Real WDVV Relations for Symplectic 4-Manifolds

A Dissertation Presented

by

Xujia Chen

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2021

Stony Brook University

The Graduate School

Xujia Chen

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation.

Aleksey Zinger – Dissertation Advisor
Professor, Department of Mathematics

Jason Starr – Chairperson of Defense
Professor, Department of Mathematics

Mark McLean
Associate Professor, Department of Mathematics

Nikita Nekrasov
Professor, Simons Center for Geometry and Physics

This dissertation is accepted by the Graduate School

Eric Wertheimer

Dean of the Graduate School

Abstract of the Dissertation

Real WDVV Relations for Symplectic 4-Manifolds

by

Xujia Chen

Doctor of Philosophy

in

Mathematics

Stony Brook University

2021

For symplectic 4-manifolds with a real structure, Welschinger (2003) showed that counts of real rational pseudo-holomorphic curves, with appropriate signs, are well-defined invariants. They are called Welschinger invariants and are analogues of Gromov-Witten invariants in the real setting. In 2007, Solomon proposed two WDVV-type relations for them, which determine these numbers recursively in many good cases. They are real analogues of the usual WDVV relation.

We establish Solomon's WDVV-type relations by implementing Georgieva's suggestion to lift homology relations from the Deligne-Mumford moduli spaces of stable real curves. This is accomplished by lifting judiciously chosen cobordisms realizing these relations. Our topological approach provides a general framework for lifting relations via morphisms between not necessarily orientable spaces.

Contents

Acknowledgements

I would like to thank my thesis advisor Professor Aleksey Zinger for introducing me to the subject of real Gromov-Witten theory and its background material, suggesting the topic of this thesis, a lot of guidance and discussions throughout the process of this work and very detailed help with the exposition. I am very grateful for him being an extremely responsible and helpful advisor. For example, he was willing to spend a lot of time discussing even the most technical, most detailed questions, and he would check every detail in my writings and help arranging them. I am also very grateful for him being a kind and caring advisor, who always encourages me and trusts my ability. I would also like to thank the National Science Foundation for supporting my work as a Research Assistant through his grants.

I am grateful to all the mathematicians who have been interested in my work and supportive of me, especially Professors Erwan Brugallé, Penka Georgieva, Jason Starr, Gang Tian, Melissa Liu and Shing-Tung Yau. I am grateful to all the mathematicians who invited me for a talk; those visits made a very memorable part of my graduate school life.

I would like to thank the professors in Stony Brook whose classes I enjoyed and learned much from, including Professors Mark McLean, Kenji Fukaya, Oleg Viro, Aleksey Zinger and Dennis Sullivan. I would like to thank Professor Mark McLean for supervising my reading on complex geometry for the preliminary exam. I would also like to thank Professor Xiuxiong Chen, as well as Professor Bin Xu at USTC, for suggesting Stony Brook to me and for their support and encouragement.

I would like to thank my friends including Yoon-Joo Kim, Dahye Cho, Nathan Chen, Tobias Shin, Mohamed El Alami, Ying Hong Tham, Jae Ho Cho, Santai Qu, Hang Yuan, Qianyu Chen, Ruijie Yang, Xuemiao Chen, Gao Chen, Alexandra Viktorova, Matthew Dannenberg, Jiasheng Teh, Anh Nguyen, Yongming Shen, Junyao Wang, Susan and Carlo Colatosti. I would like to thank all the members of the Stony Brook Math Department. I really appreciate the friendly and mathematically inspiring atmosphere here. This department feels like home to me.

At last I would like to give special thanks to my parents, for a warm and happy family, their effort in bringing me to a bigger town with more opportunities, and all their love and support.

1 Introduction

The WDVV relation [19, 22] for genus 0 Gromov-Witten invariants completely solves the classical problem of enumerating complex rational curves in the complex projective space $\mathbb{P}^n$. Invariant counts of real rational J-holomorphic curves in compact real symplectic fourfolds, now known as Welschinger invariants, were defined in [29] and interpreted in terms of counts of J-holomorphic maps from the disk $\mathbb{D}^2$ in [24]. J. Solomon announced two distinct WDVV-type relations for these counts in February 2007 and outlined an approach to their proof in the general spirit of the original proof of the complex WDVV relation in [22]. However, the outline of the proof described in [25] left a number of conceptual points mysterious and clearly required a major technical effort to implement.

The proof of the complex WDVV relation in [22] involves defining a count of J-holomorphic maps in a symplectic manifold (X,ω) for every cross-ratio ϖ of four points on $\mathbb{P}^1$ and showing that this count does not depend on ϖ. This is also the strategy used in the alternative proof of a WDVV-type relation for counts of real rational curves passing through conjugate pairs of points only (no real point constraints) in [10]. The complex WDVV relation can alternatively be viewed as a direct consequence (at least conceptually) of two specific points, ϖ_1 and ϖ_2, of the Deligne-Mumford moduli space $\overline{\mathcal{M}}_{0,4} \approx \mathbb{P}^1$ of stable complex genus 0 curves with 4 marked points determining the same element of $H_0(\overline{\mathcal{M}}_{0,4})$. This perspective is suitable for lifting homology relations in any dimension from moduli spaces of curves to moduli spaces of J-holomorphic maps and has proved instrumental to studying the structure of complex Gromov-Witten invariants as in [13, 27]. This is also the strategy used in the primary proof of a WDVV-type relation for counts of real rational curves passing through conjugate pairs of points in [10]. In all of these settings, the moduli spaces of curves and maps are closed and oriented.

1.1 Lifting homology relations

In Spring 2014, P. Georgieva suggested that WDVV-type recursions for the real genus 0 invariants of [29] might be obtainable by lifting

($\mathbb{R}$1) a zero-dimensional homology relation on the moduli space $\mathbb{R}\overline{\mathcal{M}}_{0,1,2} \approx \mathbb{R}\mathbb{P}^2$ of stable real genus 0 curves with 1 real marked point and 2 conjugate pairs of marked points and

($\mathbb{R}$2) the one-dimensional homology relation on the moduli space $\mathbb{R}\overline{\mathcal{M}}_{0,0,3}$ of stable real genus 0 curves with 3 conjugate pairs of marked points discovered in [10]

to the moduli spaces $\overline{\mathfrak{M}}_{k,l}(B;J)$ of real rational J-holomorphic maps constructed in [9]. Unlike in the complex case and in the real case considered in [10], major conceptual issues arise in lifting relations from $\mathbb{R}\overline{\mathcal{M}}_{0,1,2}$ and $\mathbb{R}\overline{\mathcal{M}}_{0,0,3}$ to $\overline{\mathfrak{M}}_{k,l}(B;J)$ and in translating any lifted relations into invariant counts of curves because the moduli spaces $\overline{\mathfrak{M}}_{k,l}(B;J)$ are generally not orientable. The present paper deals with these issues by lifting homology relations *along with* bounding cobordisms for them to cuts of $\overline{\mathfrak{M}}_{k,l}(B;J)$ along certain hypersurfaces.

We first re-interpret the disk counts of [24] in the spirit of Steenrod homology [26] in terms of counts of real J-holomorphic maps with marked points decorated by signs as in [9]. We then lift homology relations, along with *suitably chosen* bounding chains Υ, from $\mathbb{R}\overline{\mathcal{M}}_{0,1,2}$ and $\mathbb{R}\overline{\mathcal{M}}_{0,0,3}$ to the *bordered* moduli spaces $\widehat{\mathfrak{M}}_{k,l;l^*}(B;J)$ obtained by cutting $\overline{\mathfrak{M}}_{k,l}(B;J)$ along hypersurfaces that obstruct the relative orientability of the forgetful morphisms

$$\mathfrak{f}_{1,2}\colon \overline{\mathfrak{M}}_{k,l}(B;J) \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{0,1,2} \quad \text{and} \quad \mathfrak{f}_{0,3}\colon \overline{\mathfrak{M}}_{k,l}(B;J) \longrightarrow \mathbb{R}\overline{\mathcal{M}}_{0,0,3}. \tag{1.1}$$

The simple topological Lemma 3.5 expresses the wall-crossing effects on the lifted relations in $\overline{\mathfrak{M}}_{k,l}(B;J)$ in terms of the intersections of the boundary of $\widehat{\mathfrak{M}}_{k,l;l^*}(B;J)$ with Υ. This allows us to obtain the two WDVV-type relations for the map counts depicted in Figure 1 on page 10, with the left-hand sides representing the initial relations in $\mathbb{R}\overline{\mathcal{M}}_{0,1,2}$ and $\mathbb{R}\overline{\mathcal{M}}_{0,0,3}$ and the right-hand sides representing the wall-crossing corrections. The first two relations of Theorem 1.1 are obtained by using the two relations of Figure 1 with the divisors H_1 and H_2 as the first two non-real constraints and points as the remaining constraints. The last relation of Theorem 1.1 is obtained by using the second relation of Figure 1 with the divisors H_1, H_2, and H_3 as the first two non-real constraints and points as the remaining constraints.

By the comparison between the curve counts of [29] and the map counts of [24] established in [8], the relations of Theorem 1.1 are equivalent to the relations for the former stated in [6, Theorem 1]. They completely determine Welschinger invariants of $\mathbb{P}^2$ and its blowups, as shown in [25] and [14], respectively. For the ease of use, these results are summarized in [6]; the low-degree numbers obtained from the three relations for Welschinger invariants and listed in [6] agree with [2, 3, 5, 15, 16, 17, 18, 31].

The relations of [6, Theorem 1] for Welschinger invariants are the same as implied by the statements of Theorem 8, Proposition 10, and Theorem 11 in [25]. The relations of Theorem 1.1 for the map invariants in the present paper involve the same terms as the difference between equations (6) and (7) in [25] and the symmetrization of equation (5) in [25], but *different* signs. The comparison between the curve counts of [29] and the disk counts of [24] established in [8] likewise *differs* by sign from the claim of [25, Thm. 11]. The two sign discrepancies, which do not appear to be due to the formulations of the definitions of the disk invariants in the present paper and [24, 25], precisely cancel out to yield the same recursions for Welschinger invariants.

This paper presents a general approach for pulling back a relation by a morphism $\mathfrak{f}\colon \mathfrak{M} \longrightarrow \mathcal{M}$ between two spaces which is not necessarily relatively orientable. The lifted relation then acquires a correction which doubly covers a "non-orientability" hypersurface in $\mathfrak{M}$. This approach should be applicable in many other settings. It has already been used in [7] to obtain WDVV-type relations for real symplectic sixfolds.

1.2 Main theorem

Let (X,ω,ϕ) be a compact real symplectic manifold, i.e. ω is a symplectic form on X so that $\phi^*\omega = -\omega$. The fixed locus X^ϕ of the anti-symplectic involution ϕ on X is then a Lagrangian submanifold of (X,ω). We denote by $H_2(X)$ the quotient $H_2(X)$ of $H_2(X;\mathbb{Z})$

modulo torsion, by $\mathcal{J}_\omega$ the space of ω-compatible (or -tamed) almost complex structures J on X, and by $\mathcal{J}_\omega^\phi\subset\mathcal{J}_\omega$ the subspace of almost complex structures J such that $\phi^*J=-J$. Let

$$c_1(X,\omega)\equiv c_1(TX,J)\in H^2(X)$$

be the first Chern class of TX with respect to some $J\in\mathcal{J}_\omega$; it is independent of such a choice. For $B\in H_2(X)$, define

$$\ell_\omega(B)=\big\langle c_1(X,\omega),B\big\rangle-1\in\mathbb{Z},\quad \langle B\rangle_l=\begin{cases}1, & \text{if } 2l=\ell_\omega(B)-1;\\ 0, & \text{otherwise.}\end{cases}$$

For $J\in\mathcal{J}_\omega^\phi$ and $B\in H_2(X)$, a subset $C\subset X$ is a genus 0 (or rational) irreducible J-holomorphic degree B curve if there exists a simple (not multiply covered) J-holomorphic map

$$u\colon\mathbb{P}^1\longrightarrow X\qquad\text{s.t.}\quad C=u(\mathbb{P}^1),\quad u_*[\mathbb{P}^1]=B. \tag{1.2}$$

Such a subset C is called a real rational irreducible J-holomorphic degree B curve if in addition $\phi(C)=C$.

From now on, suppose that the (real) dimension of X is 4. The (tangent bundle of the) fixed locus X^ϕ then admits a Pin^--structure $\mathfrak{p}$. Let $B\in H_2(X)$ and $l\in\mathbb{Z}^{\geqslant0}$ be such that

$$k\equiv\ell_\omega(B)-2l\in\mathbb{Z}^{\geqslant0}. \tag{1.3}$$

For a generic $J\in\mathcal{J}_\omega^\phi$, there are then only finitely many real rational irreducible J-holomorphic degree B curves $C\subset X$ intersecting a connected component $\breve{X}^\phi$ of X^ϕ at infinitely many points and passing through k points in $\breve{X}^\phi$ and l points in $X-X^\phi$ in general position. According to [24, Thm. 1.3], the number of such curves counted with appropriate signs determined by $\mathfrak{p}$ is independent of the choices of J and the points. We denote this signed count of genus 0 curves by $N_{B,l}^{\phi,\mathfrak{p}}(\breve{X}^\phi)$. If the number k in (1.3) is negative, we set $N_{B,l}^{\phi,\mathfrak{p}}(\breve{X}^\phi)=0$. We denote by $N_{B,l}^{\phi,\mathfrak{p}}$ the sum of the numbers $N_{B,l}^{\phi,\mathfrak{p}}(\breve{X}^\phi)$ over the connected components $\breve{X}^\phi$ of X^ϕ.

Suppose $B\in H_2(X)$ and $\ell_\omega(B)\geqslant0$. For a generic $J\in\mathcal{J}_\omega$, there are then only finitely many rational irreducible J-holomorphic degree B curves C passing through $\ell_\omega(B)$ points in general position. The number of such curves counted with appropriate signs is independent of the choices of J and the points. This is the standard (complex) genus 0 degree B Gromov-Witten invariant of (X,ω) with $\ell_\omega(B)$ point insertions; we denote it by N_B^X. If $\ell_\omega(B)<0$, we set $N_B^X=0$.

For $B,B'\in H_2(X)$, we denote by $B\cdot_X B'\in\mathbb{Z}$ the homology intersection product of B with B' and by $B^2\in\mathbb{Z}$ the self-intersection number of B. Define

$$\mathfrak{d}\colon H_2(X)\longrightarrow H_2(X),\ \ \mathfrak{d}(B)=B-\phi_*(B),\qquad H^2(X)_-^\phi=\big\{H\in H^2(X)\colon\phi^*H=-H\big\}.$$

Theorem 1.1. *Suppose (X,ω,ϕ) is a compact real symplectic fourfold, $\mathfrak{p}$ is a Pin^--structure on X^ϕ, $\breve{X}^\phi$ is a connected component of X^ϕ, and $H_1,H_2,H_3\in H^2(X)_-^\phi$ are such that $\langle H_1H_2,X\rangle=1$ and $H_1H_3=0$. Let $l\in\mathbb{Z}^{\geqslant0}$ and $B\in H_2(X)$.*

(ℝWDVV1) If $l\geqslant 1$ and $\ell_\omega(B)-2l\geqslant 1$ (i.e. $k\geqslant 1$), then

$$\begin{aligned}
N^{\phi;\mathfrak{p}}_{B,l}(\check{X}^\phi) &= -2^{l-3}\langle B\rangle_l\langle H_1,B\rangle\langle H_2,B\rangle\sum_{\substack{B'\in H_2(X)\\ \mathfrak{d}(B')=B}}N^X_{B'}\\
&\quad-\sum_{\substack{B_0,B'\in H_2(X)-\{0\}\\ B_0+\mathfrak{d}(B')=B}}2^{l_\omega(B')}\big(B_0\cdot_X B'\big)\langle H_1,B'\rangle\langle H_2,B'\rangle\binom{l-1}{\ell_\omega(B')}N^X_{B'}N^{\phi;\mathfrak{p}}_{B_0,l-1-\ell_\omega(B')}(\check{X}^\phi)\\
&\quad+\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\ B_1+B_2=B\\ l_1+l_2=l-1,\,l_1,l_2\geqslant 0}}\langle H_1,B_1\rangle\binom{l-1}{l_1}\Bigg(\langle H_2,B_2\rangle\binom{\ell_\omega(B)-2l-1}{\ell_\omega(B_1)-2l_1-1}\\
&\qquad-\langle H_2,B_1\rangle\binom{\ell_\omega(B)-2l-1}{\ell_\omega(B_1)-2l_1}\Bigg)N^{\phi;\mathfrak{p}}_{B_1,l_1}(\check{X}^\phi)N^{\phi;\mathfrak{p}}_{B_2,l_2}(\check{X}^\phi).
\end{aligned}$$

(ℝWDVV2) If $l\geqslant 2$, then

$$\begin{aligned}
N^{\phi;\mathfrak{p}}_{B,l}(\check{X}^\phi) &= \sum_{\substack{B_0,B'\in H_2(X)-\{0\}\\ B_0+\mathfrak{d}(B')=B}}2^{l_\omega(B')-1}\big(B_0\cdot_X B'\big)\langle H_1,B'\rangle\\
&\quad\times\Bigg(\langle H_2,B_0\rangle\binom{l-2}{\ell_\omega(B')-1}-2\langle H_2,B'\rangle\binom{l-2}{\ell_\omega(B')}\Bigg)N^X_{B'}N^{\phi;\mathfrak{p}}_{B_0,l-1-\ell_\omega(B')}(\check{X}^\phi)\\
&\quad+\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\ B_1+B_2=B\\ l_1+l_2=l-2,\,l_1,l_2\geqslant 0}}\langle H_2,B_1\rangle\binom{l-2}{l_1}\Bigg(\langle H_1,B_2\rangle\binom{\ell_\omega(B)-2l}{\ell_\omega(B_1)-2l_1-1}\\
&\qquad-\langle H_1,B_1\rangle\binom{\ell_\omega(B)-2l}{\ell_\omega(B_1)-2l_1}\Bigg)N^{\phi;\mathfrak{p}}_{B_1,l_1}(\check{X}^\phi)N^{\phi;\mathfrak{p}}_{B_2,l_2+1}(\check{X}^\phi).
\end{aligned}$$

(ℝWDVV3) If $l\geqslant 1$, then

$$\begin{aligned}
\langle H_3,B\rangle N^{\phi;\mathfrak{p}}_{B,l}(\check{X}^\phi) &= \sum_{\substack{B_0,B'\in H_2(X)-\{0\}\\ B_0+\mathfrak{d}(B')=B}}2^{l_\omega(B')}\big(B_0\cdot_X B'\big)\langle H_1,B'\rangle\binom{l-1}{l_\omega(B')}\\
&\quad\times\Big(\langle H_2,B_0\rangle\langle H_3,B'\rangle-\langle H_3,B_0\rangle\langle H_2,B'\rangle\Big)N^X_{B'}N^{\phi;\mathfrak{p}}_{B_0,l-1-\ell_\omega(B')}(\check{X}^\phi)\\
&\quad+\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\ B_1+B_2=B\\ l_1+l_2=l-1,\,l_1,l_2\geqslant 0}}\langle H_1,B_1\rangle\big(\langle H_3,B_1\rangle\langle H_2,B_2\rangle-\langle H_2,B_1\rangle\langle H_3,B_2\rangle\big)\\
&\qquad\times\binom{l-1}{l_1}\binom{l_\omega(B)-2l}{l_\omega(B_1)-2l_1}N^{\phi;\mathfrak{p}}_{B_1,l_1}(\check{X}^\phi)N^{\phi;\mathfrak{p}}_{B_2,l_2}(\check{X}^\phi).
\end{aligned}$$

Taking the difference between the relations of Theorem 1.1 for $N^{\phi}_{B+B_\bullet,l}(X^\phi)$ with $\ell_\omega(B_\bullet)>0$ small yields relations involving the invariants $N^{\phi}_{B+B_\bullet,0}(X^\phi)$ without conjugate pairs of marked points. In some cases, the resulting relations determine these numbers; see [25, 14].

Remark 1.2. We define the invariants $N^{\phi;\mathfrak{p}}_{B,l}(X^\phi)$ via the moduli spaces of real maps constructed in [9]. This definition of $N^{\phi;\mathfrak{p}}_{B,l}(X^\phi)$ differs by a power of 2 from the definitions in [24, 25], but agrees in absolute value with the invariants $N^{\phi}_{B,l}(X^\phi)$ of [29].

Remark 1.3. Welschinger invariants of real symplectic fourfolds (X,ω,ϕ) with disconnected fixed loci X^ϕ often vanish; see Theorem 1.3 and Remark 1.1 in [4]. As suggested by E. Brugallé, Theorem 1.1 and its proof readily adapt to Welschinger invariants with finer notions of the curve degree B such as those taken in

$$\widetilde{H}_2(X,\check{X}^\phi)\equiv H_2(X,\check{X}^\phi;\mathbb{Z})\big/\big\{b+\phi_*(b)\colon b\in H_2(X,\check{X}^\phi;\mathbb{Z})\big\}\,.$$

As explained in [9, Section 2.2], there is a well-defined doubling map

$$\mathfrak{d}_{\check{X}^\phi}\colon \widetilde{H}_2(X,\check{X}^\phi)\longrightarrow H_2(X;\mathbb{Z})^\phi_-\,.$$

If the curve degrees B are taken in $\widetilde{H}_2(X,\check{X}^\phi)$, the sums in Theorem 1.1 should then be taken over B' and B_i in $H_2(X;\mathbb{Z})-\{0\}$ and $\widetilde{H}_2(X,\check{X}^\phi)$, respectively. The doubling map $\mathfrak{d}$ should be replaced by the composition

$$H_2(X;\mathbb{Z})\longrightarrow H_2(X,\check{X}^\phi;\mathbb{Z})\longrightarrow \widetilde{H}_2(X,\check{X}^\phi).$$

All appearances of B and B_i (but not B') inside $\langle\cdot\rangle$ and $(\cdot)$ should become $\mathfrak{d}_{\check{X}^\phi}(B)$ and $\mathfrak{d}_{\check{X}^\phi}(B_i)$, respectively. The resulting formulas yield relations in particular for the modification of Welschinger invariants which originates in [17, Prop. 1].

The relations of Theorem 1.1 correspond to the partial differential equations (4.82) and (4.76) in [1] for the generating functions given by (4.68) and (4.72), respectively, if $H^2(X)^\phi_-$ is all of $H^2(X)$. These two PDEs are the same as the differential equations (3) and (4) in [25]. The first generating function in [1] is essentially the same as the generating function Φ in [25] if $H^2(X)$ is one-dimensional (the latter does not distinguish between curve classes B with the same $\ell_\omega(B)$). However, the coefficients in the second generating function differ from the coefficients in the generating function Ω in [25] by factors of 2 and $\mathfrak{i}$, even if $H^2(X)$ is one-dimensional. The former is due to the scaling discrepancy in the definitions of the real invariants indicated in Remark 1.2, while the latter reflects the sign difference between the relations of Theorem 1.1 and their analogues in [25] mentioned in Section 1.1.

1.3 Outline of the proof

Theorem 1.1 follows from the two relations for nodal map counts represented by Figure 1 and from Propositions 5.3 and 5.7. In order to simplify the notation for the remainder for the paper, we denote $\check{X}^\phi$ by X^ϕ, $N^{\phi;\mathfrak{p}}_{B,l}(\check{X}^\phi)$ by $N^{\phi,\mathfrak{p}}_{B,l}$, and the moduli space of real rational degree B J-holomorphic maps with k real marked points and l conjugate pairs of marked points that send the fixed locus of the domain to $\check{X}^\phi$ by $\overline{\mathfrak{M}}_{k,l}(B;J)$. If the fixed locus of ϕ is connected, there is no clash with the notation above.

We denote by τ the standard conjugation on $\mathbb{P}^1$, i.e.

$$\tau\colon \mathbb{P}^1 \longrightarrow \mathbb{P}^1, \qquad \tau\big([z_0,z_1]\big) = \big[\overline{z_1},\overline{z_0}\big].$$

For every real rational irreducible J-holomorphic degree B curve contributing to $N_{B,l}^{\phi,\mathfrak{p}}$, there exists a J-holomorphic map $u\colon \mathbb{P}^1 \longrightarrow X$ as in (1.2) such that $u\circ\tau=\phi\circ u$. Thus, the number $N_{B,l}^{\phi,\mathfrak{p}}$ is a signed cardinality of the subset of the moduli space $\overline{\mathfrak{M}}_{k,l}(B;J)$ of real rational degree B J-holomorphic maps sending the k real marked points and the first points in the l conjugate pairs of marked points to generic points in X^ϕ and X, respectively.

