Cobordism for the Mapping Tori

However, a more convenient cobordism is to use the mapping torus model of the Milnor fibration. Recall that we had a small η from above where $g = f_\eta$ and we modified g to obtain $\tilde{g} = g + (f - g)\beta$. We can construct a cobordism using the mapping torus model: $E = (B_\epsilon \cap f^{-1}(\bar{D}_{\delta_1})) \setminus \tilde{g}^{-1}(\mathring{D}_{\delta_0})$. Here, $0 < \delta_0 \ll \delta_1 \ll \epsilon$ and both f and $\tilde{g}$ are submersions on E. Observe that this is also a symplectic manifold with corners; the symplectic form is inherited from $\mathbb{C}^{n+1}$ and the boundary decomposes into horizontal and vertical part.

The vertical boundary is the union of the mapping tori of f and $\tilde{g}$. We also have a symplectic collar neighborhood of the horizontal boundary because of how we constructed $\tilde{g}$.

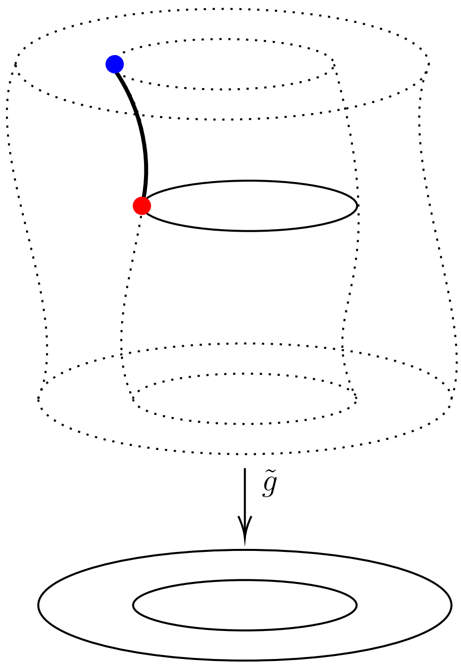


Figure 3: A cobordism between mapping tori

Remark 16.13. If we draw a tube around the nodal cubic $g^{-1}(0)$, then the picture suggests that it should form a real hypersurface that intersects itself. This immersed hypersurface, however, is *not* the mapping torus of the monodromy map on the corresponding Milnor fibration.

Another concern one may have is that the cuspidal cubic $f^{-1}(0)$ intersects the mapping torus associated to g and hence, the two mapping tori intersect. However, because they are adjacent, f, g are C^1-close and so, for small constants $\delta_0, \delta_1, \epsilon$ as above, the mapping tori do not intersect and E is an honest manifold-with-corners. For concreteness, we used these examples but this discussion is true for any pair of adjacent singularities.

Now, the situation above was such that we constructed a cobordism for the monodromy map ϕ. However, if we want to construct cobordisms for higher iterates of ϕ, we cannot use $\mathbb{C}^{n+1}$ but rather some branched covers. Again, consider $f : \mathbb{C}^{n+1} \to \mathbb{C}$ such that $f(0) = 0$ is the only singularity of f contained in $f^{-1}(0)$ and let ϕ be the associated monodromy obtained by symplectic parallel transport. We can construct more mapping tori by just taking iterates of the monodromy map. These tori no longer embed into $\mathbb{C}^n$ but do embed into $\mathbb{C}^{n+1}$ as follows. Consider $\mathbb{C}^n \setminus V(f)$, the complement of the zero locus of f. Then, it is isomorphic to the affine variety $V(wf - 1) \subset \mathbb{C}^{n+1}$ where we've added a variable w and so the coordinates are $(w, z_1, ..., z_n)$ and the map $(w, \vec{z}) \mapsto \vec{z}$ is the isomorphism since, we see that $wf(\vec{z}) = 1$ implies that neither w nor f vanish on this set and w is completely determined by f. This variety is k-fold covered by $V(w^k f - 1) \subset \mathbb{C}^{n+1}$ via the map $(w, \vec{z}) \mapsto (w^k, \vec{z})$ without any branching. Since the mapping torus for the monodromy can now be viewed as embedded in $V(wf - 1)$ via the isomorphism above, we can then take the preimage of this mapping torus to get something living in $V(w^k f - 1)$ which is abstractly, the mapping torus for the kth iterate of the monodromy.

So when I speak of a **branched mapping torus model**, I mean that I'm considering this particular model living in a branched covering of $\mathbb{C}^n$. Later on, I wish to consider a cobordism between two mapping tori; one knows that these exist since we can embed the mapping tori in some variety in $\mathbb{C}^{n+1}$ and then take the "space" between them.

17 An Induced Map on HF^*

Having constructed a cobordism, we would now like to construct a map on HF^*. Since the Milnor fiber M_g embeds into the Milnor fiber M_f, we would like to construct a chain map $CF^*(\check{\phi}_f) \to CF^*(\check{\phi}_g)$ where we view the latter chain complex as a quotient of the former chain complex. In order to do this, we need to modify the Milnor fiber of $\tilde{g}$ (which emebds into M_f) so that the fixed points of ϕ_g are seperated from the fixed points of ϕ_f. On the other hand, we do not want to change the exact symplectomorphism type of $M_{\tilde{g}}$. Since we may view $M_g \subset M_{\tilde{g}} \subset M_f$, the monodromy ϕ_g does not have fixed points in a small tubular neighborhood of L_g as it is compactly supported away from the boundary of M_g.

The key idea is inspired by neck-stretching (see, for example, [Wen16]) though we emphasize that this is not neck-stretching in the technical sense of considering a sequence of almost complex structures on longer and longer necks. However, we do glue in something like the symplectization of the link L_g. Hence, we'll call this **neck lengthening**. Since L_g is a contact hypersurface in $M_{\tilde{g}}$ and it separates $M_{\tilde{g}}$, we will neck lengthen the small tubular neighborhood $(-\epsilon, \epsilon) \times L_g$. We view $M_{\tilde{g}} = M_{\tilde{g}}^- \cup_{L_g} M_{\tilde{g}}^+$ where $M_{\tilde{g}}^+$ contains the original boundary of $M_{\tilde{g}}$. A neighborhood of L_g in $(M_{\tilde{g}}, \omega = d\theta)$ can be identified symplectically with $(N_\epsilon, \omega_\epsilon) := ((-\epsilon, \epsilon) \times L_g, d(r\theta) + \omega)$ for sufficiently small $\epsilon > 0$ (different constant from before). We then replace N_ϵ with larger collars of the form $((-T, T) \times L_g, d(q(r)\theta) + \omega)$, with C^0-small function q chosen with $q' > 0$ so that the collar can be glued in smoothly to replace $(N_\epsilon, \omega_\epsilon)$; we want q' to also be C^1-small on $(-T+1, T-1)$. This collar (or more commonly, neck) is somewhat like the symplectization of $(L_g, \theta|_{L_g})$. The symplectic manifolds constructed in this way are all exact symplectomorphic. See Figure 4.

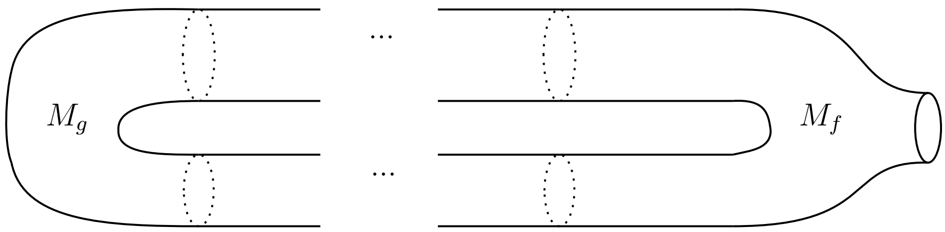
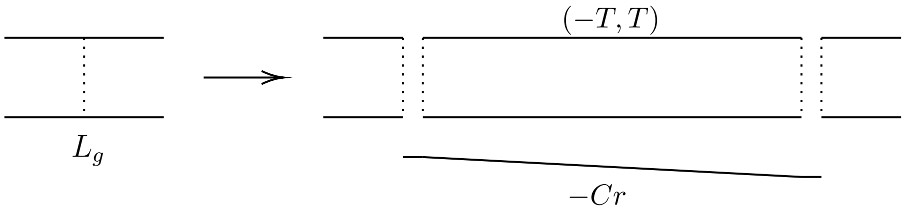


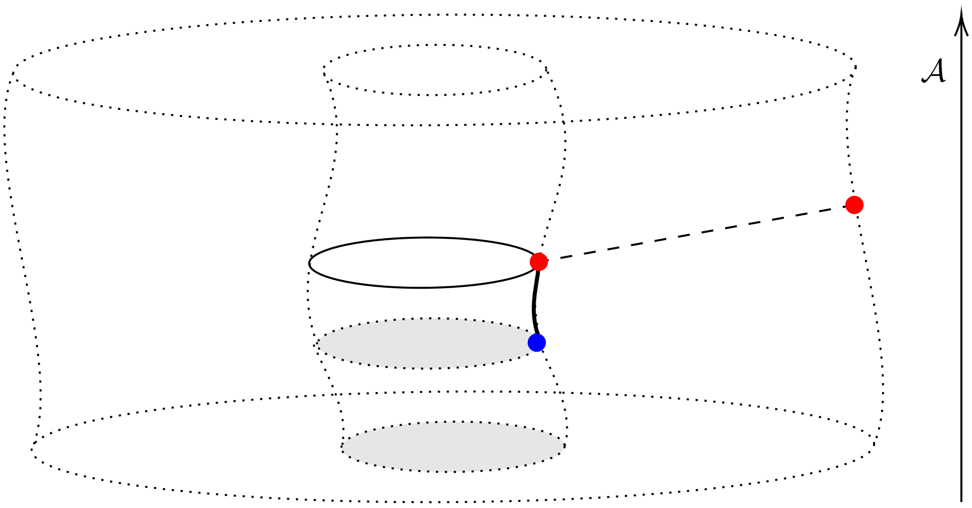
Figure 4: Neck Lengthening

Next, we modify the function $-F_{\phi_{\tilde{g}}}$ which defines the action for $\phi_{\tilde{g}}$. Let A_g be the lowest action of fixed points contained in $M^-_{\tilde{g}}$ and A_f be the highest action of fixed points contained in $M^+_{\tilde{g}}$. We then define a smooth function $-\widetilde{F}$ to equal $-F_{\phi_{\tilde{g}}}|_{M^-_{\tilde{g}}} + |A_f - A_g| + 1$ on $M^-_{\tilde{g}}$. We glue in a neck $(-(T+1), T+1) \times L_g$ such that $-\widetilde{F} = -Cr$ on $(-T, T) \times L_g$ for some $C > 0$; i.e. is linear. The symplectic structure on the neck is $d(q(r)\theta) + d\theta = q'(r)dr \wedge \theta + (q(r) + 1)d\theta$ and on the part where $F = -Cr$ is linear, the Hamiltonian vector field is simply $\pm \frac{C}{q'(r)} \cdot X_R$, where X_R is the Reeb vector field. Below is a schematic picture where we split along the separating hypersurface L_g and add in a cylinder on which there is a function $-\widetilde{F} = -Cr$.



