On the Classification of Ancient solutions to the Ricci flow with Isotropic Curvature Condition

A Dissertation presented

by

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to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2022

Stony Brook University

The Graduate School

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Abstract of the Dissertation

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2022

In this thesis, complete κ-noncollapsed ancient solutions to the Ricci flow will be studied. First, we show that every n-dimensional, κ-noncollapsed, non-compact, complete ancient solution to the Ricci flow with uniformly PIC(positive isotropic curvature) for $n = 4$ or $n \geq 12$ has bounded curvature and weakly PIC_2, which is the condition defined in [Bre10a], [Bre19]. Combining this with the results in [BN20], it implies that any such solution is isometric to either a family of shrinking cylinders (or a quotient thereof) or the Bryant soliton. Secondly, we classify all complex 2-dimensional, κ-noncollapsed, complete ancient solutions to the Kähler-Ricci flow with weakly PIC. Lastly, we will discuss about gradient Ricci shrinkers with bounded scalar curvature and uniformly PIC and show some results on its structure at infinity. Many of the results in this thesis are based on joint work with Yu Li [CL20].

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Acknowledgements

My experience since 2016 might be an empirical evidence that the world is full of great, generous people. Mathematically, I would first and foremost like to express my sincere graditude to my advisor Xiuxiong Chen for his expert guidance, encouragement, enlightenment and constant support. In the early days of graduate school when I was wandering to stack countless results in differential geometry and Ricci flow, his remarks on *How to learn slowly but accurately* had been an invaluable guide for me to build better understanding and foundations. Recently, I have learned that this lesson is not limited to learning mathematics, but rather more general methods of learning so it also helped me to be a better person overall. I would also like to thank Yu Li for his encouragement and the insights he shared during our cooperation. Also, I owe a debt of gratitude to professors Claude LeBrun, Marcus Khuri and Martin Roček for their time and for kindly accepting to be my dissertation committee members. It is a great honor for me to have them as my dissertation committee.

Additionally, I want to express my thanks to my Stony Brook friends including Xujia Chen, Dahye Cho, Yoon-Joo Kim, Eun Hye Lee, Myeongjae Lee, Jordan Rainone, Tobias Shin, Ying Hong Tham, Jiasheng Teh, Alexandra Viktorova, Ben Wu. Apparently, one of the reasons I was able to complete my Ph.D program was the mathematical and emotional support of colleagues in Mathematics department.

I would like to give special thanks to my parents, Hyun Jin Cho and Soo Hee Song, and my sister Minji Cho for their unconditional supports. I would not be where I am today without their beliefs and emotional supports on my path in life.

Last but not least, I am particularly thankful to my wife and a soulmate Ja Yeon Lee. For the past 6 years, you encouraged me to try again whenever I was exhausted from my work and did not have the courage to go back. Your uniquely positive mind is a shelter where I can take a rest from the reality, and always will be.

Chapter 1

Introduction

1.1 Historical Background

There have been many cases where parabolic partial differential equation improves geometric objects. The Ricci flow on a closed Riemannian manifold (M, g) is defined by Richard Hamilton in his seminal work [Ham82] as a family of Riemannian metrics $g(t)$ on M satisfying

$$\begin{aligned} \partial_t g(t) &= -2\mathrm{Ric}_{g(t)} \\ g(0) &= g \end{aligned}$$

After showing the short time existence, Hamilton applied the Ricci flow to the closed 3-manifold with positive Ricci curvature and showed that it converges to the round sphere under the normalized Ricci flow, which proves that any such manifolds are diffeomorphic to the round sphere or its quotient. Because of this result, Ricci flow was considered as a tool that could lead to the resolution of long-standing Poincaré and geometrization conjectures for 3-manifolds.

Even though the program turns out to be successful, it was not automatically given from Hamilton's work. As it was pointed out in [Ham93], the application of the Ricci flow in general case is concerned with singularities that arise in finite time even for the normalized Ricci flow. In the same paper, Hamilton introduced that singularities could be investigated by considering the ancient solutions to the Ricci flow. Due to this reason, the study of ancient solutions to the Ricci flow becomes one of the most important questions. About a decade later, Gregori Perelman made a breakthrough in the study of Ricci flow. By using his celebrated entropy formula in [Per02], Perelman showed that any ancient solution that could be formed by the blow-up of the high curvature region must be κ-noncollapsed if one starts the Ricci flow from a compact manifold. Therefore, it is a central issue to understand κ-noncollapsed ancient solutions to the Ricci flow. From this no local collapsing theorem, Perelman was able to classify the finite time singularity models for 3-dimensional Ricci flows and prove a canonical neighborhood theorem for high curvature parts. This observation enabled us to continue the Ricci flow through the singular time after surgeries, which consequently succeeded in proving Thurston's geometrization conjecture. Detailed arguments can be found in [Per02] [Per03b] [Per03a].

More generally, a complete classification problem of ancient solutions is still ongoing by many researchers. For 2-dimensional case, Perelman [Per02] showed that all κ-noncollapsed ancient solutions to the Ricci flow are $\mathbb{R}^2$ and shrinking spheres or their quotients. It is worth mentioning that even collapsing cases are also completely classified by [DHS12] [DS06] [Chu07]. In dimension 3, Brendle made a breakthrough [Bre20] and showed that any κ-noncollapsed ancient solution on noncompact manifolds is either the Bryant soliton or shrinking cylinders or their quotients. For compact case, it was proved by Brendle-Daskalopoulos-Sesum [BDS20] that any compact, κ-noncollapsed ancient solution is either the Perelman's solution constructed in [Per03a] or shrinking sphere or their quotients. For these low-dimensional cases, the positivity of curvature is automatically given from the evolution equation. For example, it follows from Hamilton-Ivey pinching estimate that any 3-dimensional ancient solution has positive sectional curvature. In general, Chen [Che09] proved the positivity of scaler curvature for ancient solution.

Apart from the result mentioned above, there is no natural curvature pinching for higher dimensional ancient solutions. And it makes highly difficult to investigate ancient solutions with no additional curvature assumption. So it is natural to approach this problem after adding some positivity conditions, which are preserved under the Ricci flow. One of the candidates is *positive isotropic curvature* (PIC for short). After this condition was introduced by Micallef-Moore [MM88], it has maintained a strong connection with the Ricci flow. Hamilton [Ham97] showed that this condition is preserved under the 4-dimensional Ricci flow and later, Nguyen [Ngu09] and Brendle-Schoen [BS09] independently completed the proof for the general case. In the same paper, Brendle-Schoen introduced two more conditions, PIC_1 and PIC_2, which are also preserved under the Ricci flow and closely related to the long-standing differentiable sphere theorem as the 1/4-pinched condition implies PIC_2. In dimension 4, Hamilton [Ham97] (see also [CZ06a] [CTZ12]) classified all differential structures of compact manifolds with PIC, provided that there is no essential incompressible space-form. Here, an incompressible space-form $N \subset M$ is a 3-dimensional submanifold of M that is diffeomorphic to the quotient of S^3 and $\pi_1(N)$ injects into $\pi_1(M)$. An incompressible space-form is said to be essential unless $\pi_1(N)$ is trivial, or $\pi_1(N) = \mathbb{Z}_2$ and the normal bundle is non-orientable. One of the observations found in the paper is that any ancient solution developed from the blow-up process must have a nonnegative curvature operator and uniformly PIC. In the higher dimension when $n \geq 12$, similar classification of all compact manifolds with PIC was obtained by Brendle [Bre19]. In the paper, it was proved that any ancient solution coming from a compact manifold with PIC must have weakly PIC_2 and uniformly PIC. The method used by Brendle is to construct ingeniously a family of curvature cones that pinches toward the desired curvature condition. It is worth noting that most of important properties including compactness theorem, Hamilton's differential Harnack inequality are still valid under the weakly PIC_2 condition.

1.2 Statement of Main Theorems and Ideas of Proof

After Brendle's seminal work on 3-dimensional ancient solution [Bre20], Brendle-Naff have generalized the result into the following classification result.

Theorem 1.2.1 (Corollary 1.6 of [BN20]). *Any complete, noncompact, κ-noncollapsed ancient solution to the Ricci flow with weakly* PIC_2*, uniformly* PIC *and bounded curvature is isometric to either a family of shrinking cylinders (or a quotient thereof) or to the Bryant soliton.*

The idea of Theorem 1.2.1 is worth to be mentioned. First, they started to analyze the simpler case when the solutions are rotationally symmetric with the same property and achieve the desired classification result. After then, they show a neck improvement theorem [BN20, Theorem 4.8], [Bre20, Theorem 8.5] and use it to prove that any ancient solutions with given conditions must be rotationally symmetric, which completes the proof.

In this thesis, the slight improvement of Theorem 1.2.1 is obtained by dropping two assumptions. First, we show that the uniformly PIC condition implies the weakly PIC_2 if $n = 4$ or $n \geq 12$. Secondly, we prove that the boundedness of curvature can be dropped from the statement by showing that it is implied from κ-noncollapsing and curvature conditions. As a result, we have the following statement, which is the first main result in this thesis.

Theorem 1.2.2. *Let $(M^n, g(t))_{t\in(-\infty,0]}$ be a κ-noncollapsed, noncompact, complete ancient solution to the Ricci flow with uniformly* PIC *for $n = 4$ or $n \geq 12$. Then it is isometric to either a family of shrinking cylinders (or a quotient thereof) or the Bryant soliton.*

The proof of Theorem 1.2.2 can be summarized as follows. The first step of the proof is to show that uniformly PIC implies weakly PIC_2. Note that the boundedness of curvature is not assumed here and it prevents us from directly applying the maximum principle for parabolic equations. To overcome this, we review the idea of Chen which showed that any scalar curvature is nonnegative for ancient solutions [Che09]. Since this result is obtained from the evolution inequality of scalar curvature $(\partial_t - \Delta) R \geq \frac{2}{n} R^2$, it is still applicable for parabolic equations in the form of

$$(\partial_t - \Delta) f \geq f^2. \tag{1.2.1}$$

From this approach, one can prove that any such f on the complete ancient solution to the Ricci flow(not necessarily having bounded curvature!) should be nonnegative. By this observation, the bounded curvature assumption can be easily dropped and we prove that the curvature operator is nonnegative in 4-dimensional case(Lemma 3.1.4). In the higher dimensional case, we consider Brendle's continuous family of curvature cones in [Bre19] and show that the curvature operator of our ancient solution lies in this family. One of the necessary conditions for this is to show that the Ricci tensor is 2-positive in a uniform way. In other words, we need to show that the sum of two eigenvalues obtained from two distinct eigenvectors divided by scalar curvature is bounded below by a positive constant. It can be also shown from the previous curvature improvement lemma so the bounded curvature assumption is still not needed. With the help of this continuous family, we are able to show that the ancient solution has weakly PIC_1 as the curvature cones pinch toward weakly PIC_1. Therefore, by [BCW19, Lemma 4.2] (see also [LN20, Proposition 6.2]), the ancient solution has weakly PIC_2. The second part of the proof is to show that any such ancient solution must have bounded curvature. The proof relies on a version of canonical neighborhood theorem for Ricci flows with weakly PIC_2 and uniformly PIC (Theorem 4.1.3). First, we show that any higher curvature part on a fixed time slice is close to a round cylinder. By using the fact that any manifold with positive sectional curvature cannot have infinitely small cylinders [CZ06a, Proposition 2.2], we can derive a contradiction if the curvature is unbounded on a times slice. After then, we use the positivity of curvature obtained from the original condition together with κ-noncollapsing condition to get the uniform curvature bound to a solution (Proposition 4.1.5).

Our second main result is the following classification of the ancient solutions to the Kähler Ricci flow on Kähler surfaces, which is a very natural question that follows from the first question.

Theorem 1.2.3. *Let $(M^2, g(t))_{t\in(-\infty,0]}$ be a κ-noncollapsed, complete, complex 2-dimensional ancient solution to the Kähler-Ricci flow with weakly* PIC *and bounded curvature in any compact time interval. Then it is isometrically-biholomorphic to one of the spaces $\mathbb{C}^2$, $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$, $\mathbb{C}^1 \times \mathbb{CP}^1$ equipped with standard metrics, up to scalings on each factors.*

It is well-known that a Kähler surface (i.e. Kähler manifold with complex dimension 2) has weakly PIC if and only if it has nonnegative orthogonal bisectional curvature, which is preserved under Kähler Ricci flow [GZ09]. In general, a Kähler manifold has nonnegative orthogonal bisectional curvature if it has weakly PIC. This fact is shown in Proposition 2.2.2. Also, an ancient solution to the Kähler Ricci flow with nonnegative orthogonal bisectional curvature has nonnegative bisectional curvature [LN20]. For the κ-solutions (Definition 4.2.1) to the Kähler Ricci flow, there is a parallel theory to Perelman's theory on κ-solutions to the 3-dimensional Ricci flows and many important theorems still hold for the κ-solutions to the Kähler Ricci flow [Ni05][Cao92]. Since all compact κ-solutions to the Kähler Ricci flow are classified [DZ20], the main issue is to prove the noncompact case.

To prove Theorem 1.2.3, we first prove that any κ-solutions to the Kähler Ricci flow must be of Type I (Lemma 4.2.1). From the classification of Kähler-Ricci shrinkers with nonnegative bisectional curvature [Ni05], the asymptotic behavior of the ancient solution is modeled on standard $\mathbb{C} \times \mathbb{CP}^1$. Moreover, we obtain a curvature improvement so that the ancient solution has a nonnegative curvature operator (Theorem 3.1.5). From the nonnegativity of sectional curvature, we show that the geometry at spatial infinity is also modeled on $\mathbb{C}\times\mathbb{CP}^1$. Therefore, we can prove a canonical neighborhood theorem (Proposition 4.2.9) to obtain a local S^2-fibration, and these local fibrations can be patched together to form a global fibration by following a standard argument (Theorem 4.2.17). Now the proof of Theorem 1.2.3 is complete by a topological argument.

As an independent topic, we also consider gradient Ricci shrinkers with PIC conditions, which can be regarded as a self-similar solution to Ricci flow. Note that a smooth manifold

(M, g) is a gradient Ricci shrinker if there exists a potential function $f : M \to \mathbb{R}$ such that

$$\nabla^2 f + \mathrm{Ric} = \frac{1}{2} g$$

A gradient Ricci shrinker is a very important and fundamental example in the study of differential geometry and Ricci flow for many reasons. It is a natural generalization of Einstein manifolds and it yields a self-similar ancient solution to the Ricci flow. However, the classification of gradient Ricci shrinkers is still incomplete. Li-Ni-Wang [LNW18] showed that a 4-dimensional complete gradient Ricci shrinker with PIC is either a family of round spheres or shrinking cylinders or a quotient thereof, and Naff [Naf19] showed the similar result for $n \geq 12$ with uniformly PIC condition. So it is natural to believe that the same result holds for a general complete gradient Ricci shrinker with uniformly PIC. By following the idea of Munteanu-Wang [MW15] [MW19], the following result can be made, which describes the asymptotic structure of them.

Theorem 1.2.4. *Let (M^n, f, g) be a complete, noncompact gradient Ricci shrinker with uniformly PIC and bounded scalar curvature and $n \geq 5$. Then each end E is smoothly asymptotic to either the round cylinder $\mathbb{R} \times S^{n-1}/\Gamma$ or a cone that is homeomorphic to $\mathbb{R}_+ \times S^{n-1}/\Gamma$.*

From [KW15, Corollary 1.3.], we know that two gradient Ricci shrinkers must be isometric if they are C^2-asymptotic to the same cone. So there is a possibility that this theorem can provide the answer for the classification problem of complete, noncompact gradient Ricci shrinkers with uniformly PIC. For example, if we are able to show that an asymptotic cone has to be rotationally symmetric, then (M, f, g) has to be flat due to [KW15, Corollary 1.4.]. It implies that any nonflat, complete, noncompact gradient Ricci shrinker with uniformly PIC with bounded curvature is smoothly asymptotic to the round cylinder.

Chapter 2

Preliminaries

2.1 Some Definitions for the Ricci Flow

There are many types of background materials required for the complete understanding of Ricci flow. In this section, we will provide some of standard definitions in the Ricci flow. A complete list of materials can be found in many literatures such as [Bre10a] [MT07].

Definition 2.1.1. Let $(M^n, g(t))_{t\in(-\infty,T)}$ be a complete ancient solution to the Ricci flow $\partial_t g(t) = -2\mathrm{Ric}_{g(t)}$.

(i) $(M, g(t))_{t\in(-\infty,T)}$ has bounded curvature on any compact time interval if for any compact interval $I \subset (-\infty, T)$, we have $\sup_{M\times I} |Rm(x,t)| < \infty$. In this case, T is called the singular time if $T < \infty$ and $\lim_{t\to T} \sup_{x\in M} |Rm(x,t)| = \infty$.

(ii) Given $\kappa > 0$, $(M, g(t))_{t\in(-\infty,T)}$ is said to be κ-noncollapsed (at all scales) if for any ball $B_{g(t)}(x,r)$ satisfying $|Rm(y,t)| \le r^{-2}$ for all $y \in B_{g(t)}(x,r)$, we have

$$\mathrm{Vol}(B_{g(t)}(x,r)) \ge \kappa r^n$$

Next, we recall the following curvature conditions which are used throughout the paper. For the historical background and the motivation of these conditions, readers can refer to [BS09][Bre19][Ham97].

Definition 2.1.2. *(Isotropic curvature conditions)*

(i) A Riemannian manifold (M^n, g) $(n \geq 4)$ is said to have weakly PIC (in other words, nonnegative isotropic curvature) if for any $p \in M$ and any orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$ of T_pM, we have

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0.$$

Note that when $n = 4$, this condition is equivalent to $a_1 + a_2 \geq 0$ and $c_1 + c_2 \geq 0$ using the notation in [Ham86]. This notation will be explained at the end of the section.

(ii) A Riemannian manifold (M^n, g) $(n \geq 5)$ is said to have uniformly PIC if there exists a constant $\theta > 0$ such that for any $p \in M$ and any orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$ of T_pM we have

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 4\theta R > 0.$$

Equivalently, it means that an algebraic curvature tensor $\mathrm{Rm} - \theta R I$ has weakly PIC. Similarly, a 4-dimensional Riemannian manifold (M^4, g) is said to have uniformly PIC if there exists a constant $\Lambda \geq 1$ such that

$$0 < \max(a_3, b_3, c_3) \leq \Lambda \min(a_1 + a_2, c_1 + c_2).$$

(iii) A Riemannian manifold (M^n, g) is said to have weakly PIC$_1$ if for any $\lambda \in [-1, 1]$, any $p \in M$ and any orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$ of T_pM, we have

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \geq 0$$

Equivalently, $(M, g) \times \mathbb{R}$ has weakly PIC.

(iv) A Riemannian manifold (M^n, g) is said to have weakly PIC$_2$ if for any $\lambda, \mu \in [-1, 1]$, any $p \in M$ and any orthonormal 4-frame $\{e_1, e_2, e_3, e_4\}$ of T_pM, we have

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2\mu^2 R_{2424} - 2\lambda\mu R_{1234} \geq 0.$$

Equivalently, $(M, g) \times \mathbb{R}^2$ has weakly PIC.

(v) A Ricci flow solution $(M^n, g(t))_{t\in\mathcal{I}}$ is said to have [weakly PIC, uniformly PIC, weakly PIC_1, weakly PIC_2] if $(M, g(t))$ satisfies the corresponding curvature condition for all $t \in \mathcal{I}$. For uniformly PIC, the constant θ or Λ is required to be uniform for all $t \in \mathcal{I}$.