The domain and target of the evaluation morphism

$$\mathrm{ev}\colon \overline{\mathfrak{M}}_{k,l}(B;J) \longrightarrow X_{k,l} \equiv (X^\phi)^k \times X^l \tag{1.4}$$

may not be relatively orientable, but it becomes relatively orientable after removing certain codimension 1 strata from the domain (i.e. the pull-back of the first Stiefel-Whitney class w_1 of the target is the w_1 of the domain). We cut $\mathfrak{M}_{k,l}(B;J)$ along these codimension 1 strata to obtain a bordered manifold $\widehat{\mathfrak{M}}_{k,l}(B;J)$ and give

$$\mathrm{ev}\colon \widehat{\mathfrak{M}}_{k,l}(B;J) \longrightarrow X_{k,l}$$

a relative orientation. The codimension 1 strata of $\mathfrak{M}_{k,l}(B;J)$ consist of curves with two components and one real node.

The forgetful morphisms (1.1) we encounter take values in the subspaces $\overline{\mathcal{M}}_{k',l'}^{\tau}$ of $\mathbb{R}\overline{\mathcal{M}}_{0,k',l'}$ of real curves with non-empty fixed locus; $\overline{\mathcal{M}}_{k',l'}^{\tau}$ is a proper subspace of $\mathbb{R}\overline{\mathcal{M}}_{0,k',l'}$ if and only if $k'=0$. We choose a bordered hypersurface Υ in $\overline{\mathcal{M}}_{k',l'}^{\tau}$ whose boundary consists of curves with three components and a conjugate pair of nodes and a relative orientation on the inclusion of Υ into $\overline{\mathcal{M}}_{k',l'}^{\tau}$. Let $\mathbf{C}\subset X_{k,l}$ be a generic constraint consisting of divisors and points so that the maps

$$\mathrm{ev}\times\mathfrak{f}_{k',l'}\colon \widehat{\mathfrak{M}}_{k,l}(B;J) \longrightarrow X_{k,l}\times\overline{\mathcal{M}}_{k',l'}^{\tau} \quad \text{and} \quad \iota_{\mathbf{C};\Upsilon}\colon \mathbf{C}\times\Upsilon \longhookrightarrow X_{k,l}\times\overline{\mathcal{M}}_{k',l'}^{\tau}$$

are transverse and

$$\dim\widehat{\mathfrak{M}}_{k,l}(B;J) + \dim\big(\mathbf{C}\times\Upsilon\big) = \dim\big(X_{k,l}\times\overline{\mathcal{M}}_{k',l'}^{\tau}\big) + 1\,.$$

With the relative orientations above, the signed counts of the intersection points of

(C1) $\mathrm{ev}\times\mathfrak{f}_{k',l'}$ with the boundary of $\iota_{\mathbf{C};\Upsilon}$ and

(C2) the boundary of $\mathrm{ev}\times\mathfrak{f}_{k',l'}$ with $\iota_{\mathbf{C};\Upsilon}$

are well-defined and equal.

The first count above decomposes into curve-counting invariants similarly to the complex case. The second count can also be decomposed, based on the following observations:

- most boundary strata of $\widehat{\mathfrak{M}}_{k,l}(B;J)$ get contracted by $\mathrm{ev}\times\mathfrak{f}_{k',l'}$ and thus do not contribute to (C2);

- some boundary strata that do not get contracted do not intersect Υ via $\mathfrak{f}_{k',l'}$ due to our choice of $\Upsilon \subset \overline{\mathcal{M}}^{\tau}_{k',l'}$ and thus do not contribute to (C2) either;
- intersecting the remaining boundary strata with Υ via $\mathfrak{f}_{k',l'}$ has the effect of specifying the position of the node (relative to the marked points) on the component of the curve carrying the first conjugate pair of marked points.

These statements are explained in the proof of Corollary 5.10 at the end of Section 5.3 and in the proof of Proposition 5.7 in Section 6.3. The equality of the counts (C1) and (C2) then translates into ($\mathbb{R}$WDVV1) in the case $(k', l') = (1, 2)$ and into ($\mathbb{R}$WDVV2) and($\mathbb{R}$WDVV3) in the case $(k', l') = (0, 3)$.

The paper is organized as follows. Section 2 is a detailed version of the above outline of the proof of Theorem 1.1. The notions of relative orientations, pseudocycles with relative orientations (called Steenrod pseudocycles), and intersection signs between them are defined in Section 3; this section also contains all relevant observations concerning these notions. Section 4 describes in detail the hypersurfaces Υ in the Deligne-Mumford spaces $\overline{\mathcal{M}}^{\tau}_{1,2}$ and $\overline{\mathcal{M}}^{\tau}_{0,3}$ used in the proof of Theorem 1.1. Section 5 sets up the notation relevant to the map spaces $\mathfrak{M}_{k,l}(B; J)$, states the propositions that are among the main steps in the proof of Theorem 1.1, and deduces this theorem from them and the lemmas of Section 4.4. The (somewhat technical) proofs of these propositions are deferred to Section 6.

2 Summary of the proof of Theorem 1.1

The numbers $N^{\phi,\mathfrak{p}}_{B,l} \equiv N^{\phi,\mathfrak{p}}_{B,l}(\check{X}^{\phi})$ appearing in Theorem 1.1 arise from the moduli space $\overline{\mathfrak{M}}_{k,l}(B; J)$ of genus 0 real degree B J-holomorphic maps to X that take the fixed locus of the domain to the chosen topological component X^{ϕ} of the fixed locus of ϕ. This moduli space has no boundary if

$$k + 2\mathbb{Z} \neq \langle w_2(X), B \rangle \in \mathbb{Z}_2 . \tag{2.1}$$

By [24], a Pin$^-$-structure $\mathfrak{p}$ on X^{ϕ} can be used to specify a relative orientation of the restriction of the total evaluation morphism (1.4) to the main stratum $\mathfrak{M}_{k,l}(B; J)$ of $\overline{\mathfrak{M}}_{k,l}(B; J)$ if (2.1) holds. Since $\mathfrak{M}_{k,l}(B; J)$ is generally disconnected, there are a number of systematic ways of doing so, some of which we index by $l^* \in \mathbb{Z}^{\geqslant 0}$ with $l^* \leqslant l$. By [24] again, these orientations extend across some codimension 1 boundary strata $\mathcal{S}$, but not others. In our setup, the "l^*-orientation" $\mathfrak{o}_{\mathfrak{p};l^*}$ on the restriction of (1.4) to $\mathfrak{M}_{k,l}(B; J)$ extends over a such stratum $\mathcal{S}$ if and only if a certain $\mathbb{Z}$-valued invariant $\epsilon_{l^*}(\mathcal{S})$ of $\mathcal{S}$ is congruent to 0 or 1 mod 4; see Lemma 5.1.

If $k, l \in \mathbb{Z}^{\geqslant 0}$ and $B \in H_2(X)$ satisfy (1.3), the path in $\overline{\mathfrak{M}}_{k,l}(B; J)$ determined by a generic path of collections of k points in X^{ϕ} and l points in $X - X^{\phi}$ and of almost complex structures $J_t \in \mathcal{J}^{\phi}_{\omega}$ does not cross the codimension 1 boundary strata $\mathcal{S}$ with $\epsilon_0(\mathcal{S})$ congruent to 2 or 3 mod 4. This fundamental insight, formulated in terms of moduli spaces of disk maps in [24], along with the above orientation statements established the invariance of the counts $N_{B,l}^{\phi;\mathfrak{p}}$ and has since been used to construct numerical invariants in some other settings.

The image of each codimension 1 stratum $\mathcal{S}$ with $\epsilon_0(\mathcal{S})$ congruent to 2 or 3 mod 4 under (1.4) is of smaller dimension than $\mathcal{S}$. Along with the orientation statements above, this implies that the restriction of (1.4) to the complement $\overline{\mathfrak{M}}_{k,l;0}^{\star}(B;J)$ of the closures $\overline{\mathcal{S}}$ of these strata is a codimension 0 **Steenrod pseudocycle** with respect to the orientation $\mathfrak{o}_{\mathfrak{p};0}$; see Proposition 5.2. The number $N_{B,l}^{\phi,\mathfrak{p}}$ is the **degree** $\deg(\mathrm{ev},\mathfrak{o}_{\mathfrak{p};0})$ of this pseudocycle.

The orientations on the restriction of (1.4) to $\mathfrak{M}_{k,l}(B;J)$ relevant to lifting relations from $\overline{\mathcal{M}}_{1,2}^{\tau}$ and $\overline{\mathcal{M}}_{0,3}^{\tau}$ to $\overline{\mathfrak{M}}_{k,l}(B;J)$ are the orientations $\mathfrak{o}_{\mathfrak{p};l^*}$ of Lemma 5.1 with $l^*=2,3$, as we would like to apply the lifted relations with two and three divisor insertions. The relevant restriction of (1.4) shrinks the codimension 1 strata $\mathcal{S}$ with $\epsilon_{l^*}(\mathcal{S})$ congruent to 2 or 3 mod 4, but *not* with $\epsilon_{l^*}(\mathcal{S})=2$. In order to deal with this issue, we cut $\overline{\mathfrak{M}}_{k,l}(B;J)$ along the closures $\overline{\mathcal{S}}$ of the strata $\mathcal{S}$ with $\epsilon_{l^*}(\mathcal{S})$ congruent to 2 or 3 mod 4. We obtain a moduli space $\widehat{\mathfrak{M}}_{k,l;l^*}(B;J)$ with boundary consisting of double covers $\widehat{\mathcal{S}}$ of these strata. The relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}$ extends to a relative orientation $\widehat{\mathfrak{o}}_{\mathfrak{p};l^*}$ of the total evaluation morphism

$$\mathrm{ev}\colon \widehat{\mathfrak{M}}_{k,l;l^*}(B;J)\longrightarrow X_{k,l} \tag{2.2}$$

induced by (1.4).

Suppose $k,l\in\mathbb{Z}^{\geqslant 0}$ and $B\in H_2(X)$ are as in (1.3), $k'\leqslant k$, and $l^*\leqslant l'\leqslant l+l^*-1$ so that there are well-defined forgetful morphisms

$$\mathfrak{f}_{k',l'}\colon \overline{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J)\longrightarrow \overline{\mathcal{M}}_{k',l'}^{\tau} \quad\text{and}\quad \mathfrak{f}_{k',l'}\colon \widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J)\longrightarrow \overline{\mathcal{M}}_{k',l'}^{\tau}\,. \tag{2.3}$$

An l^*-tuple $\mathbf{h}\equiv(H_1,\ldots,H_{l^*})$ of divisors in X cuts out the subspace

$$\widehat{\mathcal{Z}}_{k,l+l^*-1;\mathbf{h}}^{\star}(B;J)\subset \widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J)\times H_1\times\ldots\times H_{l^*}$$

of maps with the first l^* non-real marked points lying on $H_1,\ldots,H_{l^*}$. The relative orientation $\widehat{\mathfrak{o}}_{\mathfrak{p};l^*}$ of (2.2) and the orientation $\mathfrak{o}_{\mathbf{h}}$ on $H_1\times\ldots\times H_{l^*}$ induce a relative orientation $\widehat{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}}$ of the evaluation morphism

$$\mathrm{ev}_{\mathbf{h}}\colon \widehat{\mathcal{Z}}_{k,l+l^*-1;\mathbf{h}}^{\star}(B;J)\longrightarrow X_{k,l-1}$$

at the remaining marked points. A tuple $\mathbf{p}$ of points in $X_{k,l-1}$ and a bordered compact real hypersurface $\Upsilon\subset\overline{\mathcal{M}}_{k',l'}^{\tau}$ determine an embedding

$$f_{\mathbf{p};\Upsilon}\colon \Upsilon\longrightarrow X_{k,l-1}\times\overline{\mathcal{M}}_{k',l'}^{\tau}\,.$$

Under appropriate regularity assumptions, the fiber product $M_{(\mathrm{ev}_{\mathbf{h}},\mathfrak{f}_{k',l'}),f_{\mathbf{p};\Upsilon}}$ of

$$\big(\mathrm{ev}_{\mathbf{h}},\mathfrak{f}_{k',l'}\big)\colon \widehat{\mathcal{Z}}_{k,l+l^*-1;\mathbf{h}}^{\star}(B;J)\longrightarrow X_{k,l-1}\times\overline{\mathcal{M}}_{k',l'}^{\tau}$$

with $f_{\mathbf{p};\Upsilon}$ is a compact one-dimensional manifold with the boundary

$$\begin{split}\partial M_{(\mathrm{ev}_{\mathbf{h}},\mathfrak{f}_{k',l'}),f_{\mathbf{p};\Upsilon}} &= \widehat{\mathcal{Z}}_{k,l+l^*-1;\mathbf{h}}^{\star}(B;J)\;{}_{(\mathrm{ev}_{\mathbf{h}},\mathfrak{f}_{k',l'})}\!\times_{f_{\mathbf{p};\Upsilon}}\partial\Upsilon\\ &\qquad\sqcup\big(\partial\widehat{\mathcal{Z}}_{k,l+l^*-1;\mathbf{h}}^{\star}(B;J)\big)\;{}_{(\mathrm{ev}_{\mathbf{h}},\mathfrak{f}_{k',l'})}\!\times_{f_{\mathbf{p};\Upsilon}}\Upsilon\,.\end{split} \tag{2.4}$$

The relative orientation $\widehat{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}}$ and a co-orientation $\mathfrak{o}^c_\Upsilon$ on Υ determine signs of the points on the right-hand side of (2.4) so that

$$\begin{aligned} \big|\widehat{\mathcal{Z}}^\star_{k,l+l^*-1;\mathbf{h}}(B;J)\,{}_{(\mathrm{ev}_\mathbf{h},\mathfrak{f}_{k',l'})}\!\times_{f_{\mathbf{p};\Upsilon}}\partial\Upsilon\big|^{\pm}_{\widehat{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}},\partial\mathfrak{o}^c_\Upsilon} \\ = (-1)^{\dim\Upsilon}\big|\big(\partial\widehat{\mathcal{Z}}^\star_{k,l+l^*-1;\mathbf{h}}(B;J)\big)\,{}_{(\mathrm{ev}_\mathbf{h},\mathfrak{f}_{k',l'})}\!\times_{f_{\mathbf{p};\Upsilon}}\Upsilon\big|^{\pm}_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_\Upsilon}, \end{aligned} \tag{2.5}$$

where $|\cdot|^{\pm}$ denotes the signed cardinality; see Lemma 3.5.

Since only the strata $\widehat{\mathcal{S}}$ of $\partial\widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J)$ with $\epsilon_{l^*}(\widehat{\mathcal{S}})=2$ are not shrunk by (2.2), only the strata

$$\widehat{\mathcal{S}}_\mathbf{h} \equiv \big(\widehat{\mathcal{S}}\times H_1\times\ldots\times H_{l^*}\big)\cap\widehat{\mathcal{Z}}^\star_{k,l+1;\mathbf{h}}(B;J)$$

of $\widehat{\mathcal{Z}}^\star_{k,l+l^*-1;\mathbf{h}}(B;J)$ with $\epsilon_{l^*}(\widehat{\mathcal{S}})=2$ contribute to the right-hand side of (2.5). Since $\widehat{\mathcal{S}}_\mathbf{h}$ is a double cover of the subspace

$$\mathcal{S}_\mathbf{h}\subset\mathcal{S}\times H_1\times\ldots\times H_{l^*}$$

of maps with the first l^* non-real marked points lying on $H_1,\ldots,H_{l^*}$, we conclude that

$$\big|\big(\partial\widehat{\mathcal{Z}}^\star_{k,l+l^*-1;\mathbf{h}}(B;J)\big)\,{}_{(\mathrm{ev}_\mathbf{h},\mathfrak{f}_{k',l'})}\!\times_{f_{\mathbf{p};\Upsilon}}\Upsilon\big|^{\pm}_{\partial\widehat{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_\Upsilon} = 2\sum_{\epsilon_{l^*}(\mathcal{S})=2}\big|(\mathcal{S}_\mathbf{h})\,{}_{(\mathrm{ev}_\mathbf{h},\mathfrak{f}_{k',l'})}\!\times_{f_{\mathbf{p};\Upsilon}}\Upsilon\big|^{\pm}_{\partial\widehat{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_\Upsilon}.$$

The moduli space $\overline{\mathcal{M}}^\tau_{k',l'}$ contains codimension 1 strata S_i with $i\leqslant l'$ parametrizing marked curves with two real components so that one of the components carries only the i-th conjugate pair of marked points. We establish Theorem 1.1 by applying (2.5) with certain bordered compact hypersurfaces Υ in $\overline{\mathcal{M}}^\tau_{1,2}\approx\mathbb{RP}^2$ and in the three-dimensional orientable manifold $\overline{\mathcal{M}}^\tau_{0,3}$ so that Υ is *disjoint* from the closure $\overline{S}_1$ of S_1.

The moduli space $\overline{\mathcal{M}}^\tau_{1,2}$ contains two points $P^\pm$ corresponding to the two marked curves consisting of one real component and one conjugate pair of components; see the diagrams on the left-hand side of the first row in Figure 1. We take Υ in $\overline{\mathcal{M}}^\tau_{1,2}-\overline{S}_1$ to be a path from P^- to P^+ as in Lemma 4.4. In this case, (2.5) is represented by the first row in Figure 1. The labels $\epsilon_{l^*}(\mathcal{S})=2$ and $\cap\Upsilon$ under the diagrams on the right-hand side indicate that only "intersections" of some strata of two-component maps with Υ contribute to this relation. These intersections arise from the last part of the boundary in (2.4) and thus contribute twice each (with the same sign). The strata of two-component maps whose contributions are described as being *insignificant due to sign cancellations* in [25, p10] do not appear in our approach at all.

The one-dimensional strata of $\overline{\mathcal{M}}^\tau_{0,3}$ that parametrize marked curves consisting of one real component and one conjugate pair of components come in three pairs $\Gamma^\pm_i$ with $i=1,2,3$; see the diagrams on the left-hand side of the second row in Figure 1. The closures $\overline{\Gamma}^\pm_i$ of these strata with $i=2,3$ bound a compact oriented surface Υ in $\overline{\mathcal{M}}^\tau_{0,3}-\overline{S}_1$ as in Lemma 4.5. In this case, (2.5) is represented by the second row in Figure 1. The curves represented by the diagrams on the right-hand side in this relation again arise from the last part of the boundary in (2.4).

We apply the relations represented by Figure 1 with the divisors H_1,H_2 as the first two non-real insertions and points as the remaining insertions; we also apply the second relation

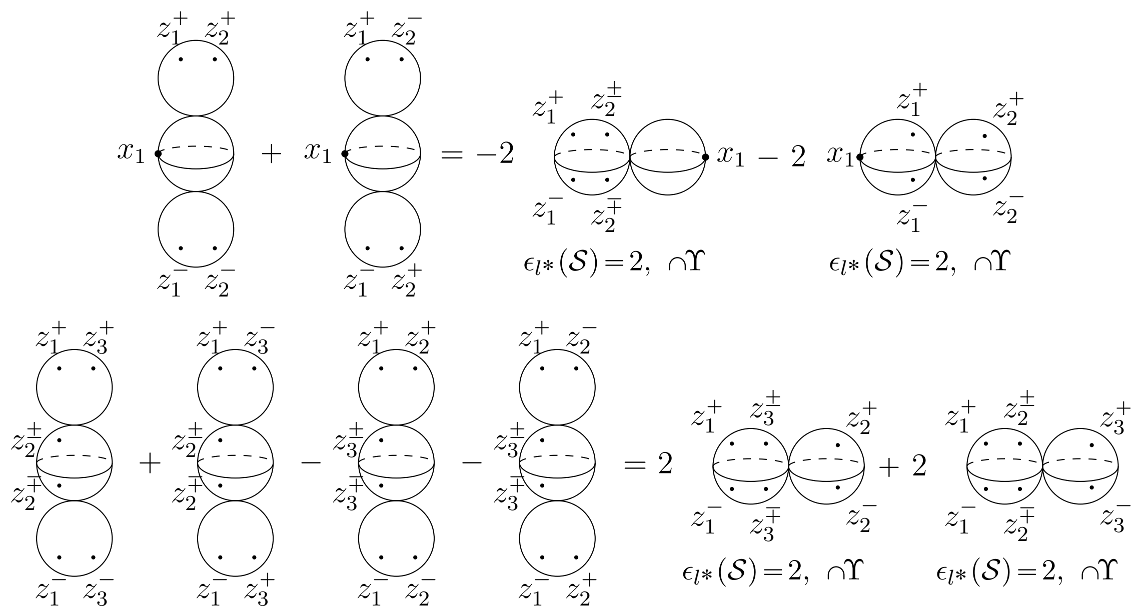

Figure 1: The relations on stable maps induced via (2.5) by lifting codimension 2 relations from $\overline{\mathcal{M}}_{1,2}^{\tau}$ and $\overline{\mathcal{M}}_{0,3}^{\tau}$; the curves on the right-hand sides of the two relations are constrained by the hypersurfaces Υ in $\overline{\mathcal{M}}_{1,2}^{\tau}$ and $\overline{\mathcal{M}}_{0,3}^{\tau}$.

with the divisors H_1, H_2, H_3 as the first three non-real insertions. The normal bundle to the strata of maps represented by the three-component curves in this figure is canonically oriented. Thus, the restriction of the total evaluation map (1.4) to these strata inherits a relative orientation from its restriction to $\mathfrak{M}_{k,l}(B;J)$. The proof of [10, Prop. 4.2] readily applies to express the associated counts of nodal maps in terms of the real map counts $N_{B,l}^{\phi,\mathfrak{p}}$ and the complex map counts N_B^X; see Proposition 5.3.

The map counts represented by the two-component curves in Figure 1 are more elaborate. Each stratum $\mathcal{S}_{\mathbf{h}}$ of such maps is the fiber product of the evaluation morphisms

$$\mathrm{ev}_{\mathrm{nd}}\colon \mathcal{Z}_1\equiv\mathcal{Z}_{k_1+1,l_1;\mathbf{h}_1}(B_1;J)\longrightarrow X^{\phi} \quad \text{and} \quad \mathrm{ev}_{\mathrm{nd}}\colon \mathcal{Z}_2\equiv\mathcal{Z}_{k_2+1,l_2;\mathbf{h}_2}(B_2;J)\longrightarrow X^{\phi}$$

at the nodal points from moduli spaces associated with the two components, for a split of $\mathbf{h}$ into an l_1^*-tuple $\mathbf{h}_1$ and an l_2^*-tuple $\mathbf{h}_2$. The condition $\epsilon_{l^*}(\mathcal{S})=2$ implies that each of the total evaluation morphisms

$$\begin{aligned} \mathrm{ev}'_{\mathbf{h}_1}\colon \mathcal{Z}'_1\equiv\mathcal{Z}_{k_1,l_1;\mathbf{h}_1}(B_1;J)&\longrightarrow X_{k_1,l_1-l_1^*} \qquad \text{and} \\ \mathrm{ev}_{\mathbf{h}_2}\equiv\big(\mathrm{ev}'_{\mathbf{h}_2},\mathrm{ev}_{\mathrm{nd}}\big)\colon \mathcal{Z}_2&\longrightarrow X_{k_2+1,l_2-l_2^*}\equiv X_{k_2,l_2-l_2^*}\times X^{\phi} \end{aligned} \tag{2.6}$$

is a map between spaces of the same dimensions. The latter implies that (2.1) with (k,B) replaced by either (k_1,B_1) and (k_2+1,B_2) holds. Thus, the maps $\mathrm{ev}'_{\mathbf{h}_1}$ and $\mathrm{ev}_{\mathbf{h}_2}$ have well-defined degrees $\deg(\mathrm{ev}'_{\mathbf{h}_1},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1})$ and $\deg(\mathrm{ev}_{\mathbf{h}_2},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_2})$ with respect to the relative orientations induced by the Pin$^-$-structure $\mathfrak{p}$ and the orientations of $H_1,\ldots,H_{l^*}$. These degrees are related to the map counts $N^{\phi,\mathfrak{p}}_{B_1,l_1-l_1^*}$ and $N^{\phi,\mathfrak{p}}_{B_2,l_2-l_2^*}$ (with k_1 and k_2+1 real point insertions, respectively) via the divisor relation (5.14).