The constant C is chosen to satisfy the requirement that $-\widetilde{F} = -F_{\phi_{\tilde{g}}}$ restricted to $M^+_{\tilde{g}}$ and that $C/q'(r)$ be smaller than P, the period of the smallest nonconstant Reeb orbit (remember $q'(r) > 0$).

In $(-(T+1), -T) \times L_g$ and $(T, T+1) \times L_g$, $-\tilde{F}$ is required to have negatives derivative in the r coordinate and to be C^∞-small. This whole construction can be viewed as choosing a Hamiltonian H such that $-\widetilde{F} = -F_{\phi_{\tilde{g}}} + H$. By choosing a large T, this separates the actions of orbits of f and of g since we can basically shift the actions of the orbits of f by at least $-2C(T-1)$ and T can be arbitrarily large. Hence, the actions of the fixed points of ϕ_g when using $-\widetilde{F}$ are all greater than the actions of the fixed points of ϕ_f when T is large. Below is a picture which basically combines the previous mapping torus cobordism picture with this kind of neck lengthening but we've flipped everything over (because we're using $-\widetilde{F}$ rather than $+\widetilde{F}$). The outer torus for the Milnor fiber of $\tilde{g}$ is "wider" to show that we've lengthened and the dashed line is meant to show that the actions of the fixed points for $\phi_{\tilde{g}}$ are all higher than before. The region between the two shaded disks is irrelevant in this situation and one could excise it if preferred.



Let a be a value between these two sets of actions. Above, we chose C to be smaller than the period of the smallest periodic Reeb orbit of $\theta|_{L_g}$. In other words, we think of H as giving a Hamiltonian symplectomorphism of small positive slope. Then, we have a perturbation $\check{\phi}_{\tilde{g}}^N$ equal to the composition of $\phi_{\tilde{g}}$ with this C^∞-small Hamiltonian symplectomorphism of small positive slope. The N stands for neck lengthening. This $CF^*(\check{\phi}_{\tilde{g}}^N)$ is isomorphic to $CF^*(\check{\phi}_{\tilde{g}})$ where $\check{\phi}_{\tilde{g}}^N$ is a small positive slope perturbation as in the definition without neck lengthening. This is because neck lengthening does not create or destroy orbits and counts of J_t-holomorphic curves are unchanged so long as J_t is cylindrical on the neck.

With this in hand, the action filtration defines a subcomplex $CF^*_{\leq a}(\check{\phi}_{\tilde{g}}^N)$ where the generators have action less than a. These generators are precisely those in M_f^+ and $CF^*(\check{\phi}_g) \cong CF^*(\check{\phi}_{\tilde{g}}^N)/CF^*_{\leq a}(\check{\phi}_{\tilde{g}}^N)$.

We may now define a chain map $\Psi : CF^*(\check{\phi}_f) \to CF^*(\check{\phi}_{\tilde{g}}) \cong CF^*(\check{\phi}_{\tilde{g}}^N)/CF^*_{\leq a}(\check{\phi}_{\tilde{g}}^N)$. The idea is to use the cobordism between mapping tori E from before and make a count of trajectories. Then we use an exact symplectomorphism $\chi : M_{\tilde{g}} \to M_{\tilde{g}}^N$ between the Milnor fiber and the neck lengthened Milnor fiber to get an associated isomorphism between mapping tori $T(\phi_{\tilde{g}}) \to T(\chi\phi_{\tilde{g}}\chi^{-1})$. This ensures that the chain map respects the action filtration. Lastly, we quotient by the subcomplex $CF^*_{\leq a}(\check{\phi}_{\tilde{g}}^N)$. Thus, the next step is to clarify how to count trajectories in E and in particular, establish compactness results to ensure finite counts.

Since $E \subset \mathbb{C}^{n+1}$, it has symplectic and almost complex structures. For a generic choice of cylindrical almost complex structure J, consider a map $u : \mathbb{R} \times \mathbb{R}/\mathbb{Z} \to E$ satisfying $\partial_s u + J\partial_t u = 0$ and $\lim_{s\to\pm\infty} u(s,t) = \gamma_\pm(t)$ where γ_+ is a simple Reeb orbit of the monodromy of ϕ_f–it corresponds to a fixed point p_+–and similary γ_- is a Reeb orbit of the monodromy of ϕ_g corresponding to a fixed point p_-. As will be shown below, because of a certain J-convex hypersurface in E near the vertical boundary, these cylinders cannot have interior tangencies to it due to the maximum principle. This allows us to apply Gromov compactness and obtain compact 0-dim oriented moduli spaces $\overline{\mathcal{M}}(E, J, p_-, p_+)$. Then $\Psi(p_+) = \sum_{p_-} \#^\pm \overline{\mathcal{M}}(E, J, p_-, p_+) \cdot p_-$ where $(-CZ(p_-)) - (-CZ(p_+)) = +1$. Standard Floer theory techniques show that this is a chain map.

18 Stable Hamiltonian Structures and Hofer Energy

In order to obtain the compactness result needed to define the map above, we will dedicate this section to some of the setup needed. In particular, we need to at least define what we mean by finite energy Floer trajectories. For a more complete survey, see ch. 6 of Wendl [Wen16].

Definition 18.1. *A* ***stable Hamiltonian structure*** *(SHS for short) on an oriented $(2n-1)$-dimensional manifold M is a pair (Λ, Ω) consisting of 1-form Λ and a closed 2-form Ω such that:*

1. $\Omega|_{\ker \Lambda}$ *is nondegenerate.*

2. $\ker \Omega \subset \ker d\Lambda$.

Moreover, a stable Hamiltonian structure (Λ, Ω) gives a co-oriented hyperplane distribution $\xi := \ker \Lambda$ and a positively transverse vector field R determined by the conditions $\Omega(R, \cdot) \equiv 0$ and $\Lambda(R) \equiv 1$. This is analogous to contact manifolds and so we'll call R the **Reeb vector field**. Indeed, contact manifolds are an example of manifolds with stable Hamiltonian structures.

Example 18.2. Suppose (M, α) is a contact manifold. Then $(\alpha, d\alpha)$ is a stable Hamiltonian structure. The second property is trivially satisfied while the first property is built into the definition for α to be a contact form. $\xi = \ker \alpha$ is the usual hyperplane distribution and R coincides with the usual Reeb vector field of contact geometry.

Example 18.3. For isolated hypersurface singularities defined by polynomial $f : \mathbb{C}^{n+1} \to \mathbb{C}$ and ϕ is the monodromy, the mapping torus lives in $\mathbb{C}^{n+1}$ and inherits lots of structure. For example, $(\pi^* d\theta, i^*\omega_0)$ as a stable Hamiltonian structure where $i^*\omega_0$ is the restriction of the standard symplectic structure ω_0 on $\mathbb{C}^n$ and $d\theta$ is a closed 1-form on S^1. Observe that for tangent vectors $v \in TM_f$, $d\pi(v) = 0$ and hence $v \in \ker \pi^* d\theta$. On the other hand, we have the linear map df (not a differential form since f is not a real function) and $\ker df|_{T_f} = \bigcup_{\theta \in S^1} TM_f(\theta)$ where $M_f(\theta) = f^{-1}(\theta)$. So $\ker \pi^* d\theta = \ker df|_{T_f}$.

On the other hand, the fibers are symplectic submanifolds with respect to ω_0 and so $\omega_0|_{\ker df}$ is nondegenerate. Moreover, $\pi^* d\theta$ is closed, so $\ker d(\pi^* d\theta) = T\mathbb{C}^n$ and hence contains everything, including $\ker \omega_0$.

Note that this stable Hamiltonian structure is not contact since the distribution $\ker df|_{T_f}$ is integrable. But there is a contact structure we could put on T_f if we use the so-called generalized Thurston-Winkelnkemper construction.

For higher iterates of the monodromy, we can use the branched covers discussed earlier to put stable Hamiltonian structures on those mapping tori.

One useful feature of a SHS on a manifold M is that there is a symplectization. The **symplectization** of (M, Λ, Ω) for any stable Hamiltonian structure (Λ, Ω) can be defined by choosing suitable diffeomorphisms of $(-\epsilon, \epsilon) \times M$ with $\mathbb{R} \times M$. Equivalently, we may consider $\mathbb{R} \times M$ with the family of symplectic forms ω_ψ defined by $\omega_\psi := d(\psi(r)\Lambda) + \Omega$ where ψ is any element of $\mathcal{T} := \{\psi \in C^\infty(\mathbb{R}, (-\epsilon, \epsilon)) : \psi' > 0\}$. Note that $\omega_\psi = \psi' dr \wedge \Lambda + \psi d\Lambda + \Omega$. If we restrict ω_ψ to $\xi = \ker \Lambda$, then the first term vanishes.