Here, we recall some notations introduced in [Ham86]. After using the self-dual and anti-self-dual decomposition of $\bigwedge^2 \mathbb{R}^4 = \bigwedge_+ \oplus \bigwedge_-$, we can write the curvature operator as

$$\mathrm{Rm} = \begin{pmatrix} A & B \\ B^t & C \end{pmatrix}.$$

Moreover, let $a_1 \le a_2 \le a_3$ and $c_1 \le c_2 \le c_3$ be eigenvalues of A and C, respectively. Also, we denote the eigenvalues of the symmetric matrix $\sqrt{BB^t}$ by $0 \le b_1 \le b_2 \le b_3$. It is clear by the Bianchi identity that

$$\mathrm{tr}(A) = a_1 + a_2 + a_3 = \frac{R}{2} = c_1 + c_2 + c_3 = \mathrm{tr}(C).$$

Now, let $\{e_1, e_2, e_3, e_4\}$ be the positively oriented orthonormal basis of $\mathbb{R}^4$. Using this, one can choose a basis for $\bigwedge_+$ and $\bigwedge_-$ as

$$\varphi_1^{\pm} = e_1 \wedge e_2 \pm e_3 \wedge e_4$$

$$\varphi_2^{\pm} = e_1 \wedge e_3 \pm e_4 \wedge e_2$$

$$\varphi_3^{\pm} = e_1 \wedge e_4 \pm e_2 \wedge e_3$$

Then we can compute

$$A_1 = \mathrm{Rm}(\varphi_1^+, \varphi_1^+) = R_{1212} + R_{3434} + 2R_{1234}$$

$$A_2 = \mathrm{Rm}(\varphi_2^+, \varphi_2^+) = R_{1313} + R_{2424} + 2R_{1342}$$

$$A_3 = \mathrm{Rm}(\varphi_3^+, \varphi_3^+) = R_{1414} + R_{2323} + 2R_{1423}$$

$$C_1 = \mathrm{Rm}(\varphi_1^-, \varphi_1^-) = R_{1212} + R_{3434} - 2R_{1234}$$

$$C_2 = \mathrm{Rm}(\varphi_2^-, \varphi_2^-) = R_{1313} + R_{2424} - 2R_{1342}$$

$$C_3 = \mathrm{Rm}(\varphi_3^-, \varphi_3^-) = R_{1414} + R_{2323} - 2R_{1423}$$

From this expression, we can see that any weakly PIC condition can be expressed as

$$A_i + A_j \geq 0$$

and

$$C_i + C_j \geq 0$$

for distinct $i, j \in \{1, 2, 3\}$. Therefore, 4-dimensional weakly PIC condition can be expressed in the following simpler way.

$$\min\{a_1 + a_2, c_1 + c_2\} \geq 0$$

It is worth noting that this expression becomes simpler for a Kähler surface. In this case, a positively oriented orthonormal basis can be chosen as $\{e_1, Je_1, e_2, Je_2\}$ where J is a complex structure. They generate self-dual and anti-self-dual two forms as we choose a basis for $\bigwedge_+$ and $\bigwedge_-$ as

$$\varphi_1^{\pm} = e_1 \wedge Je_1 \pm e_2 \wedge Je_2$$

$$\varphi_2^{\pm} = e_1 \wedge e_2 \pm Je_2 \wedge Je_1$$

$$\varphi_3^{\pm} = e_1 \wedge Je_2 \pm Je_1 \wedge e_2$$

Using this basis and Kähler condition together with Bianchi identity, one can represent the curvature operator as a 6×6 matrix

$$\begin{pmatrix} \frac{R}{2} & 0 & 0 & \rho_1 & \rho_2 & \rho_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \rho_1 & 0 & 0 & & & \\ \rho_2 & 0 & 0 & & C & \\ \rho_3 & 0 & 0 & & & \end{pmatrix}$$

with a 3×3 matrix C. Therefore, a Kähler surface has weakly PIC if and only if $c_1 + c_2 \geq 0$, where, as before, $c_1 \leq c_2 \leq c_3$ are eigenvalues of C. Readers can refer to [Ham97], [CTZ04] for more relevant results from this expression.

There are some logical implications among these curvature conditions. We can observe that

$$\text{weakly } \mathrm{PIC}_2 \Rightarrow \text{weakly } \mathrm{PIC}_1 \Rightarrow \text{weakly PIC}$$

by setting $\mu = 1$ and $\lambda = 1$ in the definition. Moreover, if we take $V, W \in \bigwedge^2$ as $V = e_1 \wedge e_3 + \lambda e_4 \wedge e_2$ and $W = \lambda e_1 \wedge e_4 + e_2 \wedge e_3$, we can express weakly PIC_1 condition as $\mathrm{Rm}(V, V) + \mathrm{Rm}(W, W) \geq 0$. Hence the weakly PIC_1 condition is implied by the 2-nonnegative of Riemannian curvature operator condition $\mathrm{Rm} \geq_2 0$. One can obtain the weakly PIC_2 condition similarly by taking $V = e_1 \wedge e_3 + \lambda\mu e_4 \wedge e_2$ and $W = \lambda e_1 \wedge e_4 + \mu e_2 \wedge e_3$, but the nonnegativity of curvature operator is required for this case since it is possible to have $W = 0$ when $\lambda = \mu = 0$. Actually, this case implies the nonnegativity of sectional curvature. Similarly, if we take $\lambda = 0$ in weakly PIC_1 condition, we can see that any sum of two sectional curvatures sharing one vector is nonnegative. In particular, it implies the nonnegativity of Ricci curvature. Also, we can obtain the nonnegativity of scalar curvature by taking a trace multiple times for weakly PIC condition. Note that in general, the nonnegativity of sectional, Ricci curvature is not preserved under the Ricci flow. So we have the following diagram. Note that the conditions on the first and second columns are preserved under the Ricci flow while those on the third are not in general.

$$\begin{tikzcd}
\mathrm{Rm} \geq 0 \arrow[r] \arrow[d] & \text{weakly } \mathrm{PIC}_2 \arrow[r] \arrow[d] & \sec \geq 0 \arrow[d] \\
\mathrm{Rm} \geq_2 0 \arrow[r] & \text{weakly } \mathrm{PIC}_1 \arrow[r] \arrow[d] & \mathrm{Ric} \geq 0 \arrow[ddl] \\
& \text{weakly PIC} \arrow[d] & \\
& \mathrm{scal} \geq 0 &
\end{tikzcd}$$

2.2 Computational Results from Curvature Conditions

There are some useful results from curvature conditions which are obtained only by algebraic computation. In this section, these results are given for the later use. In essence, the norm of curvature operator is controlled by the scalar curvature if we have weakly PIC condition.

Proposition 2.2.1. *Let (M^n, g) be a manifold with weakly PIC with $n \geq 5$. Then we have the following.*

1. $-\frac{R}{n-4} \leq \mathrm{Ric} \leq \frac{R}{2}$
2. *There exists $C > 0$ such that $|\mathrm{Rm}| \leq CR$*

Proof. Because of PIC condition, we have

$$R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl} - 2R_{ijkl} \geq 0$$

for any orthonormal 4-frame $\{e_i, e_j, e_k, e_l\}$. By interchanging e_i and e_j and adding the results, we can derive

$$R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl} \geq 0 \tag{2.2.1}$$

1. Choose an orthonormal frame $\{e_1, \cdots, e_n\}$ so that the Ricci curvature tensor is diagonalized with eigenvalues $R_{11} \leq R_{22} \leq \cdots \leq R_{nn}$. Then from 2.2.1, we have

   $$R - 2R_{nn} = \sum_{i,j=1}^{n-1} R_{ijij} \geq 0 \tag{2.2.2}$$

   which shows the upper bound in the statement. Also, by taking a sum of 2.2.1 for all $l \neq i, j, k$, we get

   $$(n-4)(R_{ikik} + R_{jkjk}) + R_{ii} + R_{jj} - 2R_{ijij} \geq 0 \tag{2.2.3}$$

   By taking a sum of for $k \neq i, j$ again and rescaling, we obtain

   $$R_{ii} + R_{jj} - 2R_{ijij} \geq 0 \tag{2.2.4}$$

By taking a sum of 2.2.4 for $j \neq i$, we finally get

$$(n-4)R_{ii} + R \geq 0 \tag{2.2.5}$$

It gives the lower bound in the statement so completes the proof.

2. From 2.2.2 and 2.2.4, we get

$$R_{ikik} \leq \frac{1}{2}(R_{ii} + R_{kk}) \leq \frac{R}{2}$$

Also, from 2.2.1, we have

$$R_{ikik} \geq -(R_{ilil} + R_{jkjk} + R_{jljl}) \geq -\frac{3}{2}R$$

It implies that $-\frac{3}{2}R \leq R_{ikik} \leq \frac{1}{2}R$ for any orthonormal 2-frame $\{e_i, e_k\}$. Now, from the computation in the proof of Berger's inequality [Ber60], we know

$$\begin{aligned} R_{ijkl} &= K(e_1+e_3, e_2+e_4) + K(e_1+e_4, e_2-e_3) + K(e_1-e_4, e_2+e_3) + K(e_1-e_3, e_2-e_4) \\ &\quad - K(e_1+e_4, e_2+e_3) - K(e_1-e_4, e_2-e_3) - K(e_1+e_3, e_2-e_4) - K(e_1-e_3, e_2+e_4) \end{aligned}$$

for any orthonormal 4-frame $\{e_i, e_j, e_k, e_l\}$. Here, we used the notation $K(v,u) = R(u,v,u,v)$. It implies that $-32R \leq R_{ijkl} \leq 32R$ and completes the proof.

$\square$

Next, we will show the following result that shows the connection between PIC condition and orthogonal bisectional curvature condition.

Proposition 2.2.2. *Let (M,g) be a Kähler manifold with $\dim_{\mathbb{C}} M = n$. If M has weakly PIC, then it has nonnegative orthogonal bisectional curvature. The converse is also true if $n = 2$.*

Proof. Recall that (M,g) has nonnegative orthogonal bisectional curvature if

$$R(X, JX, Y, JY) \geq 0$$

for all nonzero real vectors $X, Y \in TM$. Note that (M, g) has nonnegative scalar curvature if it has nonnegative orthogonal bisectional curvature as a sum of two distinct holomorphic sectional curvatures can be expressed as a sum of two orthogonal bisectional curvature components. For notational simplicity, let

$$\mathrm{IC}(A,B,C,D) = R(A,C,A,C)+R(A,D,A,D)+R(B,C,B,C)+R(B,D,B,D)-2R(A,B,C,D)$$

Then we have

$$\mathrm{IC}(A,B,C,D) = \mathrm{IC}(C,D,A,B)$$

Now, choose two orthonormal vectors e_1, e_2 such that that $\{e_1, Je_1, e_2, Je_2\}$ forms an orthonormal 4-frame of TM. Then there are 6 possible isotropic curvature components related to this frame. Using Kähler condition and Bianchi identity multiple times, we have the following result.

$$\mathrm{IC}(e_1, Je_1, e_2, Je_2) = 0$$

$$\mathrm{IC}(e_1, Je_1, Je_2, e_2) = 4R(e_1, Je_1, e_2, Je_2)$$

$$\mathrm{IC}(e_1, Je_2, Je_1, e_2) = \frac{R}{2}$$

$$\mathrm{IC}(e_1, Je_2, e_2, Je_1) = R(e_1 + e_2, J(e_1 + e_2), e_1 - e_2, J(e_1 - e_2))$$

$$\mathrm{IC}(e_1, e_2, Je_1, Je_2) = R(e_1 + Je_2, J(e_1 + Je_2), e_1 - Je_2, J(e_1 - Je_2))$$

$$\mathrm{IC}(e_1, e_2, Je_2, Je_1) = \mathrm{IC}(e_1, Je_2, Je_1, e_2) = \frac{R}{2}$$

From these expressions, we can observe that any orthogonal bisectional curvature component can be expressed in a form of isotropic curvature. Also, we know that described cases are the only possibilities when $n = 2$, which shows that two conditions are equivalent. □

2.3 Evolution of Curvature under the Ricci Flow

First, we will briefly introduce Uhlenbeck's trick that significantly simplifies the evolution equation for curvature tensor. Let $E \to M$ be a vector bundle that is isomorphic to the tangent bundle $TM \to M$ with fixed metric $g(0)$. A main idea of the trick is to extend the identity map $\imath_0 : E \to TM$ by

$$\partial_t \imath_t = \mathrm{Ric} \circ \imath_t$$

Then, this bundle map $\imath_t$ is an isometry for all t as

$$\begin{aligned}
\partial_t(\imath_t^* g(X,Y)) &= \partial_t(g_{ij}(\imath_t X)^i(\imath_t Y)^j) \\
&= -2R_{ij}(\imath_t X)^i(\imath_t Y)^j + R_{ij}(\imath_t X)^i(\imath_t Y)^j + R_{ij}(\imath_t X)^i(\imath_t Y)^j \\
&= 0
\end{aligned}$$

In other words, $\imath_t^* g_t$ is independent to t. Using this family of bundle isomorphisms, we will work with a local orthonormal basis given as follows. For each point (x_0, t_0), we choose an orthonormal basis $\{e_1, \cdots, e_n\}$ with respect to $g(t)$. Since the pull-back metric is time-independent, $\{e_1, \cdots, e_n\}$ is still an orthonormal basis at (x_0, t) for any t with respect to this pullback. By parallel transport, there exists an orthonormal frame on a spacetime neighborhood of (x_0, t_0). With the help of such a frame, locally around (x_0, t_0), we can consider $\mathrm{Rm}(x,t)$ in a fixed vector space of algebraic curvature tensors $\mathscr{C}_B(\mathbb{R}^n)$. Then we can obtain the evolution equation of the curvature tensor as

$$(\partial_t - \Delta) R_{ijkl} = Q(\mathrm{Rm})_{ijkl} := R_{ijpq}R_{klpq} + 2(R_{ipkq}R_{jplq} - R_{iplq}R_{jpkq}) \tag{2.3.1}$$

By taking a trace for indexes j and l in 2.3.1, we get

$$Q(\mathrm{Rm})_{isks} = R_{ispq}R_{kspq} + 2R_{ipkq}R_{pq} - 2R_{ipsq}R_{spkq} = 2R_{ipkq}R_{pq}$$

since

$$R_{ispq}R_{kspq} = -R_{ispq}(R_{kpqs} + R_{kqsp}) = R_{ispq}R_{sqkp} + R_{ispq}R_{pskq} = 2R_{ipsq}R_{spkq}$$

Therefore, we get the evolution equation of the Ricci curvature as

$$(\partial_t - \Delta) R_{ik} = 2R_{ipkq}R_{pq} \tag{2.3.2}$$

By taking a trace for 2.3.2 again, we can obtain the evolution equation and inequality of the scalar curvature.

$$(\partial_t - \Delta) R = 2|\mathrm{Ric}|^2 \geq \frac{2}{n} R^2 \tag{2.3.3}$$

2.4 Localization Result

It is observed that various curvature tensors satisfy parabolic partial differential equation when the metric tensor evolves by Ricci flow. Together with the maximum principle, these conditions were used to obtain the curvature improvement results for the Ricci flow on closed manifolds or complete manifold with bounded curvature, which is the crucial part for the study of Ricci flow. This standard argument becomes nontrivial if we try to extend the result for general cases since there is no guarantee that the maximum principle for parabolic PDE still holds. These cases can be handled as well after constructing a cut-off function that can localize the evolution equations. In [Che09, Proposition 2.1], this argument was given for 2.3.3 but this proposition is still applicable for more general setting. In this section, this localization result is given for the later applications.

Proposition 2.4.1. *For $r > 0$ and $A \geq 14(n-1)\frac{T}{r^2} + 2$, suppose $(M^n, g(t))_{t\in[0,T]}$ is a Ricci flow solution with a continuous function $f : M \times [0,T] \to \mathbb{R}$ satisfying the following properties.*

1) $B_{g(t)}(p, Ar)$ is compactly contained in M for any $t \in [0,T]$.

2) For any $t \in [0,T]$ and $x \in B_{g(t)}(p, r)$, $|Ric|(x,t) \leq \frac{n-1}{r^2}$.

3) For any $t \in [0,T]$ and $x \in B_{g(t)}(p, Ar)$, $(\partial_t - \Delta) f(x,t) \geq \delta f^2(x,t)$ in the barrier sense for a constant $\delta > 0$.

Then there exists a constant $C = C(n) > 0$ such that $f(x,t) \geq \min\left\{-\frac{4}{t\delta}, -\frac{C}{(Ar)^2\delta}\right\}$ for any $t \in [0,T]$ and $x \in B_{g(t)}(p, \frac{3Ar}{4})$.

Proof. By rescaling, we may assume that $\delta = 1$. Now, we will include the argument in [Che09, Proposition 2.1] to construct a cutoff function for the reader's convenience. First, let $\varphi : \mathbb{R} \to [0,1]$ be a smooth, non-increasing function such that $\varphi \equiv 1$ on $(-\infty, \frac{7}{8}]$ and $\varphi \equiv 0$ on $[1, \infty)$. For example, we can take a smooth function

$$\phi(t) = \begin{cases} e^{-\frac{1}{t}} & \text{if } t > 0 \\ 0 & \text{otherwise} \end{cases}$$

and set $\varphi(x) = \frac{\phi(8-8x)}{\phi(8-8x)+\phi(8x-7)}$. From [Per02, Lemma 8.3], we know that

$$(\partial_t - \Delta)\, d_t(p,x) \geq -\frac{5(n-1)}{3r}$$

whenever $d_{g(t)}(p,x) > r$ in the barrier sense. Now, if we define a cutoff function $\psi : M \times [0,T] \to \mathbb{R}$ by

$$\psi(x,t) := \varphi\left(\frac{d_t(p,x) + \frac{5(n-1)}{3r}t}{Ar}\right)$$

then we can check that ψ satisfies the following.

- ψ is compactly supported in $\cup_{t\in[0,T]} B_{g(t)}(p, Ar) \times \{t\}$
- $\psi(x,t) \equiv 1$ wherever $d_{g(t)}(p,x) \leq \frac{3Ar}{4}$ as

  $$\frac{5(n-1)T}{3Ar^2} \leq \frac{5}{42} < \frac{1}{8}$$

  from the choice of a constant A.
- $(\partial_t - \Delta)\,\psi + \frac{2|\nabla\psi|^2}{\psi} \leq \frac{C}{(Ar)^2}\sqrt{\psi}$ since

  $$\begin{aligned} (\partial_t - \Delta)\,\psi + \frac{2|\nabla\psi|^2}{\psi} &= \frac{\varphi'}{Ar}\left((\partial_t - \Delta)\, d_{g(t)} + \frac{5(n-1)}{3r}\right) + \frac{1}{(Ar)^2}\left(-\varphi'' + \frac{2(\varphi')^2}{\varphi}\right) \\ &\leq \frac{1}{(Ar)^2}\left|\varphi'' - \frac{2(\varphi')^2}{\varphi}\right| \\ &\leq \frac{C}{(Ar)^2}\sqrt{\psi} \end{aligned}$$

Let $u(x,t) = \psi(x,t) f(x,t)$. If $f(x,t) \geq 0$ in $B_{g(t)}\left(p, \frac{3Ar}{4}\right) \times [0,T]$, then we are done. So we may assume that there exists $t_0 \in [0,T]$ such that $\inf_{x\in M} u(x,t_0) = u(x_0,t_0) < 0$. If (x_0,t_0) is a smooth point of both $d_{g(t)}(p,x)$ and f, then from the critical point condition

$$0 = \nabla u(x_0,t_0) = f\nabla\psi + \psi\nabla f$$

we have at (x_0,t_0),

$$\begin{aligned}
(\partial_t - \Delta)\, u &= f\,(\partial_t - \Delta)\,\psi + \psi\,(\partial_t - \Delta)\, f + \frac{2f|\nabla\psi|^2}{\psi} \\
&= f\left((\partial_t - \Delta)\,\psi + \frac{2|\nabla\psi|^2}{\psi}\right) + \psi\,(\partial_t - \Delta)\, f \\
&\geq \frac{Cf}{(Ar)^2}\sqrt{\psi} + \psi f^2 \\
&= \frac{\psi f^2}{2} + \frac{1}{2}\left(\sqrt{\psi} f + \frac{C}{(Ar)^2}\right)^2 - \frac{C^2}{2(Ar)^4} \\
&\geq \frac{\psi f^2}{2} - \frac{C^2}{2(Ar)^4}.
\end{aligned}$$

From this, for $u_{\min}(t) := \inf_{x\in M} u(x,t)$, we have

$$\begin{aligned}
\frac{d^-}{dt} u_{\min}(t_0) &= \liminf_{h\searrow 0} \frac{u_{\min}(t_0+h) - u_{\min}(t_0)}{h} \\
&\geq \frac{u_{\min}(t_0)^2}{2} - \frac{C^2}{2(Ar)^4} \\
&= \frac{u_{\min}(t_0)^2}{4} + \left(\frac{u_{\min}(t_0)^2}{4} - \frac{C^2}{2(Ar)^4}\right)
\end{aligned}$$

and this inequality holds as long as $u_{\min}(t_0) \leq 0$. By integrating this inequality, we have

$$u_{\min}(t) \geq \min\left\{-\frac{4}{t}, -\frac{\sqrt{2}C}{(Ar)^2}\right\}$$

and it completes the proof since $\psi \equiv 1$ in $B_{g(t)}(p, \frac{3Ar}{4}) \times [0,T]$. This argument can be completed similarly even if $d_{g(t)}(p,x)$ or f is not smooth at (x_0,t_0) since we can choose a barrier function. $\square$

From the above localization result, one can obtain the following result, which will be used multiple times to obtain the curvature improvements.

Corollary 2.4.2. *Let $(M, g(t))_{t\in(-\infty,0]}$ be a complete ancient solution to the Ricci flow (not necessarily having bounded curvature). If a continuous function $f : M\times(-\infty, 0] \to \mathbb{R}$ satisfies the inequality*

$$(\partial_t - \Delta) f \geq \delta f^2$$

in the barrier sense for some $\delta > 0$, then $f(x,t) \geq 0$ for all $(x,t) \in M \times (-\infty, 0]$.