A crucial consequence of our choices of the hypersurfaces $\Upsilon\subset\overline{\mathcal{M}}^{\tau}_{0,3}$ is that the restriction of the first morphism in (2.3) to $\mathcal{S}_{\mathbf{h}}$ factors through a morphism

$$\mathfrak{f}_1\colon \mathcal{Z}_1 \longrightarrow \overline{\mathcal{M}}^{\tau}_{k',l'}$$

if $\epsilon_{l^*}(\mathcal{S})=2$ and $\mathcal{S}\cap\mathfrak{f}_{k',l'}^{-1}(\Upsilon)\neq\varnothing$; see Corollary 5.10. Thus,

$$(\mathcal{S}_{\mathbf{h}})\,{}_{(\mathrm{ev}_{\mathbf{h}},\mathfrak{f}_{k',l'})}\!\times_{f_{\mathbf{p};\Upsilon}}\Upsilon = \big((\mathcal{Z}_1)\,{}_{(\mathrm{ev}'_{\mathbf{h}_1},\mathfrak{f}_1)}\!\times_{f_{\mathbf{p}_1;\Upsilon}}\Upsilon\big)\,{}_{\mathrm{ev}_{\mathrm{nd}}}\!\times_{\mathrm{ev}_{\mathrm{nd}}}\mathrm{ev}_{\mathbf{h}_2}^{-1}(\mathbf{p}_2), \tag{2.7}$$

for a split of $\mathbf{p}$ into a k_1-tuple $\mathbf{p}_1$ and a k_2-tuple $\mathbf{p}_2$. The equality above holds set-theoretically; Lemma 3.4 compares the signs on the two sides. The morphism $\mathrm{ev}'_{\mathbf{h}_1}$ on the right-hand side of this equality denotes the composition of (2.6) with the natural projection

$$\mathfrak{f}\colon \mathcal{Z}_{\mathbf{h}_1} \longrightarrow \mathcal{Z}'_{\mathbf{h}_1}$$

dropping the real marked point corresponding to the node. Thus,

$$(\mathcal{Z}_1)\,{}_{(\mathrm{ev}'_{\mathbf{h}_1},\mathfrak{f}_1)}\!\times_{f_{\mathbf{p}_1;\Upsilon}}\Upsilon = \big\{u_1\in\mathcal{Z}_1|_{\mathrm{ev}'^{-1}_{\mathbf{h}_1}(\mathbf{p}_1)}\colon \mathfrak{f}_1(u_1)\in\Upsilon\big\};$$

Lemma 3.3 compares the signs on the two sides. Since this set is finite, (2.7) implies that

$$\big|(\mathcal{S}_{\mathbf{h}})\,{}_{(\mathrm{ev}_{\mathbf{h}},\mathfrak{f}_{k',l'})}\!\times_{f_{\mathbf{p};\Upsilon}}\Upsilon\big|^{\pm}_{\partial\widehat{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}} = \alpha(\mathcal{S},\Upsilon)\deg(\mathrm{ev}'_{\mathbf{h}_1},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1})\deg(\mathrm{ev}_{\mathbf{h}_2},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_2})$$

for some $\alpha(\mathcal{S},\Upsilon)\in\mathbb{Z}$ determined by $\mathcal{S}$ and Υ. This leads to a decomposition of the nodal map counts associated with the two-component diagrams in Figure 1 into sums of pairwise products of the real map counts $N^{\phi,\mathfrak{p}}_{B,l}$; see Proposition 5.7.

3 Topological preliminaries

3.1 Relative orientations

For a real vector space or vector bundle V, let $\lambda(V)\equiv\Lambda^{\mathrm{top}}_{\mathbb{R}}V$ be its top exterior power. For a manifold M, possibly with nonempty boundary ∂M, we denote by

$$\lambda(M)\equiv\lambda(TM)\equiv\Lambda^{\mathrm{top}}_{\mathbb{R}}TM\longrightarrow M$$

its orientation line bundle. An orientation of M is a homotopy class of trivializations of $\lambda(M)$. By definition, $\lambda(\mathrm{pt})=\mathbb{R}$. We identify the two orientations of any point with ±1 in the obvious way.

For submanifolds $S'\subset S\subset M$, the short exact sequences

$$0 \longrightarrow TS \longrightarrow TM|_S \longrightarrow \mathcal{N}S \equiv \frac{TM|_S}{TS} \longrightarrow 0 \qquad \text{and}$$
$$0 \longrightarrow \mathcal{N}_S S' \equiv \frac{TS|_{S'}}{TS'} \longrightarrow \mathcal{N}S' \equiv \frac{TM|_{S'}}{TS'} \longrightarrow \mathcal{N}S|_{S'} \equiv \frac{TM|_{S'}}{TS|_{S'}} \longrightarrow 0$$

of vector spaces determine isomorphisms

$$\lambda(M)\big|_S \approx \lambda(S)\otimes\lambda(\mathcal{N}S) \quad \text{and} \quad \lambda(\mathcal{N}S') \approx \lambda(\mathcal{N}_S S')\otimes\lambda(\mathcal{N}S)\big|_{S'} \tag{3.1}$$

of line bundles over S and S', respectively. A co-orientation of S in M is an orientation of $\mathcal{N}S$. We define the canonical co-orientation $\mathfrak{o}^c_{\partial M}$ of ∂M in M to be given by the outer normal direction.

For a fiber bundle $\mathfrak{f}_\mathcal{M}: \mathcal{M} \longrightarrow \mathcal{M}'$, we denote by $T\mathcal{M}^v \equiv \ker \mathrm{d}\mathfrak{f}_\mathcal{M}$ its vertical tangent bundle. The short exact sequence

$$0 \longrightarrow T\mathcal{M}^v \longrightarrow T\mathcal{M} \xrightarrow{\mathrm{d}\mathfrak{f}_\mathcal{M}} \mathfrak{f}^*_\mathcal{M} T\mathcal{M}' \longrightarrow 0 \tag{3.2}$$

of vector bundles determines an isomorphism

$$\lambda(\mathcal{M}) \approx \mathfrak{f}^*_\mathcal{M}\lambda(\mathcal{M}')\otimes\lambda(T\mathcal{M}^v) \tag{3.3}$$

of line bundles over $\mathcal{M}$. The switch of the ordering of the factors in (3.3) from (3.2) is motivated by Lemma 3.1(1) below and by the inductive construction of the orientations $\mathfrak{o}_{k,l}$ on the real Deligne-Mumford moduli spaces $\overline{\mathcal{M}}^\tau_{k,l}$ in Section 4.1.

If $f: \mathcal{Z} \longrightarrow Y$ is a continuous map between two smooth manifolds, possibly with boundary, let

$$\lambda(f) \equiv f^*\lambda(Y)^*\otimes\lambda(\mathcal{Z}) \longrightarrow \mathcal{Z}\,.$$

A relative orientation of f is an orientation on the line bundle $\lambda(f)$. For a relative orientation $\mathfrak{o}$ of f and $u\in\mathcal{Z}$, we denote by $\mathfrak{o}_u$ the associated homotopy class of trivializations of the fiber $\lambda_u(f)$ over u and the associated homotopy class of isomorphisms $\lambda_u(\mathcal{Z}) \longrightarrow \lambda_{f(u)}(Y)$. If in addition $\mathfrak{o}'$ is a relative orientation of another continuous map $g: Y \longrightarrow Z$, we denote by $\mathfrak{o}\mathfrak{o}'$ the relative orientation of $g\circ f$ corresponding to the homotopy class of the compositions

$$\lambda_u(\mathcal{Z}) \longrightarrow \lambda_{f(u)}(Y) \longrightarrow \lambda_{g(f(u))}(Z)$$

of isomorphisms in the homotopy classes $\mathfrak{o}_u$ and $\mathfrak{o}'_{f(u)}$ for each $u\in\mathcal{Z}$.

We identify an orientation $\mathfrak{o}$ on a manifold $\mathcal{Z}$ with a relative orientation of $\mathcal{Z} \longrightarrow \mathrm{pt}$ in the obvious way. For a submanifold $\mathcal{S}\subset\mathcal{Z}$, we identify a co-orientation $\mathfrak{o}^c_\mathcal{S}$ on $\mathcal{S}$ with a relative orientation of the inclusion $\iota_\mathcal{S}: \mathcal{S} \longrightarrow \mathcal{Z}$ via the first isomorphism in (3.1). If $\mathcal{S}'\subset\mathcal{S}$ is also a submanifold with a co-orientation $\mathfrak{o}^c_{\mathcal{S}'}$ in $\mathcal{S}$, then the relative orientation $\mathfrak{o}^c_{\mathcal{S}'}\mathfrak{o}^c_\mathcal{S}$ of the inclusion

$$\iota_{\mathcal{S}'}: \mathcal{S}' \longrightarrow \mathcal{S} \longrightarrow \mathcal{Z}$$

corresponds to the co-orientation of $\mathcal{S}'$ in $\mathcal{Z}$ induced by the co-orientations $\mathfrak{o}^c_\mathcal{S}$ and $\mathfrak{o}^c_{\mathcal{S}'}$ via the second isomorphism in (3.1). If $\mathfrak{f}_\mathcal{M}: \mathcal{M} \longrightarrow \mathcal{M}'$ is a fiber bundle, we similarly identify an orientation $\mathfrak{o}^v_\mathcal{M}$ of $T\mathcal{M}^v$ with a relative orientation of $\mathfrak{f}_\mathcal{M}$ via (3.3).

If f, $\mathfrak{o}$, $\mathcal{S}$, and $\mathfrak{o}^c_{\mathcal{S}}$ are as above, we denote by $\mathfrak{o}|_{\mathcal{S}}$ the restriction of the trivialization of $\lambda(f)$ determined by $\mathfrak{o}$ to $\mathcal{S}$ and define

$$\mathfrak{o}_{\mathcal{S}} \equiv \mathfrak{o}^c_{\mathcal{S}}\mathfrak{o} \tag{3.4}$$

to be **the relative orientation of $\lambda(f|_{\mathcal{S}})$ induced by $\mathfrak{o}$ and $\mathfrak{o}^c_{\mathcal{S}}$**. If $\mathcal{Z}$ is a manifold with boundary, let

$$\partial(\mathcal{Z},\mathfrak{o}) \equiv \big(\partial\mathcal{Z},\partial\mathfrak{o}\big) \equiv \big(\partial\mathcal{Z},\mathfrak{o}^c_{\partial\mathcal{Z}}\mathfrak{o}\big)\,. \tag{3.5}$$

If Y is a point (and so $\mathfrak{o}$ and $\partial\mathfrak{o}$ are orientations on $\mathcal{Z}$ and $\partial\mathcal{Z}$, respectively), this convention agrees with [28, p146] if and only if the dimension of M is odd. If $\mathcal{S}=\{P\}$ is also a point, then the projection isomorphism $T_P\mathcal{Z}\longrightarrow\mathcal{N}\mathcal{S}$ is orientation-preserving with respect to $\mathfrak{o}$ and $\mathfrak{o}^c_{\mathcal{S}}$ if and only if

$$\mathfrak{o}^c_{\mathcal{S}}\mathfrak{o} = +1\,;$$

this is the $\mathcal{M}',\Upsilon=\{\mathrm{pt}\}$ case of Lemma 3.1(1) below.

If $\mathfrak{o}$ is a relative orientation of $f\colon\mathcal{Z}\longrightarrow Y$ and $u\in\mathcal{Z}$ is such that d_uf is an isomorphism, we define

$$\mathfrak{s}_u(\mathfrak{o}) = \begin{cases} +1, & \text{if } \mathrm{d}_uf\in\mathfrak{o}_u;\\ -1, & \text{if } \mathrm{d}_uf\notin\mathfrak{o}_u. \end{cases}$$

If $g\colon Y\longrightarrow Z$ and $\mathfrak{o}'$ are also as above and $\mathrm{d}_{f(u)}g$ is an isomorphism as well, then

$$\mathfrak{s}_u(\mathfrak{o}\mathfrak{o}') = \mathfrak{s}_u(\mathfrak{o})\mathfrak{s}_{f(u)}(\mathfrak{o}')\,. \tag{3.6}$$

If $y\in Y$ is a regular value of f and the set $f^{-1}(y)$ is finite, we define

$$\big|f^{-1}(y)\big|^{\pm}_{\mathfrak{o}} = \sum_{u\in f^{-1}(y)}\mathfrak{s}_u(\mathfrak{o})\,.$$

Let $\mathfrak{f}_{\mathcal{M}}:\mathcal{M}\longrightarrow\mathcal{M}'$ be a fiber bundle. If $\Upsilon\subset\mathcal{M}$ is a submanifold and $P\in\Upsilon$, then the differential $\mathrm{d}_P(\mathfrak{f}_{\mathcal{M}}|_{\Upsilon})$ is an isomorphism if and only if the composition

$$T_P\mathcal{M}^v\equiv\ker\mathrm{d}_P\mathfrak{f}_{\mathcal{M}} \longrightarrow T_P\mathcal{M} \longrightarrow \frac{T_P\mathcal{M}}{T_P\Upsilon}\equiv\mathcal{N}_P\Upsilon \tag{3.7}$$

is. If $\mathcal{M}_2$ is another manifold, then

$$\mathfrak{f}_{\mathcal{M}}\times\mathrm{id}_{\mathcal{M}_2}\colon\mathcal{M}\times\mathcal{M}_2 \longrightarrow \mathcal{M}'\times\mathcal{M}_2$$

is also a fiber bundle and $\Upsilon\times\mathcal{M}_2\subset\mathcal{M}_1\times\mathcal{M}_2$ is a submanifold; see the first diagram in Figure 2. The differential of

$$\pi_1\colon\mathcal{M}\times\mathcal{M}_2 \longrightarrow \mathcal{M} \tag{3.8}$$

induces a commutative diagram

$$\begin{array}{ccc} \ker\{\mathfrak{f}_{\mathcal{M}}\times\mathrm{id}_{\mathcal{M}_2}\}^v & \longrightarrow & \mathcal{N}\big(\Upsilon\times\mathcal{M}_2\big)\\ \Big\downarrow{\scriptstyle \mathrm{d}\pi_1} & & \Big\downarrow{\scriptstyle \mathrm{d}\pi_1}\\ \pi_1^*T\mathcal{M}^v & \longrightarrow & \pi_1^*\mathcal{N}\Upsilon \end{array}$$

$$\begin{array}{ccccc} \Upsilon\times\mathcal{M}_2 & \hookrightarrow & \mathcal{M}\times\mathcal{M}_2 & \xrightarrow{\mathfrak{f}_{\mathcal{M}}\times\mathrm{id}_{\mathcal{M}_2}} & \mathcal{M}'\times\mathcal{M}_2 \\ \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} & & \downarrow{\scriptstyle \pi_1} \\ \Upsilon & \hookrightarrow & \mathcal{M} & \xrightarrow{\mathfrak{f}_{\mathcal{M}}} & \mathcal{M}' \end{array} \qquad \begin{array}{ccccc} f^{-1}(\Upsilon) & \hookrightarrow & \mathcal{Z} & \xrightarrow{\mathfrak{f}_{\mathcal{Z}}} & \mathcal{Z}' \\ \downarrow & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\ \Upsilon & \hookrightarrow & \mathcal{M} & \xrightarrow{\mathfrak{f}_{\mathcal{M}}} & \mathcal{M}' \end{array}$$

Figure 2: The maps of Lemmas 3.1 and 3.2.

of vector bundle homomorphisms. Since the vertical arrows above are isomorphisms, they pull back a vertical orientation $\mathfrak{o}^v_{\mathcal{M}}$ of $\mathfrak{f}_{\mathcal{M}}$ to a vertical orientation $\pi_1^*\mathfrak{o}^v_{\mathcal{M}}$ of $\mathfrak{f}_{\mathcal{M}}\times\mathrm{id}_{\mathcal{M}_2}$ and a co-orientation $\mathfrak{o}^c_{\Upsilon}$ of Υ to a co-orientation $\pi_1^*\mathfrak{o}^c_{\Upsilon}$ of $\Upsilon\times\mathcal{M}_2$. We note the following.

Lemma 3.1. *Suppose $\mathfrak{f}_{\mathcal{M}}:\mathcal{M}\longrightarrow\mathcal{M}'$ is a fiber bundle with an orientation $\mathfrak{o}^v_{\mathcal{M}}$ on $T\mathcal{M}^v$, $\Upsilon\subset\mathcal{M}$ is a submanifold with a co-orientation $\mathfrak{o}^c_{\Upsilon}$, and $P\in\Upsilon$ is such that $\mathrm{d}_P(\mathfrak{f}_{\mathcal{M}}|_{\Upsilon})$ is an isomorphism.*

(1) The isomorphism (3.7) is orientation-preserving with respect to $\mathfrak{o}^v_{\mathcal{M}}$ and $\mathfrak{o}^c_{\Upsilon}$ if and only if $\mathfrak{s}_P(\mathfrak{o}^c_{\Upsilon}\mathfrak{o}^v_{\mathcal{M}})=+1$.

(2) If $\mathcal{M}_2$ is another manifold, π_1 is as in (3.8), and $P_2\in\mathcal{M}_2$, then

$$\mathfrak{s}_{(P,P_2)}\big((\pi_1^*\mathfrak{o}^c_{\Upsilon})(\pi_1^*\mathfrak{o}^v_{\mathcal{M}})\big) = \mathfrak{s}_P\big(\mathfrak{o}^c_{\Upsilon}\mathfrak{o}^v_{\mathcal{M}}\big).$$

Suppose that $\mathfrak{f}_{\mathcal{Z}}:\mathcal{Z}\longrightarrow\mathcal{Z}'$ is another fiber bundle, f,f' are maps as in the second diagram in Figure 2 so that it commutes, and $\mathfrak{o}^v_{\mathcal{Z}}$ and $\mathfrak{o}^v_{\mathcal{M}}$ are orientations on $T\mathcal{Z}^v$ and $T\mathcal{M}^v$, respectively. If $u\in\mathcal{Z}$ is such that the restriction

$$\mathrm{d}_u f: T_u\mathcal{Z}^v\equiv\ker\mathrm{d}_u\mathfrak{f}_{\mathcal{Z}}\longrightarrow T_{f(u)}\mathcal{M}^v \tag{3.9}$$

is an isomorphism, we define $\mathfrak{s}_u(f,\mathfrak{o}^v_{\mathcal{Z}},\mathfrak{o}^v_{\mathcal{M}})$ to be $+1$ if this isomorphism is orientation-preserving with respect to the orientations $\mathfrak{o}^v_{\mathcal{Z}}$ and $\mathfrak{o}^v_{\mathcal{M}}$ and to be -1 otherwise. If $P\in\Upsilon$ are as above, $u\in f^{-1}(P)$, and the homomorphisms (3.7) and (3.9) are isomorphisms, then f is transverse to Υ at u, $f^{-1}(\Upsilon)\subset\mathcal{Z}$ is a smooth submanifold near u, the composition

$$T_u\mathcal{Z}^v\equiv\ker\mathrm{d}_u\mathfrak{f}_{\mathcal{Z}}\longrightarrow T_u\mathcal{Z}\longrightarrow\frac{T_u\mathcal{Z}}{T_uf^{-1}(\Upsilon)}\equiv\mathcal{N}_uf^{-1}(\Upsilon)$$

is an isomorphism, and d_ug descends to an isomorphism

$$\mathrm{d}_ug:\mathcal{N}_uf^{-1}(\Upsilon)\equiv\frac{T_u\mathcal{Z}}{T_uf^{-1}(\Upsilon)}\longrightarrow\frac{T_P\mathcal{M}}{T_P\Upsilon}\equiv\mathcal{N}_P\Upsilon$$

and thus pulls back a co-orientation $\mathfrak{o}^c_{\Upsilon}$ on $\Upsilon\subset\mathcal{M}$ to a co-orientation $f^*\mathfrak{o}^c_{\Upsilon}$ on $f^{-1}(\Upsilon)\subset\mathcal{Z}$ near u.

Lemma 3.2. *Let $\mathfrak{f}_{\mathcal{M}}$, Υ, P, $\mathfrak{o}^v_{\mathcal{M}}$, and $\mathfrak{o}^c_{\Upsilon}$ be as in Lemma 3.1. Suppose in addition that $\mathfrak{f}_{\mathcal{Z}}:\mathcal{Z}\longrightarrow\mathcal{Z}'$ is another fiber bundle with an orientation $\mathfrak{o}^v_{\mathcal{Z}}$ on $T\mathcal{Z}^v$, f,f' are maps so that the second diagram in Figure 2 commutes, and $u\in f^{-1}(P)$. If the homomorphism (3.9) is an isomorphism, then*

$$\mathfrak{s}_u\big((f^*\mathfrak{o}^c_{\Upsilon})\mathfrak{o}^v_{\mathcal{Z}}\big) = \mathfrak{s}_u(f,\mathfrak{o}^v_{\mathcal{Z}},\mathfrak{o}^v_{\mathcal{M}})\mathfrak{s}_P\big(\mathfrak{o}^c_{\Upsilon}\mathfrak{o}^v_{\mathcal{M}}\big)\,.$$

3.2 Intersection signs

For continuous maps $f\colon \mathcal{Z}\longrightarrow Y$ and $g\colon \Upsilon\longrightarrow Y$ between manifolds with boundary, define

$$\begin{gathered} M_{f,g}\equiv \mathcal{Z}_f\!\times_g\Upsilon=\big\{(u,P)\in\mathcal{Z}\times\Upsilon-(\partial\mathcal{Z})\times(\partial\Upsilon)\colon f(u)=g(P)\big\},\\ f\times_Y g\colon M_{f,g}\longrightarrow Y, \qquad f\times_Y g(u,P)=f(u)=g(P). \end{gathered}$$

We call two such maps f and g strongly transverse if they are smooth and the maps f and $f|_{\partial\mathcal{Z}}$ are transverse to the maps g and $g|_{\partial\Upsilon}$. The space $M_{f,g}$ is then a smooth manifold and

$$\begin{gathered} \dim M_{f,g}+\dim Y=\dim\mathcal{Z}+\dim\Upsilon\,,\\ \partial M_{f,g}=\big(\mathcal{Z}-\partial\mathcal{Z}\big)\,{}_f\!\times_g(\partial\Upsilon)\sqcup(\partial\mathcal{Z})\,{}_f\!\times_g\big(\Upsilon-\partial\Upsilon\big)\,. \end{gathered} \tag{3.10}$$

Suppose in addition that

$$f=(f_1,f_2)\colon\mathcal{Z}\longrightarrow Y\equiv X\times\mathcal{M}, \quad g=(g_1,g_2)\colon\Upsilon\longrightarrow X\times\mathcal{M}, \tag{3.11}$$

$\mathfrak{o}_1$ is a relative orientation of f_1, and $\mathfrak{o}_2$ is a relative orientation of g_2; see the second rows in the diagrams of Figure 3. For $(u,P)\in M_{f,g}$ such that the homomorphism

$$T_u\mathcal{Z}\oplus T_P\Upsilon\longrightarrow T_{f(u)}Y=T_{g(P)}Y, \quad (v,w)\longrightarrow \mathrm{d}_uf(v)+\mathrm{d}_Pg(w), \tag{3.12}$$

is an isomorphism, we define $(f,\mathfrak{o}_1)_u\cdot_P(g,\mathfrak{o}_2)$ to be $+1$ if the top exterior power $\Lambda^{\mathrm{top}}_{\mathbb{R}}$ of this isomorphism lies in the homotopy class determined by $(\mathfrak{o}_1)_u$ and $(\mathfrak{o}_2)_P$ and to be -1 otherwise. If $M_{f,g}$ is also finite, let

$$\big|M_{f,g}\big|^{\pm}_{\mathfrak{o}_1,\mathfrak{o}_2}=\sum_{(u,P)\in M_{f,g}}(f,\mathfrak{o}_1)\,{}_u\!\cdot_P(g,\mathfrak{o}_2)\,.$$

Suppose $\Upsilon\subset\mathcal{M}$ is a bordered submanifold with co-orientation $\mathfrak{o}^c_\Upsilon$, $g_1:\Upsilon\longrightarrow X$ is a constant map, and $g_2\colon\Upsilon\longrightarrow\mathcal{M}$ is the inclusion. Then,

$$M_{f,g}=\big\{\big(u,f_2(u)\big)\colon u\in f_2^{-1}(\Upsilon)\cap f_1^{-1}(g_1(\Upsilon))-(\partial\mathcal{Z})\cap f_2^{-1}(\partial\Upsilon)\big\}.$$

If in addition f is strongly transverse to g, then $f_2^{-1}(\Upsilon)\subset\mathcal{Z}$ is a smooth submanifold with co-orientation $f^*\mathfrak{o}^c_\Upsilon$ and the restriction

$$f_1\colon f_2^{-1}(\Upsilon)\longrightarrow X$$

is a submersion. Along with the relative orientation $\mathfrak{o}_1$ on f_1, $f_2^*\mathfrak{o}^c_\Upsilon$ induces a relative orientation $(f_2^*\mathfrak{o}^c_\Upsilon)\mathfrak{o}_1$ on this restriction as in (3.4). If in addition $S\subset\mathcal{M}$ is another submanifold strongly transverse to Υ, then the homomorphism

$$\mathcal{N}_S(\Upsilon\cap S)\equiv\frac{TS|_{\Upsilon\cap S}}{T(\Upsilon\cap S)}\longrightarrow\frac{T\mathcal{M}|_{\Upsilon\cap S}}{T\Upsilon|_{\Upsilon\cap S}}\equiv\mathcal{N}\Upsilon|_{\Upsilon\cap S}$$

of vector bundles is an isomorphism. The co-orientation $\mathfrak{o}^c_\Upsilon$ of Υ in $\mathcal{M}$ then restricts to a co-orientation $\mathfrak{o}^c_\Upsilon|_{\Upsilon\cap S}$. In such a case,

$$M_{f,g}=M_{f,g|_{\Upsilon\cap S}}\,.$$

The following observations are straightforward.