Having a way to define a symplectization in hand, it's natural to consider J-holomorphic cylinders in the symplectization. But what sort of J do we consider?

Definition 18.4. *Let r be the coordinate on $(-\epsilon, \epsilon)$. Then J is an* ***admissible*** *almost complex structure if*

1. *J is invariant under translation in the $(-\epsilon, \epsilon)$ factor.*

2. *$J\partial_r = R$ (Reeb vector field)*

3. *$J(\xi) = \xi$ (the hyperplane distribution is J-invariant)*

4. *$J|_\xi$ is compatible with the symplectic vector bundle structure $\Omega|_\xi$.*

An alternative nomenclature is to call such J ***cylindrical*** *since they bear much similarity to the contact case.*

These properties are enough to show that an admissible J is tamed by every ω_ψ. Thus, we may define the **Hofer energy** of a J-holomorphic curve $u : (\Sigma, j) \to (\mathbb{R} \times M, J)$ where J is admissible:

$$E(u) := \sup_{\psi \in \mathcal{T}} \int_\Sigma u^* \omega_\psi.$$

One can show that each ω_ψ tames an admissible J and hence, $E(u) \geq 0$ with equality if and only if u is a constant map. One important remark that Wendl points out is that this notion of energy is different from some other notions that appear in symplectic geometry, such as Hofer energy. However, for the purposes of getting uniform bounds in order to have compactness of moduli spaces, this notion of energy is sufficient.

Note: One reason to define the Hofer energy in this way is for the following reason. Consider a contact manifold (M, α) and a Reeb orbit γ of period $T > 0$. A **trivial cylinder** $u : \mathbb{R} \times S^1 \to (\mathbb{R} \times M, d(e^t \alpha))$ is $u(s,t) = (Ts, \gamma(Tt))$ and if we use $\int_{\mathbb{R} \times S^1} u^* d(e^t \alpha)$ as the energy, the energy would be

$$\lim_{s \to +\infty} \int_{S^1} u^* e^t \alpha - \lim_{s \to -\infty} \int_{S^1} u^* e^t \alpha = \infty.$$

The point of choosing $\psi : \mathbb{R} \to (-\epsilon, \epsilon)$ is get a finite quantity (without changing the symplectomorphism type); we then take a supremum since there is no canonical choice of ψ.

In the mapping torus case, we have that $\omega_\psi = d(\psi(r)\pi^* d\theta + i^* \alpha_0)$ where α_0 is a primitive 1-form for ω_0; say $\alpha_0 = \frac{1}{2} \sum y_k dx_k - x_k dy_k$. Then, if we look at finite energy curves $u : \Sigma \to \mathbb{R} \times T_f$, they have to limit to positive and negative orbits. We'll denote the positive and negative ends with $\Gamma^\pm$. So the energy is

$$E(u) := \sum_{\gamma_+ \in \Gamma^+} \lim_{r \to +\infty} \int_{S^1} \gamma_+^* (\psi(r) \pi^* d\theta + i^* \alpha_0) - \sum_{\gamma_- \in \Gamma^-} \lim_{r \to -\infty} \int_{S^1} \gamma_-^* (\psi(r) \pi^* d\theta + i^* \alpha_0).$$

Supposing $\psi \to \pm\epsilon$ as $r \to \pm\infty$, the energy becomes

$$E(u) := \epsilon \sum_{\gamma_+ \in \Gamma^+} \int_{S^1} \gamma_+^* (\pi^* d\theta + i^* \alpha_0) - \epsilon \sum_{\gamma_- \in \Gamma^-} \int_{S^1} \gamma_-^* (\pi^* d\theta + i^* \alpha_0).$$

The $\gamma^* \pi^* d\theta$ gives the winding number whereas $\gamma^* i^* \alpha_0 = \gamma^* \alpha_0$ (since the image of γ is in T_f) gives the action or length of the orbit. If the orbit is near the horizontal boundary of T_f, it will have very large action. At least, on a Milnor fiber which is Stein, as we go to infinity, the volume increases exponentially. Since the mapping torus, near infinity, is a circles worth of Stein manifolds, the energy is increasing.

19 Compactness of Moduli of Floer Trajectories

We're now prepared to prove the following:

Theorem 19.1. *Let p_-, p_+ be two fixed points of the monodromy symplectomorphism with $(-CZ(p_-)) - (-CZ(p_+)) = 1$. Then, the 0-dim oriented moduli space $\overline{\mathcal{M}}(E, J, p_-, p_+)$ of finite energy Floer trajectories that limit to the Reeb orbits corresponding to p_- and p_+ is compact and hence, finite.*

Proof. To get compactness results, we first note that in the cobordism between mapping tori T_f and T_g and also in the symplectizations, because the monodromy is compactly supported, there is a hypersurface of the form $\mathbb{R} \times L \times S^1$ where L is the link and $L \times S^1 \subset T_f$ (again, the monodromy being compactly supported means that the mapping torus is trivial near the boundary). Note that since $L \subset T_f$ is a contact submanifold, $\mathbb{R} \times L$ is a symplectic submanifold of $\mathbb{R} \times T_f$ and that $\mathbb{R} \times L \times S^1$ is a union of all the slices of $\mathbb{R} \times L$ transported by the Reeb flow. Let ψ be the Stein Morse function on a Milnor fiber and let Ψ be the trivial extension of it to the mapping torus T_f near the boundary (again, the boundary is trivial because the monodromy is compactly supported). Then in a neighborhood of the boundary, the level sets of Ψ are $L \times S^1$. Taking $pr_2 : \mathbb{R} \times T_f \to T_f$, the map $\Psi \circ pr_2$ has $\mathbb{R} \times L \times S^1$ as a regular level set and $dd^c(\Psi \circ pr_2) \leq 0$ since $dd^c\psi \leq 0$. Hence, $\mathbb{R} \times L \times S^1$ is a J_0-convex hypersurface. We now state a lemma (from Oancea's survey, Lemma 1.4, [Oan04]).

Lemma 19.2. *Let $S \subset M$ be a J-convex hypersurface and ψ a (local) function of definition. No J-holomorphic curve $u : (D^2(0,1), i) \to M$ can have an interior strict tangency point with S; i.e. $\psi \circ u$ cannot have a strict local maximum.*

$\mathbb{R} \times L \times S^1$ is a J_0-convex hypersurface with $\Psi \circ pr_2$ as the function of definition. Since the trajectories converge to Reeb orbits lying in a compact set, any trajectory has its ends contained in the compact set. This means that a sequence of trajectories that escape towards the boundary will give a trajectory with an interior tangency point. This contradicts the lemma and hence, we cannot actually have a sequence of trajectories escaping. In other words, the image of the Floer trajectories of finite energy limiting to the given Reeb orbits must all lie in a compact set in the target. Then the two conditions of Gromov compactness are fulfilled: finite energy and images lying in a compact set. We conclude that the moduli space is compact. □

20 (1+1) TQFT Equipped with Partial Lefschetz Fibrations

Having shown that the maps we defined for the concrete cobordisms are sensible because we have compact moduli spaces, we can then ask various questions about their properties. For example, do the maps depend on the open book decomposition or perhaps only on data that comes from our polynomial f? In this section, we'll develop a bit of theory to address this question but provide more than we need because this theory may be of independent interest. In some of the discussion here, we will speak of Floer homology as opposed to *co*homology but there is really no difference. We discuss homology because in [Sei08c], Seidel presents Floer *homology* of a symplectomorphism $\phi : M \to M$ alternatively as the Floer homology of the mapping torus $M \times [0,1]/\sim$

where $(x,t) \sim (\phi(x), t+1)$. This is a natural fibration over an oriented circle Z (to keep with his notation). However, the formalism presented works for any symplectic fibration $F \to Z$. Moreover, when we have a connected, compact, oriented surface S with $p+q$ boundary circles divided into p positive and q negative ends, this induces maps on tensor products of the homology. In more detail, let $\partial S = \bar{Z}_1^- \cup ... \cup \bar{Z}_p^- \cup Z_{p+1}^+ \cup ... \cup Z_{p+q}^+$. Then, given a symplectic fibration $E \to S$ with fiber M, the restrictions $F_k^{\pm} = E|_{Z_k^{\pm}}$ are symplectic fibrations and we have a relative Gromov invariant

$$G(S,E) : \bigotimes_{k=1}^{p} HF_*(Z_k^-, F_k^-) \longrightarrow \bigotimes_{k=p+1}^{p+q} HF_*(Z_k^+, F_k^+).$$

Remark 20.1. In Seidel's lecture notes, he sometimes assumes that $H^1(M,\mathbb{R}) = 0$ in order to use the C^∞-topology on $\mathrm{Symp}(M,\omega)$. For us, when $n \geq 3$, the homotopy type of the Milnor fiber M_f of $f : \mathbb{C}^{n+1} \to \mathbb{C}$ is that of $\bigvee^{\mu} S^n$ where μ is the Milnor number. Hence, $H^1(M_f, \mathbb{R}) = 0$. When $n = 1$, the fiber is finitely many points and when $n = 2$, the fiber is a curve, a well-studied situation.

For our purposes, we mostly care about $S = S^1 \times [0,1]$ with a positive and negative end. However, we'll like to extend this formalism to include maps between fibrations with different fibers. In order to do this, we need to extend the theory to that of partial Lefschetz fibrations (see McLean's preprint [McL20a]).