Proof. For each $p \in M$, we can choose $r > 0$ small enough so all assumptions in Proposition 2.4.1 are satisfied. By taking $A \to \infty$ and translating the initial time by $-\tau_0$, we get

$$f(x,t) \geq -\frac{4}{(t+\tau_0)\delta}.$$

Then the result is obtained by taking $\tau_0 \to \infty$. □

2.5 Gradient Ricci Shrinker

In this section, we will review some of standard formulas for gradient Ricci shrinkers for the later use. First, we have the following equation from the definition.

$$\nabla^2 f + \mathrm{Ric} = \frac{1}{2} g$$

By taking a trace, we have

$$\Delta f + R = \frac{n}{2}$$

Now, by combining the shrinker equation with Bochner's formula and Ricci identity, we can derive

$$-\frac{1}{2}\nabla_j R = -\nabla_i R_{ij} = \nabla_i \nabla_i \nabla_j f = \nabla_j \nabla_i \nabla_i f + R_{jk}\nabla_k f = -\nabla_j R + R_{jk}\nabla_k f$$

which implies

$$\nabla_j R = 2R_{jk}\nabla_k f$$

Using this, one can observe the following formula by adding a constant to f if necessary

$$R + |\nabla f|^2 = f$$

since

$$\begin{aligned}\nabla_i(R+|\nabla f|^2) &= \nabla_i R + 2\nabla_i\nabla_j f\nabla_j f\\ &= \nabla_i R + \nabla_i f - 2R_{ij}\nabla_j f\\ &= \nabla_i f\end{aligned}$$

Now, we define the f-Laplacian to be

$$\Delta_f = \Delta - \nabla f\cdot\nabla$$

Then, we have

$$\Delta_f f = \Delta f - |\nabla f|^2 = \frac{n}{2} - R - |\nabla f|^2 = \frac{n}{2} - f$$

We can also derive the following.

$$\Delta R = 2\nabla_j(R_{jk}\nabla_k f) = \nabla R\cdot\nabla f + 2R_{jk}\nabla_j\nabla_k f = \nabla R\cdot\nabla f + R - 2|\mathrm{Ric}|^2$$

From this, we can derive the shrinker version of the evolution equation $(\partial_t - \Delta)\, R = 2|\mathrm{Ric}|^2$ under the Ricci flow.

$$\Delta_f R = \Delta R - \nabla R\cdot\nabla f = R - 2|\mathrm{Ric}|^2$$

Also, one of the properties for the potential function f is that it is in fact uniformly equivalent to the square of distance function. This result is first observed in [CZ10] and later optimized in [HM11].

Lemma 2.5.1 (Lemma 2.1 of [HM11]). *Let (M^n, g, f) be a gradient Ricci shrinker. Then there exists a point $p\in M$ where f attains its infimum and f satisfies the quadratic growth estimate*

$$\frac{1}{4}(d(x,p)-5n)_+^2 \le f(x) \le \frac{1}{4}(d(x,p)+\sqrt{2n})^2$$

for all $x\in M$.

Chapter 3

Curvature Improvement

3.1 4-dimensional Case

3.1.1 Ricci flow with uniformly PIC

In this section, we consider a complete 4-dimensional ancient solution to the Ricci flow with uniformly PIC and prove the curvature improvement. It is notable that the same result is given in [Bre14, Theorem 1.3] for steady solitons. But, every steady soliton has bounded scalar curvature as $R + |\nabla f|^2$ is constant and hence every steady soliton with weakly PIC must have bounded curvature due to Proposition 2.2.1. As the boundedness of curvature is not assumed here, we need to use Corollary 2.4.2 to obtain the result.

First, remark that M^4 has weakly PIC if and only if $a_1 + a_2 \geq 0$ and $c_1 + c_2 \geq 0$. In other words, the curvature tensors restricted to self-dual, anti-self-dual 2 forms are 2-positive. For ancient solutions, this 2-positivity can be improved as the positivity as follows.

Lemma 3.1.1. *Let $(M^4, g(t))_{t\in(-\infty,0]}$ be a complete 4-dimensional ancient solution to the Ricci flow with weakly* PIC. *Then we have*

$$a_1 \geq 0 \quad \textit{and} \quad c_1 \geq 0$$

on $M \times (-\infty, 0]$.

Proof. From the evolution equation of the curvature operator (see [Ham97]), we have

$$(\partial_t - \Delta)\, a_1 \geq a_1^2 + b_1^2 + 2a_2a_3 \geq a_1^2 + 2a_2a_3$$
$$(\partial_t - \Delta)\, c_1 \geq c_1^2 + b_1^2 + 2c_2c_3 \geq c_1^2 + 2c_2c_3$$

From the weakly PIC condition, we know that

$$0 \leq \frac{1}{2}(a_1 + a_2) \leq a_2 \leq a_3$$
$$0 \leq \frac{1}{2}(c_1 + c_2) \leq c_2 \leq c_3$$

Therefore, we have

$$(\partial_t - \Delta)\, a_1 \geq a_1^2$$
$$(\partial_t - \Delta)\, c_1 \geq c_1^2$$

Now the result follows from Corollary 2.4.2. □

Next, we focus on uniformly PIC condition which implies that all curvature components are controlled by $a_1 + a_2$ and $c_1 + c_2$. Starting from this condition, we show that a_3 and c_3 are controlled solely by a_1 and c_1, respectively.

Lemma 3.1.2. *Let $(M^4, g(t))_{t\in(-\infty,0]}$ be a complete 4-dimensional ancient solution to the Ricci flow with uniformly* PIC*. Then we have*

$$a_3 \leq (6\Lambda^2 + 1)a_1$$

$$c_3 \leq (6\Lambda^2 + 1)c_1$$

on $M \times (-\infty, 0]$.

Proof. First, we have

$$(\partial_t - \Delta)\, a_3 \leq a_3^2 + b_3^2 + 2a_1a_2$$
$$(\partial_t - \Delta)\, a_1 \geq a_1^2 + b_1^2 + 2a_2a_3$$

Since we have $\max(a_3, b_3) \le \Lambda(a_1 + a_2)$ and $(a_1 + a_2)^2 \le 4a_2^2 \le 4a_2 a_3$, we have

$$
\begin{aligned}
&(\partial_t - \Delta)\left(a_3 - (6\Lambda^2 + 1)a_1\right) \\
\le\, & a_3^2 + b_3^2 + 2a_1a_2 - (6\Lambda^2 + 1)(a_1^2 + b_1^2 + 2a_2a_3) \\
\le\, & a_3^2 + b_3^2 + 2a_2(a_1 - a_3) - 12\Lambda^2 a_2 a_3 \\
\le\, & a_3^2 + b_3^2 - 12\Lambda^2 a_2 a_3 \\
\le\, & a_3^2 + b_3^2 - 3\Lambda^2(a_1 + a_2)^2 \\
\le & - a_3^2.
\end{aligned}
$$

Moreover, if $(a_3 - (6\Lambda^2 + 1)a_1)(x_0, t_0) \ge 0$ for some $(x_0, t_0) \in M \times (-\infty, 0]$, then we have at (x_0, t_0),

$$
(\partial_t - \Delta)\left(a_3 - (6\Lambda^2 + 1)a_1\right) \le -(a_3 - (6\Lambda^2 + 1)a_1)^2
$$

Now the result follows from Corollary 2.4.2, by choosing $f = (-a_3 + (6\Lambda^2 + 1)a_1)^-$. Similarly, the conclusion for c_1 and c_3 also holds. $\square$

We continue to show the following pinching condition.

Lemma 3.1.3. *Let $(M^4, g(t))_{t\in(-\infty,0]}$ be a complete 4-dimensional ancient solution to the Ricci flow with uniformly* PIC. *Then we have*

$$
\frac{b_3^2}{(a_1 + a_2)(c_1 + c_2)} \le \frac{1}{4}
$$

on $M \times (-\infty, 0]$.

Proof. First, from the uniformly PIC condition, we know that

$$
\gamma := \sup_{M\times(-\infty,0]} \frac{b_3}{\sqrt{(a_1 + a_2)(c_1 + c_2)}} \le \Lambda < \infty
$$

Now, suppose that the statement is not true, then we have $\gamma > \frac{1}{2}$. From [Ham86], we have the following evolution equations

$$\begin{aligned}(\partial_t - \Delta)\, b_3 \leq& b_3(a_3 + c_3) + 2b_1 b_2,\\ (\partial_t - \Delta)\,(a_1 + a_2) \geq& a_1^2 + a_2^2 + 2a_3(a_1 + a_2) + b_1^2 + b_2^2,\\ (\partial_t - \Delta)\,(c_1 + c_2) \geq& c_1^2 + c_2^2 + 2c_3(c_1 + c_2) + b_1^2 + b_2^2.\end{aligned}$$

From direct calculations, we get

$$(\partial_t - \Delta)\sqrt{(a_1 + a_2)(c_1 + c_2)} \geq \sqrt{(a_1 + a_2)(c_1 + c_2)}(G + E)$$

where

$$G = \frac{1}{4}|\nabla \log(a_1 + a_2) - \nabla \log(c_1 + c_2)|^2 \geq 0$$

and

$$\begin{aligned}E =& \frac{1}{2}\left(\frac{a_1^2 + a_2^2 + b_1^2 + b_2^2}{a_1 + a_2} + \frac{c_1^2 + c_2^2 + b_1^2 + b_2^2}{c_1 + c_2}\right) + a_3 + c_3\\ =& \frac{1}{2}\left(\frac{(a_1 - b_1)^2 + (a_2 - b_2)^2}{a_1 + a_2} + \frac{(c_1 - b_1)^2 + (c_2 - b_2)^2}{c_1 + c_2}\right)\\ &+ a_3 + c_3 + 2b_1 + \frac{a_2(b_2 - b_1)}{a_1 + a_2} + \frac{c_2(b_2 - b_1)}{c_1 + c_2}.\end{aligned}$$

Now, set $u(x,t) = b_3 - \gamma\sqrt{(a_1 + a_2)(c_1 + c_2)}$. Then $u(x,t) \leq 0$ globally because of the definition of γ. Furthermore, this function satisfies the following evolution inequality.

$$\begin{aligned}(\partial_t - \Delta)\, u(x,t) &\leq b_3(a_3 + c_3) + 2b_1 b_2 - \gamma\sqrt{(a_1 + a_2)(c_1 + c_2)}E\\ &\leq u(x,t)(a_3 + c_3) + 2b_1(b_2 - \gamma\sqrt{(a_1 + a_2)(c_1 + c_2)}) - \gamma\sqrt{(a_1 + a_2)(c_1 + c_2)}K\end{aligned}$$

where

$$K = \frac{1}{2}\left(\frac{(a_1 - b_1)^2 + (a_2 - b_2)^2}{a_1 + a_2} + \frac{(c_1 - b_1)^2 + (c_2 - b_2)^2}{c_1 + c_2}\right) + \frac{a_2(b_2 - b_1)}{a_1 + a_2} + \frac{c_2(b_2 - b_1)}{c_1 + c_2}.$$

Now, since

$$b_2 - \gamma\sqrt{(a_1 + a_2)(c_1 + c_2)} \leq u(x,t) \leq 0$$

and $K \geq 0$, it is clear that $(\partial_t - \Delta)\, u(x,t) \leq 0$ everywhere. Moreover, it can be zero only when $K = u(x,t) = 0$ and $b_2 = b_3$. But this implies

$$a_1 = b_1 = c_1 = a_2 = b_2 = c_2 = b_3$$

and

$$0 = u(x,t) = b_3 - \gamma\sqrt{(a_1+a_2)(c_1+c_2)} = (1-2\gamma)b_3$$

Since $\gamma > \frac{1}{2}$ from the assumption, it can only happen when all terms are equal to 0, which is impossible since $a_1 + a_2 > 0$. Moreover, it follows from the uniformly PIC and Lemma 3.1.2 that $(\partial_t - \Delta)\, u(x,t) \leq -6\delta|\mathrm{Ric}|^2$ for a small constant $\delta = \delta(\Lambda,\gamma) > 0$. With this fact, we obtain

$$(\partial_t - \Delta)\,(u + \delta R) \leq -4\delta|\mathrm{Ric}|^2 \leq -\delta R^2.$$

Therefore, we have

$$(\partial_t - \Delta)\,(u + \delta R) \leq -\delta^{-1}(u+\delta R)^2$$

wherever $u + \delta R \geq 0$. From Corollary 2.4.2, it implies that $u(x,t) + \delta R(x,t) \leq 0$ everywhere. From this, we have

$$0 \geq \frac{b_3}{\sqrt{(a_1+a_2)(c_1+c_2)}} - \gamma + \frac{\delta R}{\sqrt{(a_1+a_2)(c_1+c_2)}} \geq \frac{b_3}{\sqrt{(a_1+a_2)(c_1+c_2)}} - \gamma + \delta'$$

for a small constant $\delta' = \delta'(\Lambda,\gamma) > 0$. However, it implies that $\frac{b_3}{\sqrt{(a_1+a_2)(c_1+c_2)}} \leq \gamma - \delta'$ everywhere, which contradicts the choice of γ. So we conclude that $\gamma \leq \frac{1}{2}$, which completes the proof. $\square$

One can check that Lemma 3.1.3 actually implies $(M, g(t))_{t\in(-\infty,0]}$ has weakly PIC_1. However, this condition can be further improved in the following way.

Lemma 3.1.4. *Let $(M^4, g(t))_{t\in(-\infty,0]}$ be a 4-dimensional, complete ancient solution to the Ricci flow with uniformly* PIC. *Then it has nonnegative curvature operator.*

Proof. To this end, we will show that

$$\frac{b_3^2}{a_1c_1} \leq 1$$

for all $(x,t) \in M \times (-\infty, 0]$, which can be proven similarly as follows. If it is not true, then we have

$$\eta := \sup_{M\times(-\infty,0]} \frac{b_3}{\sqrt{a_1 c_1}} > 1.$$

Notice that η is finite by Lemma 3.1.2. If we set $v(x,t) = b_3 - \eta\sqrt{a_1c_1}$, then we get $v(x,t) \le 0$ everywhere and we have the following evolution equation.

$$(\partial_t - \Delta)\, v(x,t) \le b_3(a_3 + c_3) + 2b_1b_2 - \eta\sqrt{a_1c_1}E$$

where

$$\begin{aligned} E &= \frac{1}{2}\left(\frac{a_1^2 + b_1^2 + 2a_2a_3}{a_1} + \frac{c_1^2 + b_1^2 + 2c_2c_3}{c_1}\right) \\ &= \frac{1}{2}\left(\frac{(a_1-b_1)^2 + 2a_3(a_2-a_1)}{a_1} + \frac{(c_1-b_1)^2 + 2c_3(c_2-c_1)}{c_1}\right) + 2b_1 + a_3 + c_3 \\ &=: F + 2b_1 + a_3 + c_3. \end{aligned}$$

Therefore, we obtain

$$(\partial_t - \Delta)\, v(x,t) \le v(x,t)(a_3 + c_3) + 2b_1(b_2 - \eta\sqrt{a_1c_1}) - \eta\sqrt{a_1c_1}F \le 0$$

since $F \ge 0$ and $v(x,t) \le 0$. Moreover, the equality case is obtained if and only if $a_1 = b_1 = c_1 = a_2 = c_2$, $b_2 = b_3$ and $b_3 = \eta\sqrt{a_1c_1}$. Since we already know that $\frac{b_3}{\sqrt{(a_1+a_2)(c_1+c_2)}} \le \frac{1}{2}$ from Lemma 3.1.3, we get $b_3 \le \frac{1}{2}\sqrt{(a_1+a_2)(c_1+c_2)} = b_1$. Therefore the equality case is obtained when $a_1 = b_1 = c_1 = a_2 = c_2 = b_2 = b_3$ and $b_3 = \eta\sqrt{a_1c_1} > \sqrt{a_1c_1} = b_3$. It shows that $a_1 = c_1 = 0$ in the equality case, which contradicts to the curvature condition. As before, $(\partial_t - \Delta)\, v(x,t) \le -6\delta|\mathrm{Ric}|^2$ for a small constant $\delta = \delta(\Lambda, \eta) > 0$. From the similar argument used in Lemma 3.1.3, we obtain a contradiction.

Now we show that $\frac{b_3}{\sqrt{a_1c_1}} \le 1$ implies the nonnegativity of the curvature operator. To do so, let $\varphi_i^{\pm} \in \bigwedge^2_{\pm}$ $(i = 1,2,3)$ be bases of a self-dual vector space $\bigwedge_+$ and an anti-self-dual vector space $\bigwedge_-$ that $\mathrm{Rm}|_{\bigwedge^2_{\pm}}$ is diagonalized, respectively. Let $\varphi = \sum_{i=1}^3 p^i\varphi_i^+ + \sum_{j=1}^3 q^j\varphi_j^-$. From the definition, we have $\mathrm{Rm}(\varphi_i^+, \varphi_i^+) \ge a_1$ and $\mathrm{Rm}(\varphi_i^-, \varphi_i^-) \ge c_1$ and $\mathrm{Rm}(\varphi_i^+, \varphi_j^-) \ge -b_3$

for all $i, j = 1, 2, 3$. Therefore we have

$$\begin{aligned}
\mathrm{Rm}(\varphi, \varphi) &\geq 3a_1 \sum_{i=1}^{3} (p^i)^2 + 3c_1 \sum_{j=1}^{3} (q^j)^2 - 6b_3 \sum_{i,j=1}^{3} |p^i||q^j| \\
&\geq 3a_1 \sum_{i=1}^{3} (p^i)^2 + 3c_1 \sum_{j=1}^{3} (q^j)^2 - 6\sqrt{a_1 c_1} \sum_{i,j=1}^{3} |p^i||q^j| \\
&= 3 \sum_{i,j}^{3} (|p^i|\sqrt{a_1} - |q_j|\sqrt{c_1})^2 \geq 0.
\end{aligned}$$

Therefore, the proof is complete. $\square$

Consequently, for a 4-dimensional, complete, noncompact ancient solution to the Ricci flow $(M^4, g(t))_{t\in(-\infty,0]}$ with uniformly PIC, there exists a constant $K = K(\Lambda) > 0$ such that

$$a_3 \leq K a_1, \ c_3 \leq K c_1, \ b_3^2 \leq a_1 c_1. \tag{3.1.1}$$

In other words, it satisfies the restricted isotropic curvature pinching condition in [CZ06a].

3.1.2 Kähler Ricci flow with weakly PIC

In this section, we consider a complete (complex) 2-dimensional ancient solution to the Kähler Ricci flow with weakly PIC. See also [CTZ04, Lemma 3.1].

Lemma 3.1.5. *Let $(M, g(t))_{t\in(-\infty,0]}$ be a complete, complex 2-dimensional ancient solution to Kähler-Ricci flow with weakly* PIC. *Then it has nonnegative curvature operator.*

Proof. From Lemma 3.1.1, we obtain $c_1 \geq 0$ on $M \times (-\infty, 0]$. Moreover, we have the evolution equation

$$(\partial_t - \Delta)\,\mathrm{Rm} = \mathrm{Rm}^2 + \mathrm{Rm}^\sharp = \mathrm{Rm}^2 + 2\begin{pmatrix} 0 & 0 \\ 0 & C^\sharp \end{pmatrix}, \tag{3.1.2}$$

where $C^\sharp$, which is the adjoint matrix of C, has eigenvalues $\{c_1c_2, c_2c_3, c_3c_1\}$. Therefore, we know that $C^\sharp$ is positive definite. So if we set λ to be the smallest eigenvalue of Rm, then it follows from (3.1.2) that

$$(\partial_t - \Delta)\,\lambda \geq \lambda^2.$$

Therefore, we conclude from Corollary 2.4.2 that $\mathrm{Rm} \geq 0$ on $M \times (-\infty, 0]$. $\square$

3.2 Higher Dimensional Case

In this section, we will show that every n-dimensional, complete ancient solution to the Ricci flow(not necessarily having bounded curvature) with $n \geq 12$ and uniformly PIC automatically has weakly PIC2. This result can be obtained in two steps. First, we will show the uniform version of [LN20, Proposition 5.2], which shows that any ancient solution to the Ricci flow with weakly PIC should have 2-nonnegative Ricci curvature; that is, a sum of two smallest eigenvalues of Ricci curvature should be nonnegative. After then, we will use two continuous families of cones in a space of algebraic curvature tensors constructed by Brendle.

Proposition 3.2.1. *Let $(M^n, g(t))_{t \in (-\infty, 0]}$ be a complete ancient solution to the Ricci flow with uniformly* PIC *for $n \geq 5$. Then there exists $\delta = \delta(n, \theta) > 0$ such that*

$$\lambda_1 + \lambda_2 \geq \delta R$$

on $M \times (-\infty, 0]$, where $\lambda_1 + \lambda_2$ is a sum of the two smallest eigenvalues of Ric.