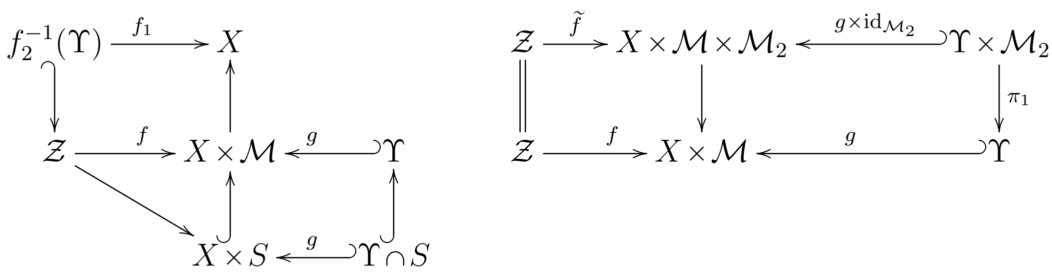

Figure 3: The maps of Lemma 3.3 with $g=(x,g_2)$ for some $x\in X$.

Lemma 3.3. *Suppose $\mathcal{Z},\Upsilon,X,\mathcal{M},f,g,f_i,g_i,\mathfrak{o}_1,\mathfrak{o}_\Upsilon^c$ are as above (with $\Upsilon\subset\mathcal{M}$ and g_1 constant),*

$$\dim\mathcal{Z}+\dim\Upsilon = \dim X+\dim\mathcal{M}\,,$$

and f is strongly transverse to g.

(1) For every $u\in f_2^{-1}(\Upsilon)\cap f_1^{-1}(g_1(\Upsilon))$,

$$(f,\mathfrak{o}_1)\,{}_u\!\cdot_{f_2(u)}\big(g,\mathfrak{o}_\Upsilon^c\big) = (-1)^{(\dim\Upsilon)(\mathrm{codim}\,\Upsilon)}\mathfrak{s}_u\big((f_2^*\mathfrak{o}_\Upsilon^c)\mathfrak{o}_1\big).$$

(2) Suppose $S\subset\mathcal{M}$ is another submanifold strongly transverse to Υ and $f_2(\mathcal{Z})\subset S$. For every $(u,P)\in M_{f,g|_{\Upsilon\cap S}}$,

$$(f,\mathfrak{o}_1)\,{}_u\!\cdot_P\big(g,\mathfrak{o}_\Upsilon^c\big) = (-1)^{(\mathrm{codim}\,S)(\mathrm{codim}\,\Upsilon)}(f,\mathfrak{o}_1)\,{}_u\!\cdot_P\big(g|_{\Upsilon\cap S},\mathfrak{o}_\Upsilon^c|_{\Upsilon\cap S}\big)$$

with the intersection on the right-hand side above taken in $X\times S$.

(3) Suppose $\mathcal{M}_2$, π_1, and $\pi_1^*\mathfrak{o}_\Upsilon^c$ are as in Lemma 3.1(2) and the second diagram in Figure 3 commutes. For every $(u,\widetilde{P})\in M_{\widetilde{f},g\times\mathrm{id}_{\mathcal{M}_2}}$,

$$(\widetilde{f},\mathfrak{o}_1)\,{}_u\!\cdot_{\widetilde{P}}\big(g\times\mathrm{id}_{\mathcal{M}_2},\pi_1^*\mathfrak{o}_\Upsilon^c\big) = (-1)^{(\dim\mathcal{M}_2)(\mathrm{codim}\,\Upsilon)}(f,\mathfrak{o}_1)\,{}_u\!\cdot_{\pi_1(\widetilde{P})}\big(g,\mathfrak{o}_\Upsilon^c\big).$$

Let $e_1\colon\mathcal{Z}_1\longrightarrow X'$ and $e_2\colon\mathcal{Z}_2\longrightarrow X'$ be strongly transverse maps so that

$$\mathcal{Z}\equiv M_{e_1,e_2}\equiv\big\{(u_1,u_2)\in\mathcal{Z}_1\times\mathcal{Z}_2-(\partial\mathcal{Z}_1)\times(\partial\mathcal{Z}_2)\colon e_1(u_1)=e_2(u_2)\big\}\subset\mathcal{Z}_1\times\mathcal{Z}_2$$

is a smooth submanifold. For each $u\equiv(u_1,u_2)\in\mathcal{Z}$, the short exact sequence

$$\begin{gathered}0\longrightarrow T_u\mathcal{Z}\longrightarrow T_{u_1}\mathcal{Z}_1\oplus T_{u_2}\mathcal{Z}_2\longrightarrow T_{e_1(u_1)}X'=T_{e_2(u_2)}X'\longrightarrow 0,\\ (v_1,v_2)\longrightarrow \mathrm{d}_{u_2}e_2(v_2)-\mathrm{d}_{u_1}e_1(v_1),\end{gathered}$$

of vector spaces induces an isomorphism

$$\lambda_u(\mathcal{Z})\otimes\lambda\big(T_{e_2(u_2)}X'\big)\approx\lambda_{u_1}(\mathcal{Z}_1)\otimes\lambda_{u_2}(\mathcal{Z}_2)\,.$$

Combined with relative orientations $\mathfrak{o}_{11}$ and $\mathfrak{o}_{12}$ of

$$f_{11}\colon\mathcal{Z}_1\longrightarrow X_1\qquad\text{and}\qquad(e_2,f_{12})\colon\mathcal{Z}_2\longrightarrow X'\times X_2,$$

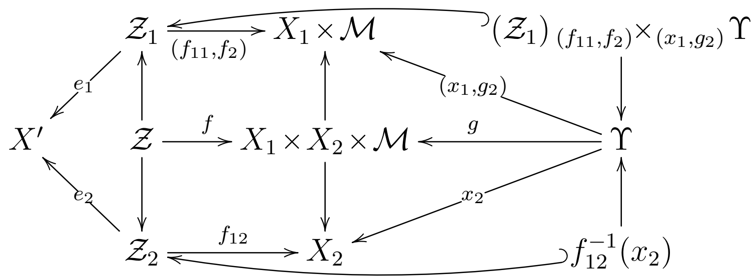

Figure 4: The maps of Lemma 3.4 with $x_i\in X_i$ corresponding to the constant map g_{1i}.

this isomorphism determines a homotopy class of isomorphisms

$$\begin{aligned}\lambda_u(\mathcal{Z})\otimes\lambda_{e_2(u_2)}\big(TX'\big) &\longrightarrow \lambda_{f_{11}(u_1)}(X_1)\otimes\lambda_{(e_2(u_2),f_{12}(u_2))}(X'\times X_2)\\ &\longrightarrow \lambda_{f_{11}(u_1)}(X_1)\otimes\lambda_{f_{12}(u_2)}(X_2)\otimes\lambda\big(T_{e_2(u_2)}X'\big)\end{aligned}\tag{3.13}$$

The homotopy class of trivializations in (3.13) corresponds to a relative orientation $(\mathfrak{o}_{11})\,{}_{e_1}\!\times_{e_2}\mathfrak{o}_{12}$ of the restriction

$$f_1\equiv(f_{11}\times f_{12})\big|_{\mathcal{Z}}\colon\mathcal{Z}\longrightarrow X_1\times X_2\,.$$

For a map $f_2:\mathcal{Z}_1\longrightarrow\mathcal{M}$, let $f_2:\mathcal{Z}\longrightarrow\mathcal{M}$ also denote the composition of f_2 with the projection to $\mathcal{Z}_1$. If in addition $g_2\colon\Upsilon\longrightarrow\mathcal{M}$ is the embedding of a (possibly bordered) submanifold, $g_{1i}\colon\Upsilon\longrightarrow X_i$ are constant maps with values x_1 and x_2, respectively,

$$f\equiv(f_{11},f_{12},f_2)\colon\mathcal{Z}\longrightarrow X_1\times X_2\times\mathcal{M},\quad\text{and}\quad g\equiv(g_{11},g_{12},g_2)\colon\Upsilon\longrightarrow X_1\times X_2\times\mathcal{M},$$

then

$$M_{f,g}=\big\{(u_1,u_2,P)\colon((u_1,P),u_2)\in M_{e_1|_{M_{(f_{11},f_2),(g_{11},g_2)}},e_2|_{f_{12}^{-1}(x_2)}}\big\};$$

see the diagram in Figure 4.

Lemma 3.4. *Suppose* $\mathcal{Z}_1,\mathcal{Z}_2,\mathcal{Z},X',X_1,X_2,\mathcal{M},\Upsilon,e_i,f_{1i},f_2,f,g_{1i},g_2,g,\mathfrak{o}_{1i}$ *are as above,* $\mathfrak{o}_\Upsilon^c$ *is a co-orientation on* Υ*,*

$$\dim\mathcal{Z}_1+\dim\Upsilon=\dim X_1+\dim\mathcal{M},\qquad\dim\mathcal{Z}_2=\dim X'+\dim X_2,$$

the maps e_1 *and* e_2 *are strongly transverse, and the maps* f *and* g *are strongly transverse. If*

$$\big((u_1,P),u_2\big)\in\big((\mathcal{Z}_1)\,{}_{(f_{11},f_2)}\!\times_{(g_{11},g_2)}\Upsilon\big)\,{}_{e_1}\!\times_{e_2}f_{12}^{-1}\big(g_{12}(\Upsilon)\big),$$

then

$$\begin{aligned}&\big(f,(\mathfrak{o}_{11})\,{}_{e_1}\!\times_{e_2}\mathfrak{o}_{12}\big)\,{}_{(u_1,u_2)}\!\cdot_{f_2(u_1)}\!\big(g,\mathfrak{o}_\Upsilon^c\big)\\ &\qquad=(-1)^{(\dim X_2)(\operatorname{codim}\Upsilon+\dim X')}\big(((f_{11},f_2),\mathfrak{o}_{11})\,{}_{u_1}\!\cdot_{f_2(u_1)}\!((g_{11},g_2),\mathfrak{o}_\Upsilon^c)\big)\mathfrak{s}_{u_2}\big(\mathfrak{o}_{12}\big).\end{aligned}$$

3.3 Steenrod pseudocycles

Let Y be a smooth manifold, possibly with boundary. For a continuous map $f\colon \mathcal{Z}\longrightarrow Y$, let

$$\Omega(f)=\bigcap_{K\subset\mathcal{Z}\ \text{cmpt}}\overline{f(\mathcal{Z}-K)}$$

be the limit set of f. A $\mathbb{Z}_2$-pseudocycle into Y is a continuous map $f\colon\mathcal{Z}\longrightarrow Y$ from a manifold, possibly with boundary, so that the closure of $f(\mathcal{Z})$ in Y is compact and there exists a smooth map $h\colon\mathcal{Z}'\longrightarrow Y$ such that

$$\dim\mathcal{Z}'\leqslant\dim\mathcal{Z}-2,\qquad \Omega(f)\subset h(\mathcal{Z}'),\qquad f(\partial\mathcal{Z})\subset(\partial Y)\cup h(\mathcal{Z}')\,.$$

The codimension of such a $\mathbb{Z}_2$-pseudocycle is $\dim Y-\dim\mathcal{Z}$. A continuous map $\widetilde{f}\colon\widetilde{\mathcal{Z}}\longrightarrow Y$ from a manifold, possibly with boundary, is a bordered $\mathbb{Z}_2$-pseudocycle with boundary $f:\mathcal{Z}\longrightarrow Y$ if the closure of $\widetilde{f}(\widetilde{\mathcal{Z}})$ in Y is compact,

$$\mathcal{Z}\subset\partial\widetilde{\mathcal{Z}},\qquad \widetilde{f}|_{\mathcal{Z}}=f,$$

and there exists a smooth map $\widetilde{h}\colon\widetilde{\mathcal{Z}}'\longrightarrow Y$ such that

$$\dim\widetilde{\mathcal{Z}}'\leqslant\dim\widetilde{\mathcal{Z}}-2,\qquad \Omega(\widetilde{f})\subset\widetilde{h}(\widetilde{\mathcal{Z}}'),\qquad \widetilde{f}\big(\partial\widetilde{\mathcal{Z}}-\mathcal{Z}\big)\subset(\partial Y)\cup\widetilde{h}(\widetilde{\mathcal{Z}}')\,.$$

Given $\widetilde{f}$ as above, the choice of $\mathcal{Z}\subset\partial\widetilde{\mathcal{Z}}$ is generally not unique, and the restriction of $\widetilde{f}$ to any such $\mathcal{Z}$ need not be a $\mathbb{Z}_2$-pseudocycle. If $\widetilde{\mathcal{Z}}$ is one-dimensional, then $\widetilde{\mathcal{Z}}$ is compact and $\widetilde{f}(\partial\widetilde{\mathcal{Z}}-\mathcal{Z})\subset\partial Y$.

Two bordered $\mathbb{Z}_2$-pseudocycles $\widetilde{f}_1\colon\widetilde{\mathcal{Z}}_1\longrightarrow Y$ and $\widetilde{f}_2\colon\widetilde{\mathcal{Z}}_2\longrightarrow Y$ as above are transverse if

- the maps $\widetilde{f}_1$ and $\widetilde{f}_2$ are strongly transverse and
- there exist smooth maps $\widetilde{h}_1:\widetilde{\mathcal{Z}}_1'\longrightarrow Y$ and $\widetilde{h}_2:\widetilde{\mathcal{Z}}_2'\longrightarrow Y$ such that $\widetilde{h}_1$ is transverse to $\widetilde{f}_2$ and $\widetilde{f}_2|_{\partial\widetilde{\mathcal{Z}}_2}$, $\widetilde{h}_2$ is transverse to $\widetilde{f}_1$ and $\widetilde{f}_1|_{\partial\widetilde{\mathcal{Z}}_1}$, and

$$\dim\widetilde{\mathcal{Z}}_1'\leqslant\dim\widetilde{\mathcal{Z}}_1-2,\quad \dim\widetilde{\mathcal{Z}}_2'\leqslant\dim\widetilde{\mathcal{Z}}_2-2,\quad \Omega(\widetilde{f}_1)\subset\widetilde{h}_1(\widetilde{\mathcal{Z}}_1'),\quad \Omega(\widetilde{f}_2)\subset\widetilde{h}_2(\widetilde{\mathcal{Z}}_2').$$

In such a case,

$$\widetilde{f}_1\times_Y\widetilde{f}_2\colon M_{\widetilde{f}_1,\widetilde{f}_2}\longrightarrow Y$$

is a bordered $\mathbb{Z}_2$-pseudocycle with boundary (3.10).

A Steenrod pseudocycle into Y is a $\mathbb{Z}_2$-pseudocycle $f\colon\mathcal{Z}\longrightarrow Y$ along with a relative orientation $\mathfrak{o}$ of f. A bordered $\mathbb{Z}_2$-pseudocycle $\widetilde{f}\colon\widetilde{\mathcal{Z}}\longrightarrow Y$ with boundary f and a relative orientation $\widetilde{\mathfrak{o}}$ of $\widetilde{f}$ is a bordered Steenrod pseudocycle with boundary $(f,\mathfrak{o})$ if $\partial\widetilde{\mathfrak{o}}=\mathfrak{o}$. If $(f,\mathfrak{o})$ is a codimension 0 Steenrod pseudocycle, then the number

$$\deg(f,\mathfrak{o})\equiv\sum_{u\in f^{-1}(y)}\mathfrak{s}_u(\mathfrak{o})\in\mathbb{Z}\tag{3.14}$$

is well-defined for a generic choice of $y\in Y$ and is independent of such a choice. We call this number the degree of $(f,\mathfrak{o})$. It vanishes if $(f,\mathfrak{o})$ bounds a bordered Steenrod pseudocycle $(\widetilde{f},\widetilde{\mathfrak{o}})$.

Lemma 3.5. *Suppose $\mathcal{Z}, \Upsilon, X, \mathcal{M}, Y, f, g, f_i, g_i, \mathfrak{o}_1, \mathfrak{o}_2$ are as in (3.11) and just below and such that*

$$\dim \mathcal{Z} + \dim \Upsilon = \dim Y + 1\,.$$

If f and g are transverse bordered $\mathbb{Z}_2$-pseudocycles, then $\mathcal{Z}_f\times_g(\partial\Upsilon)$ and $(\partial\mathcal{Z})_f\times_g\Upsilon$ are finite sets and

$$\big|\mathcal{Z}_f\times_g(\partial\Upsilon)\big|^{\pm}_{\mathfrak{o}_1,\partial\mathfrak{o}_2} = (-1)^{\dim\Upsilon}\big|(\partial\mathcal{Z})_f\times_g\Upsilon\big|^{\pm}_{\partial\mathfrak{o}_1,\mathfrak{o}_2}\,. \tag{3.15}$$

Proof. By the transversality and dimension assumptions, $M_{f,g}$ is a compact one-dimensional manifold and

$$\partial M_{f,g} = \mathcal{Z}\,{}_f\times_g(\partial\Upsilon) \sqcup (\partial\mathcal{Z})\,{}_f\times_g\Upsilon.$$

In particular, the two sets on the right-hand side above are finite. By a direct computation, this equality respects the orientations with the orientation on the last fiber product modified by $(-1)^{\mathrm{codim}\,\mathcal{Z}}$ and for a suitably chosen orientation on the left-hand side. Alternatively, (3.15) is equivalent to

$$\big|\mathcal{Z}_f\times_g(\partial\Upsilon)\big|^{\pm}_{\mathfrak{o}_1,\partial\mathfrak{o}_2} = (-1)^{(\dim\mathcal{Z})(\dim\Upsilon)}\big|\Upsilon_g\times_f(\partial\mathcal{Z})\big|^{\pm}_{\mathfrak{o}_2,\partial\mathfrak{o}_1}\,.$$

The sign in this statement must be symmetric in $\dim\mathcal{Z}$ and $\dim\Upsilon$, depend only on their parity, be $+1$ if both dimensions or codimensions are even, and be -1 for linear maps from intervals to $\mathbb{R}$. $\square$

4 Moduli spaces of stable curves

4.1 Main stratum and orientations

For $k\in\mathbb{Z}^{\geqslant 0}$, let $[k]=\{1,\ldots,k\}$. If in addition $k\geqslant 3$, we denote by $\overline{\mathcal{M}}_{0,k}$ the Deligne-Mumford moduli space of stable rational curves with k marked points. For $k,l\in\mathbb{Z}^{\geqslant 0}$ with $k+2l\geqslant 3$, we denote by $\overline{\mathcal{M}}^{\tau}_{k,l}$ the Deligne-Mumford moduli space of stable real genus 0 curves

$$\mathcal{C} \equiv \big(\Sigma, (x_i)_{i\in[k]}, (z_i^+, z_i^-)_{i\in[l]}, \sigma\big) \tag{4.1}$$

with k real marked points, l conjugate pairs of marked points, and an anti-holomorphic involution σ with separating fixed locus. This space is a smooth manifold of dimension $k+2l-3$, without boundary if $k\geqslant 1$ and with boundary if $k=0$. The boundary of $\overline{\mathcal{M}}^{\tau}_{0,l}$ parametrizes the curves with no irreducible component fixed by the involution; the fixed locus of the involution on a curve in $\partial\overline{\mathcal{M}}^{\tau}_{0,l}$ is a single node. The strata of $\overline{\mathcal{M}}^{\tau}_{0,l}$ parametrizing curves with two invariant irreducible components sharing a real node are of codimension 1, but are not part of $\partial\overline{\mathcal{M}}^{\tau}_{0,l}$. The moduli space $\overline{\mathcal{M}}^{\tau}_{k,l}$ is orientable if and only if $k=0$ or $k+2l\leqslant 4$; see [9, Prop. 1.5].

The main stratum $\mathcal{M}^{\tau}_{k,l}$ of $\overline{\mathcal{M}}^{\tau}_{k,l}$ is the quotient of

$$\Big\{\big((x_i)_{i\in[k]}, (z_i^+, z_i^-)_{i\in[l]}\big):\ x_i\in S^1,\ z_i^{\pm}\in\mathbb{P}^1-S^1,\ z_i^+=\tau(z_i^-),\\ x_i\neq x_j,\ z_i^+\neq z_j^+, z_j^-\ \forall\, i\neq j\Big\}$$

by the natural action of the subgroup $\mathrm{PSL}_2^\tau\mathbb{C}\subset\mathrm{PSL}_2\mathbb{C}$ of automorphisms of $\mathbb{P}^1$ commuting with τ. The topological components of $\mathcal{M}_{k,l}^\tau$ are indexed by the possible distributions of the points z_i^+ between the interiors of the two disks cut out by the fixed locus S^1 of the standard involution τ on $\mathbb{P}^1$ and by the orderings of the real marked points x_i on S^1.

If $k+2l\geqslant 4$ and $i\in[k]$, let

$$\mathfrak{f}_{k,l;i}^{\mathbb{R}}\colon \overline{\mathcal{M}}_{k,l}^\tau \longrightarrow \overline{\mathcal{M}}_{k-1,l}^\tau \tag{4.2}$$

be the forgetful morphism dropping the i-th real marked point. The restriction of $\mathfrak{f}_{k,l;i}^{\mathbb{R}}$ to the preimage of $\mathcal{M}_{k-1,l}^\tau$ is an S^1-fiber bundle. The associated short exact sequence (3.2) induces an isomorphism

$$\lambda\big(\mathcal{M}_{k,l}^\tau\big) \approx \mathfrak{f}_{k,l;i}^{\mathbb{R}*}\lambda\big(\mathcal{M}_{k-1,l}^\tau\big)\big|_{\mathcal{M}_{k,l}^\tau}\otimes\big(\ker \mathrm{d}\mathfrak{f}_{k,l;i}^{\mathbb{R}}\big)\big|_{\mathcal{M}_{k,l}^\tau}. \tag{4.3}$$

If $k+2l\geqslant 5$ and $i\in[l]$, we similarly denote by

$$\mathfrak{f}_{k,l;i}\colon \overline{\mathcal{M}}_{k,l}^\tau \longrightarrow \overline{\mathcal{M}}_{k,l-1}^\tau \tag{4.4}$$

the forgetful morphism dropping the i-th conjugate pair of marked points. The restriction of $\mathfrak{f}_{k,l;i}$ to $\mathcal{M}_{k,l}^\tau$ is a dense open subset of a $\mathbb{P}^1$-fiber bundle and thus induces an isomorphism

$$\lambda\big(\mathcal{M}_{k,l}^\tau\big) \approx \mathfrak{f}_{k,l;i}^{*}\lambda\big(\mathcal{M}_{k,l-1}^\tau\big)\big|_{\mathcal{M}_{k,l}^\tau}\otimes\lambda\big(\ker \mathrm{d}\mathfrak{f}_{k,l;i}\big)\big|_{\mathcal{M}_{k,l}^\tau}. \tag{4.5}$$

For each $\mathcal{C}\in\mathcal{M}_{k,l}^\tau$ as in (4.1),

$$\ker \mathrm{d}_{\mathcal{C}}\mathfrak{f}_{k,l;i} \approx T_{z_i^+}\mathbb{P}^1$$

is canonically oriented by the complex orientation of the fiber $\mathbb{P}^1$ at z_i^+. We denote the resulting orientation of the last factor in (4.5) by $\mathfrak{o}_i^+$.