Definition 20.2. *A* ***partial Lefschetz fibration*** *is comprised of a quadruple (E, S, π, K) where E is a smooth manifold with corners, $K \subset Int(E)$ is a compact subset in the interior of E, $\pi : E \setminus K \to S$ is a map between manifolds. Moreover, E consists of two codimension 1 boundary components $\partial_h E$ (horizontal boundary) and $\partial_v E$ (vertical boundary) meeting in a codimension 2 component. There is a 1-form θ_E on E making E into a Liouville domain after smoothing the corners. The map π must satisfy the following properties:*

1. *A neighborhood of $\partial_h E$ is diffeomorphic to $S \times (1-\epsilon, 1] \times \partial F$ where F is some Liouville domain called the* ***fiber*** *of π. Here $\theta_E = \theta_S + r\alpha_F$ where θ_S is a Liouville form on S and r parameterizes the interval. The 1-form α_F is the contact form on ∂F. The map π is the projection map to S in this neighborhood.*

2. *θ_E restricted to the fibers of π is non-degenerate away from the singularities of π.*

3. *The restriction $\pi|_{\partial_v E}$ is a fibration whose fibers are exact symplectomorphic to F such that the fibers of π are either disjoint from $\partial_V E$ or entirely contained in $\partial_v E$.*

4. *There are only finitely many singularities of π and they are all disjoint from the boundary ∂E. They are modeled on non-degenerate holomorphic singularities.*

The most general theory would not require a global Liouville form on $E \setminus K$ but only when restricted to the fibers. The way to think about such a definition is perhaps to first begin with $K = \varnothing$ and $S = D \subset \mathbb{C}$ which reduces the situation to the usual Lefschetz fibrations that the reader may be accustomed to. In all situations, the preimage $\pi^{-1}(\partial S)$ is the vertical boundary but in particular when S is the unit disk, $\pi|_{\partial_v E} : \partial_v E \to S^1$ is part of the data of an open book decomposition.

When the compact set $K \neq \varnothing$, one might think of it as "hiding" the critical points of any extension of π to E; these critical points can be quite pathological and nonisolated, which allows us to consider, for example, cobordisms between mapping tori with different fibers. These cobordisms can be treated with classical Morse theory where we think of handle attachment as occuring when we traverse pass a critical value. Lefschetz theory, being over $\mathbb{C}$, cannot recover this since one can always go around a critical value. In fact, we can take K to be larger so that π has no singularities at all.

Also, near the boundary $\partial_v E$, we have a connection given by the ω_E-orthogonal plane field to the fibers. Because the fibration is a product near $\partial_h E$, the parallel transport maps associated to this connection are well defined and are compactly supported if we transport around a loop. The symplectomorphism $\phi : F \to F$ given by parallel transporting around a loop on ∂S is called, without surprise, the monodromy symplectomorphism around this boundary component.

20.1 Unique Analytic Continuation ⇒ HF^* Depends Only On the Framed Binding

This theory may seem unsuitable for studying holomorphic sections since such curves seem uncontrolled within K. However, this is not the case. Let

$$\mathcal{M}_{E,K} = \{u : S \to E : \bar{\partial}_J u = 0, \pi \circ (u|_{u^{-1}(E \setminus K)}) : u^{-1}(E \setminus K) \to S \text{ is the natural injection}\}$$

When a holomorphic curve is injective on an open set (and in fact, the identity), then there is a unique analytic continuation of the curve due to the identity theorem: a holomorphic map is completely determined by its restriction to open sets and in the case of curves, determined even just by a sequence of points with an accumulation point. Hence, the moduli space above does not depend on K.

Within this formalism, we're able to show that our maps do not depend on the open book decomposition of the contact manifold (the mapping tori) but only on the binding and its normal framing. Recall that W is the section of the Milnor fiber of $\tilde{g}$ which has links L_g and L_f as boundary.

Lemma 20.3. *Let $M \to Z$ be an open book decomposition over an oriented circle Z with page F being a Liouville domain and B the codim 2 binding. Then, up to isomorphism, $HF_*(Z, M)$ only depends on B and a choice of normal framing.*

Proof. Let the total space M be given two open book decompositions with pages F_1, F_2 and the same binding B and trivialization of the normal bundle. We then have two trivial cobordisms on the total space M, call them E_{12} and E_{21}. Note that this does not affect the orientation of M. In each of these, we can choose compact sets K_{12}, K_{21} such that their complements are basically small neighborhoods of the boundary of each E_{12}, E_{21}. For each situation, the boundary is a union of vertical and horizontal components. The horizontal boundary is, under the trivialization, $[0,1] \times \partial F_i$, $i = 1, 2$. But $\partial F_1 = \partial F_2 = B$.

Thus, we have two partial Lefschetz fibrations. The gluing of E_{12} and E_{21} gives a trivial cobordism between M and itself equipped with the same open book. This induces the identity map which means that the composition of the maps induced by E_{12} and E_{21} must be inverses. The same is true if we glue in the opposite way: E_{21} to E_{12}. Hence, each induced map is an isomorphism and $HF_*(Z, M)$ does not depend on the choice of open book decomposition, so long as we fix the binding and choice of normal framing. □

The argument for this lemma is another proof of some of the results of Appendix B [McL19]. Moreover, it implies the following:

Corollary 20.4. *Let $M_f, M_{\tilde{g}}$ be the mapping tori from before which fiber over $Z_f, Z_{\tilde{g}}$, respectively. Then, if $E \to A$ is the cobordism over the annulus from Section 16.3, using this notation, we have an induced map Ψ of Section 17 which passes to cohomology $HF^*(Z_f, M_f) \to HF^*(Z_{\tilde{g}}, M_{\tilde{g}})$. This Ψ does not depend on the choice of open books but only on the horizontal boundary of E which is given by $W \times A$, under the canonical normal framing.*

21 Novel Proof of Zariski's Conjecture

Now that we've established some properties of this map, let's apply it towards addressing a conjecture of Zariski. Recall that the Milnor number of an analytic hypersurface singularity represented by f can be algebraically defined as $\mu := \dim_{\mathbb{C}} \mathcal{O}/\mathrm{Jac}(f)$. It is a fact that the singularity is isolated if and only if $\mu < \infty$. In the work of [dBP22], they prove results which imply the following:

Theorem 21.1. *If a family of isolated hypersurface singularities has constant Milnor number μ, then they also have constant multiplicity.*

This is a weaker form of Zariski's conjecture [Zar71] which states: If $f, g : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ define singularities with the same topological type; i.e. there is a homeomorphism $\Phi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n+1}, 0)$ such that $\Phi(V_f) = V_g$, then the singularities have the same multiplicity.

In the case that the singularities are isolated, Milnor proved that the Milnor number is a topological invariant and hence, de Bobadilla-Pełka's work proves Zariski's conjecture for families of isolated hypersurface singularities. However, as of 2022, the conjecture is open for pairs of singularities that do not fit into a family and is also open for general classes of nonisolated singularities though Massey pointed out that their work applies to certain types of nonisolated singularities. I believe there is also a notion of multiplicity for singularities that are not hypersurfaces but for complete intersections and maybe quotient singularities. So one can formulate a Zariski-type conjecture for these as well.

The de Bobadilla-Pełka proof uses log resolutions and builds something called an A'Campo space for it which involves some tropical geometry. They also use McLean's work on fixed-point Floer cohomology for Milnor fibrations and multiplicity. The main technical result is an extension of McLean's spectral sequence to include the data of fixed points at infinity.

In this section, we provide an alternative and somewhat simpler proof (which still relies on McLean's work and hence, on his spectral sequence).

Consider polynomials $f, g : \mathbb{C}^{n+1} \to \mathbb{C}$ with isolated singularities and suppose that they fit into an adjacent family. We can define the Milnor fibration for f using radius δ circle and cutoff the total space with an ϵ ball. For the Milnor fibration of g, use constants δ', ϵ' where these are much smaller than their respective counterparts for f. Then, the Milnor fiber M_g embeds into the Milnor fiber of M_f and the extra part is a cobordism $W = M_f \setminus M_g$. Here's the picture from before:

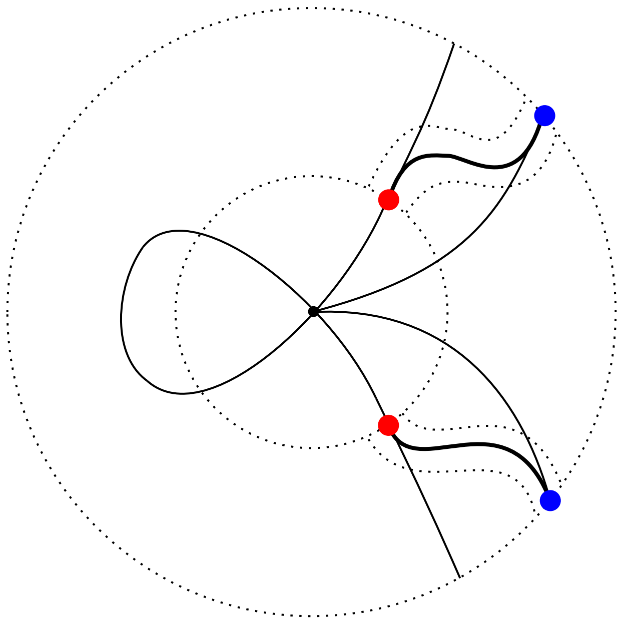
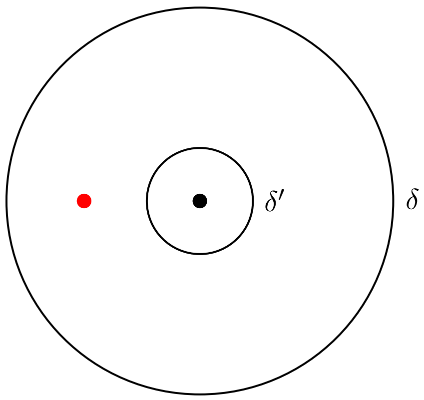


Figure: A cobordism between Milnor fibrations

In this picture, there's a modification of g to some $\tilde{g}$ so that it agrees with f near the boundary of B_ϵ. There may be some extra topology appearing when we enlarge the ball but the radius δ' monodromy ϕ_g^k won't do anything to the extra part. However, if we enlarge both the ball and the circle for the monodromy of g to radii ϵ and δ, respectively, then it may see extra singularities. In the picture below, when we perturb f to g, some critical value(s) split off (depicted in red) and the monodromy ϕ_g^k does not see them.