Proof. Let $f = \lambda_1 + \lambda_2 - \delta R$ with $\delta > 0$ determined later. Here, we choose an orthonormal frame $\{e_1, e_2, \cdots, e_n\}$ such that the Ricci curvature is diagonalized with eigenvalues $R_{11} \leq R_{22} \leq \cdots \leq R_{nn}$. From 2.3.2 and 2.3.3, we know that the function f satisfies the following evolution equation in the barrier sense.

$$(\partial_t - \Delta) f \geq 2(R_{1i1i} + R_{2i2i})R_{ii} - 2\delta |\mathrm{Ric}|^2 \tag{3.2.1}$$

From Proposition 2.2.1, we have $|\mathrm{Ric}|^2 \leq nR^2$. So (3.2.1) becomes

$$(\partial_t - \Delta) f \geq 2(R_{1i1i} + R_{2i2i})R_{ii} - 2n\delta R^2. \tag{3.2.2}$$

Also, we can rewrite

$$2(R_{1i1i} + R_{2i2i})R_{ii} = \sum_{i \geq 3} (R_{1i1i} + R_{2i2i})(2R_{ii} - R_{11} - R_{22}) + (R_{11} + R_{22})^2 \tag{3.2.3}$$

Now, we will show that there exists a constant $\theta > 0$ such that

$$\sum_{i\geq 3}(R_{1i1i}+R_{2i2i})(2R_{ii}-R_{11}-R_{22}) \geq 3\theta R^2$$

To do so, we can first observe that this expression can be decomposed as follows.

$$\begin{aligned}&\sum_{i\geq 3}(R_{1i1i}+R_{2i2i})(2R_{ii}-R_{11}-R_{22})\\ =&2\theta R(2R-n(R_{11}+R_{22}))+\sum_{i\geq 3}(R_{1i1i}+R_{2i2i}-2\theta R)(2R_{ii}-R_{11}-R_{22}).\end{aligned}$$

If $f(x,t)\leq 0$, then we additionally have

$$2\theta R(2R-n(R_{11}+R_{22}))\geq 2\theta(2-n\delta)R^2.$$

Since $(M,g(t))$ has uniformly PIC, we know that there is at most one $i\geq 3$ such that $R_{1i1i}+R_{2i2i}-2\theta R<0$ as a sum of two components in such a form has to be nonnegative. Based on this observation, there are 3 possibilities.

1. If $R_{1i1i}+R_{2i2i}-2\theta R\geq 0$ for all $i=3,\cdots,n$, then we have

$$\sum_{i\geq 3}(R_{1i1i}+R_{2i2i}-2\theta R)(2R_{ii}-R_{11}-R_{22})\geq 0. \tag{3.2.4}$$

2. If $R_{1i1i}+R_{2i2i}-2\theta R<0$ for some $3\leq i<n$, then we still have 3.2.4 since

$$\begin{aligned}&(R_{1i1i}+R_{2i2i}-2\theta R)(2R_{ii}-R_{11}-R_{22})+(R_{1n1n}+R_{2n2n}-2\theta R)(2R_{nn}-R_{11}-R_{22})\\ &\geq(R_{1n1n}+R_{2n2n}-2\theta R)(2R_{nn}-2R_{ii})\geq 0\end{aligned}$$

3. If $R_{1n1n}+R_{2n2n}-2\theta R<0$, then there exists a constant $C=C(n,\theta)>0$ such that

$$\begin{aligned}&\sum_{i\geq 3}(R_{1i1i}+R_{2i2i}-2\theta R)(2R_{ii}-R_{11}-R_{22})\\ \geq&(2\theta R-R_{1n1n}-R_{2n2n})\left(\sum_{i=3}^{n-1}(2R_{ii}-R_{11}-R_{22})+R_{11}+R_{22}-2R_{nn}\right)\\ =&(2\theta R-R_{1n1n}-R_{2n2n})(2R-4R_{nn}-(n-2)(R_{11}+R_{22}))\geq -\delta CR^2\end{aligned}$$

where the last inequality holds since

$$2R - 4R_{nn} - (n-2)(R_{11} + R_{22}) \geq -(n-2)(R_{11} + R_{22}) \geq -(n-2)\delta R$$

and

$$2\theta R - R_{1n1n} - R_{2n2n} \leq 2\theta R + |R_{1n1n}| + |R_{2n2n}| \leq (2\theta + C_1)R,$$

where $C_1 = 2C > 0$ with the constant $C > 0$ from Proposition 2.2.1.

Therefore, in any case, we have

$$\sum_{i\geq 3}(R_{1i1i} + R_{2i2i})(2R_{ii} - R_{11} - R_{22}) \geq 2\theta(2 - n\delta)R^2 - \delta C R^2 \geq 3\theta R^2 \tag{3.2.5}$$

by taking $\delta > 0$ small enough. By combining (3.2.2), (3.2.3) and (3.2.5), we finally obtain that if $f(x,t) \leq 0$,

$$(\partial_t - \Delta) f \geq (R_{11} + R_{22})^2 + 3\theta R^2 - 2n\delta R^2 \geq (R_{11} + R_{22})^2 + \delta^2 R^2$$

by decreasing $\delta > 0$ if necessary. It implies that

$$(\partial_t - \Delta) f^- \geq \frac{1}{2}(f^-)^2$$

where $f^- := \min\{f, 0\}$. Now the proof is complete by Corollary 2.4.2. □

Now we are ready to prove the main theorem in the section.

Theorem 3.2.2. *Let $(M^n, g(t))_{t\in(-\infty,0]}$ be a complete ancient solution to the Ricci flow with uniform* PIC *for $n \geq 12$. Then it has weakly* PIC$_2$.

Proof. In [Bre19, Definition 3.1] and [Bre19, Definition 4.1], Brendle has constructed two continuous families of closed, convex, $O(n)$-invariant cones $\mathcal{C}(b)$ and $\tilde{\mathcal{C}}(b)$ in $\mathscr{C}_B(\mathbb{R}^n)$ for $n \geq 12$. Moreover, the following properties hold:

- $\mathcal{C}(b)$ is defined for all $b \in (0, b_{\max}]$.
- $\tilde{\mathcal{C}}(b)$ is defined for all $b \in (0, \tilde{b}_{\max}]$.

- $\lim_{b\to 0}\tilde{\mathcal{C}}(b) = \mathcal{C}(b_{\max}) \cap \mathrm{PIC}_1$.

- $\mathcal{C}(b_{\max}) = \tilde{\mathcal{C}}(\tilde{b}_{\max})$.

As in [Bre19, Section 5], we define

$$\hat{\mathcal{C}}(b) := \begin{cases} \mathcal{C}(b) & \text{if } b \in [0, b_{\max}) \\ \tilde{\mathcal{C}}(b_{\max} + \tilde{b}_{\max} - b) & \text{if } b \in [b_{\max}, b_{\max} + \tilde{b}_{\max}). \end{cases}$$

From the construction, $I = \frac{1}{2}\mathrm{id} \owedge \mathrm{id} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ is in the interior of $\hat{\mathcal{C}}(b)$, where $\owedge$ is the Kulkarni-Nomizu product. Moreover, if $\mathrm{Rm} \in \hat{\mathcal{C}}(b)$, then $Q(\mathrm{Rm})$ lies in the interior of $T_{\mathrm{Rm}}\hat{\mathcal{C}}(b)$. The readers can refer to [Bre19, Theorem 3.2] for more details. Here $T_{\mathrm{Rm}}\hat{\mathcal{C}}(b)$ is the tangent cone to $\hat{\mathcal{C}}(b)$ at Rm. Note that for a finite dimensional inner product space X and a closed, convex subset $F \subset X$, the tangent cone T_yF to F at $y \in F$ is defined as

$$T_yF = \{x \in X \mid x \cdot z \geq 0 \text{ for all } z \in N_yF\}$$

where N_yF is the normal cone to F at y, which is given as

$$N_yF = \{z \in X \mid x \cdot z \geq y \cdot z \text{ for all } x \in F\}$$

Claim 1. There exists $b_0 > 0$ such that the curvature tensor of $(M, g(t))$ is contained in $\hat{\mathcal{C}}(b_0)$ for all $t \in (-\infty, 0]$.

Recall the transformation $l_{a,b}$, defined in [BW08], so that there exists $S \in \mathscr{C}_B(\mathbb{R}^n)$ satisfying

$$l_{a,b}(S) := S + b\mathrm{Ric}(S) \owedge \mathrm{id} + \left(\frac{2(a-b)}{n}\mathrm{scal}(S)\right) I = \mathrm{Rm}. \tag{3.2.6}$$

Clearly, $l_{a,b}(S)$ is close to S if a, b are close to 0. Note that $\mathcal{C}(b)$ is defined as an image of $\mathcal{E}(b)$ under $l_{a,b}$ where $a = \frac{(2+(n-2)b)^2}{2(2+(n-3)b)}b$. Therefore, by assuming that b is small enough, it is enough to show that Rm is contained in $\mathcal{E}(b)$. In other words, we need to find an algebraic curvature tensor T satisfying

1. $T \geq 0$

2. $\mathrm{Rm} - T$ has weakly PIC.

3. $R_{11} + R_{22} \geq 0$

4. $R_{22} - R_{11} \leq \frac{C}{b}\sqrt{R}\left(\sum_{p=3}^{n}(T_{1p1p} + T_{2p2p})\right)^{\frac{1}{2}}$ for some $C > 0$.

Note that the condition 3,4 are the weaker version of the original in the paper. The readers can refer to [Bre19, Definition 3.1] for the full definition of the curvature cone. Now, let $T = \frac{\theta}{2}RI$. Since Rm is a curvature tensor of ancient solution, we have $R \geq 0$ so the condition 1 is satisfied. Also, the condition 2 is guaranteed as Rm has uniform PIC. And the condition 3,4 hold because of Proposition 3.2.1 and Proposition 2.2.1. Therefore, we can conclude that $\mathrm{Rm} \in \hat{\mathcal{C}}(b_0)$ if b_0 is sufficiently small. It verifies Claim 1.

Now, we prove that if the curvature operator Rm of $(M, g(t))_{t\in(-\infty,0]}$ is contained in $\hat{\mathcal{C}}(b)$ for some $b \in (0, b_{\max} + \tilde{b}_{\max})$, then $\mathrm{Rm} \in \hat{\mathcal{C}}(b')$ if b' is sufficiently close to b. To do so, we need the following claim.

Claim 2. There exists a constant $\tau \in (0, 1)$ depending only on n and b such that $Q(\mathrm{Rm}) - \tau R^2 I \in T_{\mathrm{Rm}}\hat{\mathcal{C}}(b)$.

Suppose that Claim 2 is false. Then since $\hat{C}(b)$ is a cone and $\frac{1}{R^2}Q(\mathrm{Rm})$ is scaling invariant, one can choose a sequence of counterexamples $\mathrm{Rm}_k \in \partial\hat{C}(b)$ such that $|\mathrm{Rm}_k| = 1$ and $Q(\mathrm{Rm}_k) - \frac{1}{k^2}R_k^2 I$ is on the boundary of $T_{\mathrm{Rm}_k}\hat{\mathcal{C}}(b)$ for all $k > 0$. By taking a subsequence, Rm_k converges to $\mathrm{Rm}_\infty \in \partial\hat{\mathcal{C}}(b)$ and $Q(\mathrm{Rm}_\infty)$ is on the boundary of $T_{\mathrm{Rm}_\infty}\hat{\mathcal{C}}(b)$. However, this contradicts the transversality of $\hat{\mathcal{C}}(b)$. It verifies Claim 2.

Now we will show the aforementioned statement. Suppose that the curvature operator of $(M, g(t))_{t\in(-\infty,0]}$ is contained in $\hat{\mathcal{C}}(b)$ for some $b \in (0, b_{\max} + \tilde{b}_{\max})$. To complete the proof of the statement, it is enough to show that there exists a constant $\delta > 0$ so that $\mathrm{Rm} - \delta R I \in \hat{\mathcal{C}}(b)$ since it will imply that Rm lies in the interior of the cone, which makes the statement true.

With this objective in mind, we define a function λ as follows. For any spacetime point $(x,t) \in M \times (-\infty, 0]$, let $\lambda(x,t)$ be the smallest number so that the curvature operator $S := \mathrm{Rm} + (\lambda - \delta R)I$ lies on the boundary of $\hat{C}(b)$, where the constant $\delta > 0$ will be determined later. Here, we may assume λ to be locally smooth since the general case will be handled at the end of the proof. Now, we will show that $\lambda(x,t) \leq 0$ for all $(x,t) \in M \times (-\infty, 0]$ using the localization result obtained earlier.

For any $(x,t) \in M \times (-\infty, 0]$, we know that $\mathrm{Rm}(x,t)$ is contained in $\hat{\mathcal{C}}(b)$ and hence $\lambda(x,t) - \delta R(x,t) \leq 0$. Now, we will investigate the evolution equation for the curvature operator S. First, one can compute the quadratic term $Q(S)$ of S as follows. Note that the definition of Q is given in (2.3.1).

$$Q(S) = Q(\mathrm{Rm}) + 2(\lambda - \delta R)\mathrm{Ric} \owedge \mathrm{id} + 2(n-1)(\lambda - \delta R)^2 I.$$

Therefore, we can derive the following evolution equation

$$\begin{aligned}(\partial_t - \Delta) S &= Q(\mathrm{Rm}) + ((\partial_t - \Delta)\lambda) I - 2\delta |\mathrm{Ric}|^2 I \\ &= Q(S) - 2(\lambda - \delta R)\mathrm{Ric} \owedge \mathrm{id} - 2(n-1)(\lambda - \delta R)^2 I - 2\delta|\mathrm{Ric}|^2 I + ((\partial_t - \Delta)\lambda) I.\end{aligned} \tag{3.2.7}$$

Now, wherever $\lambda(x,t) \geq 0$, we know that

$$\mathrm{scal}(S) = R + n(n-1)(\lambda - \delta R) \geq \frac{R}{2}$$

for small enough $\delta > 0$ and

$$|\lambda - \delta R| = \delta R - \lambda \leq \delta R$$

Together with Proposition 2.2.1, we have

$$-2(\lambda - \delta R)\mathrm{Ric} \owedge \mathrm{id} - 2(n-1)(\lambda - \delta R)^2 I - 2\delta |\mathrm{Ric}|^2 I \geq -C_1 \delta R^2 I$$

for a constant $C_1 = C_1(n) > 0$. Therefore, we obtain from (3.2.7)

$$(\partial_t - \Delta) S \geq ((\partial_t - \Delta) \lambda) I + Q(S) - C_1 \delta R^2 I,$$

wherever $\lambda(x,t) \geq 0$. Also, since $\lambda - \delta R \leq 0$ and $R > 0$, we have

$$\lambda^2(x,t) \leq \delta^2 R^2(x,t)$$

if $\lambda(x,t) \geq 0$. This implies

$$-C_1 \delta R^2 - \lambda^2 \geq -(C_1 \delta + \delta^2) R^2 \geq -C_2 \delta \mathrm{scal}(S)^2,$$

where $C_2 = 4(C_1 + 1)$. As a result, we obtain

$$(\partial_t - \Delta) S \geq \left((\partial_t - \Delta) \lambda + \lambda^2\right) I + Q(S) - C_2 \delta \mathrm{scal}^2(S) \qquad (3.2.8)$$

wherever $\lambda(x,t) \geq 0$. Now we observe the following.

Claim 3. $(\partial_t - \Delta) S$ is not contained in the interior of $T_S \hat{\mathcal{C}}(b)$.

To show Claim 3, we fix a point $(x_0, t_0) \in M \times (-\infty, 0]$ and choose a supporting plane H of $\hat{\mathcal{C}}(b)$ at $S(x_0, t_0)$ with a normal vector ν pointing toward a half-space containing $\hat{\mathcal{C}}(b)$. Then we can see that $F(p) = \langle p, \nu \rangle$ is a distance function between p and H. So the function $F(S(x_0, t))$ has its local minimum at $t = t_0$. It implies that

$$0 = \frac{\partial}{\partial t} F(S(x_0, t)) \bigg|_{t=t_0} = \left\langle \frac{\partial}{\partial t} S(x_0, t_0), \nu \right\rangle.$$

Also, if we choose any tangent vector $u \in T_{x_0} M$ and consider a geodesic $\gamma(s)$ starting from x_0 with a directional vector u, then $F(S(\gamma(s), t_0))$ has its local minimum at $s = 0$. So we have

$$0 \leq \frac{d^2}{ds^2} F(S(\gamma(s), t_0)) \bigg|_{s=0} = \langle \nabla_u \nabla_u S(x_0, t_0), \nu \rangle.$$

After taking the sum for u, we get $\langle \Delta S(x_0, t_0), \nu \rangle \geq 0$. By combining this two results, we get

$$\langle (\partial_t - \Delta) S(x_0, t_0), \nu \rangle \leq 0$$

which shows that $(\partial_t - \Delta) S(x_0, t_0)$ can not be contained in the interior of $T_S\hat{\mathcal{C}}(b)$. It verifies the Claim 3.

Since S is contained of $\hat{\mathcal{C}}(b)$, we know from Claim 2 that $Q(S) - \tau \mathrm{scal}^2(S)I \in T_S\hat{\mathcal{C}}(b)$. Therefore, if we choose $\delta > 0$ small enough so that $\delta < \frac{\tau}{2C_2}$, then $Q(S) - C_2\delta\mathrm{scal}^2(S)I$ is contained in the interior of $T_S\hat{\mathcal{C}}(b)$. Combining this with Claim 3, we conclude that the first term $((\partial_t - \Delta)\lambda + \lambda^2) I$ on the right side of (3.2.8) should not be contained in $T_S\hat{\mathcal{C}}(b)$. Therefore, we obtain

$$(\partial_t - \Delta)\lambda + \lambda^2 \leq 0 \tag{3.2.9}$$

wherever $\lambda(x,t) \geq 0$. By applying Corollary 2.4.2 on $(-\lambda)^-$, we obtain $\lambda(x,t) \leq 0$ for all $(x,t) \in M \times (-\infty, 0]$. It completes the proof of statement when λ is locally smooth at (x_0, t_0).

Now, let us consider the case when λ is not smooth at (x_0, t_0). Note that $S(x_0, t_0) = \mathrm{Rm}(x_0, t_0) + (\lambda(x_0, t_0) - \delta R(x_0, t_0))I$ is on the boundary of $\hat{\mathcal{C}}(b)$. So we can choose a supporting hyperplane H of $\hat{\mathcal{C}}(b)$ at $S(x_0, t_0)$. Using this hyperplane, we can define a new function $\tilde{\lambda}$ which is the smallest number so that $\tilde{S} := \mathrm{Rm} + (\tilde{\lambda} - \delta R)I$ lies on H. Clearly, $\tilde{\lambda}(x,t)$ is smooth and $\tilde{\lambda}(x_0, t_0) = \lambda(x_0, t_0)$. Also, since I is in the interior of $\hat{\mathcal{C}}(b)$, we have $\tilde{\lambda}(x,t) \leq \lambda(x,t)$ for all (x,t) in a small neighborhood of (x_0, t_0). In other words, the function $\tilde{\lambda}$ is a lower barrier function of λ from below. Also, since $F(p) = \langle p, \nu \rangle$ is constant for $p \in H$, we can use the proof of Claim 3 again to show that $(\partial_t - \Delta)\tilde{S}(x_0, t_0)$ has to be contained in H. From the same argument, $\tilde{\lambda}$ satisfies the differential inequality

$$(\partial_t - \Delta)\tilde{\lambda} + \tilde{\lambda}^2 \leq 0$$

at (x_0, t_0), wherever $\tilde{\lambda}(x_0, t_0) \geq 0$. Therefore, (3.2.9) holds in the barrier sense, which is sufficient to apply Corollary 2.4.2.

Now we set

$$\mathcal{I} := \{b \in (0, b_{\max} + \tilde{b}_{\max}) \mid \mathrm{Rm}(x,t) \text{ is contained in } \hat{\mathcal{C}}(b) \text{ for all } (x,t) \in M \times (-\infty, 0]\}.$$

Then $\mathcal{I}$ is nonempty since $b_0 \in \mathcal{I}$ and open from the previous argument. Also, it is closed since $\hat{\mathcal{C}}(b)$ is a continuous family. Therefore, it implies that $\mathrm{Rm}(x,t)$ is contained in $\hat{\mathcal{C}}(b)$ for all

$b \in (0, b_{\max} + \tilde{b}_{\max})$. By taking $b \to (b_{\max} + \tilde{b}_{\max})$, we can conclude that $(M, g(t))_{t\in(-\infty,0]}$ has weakly PIC_1. Then by [BCW19, Lemma 4.2](see also [LN20, Proposition 6.2]), we conclude that $(M, g(t))_{t\in(-\infty,0]}$ has weakly PIC_2 and it completes the proof of Theorem 3.2.2. $\square$

Chapter 4

Proof of Main Theorems

In this chapter, we will complete the proof of three main theorems: Theorem 1.2.2, 1.2.3 and 1.2.4.