Suppose $l\in\mathbb{Z}^+$ and $\mathcal{C}\in\mathcal{M}_{k,l}^\tau$ is as in (4.1) with $\Sigma=\mathbb{P}^1$. Let $\mathbb{D}_+^2\subset\mathbb{C}\subset\mathbb{P}^1$ be the disk cut out by the fixed locus S^1 of τ which contains z_1^+. We orient $S^1\subset\mathbb{D}_+^2\subset\mathbb{C}$ in the standard way (this is the opposite of the boundary orientation of $\mathbb{D}_+^2$ as defined in Section 3.1). If $k+2l\geqslant 4$ and $i\in[k]$, this determines an orientation $\mathfrak{o}_i^{\mathbb{R}}$ of the fiber

$$\ker \mathrm{d}_{\mathcal{C}}\mathfrak{f}_{k,l;i}^{\mathbb{R}} \approx T_{x_i}S^1$$

of the last factor in (4.3) over $\mathfrak{f}_{k,l;i}^{\mathbb{R}}(\mathcal{C})$. This orientation extends over the subspace

$$\overline{\mathcal{M}}_{k,l;i}^{\tau;\bigstar} \subset \overline{\mathcal{M}}_{k,l}^\tau$$

consisting of curves $\mathcal{C}$ as in (4.1) such that the real marked point x_i of $\mathcal{C}$ lies on the same irreducible component of Σ as the marked point z_1^+.

Let $(x_1,x_{j_2(\mathcal{C})},\ldots,x_{j_k(\mathcal{C})})$ be the ordering of the real marked points of $\mathcal{C}$ starting with x_1 and going in the direction of the standard orientation of S^1. We denote by $\delta_{\mathbb{R}}(\mathcal{C})\in\mathbb{Z}_2$ the sign of the permutation sending

$$\varpi_{\mathcal{C}}\colon \big\{2,\ldots,k\big\}\longrightarrow\big\{2,\ldots,k\big\}, \quad \varpi_{\mathcal{C}}(i)=j_i(\mathcal{C}).$$

If $k=0$, we take $\delta_{\mathbb{R}}(\mathcal{C})=0$. For $l^*\in[l]$, let

$$\delta_{l^*}^c(\mathcal{C}) = \big|\big\{i\in[l]-[l^*]\colon z_i^+\notin\mathbb{D}_+^2\big\}\big| + 2\mathbb{Z}\in\mathbb{Z}_2.$$

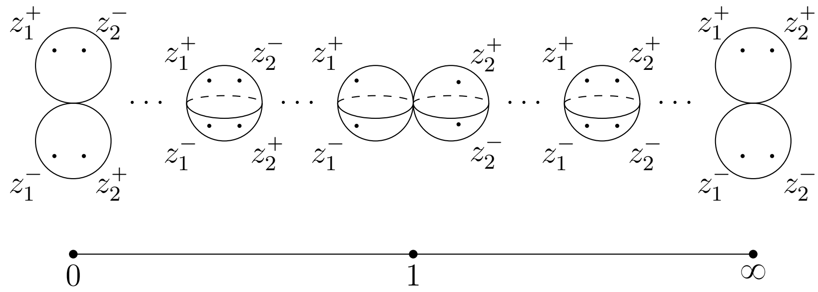

Figure 5: The structure of $\overline{\mathcal{M}}_{0,2}^{\tau}$

In particular, $\delta_{\mathbb{R}}(\mathcal{C})=0$ if $k\leqslant 2$ and $\delta_l^c(\mathcal{C})=0$. The functions $\delta_{\mathbb{R}}$ and δ_{l*} are locally constant on $\mathcal{M}_{k,l}^{\tau}$.

The space $\mathcal{M}_{1,1}^{\tau}=\overline{\mathcal{M}}_{1,1}^{\tau}$ is a single point; we take $\mathfrak{o}_{1,1}\equiv+1$ to be its orientation as a plus point. We identify the one-dimensional space $\overline{\mathcal{M}}_{0,2}^{\tau}$ with $[0,\infty]$ via the cross ratio

$$\varphi_{0,2}\colon \overline{\mathcal{M}}_{0,2}^{\tau}\longrightarrow[0,\infty],\quad \varphi\big([(z_1^+,z_1^-),(z_2^+,z_2^-)]\big)=\frac{z_2^+-z_1^-}{z_2^--z_1^-}:\frac{z_2^+-z_1^+}{z_2^--z_1^+}=\frac{|1-z_1^+/z_2^-|^2}{|z_1^+-z_2^+|^2}; \quad (4.6)$$

see Figure 5. This identification, which is the *opposite* of [10, (3.1)] and [12, (1.12)], determines an orientation $\mathfrak{o}_{0,2}$ on $\overline{\mathcal{M}}_{0,2}^{\tau}$.

We now define an orientation $\mathfrak{o}_{k,l}$ on $\mathcal{M}_{k,l}^{\tau}$ for $l\in\mathbb{Z}^+$ and $k+l\geqslant 3$ inductively. If $k\geqslant 1$, we take $\mathfrak{o}_{k,l}$ to be so that the $i=k$ case of the isomorphism (4.3) is compatible with the orientations $\mathfrak{o}_{k,l}$, $\mathfrak{o}_{k-1,l}$, and $\mathfrak{o}_k^{\mathbb{R}}$ on the three line bundles involved. If $l\geqslant 2$, we take $\mathfrak{o}_{k,l}$ to be so that the $i=l$ case of the isomorphism (4.5) is compatible with the orientations $\mathfrak{o}_{k,l}$, $\mathfrak{o}_{k,l-1}$, and $\mathfrak{o}_l^+$. By a direct check, the orientations on $\mathcal{M}_{1,2}^{\tau}$ induced from $\mathcal{M}_{0,2}^{\tau}$ via (4.3) and $\mathcal{M}_{1,1}^{\tau}$ via (4.5) are the same. Since the fibers of $\mathfrak{f}_{k,l;l}|_{\mathcal{M}_{k,l}^{\tau}}$ are even-dimensional, it follows that the orientation $\mathfrak{o}_{k,l}$ on $\mathcal{M}_{k,l}^{\tau}$ is well-defined for all $l\in\mathbb{Z}^+$ and $k\in\mathbb{Z}^{\geqslant 0}$ with $k+2l\geqslant 3$. This orientation is as above [9, Lemma 5.4].

For $l^*\in[l]$, we denote by $\mathfrak{o}_{k,l;l^*}$ the orientation on $\mathcal{M}_{k,l}^{\tau}$ which equals $\mathfrak{o}_{k,l}$ at $\mathcal{C}$ if and only if $\delta_{\mathbb{R}}(\mathcal{C})=\delta_{l^*}^c(\mathcal{C})$. The next statement is straightforward.

Lemma 4.1. *The orientations $\mathfrak{o}_{k,l;l^*}$ on $\mathcal{M}_{k,l}^{\tau}$ with $k,l\in\mathbb{Z}^{\geqslant 0}$ and $l^*\in[l]$ such that $k+2l\geqslant 3$ satisfy the following properties:*

($\mathfrak{o}_{\mathcal{M}}1$) *if $\mathcal{C}\in\mathcal{M}_{k+1,l}^{\tau}$, the isomorphism (4.3) with (k,i) replaced by $(k+1,j_{k+1}(\mathcal{C}))$ respects the orientations $\mathfrak{o}_{k+1,l;l^*}$, $\mathfrak{o}_{k,l;l^*}$, and $\mathfrak{o}_{k+1}^{\mathbb{R}}$ at $\mathcal{C}$;*

($\mathfrak{o}_{\mathcal{M}}2$) *the isomorphism (4.5) with (l,i) replaced by $(l+1,l^*+1)$ respects the orientations $\mathfrak{o}_{k,l+1;l^*+1}$, $\mathfrak{o}_{k,l;l^*}$, and $\mathfrak{o}_{l^*+1}^+$;*

($\mathfrak{o}_{\mathcal{M}}3$) *the interchange of two real points x_i and x_j with $2\leqslant i,j\leqslant k$ preserves $\mathfrak{o}_{k,l;l^*}$;*

($\mathfrak{o}_{\mathcal{M}}4$) *if $\mathcal{C}\in\mathcal{M}_{k,l}^{\tau}$, the interchange of the real points x_1 and $x_{j_i(\mathcal{C})}$ with $2\leqslant i\leqslant k$ preserves $\mathfrak{o}_{k,l;l^*}$ at $\mathcal{C}$ if and only if $(k-1)(i-1)\in 2\mathbb{Z}$;*

($\mathfrak{o}_{\mathcal{M}}5$) *if* $\mathcal{C}\in\mathcal{M}^{\tau}_{k,l}$ *and the marked points* z_i^+ *and* z_j^+ *are not separated by the fixed locus* S^1 *of* $\mathcal{C}$*, then the interchange of the conjugate pairs* (z_i^+,z_i^-) *and* (z_j^+,z_j^-) *preserves* $\mathfrak{o}_{k,l;l^*}$ *at* $\mathcal{C}$*;*

($\mathfrak{o}_{\mathcal{M}}6$) *the interchange of the points in a conjugate pair* (z_i^+,z_i^-) *with* $l^*<i\leqslant l$ *preserves* $\mathfrak{o}_{k,l;l^*}$*;*

($\mathfrak{o}_{\mathcal{M}}7$) *the interchange of the points in a conjugate pair* (z_i^+,z_i^-) *with* $1^*<i\leqslant l^*$ *reverses* $\mathfrak{o}_{k,l;l^*}$*;*

($\mathfrak{o}_{\mathcal{M}}8$) *the interchange of the points in the conjugate pair* (z_1^+,z_1^-) *preserves* $\mathfrak{o}_{k,l;l^*}$ *if and only if*

$$k\neq 0 \quad \textit{and} \quad l-l^*\cong\binom{k}{2} \bmod 2 \qquad \textit{or} \qquad k=0 \quad \textit{and} \quad l-l^*\cong 1 \bmod 2.$$

4.2 Codimension 1 strata and degrees

The (open) codimension 1 strata of $\overline{\mathcal{M}}^{\tau}_{k,l}-\partial\overline{\mathcal{M}}^{\tau}_{k,l}$ correspond to the sets $\{(K_1,L_1),(K_2,L_2)\}$ such that

$$[k]=K_1\sqcup K_2,\quad [l]=L_1\sqcup L_2,\quad |K_1|+2|L_1|\geqslant 2,\quad |K_2|+2|L_2|\geqslant 2\,.$$

The stratum S corresponding to such a set parametrizes marked curves $\mathcal{C}$ as in (4.1) so that the underlying surface Σ consists of two real irreducible components with one of them carrying the real marked points x_i with $i\in K_1$ and the conjugate pairs of marked points (z_i^+,z_i^-) with $i\in L_1$ and the other component carrying the other marked points. A closed codimension 1 stratum $\overline{S}$ is the closure of such an open stratum S. Thus,

$$S\approx\mathcal{M}^{\tau}_{|K_1|+1,|L_1|}\times\mathcal{M}^{\tau}_{|K_2|+1,|L_2|},\qquad \overline{S}\approx\overline{\mathcal{M}}^{\tau}_{|K_1|+1,|L_1|}\times\overline{\mathcal{M}}^{\tau}_{|K_2|+1,|L_2|}. \tag{4.7}$$

Let $l\in\mathbb{Z}^+$. If S is a codimension 1 stratum of $\overline{\mathcal{M}}^{\tau}_{k,l}-\partial\overline{\mathcal{M}}^{\tau}_{k,l}$ and $\mathcal{C}\in S$, we denote by $\mathbb{P}^1_1$ the irreducible component of $\mathcal{C}$ containing the marked points $z_1^{\pm}$, by $\mathbb{P}^1_2$ the other irreducible component, and by $S^1_1\subset\mathbb{P}^1_1$ and $S^1_2\subset\mathbb{P}^1_2$ the fixed loci of the involutions on these components. For $r=1,2$, we then take $K_r(S)$ and $L_r(S)$ to be the set of real marked points and the set of conjugate pairs of marked points, respectively, carried by $\mathbb{P}^1_r$ and define

$$k_r(S)=\big|K_r(S)\big| \qquad\text{and}\qquad l_r(S)=\big|L_r(S)\big|.$$

For $i\in[l]$, we denote by

$$S_i\subset\overline{\mathcal{M}}^{\tau}_{k,l}\qquad\text{and}\qquad \overline{S}_i\subset\overline{\mathcal{M}}^{\tau}_{k,l}$$

the open codimension 1 stratum parametrizing marked curves consisting of two real spheres with the marked points $z_i^{\pm}$ on one of them and all other marked points on the other sphere and its closure, respectively.

If $\overline{S}\subset\overline{\mathcal{M}}^{\tau}_{k,l}-\partial\overline{\mathcal{M}}^{\tau}_{k,l}$ is a closed codimension 1 stratum different from $\overline{S}_1$, let

$$\mathfrak{f}_{S;1}\colon\overline{S}\longrightarrow\overline{\mathcal{M}}^{\tau}_{k_1(S),l_1(S)}\times\overline{\mathcal{M}}^{\tau}_{k_2(S)+1,l_2(S)} \tag{4.8}$$

denote the composition of the second identification in (4.7) with the forgetful morphism

$$\mathfrak{f}^{\mathbb{R}}_{\mathrm{nd}}\colon \overline{\mathcal{M}}^{\tau}_{k_1(S)+1,l_1(S)} \longrightarrow \overline{\mathcal{M}}^{\tau}_{k_1(S),l_1(S)}$$

as in (4.2) dropping the marked point nd corresponding to the node. The vertical tangent bundle of $\mathfrak{f}_{S;1}|_S$ is a pullback of the vertical tangent bundle of $\mathfrak{f}^{\mathbb{R}}_{\mathrm{nd}}|_{\mathcal{M}^{\tau}_{k_1(S)+1,l_1(S)}}$ and thus inherits an orientation from the orientation $\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}$ of the latter specified in Section 4.1; we denote the induced orientation also by $\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}$. It extends over the subspace

$$S^{\star}\subset\overline{S}\subset\overline{\mathcal{M}}^{\tau}_{k,l}$$

of curves $\mathcal{C}$ so that the marked point nd of the first component of the image of $\mathcal{C}$ under (4.7) lies on the same irreducible component of the domain as the marked point corresponding to z_1^+.

Let $\Upsilon\subset\overline{\mathcal{M}}^{\tau}_{k,l}$ be a bordered hypersurface. If $k+2l\geqslant 4$ and $i\in[k]$, we call Υ regular with respect to $\mathfrak{f}^{\mathbb{R}}_{k,l;i}$ if $\Upsilon\subset\overline{\mathcal{M}}^{\tau;\star}_{k,l;i}$, $\mathfrak{f}^{\mathbb{R}}_{k,l;i}(\overline{\Upsilon}-\Upsilon)$ is contained in the strata of codimension at least 2, i.e. the subspace of $\overline{\mathcal{M}}^{\tau}_{k-1,l}$ parametrizing curves with at least two nodes, and $\mathfrak{f}^{\mathbb{R}}_{k,l;i}(\partial\Upsilon)$ is contained in the union of $\partial\overline{\mathcal{M}}^{\tau}_{k-1,l}$ and the strata of codimension at least 2. By the last two assumptions, $\mathfrak{f}^{\mathbb{R}}_{k,l;i}|_{\Upsilon}$ is a $\mathbb{Z}_2$-pseudocycle of codimension 0; see Section 3.3. By the first assumption, the orientation $\mathfrak{o}^{\mathbb{R}}_i$ of the last factor in (4.3) and a co-orientation $\mathfrak{o}^c_{\Upsilon}$ on Υ induce a relative orientation $\mathfrak{o}^c_{\Upsilon}\mathfrak{o}^{\mathbb{R}}_i$ of $\mathfrak{f}^{\mathbb{R}}_{k,l;i}|_{\Upsilon}$; see the paragraph above Lemma 3.1. Let

$$\deg^{\mathbb{R}}_i\big(\Upsilon,\mathfrak{o}^c_{\Upsilon}\big)\equiv\deg\big(\mathfrak{f}^{\mathbb{R}}_{k,l;i}|_{\Upsilon},\mathfrak{o}^c_{\Upsilon}\mathfrak{o}^{\mathbb{R}}_i\big)$$

be the degree of the Steenrod pseudocycle $(\mathfrak{f}^{\mathbb{R}}_{k,l;i}|_{\Upsilon},\mathfrak{o}^c_{\Upsilon}\mathfrak{o}^{\mathbb{R}}_i)$; see (3.14).

Suppose in addition that $S\subset\overline{\mathcal{M}}^{\tau}_{k,l}-\partial\overline{\mathcal{M}}^{\tau}_{k,l}$ is a codimension 1 stratum. We call Υ regular with respect to S if Υ and $\partial\Upsilon$ are transverse to $\overline{S}$ in $\overline{\mathcal{M}}^{\tau}_{k,l}$,

$$\Upsilon\cap\overline{S}\approx\Upsilon_1\times\overline{\mathcal{M}}^{\tau}_{k_2(S)+1,l_2(S)}$$

under the second identification in (4.7) for some $\Upsilon_1\subset\overline{\mathcal{M}}^{\tau;\star}_{k_1(S)+1,l_1(S);\mathrm{nd}}$, $\mathfrak{f}_{S;1}((\overline{\Upsilon}-\Upsilon)\cap\overline{S})$ is contained in the strata of codimension at least 2 of the target of $\mathfrak{f}_{S;1}$, and $\mathfrak{f}_{S;1}(\partial\Upsilon\cap\overline{S})$ is contained in the union of the boundary and the strata of codimension at least 2 of the target of $\mathfrak{f}_{S;1}$. By the first and the last two assumptions, $\mathfrak{f}_{S;1}|_{\Upsilon\cap\overline{S}}$ is a $\mathbb{Z}_2$-pseudocycle of codimension 0. By the first assumption, a co-orientation $\mathfrak{o}^c_{\Upsilon}$ on Υ in $\overline{\mathcal{M}}^{\tau}_{k,l}$ determines a co-orientation

$$\mathfrak{o}^c_{\Upsilon\cap S}\equiv\mathfrak{o}^c_{\Upsilon}\big|_{\Upsilon\cap\overline{S}}$$

on $\Upsilon\cap\overline{S}$ in $\overline{S}$. By the second assumption, $\Upsilon\cap\overline{S}\subset S^{\star}$. By the first two assumptions, $S\neq S_1$ if $\Upsilon\cap\overline{S}\neq\varnothing$ and that $\mathfrak{o}^c_{\Upsilon}$ and the orientation $\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}$ of the fibers of the restriction of (4.8) to S specified above induce a relative orientation $\mathfrak{o}^c_{\Upsilon\cap S}\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}$ of $\mathfrak{f}_{S;1}|_{\Upsilon\cap\overline{S}}$. Let

$$\deg_S\big(\Upsilon,\mathfrak{o}^c_{\Upsilon}\big)\equiv\deg\big(\mathfrak{f}_{S;1}|_{\Upsilon\cap\overline{S}},\mathfrak{o}^c_{\Upsilon}\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}\big)\equiv\deg\big(\mathfrak{f}_{S;1}|_{\Upsilon\cap\overline{S}},\mathfrak{o}^c_{\Upsilon\cap S}\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}\big).$$

We call a bordered hypersurface $\Upsilon\subset\overline{\mathcal{M}}^{\tau}_{k,l}$ regular if $\overline{\Upsilon}-\Upsilon$ is contained in the strata of codimension at least 2 and Υ is regular with respect to the forgetful morphism $\mathfrak{f}^{\mathbb{R}}_{k,l;i}$ for every $i\in[k]$ and with respect to every codimension 1 stratum $S\subset\overline{\mathcal{M}}^{\tau}_{k,l}-\partial\overline{\mathcal{M}}^{\tau}_{k,l}$. For such a hypersurface, $\Upsilon\cap\overline{S}_1=\varnothing$.

4.3 Strata orientations

Suppose $l \geqslant 2$ and $k+2l \geqslant 5$. The moduli space $\overline{\mathcal{M}}^{\tau}_{k,l}$ contains codimension 2 strata Γ that parametrize marked curves $\mathcal{C}$ as in (4.1) so that the underlying surface Σ consists of one real component $\mathbb{P}^1_0$ and one pair $\mathbb{P}^1_{\pm}$ of conjugate components; see Figure 1. We do not distinguish these strata based on the ordering of the marked points on the fixed locus $S^1_1 \subset \mathbb{P}^1_0$ of the involution. For such a stratum Γ, let $l_0(\Gamma), l_{\mathbb{C}}(\Gamma) \in \mathbb{Z}^{\geqslant 0}$ be the number of conjugate pairs of marked points carried by $\mathbb{P}^1_0$ and $\mathbb{P}^1_- \cup \mathbb{P}^1_+$, respectively. In particular,

$$l_{\mathbb{C}}(\Gamma) \geqslant 2 \qquad \text{and} \qquad l_0(\Gamma)+l_{\mathbb{C}}(\Gamma) = l.$$

The closure $\overline{\Gamma}$ of Γ decomposes as

$$\overline{\Gamma} \approx \overline{\mathcal{M}}^{\tau}_{k,l_0(\Gamma)+1} \times \overline{\mathcal{M}}_{0,l_{\mathbb{C}}(\Gamma)+1} . \tag{4.9}$$

We call a codimension 2 stratum as above primary if the marked point z_1^+ of the curves $\mathcal{C}$ in Γ is carried by $\mathbb{P}^1_- \cup \mathbb{P}^1_+$.

For a primary codimension 2 stratum Γ and $\mathcal{C} \in \Gamma$, we denote by $\mathbb{P}^1_+$ the irreducible component of $\mathcal{C}$ carrying the marked point z_1^+. In this case, we choose the identification (4.9) so that

($\mathfrak{o}_{\Gamma}1$) the second factor on the right-hand side parametrizes the irreducible component $\mathbb{P}^1_+$ with its marked points so that the node $z_{\mathbb{C}}$ separating it from $\mathbb{P}^1_0$ is the *first* marked point,

($\mathfrak{o}_{\Gamma}2$) the node $z^+_{\mathbb{R}}$ separating $\mathbb{P}^1_0$ from $\mathbb{P}^1_+$ is the *first* marked point in the *first* conjugate pair of marked points in the corresponding element in the first factor on the right-hand side, and

($\mathfrak{o}_{\Gamma}3$) the remaining conjugate pairs of points and the real points in the first factor on the right-hand side are numbered in the same order as on the left-hand side.

If in addition $l^* \in [l]$, let $l^*_0(\Gamma)$ (resp. $l^*_-(\Gamma)$) be the number of marked points z_i^- with $i \in [l^*]$ carried by $\mathbb{P}^1_0$ (resp. $\mathbb{P}^1_+$). The second factor in (4.9) is canonically oriented (being a complex manifold). We denote by $\mathfrak{o}_{\Gamma;l^*}$ the orientation on Γ obtained via the identification (4.9) from the orientation $\mathfrak{o}_{k,l_0(\Gamma)+1;l^*_0(\Gamma)+1}$ on $\mathcal{M}^{\tau}_{k,l_0(\Gamma)+1}$ times $(-1)^{l^*_-(\Gamma)}$.