But if we enlarge the circle to radius δ and consider a new monodromy, denote it as ψ_g^k, then it will take into account the extra critical values and have some new and nontrivial action on B_ϵ. Note that if we used a smaller cutoff $B_{\epsilon'}$; i.e. restrict ψ_g^k to a smaller ball, then $\psi_g^k|_{B_{\epsilon'}} = \phi_g^k$ because we've taken away the piece on which it would perform its new monodromy. So to **recap**: if we only enlarge the ball, we see extra topology but the monodromy is trivial on the extra part. If we only enlarge the circle, the monodromy will encircle more critical values but not do anything on a restricted ball of small radius. However, if we enlarge both, then there is both extra topology and extra critical values so then the monodromy ψ_g^k is truly different.

Lemma 21.2. *Let $W = g^{-1}(\delta) \cap (B_\epsilon \setminus B_{\epsilon'})$; this is the extra part of the Milnor fiber when we enlarge the ball. Then there is an exact triangle given by action filtration of orbits:*

$$\begin{array}{ccc} HF^*(\psi_g^k) & \longrightarrow & HF^*(\phi_g^k) \\ & \nwarrow \qquad \swarrow & \\ & H^*(W, \partial W) & \end{array}$$

Proof. Above in Section 17, we made an action filtering/lengthening the cylinder argument to define the chain map. The discussion there is easily adapted to our current situation basically verbatim to show how to use the action and lengthening to separate out the fixed points of ψ_g^k which are not also fixed points of ϕ_g^k; let b be a value in between these two sets of actions. However, unlike that situation, we do have extra fixed points due to enlarging the radius of the circle to take into account extra critical values which are seen by the monodromy ψ_g^k. These fixed points live in W, are made to have much higher action by the lengthening argument, and compute the relative cohomology: $H^*(W, \partial W)$. There is an injective chain map $C^*(W, \partial W) \hookrightarrow CF^*(\psi_g^k)$.

Next, we define a chain map on the subcomplex $CF^*(\psi_g^k)_{\leq b} \to CF^*(\phi_g^k)$ in a similar way as in Section 17 via a count of Floer trajectories. We then extend the map by zero to the full complex $CF^*(\psi_g^k)$; it is clear that the kernel is exactly $C^*(W, \partial W)$. Lastly, the map from $CF^*(\phi_g^k) \to C^*(W, \partial W)$ is just the zero map. This is because the geometric generators for each complex (fixed points or Morse critical points, for example) live in disjoint subsets and moreover, the actions of the generators are also disjointed. Hence, we get an exact triangle but it in fact splits up into a series of short exact sequences. □

Now, Lê-Ramanujam [LR76] (or see de Bobadilla-Pelka, Prop. 5.22), proved:

Theorem 21.3 (Lê-Ramanujam)**.** *Given a family of isolated singularities $f_t : \mathbb{C}^{n+1} \to \mathbb{C}$ with $n \neq 2$ such that the Milnor number for these is constant (independent of the parameter t), then the diffeotype of the Milnor fibrations for the f_t is also independent of t.*

Their proof relies on the h-cobordism theorem and so when $n = 2$, the homeotype, thought not diffeotype, is also independent of t by the work of Freedman. What are the consequences of this theorem?

Given a μ-constant family, the members of the family are adjacent to each other. When we perturb one to the other, there will not be any critical values shooting off because the number of those which can shoot off is bounded above by the difference in Milnor number which, in this case, is zero. So the extended monodromy ψ_g^k will not see any extra critical values and hence, not change. This is because the monodromy on a smoothing of the singular fiber is only nontrivial in the region very near the singularity. So ψ_g^k extends ϕ_g^k by the identity map.

What about the extra piece that comes from enlarging to B_ϵ? Lê-Ramanujam's result tells us that no extra topology appears so this W is smoothly trivial when $n \neq 2$: $W = [0, 1] \times L_g$ where L_g is the link of g and boundary of M_g. Again, when $n = 2$, then it is topologically trivial.

Remark 21.4. We emphasize that it's possible that there are values of $r \in [\epsilon', \epsilon]$ such that $g^{-1}(\delta)$ does not intersect S_r^{2n+1} transversally which allows for the possibility of being nontrivial as a Stein cobordism. If all the critical points of the Liouville vector field are subcritical, we can perform some handlemoves to cancel the points and obtain something in the same Weinstein class. But if there are critical points (and the cobordism isn't flexible), then it may very well have some interesting symplectic topology that the monodromy maps completely overlooks.

By Lemma 21.2, we already know that $HF^*(\phi_g^k) \cong HF^*(\psi_g^k)$. We now want to show that $HF^*(\psi_g^k) \cong HF^*(\phi_f^k)$. The indirect way to do this is to take the branched mapping torus (embedded in some k-fold branch cover of $\mathbb{C}^{n+1}$) of each and observe that it is graded contactomorphic to the abstract mapping torus given by a page and self-map: $(M_g(\delta, \epsilon), \psi_g^k)$. This is the content of Giroux's work; see Remark 25.3 in Appendix A.

This abstract mapping torus, in turn, is graded contactomorphic to the boundary of the Lefschetz fibration we get when we Morsify g which is an open book decomposition; by Morsify, we mean that we take a very small perturbation of g to make it a Morse function. The reason for this is due to Picard-Lefschetz theory which tells us that the monodromy is a composition of generalized Dehn twists along the vanishing cycles (the Lagrangian spheres). The Morsification of f is essentially the same as the Morsification of g and hence, gives the same Lefschetz fibration (up to deformation equivalence) if we use the same δ and ϵ. If we use the larger ones, then Remark 21.4 says there may be some symplectic subtleties that go undetected. However, fixed point Floer cohomology is blind to those differences and is even insensitive to the deformation equivalence because it is only sensitive to the monodromies. So as far as HF^* is concerned, the mapping tori for ψ_g^k and ϕ_f^k use the same δ and ϵ by comparing them to the boundary of the "same" Lefschetz fibration.

The final step is to see that since Lê-Ramanujam theorem (plus the $n = 2$ case) implies that $W \cong_{\text{homeo}} [0,1] \times L_g$ which means that $H^*(W, \partial W) = 0$ for any n. Therefore, our exact triangle above produces for us a symplectically-constructed isomorphism: $HF^*(\phi_f^k) \cong HF^*(\phi_g^k)$ for every k. By [McL19], we are able to recover the multiplicity of f simply by looking for the smallest k such that $HF^*(\phi_f^k) \neq 0$. Moreover, we also recover the log canonical threshold of f by the formula:

$$\operatorname{lct}_0(f) = \liminf_{k \to \infty} \left(\inf \left\{ -\frac{\alpha}{2k} : HF^\alpha(\phi_f^k, +) \neq 0 \text{ or } -\frac{\alpha}{2k} = 1 \right\} \right)$$

To summarize, we have the following:

Theorem 21.5. *If a family of isolated hypersurface singularities is μ-constant, the multiplicity and log canonical threshold are also constant in the family as a result of symplectic considerations.*

Remark 21.6. We highlight the fact that the Milnor number is a smooth topological invariant and the multiplicity and log canonical threshold are invariants for graded contactomorphic pairs by the work of McLean [McL19]. By comparison, the theorem here is about families with constant Milnor number and we show that the family has the property that each member has the same multiplicity and log canonical threshold and moreover, this is a *symplectic property.* The result was previously proven for the log canonical threshold by Varchenko [Var82] in 1982 and for multiplicity in 2022 by de Bobadilla-Pełka [dBP22]. So our work is a new proof and also illustrates that these are not just algebro-geometric properties but symplectic properties.

22 A Brief Remark about the Fukaya-Seidel Category

The above section studied the algebraic invariants for μ-constant families of isolated singularities. We may also ask about the symplectic invariants for the Milnor fibers of such families. In particular, the Fukaya-Seidel category is a symplectic invariant. In our situation, it is an A_∞ category generated by Lefschetz thimbles that are associated to an ordered set of paths that are assigned to a Morsification of f. We won't give the definition here but perhaps one thing to point out is that this category is not the "full" Fukaya category since there certainly are more exact Lagrangians than just the thimbles. For example, Keating constructed exact Lagrangian tori for the Milnor fibers of singularities with positive modality [Kea15] which is in contrast to the result of Ritter that the only exact Lagrangians in Milnor fibers for modality zero singularities (ADE type) of real dimension 4 are spheres [Rit10].

At any rate, the Lefschetz fibrations for two adjacent isolated singularities with the same Milnor number are deformation equivalent and hence, they have equivalent Fukaya-Seidel categories.

23 Follow-up Questions Suggested by Theorem 21.5

In this section, we pose some questions related to the proof of Theorem 21.5 and also discuss related notions and examples.

Question 23.1. If f, g belong in a μ-constant family, do they have graded contactomorphic links? If so, then this would immediately imply the theorem above. But this is either false or unknown because otherwise, de Bobadilla-Pełka would have given a shorter proof. Ailsa Keating pointed out to us an example of Pham which shows that the topological type of a generic discriminant curve can change for a family of plane singularities even if they have constant μ. The example will appear below in Section 24 but the discriminant curve still seems like an analytic invariant to rather than a topological one. Other interesting examples/facts from the literature will also be enumerated.