4.1 Proof of Theorem 1.2.2

First, we recall the definition of κ-solutions in [BN20, Definition 1.1].

Definition 4.1.1. [κ-solutions to the Ricci flow] For $n \geq 4$, a complete noncompact ancient solution $(M^n, g(t))_{t\in(-\infty,0]}$ to the Ricci flow is called a κ-solution if it satisfies the following conditions.

1. (Curvature condition) Uniformly PIC, weakly PIC_2, uniformly bounded curvature

2. (Noncollapsing condition) κ-noncollapsed

Note that all κ-solutions are completely classified in [BN20].

Theorem 4.1.1 (Corollary 1.6 of [BN20])**.** *Any κ-solution $(M^n, g(t))_{t\in(-\infty,0]}$ is isometric to either a family of shrinking cylinders (or a quotient thereof) or to the Bryant soliton.*

Now, we prove a canonical neighborhood theorem on Ricci flows with possibly unbounded curvature. Naively speaking, it helps us model all higher curvature region using κ-solutions, which are completely classified. To do so, we will collect some of Perelman's tools that are still valid in our setting. First, one can observe that [Per02, Corollary 11.6] can be easily generalized as follows.

Lemma 4.1.2 (Perelman)**.** *For every $w > 0$, there exist constants $C = C(w) < \infty$ and $\tau = \tau(w) > 0$ with the following properties. Let $(M^n, g(t))_{t\in[-T,0]}$ be a (possibly incomplete) Ricci flow solution with weakly* PIC_2*. Suppose $B_{g(0)}(x_0, r_0)$ is compactly contained in M such that* $\operatorname{Vol} B_{g(0)}(x_0, r_0) \geq w r_0^n$ *and $T \geq 2\tau r_0^2$. Then*

$$R(x,t) \leq C r_0^{-2}$$

for $(x,t) \in B_{g(0)}(x_0, r_0/4) \times [-\tau r_0^2, 0]$.

Proof. The proof is almost identical with the original proof of [Per02, Corollary 11.6]. The only difference is we need to replace [Per02, Proposition 11.4] to [CW15, Lemma 4.5]. □

Simply speaking, the κ-noncollapsing condition implies that one can obtain a noncollapsing ball from the curvature bound. And the above lemma gives the way to obtain the curvature bound when the ball is noncollapsing. Using this result, we are able to show a variation of canonical neighborhood theorem whose statement is given as follows.

Theorem 4.1.3. *Let $(M^n, g(t))_{t\in[0,2]}$ be a complete noncompact κ-noncollapsed Ricci flow solution with uniformly* PIC *and weakly* PIC_2*. For any $\epsilon > 0$, there exists a small number $\bar{r} > 0$ satisfying the following property.*

Suppose $(\bar{x}, \bar{t}) \in M \times [1,2]$ and $R(\bar{x}, \bar{t}) = r^{-2} \geq \bar{r}^{-2}$, then after rescaling the metric by the factor r^{-2}, the parabolic neighborhood $B_{g(\bar{t})}(\bar{x}, \epsilon^{-1} r) \times [\bar{t} - \epsilon^{-1} r^2, \bar{t}]$ is ϵ-close in $C^{[\epsilon^{-1}]}$-topology to a κ-solution.

Proof. The proof is similar to the canonical neighborhood theorem for compact Ricci flows. See [Per03b][KL+08][MT07][CZ06a][Bre19] for the original proof. The argument here is easier since we assume uniformly PIC and weakly PIC_2 conditions, which in particular implies the nonnegative Ricci curvature. We sketch the proof for the reader's convenience.

Assume that there exists an $\bar{\epsilon} > 0$ such that the conclusion does not hold for a sequence $(x_k, t_k) \in M \times [1, 2]$ with $Q_k = R(x_k, t_k) \to \infty$. By a standard point-picking argument, we can replace this sequence if necessary and assume that for any $A > 0$, the conclusion of the theorem holds for all $(y, t) \in B_{g(t_k)}(x_k, AQ_k^{-1/2}) \times [t_k - AQ_k^{-1}, t_k]$ with $R(y, t) \geq 2Q_k$. In other words, we can say that the theorem is true for all points in the parabolic neighborhood of (x_k, t_k) whose curvature is larger than the base point (x_k, t_k). Indeed, otherwise for a fixed k there exists a spacetime sequence (y_i, s_i) with $(y_0, s_0) = (x_k, t_k)$ satisfying

1. The conclusion fails.

2. $R(y_{i+1}, s_{i+1}) \geq 2R(y_i, s_i)$

3. $(y_{i+1}, s_{i+1}) \in B_{g(s_i)}(y_i, AR(y_i, s_i)^{-1/2}) \times [s_i - AR(y_i, s_i)^{-1}, s_i]$.

Also, note that the distance is nonincreasing for t as we have nonnegative Ricci curvature. Therefore, we have

$$d_{g(t_k)}(x_k, y_I) \leq \sum_{i=0}^{I-1} d_{g(s_i)}(y_i, y_{i+1}) \leq \sum_{i=0}^{I-1} AR(y_i, s_i)^{-1/2} \leq 4AR(y_0, s_0)^{-1/2},$$
$$t_k - s_I = \sum_{i=0}^{I-1}(s_i - s_{i+1}) \leq \sum_{i=0}^{I-1} AR(y_i, s_i)^{-1} \leq 2AR(y_0, s_0)^{-1/2}.$$

Therefore, the process must be terminated in finitely many steps, which is a contradiction. By taking a diagonal sequence, we consider the spacetime limit of $(M, g_k(t), x_k)$ for $t \leq 0$, where $g_k(t) = Q_k g(Q_k^{-1} t + t_k)$. To derive a contradiction, we only need to prove the limit is a κ-solution.

Step 1: We claim there exists a sequence $H_k \to \infty$ and constants $\eta_m > 0, c > 0$ satisfying the following. For any $(\bar{x}, \bar{t}) \in B_{g(t_k)}(x_k, H_k Q_k^{-1/2}) \times [t_k - H_k Q_k^{-1}, t_k]$, if we set $\bar{Q} = Q_k + R(\bar{x}, \bar{t})$, then on the parabolic neighborhood $P = B_{g(\bar{t})}(\bar{x}, c\bar{Q}^{-1/2}) \times [\bar{t} - c\bar{Q}^{-1}, \bar{t}]$ we have

$$|\nabla^m R| \leq \eta_m \bar{Q}^{\frac{m}{2}+1} \tag{4.1.1}$$

for $m \geq 0$. Indeed, if $\bar{Q} \geq 3Q_k$, then by our assumption $(\bar{x}, \bar{t})$ has a canonical neighborhood and hence $|\partial_t R^{-1}| + |\nabla R^{-1/2}| \leq C$. Therefore, the local geometry around $(\bar{x}, \bar{t})$ is well-controlled. Moreover, the higher curvature estimates follow from Shi's local estimates. For more details, see [CZ06a, Theorem 4.1, Step 1] or [KL+08, Lemma 52.11].

Step 2: Next, we prove that $(M, g_k(0), x_k)$ converges smoothly to a complete smooth Riemannian manifold $(M_\infty, g_\infty, x_\infty)$. We have $R \leq 2\eta_0 Q_k$ on a parabolic neighborhood of (x_k, t_k) if we take $(\bar{x}, \bar{t}) = (x_k, t_k)$ in Step 1. Therefore, there exist constants $c_1 > 0, C_1 > 0$ such that $R_{g_k(0)} \leq C_1$ on $B_{g_k(0)}(x_k, c_1)$. From the κ-noncollapsing condition, it implies that $\operatorname{Vol} B_{g_k(0)}(x_k, 1) \geq v_0 > 0$. From the Bishop-Gromov volume comparison theorem, for any $L > 0$ and $y \in B_{g_k(0)}(x_k, L)$, we have

$$\operatorname{Vol} B_{g_k(0)}(y, 1) \geq \frac{\operatorname{Vol} B_{g_k(0)}(y, L+1)}{(L+1)^n} \geq \frac{\operatorname{Vol} B_{g_k(0)}(x_k, 1)}{(L+1)^n} \geq \frac{v_0}{(L+1)^n} =: v_1$$

Therefore, it follows from Lemma 4.1.2 that there exist $C_2 = C_2(v_1) > 0$ and $\tau = \tau(v_1) > 0$ such that

$$R(x, t) \leq C_2 \tag{4.1.2}$$

for $(x, t) \in B_{g_k(0)}(y, 1/4) \times [-\tau, 0]$. Combining (4.1.1) and (4.1.2), one easily concludes that the limit $(M_\infty, g_\infty, x_\infty)$ of a sequence $(M, g_k(0), x_k)$ is a complete smooth Riemannian manifold, which has uniformly PIC_1, weakly PIC_2, and is κ-noncollapsed.

Step 3: Next, we show that the curvature of the limit $(M_\infty, g_\infty, x_\infty)$ must be uniformly bounded. From the proof of Step 2, we can assume $(M_\infty, g_\infty, x_\infty)$ is defined on a spacetime open neighborhood of $M_\infty \times (-\infty, 0]$ which contains $M_\infty \times \{0\}$. If the curvature operator of $g_\infty(0)$ lies on the boundary of PIC_2 cone somewhere, then it follows from [Bre19, Proposition 6.6] that the universal covering $(\tilde{M}_\infty, g_\infty(0))$ is isometric to $(N \times \mathbb{R}, g_1 \times g_E)$, where (N, g_1) is a complete Riemannian manifold with uniformly PIC_1 and weakly PIC_2. We claim that N has bounded curvature. Otherwise there exists a sequence $q_k \in N$ such that $R_{g_1}(q_k) \to \infty$. By our assumption of the canonical neighborhood, (N, g_1, q_k) is $2\bar{\epsilon}$-close to the standard S^{n-1}/Γ and hence N is compact, which is a contradiction.

Therefore, we may assume (M_∞, g_∞) has strictly PIC_2. If the curvature is not bounded, we can choose $z_k \in M_\infty$ such that $R_{g_\infty}(z_k) \to \infty$. Since M_∞ has a positive sectional curvature, we know that $(M_\infty, R(z_k)g_\infty, z_k)$ subconverges to the standard $(S^{n-1}/\Gamma) \times \mathbb{R}$. In addition, it follows from [Bre18, Proposition A.2] that $\Gamma = \{1\}$. However, it contradicts [CZ06a, Proposition 2.2].

Step 4: Now, $(M_\infty, g_\infty(t))$ can be extended backwards to an ancient solution with uniformly bounded curvature. The proof of this claim is given similarly to [CZ06a, Theorem 4.1, Step 4]. See also [KL+08, Step 4 in Page 2705] and [Bre19, Theorem 7.2, Step 6]. In sum, we have proved that $(M_\infty, g_\infty(t))_{t\in(-\infty,0]}$ is a κ-solution, which contradicts our assumptions on x_k. $\square$

Next, we prove the following lemma, see also [Che09, Theorem 3.6].

Lemma 4.1.4. *Let $(M^n, g(t))_{t\in[0,T]}$ be a complete, κ-noncollapsed solution of the Ricci flow with weakly PIC_2 and $(M, g(t))$ has bounded curvature for each $t \in [0, T]$. Then $(M, g(t))_{t\in[0,T]}$ has uniformly bounded curvature.*

Proof. We claim that for any $t_0 \in [0, T]$, there exists an $\epsilon > 0$ such that R is uniformly bounded on $M \times [t_0, t_0 + \epsilon)$. Indeed, since $(M, g(t_0))$ has bounded curvature and is κ-noncollapsed, it clear that there exists a $v_0 > 0$ such that $\mathrm{Vol}\, B_{g(t_0)}(x, 1) \geq v_0$ for any $x \in M$. Therefore by [CW15, Corollary 1.3], there exists an $\epsilon > 0$ such that $R \leq C(t - t_0)^{-1}$ for $t \in [t_0, t_0 + \epsilon)$. Now the claim follows from [Che09, Theorem 3.1].

Next we define $I := \{t \in [0, T] \mid \sup_{M\times[0,t]} R < \infty\}$. It is clear from the claim that I is open and nonempty. On the other hand, for any $t_i \in I$ such that $\lim_{i\to\infty} t_i = \bar{t}$, we know from our definition that $(M, g(t))_{t\in[0,\bar{t})}$ has bounded curvature in any compact time interval. Therefore, the trace Harnack inequality holds, see [Bre09]. In particular, R is uniformly bounded on $M \times [0, \bar{t}]$ since

$$tR(x,t) \leq \bar{t}\left(\sup_{M\times\{\bar{t}\}} R\right)$$

for any $x \in M$ and $t \leq \bar{t}$. Since I is both open and closed, $T \in I$ and the proof is complete. $\square$

Now, we prove the main result of the section, which states that the assumption of the trace Harnack inequality in [Bre19, Proposition 6.11] can be dropped.

Proposition 4.1.5. *Let* $(M^n, g(t))_{t\in(-\infty,0]}$ *be an* n*-dimensional,* κ*-noncollapsed, noncompact complete ancient solution to the Ricci flow with uniformly* PIC *and weakly* PIC_2*. Then the curvature of* $(M, g(t))_{t\in(-\infty,0]}$ *is uniformly bounded. In particular, under the same assumption, the trace Harnack inequality holds for* $(M, g(t))_{t\in(-\infty,0]}$*.*

Proof. If there exists $(x_0, t_0) \in M \times (-\infty, 0]$ such that the curvature operator $\mathrm{Rm}(x_0, t_0)$ lies on the boundary of PIC_2 cone, then the universal cover $\tilde{M}$ of M splits off a line by [Bre19, Proposition 6.6]. We assume $\tilde{M}$ is isometric to $N \times \mathbb{R}$ where $(N, g_1(t))_{t\in(-\infty,0]}$ is $(n-1)$-dimensional, κ-noncollapsed, complete ancient solution to the Ricci flow with uniformly PIC_1, see [Bre10a, Proposition 7.14]. Therefore, it follows from [Bre19, Theorem 6.4] (see also [Yok17]) that N is homothetic to S^{n-1}. In this case, the conclusion is obviously true.

Therefore, we may assume that M has strictly PIC_2. From the Lemma 4.1.4, we only need to prove the curvature is bounded for each time slice. Fix a $t_0 \leq 0$, if the curvature at t_0 is unbounded, there exists a sequence p_i with $Q_i = R(p_i, t_0) \to \infty$. By applying Theorem 4.1.3 on $M \times [t_0 - 2, t_0]$, we conclude that $(M, Q_i g(t_0), p_i)$ converges smoothly to a κ-solution. Since the limit must contain a splitting direction, by Toponogov's splitting theorem, we may assume the limit is the standard $(S^{n-1}/\Gamma) \times \mathbb{R}$. By our assumption, it follows from [Bre18, Proposition A.2] that $\Gamma = \{1\}$. However, we obtain a contradiction by [CZ06a, Proposition 2.2]. Together with Lemma 4.1.4, it implies that $(M^n, g(t))_{t\in(-\infty,0]}$ has bounded curvature on each compact time interval and hence the trace Harnack inequality holds, see [Bre09]. Therefore, the curvature is uniformly bounded since R is nondecreasing in t. $\square$

By combining Theorem 4.1.1 and Proposition 4.1.5, we obtain the following theorem.

Theorem 4.1.6. *Suppose* $(M^n, g(t))_{t\in(-\infty,0]}$ *is a complete, noncompact,* κ*-noncollapsed ancient solution to the Ricci flow with uniformly* PIC *and weakly* PIC_2*. Then it is isometric to either a family of shrinking cylinders (or a quotient thereof) or to the Bryant soliton.*

Remark 4.1.7. The conclusion of Theorem 4.1.6 also holds for any 3-dimensional, κ-noncollapsed, noncompact complete ancient solution to the Ricci flow, based on Brendle's breakthrough *[Bre20, Theorem* 1.3*]*. On the other hand, it is not clear if there is any 3-dimensional noncompact complete ancient solution to the Ricci flow which has unbounded curvature. It is worth noting that the counterexample is given in *[CW15, Theorem* 1.4*]* for immortal solutions with nonnegative curvature operator and unbounded curvature.

Proof of Theorem 1.2.2: Theorem 1.2.2 follows immediately from Theorem 4.1.6 and the curvature improvements obtained from Theorem 3.2.2 and Lemma 3.1.4.

4.2 Proof of Theorem 1.2.3

4.2.1 κ-solutions to the Kähler Ricci flow

We first recall the following definition of κ-solutions in the Kähler setting.

Definition 4.2.1. [κ-solutions to the Kähler Ricci flow] A complete ancient solution $(M^n, g(t))_{t\in(-\infty,0]}$ to the Kähler Ricci flow is called a κ-solution if it satisfies the following conditions.

1. (Curvature condition) Nonnegative bisectional curavture, uniformly bounded curvature
2. (Noncollapsing condition) κ-noncollapsed

Now, we will show that any κ-solution must be of Type I. See also [DZ20, Lemma 2.5].

Lemma 4.2.1. *For any κ-solution $(M^n, g(t))_{t\in(-\infty,0)}$ such that $t = 0$ is the singular time, there exists a constant $C_0 > 0$ such that $|t||R(x,t)| \le C_0$ for all $(x,t) \in M \times (-\infty, 0)$.*

Proof. Suppose that it is not true. We take $T_i \to -\infty$ and $\epsilon_i \to 0^-$ and choose $(x_i, t_i) \in M \times (T_i, \epsilon_i)$ so that

$$(\epsilon_i - t_i)(t_i - T_i)R(x_i, t_i) = (1-\delta_i) \sup_{M\times[T_i,\epsilon_i]} (\epsilon_i - t)(t - T_i)R(x,t),$$

where $\delta_i \to 0$. Then with a rescaled metric $\tilde{g}_i(t) := R_i g(t_i + Q_i^{-1}t)$ where $Q_i := R(x_i, t_i)$, we get

$$\begin{aligned}\tilde{R}_i(x,t) = Q_i^{-1}R(x, t_i + Q_i^{-1}t) &\le Q_i^{-1}\frac{(\epsilon_i - t_i)(t_i - T_i)Q_i}{(1-\delta_i)(\epsilon_i - t_i - Q_i^{-1}t)(t_i + Q_i^{-1}t - T_i)} \\ &= \frac{(\epsilon_i - t_i)(t_i - T_i)Q_i^2}{(1-\delta_i)((\epsilon_i - t_i)Q_i - t)((t_i - T_i)Q_i + t)} \\ &= \frac{1}{1-\delta_i}\cdot\frac{(\epsilon_i - t_i)Q_i}{(\epsilon_i - t_i)Q_i + t}\cdot\frac{(t_i - T_i)Q_i}{(t_i - T_i)Q_i + t}\end{aligned}$$

for all $x \in M$ and $t \in (Q_i(T_i - t_i), Q_i(\epsilon_i - t_i))$. In particular, we have $\tilde{R}_i(x, 0) \le (1-\delta_i)^{-1}$ and $\tilde{R}_i(x_i, 0) = 1$. If we set $\alpha_i : Q_i(\epsilon_i - t_i)$ and $\beta_i := Q_i(T_i - t_i)$, then we know $\beta_i \le 0 \le \alpha_i$ and

$$\begin{aligned}\frac{1}{\alpha_i^{-1} - \beta_i^{-1}} = \frac{(\epsilon_i - t_i)(t_i - T_i)Q_i}{\epsilon_i - T_i} &\ge \frac{1}{2(\epsilon_i - T_i)}\sup_{M\times[T_i,\epsilon_i]}(\epsilon_i - t)(t - T_i)R(x,t) \\ &\ge \frac{1}{4}\sup_{M\times[T_i/2, 2\epsilon_i]}(\epsilon_i - t)R(x,t) \\ &\ge \frac{1}{8}\sup_{M\times[T_i/2, 2\epsilon_i]}|t|R(x,t) \to \infty\end{aligned}$$

because of the assumption. It implies that $\alpha_i \to \infty$ and $\beta_i \to -\infty$. From the previous curvature bound and a κ-noncollapsing property, the sequence of pointed manifolds $(M, \tilde{g}_i(t), x_i)_{t\in(\beta_i,\alpha_i)}$ smoothly subconverges to $(M_\infty, g_\infty(t), x_\infty)_{t\in(-\infty,\infty)}$ which is a nonflat κ-noncollapsed eternal solution. In particular, this limit has a nonnegative bisectional curvature which is bounded in any compact time interval. Also, since $\delta_i \searrow 0$, we get $R_\infty(x, 0) \le 1$ for all $x \in M_\infty$ and $R_\infty(x_\infty, 0) = 1$. After considering the universal cover of M_∞ and Cao's dimension reduction argument [Cao04, Theorem 2.1], we may assume that M_∞ is simply connected and has positive Ricci curvature. By using the result in [Cao97, Theorem 1.3], we conclude that (M_∞, g_∞) is a nonflat, κ-noncollapsed Kähler Ricci steady soliton with nonnegative bisectional curvature. However, such a Kähler steady soliton does not exist by [DZ19, Theorem 1.2]. $\square$

Notice that the Kähler Ricci shrinker with nonnegative bisectional curvature is an important type of κ-solution. It is notable that their classification is given as follows.