With the identification as above, let

$$\pi_1, \pi_2 \colon \overline{\Gamma} \longrightarrow \overline{\mathcal{M}}^{\tau}_{k,l_0(\Gamma)+1}, \overline{\mathcal{M}}_{0,l_{\mathbb{C}}(\Gamma)+1}$$

be the projections to the two factors. Denote by

$$\mathcal{L}^{\mathbb{R}}_{\Gamma} \longrightarrow \overline{\mathcal{M}}^{\tau}_{k,l_0(\Gamma)+1} \qquad \text{and} \qquad \mathcal{L}^{\mathbb{C}}_{\Gamma} \longrightarrow \overline{\mathcal{M}}_{0,l_{\mathbb{C}}(\Gamma)+1}$$

the universal tangent line bundles at the first point of the first conjugate pair of marked points and at the first marked point, respectively. The normal bundle $\mathcal{N}\Gamma$ consists of conjugate smoothings of the two nodes of the curves in Γ. Thus, it is canonically isomorphic to the complex line bundle

$$\mathcal{L}_\Gamma \equiv \pi_1^*\mathcal{L}_\Gamma^{\mathbb{R}}\otimes_{\mathbb{C}}\pi_2\mathcal{L}_\Gamma^{\mathbb{C}} \longrightarrow \Gamma\,.$$

The next observation is straightforward.

Lemma 4.2. *Suppose $k,l\in\mathbb{Z}^{\geqslant 0}$ and $l^*\in[l]$ are such that $k+2l\geqslant 3$. Let $\Gamma\subset\overline{\mathcal{M}}_{k,l}^\tau$ be a primary codimension 2 stratum. The orientation $\mathfrak{o}_\Gamma^c$ on $\mathcal{N}\Gamma$ induced by the orientations $\mathfrak{o}_{k,l;l^*}$ on $\mathcal{M}_{k,l}^\tau$ and $\mathfrak{o}_{\Gamma;l^*}$ on Γ agrees with the complex orientation of $\mathcal{L}_\Gamma$.*

Suppose now that $l\in\mathbb{Z}^+$ and S is a codimension 1 stratum of $\overline{\mathcal{M}}_{k,l}^\tau-\partial\overline{\mathcal{M}}_{k,l}^\tau$. For $r=1,2$, let

$$K_r(S)\subset[k], \qquad L_r(S)\subset[l], \qquad \text{and} \quad k_r(S),l_r(S)\in\mathbb{Z}^{\geqslant 0}$$

be as in Section 4.2. Define

$$r(S)=\begin{cases}1, & \text{if } k=0 \text{ or } 1\in K_1(S);\\ 2, & \text{if } 1\in K_2(S);\end{cases}$$

thus, the real marked point x_1 lies on $S^1_{r(S)}$ if $k\geqslant 1$.

For $l^*\in[l]$ and $r=1,2$, define

$$L_r^*(S)=L_r(S)\cap[l^*], \qquad l_r^*(S)\equiv\big|L_r^*(S)\big|.$$

An orientation $\mathfrak{o}^c_{S;\mathcal{C}}$ of the normal bundle $\mathcal{N}_{\mathcal{C}}S$ of S in $\overline{\mathcal{M}}_{k,l}^\tau$ at $\mathcal{C}\in S$ determines a direction of degeneration of elements of $\mathcal{M}_{k,l}^\tau$ to $\mathcal{C}$. The orientation $\mathfrak{o}_{k,l;l^*}$ on $\mathcal{M}_{k,l}^\tau$ limits to an orientation $\mathfrak{o}_{k,l;l^*;\mathcal{C}}$ of $\lambda_{\mathcal{C}}(\overline{\mathcal{M}}_{k,l}^\tau)$ obtained by approaching $\mathcal{C}$ from this direction. Along with $\mathfrak{o}^c_{S;\mathcal{C}}$, $\mathfrak{o}_{k,l;l^*;\mathcal{C}}$ determines an orientation $\partial_{\mathfrak{o}^c_{S;\mathcal{C}}}\mathfrak{o}_{k,l;l^*;\mathcal{C}}$ of $\lambda_{\mathcal{C}}(S)$ via the first isomorphism in (3.1). If in addition $l_2^*(S)\geqslant 1$, let $i^*\in L_2^*(S)$ be the smallest element. The two directions of degeneration of elements of $\mathcal{M}_{k,l}^\tau$ to $\mathcal{C}$ are then distinguished by whether the marked points z_1^+ and $z_{i^*}^+$ of the degenerating elements lie on the same disk $\mathbb{D}_+^2$ or not. We denote by $\mathfrak{o}^{c;+}_{S;\mathcal{C}}$ the orientation of $\mathcal{N}_{\mathcal{C}}S$ which corresponds to the direction of degeneration for which $z_1^+,z_{i^*}^+\in\mathbb{D}_+^2$ and by $\mathfrak{o}^{c;-}_{S;\mathcal{C}}$ the opposite orientation. Let $\mathfrak{o}^{\pm}_{k,l;l^*;\mathcal{C}}$ and $\mathfrak{o}^{\pm}_{S;l^*;\mathcal{C}}$ be the orientations of $\lambda_{\mathcal{C}}(\overline{\mathcal{M}}_{k,l}^\tau)$ and $\lambda_{\mathcal{C}}(S)$, respectively, induced by $\mathfrak{o}^{c;\pm}_{S;\mathcal{C}}$ as above.

A topological component S_* of S is characterized by the distribution of the points z_i^+ with $i\in L_r(S)$ between the interiors of the two disks cut out by the fixed locus S^1_r in each component $\mathbb{P}^1_r$ of the domain of the curves in S and by the orderings of the real marked points x_i with $i\in K_r(S)$ on S^1_r. Thus,

$$S_*\approx\mathcal{M}_1\times\mathcal{M}_2\subset\mathcal{M}^\tau_{k_1(S)+1,l_1(S)}\times\mathcal{M}^\tau_{k_2(S)+1,l_2(S)} \tag{4.10}$$

for some topological components $\mathcal{M}_1$ and $\mathcal{M}_2$ of the moduli spaces on the right-hand side above. We choose this identification so that

($\mathfrak{o}_S$1) the orderings of the conjugate pairs of marked points on the two sides are consistent,

($\mathfrak{o}_S$2) the nodal point on each of the irreducible components on the left-hand side corresponds to the *first* real marked point in the associated factor on the right-hand side.

If in addition $l_2^*(S)\geqslant 1$ and $i^*\in L_2^*(S)$ is the smallest element as before, we denote by $\mathfrak{o}_{S;l^*}$ the orientation on S obtained via the identification (4.10) from the orientations $\mathfrak{o}_{k_1(S)+1,l_1(S);l_1^*(S)}$ on $\mathcal{M}^\tau_{k_1(S)+1,l_1(S)}$ and $\mathfrak{o}_{k_2(S)+1,l_2(S);l_2^*(S)}$ on $\mathcal{M}^\tau_{k_2(S)+1,l_2(S)}$. The orientation $\mathfrak{o}_{S;l^*}$ does not depend on the orderings of the real points on S_1^1 and S_2^1. In this case, both fixed loci $S_r^1\subset\mathbb{P}_r^1$ are canonically oriented. For a topological component S_* of S, let $j_1'(S_*)\in\mathbb{Z}^{\geqslant 0}$ be the number of real marked points that lie on the oriented arc of $S^1_{r(S)}$ between the nodal point of $\mathbb{P}^1_{r(S)}$ and the real marked point x_1 of any $\mathcal{C}\in S_*$; if $k=0$, we take $j_1'(S_*)=0$. Define

$$\delta^+_{\mathbb{C};l^*}(S)=1,\qquad \delta^+_{\mathbb{R}}(S_*)=(k-1)j_1'(S_*)+\big(r(S)-1\big)k_1(S)k_2(S),$$
$$\delta^-_{\mathbb{C};l^*}(S)=l_2(S)-l_2^*(S),\quad \delta^-_{\mathbb{R}}(S_*)=(k-1)j_1'(S_*)+\binom{k_2(S)+1}{2}+\big(r(S)-1\big)(k-1).$$

Lemma 4.3. *Suppose $k,l\in\mathbb{Z}^{\geqslant 0}$ and $l^*\in[l]$ are such that $k+2l\geqslant 3$. Let $S_*\subset\overline{\mathcal{M}}^\tau_{k,l}-\partial\overline{\mathcal{M}}^\tau_{k,l}$ be a topological component of codimension 1 stratum such that $l_2^*(S)\geqslant 1$. The orientations $\mathfrak{o}^\pm_{S;l^*}$ and $\mathfrak{o}_{S;l^*}$ on $\lambda(S)|_{S_*}$ are the same if and only if $\delta^\pm_{\mathbb{C};l^*}(S)\cong k+\delta^\pm_{\mathbb{R}}(S_*)$ mod 2.*

Proof. For $r=1,2$, let

$$l_r=l_r(S),\qquad l_r^*=l_r^*(S),\qquad k_r=k_r(S),\qquad j_1'=j_1'(S_*).$$

If $l^*=l=2$ and $k=0$, $S=S_1=S_2$ is a point and $\mathfrak{o}_{S;l^*}=+1$. The claim in this case thus holds by the definition of the orientations $\mathfrak{o}_{0,2;2}=\mathfrak{o}_{0,2}$ on $\mathcal{M}^\tau_{0,2}$ and $\mathfrak{o}^{c;\pm}_{S;\mathcal{C}}$ on $\mathcal{N}S$. Since the orientation $\mathfrak{o}_{0,l;l}\equiv\mathfrak{o}_{0,l}$ with $l\geqslant 3$ (resp. $\mathfrak{o}_{1,l;l}\equiv\mathfrak{o}_{1,l}$ with $l\geqslant 2$) is obtained from the orientations $\mathfrak{o}_{0,l-1;l-1}$ (resp. $\mathfrak{o}_{1,l-1;l-1}$) and $\mathfrak{o}_l^+$, it follows that the claim holds whenever $l^*=l$ and $k=0$.

Let $\mathcal{C}\in S_*$ be as in (4.1). Suppose $l^*<l$ and $k=0$. Let l_1^c and l_2^c be the numbers of the marked points z_i^- of $\mathcal{C}$ with $i\in[l]-[l^*]$ on the same disk as z_1^+ and on the same disk as $z_{i^*}^+$, respectively. By definition,

$$\mathfrak{o}_{1,l_1;l_1^*}\big|_{\mathcal{M}_1}=(-1)^{l_1^c}\mathfrak{o}_{1,l_1;l_1}\big|_{\mathcal{M}_1},\quad \mathfrak{o}^+_{S;l^*}=(-1)^{l_1^c+l_2^c}\mathfrak{o}^+_{S;l},$$
$$\mathfrak{o}_{1,l_2;l_2^*}\big|_{\mathcal{M}_2}=(-1)^{l_2^c}\mathfrak{o}_{1,l_2;l_2}\big|_{\mathcal{M}_2},\quad \mathfrak{o}^-_{S;l^*}=(-1)^{l_1^c+(l_2-l_2^*-l_2^c)}\mathfrak{o}^-_{S;l}\,.$$

Thus, the claim in this case follows from the $l^*=l$ case above.

Suppose $k>0$, $S'\subset\mathcal{M}^\tau_{0,l}$ is the image of S under the forgetful morphism

$$\mathfrak{f}\colon\mathcal{M}^\tau_{k,l}\longrightarrow\mathcal{M}^\tau_{0,l}$$

dropping all real marked points, $\mathcal{C}'=\mathfrak{f}(\mathcal{C})$, and $(\mathcal{C}_1',\mathcal{C}_2')\in\mathcal{M}_1'\times\mathcal{M}_2'$ is the corresponding pair of marked irreducible components (with 1 real marked point each). Let $(x_{i_1},\ldots,x_{i_{k_1}})$ be the ordering of the real marked points on S_1^1 along its canonical direction starting from the first point after the node and $(x_{j_1},\ldots,x_{j_{k_2}})$ be the analogous ordering of the real marked points on S^1_2. The orientation $\mathfrak{o}_{S;l*}$ on $T_{\mathcal{C}}S$ is obtained via isomorphisms

$$\begin{aligned}\big(T_{\mathcal{C}}S,\mathfrak{o}_{S;l*}\big) &\approx \big(T_{\mathcal{C}'_1}\mathcal{M}'_1,\mathfrak{o}_{1,l_1;l_1^*}\big)\oplus\bigoplus_{m=1}^{k_1}T_{x_{i_m}}S^1_1\oplus\big(T_{\mathcal{C}'_2}\mathcal{M}'_2,\mathfrak{o}_{1,l_2;l_2^*}\big)\oplus\bigoplus_{m=1}^{k_2}T_{x_{j_m}}S^1_2\\ &\approx \big(T_{\mathcal{C}'_1}\mathcal{M}'_1,\mathfrak{o}_{1,l_1;l_1^*}\big)\oplus\big(T_{\mathcal{C}'_2}\mathcal{M}'_2,\mathfrak{o}_{1,l_2;l_2^*}\big)\oplus\bigoplus_{m=1}^{k_1}T_{x_{i_m}}S^1_1\oplus\bigoplus_{m=1}^{k_2}T_{x_{j_m}}S^1_2\\ &\approx \big(T_{\mathcal{C}'}S',\mathfrak{o}_{S';l*}\big)\oplus\bigoplus_{m=1}^{k_1}T_{x_{i_m}}S^1_1\oplus\bigoplus_{m=1}^{k_2}T_{x_{j_m}}S^1_2\end{aligned}\tag{4.11}$$

from the standard orientations on S^1_1 and S^1_2 determined by the marked points z_1^+ and z_{i*}^+. The second isomorphism above is orientation-preserving because the dimension of $T_{\mathcal{C}'_2}\mathcal{M}'_2$ is even.

Let $\widetilde{\mathcal{C}}\in\mathcal{M}^\tau_{k,l}$ be a smooth marked curve close to $\mathcal{C}$ from the direction of degeneration determined by $\mathfrak{o}_S^{c;\pm}$ and $\widetilde{\mathcal{C}}'=\mathfrak{f}(\widetilde{\mathcal{C}})$. Let $(x_1,x_{i_2^\pm},\ldots,x_{i_k^\pm})$ be the ordering of the real marked points of $\widetilde{\mathcal{C}}$ along the standard direction of S^1 determined by $z_1^+(\widetilde{\mathcal{C}})$. The orientation $\mathfrak{o}^\pm_{S;l*}$ at $\mathcal{C}$ is obtained via isomorphisms

$$\begin{aligned}\big(T_{\mathcal{C}}S,\mathfrak{o}^\pm_{S;l*}\big)\oplus\big(\mathcal{N}_{\mathcal{C}}S,\mathfrak{o}_S^{c;\pm}\big) &\approx \big(T_{\widetilde{\mathcal{C}}}\mathcal{M}^\tau_{k,l},\mathfrak{o}_{k,l;l*}\big)\approx\big(T_{\widetilde{\mathcal{C}}'}\mathcal{M}^\tau_{0,l},\mathfrak{o}_{0,l;l*}\big)\oplus\bigoplus_{m=1}^{k}T_{x_{i_m^\pm}}S^1\\ &\approx \big(T_{\mathcal{C}'}S',\mathfrak{o}^\pm_{S';l*}\big)\oplus\big(\mathcal{N}_{\mathcal{C}'}S',\mathfrak{o}_{S'}^{c;\pm}\big)\oplus\bigoplus_{m=1}^{k}T_{x_{i_m^\pm}}S^1\\ &\approx (-1)^k\big(T_{\mathcal{C}'}S',\mathfrak{o}^\pm_{S';l*}\big)\oplus\bigoplus_{m=1}^{k}T_{x_{i_m^\pm}}S^1\oplus\big(\mathcal{N}_{\mathcal{C}}S,\mathfrak{o}_S^{c;\pm}\big).\end{aligned}\tag{4.12}$$

By (4.11), (4.12), and the $k=0$ case above, the claim in the general case holds if $\delta^\pm_{\mathbb{R}}(S_*)$ has the same parity as the parity of the permutation

$$\big(i_1,\ldots,i_{k_1},j_1,\ldots,j_{k_2}\big)\longrightarrow\big(i_1^\pm=1,i_2^\pm,\ldots,i_k^\pm\big)\tag{4.13}$$

plus the parity of k_2 in the minus case, since the tangent spaces $T_{x_{j_m}}S^1_2$ then enter with the reversed orientations.

Suppose $r(S)=1$. The plus case of (4.13) then moves the indices $(i_1,\ldots,i_{j'_1})$ to the end preserving their order. The parity of this permutation is

$$\delta^+_{\mathbb{R}}(S_*)=j'_1\big(k-j'_1\big)\cong(k-1)j'_1\mod 2\,.$$

The minus case of (4.13) is the composition of the permutation

$$\big(j_1,\ldots,j_{k_2}\big)\longrightarrow\big(j_{k_2},\ldots,j_1\big)\tag{4.14}$$

with the transposition in the plus case. This adds an extra $k_2(k_2-1)/2$ to the parity.

Suppose $r(S)=2$. The plus case of (4.13) then moves $(j_{j_1'+1}=1,\ldots,j_{k_2})$ to the front preserving their order. The parity of this permutation is

$$\delta^+_{\mathbb{R}}(S_*)=\big(k_2-j_1'\big)\big(k_1+j_1'\big)\cong(k-1)j_1'+k_1k_2 \mod 2\,.$$

The minus case of (4.13) consists of the permutation (4.14) followed by moving $(j_{j_1'+1},\ldots,j_1)$ to the front of the entire k tuple. The parity of this permutation plus k_2 is

$$\delta^-_{\mathbb{R}}(S_*)=\binom{k_2+1}{2}+\big(j_1'+1\big)\big(k-1-j_1'\big)\cong(k-1)j_1'+k-1+\binom{k_2+1}{2}\,.$$

This establishes the claim. □

4.4 Bordisms in $\overline{\mathcal{M}}^\tau_{1,2}$ and $\overline{\mathcal{M}}^\tau_{0,3}$

The two relations of Theorem 1.1 are proved by applying (2.5) with the hypersurfaces $\Upsilon\subset\overline{\mathcal{M}}^\tau_{1,2}$ and $\Upsilon\subset\overline{\mathcal{M}}^\tau_{0,3}$ of Lemmas 4.4 and 4.5 below. These hypersurfaces are regular, in the sense defined at the end of Section 4.2, and in particular are disjoint from the codimension 1 stratum S_1 of the moduli space. We determine the degrees of these hypersurfaces with respect to the other non-boundary codimension 1 strata and with respect to the forgetful morphism $\mathfrak{f}^{\mathbb{R}}_{1,2;1}$ in the first case. These degrees are essential for computing the right-hand side of (2.5); see Proposition 5.7.

Orientations are interpreted below as relative orientations of maps to a point; see Section 3.1. All notation for the codimension 1 strata and the degrees is as in Section 4.2. For a primary codimension 2 stratum Γ of $\overline{\mathcal{M}}^\tau_{k,l}$, we denote by $\mathfrak{o}^c_\Gamma$ the canonical orientation on $\mathcal{N}\Gamma$ as in Lemma 4.2 and by $\mathfrak{o}_{\Gamma;l}$ the orientation on Γ as in the first half of Section 4.3. Since $\mathfrak{o}_{k,l}=\mathfrak{o}_{k,l;l}$ for $k=0,1$,

$$\mathfrak{o}_{\Gamma;l}=\mathfrak{o}^c_\Gamma\mathfrak{o}_{k,l} \tag{4.15}$$

in the cases of Lemmas 4.4 and 4.5. Let $P^\pm\in\overline{\mathcal{M}}^\tau_{1,2}$ be the three-component curve so that z_1^+ and $z_2^\pm$ lie on the same irreducible component.

Lemma 4.4. *There exists an embedded closed path $\Upsilon\subset\overline{\mathcal{M}}^\tau_{1,2}$ with a co-orientation $\mathfrak{o}^c_\Upsilon$ so that Υ is a regular hypersurface and*

$$\partial\big(\Upsilon,\mathfrak{o}^c_\Upsilon\big)=\big(P^+,\mathfrak{o}^c_{P^+}\big)\sqcup\big(P^-,\mathfrak{o}^c_{P^-}\big),\quad \deg^{\mathbb{R}}_1\big(\Upsilon,\mathfrak{o}^c_\Upsilon\big)=1,\quad \deg_{S_2}\big(\Upsilon,\mathfrak{o}^c_\Upsilon\big)=-1\,. \tag{4.16}$$

Proof. Since $(P^+,\mathfrak{o}_{P^\pm;2})$ is a $\pm$-point, (4.15) gives

$$\mathfrak{o}^c_{P^\pm}\mathfrak{o}_{1,2}=\pm1. \tag{4.17}$$

Let $\breve{\mathcal{M}}^\tau_{1,2}\approx S^2$ be the space obtained by contracting S_1 to a point P_0. By [9, Lemma 5.4], the orientation $\mathfrak{o}_{1,2}$ on $\mathcal{M}^\tau_{1,2}$ extends over $\breve{\mathcal{M}}^\tau_{1,2}$; this can also be readily seen from the definitions. The morphisms $\mathfrak{f}^{\mathbb{R}}_{1,2;1}$ and $\mathfrak{f}_{1,2;2}$ descend to smooth maps

$$\mathfrak{f}^{\mathbb{R}}_{1,2;1}\colon\breve{\mathcal{M}}^\tau_{1,2}\longrightarrow\overline{\mathcal{M}}^\tau_{0,2}\qquad\text{and}\qquad\mathfrak{f}_{1,2;2}\colon\breve{\mathcal{M}}^\tau_{1,2}\longrightarrow\overline{\mathcal{M}}^\tau_{1,1}\,.$$

We can identify $\widetilde{\mathcal{M}}^{\tau}_{1,2}$ with $S^2\subset\mathbb{R}^3$ and $\overline{\mathcal{M}}^{\tau}_{0,2}$ with $[-1,1]$ so that $P^{\pm}=(\pm1,0,0)$ and $\mathfrak{f}^{\mathbb{R}}_{1,2;1}$ is the height function. The fibers of $\mathfrak{f}^{\mathbb{R}}_{1,2;1}$ over $\mathcal{M}^{\tau}_{0,2}$ are then the circles of constant latitude. The orientation $\mathfrak{o}^{\mathbb{R}}_1$ of the fibers of $\mathfrak{f}^{\mathbb{R}}_{1,2;1}|_{\mathcal{M}^{\tau}_{1,2}}$ specified in Section 4.1 extends over the equator $\overline{S}_2\approx S^1$. By Lemma 4.1($\mathfrak{o}_{\mathcal{M}}1$),

$$\mathfrak{o}_{1,2}\big|_{\mathcal{M}^{\tau}_{1,2}}=\big(\mathfrak{o}^{\mathbb{R}}_1\mathfrak{o}_{0,2}\big)\big|_{\mathcal{M}^{\tau}_{1,2}}. \tag{4.18}$$

Let $\Upsilon'\subset\widetilde{\mathcal{M}}^{\tau}_{1,2}$ be a meridian running from P^- to P^+ disjoint from P_0 and $\mathfrak{o}_{\Upsilon'}$ be its canonical orientation. Thus, the restriction

$$\mathfrak{f}^{\mathbb{R}}_{1,2;1}\colon\big(\Upsilon',\mathfrak{o}_{\Upsilon'}\big)\longrightarrow\big(\overline{\mathcal{M}}^{\tau}_{0,2},\mathfrak{o}_{0,2}\big)$$

is an orientation-preserving diffeomorphism. We take $\mathfrak{o}^c_{\Upsilon'}$ to be the orientation of $\mathcal{N}\Upsilon'$ so that the projection

$$\big(\ker\mathrm{d}\mathfrak{f}^{\mathbb{R}}_{1,2;1},\mathfrak{o}^{\mathbb{R}}_1\big)\longrightarrow\big(\mathcal{N}\Upsilon',\mathfrak{o}^c_{\Upsilon'}\big)$$

is an orientation-preserving isomorphism. By (4.18) and Lemma 3.1(1),

$$\mathfrak{o}_{\Upsilon'}=\mathfrak{o}^c_{\Upsilon'}\mathfrak{o}_{1,2}\qquad\text{and}\qquad\deg^{\mathbb{R}}_1\big(\Upsilon',\mathfrak{o}^c_{\Upsilon'}\big)\equiv\deg\big(\mathfrak{f}^{\mathbb{R}}_{1,2;1}|_{\Upsilon'},\mathfrak{o}^c_{\Upsilon'}\mathfrak{o}^{\mathbb{R}}_1\big)=1. \tag{4.19}$$

By Lemma 4.1($\mathfrak{o}_{\mathcal{M}}2$), the orientation $\mathfrak{o}_{1,2}$ corresponds to the natural orientation of the complex coordinate z_2^+ with $z_1^+=0$ and $x_1=1$ fixed. Thus, $\mathfrak{o}^c_{\Upsilon'}$ is the negative rotation in the z_2^+-coordinate. Along $\overline{S}_2$, it corresponds to the negative rotation of the node. Thus,

$$\deg_{S_2}\big(\Upsilon',\mathfrak{o}^c_{\Upsilon'}\big)\equiv\deg\big(\mathfrak{f}_{0,2;2}|_{\Upsilon'\cap\overline{S}_2},\mathfrak{o}^c_{\Upsilon'}\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}\big)=-1.$$

Since the outer normal co-orientation $\mathfrak{o}^c_{\partial\Upsilon'}$ of $\partial\Upsilon'$ agrees with the restriction of $\pm\mathfrak{o}_{\Upsilon'}$ at $P^{\pm}$, i.e.

$$\big(\mathfrak{o}^c_{\partial\Upsilon'}\mathfrak{o}_{\Upsilon'}\big)\big|_{P^{\pm}}=\pm1,$$

the first statement in (4.19) gives

$$\big(\mathfrak{o}^c_{\partial\Upsilon'}\mathfrak{o}^c_{\Upsilon'}\big)\big|_{P^{\pm}}\mathfrak{o}_{1,2}\big|_{P^{\pm}}=\mathfrak{o}^c_{\partial\Upsilon'}\big|_{P^{\pm}}\mathfrak{o}_{\Upsilon'}\big|_{P^{\pm}}=\pm1.$$

Comparing with (4.18), we conclude that

$$\big(\mathfrak{o}^c_{\partial\Upsilon'}\mathfrak{o}_{\Upsilon'}\big)\big|_{P^{\pm}}=\mathfrak{o}^c_{P^{\pm}},$$

i.e. the first equality in (4.16) with Υ replaced by Υ' holds as well.