Question 23.2. The following singularities all have the same Milnor number: A_7, D_7, E_7. They are not adjacent as there is no Dynkin diagram embedding of any one of these into any of the other. Do the singularities admit different monodromies (and hence, difference Lefschetz fibrations)? This had better be true since the multiplicity of A_7 is 2 and for the other two, it is 3.

Question 23.3. There is a graded ring structure on $\bigoplus_{k\geq 0} HF^*(\phi^k, +)$ where $\phi^0 = \mathrm{id}$ and the product is the pair-of-pants product $HF^*(\phi^k, +) \otimes HF^*(\phi^\ell, +) \to HF^*(\phi^{k+\ell}, +)$. This map is a particular example of the relative Gromov-Witten invariants discussed in Section 20. Can this structure be utilized to prove algebraic results such as detecting changes of multiplicity within a family?

Question 23.4. If we use the mapping torus model to study a pair of adjacent singularities $[f] \to [g]$, we can just extend the torus to a (singular) fibration over a disk $D \subset \mathbb{C}$ and count multisections of this with one interior marked point. We obtain an augmentation $HF^*(\phi_f^k) \to \mathbb{Z}$ and also one for g; in the language of TQFTs, it is a co-unit. Moreover, the augmentation for f factors through that of g as in the following commuting diagram.

$$\begin{array}{ccc} HF^*(\phi_f^k) & \longrightarrow & HF^*(\phi_g^k) \\ & \searrow & \downarrow \\ & & \mathbb{Z} \end{array}$$

This diagram commutes by the functoriality of TQFTs. Now, if we don't consider this filling and capping disks, there is still a way to obtain this diagram. We interpret $\mathbb{Z} = HF^*(\mathrm{id}) = CF^*(\mathrm{id})$ for a polynomial which is smooth at 0. For example, we can just shift g by some constant so that its isolated singularity is now far away and the link is just a sphere and the monodromy is trivial. Again, by functoriality properties of HF, we get the same diagram as before, just writing $\mathbb{Z} = HF^*(\mathrm{id})$ which is concentrated in a single degree. The Morse index is 0 (the minimum of a Morse function on a ball) and so the Conley-Zehnder index is some shift of this. What's the usefulness of this? We see that if the multiplicity of f, g is the same and that the two maps to $\mathbb{Z}$ are also nontrivial, then the horizontal map above has to be nontrivial. So the question is: how can we prove the augmentation type maps are nonzero except for multiplicity reasons? If we can, then the triviality of horizontal map implies that the multiplicity is not the same.

Example 23.5. An **unfolding** of a singularity defined by any polynomial f can be viewed as a deformation space. For example, the miniversal deformation space is an unfolding. If S is a semi-universal unfolding (I don't know what the semi-universal means but S can be viewed as a ball inside some $\mathbb{C}$-vector space), then $f_s = f + \sum s_j \phi_j$ is a deformation of f. For each s, we can consider $V(f_s)$, the zero locus of f_s. The space $D \subset S$ is comprised of those s such that $V(f_s)$ has singularities. If $H_0 \subset S$ is a generic 2-plane passing through the origin, then $H_0 \cap D$ is a curve called a **generic discriminant curve**. If H is a parallel 2-plane to H_0, then $H \cap D$ is called a **generic unfolded discriminant curve**.

For plane curves defined by $f(x, y) = 0$, we have the Milnor number μ as well as an analytic invariant $\sigma := 1 + \dim \mathbb{C}[x, y]/I$ where I is the ideal generated by f, its 1st partial derivatives, and its 2nd partial derivatives.
Fact: Suppose $f = y^3 - P(x)y + Q(x)$. Then the topological type of $H_0 \cap D$ is determined by μ and σ. This fact implies that the topological type of the unfolded discriminant curve $H \cap D$ only depends on μ and σ.

Now for Pham's example, originally appearing in [Pha73]; another reference is [LÏ8]. Let $f(x, y) = y^3 + x^k + 1$ and choose $m \geq 2$ such that $2k \leq 3m - 2$ and $m \leq k$. He showed there is a 3-parameter unfolding $F(x, y; u, t, s)$ so that for $u = t = 0$, μ is constant. For a fixed sufficiently small s_0, the discriminant curve of $F(x, y, ; u, t, s_0)$, denoted $D_{s_0} := \{s = s_0\} \cap D \subset \mathbb{C}^2_{u,t}$, is reduced (all irreducible components have multiplicity 1) and is topologically equivalent to D_0. Observe that our 2-plane $\{s = s_0\}$ is not generic. On the other hand, $\sigma = k$ for $s = 0$ and $\sigma = m$ for sufficiently small $s \neq 0$.

Since the topological type of the (unfolded) generic discriminant curve is determined by μ and σ, this F gives us examples where μ is constant but the topological type of the generic discriminant curve changes.

Example 23.6. Let $f_p = z_0^p + \sum_{j=1}^{2m+1} z_j^2$ be a weighted homogeneous polynomial with $p \equiv \pm 1 \pmod 8$ and $m \geq 1$. It's Milnor number is $p - 1$. Brieskorn [Bri66] proved that the link of this polynomial, denoted $\Sigma(p, 2, ..., 2)$ is diffeomorphic to the standard S^{4m+1}.

In [Ust99], Ustilovsky showed that for each p, we get a contact structure ξ_p on S^{4m+1} and these are all pairwise nonisomorphic, distinguished by contact homology. In fact, for each homotopy class of almost contact structure on S^{4m+1}, there are infinitely many pairwise nonisomorphic contact structures.

This is in contrast to the result of Caubel-Nemethi-Popescu-Pampu [CNPP06] which says that if we fix a smooth, oriented 3-manifold Y, then it admits at most one contact structure ξ which is Milnor fillable; i.e. (Y, ξ) is contactomorphic to the link L of some isolated surface singularity. In fact, if we are only interested in $Y = S^3$, then Mumford showed that S^3 can only arise as the link of a smooth point. He does so by showing how to compute $\pi_1(L)$ given the singularity and that $\pi_1(L) = 0$ if and only if the link is of a smooth point and therefore, must be S^3. We note that Mumford basically proved the Poincaré conjecture in the setting of links of isolated surface singularities.

24 Examples of μ-Constant Families

We are grateful to Jason Starr for reminding us of this first example. Consider the union of 4 lines in $\mathbb{C}^2$, say $xy(x-y)(x-ty) = 0$ where $t \in \mathbb{C} \setminus \{0, 1\}$ This has the same topological type as the union of four lines $x^4 - y^4 = (x+y)(x-y)(x+iy)(x-iy)$. The Milnor number is a topological invariant and while it's cumbersome to compute the Milnor number for the family above, we can easily see from the other example that $\mu = 9$.

Next, if we take the line $x + y = 2$ in $\mathbb{C}^2$, we can compute the points of intersection with the 4 lines; they are $(0, 2), (1, 1), (2, 0), (\frac{2t}{t+1}, \frac{2}{t+1})$. We can then use this $x + y = 2$ line and some change of coordinates to treat these 4 collinear points as $-1, 0, 1, \frac{t-1}{t+1}$ in $\mathbb{C}$. The cross-ratio of the four points is therefore $\frac{t-1}{t}$ (a minor calculation) and of course, $t \neq 0, 1$ from our conditions above since we didn't want the 4 lines to degenerate to 3 lines with one of them having multiplicity (it should be said that $t \neq \infty$ as well).

If we now compactify to $\mathbb{CP}^2$, there is the following theorem:

Theorem 24.1. *Let E be an arbitrary elliptic curve in $\mathbb{CP}^2$ and $p \in E$. Then there are exactly four lines passing through p which are each tangent to E. Conversely, given four lines that all pass through a common point, there is a unique elliptic curve lying tangent to the four prescribed lines.*

Lemma 24.2. *Every elliptic curve $(E, 0)$ embeds into $\mathbb{CP}^2$.*

Now, intersect the four lines from the theorem with a fifth line (not through the marked point) and let λ be the cross ratio of the four intersection points. Then the j-invariant of the elliptic curve is

$$j = 256\frac{(\lambda^2 - \lambda + 1)^3}{\lambda^2(\lambda - 1)^2}.$$

The j-invariant is an intrinsic analytic invariant for elliptic curves and if $\lambda = \frac{t-1}{t}$, the algebra works out so that j-invariant is actually of the same form:

$$j = 256\frac{(t^2 - t + 1)^3}{t^2(t - 1)^2}.$$

This means that the analytic type of the singularities in this family are distinguished by their cross-ratio which is in fact, *intrinsic*, due to this reformulation with the j-invariant.

Proof. (Sketch of lemma) Let $L \to E$ be a line bundle and K the canonical bundle on E. Riemann-Roch says:

$$h^0(E, L) - h^0(X, L^{-1} \otimes K) = \deg(L) + 1 - g.$$

Since $g = 1$ and K is trivial for elliptic curves, we see that $h^0(E, L) \geq \deg(L)$. So if we want a section with vanishing of order 2 or 3 near the marked point $0 \in E$, such sections exist. Call these sections x, y. We use them to define an embedding into $\mathbb{CP}^2$; first we use x, y as coordinates to show that the elliptic curve satisfies the following Weierstrass normal form equation $y^2 = x^3 + ax^2 + bx + c$. Then projectivize. Note, by the way, that the y^2 term on the left tells us there is an involution on the curve. Of course, the group law on E says we have an involution $x \mapsto -x$. □

Remark 24.3. One thing that may be of secondary interest is that picking a point on an elliptic curve is like choosing the identity element for the group law and the tangent lines give us the 2-torsion points (on $\mathbb{R}^2/\mathbb{Z}^2$ for example, there are four 2-torsion points: $(0,0), (0,1/2), (1/2,0)$, and $(1/2,1/2)$). In Weierstrass normal form, the inverse to a point (x, y) on the curve is $(x, -y)$ and a 2-torsion point then has $y = 0$. Hence, we're looking for the three roots of $x^3 + ax^2 + bx + c$; the fourth 2-torsion point is at infinity and is the identity. In any abelian group, the subset of 2-torsion points forms a subgroup and here, it is $\mathbb{Z}/2 \times \mathbb{Z}/2$. The sum of any two of these points is equal to a third in the group which is obvious from the fact that the three roots are collinear.