Theorem 4.2.2 (Theorem 3 of [Ni05]). *Let (M^n, g, f) be a Kähler Ricci shrinker with nonnegative bisectional curvature. Then (M, g) is isometrically biholomorphic to a quotient of $N^k \times \mathbb{C}^{n-k}$, where N is a compact Hermitian symmetric spaces.*

Also, all compact κ-solutions are completely classified.

Theorem 4.2.3 (Proposition 2.8 of [DZ20]). *Let $(M, g(t))_{t\in(-\infty,0]}$ be a compact κ-solution. Then it must be isometrically-biholomorphic to a quotient of compact Hermitian symmetric space.*

In particular, this result implies that any compact, complex 2-dimensional κ-solution has to be isometrically-biholomorphic to either $\mathbb{CP}^2$ or $\mathbb{CP}^1 \times \mathbb{CP}^1$, up to scalings on each factor. From now on, we will consider the noncompact case. First, we prove the following which implies that there is no κ-solution given as a nontrivial quotient of $\mathbb{C} \times \mathbb{CP}^1$.

Lemma 4.2.4. *Let $(M^2, g(t))$ be a κ-solution whose universal cover is $\mathbb{C} \times \mathbb{CP}^1$. Then it is isometrically-biholomorphic to $\mathbb{C} \times \mathbb{CP}^1$ itself.*

Proof. First, we investigate an isometry of $\mathbb{C} \times \mathbb{CP}^1$. Let $\Phi : \mathbb{C} \times \mathbb{CP}^1 \to \mathbb{C} \times \mathbb{CP}^1$ and v_1 is a vector field tangent to $\mathbb{C}$. Then $\Phi_*(v_1)$ is also tangent to $\mathbb{C}$ since it is parallel. Therefore, Φ preserves the product structure and we can write $\Phi = (\Phi_1, \Phi_2)$ where $\Phi_1 : \mathbb{C} \to \mathbb{C}$ and $\Phi_2 : \mathbb{CP}^1 \to \mathbb{CP}^1$ are isometries. Now, let $f : \mathbb{C} \times \mathbb{CP}^1 \to \mathbb{C} \times \mathbb{CP}^1$ be an isometry corresponding to the covering map $\mathbb{C} \times \mathbb{CP}^1 \to M$. If f is not trivial, we can write $f = (f_1, f_2)$ where f_1 is an orientation preserving rigid motion on $\mathbb{R}^2$ and $f_2 \in \mathrm{SO}(3)$. In particular, we know that $f_2 \in \mathrm{SO}(3)$ has a fixed point. Therefore, f_1 has no fixed point and hence generates a infinite group. Now, we will show that M has no maximal volume growth. Once this result is proven, then we get a contradiction since $(M, g(t))_{t\in(-\infty,0)}$ is κ-noncollapsed. For the result, it is enough to show the following claim.

Claim. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be an orientation preserving rigid motion with no fixed point. Then the quotient space $\mathbb{R}^n/\langle f \rangle$ has no maximal volume growth, i.e., we have

$$\lim_{r\to\infty} \frac{\mathrm{Vol}(B(x,r))}{r^n} = 0$$

This claim can be proven as follows. We know that f has an infinite order since otherwise the center of mass of any orbit must be a fixed point. Let $f(x) = Ax + b$ for $A \in \mathrm{SO}(n)$ and $b \in \mathbb{R}^n$ and consider the following function $r(x) := |f(x) - x|^2 = |(A-I)x + b|^2$. Then we can check that for $x = tv$ with a unit vector v, $r(tv) \to \infty$ as $t \to \infty$ unless $(A - I)v = 0$. In this exceptional case, we get $r(tv) = |b|^2$ for all t. Therefore, we can conclude that there exists $p \in \mathbb{R}^n$ such that $r(p) = \min_{x\in\mathbb{R}^n} r(x)$. Note that $r(p) \neq 0$ since otherwise p is a fixed point of f. Now let H be a hyperplane in $\mathbb{R}^n$ passing through p with a normal vector $f(p) - p$. Since p is a critical point of $r(x)$, for any $V \in \mathbb{R}^n$, we have $Dr_p(V) = 2\langle f(p) - p, Df_p(V) - V\rangle = 0$. Moreover, for any tangent vector V on H at p, we have $\langle f(p) - p, V\rangle = 0$ so we finally have $\langle f(p) - p, Df_p(V)\rangle = 0$ which implies that $f(H)$ is parallel to H. Let $S^i \subset \mathbb{R}^n$ be a slit domain bounded by $f^i(H)$ and $f^{i+1}(H)$. Then $\{S^i\}_{i\in\mathbb{Z}}$ forms a partition of $\mathbb{R}^n$ and moreover we can take a fundamental domain of $\mathbb{R}^n/\langle f \rangle$ contained in S^0. After choosing $r \geq r(p)$, we eventually get

$$\frac{\mathrm{Vol}(B(x,r))}{r^n} \leq \frac{Cr(p)r^{n-1}}{r^n} \to 0$$

as $r \to \infty$. It completes the proof of the claim so the proof of Lemma 4.2.4 is also obtained. $\square$

Now, we will recall the properties of Type I ancient solution for later applications.

Theorem 4.2.5. *Let $(M, g(-\tau))_{\tau\in(0,\infty)}$ be a κ-solution and $R(x, -\tau) < \frac{C_0}{\tau}$ for any $x \in M$ and $\tau \in (0, \infty)$. Then we have the following.*

(i) For any $0 < \tau_1 < \tau_2$ and $p, q \in M$, we have

$$d_{g(-\tau_2)}(p, q) - 8(n-1)C_0(\sqrt{\tau_2} - \sqrt{\tau_1}) \leq d_{g(-\tau_1)}(p, q) \leq d_{g(-\tau_2)}(p, q).$$

(ii) For any $p, q \in M$, if we define a reduced distance $l(q, \tau)$ based on $(p, 0)$, then it satisfies

$$\frac{1}{4\sqrt{3}}\left(\frac{d_{g(-\tau)}(p, q)}{\sqrt{\tau}} - 8(n-1)C_0\right) \leq \sqrt{l}(q, \tau) \leq \frac{d_{g(-\tau)}(p, q)}{\sqrt{\tau}} + \sqrt{n(n-1)C_0}$$

(iii) For $g_i(-\tau) = \tau_i^{-1} g(-\tau_i \tau)$ with $\tau_i \to \infty$, the sequence of pointed manifolds $(M, g_i(-\tau), p_i)_{\tau \in (0,\infty)}$ subsequentially converges to $(M_\infty, g_\infty(-\tau), p_\infty)_{\tau \in (0,\infty)}$ in Cheeger-Gromov sense.

(iv) For a reduced distance function $l_i(q,\tau) = l^{g_i}_{(p_i,0)}(q,\tau) = l^{g}_{(p_i,0)}(q, \tau_i \tau)$ defined on M, we have $l_i(q,\tau) \to l_\infty(q,\tau)$ on $M_\infty \times (0,\infty)$ in the Cheeger-Gromov convergence.

(v) If we define

$$\hat{V}_\infty(\tau) = \int_{M_\infty} \tau^{-n/2} e^{-l_\infty(q,\tau)} dv_{g_\infty(-\tau)}$$

for $\tau \in (0,\infty)$, then this quantity is given as a limit of the sequence of reduced volumes with respect to g_i. In other words, for $\tilde{V}_i(\tau) = \tilde{V}^{g_i}_{(p_i,0)}(\tau) = \tilde{V}^{g}_{(p_i,0)}(\tau_i \tau)$, we have $\lim_{i\to\infty} \tilde{V}_i(\tau) = \hat{V}_\infty(\tau)$ for all $\tau \in (0,\infty)$. Moreover, when $p_i = p$ for all i, then $\hat{V}_\infty(\tau)$ is a constant which implies that the limit is a Ricci shrinker.

(vi) When $p_i = p$ for all i, then the limit doesn't depend on the choice of p. In particular, for any $p \in M$ and $\tau_i \to \infty$, we have $(M, g_i(-\tau), p) \to (M_\infty, g_\infty(-\tau), p_\infty)_{\tau \in (0,\infty)}$ in Cheeger-Gromov sense with a limit which is a non-flat Ricci shrinker.

Proof. **(i)** The second inequality is given from the fact that the solution has nonnegative Ricci curvature. Also, the first inequality is given from the time derivative estimate of the distance function in [Per02, Lemma 8.3].

(ii) From [Per02], we know the following estimate

$$|\nabla l|^2(x,\tau) \leq \frac{3}{\tau} l(x,\tau)$$

which is equivalent to

$$|\nabla \sqrt{l}|(x,\tau) \leq \sqrt{\frac{3}{4\tau}} \leq \frac{1}{\sqrt{\tau}}$$

By integrating both sides along the geodesic connecting p and q, we have

$$|\sqrt{l}(q,\tau) - \sqrt{l}(p,\tau)| \leq \frac{d_{g(-\tau)}(p,q)}{\sqrt{\tau}}$$

Let $\gamma : [0,\tau] \to M$ be a constant path so that $\gamma(\eta) = p$ for all $\eta \in [0,\tau]$. Then we can estimate $l(p,\tau)$ using γ and type I condition by

$$l(p,\tau) \leq \frac{1}{2\sqrt{\tau}}\mathcal{L}(\gamma) = \frac{1}{2\sqrt{\tau}}\int_0^\tau \sqrt{\eta}R(p,\eta)d\eta \leq \frac{1}{2\sqrt{\tau}}\int_0^\tau \frac{n(n-1)C_0}{\sqrt{\eta}}d\eta = n(n-1)C_0$$

so we can get

$$\sqrt{l}(q,\tau) \leq \frac{d_{g(-\tau)}(p,q)}{\sqrt{\tau}} + \sqrt{n(n-1)C_0}$$

For the other direction, let $\tilde{\gamma} : [0,\tau] \to M$ be a $\mathcal{L}$-minimizing geodesic connecting p and q. Then we have

$$d_{g(-\tau)}(p,q) = \int_0^\tau \frac{d}{d\eta}d(p,\tilde{\gamma}(\eta),\eta)d\eta = \int_0^\tau \left(\nabla d \cdot \tilde{\gamma}'(\eta) + \frac{\partial}{\partial\eta}d(p,\tilde{\gamma}(\eta),\eta)\right)d\eta$$

Again, from [Per02], we know

$$|\tilde{\gamma}'(\eta)| = \left|\frac{1}{2\sqrt{\eta}}\nabla L(\tilde{\gamma}(\eta),\eta)\right| = |\nabla l(\tilde{\gamma}(\eta),\eta)|$$

Using the similar argument, we can estimate the right hand side as follows.

$$|\nabla l(\tilde{\gamma}(\eta),\eta)| \leq \sqrt{\frac{3l(\tilde{\gamma}(\eta),\eta)}{\eta}} = \sqrt{\frac{3L(\tilde{\gamma}(\eta),\eta)}{2\eta^{3/2}}} \leq \sqrt{\frac{3L(q,\tau)}{2\eta^{3/2}}} = \sqrt{3}\tau^{1/4}\eta^{-3/4}\sqrt{l}(q,\tau)$$

We also get the following due to the distance estimate for type I condition.

$$\frac{\partial}{\partial\eta}d \leq \frac{4(n-1)C_0}{\sqrt{\eta}}$$

Combining all these together, we can derive

$$d_{g(-\tau)}(p,q) \leq \int_0^\tau \left(\sqrt{3}\tau^{1/4}\eta^{-3/4}\sqrt{l}(q,\tau) + \frac{4(n-1)C_0}{\sqrt{\eta}}\right)d\eta = \sqrt{\tau}(4\sqrt{3}\sqrt{l}(q,\tau) + 8(n-1)C_0)$$

which completes the proof.

(iii) From the Type I condition, we have

$$R_{g_i}(x,-\tau) = \tau_i R(x,-\tau_i\tau) \leq \tau_i\frac{n(n-1)C_0}{\tau_i\tau} = \frac{n(n-1)C_0}{\tau}$$

on $M \times (0,\infty)$ for all i. With κ-noncollapsing property, it completes the proof.

(iv) Now, for any i and $\tau \in [A^{-1}, A]$ and $x \in B_{g_i(-1)}(p_i, r) = B_{g(-\tau_i)}(p_i, r\sqrt{\tau_i})$ for some $r > 0$, we can apply (i) to obtain the following.

$$\begin{aligned}
\frac{d_{g(-\tau_i\tau)}(p_i, x)}{\sqrt{\tau_i\tau}} &\leq \frac{1}{\sqrt{\tau_i\tau}}\left(d_{g(-\tau_i)}(p_i, x) + 8(n-1)C_0\sqrt{\tau_i}(\sqrt{\tau}-1)^+\right) \\
&\leq \frac{1}{\sqrt{\tau}}\left(r + 8(n-1)C_0(\sqrt{\tau}-1)^+\right) \\
&\leq C(r, A, n, C_0)
\end{aligned}$$

Hence, from (ii), we get

$$0 \leq l_i(x, \tau) = l_{(p_i,0)}(q, \tau_i\tau) \leq C_1(r, A, n, C_0)$$

for all i. Also, from [Per02], l_i satisfies

$$\begin{aligned}
&|\nabla_{g_i(-\tau)} l_i(x,\tau)|^2_{g_i(-\tau)} \leq |\nabla_{g_i(-\tau)} l_i(x,\tau)|^2_{g_i(-\tau)} + R_{g_i}(x, -\tau) \leq \frac{3l_i(x,\tau)}{\tau} \leq C_2(r, A, n, C_0) \\
&\left|\frac{\partial l_i}{\partial \tau}\right| \leq \frac{2l_i(x,\tau)}{\tau} \leq C_3(r, A, n, C_0)
\end{aligned}$$

Now, let $\Phi_i : U_i \subset M_\infty \to \Phi_i(U_i) \subset M$ be diffeomorphisms yielding the convergence $(M, g_i(-\tau), p_i) \to (M_\infty, g_\infty(-\tau), p_\infty)$. Then for any $0 < r_0 < r$, we can take i large enough so that we have a sequence of uniformly Lipschitz functions $\Phi_i^* l_i$ on $B_{g_\infty(-1)}(p_\infty, r_0)$. Since $r > 0$ is chosen arbitrarily, we can see that $l_i(q,\tau) \to l_\infty(q,\tau)$ on $M_\infty \times (0,\infty)$ in the Cheeger-Gromov convergence by the Arzela-Ascoli theorem and a diagonalization argument. Since $l_\infty(q,\tau)$ has to be a locally Lipschitz function on $M_\infty \times (0,\infty)$, we also know that $\nabla_{g_\infty(-\tau)} l_\infty(x,\tau)$ and $\frac{\partial l_\infty}{\partial \tau}(x,\tau)$ exist a.e. on $M_\infty \times (0,\infty)$.

(v) From the monotonicity formula from [Per02], we can show the integrand of $\tilde{V}_i(\tau)$ is always bounded by the integrable function in any compact interval in $(0,\infty)$. Now the result is obtained from the Lebesgue's dominated convergence theorem. For the case when $p_i = p$ for all i, from the monotonicity we know that $\lim_{i\to\infty} \tilde{V}_i(\tau)$ doesn't depend on the choice of τ since $\tau_i \to \infty$. Therefore, $\hat{V}_\infty(\tau)$ is a constant and in particular, this value doesn't depend on the choice of τ_i as long as $\tau_i \to \infty$.

Now, we know that

$$\frac{\partial l_i}{\partial \tau} - \Delta_{g_i(-\tau)} l_i + |\nabla_{g_i(-\tau)} l_i|^2_{g_i(-\tau)} - R_{g_i}(-\tau) + \frac{n}{2\tau} \geq 0$$

in the weak sense. i.e. for any nonnegative Lipschitz function ϕ compactly supported on $M \times [\tau_1, \tau_2]$ with $0 < \tau_1 < \tau_2$, we have

$$\int_{\tau_1}^{\tau_2} \int_M \left(\nabla_{g_i(-\tau)} l_i \cdot \nabla_{g_i(-\tau)} \phi + \left(\frac{\partial l_i}{\partial \tau} + |\nabla_{g_i(-\tau)} l_i|^2_{g_i(-\tau)} - R_{g_i}(-\tau) + \frac{n}{2\tau} \right) \phi \right) \tau^{-n/2} dv_{g_i(-\tau)} d\tau \geq 0$$

Again, from the Lebesgue's dominated convergence theorem, we have

$$\frac{\partial l_\infty}{\partial \tau} - \Delta_{g_\infty(-\tau)} l_\infty + |\nabla_{g_\infty(-\tau)} l_\infty|^2_{g_\infty(-\tau)} - R_{g_\infty}(-\tau) + \frac{n}{2\tau} \geq 0$$

in the weak sense. Now we want to show that it is still true even though we take the test function as e^{-l_∞}. Since the result (ii) is invariant under the parabolic rescaling, we have

$$\frac{1}{4\sqrt{3}} \left(\frac{d_{g_i(-\tau)}(p_i, q)}{\sqrt{\tau}} - 8(n-1)C_0 \right) \leq \sqrt{l_i}(q, \tau) \leq \frac{d_{g_i(-\tau)}(p_i, q)}{\sqrt{\tau}} + \sqrt{n(n-1)C_0}$$

Now, let $\phi_k = \psi_k e^{-l_\infty}$ where ψ_k is a Lipschitz functions compactly supported on $B_{g_\infty(-\tau)}(p_\infty, k+1)$ so that $0 \leq \psi_k \leq 1$, $|\nabla \psi_k| \leq 1$ and $\psi_k \equiv 1$ on $B_{g_\infty(-\tau)}(p_\infty, k)$. Using ϕ_k as a test function, we have

$$0 \leq \int_{\tau_1}^{\tau_2} \int_{B_{g_\infty(-\tau)}(p_\infty, k)} \left(\frac{\partial l_\infty}{\partial \tau} - R_{g_\infty}(-\tau) + \frac{n}{2\tau} \right) e^{-l_\infty} \tau^{-n/2} dv_{g_\infty(-\tau)} d\tau + A + B$$

where

$$A = \int_{\tau_1}^{\tau_2} \int_{B_{g_\infty(-\tau)}(p_\infty, k+1) \setminus B_{g_\infty(-\tau)}(p_\infty, k)} \left(\frac{\partial l_\infty}{\partial \tau} - R_{g_\infty}(-\tau) + \frac{n}{2\tau} \right) \psi_k e^{-l_\infty} \tau^{-n/2} dv_{g_\infty(-\tau)} d\tau$$

$$B = \int_{\tau_1}^{\tau_2} \int_{B_{g_\infty(-\tau)}(p_\infty, k+1) \setminus B_{g_\infty(-\tau)}(p_\infty, k)} e^{-l_\infty} \nabla_{g_\infty(-\tau)} l_\infty \cdot \nabla_{g_\infty(-\tau)} \psi_k \tau^{-n/2} dv_{g_\infty(-\tau)} d\tau$$

By taking k large enough, from (i) we have a quadratic bound of l_∞ by

$$C_1 \frac{d_{g_\infty(-\tau)}(p_\infty, q)^2}{\tau} \leq l_\infty(q, \tau) \leq C_2 \frac{d_{g_\infty(-\tau)}(p_\infty, q)^2}{\tau}$$

with a fixed constant $C_1, C_2 > 0$. Since the Ricci curvature is nonnegative in our case, we can use the volume comparison theorem between M_∞ and the flat Euclidean space $(\mathbb{R}^n, h)$.