We take $\Upsilon\subset\overline{\mathcal{M}}^{\tau}_{1,2}$ to be the preimage of Υ' under the blowdown map (which is a diffeomorphism on a neighborhood of Υ) and $\mathfrak{o}^c_{\Upsilon}$ to be the pullback of $\mathfrak{o}^c_{\Upsilon'}$. $\square$

The moduli space $\overline{\mathcal{M}}^{\tau}_{0,3}$ is a 3-manifold with the boundary

$$\partial\overline{\mathcal{M}}^{\tau}_{0,3}=\overline{S}^{++}_{23}\sqcup\overline{S}^{+-}_{23}\sqcup\overline{S}^{-+}_{23}\sqcup\overline{S}^{--}_{23},$$

where

$$S^{\pm\pm}_{ij}\approx\overline{\mathcal{M}}_{0,4}\approx S^2$$

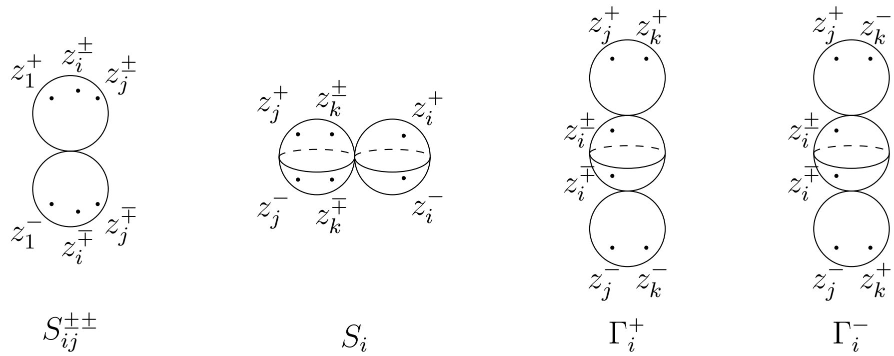

Figure 6: Elements of open codimension 1 and 2 strata of $\overline{\mathcal{M}}_{0,3}^{\tau}$, with $\{i,j\}=\{2,3\}$ in the first diagram and $\{i,j,k\}=\{1,2,3\}$ in the other four.

is the closure of the open codimension 1 stratum $S_{ij}^{\pm\pm}$ of curves consisting of a pair of conjugate spheres with the marked points $z_i^{\pm}$ and $z_j^{\pm}$ on the same sphere as z_1^+; see [10, Fig. 4] and the first diagram in Figure 6. There are four primary codimension 2 strata $\Gamma_i^{\pm}$, with $i=2,3$, in $\overline{\mathcal{M}}_{0,3}^{\tau}$. The closed interval $\overline{\Gamma}_i^+$ (resp. $\overline{\Gamma}_i^-$) is the closure of the open codimension 2 stratum Γ_i^+ (resp. Γ_i^-) of curves consisting of one real sphere and a conjugate pair of spheres so that the real sphere carries the marked points $z_i^{\pm}$ and the decorations $^{\pm}$ of the marked points on each of the conjugate spheres are the same (resp. different); see the last pair of diagrams in Figure 6. Let

$$\mathring{\Gamma}_i^+ = \Gamma_i^+ \cup \big(\overline{\Gamma}_i^+ \cap \overline{S}_i\big) \subset \overline{\Gamma}_i^+$$

be the complement of the endpoints of $\overline{\Gamma}_i^+$.

Lemma 4.5. *There exist a bordered surface $\Upsilon \subset \overline{\mathcal{M}}_{0,3}^{\tau}$ with a co-orientation $\mathfrak{o}_{\Upsilon}^c$ and a one-dimensional manifold $\gamma' \subset \overline{\mathcal{M}}_{0,3}^{\tau}$ with a co-orientation $\mathfrak{o}_{\gamma'}^c$ so that Υ is transverse to all open strata of $\overline{\mathcal{M}}_{0,3}^{\tau}$ not contained in any $\overline{\Gamma}_i^{\pm}$ with $i=2,3$, Υ is a regular hypersurface, and*

$$\begin{gathered}\partial\big(\Upsilon,\mathfrak{o}_{\Upsilon}^c\big) = \big(\mathring{\Gamma}_2^+,\mathfrak{o}_{\Gamma_2^+}^c\big) \cup \big(\mathring{\Gamma}_3^+,-\mathfrak{o}_{\Gamma_3^+}^c\big) \cup \big(\mathring{\Gamma}_2^-,\mathfrak{o}_{\Gamma_2^-}^c\big) \cup \big(\mathring{\Gamma}_3^-,-\mathfrak{o}_{\Gamma_3^-}^c\big) \cup \big(\gamma',\mathfrak{o}_{\gamma'}^c\big), \\ \gamma' \subset \partial\overline{\mathcal{M}}_{0,3}^{\tau}, \qquad \deg_{S_2}\big(\Upsilon,\mathfrak{o}_{\Upsilon}^c\big)=1, \qquad \deg_{S_3}\big(\Upsilon,\mathfrak{o}_{\Upsilon}^c\big)=-1.\end{gathered} \tag{4.20}$$

Proof. For $i=2,3$, z_i^+ moves in $(\Gamma_i^+,\mathfrak{o}_{\Gamma_i^+;3})$ (resp. $(\Gamma_i^-,\mathfrak{o}_{\Gamma_i^-;3})$) from the node separating the sphere carrying z_1^- (resp. z_1^+) to the other node. Each closed interval $\overline{\Gamma}_i^{\pm}$ intersects $\overline{S}_i$ transversally at one point $P_i^{\pm}$ and does not intersect $\overline{S}_j$ for $j=1,2,3$ with $j\neq i$. It intersects $\partial\overline{\mathcal{M}}_{0,3}^{\tau}$ at its endpoints; we denote the starting point by $P_i^{\pm-}$ and the ending point by $P_i^{\pm+}$. By [10, Section 3], the orientation $\mathfrak{o}_{0,3}=\mathfrak{o}_{0,3;3}$ on $\mathcal{M}_{0,3}^{\tau}$ extends over $\overline{\mathcal{M}}_{0,3}^{\tau}$.

By [10, Remark 3.5], $\overline{\mathcal{M}}_{0,3}^{\tau}$ is the blowup of a bordered manifold $\breve{\mathcal{M}}_{0,3}^{\tau}$ at a point P_0 with the exceptional divisor $\overline{S}_1$. Denote by

$$p:\overline{\mathcal{M}}_{0,3}^{\tau} \longrightarrow \breve{\mathcal{M}}_{0,3}^{\tau}$$

the blowdown map. The morphisms $\mathfrak{f}_{0,3;2}$ and $\mathfrak{f}_{0,3;3}$ descend to smooth maps

$$\mathfrak{f}_{0,3;2}\colon \breve{\mathcal{M}}^\tau_{0,3} \longrightarrow \overline{\mathcal{M}}^\tau_{0,2} \qquad\text{and}\qquad \mathfrak{f}_{0,3;3}\colon \breve{\mathcal{M}}^\tau_{0,3} \longrightarrow \overline{\mathcal{M}}^\tau_{0,2}\,. \tag{4.21}$$

Since $\overline{S}_1$ is disjoint from the four spheres of $\partial\overline{\mathcal{M}}^\tau_{0,3}$ and the four intervals $\overline{\Gamma}^\pm_i$ with $i=2,3$, p is a diffeomorphism on neighborhoods of these spaces. We denote the images of these intervals and the twelve points $P_i^\pm, P_i^{\pm\pm}$ on them under p in the same way. The spaces

$$\breve{S}_2\equiv p(\overline{S}_2)\approx \breve{\mathcal{M}}^\tau_{1,2}\approx S^2 \qquad\text{and}\qquad \breve{S}_3\equiv p(\overline{S}_3)\approx \breve{\mathcal{M}}^\tau_{1,2}$$

are the fibers of $\mathfrak{f}_{0,3;3}$ and $\mathfrak{f}_{0,3;2}$, respectively, over the curve consisting of two real components, which corresponds to $1\in[0,\infty]$ under the identification $\varphi_{0,2}$ in (4.6).

Setting $(z_1^+,z_1^-)=(0,\infty)$, we obtain a natural identification

$$\begin{gathered}\breve{\mathcal{M}}^\tau_{0,3}-\partial\breve{\mathcal{M}}^\tau_{0,3}\approx \big\{\big((z_2^+,z_2^-),(z_3^+,z_3^-)\big)\in(\mathbb{P}^1)^4\colon z_i^+=\tau(z_i^-),\ (z_2^\pm,z_3^\pm)\neq(0,0)\big\}/\!\sim,\\ \big((z_2^+,z_2^-),(z_3^+,z_3^-)\big)\sim\big((zz_2^+,zz_2^-),(zz_3^+,zz_3^-)\big)\quad \forall\,z\in S^1.\end{gathered}$$

The condition $z_i^+=\tau(z_i^-)$ implies that the points z_i^+ and z_i^- lie on a great arc through the poles $z_1^+\equiv0$ and $z_1^-\equiv\infty$ (or lie at $z_1^\pm$). The blowup point P_0 in this identification corresponds to the point $[(1,1),(1,1)]$. The projections (4.21) in this identification are given by

$$\mathfrak{f}_{0,3;i}\big([(z_2^+,z_2^-),(z_3^+,z_3^-)]\big)=\big[(0,\infty),(z_{5-i}^+,z_{5-i}^-)\big].$$

In particular, the fiber of $\mathfrak{f}_{0,3;i}$ over a point of $[(0,\infty),(z_{5-i}^+,z_{5-i}^-)]$ of $\overline{\mathcal{M}}^\tau_{0,2}-\partial\overline{\mathcal{M}}^\tau_{0,2}$ can be identified via z_i^+ with $\mathbb{P}^1$ by choosing $z_{5-i}^+\in\mathbb{R}^+$.

The space $\breve{\mathcal{M}}^\tau_{0,3}-\partial\breve{\mathcal{M}}^\tau_{0,3}$ is covered by two charts

$$\begin{aligned}\mathbb{R}^+\times\mathbb{P}^1&\longrightarrow\breve{\mathcal{M}}^\tau_{0,3}-\mathfrak{f}_{0,3;2}^{-1}(\partial\overline{\mathcal{M}}^\tau_{0,2}),\quad (r_2,z_3)\longrightarrow\big[(r_2,1/r_2),(z_3,1/\overline{z_3})\big],\\ \mathbb{P}^1\times\mathbb{R}^+&\longrightarrow\breve{\mathcal{M}}^\tau_{0,3}-\mathfrak{f}_{0,3;3}^{-1}(\partial\overline{\mathcal{M}}^\tau_{0,2}),\quad (z_2,r_3)\longrightarrow\big[(z_2,1/\overline{z_2}),(r_3,1/r_3)\big].\end{aligned}\tag{4.22}$$

In these charts,

$$\begin{aligned}\varphi_{0,2}\big(\mathfrak{f}_{0,3;3}(r_2,z_3)\big)&=1/r_2^2, &\quad \breve{S}_2&=\big\{(r_2,z_3)\in\mathbb{R}^+\times\mathbb{P}^1\colon r_2=1\big\},\\ \varphi_{0,2}\big(\mathfrak{f}_{0,3;2}(z_2,r_3)\big)&=1/r_3^2, &\quad \breve{S}_3&=\big\{(z_2,r_3)\in\mathbb{P}^1\times\mathbb{R}^+\colon r_3=1\big\}.\end{aligned}\tag{4.23}$$

The overlap map between the two charts,

$$\mathbb{R}^+\times\mathbb{C}^*\longrightarrow\mathbb{C}^*\times\mathbb{R}^+,\qquad \big(r_2,r_3\mathrm{e}^{\mathrm{i}\theta}\big)\longrightarrow\big(r_2\mathrm{e}^{-\mathrm{i}\theta},r_3\big),$$

is orientation-preserving with respect to the standard orientations $\mathfrak{o}_{\mathbb{R}^+}$ on $\mathbb{R}^+$ and $\mathfrak{o}_{\mathbb{P}^1}$ on $\mathbb{P}^1$. We take $\breve{\mathfrak{o}}_{0,3}$ to be the orientation on $\breve{\mathcal{M}}^\tau_{0,3}$ *opposite* to the orientation determined by $\mathfrak{o}_{\mathbb{R}^+}$ and $\mathfrak{o}_{\mathbb{P}^1}$ via the two charts in (4.22). Since the map

$$\big(\mathbb{R}^+,\mathfrak{o}_{\mathbb{R}^+}\big)\longrightarrow\big(\mathbb{R}^+,-\mathfrak{o}_{\mathbb{R}^+}\big),\qquad r_2\longrightarrow\varphi_{0,2}\big(\mathfrak{f}_{0,3;3}(r_2,z_3)\big),\tag{4.24}$$

is orientation-preserving for each $z_3^+\in\mathbb{P}^1$ fixed, Lemma 4.1($\mathfrak{o}_{\mathcal{M}}2$) and (4.15) give

$$\mathfrak{o}_{0,3}\big|_{\overline{\mathcal{M}}_{0,3}^\tau-\overline{S}_1}=p^*\breve{\mathfrak{o}}_{0,3}\big|_{\overline{\mathcal{M}}_{0,3}^\tau-\overline{S}_1}, \qquad \mathfrak{o}_{\Gamma_i^\pm;3}=\mathfrak{o}_{\Gamma_i^\pm}^c\breve{\mathfrak{o}}_{0,3}\,. \tag{4.25}$$

The moduli space $\overline{\mathcal{M}}_{0,3}^\tau$ is a submanifold of $\overline{\mathcal{M}}_{0,6}$. By [21, Appendix D.4,5], the four cross-ratios

$$\mathrm{CR}_{\pm\pm}^\tau\colon \mathcal{M}_{0,3}^\tau\longrightarrow\mathbb{P}^1, \quad \mathrm{CR}_{\pm\pm}^\tau\big([(z_i^+,z_i^-)_{i\in[3]}]\big)=\frac{z_2^\pm-z_1^-}{z_3^\pm-z_1^-}:\frac{z_2^\pm-z_1^+}{z_3^\pm-z_1^+}\,,$$

extend over $\overline{\mathcal{M}}_{0,3}^\tau$ and descend to smooth maps from $\breve{\mathcal{M}}_{0,3}^\tau$. The subspace

$$\widetilde{\Upsilon}'\subset\breve{\mathcal{M}}_{0,3}^\tau-\{P_2^{\pm\pm},P_3^{\pm\pm}\}$$

where all four cross-ratios take values in

$$\mathbb{RP}^1\equiv[-\infty,\infty]/-\infty\sim\infty \tag{4.26}$$

is an orientable surface, as explained in the next paragraph. The boundary of $\widetilde{\Upsilon}'$ consists of the complement of two points in a circle on each boundary sphere of $\partial\breve{\mathcal{M}}_{0,3}^\tau$.

The intersections of $\widetilde{\Upsilon}'$ with the charts (4.22) are given by

$$\begin{aligned}\mathbb{R}^+\times\mathbb{RP}^1&\longrightarrow\widetilde{\Upsilon}'-\mathfrak{f}_{0,3;2}^{-1}(\partial\overline{\mathcal{M}}_{0,2}^\tau), \quad (r_2,r_3)\longrightarrow\big[(r_2,1/r_2),(r_3,1/r_3)\big],\\ \mathbb{RP}^1\times\mathbb{R}^+&\longrightarrow\widetilde{\Upsilon}'-\mathfrak{f}_{0,3;3}^{-1}(\partial\overline{\mathcal{M}}_{0,2}^\tau), \quad (r_2,r_3)\longrightarrow\big[(r_2,1/r_2),(r_3,1/r_3)\big].\end{aligned} \tag{4.27}$$

An element $[(z_2^+,z_2^-),(z_3^+,z_3^-)]$ of $\breve{\mathcal{M}}_{0,3}^\tau-\partial\breve{\mathcal{M}}_{0,3}^\tau$ belongs to $\widetilde{\Upsilon}'$ if and only if all four points $z_i^\pm\in\mathbb{P}^1$ with $i=2,3$ lie on a great circle through z_1^- and z_1^+. The structure of $\widetilde{\Upsilon}'$ along $\partial\breve{\mathcal{M}}_{0,3}^\tau$ is described by the local coordinates of [10, Remark 3.5] with $z\in\mathbb{R}$. The overlap map between the charts (4.27),

$$\mathbb{R}^+\times\mathbb{R}^*\longrightarrow\mathbb{R}^*\times\mathbb{R}^+, \qquad (r_2,r_3)\longrightarrow\begin{cases}(r_2,r_3), & \text{if } r_3\in\mathbb{R}^+;\\ (-r_2,-r_3), & \text{if } r_3\in\mathbb{R}^-;\end{cases}$$

is orientation-preserving with respect to the orientation $\mathfrak{o}_{\mathbb{R}^+}$ on $\mathbb{R}^+$ and the orientation $\mathfrak{o}_{\mathbb{RP}^1}$ on $\mathbb{RP}^1$ induced by the standard orientation of $[-\infty,\infty]$ via (4.26). We take $\mathfrak{o}_{\widetilde{\Upsilon}'}$ to be the orientation on $\widetilde{\Upsilon}'$ determined by $\mathfrak{o}_{\mathbb{R}^+}$ and $\mathfrak{o}_{\mathbb{RP}^1}$ via the two charts in (4.27).

The surface $\widetilde{\Upsilon}'$ contains the four open intervals

$$\mathring{\Gamma}_i^\pm\equiv\overline{\Gamma}_i^\pm-\{P_i^{\pm+},P_i^{\pm-}\}$$

with $i=2,3$; the closures of these intervals connect the components of the closure of $\partial\widetilde{\Upsilon}'$. In the two charts (4.22),

$$\begin{aligned}\mathring{\Gamma}_2^+&=\big\{(r_2,z_3)\in\mathbb{R}^+\times\mathbb{P}^1\colon z_3=0\big\}, \quad \mathring{\Gamma}_2^-=\big\{(r_2,z_3)\in\mathbb{R}^+\times\mathbb{P}^1\colon z_3=\infty\big\},\\ \mathring{\Gamma}_3^+&=\big\{(z_2,r_3)\in\mathbb{P}^1\times\mathbb{R}^+\colon z_2=0\big\}, \quad \mathring{\Gamma}_3^-=\big\{(z_2,r_3)\in\mathbb{P}^1\times\mathbb{R}^+\colon z_2=\infty\big\}.\end{aligned}$$

The cut $\widehat{\Upsilon}'$ of $\widetilde{\Upsilon}'$ along the four open intervals has two components, $\widehat{\Upsilon}^+$ and $\widehat{\Upsilon}^-$. They are distinguished by whether $\mathrm{CR}_{23}^{++}(\mathcal{C})$ lies in $\mathbb{R}^+$ or $\mathbb{R}^-$ for the elements $\mathcal{C}$ of $\widehat{\Upsilon}'-\partial\widehat{\Upsilon}'$, i.e. whether the points $z_2^{\pm}$ lie on the same great arc through z_1^- and z_1^+ as the points $z_3^{\pm}$ or on the opposite arc.

Let $\Upsilon'=\widehat{\Upsilon}^+$ and $\gamma'=\Upsilon'\cap\partial\widetilde{\Upsilon}'$. The former is a surface with boundary

$$\partial\Upsilon' = \mathring{\Gamma}_2^+\cup\mathring{\Gamma}_3^+\cup\mathring{\Gamma}_2^-\cup\mathring{\Gamma}_3^-\cup\gamma'\,.$$

By (4.23) and (4.27), this surface intersects $\breve{S}_2$ and $\breve{S}_3$ transversely along the closed line segments given by

$$\begin{aligned}\breve{\Gamma}_2 &\equiv \Upsilon'\cap\breve{S}_2 = \big\{(r_2,z_3)\in\mathbb{R}^+\times\mathbb{P}^1\colon r_2=1,\ z_3\in[0,\infty]\big\},\\ \breve{\Gamma}_3 &\equiv \Upsilon'\cap\breve{S}_3 = \big\{(z_2,r_3)\in\mathbb{P}^1\times\mathbb{R}^+\colon r_3=1,\ z_2\in[0,\infty]\big\}\end{aligned}$$

in the charts (4.22). For $i=2,3$, let $\mathfrak{o}_{\breve{\Gamma}_i}$ be the *opposite* of the orientation on $\breve{\Gamma}_i$ given by the $r_{5-i}\equiv|z_{5-i}|$ coordinate. The restriction

$$\mathfrak{f}_{0,3;i}\colon \big(\breve{\Gamma}_i,\mathfrak{o}_{\breve{\Gamma}_i}\big)\longrightarrow\big(\overline{\mathcal{M}}_{0,2}^{\tau},\mathfrak{o}_{0,2}\big) \tag{4.28}$$

is then an orientation-preserving diffeomorphism (because (4.24) is orientation-preserving).