Also, the example of $x^4 + y^4$ is discussed on the level of directed Fukaya categories by Keating [Kea16].

Another example of constant Milnor number and variation of moduli: take a family of elliptic surface singularities such as $x^3 + y^3 + z^3 - txyz = 0$, where $t^3 \neq 27$ (without this condition, the singularities are nonisolated). Of course this is related to cross-ratio, since this affine equation in $\mathbb{C}^3$ is just the affine cone over the elliptic curve with the same equation in $\mathbb{CP}^2$.

25 Appendix A: Gradings and the Conley-Zehnder Index for Fixed-Point Floer Cohomology

Much of this reiterates what is found in [McL19] and [Sei00] and is also related to Section 8. The point of gradings is that we would otherwise only have a relative grading on HF^* (or the E^1 page of the spectral sequence); i.e. we would have to make a choice of what counts as HF^0.

What we say here is less detailed. Let $(\mathbb{R}^{2n}, \omega_{st})$ be the standard symplectic $\mathbb{R}^{2n}$ and $Sp(2n)$ the group of linear symplectomorphisms. Let $p : \widetilde{Sp}(2n) \to Sp(2n)$ denote its universal cover, recalling that $\pi_1(Sp(2n)) = \mathbb{Z}$. If we have a symplectic vector bundle $\pi : (E, \omega_E) \to X$ of rank $2n$, then we can form the symplectic frame bundle $Fr(E) \to V$ whose fiber is isomorphic to $Sp(2n)$. The fiber over $x \in V$ is the group of linear symplectomorphisms between $(\mathbb{R}^{2n}, \omega_{st})$ and $(\pi^{-1}(x), \omega_E|_x)$.

Definition 25.1. *A* ***grading*** *on a bundle* $\pi : E \to X$ *is a principal* $\widetilde{Sp}(2n)$*-bundle* $\widetilde{Fr}(E) \to X$ *together with an isomorphism of principal* $Sp(2n)$*-bundles:*

$$\iota : \widetilde{Fr}(E) \times_{\widetilde{Sp}(2n)} Sp(2n) \cong Fr(E).$$

Recall that $\widetilde{Fr}(E) \times_{\widetilde{Sp}(2n)} Sp(2n)$ is a quotient $\widetilde{Fr}(E) \times Sp(2n)/\sim$ where $(x \cdot g, h) \sim (x, p(g)h)$. For example, if $g \in p^{-1}(1) \cong \mathbb{Z}$, then the multiplication $p(g)h = h$ and hence $(x \cdot g, h) \sim (x, h)$.

Example 25.2. A grading of a symplectic manifold (X, ω) is a grading of $(TX, \omega) \to X$.

Now suppose we have the following commutative diagram:

$$\begin{array}{ccc} (E_1,\omega_1) & \xrightarrow{F} & (E_2,\omega_2) \\ \big\downarrow{\scriptstyle \pi_2} & & \big\downarrow{\scriptstyle \pi_2} \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

where $\pi_i : (E_i, \omega_i) \to X_i$ are symplectic vector bundles of rank $2n$ and F is a map which covers a diffeomorphism f and is fiberwise a linear symplectomorphism. Then there is an induced map on frame bundles $Fr(F) : Fr(E_1) \to Fr(E_2)$ defined fiberwise as follows. Let $x \in X_1$. Then for an element $A \in Sp(2n)_x$ in the fiber over x, we may compose it with $F_x : E_1|_x \to E_2|_{f(x)}$. This gives an element $F_x \circ A \in Sp(2n)_{f(x)}$. So $Fr(F)$ sends $A \mapsto F_x \circ A$.

A grading of F is a map $\widetilde{F} : \widetilde{Fr}(E_1) \to \widetilde{Fr}(E_2)$ which covers $Fr(F) : Fr(E_1) \to Fr(E_2)$. Note that if we have a symplectomorphism $\phi : (X_1, \omega_1) \to (X_2, \omega_2)$, then we can take $d\phi : (TX_1, \omega_1) \to (TX_2, \omega_2)$ to be our F which covers ϕ. So it is in this sense that we say ϕ is graded.

We also want to consider gradings of co-oriented contact manifolds. Suppose (C, ξ) is co-oriented; i.e. $TC/\xi \cong \mathbb{R}$ is oriented. Choose a 1-form α such that $\ker \alpha = \xi$ and $\alpha > 0$ on TC/ξ. Then $(\xi, d\alpha|_\xi)$ is a symplectic vector bundle which can be given a grading. Up to isotopy, this grading does not depend on the choice of α so we may sensible define this to be a grading for the co-oriented contact manifold (C, ξ). A grading on a co-orientation preserving contactomorphism $\Psi : (C_1, \xi_1) \to (C_2, \xi_2)$ is a grading on $d\Psi|_{\xi_1} : (\xi_1, d\alpha_1) \to (\xi_2, d\alpha_2)$. For more on isotopies, see [McL19].

Next, suppose that $B \subset C$ is a contact submanifold of codimension 2. The normal bundle $N_C B := TC|_B/TB \to B$ is symplectic and there is a natural bundle isomorphism $N_C B \cong T^\perp B := \{v \in \xi : d\alpha(v, w) = 0, \forall w \in TB \cap \xi\}$. In this case where B is codim 2, the tuple $(B \subset C, \xi, \Phi)$ is a contact pair and $\Phi : N_C B \to B \times \mathbb{C}$ is an isomorphism. We call this tupple a **contact pair with normal bundle data**. A grading for such a pair is a grading in the sense above on $C \setminus B$.

A contactomorphism between two such triples $(B_1 \subset C_1, \xi_1, \Phi_1), (B_2 \subset C_2, \xi_2, \Phi_2)$ is a contactomorphism $\Psi : C_1 \to C_2$ sending B_1 to B_2 so that the composition

$$N_{C_1} B_1 \xrightarrow{d\Psi|_{B_1}} N_{C_2} B_2 \xrightarrow{\Phi_2} B_2 \times \mathbb{C} \xrightarrow{(\Psi|_{B_1})^{-1} \times \mathrm{id}_{\mathbb{C}}} B_1 \times \mathbb{C}$$

is homotopic through symplectic bundle trivializations to Φ_1. If we have such a Ψ, we may additionally consider a grading on its restriction $\Psi|_{C_1 \setminus B_1} : C_1 \setminus B_1 \to C_2 \setminus B_2$.

The definition for an abstract contact open book (M, θ, ϕ) is in [McL19] and was mentioned in Section 13. To briefly recall, (M, θ) is a Liouville domain and $\phi : M \to M$ is an exact symplectomorphism supported away from the boundary ∂M. From an abstract contact open book, we can construct the mapping torus $T(\phi)$ and then obtain a contact open book: $C_\phi := (\partial M \times D(\delta)) \sqcup T(\phi)/\sim$ where we glue in a small thickening $\partial M \times D(\delta)$ to the mapping torus. As mentioned above, this is the generalized Thurston-Winkelnkemper construction.

We can give $(M, d\theta)$ a grading as a symplectic manifold and also ϕ as a symplectomorphism. Suppose that (M, θ, ϕ) is graded. Then this means we have an isomorphism

$$\iota : \widetilde{Fr}(E) \times_{\widetilde{Sp}(2n)} Sp(2n) \cong Fr(E).$$

Remark 25.3. Briefly, we remark that we may then define what it means for graded abstract contact open books to be isotopic as well as what it means for graded contact open books to be isotopic. One can prove that if two graded abstract contact open books are isotopic, then their associated graded contact open books will also be isotopic. This gives us a map

$$\{\text{(graded) abstract contact open books}\}/\text{isotopy} \longrightarrow \{\text{(graded) open books}\}/\text{isotopy}$$

which was shown to be a bijection by Giroux [Gir02].

Returning to gradings, if we identify a fiber $\widetilde{Fr}(TM)_p \cong \widetilde{Sp}(2n)$, then the grading $\widetilde{\phi} : \widetilde{Fr}(TM)_p \to \widetilde{Fr}(TM)_{\phi(p)}$ can be viewed as a map $\widetilde{\phi} : \widetilde{Sp}(2n) \to \widetilde{Sp}(2n)$. Elements of $\widetilde{Sp}(2n)$ are equivalence classes of paths starting at a basepoint where the equivalence is $\alpha \sim \beta$ if and only if α, β have the same endpoints and $\alpha * -\beta$ is a contractible loop. Here, the $*$ means concatenation. Then, $\widetilde{\phi}(\mathrm{id})$ is a path in $Sp(2n)$. The Conley-Zehnder index, which we explain below, assigns a number to any such path so we will take $CZ(\widetilde{\phi}(\mathrm{id}))$ to define the grading.

For any path of symplectic matrices $(A_t)_{t\in[a,b]}$ we can assign an index $CZ(A_t)$ called its Conley-Zehnder index. The Conley-Zehnder index was originally defined only for certain paths of symplectic matrices A_t but now it's been done in general in [RS93] or [Gut14]. We will not define it here but only list some of its properties (see [Gut14], Prop. 8):

1. $CZ((e^{it})_{t\in[0,2\pi]}) = 2$.
2. $(-1)^{n-CZ((A_t)_{t\in[0,1]})} = \operatorname{sign}\det_{\mathbb{R}}(\mathrm{id} - A_1)$ for any path of symplectic matricies $(A_t)_{t\in[0,1]}$.
3. $CZ(A_t \oplus B_t) = CZ(A_t) + CZ(B_t)$.
4. The Conley-Zehnder index of the catenation of two paths is the sum of their Conley-Zehnder indices.
5. If A_t and B_t are two paths of symplectic matrices which are homotopic relative to their endpoints then they have the same Conley-Zehnder index. Also the index only depends on the path up to orientation-preserving reparameterization.