Using Type I condition, we get

$$\begin{aligned}
&\left|\int_{\tau_1}^{\tau_2}\int_{B_{g_\infty(-\tau)}(p_\infty,k+1)\backslash B_{g_\infty(-\tau)}(p_\infty,k)}\left(-R_{g_\infty}(-\tau)+\frac{n}{2\tau}\right)e^{-l_\infty}\tau^{-n/2}dv_{g_\infty(-\tau)}d\tau\right| \\
&\le \int_{\tau_1}^{\tau_2}\int_{B_{g_\infty(-\tau)}(p_\infty,k+1)\backslash B_{g_\infty(-\tau)}(p_\infty,k)}\frac{C_3}{\tau^{n/2+1}}e^{-C_1\frac{d_{g_\infty(-\tau)}(p_\infty,q)^2}{\tau}}dv_{g_\infty(-\tau)}d\tau \\
&\le \int_{\tau_1}^{\tau_2}\int_{B_h(O,k+1)\backslash B_h(O,k)}\frac{C_3}{\tau^{n/2+1}}e^{-C_1\frac{d_h(O,q)^2}{\tau}}dv_h d\tau \\
&\le C_4\int_{\tau_1}^{\tau_2}\int_k^{k+1}\frac{e^{-C_1\frac{r^2}{\tau}}}{\tau^{n/2+1}}rdrd\tau \le C_5e^{-k}
\end{aligned}$$

Also, since we have

$$\left|\frac{\partial l_\infty}{\partial\tau}\right| \le \frac{2l_\infty}{\tau} \le 2C_2\left(\frac{d_{g_\infty(-\tau)}(p_\infty,q)}{\tau}\right)^2$$

and

$$|\nabla_{g_\infty(-\tau)}l_\infty| \le \sqrt{\frac{3l_\infty}{\tau}} \le \sqrt{3C_2}\frac{d_{g_\infty(-\tau)}(p_\infty,q)}{\tau}$$

we can do the similar computation to conclude that $A, B \to 0$ as $k \to \infty$. Consequently, we get

$$\int_{\tau_1}^{\tau_2}\int_{M_\infty}\left(\frac{\partial l_\infty}{\partial\tau}-R_{g_\infty}(-\tau)+\frac{n}{2\tau}\right)e^{-l_\infty}\tau^{-n/2}dv_{g_\infty(-\tau)}d\tau \ge 0$$

Now, observe that

$$\frac{\partial}{\partial\tau}\hat{V}_\infty(\tau) = -\int_{M_\infty}\left(\frac{\partial l_\infty}{\partial\tau}-R_{g_\infty}(-\tau)+\frac{n}{2\tau}\right)e^{-l_\infty}\tau^{-n/2}dv_{g_\infty(-\tau)}d\tau$$

Since we know that $\hat{V}_\infty(\tau)$ is constant, we can conclude that

$$\frac{\partial l_\infty}{\partial\tau}-R_{g_\infty}(-\tau)+\frac{n}{2\tau}=0$$

Using the similar argument, we can also get

$$2\Delta_{g_\infty(-\tau)}l_\infty-|\nabla_{g_\infty(-\tau)}l_\infty|^2+R_{g_\infty(-\tau)}+\frac{l_\infty-n}{\tau}=0$$

Now, from the direct computation given in [Per02], if we set $u=\tau^{-n/2}e^{-l_\infty}$, we can see that u satisfies $\Box^*u=(\partial_\tau-\Delta+R)u=0$. Then for

$$v=(\tau(2\Delta_{g_\infty(-\tau)}l_\infty-|\nabla_{g_\infty(-\tau)}l_\infty|^2+R_{g_\infty(-\tau)})+l_\infty-n)u$$

we already proved that $v = 0$ so obviously $\Box^* v = 0$. But it implies that l_∞ should satisfy $\mathrm{Ric}_{g_\infty(-\tau)} + \nabla^2_{g_\infty(-\tau)} l_\infty - \frac{1}{2\tau} g_\infty(-\tau) = 0$, which implies that M_∞ is a Ricci shrinker.

(vi) From [CZ11], there exists a point $x_0 \in M$ such that $(M, g_i(-\tau), x_0)$ subsequentially converges to $(M_\infty, g_\infty(-\tau), x_\infty)$ which is a non-flat Ricci shrinker. Now pick any $q \in M$ and denote $r = d_{g(-1)}(x_0, q)$. From (i), we have

$$
\begin{aligned}
d_{g_i(-\tau)}(x_0, q) \le \tau_i^{-\frac{1}{2}} d_{g(-\tau_i \tau)}(x_0, q) &\le \tau_i^{-\frac{1}{2}} (r + 8(n-1) C_0 \sqrt{\tau_i \tau}) \\
&\le 9(n-1) C_0 \sqrt{\tau}
\end{aligned}
$$

Therefore, for given τ, even though we change the basepoint as q, which is away from x_0 by uniformly bounded distance, we get a limit isometric to $(M_\infty, g_\infty(-\tau))$, which is a non-flat Ricci shrinker. It completes the proof. $\Box$

Remark 4.2.6. Note that the result (iii), (iv), (v) are still true for $\tau_i \to 0$. For (vi), we still have a uniqueness of the limit for a fixed point but we don't know whether it is non-flat.

Next, we prove the following characterization of the behavior near the singular time. Here, a Kähler manifold M is said to be irreducible if its universal cover is not isometrically-biholomorphic to a product of two Kähler manifolds of smaller dimensions. From the uniqueness of the Ricci flow [CZ06b] and [Kot14, Corollary 1.2], we know that a Kähler Ricci flow is irreducible if and only if any time slice is irreducible.

Proposition 4.2.7. *Let $(M, g(-\tau))_{\tau \in (0,\infty)}$ be an irreducible κ-solution and let $\tau = 0$ be the singular time. Then the followings are equivalent.*

(a) M is isometrically-biholomorphic to a quotient of compact Hermitian symmetric space.

(b) There exists $p \in M$ such that $\lim_{\tau \to 0} R(p, -\tau) = \infty$.

(c) For all $p \in M$, we have $\lim_{\tau \to 0} R(p, -\tau) = \infty$.

Proof. Clearly, we can see that (a) implies (b). For the direction from (b) to (c), if it is not true, then there exists $p, q \in M$ such that for $\tau_i \to 0$, we have $R(p, -\tau_i) \to \infty$ but $R(q, -\tau_i) \leq C$ for some $C > 0$. For $Q_i = R(p, -\tau_i)$, let $g_i(-\tau) = Q_i g(-Q_i^{-1}\tau - \tau_i)$. Then the sequence of manifolds $(M, g_i(-\tau), p)_{\tau\in[0,\infty)}$ subsequentially converges to $(M_\infty, g_\infty(-\tau), p_\infty)_{\tau\in[0,\infty)}$ and $R_{g_\infty}(p_\infty, 0) = 1$. In particular, this limit is non-flat. From (vi) of Theorem 4.2.5, we know that this limit is isometric to the limit of $(M, g_i(-\tau), q)_{\tau\in[0,\infty)}$. But the latter must converge to the flat space since $R_{g_i}(q, 0) = R(q, -\tau_i)/Q_i \to 0$ as $i \to \infty$. This is a contradiction. For the direction from (c) to (a), fix $\tau > 0$. Then for any $\tau_0 \in (0, \tau)$ and $p \in M$, from the estimate

$$\left|\frac{\partial R}{\partial \tau}\right| \leq \eta R^2$$

we obtain

$$\frac{1}{R(p, -\tau)} - \frac{1}{R(p, -\tau_0)} \leq \eta(\tau - \tau_0).$$

Now, by taking $\tau_0 \to 0$, since $R(p, -\tau_0) \to \infty$, we have $R(p, -\tau) \geq \frac{1}{\eta\tau}$. Therefore, the scalar curvature of $(M, g(-\tau))$ has a positive lower bound. If M is noncompact, it contradicts the average curvature decay in [NT03, Theorem 0.4]. So M has to be compact. Now the result follows from Theorem 4.2.3 $\square$

Next, we prove the following result about the asymptotic scalar curvature ratio. Later, this result will be used to get a splitting direction of the rescaling limit.

Lemma 4.2.8. *Let $(M, g(-\tau))_{\tau\in(0,\infty)}$ be a κ-solution. Then for any $\tau \in (0, \infty)$ and $p \in M$, we have*

$$\liminf_{d_{g(-\tau)}(x,p)\to\infty} R(x, -\tau) d^2_{g(-\tau)}(x, p) = \infty.$$

Proof. If the statement is not true, then we may assume it does not hold when $\tau = 1$. So there exists a sequence $\{p_i\} \subset M$ such that $d_{g(-1)}(p, p_i) \to \infty$ but $R(p_i, -1)d^2_{g(-1)}(p, p_i) \leq C$ for some $C > 0$. If we take $\rho_i = d^2_{g(-1)}(p, p_i)$ and consider $g_i(-\tau) = \rho_i^{-1} g(\rho_i(1 - \tau) - 1)$, then from the compactness of κ-solutions, a sequence $(M, g_i(-\tau), p_i)_{\tau\in[1,\infty)}$ subsequentially converges to $(M_\infty, g_\infty(-\tau), p_\infty)_{\tau\in[1,\infty)}$. Moreover, since $\rho_i \to \infty$, one can use Theorem 4.2.5

to show that $(M, g_i(-\tau), p)_{\tau\in(1,\infty)}$ subsequentially converges to a nonflat Kähler Ricci shrinker $(M'_\infty, g'_\infty(1-\tau), p'_\infty)_{\tau\in(1,\infty)}$. But since $d_{g_i(-1)}(p, p_i) = 1$ for all i, we know that both limits are isometric. Note that

$$R_\infty(p_\infty, -1) = \lim_{i\to\infty} R_i(p_i, -1) = \lim_{i\to\infty} d^2_{g(-1)}(p, p_i)R(p_i, -1) \leq C$$

from the assumption. However, since $\tau = 1$ is the singular time of the Kähler Ricci shrinker, we have $R_{g_\infty}(q, -\tau) \to \infty$ for any $q \in M_\infty$ as $\tau \to 1$, which is a contradiction. $\square$

Using this result, we can investigate the asymptotic behavior of the ancient solution in the following way.

Proposition 4.2.9. *Let $(M^2, g(-\tau))_{\tau\in[0,\infty)}$ be a noncompact complex 2-dimensional nonflat κ-solution. Then for any sequence $(p_i, \tau_i) \in M \times (0, \infty)$ with $\tau_i \to \infty$ and $Q_i = R(p_i, -\tau_i)$, the sequence $(M, Q_i g(-\tau_i), p_i)$ subsequentially converges to $\mathbb{C} \times \mathbb{CP}^1$ in the Cheeger-Gromov sense. Here we assume the scalar curvature of $\mathbb{C} \times \mathbb{CP}^1$ is identically* 1.

Proof. We first consider the case when $p_i = p$ is a fixed point. Let $g_i(-\tau) = \tau_i^{-1} g(-\tau_i\tau)$. Then by Theorem 4.2.5, we know that the limit of $(M, g_i(-\tau), p)_{\tau\in(0,\infty)}$ exists as a nonflat, noncompact, Kähler Ricci shrinker with nonnegative bisectional curvature. And this limit has to be a shrinking $(\mathbb{C} \times \mathbb{CP}^1, g_\infty(-\tau))_{\tau\in(0,\infty)}$ because of Theorem 4.2.2 and Lemma 4.2.4. Because of type I condition, there are 2 possible cases.

Case 1. $R(p, -\tau_i)\tau_i \to L > 0$.

Then the limit of $(M, g_i(-1), p)$ is homothetic to the limit of $(M, \tilde{g}_i(-1), p)$ where $\tilde{g}_i(-1) = Q_i g(-\tau_i)$. Therefore, the conclusion holds.

Case 2. $R(p, -\tau_i)\tau_i \to 0$.

From

$$\left|\frac{\partial R}{\partial \tau}\right| \leq \eta R^2$$

we have

$$\frac{1}{R(p,-\tau_i)} - \frac{1}{R(p,0)} \leq \eta\tau_i$$

This inequality implies that

$$1 - \eta\tau_i R(p,-\tau_i) \leq \frac{R(p,-\tau_i)}{R(p,0)} \leq 1.$$

So we get $\frac{R(p,-\tau_i)}{R(p,0)} \to 1$ as $i \to \infty$. But then the trace Harnack inequality implies that $R(p,-\tau)$ is constant in τ. However, this implies that M is flat, which is a contradiction. It completes the proof when the sequence $\{p_i\}$ is a fixed point.

Now we will handle general cases. If there exists a point $p \in M$ such that $\frac{d_{g(0)}(p,p_i)}{\sqrt{\tau_i}} \leq D$ for all i, then the limits of $(M, g_i(-1), p)$ and $(M, g_i(-1), p_i)$ are isometric as $d_{g_i(-1)}(p, p_i)$ will be bounded due to (i) in Theorem 4.2.5. Therefore, we may assume $\frac{d_{g(0)}(p,p_i)}{\sqrt{\tau_i}} \to \infty$. There are two possibilities.

Case 1. $Q_i\tau_i \to 0$.

Let $\tilde{Q}_i = R(p_i, 0)$. By integrating $\left|\frac{\partial R}{\partial \tau}\right| \leq \eta R^2$ and multiplying Q_i on both sides, we get

$$0 \leq 1 - \frac{Q_i}{\tilde{Q}_i} \leq \eta Q_i \tau_i \to 0.$$

So we have $\frac{Q_i}{\tilde{Q}_i} \to 1$ as $i \to \infty$. In particular, we have $\tilde{Q}_i \to 0$. It implies that the limit, denoted by $(M_\infty, g_\infty(-\tau), p_\infty)_{\tau\in(0,\infty)}$, of $(M, Q_i g(-Q_i^{-1}\tau), p_i)_{\tau\in(0,\infty)}$ is isometric to that of $(M, \tilde{Q}_i g(-\tilde{Q}_i^{-1}\tau), p_i)_{\tau\in(0,\infty)}$. Note that we know $\tilde{Q}_i d^2_{g(0)}(p, p_i) \to \infty$ from Lemma 4.2.8 and $\tilde{Q}_i g(-\tilde{Q}_i^{-1}\tau)$ has a nonnegative curvature operator from Lemma 3.1.5, we can apply Toponogov's splitting theorem to conclude that the limit of $(M, \tilde{Q}_i g(-\tilde{Q}_i^{-1}\tau), p_i)$ splits a line generated by the vector field V. Since the limit is equipped with a complex structure J_∞, the vector field $J_\infty V$ generates another splitting direction. Therefore, the universal covering $(\tilde{M}_\infty, \tilde{g}_\infty(\tau), p_\infty)$ is isometrically biholomorphic to $\mathbb{C} \times \mathbb{CP}^1$, where we have used the fact that every real 2-dimensional nonflat κ-noncollapsed ancient solution is a shrinking sphere

[Per02, Corollary 11.3]. Therefore, it follows from Lemma 4.2.4 that $(M_\infty, g_\infty(-\tau), p_\infty)_{\tau\in(0,\infty)}$ is isometrically biholomorphic to a family of shrinking $(\mathbb{C}\times\mathbb{CP}^1, g_\infty(-\tau))_{\tau\in[0,\infty)}$ with a unit scalar curvature at $\tau = 1$. Therefore, $(M, Q_i g(-\tau_i), p_i)$ converges smoothly to $\mathbb{C}\times\mathbb{CP}^1$.

Case 2. $Q_i\tau_i \to C > 0$.

For simplicity, we assume that $C = 1$. Then there are 2 possible subcases.

Subcase 1. $\tilde{Q}_i\tau_i \to L > 0$

Then it implies that the limit of $(M, Q_i g(-Q_i^{-1}\tau), p_i)_{\tau\in(0,\infty)}$ and $(M, \tilde{Q}_i g(-\tilde{Q}_i^{-1}\tau), p_i)_{\tau\in(0,\infty)}$ are homothetic. From the previous argument, we know that the limit of $(M, \tilde{Q}_i g(-\tilde{Q}_i^{-1}\tau), p_i)_{\tau\in(0,\infty)}$ is isometric to $(\mathbb{C}\times\mathbb{CP}^1, g_\infty(-\tau))_{\tau\in(0,\infty)}$. Therefore, it is clear that the limit of $(M, Q_i g(-\tau_i), p_i)$ is also $(\mathbb{C}\times\mathbb{CP}^1, g_\infty(-1))$.

Subcase 2. $\tilde{Q}_i\tau_i \to \infty$

From the compactness, for $\hat{g}_i := \tau_i^{-1} g(-\tau_i\tau)$, a sequence $(M, \hat{g}_i(-\tau), p_i)_{\tau\in(0,\infty)}$ subsequentially converges to a noncompact κ-solution $(M_\infty, \hat{g}_\infty(-\tau), p_\infty)_{\tau\in(0,\infty)}$. If M_∞ is reducible, then we can argue as above that the limit is $\mathbb{C}\times\mathbb{CP}^1$. In this case, since we already have $Q_i\tau_i \to 1$, the limit of $(M, \hat{g}_i(-\tau), p_i)_{\tau\in(0,\infty)}$ is isometric to the limit of $(M, Q_i g(-\tau_i\tau), p_i)_{\tau\in(0,\infty)}$ which verifies the statement by taking $\tau = 1$. Therefore, we may assume that M_∞ is irreducible. By integrating $\left|\frac{\partial R}{\partial \tau}\right| \le \eta R^2$ from 0 to $-\tau_i\tau$ and dividing both sides by τ_i, we obtain

$$\frac{1}{\tau_i R(p_i, -\tau_i\tau)} - \frac{1}{\tilde{Q}_i\tau_i} \le \eta\tau$$

Since $\tilde{Q}_i\tau_i \to \infty$ from the assumption, it implies that $\hat{R}_i(p_i, -\tau) = \tau_i R(p_i, -\tau_i\tau) \ge \frac{1}{2\eta\tau}$ for large enough i. In particular, we can deduce that $\lim_{\tau\to 0} \hat{R}_\infty(p_\infty, -\tau) = \infty$. From Proposition 4.2.7, it implies that M_∞ has to be compact, which is a contradiction. $\square$

4.2.2 Construction of the Fibration

From Proposition 4.2.9, we are able to reconstruct M as a total space of S^2-fibration over a noncompact Riemann surface. This construction will be given in this subsection. From now on, we will drop the time parameter $-\tau$ for simplicity unless there is confusion. For all the spheres S^2, we assume the scalar curvature is identically 1. Here, the argument in [CL20] is repeated.

Proposition 4.2.10. *Let $(M^2, g(-\tau))_{\tau\in[0,\infty)}$ be a noncompact complex 2-dimensional nonflat κ-solution. For any $\epsilon > 0$, there exists a $\bar{\tau} > 0$ such that for any $(x, -\tau) \in M \times (-\infty, -\bar{\tau}]$, there exists an open neighborhood $\Omega_x \ni x$ with a diffeomorphism $\psi_x : \Omega_x \to B(0, 100) \times S^2 \subset \mathbb{R}^2 \times S^2$ such that*

(a) $\psi_x(x) = (0, \bar{s})$, where $\bar{s}$ is the north pole of S^2.

(b) For the standard metric g_0 on $\mathbb{R}^2 \times S^2$ and $i \in [0, \epsilon^{-1}]$ and $g_x = R(x)g$, we have

$$\sup_{\Omega_x} |\nabla^i_{g_x}(g_x - \psi_x^* g_0)|_{g_x} \leq \epsilon.$$

(c) The map $\varphi_x = \pi_1 \circ \psi_x : (\Omega_x, g_x) \to (B(0, 99), g_0|_{B(0,99)})$ is an ϵ-Riemannian submersion.

Now, we will construct a transition map $\varphi_{x,y}$ between two local fibrations φ_x and φ_y defined in Proposition 4.2.10. This strategy originates in [CG90; CFG92], see also [CL21]. In the following, the function $\delta(\epsilon) \to 0$ as $\epsilon \to 0$ and $\delta(\epsilon)$ may be different line by line.

Proposition 4.2.11. *With the same assumptions as in Proposition 4.2.10, for any $x, y \in M$ with $d_g(x, y) \leq 10r$ with $r = r(x, y) = 1/\sqrt{\max(R(x), R(y))}$, if we set $\Omega_{x,y} = \Omega_x \cap \Omega_y$, then there is a $\delta(\epsilon)$-almost isometry $\varphi_{x,y} : \varphi_y(\Omega_{x,y}) \to \varphi_x(\Omega_{x,y})$. Moreover, it satisfies the following properties.*

(a) $|\varphi_x - \varphi_{x,y} \circ \varphi_y| \leq \delta(\epsilon) r$.

(b) $|D\varphi_x - D\varphi_{x,y} \circ D\varphi_y| \leq \delta(\epsilon)$.

Proof. From the assumption, we have $d_{g_y}(x,y) = \sqrt{R(y)}d_g(x,y) \leq 10$ and hence $x \in \Omega_y$. It follows from Proposition 4.2.10 that

$$\left|\frac{R(x)}{R(y)} - 1\right| \leq \delta(\epsilon). \tag{4.2.1}$$

Moreover, if we set $g_1 = (\psi_x \circ \psi_y^{-1})^* g_0$, then it follows from Proposition 4.2.10 and (4.2.1) that g_1 and g_0 are C^2-close on $\psi_y(\Omega_{x,y})$. More precisely, on $\psi_y(\Omega_{x,y})$ one has

$$|g_1 - g_0| + |\nabla_{g_0} g_1| + |\nabla^2_{g_0} g_1| \leq \delta(\epsilon), \tag{4.2.2}$$

where the norms are with respect to g_0. Next, we prove the map $\psi_x \circ \psi_y^{-1}$ almost preserves the product structure. Indeed, if V is a parallel vector field along $\mathbb{R}^2$, with respect to g_0, and $V_1 = (\psi_x \circ \psi_y^{-1})_* V$, then from (4.2.2) we have

$$|\nabla_{g_0} V_1| \leq \delta(\epsilon), \tag{4.2.3}$$

and hence V_1 is almost tangent to $\mathbb{R}^2$ in the sense that

$$|(\pi_1)_* V_1 - V_1| \leq \delta(\epsilon), \tag{4.2.4}$$

where $\pi_1 : \mathbb{R}^2 \times S^2 \to \mathbb{R}^2$ is the projection map. Similarly, if V_2 is tangent to S^2, then we have

$$|(\pi_2)_* V_2 - V_2| \leq \delta(\epsilon). \tag{4.2.5}$$

Now we define $\varphi_{x,y} : \varphi_y(\Omega_{x,y}) \to \varphi_x(\Omega_{x,y})$ by $\varphi_{x,y}(p) = \pi_1 \circ \psi_x \circ \psi_y^{-1}(p, \bar{s})$. It is clear from the definition that $\varphi_{x,y}$ is a $\delta(\epsilon)$-almost isomtery on $\varphi_y(\Omega_{x,y})$. We claim that $\varphi_{x,y}$ satisfies all required property. Indeed, for any $z \in \Omega_{x,y}$, we set $\psi_y(z) = (p_1, s_1)$ and $\psi_y(z) = (p_2, s_2)$. We consider a geodesic segment γ such that $\gamma(0) = (p_1, s)$ and $\gamma(1) = (p_1, \bar{s})$. If we set $\tilde{\gamma} = \psi_x \circ \psi_y^{-1} \circ \gamma$, then it is clear from (4.2.4) that $|\pi_1(\tilde{\gamma}(0)) - \pi_1(\tilde{\gamma}(1))| \leq \delta(\epsilon) r$. Therefore, the property (a) is proved by our definition of $\varphi_{x,y}$. The property (b) can be proved similarly by (4.2.4) and (4.2.5). □

Remark 4.2.12. To make the proof of Proposition 4.2.11 rigorous, one needs to slightly shrink Ω_x, Ω_y and $\Omega_{x,y}$ such that the new sets contain all S^2 fibers. In the following, we will not mention this explicitly.

Next, we will show that the map $\varphi_{x,y}$ which is constructed previously almost satisfies the cocycle condition $\varphi_{x_1,x_3} = \varphi_{x_1,x_2} \circ \varphi_{x_2,x_3}$.

Proposition 4.2.13. *For any $x_1, x_2, x_3 \in M$ with $d_g(x_i, x_j) \le 10r$ with $r = r(x_1, x_2, x_3) = 1/\sqrt{\max(R(x_1), R(x_2), R(x_3))}$ for any pair $i, j \in \{1, 2, 3\}$, wherever it makes sense, we have the following.*

(a) $|\varphi_{x_1,x_3} - \varphi_{x_1,x_2} \circ \varphi_{x_2,x_3}| \le \delta(\epsilon)r$.

(b) $|D\varphi_{x_1,x_3} - D\varphi_{x_1,x_2} \circ D\varphi_{x_2,x_3}| \le \delta(\epsilon)$.

Proof. From Proposition 4.2.11, wherever it makes sense, we have

$$|\varphi_{x_1} - \varphi_{x_1,x_3} \circ \varphi_{x_3}| \le \delta(\epsilon)r \text{ and } |\varphi_{x_2} - \varphi_{x_2,x_3} \circ \varphi_{x_3}| \le \delta(\epsilon)r.$$

Since φ_{x_1,x_2} is $\delta(\epsilon)$-almost isometry, we have

$$|\varphi_{x_1} - \varphi_{x_1,x_2} \circ \varphi_{x_2,x_3} \circ \varphi_{x_3}| \le |\varphi_{x_1} - \varphi_{x_1,x_2} \circ \varphi_{x_2}| + |\varphi_{x_1,x_2} \circ (\varphi_{x_2} - \varphi_{x_2,x_3} \circ \varphi_{x_3})| \le \delta(\epsilon)r.$$

Using the inequality $|\varphi_{x_1} - \varphi_{x_1,x_3} \circ \varphi_{x_3}| \le \delta(\epsilon)r$, we have

$$|(\varphi_{x_1,x_3} - \varphi_{x_1,x_2} \circ \varphi_{x_2,x_3}) \circ \varphi_{x_3}| \le \delta(\epsilon)r.$$

Since φ_{x_3} is surjective, (a) is proved. Similarly, (b) can be proved by the same argument. $\square$

Now we want to modify the local fibrations φ_x to make them compatible with a transition map $\varphi_{x,y}$. To do so, we need the following lemma.

Lemma 4.2.14. *For any $x, y \in M$ and $r = r(x, y) = 1/\sqrt{\max(R(x), R(y))}$ with $2r \le d_g(x, y) \le 4r$, we assume that $B_g(x, 2r) \cap B_g(y, 2r) \ne \emptyset$. Then there exista a new fibrations $\tilde{\varphi}_x$ on $B_g(x, 2r)$ such that*

$$\tilde{\varphi}_x = \varphi_{x,y} \circ \varphi_y$$

on $B_g(x, 2r) \cap B_g(y, 2r)$. Moreover, it has same estimates with those of φ_x in Proposition 4.2.10 and coincides with φ_x on $B_g(y, 4r(x, y))$ wherever $\varphi_x = \varphi_{x,y} \circ \varphi_y$.

Proof. Let $\theta(z)$ be a cut-off function on $\mathbb{R}^2$ such that $\theta(z) \equiv 1$ on $B(0, 2)$ and $\theta(z) \equiv 0$ outside $B(0, 4)$ and $\phi(p) := \theta\left(\frac{\varphi_y(p)}{r}\right)$. Now define a map $\tilde{\varphi}_x : \Omega_{x,y} \to \mathbb{R}^2$ by $\tilde{\varphi}_x(z) = \phi(z)(\varphi_{x,y} \circ \varphi_y(z)) + (1 - \phi(z))\varphi_x(z)$. Clearly, $\tilde{\varphi}_x = \varphi_x$ wherever $\varphi_x = \varphi_{x,y} \circ \varphi_y$. Also since $\phi(z) \equiv 1$ for any $z \in B_g(x, 2r) \cap B_g(y, 2r)$, it satisfies the property. Now the estimates follow from $\tilde{\varphi}_x(z) - \varphi_x(z) = \phi(z)(\varphi_{x,y} \circ \varphi_y(z) - \varphi_x(z))$ and the Proposition 4.2.10. □

Lemma 4.2.15. *For any $x_1, x_2, x_3 \in M$ and $r = r(x_1, x_2, x_3) = 1/\sqrt{\max(R(x_1), R(x_2), R(x_3))}$ with $2r \leq d_g(x_i, x_j) \leq 4r$ for any pair $i, j \in \{1, 2, 3\}$, we assume that $B_g(x_1, 2r)$, $B_g(x_2, 2r)$ and $B_g(x_3, 2r)$ have a nonempty intersection. Then there exist a new diffeomorphism $\tilde{\varphi}_{x_1,x_3}$ on $\varphi_{x_3}(B_g(x_1, 2r) \cap B_g(x_3, 2r))$ such that*

$$\tilde{\varphi}_{x_1,x_3} = \varphi_{x_1,x_2} \circ \varphi_{x_2,x_3}$$

on $\varphi_{x_3}(B_g(x_1, 2r) \cap B_g(x_2, 2r) \cap B_g(x_3, 2r))$. Moreover, it has the same estimates with those of φ_{x_1,x_3} in Proposition 4.2.11 and coincides with φ_{x_1,x_3} wherever $\varphi_{x_1,x_3} = \varphi_{x_1,x_2} \circ \varphi_{x_2,x_3}$.

Proof. Let $\theta(z)$ be the cut-off function defined in Lemma 4.2.14, then we define $\tilde{\varphi}_{x_1,x_3}(z) = \phi(z)(\varphi_{x_1,x_2} \circ \varphi_{x_2,x_3}(z)) + (1 - \phi(z))\varphi_{x_1,x_3}(z)$. Then one can check that $\tilde{\varphi}_{x_1,x_3}$ satisfies all properties mentioned in the statement. □

Now we can construct a global fibration on M.

Proposition 4.2.16. *Let $(M^2, g(t))_{t\in(-\infty,0]}$ be a noncompact complex 2-dimensional nonflat κ-solution. Then there exists a smooth S^2-fibration $p : M \to \mathcal{S}$ where $\mathcal{S}$ is a noncompact Riemann surface.*

Proof. From Proposition 4.2.10, there exist local fibrations for all points on M, if $-t$ is sufficiently large. By using Lemma 4.2.14 and Lemma 4.2.15. we can follow the standard technique in [CG90; CFG92] to modify all local fibrations to be compatible. For details, the reader can refer to [CL21, Theorem 5.16]. □

Now we can prove the following classification of the noncompact κ-solutions on Kähler surface.

Theorem 4.2.17. *Let $(M^2, g(t))_{t\in(-\infty,0]}$ be a nonflat, noncompact κ-solution. Then it is isometrically-biholomorphic to a family of shrinking $\mathbb{C}\times\mathbb{CP}^1$.*

Proof. By considering the universal cover, we may assume that M is simply connected. From Lemma 3.1.5, we know that $(M, g(-\tau))$ has nonnegative curvature operator. From [NT03, Theorem 5.3], we conclude that $(M, g(0))$ is isometrically-biholomorphic to $N\times L$ where N is a compact Hermitian symmetric space and L is diffeomorphic to $\mathbb{R}^{2k}$ where $k = \dim_{\mathbb{C}} L$. If $k = 1$, then $(M, g(t))$ is isometrically-biholomorphic to the product of a real 2-dimensional κ-solutions L and $\mathbb{CP}^1$. From the result in [Per02], it implies that M is isometrically-biholomorphic to $\mathbb{C}\times\mathbb{CP}^1$ that verifies the statement.

If $k = 2$, then $M = L$ and in particular M is diffeomorphic to $\mathbb{R}^4$. From Proposition 4.2.16, there exists a global S^2-fibration over a noncompact Riemann surface $\mathcal{S}$. Applying the long exact sequence of homotopy groups on the fibration $S^2\to\mathbb{R}^4\to\mathcal{S}$, we have

$$\pi_1(S^2)\to(\pi_1(\mathbb{R}^4)=0)\to\pi_1(\mathcal{S})\to(\pi_0(S^2)=0)$$

which shows $\pi_1(\mathcal{S}) = 0$, i.e., $\mathcal{S}$ is simply connected. By the classification of noncompact surfaces, $\mathcal{S}$ is diffeomorphic to $\mathbb{R}^2$. In particular, the base space of this fibration is contractible. Therefore, this fibration is trivial so the total space has to be diffeomorphic to $S^2\times\mathbb{R}^2$, which is a contradiction. From Lemma 4.2.4, the proof is complete. $\square$

Now the proof of Theorem 1.2.3 is immediate.

Proof of Theorem 1.2.3: Theorem 1.2.3 follows from Theorem 4.2.3 and Theorem 4.2.17.

4.3 Proof of Theorem 1.2.4

In this section, we will show the following proposition whose idea is given in [MW15].

Proposition 4.3.1. *Let (M^n, f, g) be a complete, noncompact gradient Ricci shrinker with weakly PIC and $n \geq 5$. If its scalar curvature converges to 0 at infinity, then (M, g) is smoothly asymptotic to a cone at infinity.*

Proof. The proof is obtained from a slight modification of [MW15, Theorem 0.3] after using the result in Proposition 2.2.1. However, we include the sketch of proof for reader's convenience.

From [CLY11], we know that there exists a constant $C_0 > 0$ such that

$$C_0 \leq Rf$$

Now, the proof is completed from [KW15] after showing that there exists a constant $C_1 > 0$ such that

$$Rf \leq C_1 \tag{4.3.1}$$

It is enough to show that (4.3.1) holds in $K_r := \{f \geq r\}$ for large enough $r > 0$. From Proposition 2.2.1, we have $|\mathrm{Ric}|^2 \leq nR^2$ and this implies

$$\Delta_f R = R - 2|\mathrm{Ric}|^2 \geq R - 2nR^2 \tag{4.3.2}$$

Also, since $\Delta_f f = \frac{n}{2} - f$, we can compute

$$\begin{aligned} \Delta_f\left(\frac{1}{f^k}\right) &= -\frac{k}{f^{k+1}}(\Delta_f f) + \frac{k(k+1)}{f^{k+2}}|\nabla f|^2 \\ &= \frac{k}{f^{k+2}}\left(f^2 - \frac{nf}{2} + (k+1)|\nabla f|^2\right) \end{aligned}$$

for all $k \geq 1$. In particular, using $|\nabla f|^2 \leq R + |\nabla f|^2 = f$, we can derive

$$\Delta_f\left(\frac{1}{f}\right) \leq \frac{1}{f} \text{ in } M \tag{4.3.3}$$

Remark that the potential function f is equivalent to the square of distance function and $R + |\nabla f|^2 = f$. Since the scalar curvature of M is asymptotically vanishing at infinity, we can choose $r_0 > n$ large enough so that all of the followings are satisfied in K_{r_0}.

1. $R < \frac{1}{8n}$

2. $\frac{f^2}{4} - \frac{nf}{2} + 3|\nabla f|^2 \geq 0$

In particular, the second condition implies that

$$\Delta_f \left(\frac{1}{f^2} \right) = \frac{2}{f^4} \left(f^2 - \frac{nf}{2} + 3|\nabla f|^2 \right) \geq \frac{3}{2f^2} \tag{4.3.4}$$

in K_{r_0}. Now, define a function

$$u := R - \frac{r_0}{8nf} + \frac{r_0^2}{16nf^2}$$

which is negative on ∂K_{r_0}. Also, by combining (4.3.2), (4.3.3) and (4.3.4), we have

$$\begin{aligned}
\Delta_f u &\geq R - 2nR^2 - \frac{r_0}{8nf} + \frac{3r_0^2}{32nf^2} \\
&= u - 2n \left(R^2 - \frac{r_0^2}{64n^2 f^2} \right) \\
&= u - 2n \left(R - \frac{r_0}{8nf} \right) \left(R + \frac{r_0}{8nf} \right) \\
&\geq u - 2nu \left(R + \frac{r_0}{8nf} \right) \\
&= u \left(1 - 2nR - \frac{r_0}{4f} \right)
\end{aligned} \tag{4.3.5}$$

in K_{r_0}. Now we claim that $u \leq \frac{C_2}{f^2}$ for some constant $C_2 > 0$, which verifies that (4.3.1) holds in K_{r_0}. Note that we may assume u is positive somewhere since otherwise the statement becomes trivial. To get this result, let us choose a cutoff function $\psi : M \to \mathbb{R}$ defined by

$$\psi(x) = \begin{cases} \frac{l - f(x)}{l} & \text{for } 0 \leq f(x) \leq l \\ 0 & \text{for } f(x) \geq l \end{cases}$$

with a constant $l > 2r_0$. Then we have

$$\Delta_f \psi^2 = 2\psi \Delta_f \psi + 2|\nabla \psi|^2 = \frac{2\psi(f - \frac{n}{2})}{l} + \frac{2|\nabla f|^2}{l^2} \tag{4.3.6}$$

Now, let $G := \psi^2 u$ and $p \in M$ be a point where G achieves its positive maximum on K_{r_0}. Clearly, p should be the interior point of K_{r_0} as G is negative on ∂K_{r_0}. So we have

$$0 = \nabla G = u \nabla(\psi^2) + \psi^2 \nabla u$$

at p. By combining (4.3.5) and (4.3.6), we get the following inequality at p.

$$
\begin{aligned}
0 \geq \Delta_f G &= \Delta_f(\psi^2)u + \psi^2 \Delta_f u + 2\nabla(\psi^2) \cdot \nabla u \\
&\geq 2u\left(\frac{\psi(f-\frac{n}{2})}{l} + \frac{|\nabla f|^2}{l^2}\right) + G\left(1 - 2nR - \frac{r_0}{4f}\right) - 8u|\nabla\psi|^2 \\
&= G\left(\frac{2(f-\frac{n}{2})}{l\psi} + \frac{2|\nabla f|^2}{\psi^2 l^2} + 1 - 2nR - \frac{r_0}{4f} - 8\frac{|\nabla\psi|^2}{\psi^2}\right) \\
&= G\left(\frac{2(f-\frac{n}{2})}{l\psi} + 1 - 2nR - \frac{r_0}{4f} - 6\frac{|\nabla f|^2}{\psi^2 l^2}\right)
\end{aligned}
$$

Since $G(p) > 0$, it implies that

$$\frac{2(f-\frac{n}{2})}{l\psi} + 1 - 2nR - \frac{r_0}{4f} - 6\frac{|\nabla f|^2}{\psi^2 l^2} \leq 0 \tag{4.3.7}$$

at p. Moreover, from the choice of r_0, we know that $R(p) \leq \frac{1}{8n}$. It implies that

$$1 - 2nR - \frac{r_0}{4f} \geq 1 - \frac{1}{4} - \frac{1}{4} > 0$$

at p. From (4.3.7), we get

$$\frac{2(f-\frac{n}{2})}{l\psi} - 6\frac{|\nabla f|^2}{\psi^2 l^2} \leq 0$$

at p. Also, since $r_0 > n$ and $|\nabla f|^2 \leq f$, we have

$$\frac{2(f-\frac{n}{2})}{l\psi} \geq \frac{f}{l\psi}$$

and

$$\frac{|\nabla f|^2}{l^2\psi^2} \leq \frac{f}{l^2\psi^2}$$

From these inequalities, we get

$$0 \geq \frac{2(f-\frac{n}{2})}{l\psi} - 6\frac{|\nabla f|^2}{l^2\psi^2} \geq \frac{f}{l\psi}\left(1 - \frac{6}{l\psi}\right)$$

hence we finally get $l\psi(p) \leq 6$, which implies

$$G(p) = \psi(p)^2 u(p) \leq \frac{36u(p)}{l^2}$$

Since $\psi(x) \geq \frac{1}{2}$ in $\{f \leq \frac{l}{2}\}$, we obtain the following for a set $V_l = \{x \in M \mid r_0 \leq f(x) \leq \frac{l}{2}\}$.

$$\sup_{V_l} u \leq 4\sup_{V_l} G \leq 4\sup_{K_{r_0}} G = 4G(p) \leq \frac{144u(p)}{l^2}$$

which completes the proof of claim. As a result, we have

$$R - \frac{r_0}{8nf} + \frac{r_0^2}{16nf^2} \leq \frac{C_2}{f^2}$$

for some constant $C_2 > 0$. It completes the proof since

$$Rf \leq \frac{r_0}{8n} + \frac{C_2}{f} \leq \frac{r_0}{8n} + C_2$$

in K_{r_0}.

$\square$

Proof of Theorem 1.2.4: Let (M, f, g) be a complete, noncompact gradient Ricci shrinker with uniformly PIC and bounded curvature. For each end E, we first consider the case when there exists a sequence $x_i \to \infty$ along E such that $\lim_{i\to\infty} R(x_i) = c > 0$. Then by [Nab10, Proposition 4.1.], we know that a sequence (M, g, x_i) subconverges to a nonflat gradient Ricci shrinker $(M_\infty, g_\infty, x_\infty)$. Also, since $x_i \to \infty$, we know that M_∞ splits isometrically as $\mathbb{R} \times N_\infty$ where N_∞ is $(n-1)$-dimensional gradient Ricci shrinker with uniformly PIC_1. So we know that N_∞ is isometric to S^{n-1}/Γ by [Bre19, Theorem 6.4] (see also [Yok17]). Now, by [MW19, Theorem 1.7.], we know that E is smoothly asymptotic to the same round cylinder.

So we may assume that the scalar curvature converges to 0 at infinity along E. Then by Proposition 4.3.1, E is smoothly asymptotic to a Riemannian cone $[r_0, \infty) \times \Sigma$ with a metric $g_\infty = dr^2 + r^2 g_\Sigma$ where (Σ, g_Σ) is a closed $(n-1)$-dimensional Riemannian manifold and $r_0 \geq 0$. Note that g_∞ also has weakly PIC as well.

Now we will investigate the metric g_∞ closely. If we set $g_r := r^2 g_\Sigma$, then we get

$$\nabla^2 r = \frac{1}{2} L_{\partial_r} g = r g_\Sigma = \frac{1}{r} g_r$$

So the Gauss equation implies that

$$\begin{aligned} R^r_{ijkl} &= R_{ijkl} + (\nabla_i \nabla_k r)(\nabla_j \nabla_l r) - (\nabla_i \nabla_l r)(\nabla_j \nabla_k r) \\ &= R_{ijkl} + \frac{1}{r^2}((g_r)_{ik}(g_r)_{jl} - (g_r)_{il}(g_r)_{jk}) \end{aligned}$$

In particular, we have

$$R^r_{ijij} = R_{ijij} + \frac{1}{r^2} > R_{ijij}$$

As a result, for any orthonormal 4-frame $\{e_i, e_j, e_k, e_l\}$ of Σ, we have

$$R^r_{ikik} + R^r_{ilil} + R^r_{jkjk} + R^r_{jljl} - 2R^r_{ijkl} > R_{ikik} + R_{ilil} + R_{jkjk} + R_{jljl} \geq 0$$

since g_∞ has weakly PIC. Therefore, we know that Σ is a closed manifold with PIC. From the result in [MM88], it implies that Σ is homeomorphic to S^{n-1}/Γ and completes the proof.

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