26 Appendix B: Relating Fixed-Point Floer Cohomology and Symplectic Cohomology

Let $\pi : E \to D$ be a proper symplectic Lefschetz fibration where $D \subset \mathbb{C}$ is the open unit disc, and whose total space is a $2n$-dimensional symplectic manifold. For the sake of gradings on symplectic cohomology, one often requires that $c_1(E) = 0$ and also makes a choice of a trivialization of the anticanonical bundle, up to homotopy. The smooth fibers are closed $(2n-2)$-dimensional symplectic manifolds, again with trivialized anticanonical bundle.

To this Lefschetz fibration, we may associate a family of Floer cohomology groups, $SH^*(E, \lambda)$ where $\lambda \in \mathbb{R}$. We choose this notation because when defining symplectic cohomology, one of the options is to use Hamiltonians H_λ that are linear at infinity with slope λ. Everything is normalized so that, if the slope is an integer, the time-1 Hamiltonian flow ϕ_1^λ satisfies $\pi = \pi \circ \phi_1^\lambda$. So one can view the flow as winding a point around the disk D λ times. Because of continuation maps, we may define symplectic cohomology as $SH^*(E) := \varinjlim_\lambda SH^*(E, \lambda)$.

In [Sei19], Seidel considers the situation when $\lambda \in \mathbb{Z}$ and shows there is an exact triangle:

$$\begin{array}{ccc} SH^*(E,\lambda) & \longrightarrow & SH^*(E,\lambda+k) \\ \nwarrow & & \swarrow^{[1]} \\ & HF^*(\phi^k) & \end{array}$$

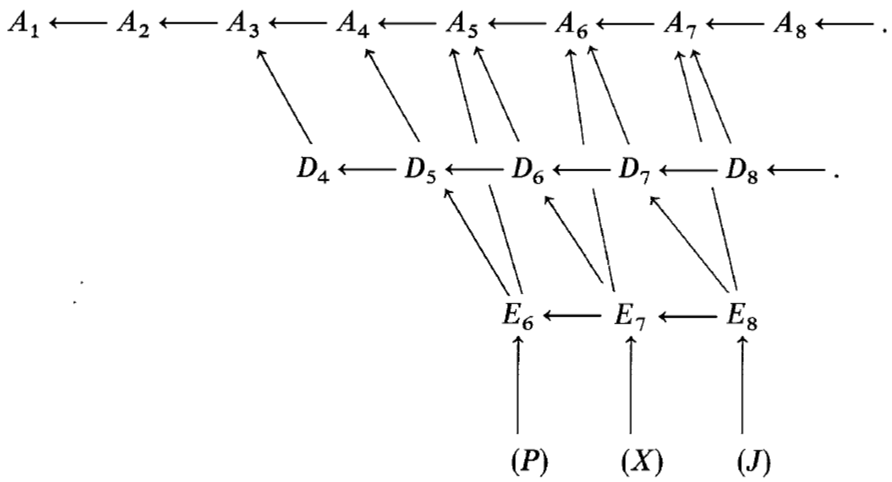
Here, $HF^*(\phi^k)$ is the fixed-point Floer cohomology of the kth interate of the monodromy map ϕ which is defined by parallel transporting around a loop in D that circles all the critical values. Technically, he shows it for $k = 1$ but the result holds for larger integers k. If we want to work with non-closed fibers and get similar results, we need the monodromy to be compactly supported. Seidel deduce a cohomological spectral sequence which appears in [Sei17]; it is similar to a version of the homological spectral sequence found in [McL12] where the terms of the E^1 page are not exactly fixed-point Floer homology of ϕ^k but something similarly defined from fixed points and looking at invariants of a $\mathbb{Z}/k\mathbb{Z}$ action.

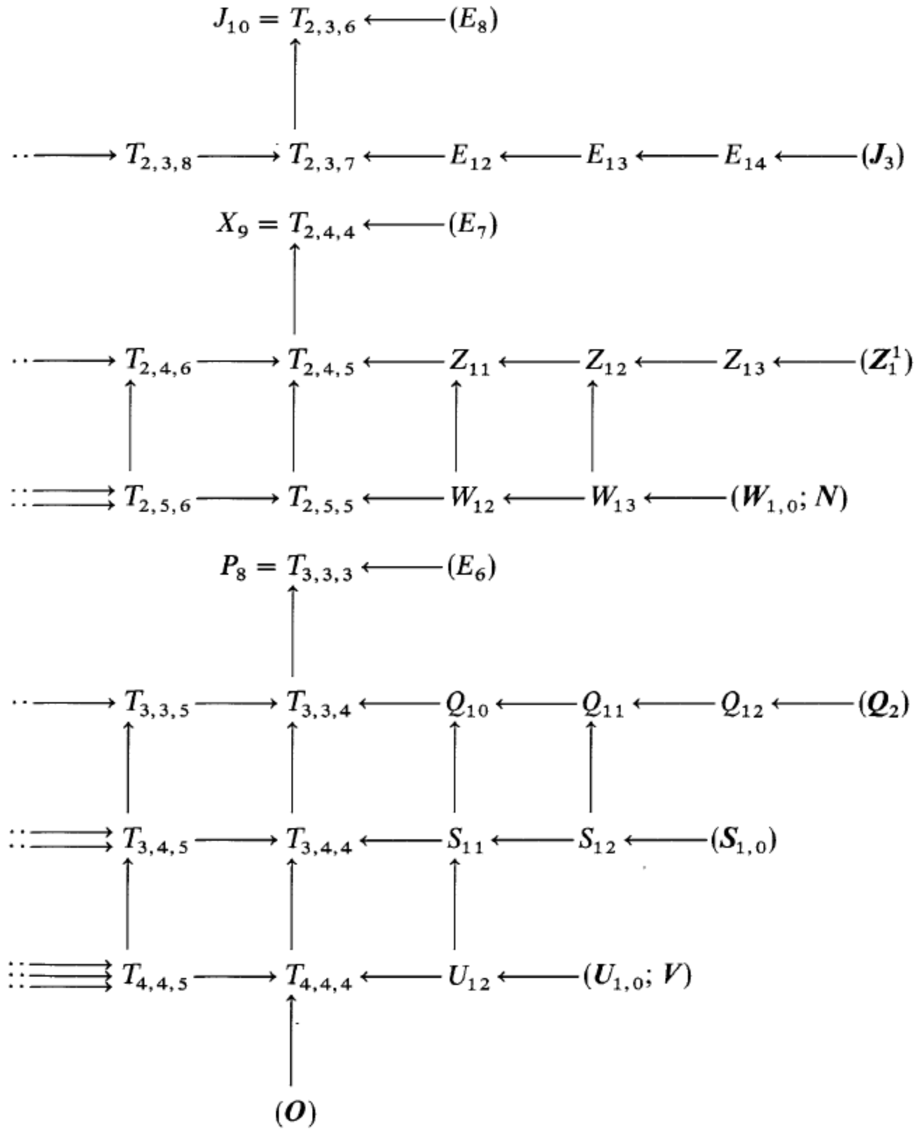
It should be noted that these tools are really useful for understanding the total space of the Lefschetz fibration E but the Hamiltonians mainly interact with the vertical boundary of E rather than the horizontal one. As such, it doesn't tell us much about the fibers of the fibration. For example, if $E = M \times D^2$, then $SH^*(E) = 0$ regardless of what $SH^*(M)$ is.

27 Appendix C: Systems of Adjacent Singularities

When introducing adjacencies, we emphasized that it's not a symmetric relationship and in terms of symplectic geometry, the embedding of Milnor fibers only goes one way (which was also proved in [Kea15]). An important observation in this work is that these embeddings give us cobordisms between links. For example, if f and g are adjacent (g is the "better" singularity), then the embedding gives a cobordism and hence, surgery description for obtaining L_f from L_g.

One can go further and observe that if two singularities are adjacent, we have a cobordism between their associated mapping tori. What does this afford us? It tell us that for any diagram of adjacencies, there is a corresponding commuting diagram of HF since HF is functorial with respect to adjacencies. Here are some sample diagrams of adjacencies from Arnold where the relations are directed.


A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8
D_4 D_5 D_6 D_7 D_8
E_6 E_7 E_8
(P) (X) (J)


$J_{10} = T_{2,3,6}$ (E_8)
$T_{2,3,8}$ $T_{2,3,7}$ E_{12} E_{13} E_{14} $(\boldsymbol{J_3})$
$X_9 = T_{2,4,4}$ (E_7)
$T_{2,4,6}$ $T_{2,4,5}$ Z_{11} Z_{12} Z_{13} $(\boldsymbol{Z_1^1})$
$T_{2,5,6}$ $T_{2,5,5}$ W_{12} W_{13} $(\boldsymbol{W_{1,0}}; \boldsymbol{N})$
$P_8 = T_{3,3,3}$ (E_6)
$T_{3,3,5}$ $T_{3,3,4}$ Q_{10} Q_{11} Q_{12} $(\boldsymbol{Q_2})$
$T_{3,4,5}$ $T_{3,4,4}$ S_{11} S_{12} $(\boldsymbol{S_{1,0}})$
$T_{4,4,5}$ $T_{4,4,4}$ U_{12} $(\boldsymbol{U_{1,0}}; \boldsymbol{V})$
$(\boldsymbol{O})$

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