We denote by $\mathfrak{o}_{\Upsilon'}$ and $\mathfrak{o}_{\Upsilon'}^c$ the restrictions of the orientation $\mathfrak{o}_{\widetilde{\Upsilon}'}$ and the co-orientation $\mathfrak{o}_{\widetilde{\Upsilon}'}^c$ to Υ', by $\mathfrak{o}_{\gamma'}$ the boundary orientation on γ' induced by $\mathfrak{o}_{\Upsilon'}$, and by $\mathfrak{o}_{\gamma'}^c$ the orientation on $\mathcal{N}\gamma'$ determined by $\breve{\mathfrak{o}}_{0,3}$ and $\mathfrak{o}_{\gamma'}$. Thus,

$$\mathfrak{o}_{\Upsilon'}=\mathfrak{o}_{\Upsilon'}^c\breve{\mathfrak{o}}_{0,3},\qquad \mathfrak{o}_{\gamma'}=\big(\mathfrak{o}_{\partial\Upsilon'}^c\mathfrak{o}_{\Upsilon'}\big)\big|_{\gamma'}=\mathfrak{o}_{\gamma'}^c\breve{\mathfrak{o}}_{0,3}\,. \tag{4.29}$$

At the point $P_2^+\in\overline{\Gamma}_2^+,\breve{\Gamma}_2$, the orientation $\mathfrak{o}_{\Gamma_2^+;3}$ on $\overline{\Gamma}_2^+$ is the opposite of the orientation given by the r_2-coordinate (because z_2^+ moves from $z_1^-=\infty$ to $z_1^+=0$); see Figure 7. Since the natural isomorphisms

$$\big(T_{P_2^+}\breve{\Gamma}_2,\mathfrak{o}_{\breve{\Gamma}_2}\big)\longrightarrow\big(\mathcal{N}_{\Upsilon'}\partial\Upsilon'|_{P_2^+},\mathfrak{o}_{\partial\Upsilon'}^c\big)\quad\text{and}\quad\big(T_{P_2^+}\overline{\Gamma}_2^+,\mathfrak{o}_{\Gamma_2^+;3}\big)\oplus\big(T_{P_2^+}\breve{\Gamma}_2,\mathfrak{o}_{\breve{\Gamma}_2}\big)\longrightarrow\big(T_{P_2^+}\Upsilon',\mathfrak{o}_{\Upsilon'}\big)$$

are orientation-preserving,

$$\partial\mathfrak{o}_{\Upsilon'}|_{P_2^+}\equiv\big(\mathfrak{o}_{\partial\Upsilon'}^c\mathfrak{o}_{\Upsilon'}\big)\big|_{P_2^+}=\mathfrak{o}_{\Gamma_2^+;3}|_{P_2^+}.$$

Since the right-hand side of

$$\partial\big(\Upsilon',\mathfrak{o}_{\Upsilon'}\big)=\big(\mathring{\Gamma}_2^+,\mathfrak{o}_{\Gamma_2^+;3}\big)\cup\big(\mathring{\Gamma}_3^+,-\mathfrak{o}_{\Gamma_3^+;3}\big)\cup\big(\mathring{\Gamma}_2^-,\mathfrak{o}_{\Gamma_2^-;3}\big)\cup\big(\mathring{\Gamma}_3^-,-\mathfrak{o}_{\Gamma_3^-;3}\big)\cup\big(\gamma',\mathfrak{o}_{\gamma'}\big)$$

is an oriented loop and the equality above respects the orientations at P_2^+, it follows that this equality respects the orientations everywhere. Combining it with the second equality in (4.25) and the first and last equalities in (4.29), we obtain (4.20) with Υ replaced by Υ'.

We now compute the degree

$$\deg\big(\mathfrak{f}_{0,3;i}|_{\breve{\Gamma}_i},\mathfrak{o}_{\Upsilon'}^c|_{\breve{\Gamma}_i}\mathfrak{o}_{\mathrm{nd}}^{\mathbb{R}}\big)\in\mathbb{Z} \tag{4.30}$$

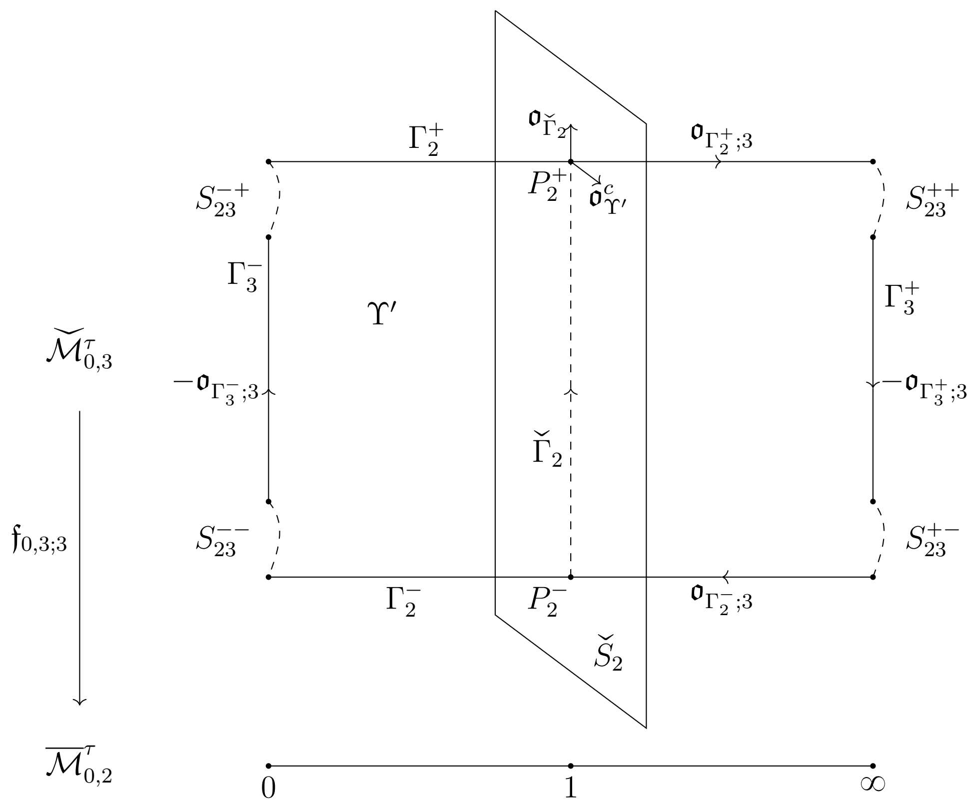

Figure 7: The surfaces Υ' and $\breve{S}_2$ in $\breve{\mathcal{M}}^\tau_{0,3}$; the dotted arcs indicate the four components of $\gamma'\subset\partial\Upsilon'$.

of $\mathfrak{f}_{0,3;i}|_{\breve{\Gamma}_i}$ with respect to the co-orientation $\mathfrak{o}^c_{\Upsilon'}|_{\breve{\Gamma}_i}$ on $\breve{\Gamma}_i$ in $\breve{S}_i$ and the natural orientation $\mathfrak{o}^{\mathbb{R}}_{\rm nd}$ of the fibers of

$$\mathfrak{f}_{0,3;i}|_{\breve{S}_i}\approx\mathfrak{f}^{\mathbb{R}}_{1,2;1}: \breve{S}_i\approx\breve{\mathcal{M}}^\tau_{1,2}\longrightarrow\overline{\mathcal{M}}^\tau_{0,2}$$

over $\mathcal{M}^\tau_{0,2}$ as in the proof of Lemma 4.4. By Lemma 4.1($\mathfrak{o}_{\mathcal{M}}2$), the orientation $\mathfrak{o}_{1,2}$ on $\breve{S}_i\cap\mathfrak{f}^{-1}_{0,3;i}(\mathcal{M}^\tau_{0,2})$ is given by the z_{5-i}-coordinate under the corresponding identification in (4.23). Since the diffeomorphism (4.28) is orientation-preserving, it follows that the vertical orientation $\mathfrak{o}^{\mathbb{R}}_{\rm nd}$ on $\breve{S}_i$ is given by the *negative* rotation in the z_{5-i}-coordinate. Since the charts (4.22) are orientation-reversing with respect to $\breve{\mathfrak{o}}_{0,3}$ and the charts (4.27) are orientation-preserving with respect to $\mathfrak{o}_{\tilde{\Upsilon}'}$, the orientation $\mathfrak{o}^c_{\Upsilon'}$ on $\mathcal{N}\Upsilon'$ is given by the *negative* rotation in the z_3-coordinate in the first chart in (4.22) and the *positive* rotation in the z_2-coordinate in the second chart in (4.22). Thus, the projection

$$\big(\ker \mathrm{d}\{\mathfrak{f}_{0,3;i}|_{\breve{S}_i}\},\mathfrak{o}^{\mathbb{R}}_{\rm nd}\big)\big|_{\breve{\Gamma}_i-\{P_i^\pm\}}\longrightarrow\big(\mathcal{N}\Upsilon',(-1)^i\mathfrak{o}^c_{\Upsilon'}\big)\big|_{\breve{\Gamma}_i-\{P_i^\pm\}}$$

is an orientation-preserving isomorphism and the number in (4.30) is $(-1)^i$; see Lemma 3.1(1).

The surface Υ' is transverse to $\breve{S}_1\cap\breve{S}_2$, but passes through P_0. Let $(\Upsilon'',\mathfrak{o}^c_{\Upsilon''})$ be a co-oriented surface in $\breve{\mathcal{M}}^\tau_{0,3}$ obtained from $(\Upsilon',\mathfrak{o}^c_{\Upsilon'})$ by a small deformation around P_0 so that Υ'' is still transverse to $\breve{S}_1\cap\breve{S}_2$ and $P_0\notin\Upsilon''$. We take $\Upsilon\subset\overline{\mathcal{M}}^\tau_{0,3}$ to be the preimage of Υ'' under the blowdown map p (which is a diffeomorphism on a neighborhood of Υ) and $\mathfrak{o}^c_\Upsilon=p^*\mathfrak{o}^c_{\Upsilon''}$. $\square$

Remark 4.6. We could have taken $\Upsilon' = \widehat{\Upsilon}^-$ in the proof of Lemma 4.5 to avoid P_0, but with $\mathfrak{o}_{\Upsilon'} = -\mathfrak{o}_{\tilde{\Upsilon}'}|_{\Upsilon'}$.

5 Real GW-invariants

5.1 Moduli spaces of stable maps

Let (X,ω,ϕ) be a real symplectic manifold and $k,l\in\mathbb{Z}^{\geqslant 0}$ with $k+2l\geqslant 3$. We denote by $\mathcal{H}_{k,l}^{\omega,\phi}$ the space of pairs (J,ν) consisting of $J\in\mathcal{J}_\omega^\phi$ and a real perturbation ν of the $\bar\partial_J$-equation as in [9, Section 2]. For $(J,\nu)\in\mathcal{H}_{k,l}^{\omega,\phi}$, a real genus 0 (J,ν)-map with k real marked points and l conjugate pairs of marked points is a tuple

$$\mathbf{u} = \big(u\colon \Sigma \longrightarrow X, (x_i)_{i\in[k]}, (z_i^+, z_i^-)_{i\in[l]}, \sigma\big) \tag{5.1}$$

such that

$$\mathcal{C} \equiv \big(\Sigma, (x_i)_{i\in[k]}, (z_i^+, z_i^-)_{i\in[l]}, \sigma\big) \tag{5.2}$$

is a real genus 0 nodal curve with complex structure $\mathfrak{j}$, k real marked points, and l conjugate pairs of marked points and u is a smooth map satisfying

$$u\circ\sigma = \phi\circ u, \qquad \bar\partial_J u|_z \equiv \frac{1}{2}\big(\mathrm{d}_z u + J\circ \mathrm{d}_z u\circ\mathfrak{j}\big) = \nu\big(z,u(z)\big) \quad \forall\ z\in\Sigma.$$

Such a map is called simple if the restriction of u to each unstable irreducible component of the domain is simple (i.e. not multiply covered) and no two such restrictions have the same image.

For $B\in H_2(X)$ and $(J,\nu)\in\mathcal{H}_{k,l}^{\omega,\phi}$, we denote by $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ the moduli space of the equivalence classes of stable real genus 0 degree B (J,ν)-maps with k real marked points and l conjugate pairs of marked points that take the fixed locus of the domain to the chosen topological component X^ϕ of the fixed locus of ϕ. modulo the reparametrizations. Let

$$\overline{\mathfrak{M}}_{k,l}^*(B;J,\nu) \subset \overline{\mathfrak{M}}_{k,l}(B;J,\nu) \quad \text{and} \quad \mathfrak{M}_{k,l}(B;J,\nu) \subset \overline{\mathfrak{M}}_{k,l}^*(B;J,\nu)$$

be the subspace of simple maps and the (virtually) main stratum, i.e. the subspace consisting of maps as in (5.1) from smooth domains Σ, respectively.

The forgetful morphisms

$$\mathfrak{f}_{k+1,l;i}^{\mathbb{R}}\colon \overline{\mathcal{M}}_{k+1,l}^\tau \longrightarrow \overline{\mathcal{M}}_{k,l}^\tau, \quad i\in[k+1], \quad \text{and} \quad \mathfrak{f}_{k,l+1;i}\colon \overline{\mathcal{M}}_{k,l+1}^\tau \longrightarrow \overline{\mathcal{M}}_{k,l}^\tau, \quad i\in[l+1],$$

induce maps

$$\mathfrak{f}_{k+1,l;i}^{\mathbb{R}*}\colon \mathcal{H}_{k,l}^{\omega,\phi} \longrightarrow \mathcal{H}_{k+1,l}^{\omega,\phi} \qquad \text{and} \qquad \mathfrak{f}_{k,l+1;i}^{*}\colon \mathcal{H}_{k,l}^{\omega,\phi} \longrightarrow \mathcal{H}_{k,l+1}^{\omega,\phi},$$

respectively. For each $\nu\in\mathcal{H}_{k,l}^{\omega,\phi}$, we also denote by

$$\begin{aligned} \mathfrak{f}_{k+1,l;i}^{\mathbb{R}}\colon \overline{\mathfrak{M}}_{k+1,l}(B;J,\mathfrak{f}_{k+1,l;i}^{\mathbb{R}*}\nu) &\longrightarrow \overline{\mathfrak{M}}_{k,l}(B;J,\nu), \\ \mathfrak{f}_{k,l+1;i}\colon \overline{\mathfrak{M}}_{k,l+1}(B;J,\mathfrak{f}_{k,l+1;i}^{*}\nu) &\longrightarrow \overline{\mathfrak{M}}_{k,l}(B;J,\nu) \end{aligned} \tag{5.3}$$

the forgetful morphisms dropping the i-th real marked point and the i-th conjugate pair of marked points, respectively. The restriction of the second morphism in (5.3) to $\mathfrak{M}_{k,l+1}(B;J,\mathfrak{f}^*_{k,l+1;i}\nu)$ is a dense open subset of a $\mathbb{P}^1$-fiber bundle. We denote by $\mathfrak{o}_i^+$ the relative orientation of this restriction induced by the position of the marked point z_i^+.

For $\mathfrak{c}\in\mathbb{Z}^+$, a (virtually) codimension $\mathfrak{c}$ stratum $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ is a subspace of maps from domains Σ with precisely $\mathfrak{c}$ nodes and thus with $\mathfrak{c}$+1 irreducible components isomorphic to $\mathbb{P}^1$. It is characterized by the distributions of

- the degree B of the map components u of its elements $\mathbf{u}$ as in (5.1),
- the k real marked points, and
- the l conjugate pairs of marked points

between the irreducible components of Σ. There are two types of codimension 1 strata distinguished by whether the fixed locus Σ^σ of (Σ,σ) consists of a single point or a wedge of two circles. These two types are known as sphere bubbling and disk bubbling, respectively. If k and B satisfy (2.1), as is the case if (1.3) holds, then the fixed locus Σ^σ of the domain (Σ,σ) of every element (5.1) of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ is a circle or a tree of two or more circles. In this case, sphere bubbling does not occur.

Suppose $l\in\mathbb{Z}^+$ and $\mathcal{S}$ is a codimension 1 disk bubbling stratum of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$. We define

$$K_1(\mathcal{S}),K_2(\mathcal{S})\subset[k],\quad L_1(\mathcal{S}),L_2(\mathcal{S})\subset[l],\quad k_1(\mathcal{S}),k_2(\mathcal{S}),l_1(\mathcal{S}),l_2(\mathcal{S})\in\mathbb{Z}^{\geqslant0}$$

analogously to $K_r(S),L_r(S),k_r(S),l_r(S)$ in Section 4.2. We denote by $B_1(\mathcal{S})\in H_2(X)$ the degree of the restriction of the map components u of the elements $\mathbf{u}$ of $\mathcal{S}$ to the irreducible component $\mathbb{P}^1_1$ of the domain carrying the marked points $z_1^\pm$ and by $B_2(\mathcal{S})\in H_2(X)$ the degree of the restriction of u to the other irreducible component $\mathbb{P}^1_2$ of the domain. Let

$$\overline{\mathcal{S}}\subset\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$$

be the virtual closure of $\mathcal{S}$, i.e. the subspace of maps $\mathbf{u}$ as in (5.1) so that the domain Σ can be split at a node into two connected (possibly reducible) surfaces, Σ_1 and Σ_2, so that the degree of the restriction of the map component u of $\mathbf{u}$ to Σ_1 is $B_1(\mathcal{S})$, the real marked points x_i with $i\in K_1(\mathcal{S})$ lie on Σ_1, and so do the conjugate pairs of marked points $z_i^\pm$ with $i\in L_1(\mathcal{S})$.

If in addition $l^*\in[l]$, let

$$\begin{gathered}L_1^*(\mathcal{S})=L_1(\mathcal{S})\cap[l^*],\quad L_2^*(\mathcal{S})=L_2(\mathcal{S})\cap[l^*],\quad l_1^*(\mathcal{S})=\big|L_1^*(\mathcal{S})\big|,\quad l_2^*(\mathcal{S})=\big|L_2^*(\mathcal{S})\big|,\\ \varepsilon_{l^*}(\mathcal{S})=\big\langle c_1(X,\omega),B_2(\mathcal{S})\big\rangle-\big(k_2(\mathcal{S})+2\big(l_2(\mathcal{S})-l_2^*(\mathcal{S})\big)\big)\,.\end{gathered}$$

In particular,

$$\begin{gathered}\ell_\omega\big(B_1(\mathcal{S})\big)+\ell_\omega\big(B_2(\mathcal{S})\big)=\ell_\omega(B)-1,\quad 1\leqslant l_1^*(\mathcal{S})\leqslant l_1(\mathcal{S}),\quad l_2^*(\mathcal{S})\leqslant l_2(\mathcal{S}),\\ k_1(\mathcal{S})+k_2(\mathcal{S})=k,\quad l_1(\mathcal{S})+l_2(\mathcal{S})=l,\quad l_1^*(\mathcal{S})+l_2^*(\mathcal{S})=l^*.\end{gathered}$$

We denote by

$$\mathfrak{M}^\star_{k,l;l^*}(B;J,\nu)\subset\overline{\mathfrak{M}}^*_{k,l}(B;J,\nu)$$

the subspace of simple maps that have no nodes, or lie in a codimension 1 stratum $\mathcal{S}$ with $\varepsilon_{l^*}(\mathcal{S})\cong 0,1$ mod 4, or have only one conjugate pair of nodes. Let $\widehat{\mathfrak{M}}_{k,l;l^*}(B;J,\nu)$ be the space obtained by cutting $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ along the closures $\overline{\mathcal{S}}$ of the codimension 1 strata $\mathcal{S}$ with $\varepsilon_{l^*}(\mathcal{S})\cong 2,3$ mod 4. Thus, $\widehat{\mathfrak{M}}_{k,l;l^*}(B;J,\nu)$ contains a double cover of $\overline{\mathcal{S}}$ for each codimension 1 stratum $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ with $\varepsilon_{l^*}(\mathcal{S})\cong 2,3$ mod 4; the union of these covers forms the (virtual) boundary of $\widehat{\mathfrak{M}}_{k,l;l^*}(B;J,\nu)$. Let

$$q\colon \widehat{\mathfrak{M}}_{k,l;l^*}(B;J,\nu)\longrightarrow \overline{\mathfrak{M}}_{k,l}(B;J,\nu) \tag{5.4}$$

be the quotient map. We denote by

$$\widehat{\mathfrak{M}}^{\star}_{k,l;l^*}(B;J,\nu)\subset \widehat{\mathfrak{M}}_{k,l;l^*}(B;J,\nu) \tag{5.5}$$

the subspace of simple maps that

- have no nodes, or
- have only one real node, or
- have only one conjugate pair of nodes.

The boundary $\partial\widehat{\mathfrak{M}}^{\star}_{k,l;l^*}(B;J,\nu)$ of this subspace consists of double covers $\hat{\mathcal{S}}^*$ of the subspaces $\mathcal{S}^*$ of simple maps of the codimension 1 strata $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ with $\varepsilon_{l^*}(\mathcal{S})\cong 2,3$ mod 4.

For each $i\in[k]$, let

$$\mathrm{ev}_i^{\mathbb{R}}\colon \overline{\mathfrak{M}}_{k,l}(B;J,\nu)\longrightarrow X^{\phi},\quad \mathrm{ev}_i^{\mathbb{R}}\big([u,(x_j)_{j\in[k]},(z_j^+,z_j^-)_{j\in[l]},\sigma]\big)=u(x_i),$$

be the evaluation morphism for the i-th real marked point. For each $i\in[l]$, let

$$\mathrm{ev}_i^{+}\colon \overline{\mathfrak{M}}_{k,l}(B;J,\nu)\longrightarrow X,\quad \mathrm{ev}_i^{+}\big([u,(x_j)_{j\in[k]},(z_j^+,z_j^-)_{j\in[l]},\sigma]\big)=u(z_i^+),$$

be the evaluation morphism for the positive point of the i-th conjugate pair of marked points. Let

$$\mathrm{ev}\equiv\prod_{i=1}^{k}\mathrm{ev}_i^{\mathbb{R}}\times\prod_{i=1}^{l}\mathrm{ev}_i^{+}\colon \overline{\mathfrak{M}}_{k,l}(B;J,\nu)\longrightarrow X_{k,l}\equiv(X^{\phi})^k\times X^l \tag{5.6}$$

be the total evaluation map. We also denote by

$$\mathrm{ev}_i^{\mathbb{R}}\colon \widehat{\mathfrak{M}}_{k,l;l^*}(B;J,\nu)\longrightarrow X^{\phi},\qquad \mathrm{ev}_i^{+}\colon \widehat{\mathfrak{M}}_{k,l;l^*}(B;J,\nu)\longrightarrow X,$$
$$\mathrm{ev}\colon \widehat{\mathfrak{M}}_{k,l;l^*}(B;J,\nu)\longrightarrow X_{k,l} \tag{5.7}$$

the compositions of the evaluation maps above with the quotient map q in (5.4). We will use the same notation for the compositions of the first three evaluation maps with all obvious maps to $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$.

For $l^*\in[l]$ and a tuple

$$\mathbf{h}\equiv(h_i\colon H_i\longrightarrow X)_{i\in[l^*]} \tag{5.8}$$

of maps, define

$$f_{\mathbf{h}}\colon M_{\mathbf{h}} \equiv \prod_{i=1}^{l^*} H_i \longrightarrow X^{l^*}, \qquad f_{\mathbf{h}}\big((y_i)_{i\in[l^*]}\big) = \big(h_i(y_i)\big)_{i\in[l^*]}, \tag{5.9}$$
$$\mathcal{Z}^{\star}_{k,l;\mathbf{h}}(B;J,\nu) = \big\{\big(\mathbf{u},(y_i)_{i\in[l^*]}\big)\in\mathfrak{M}^{\star}_{k,l;l^*}(B;J,\nu)\times M_{\mathbf{h}}\colon \mathrm{ev}_i^+(\mathbf{u})=h_i(y_i)\ \forall\, i\in[l^*]\big\}.$$

We denote by

$$\mathrm{ev}_{k,l;\mathbf{h}}\colon \mathcal{Z}^{\star}_{k,l;\mathbf{h}}\big(B;J,\nu\big) \longrightarrow X_{k,l-l^*} \tag{5.10}$$

the map induced by (5.6). Orientations on H_i determine an orientation $\mathfrak{o}_{\mathbf{h}}$ on $M_{\mathbf{h}}$. Along with the symplectic orientation $\mathfrak{o}_\omega$ of X and a relative orientation $\mathfrak{o}_{\mathrm{ev}}$ of

$$\mathrm{ev}\colon \mathfrak{M}^{\star}_{k,l;l^*}(B;J,\nu) \longrightarrow X_{k,l}\,, \tag{5.11}$$

the orientation $\mathfrak{o}_{\mathbf{h}}$ determines a relative orientation $\mathfrak{o}_{\mathrm{ev}}\mathfrak{o}_{\mathbf{h}}$ of (5.10).

A dimension n pseudocycle $h: H \longrightarrow X$ in the usual sense determines an element $[h]$ of $H_n(X;\mathbb{Z})$; see [32]. If in addition B is a homology class in X in the complementary dimension, let

$$h\cdot_X B \equiv \big\langle \mathrm{PD}_X([h]), B\big\rangle \in \mathbb{Z}$$

denote the homology intersection product of $[h]$ with B. If h and B are not of complementary dimensions, we set $h\cdot_X B=0$. The next two statements follow readily from [24]; see Section 6.2.

Lemma 5.1. *Suppose (X,ω,ϕ) is a real symplectic fourfold, $k,l\in\mathbb{Z}^{\geqslant 0}$ with $k+2l\geqslant 3$, $l^*\in[l]$, $B\in H_2(X)$, and $(J,\nu)\in\mathcal{H}^{\omega;\phi}_{k,l}$ is generic. If k and B satisfy (2.1), then a Pin^--structure $\mathfrak{p}$ on X^ϕ determines relative orientations $\mathfrak{o}_{\mathfrak{p};l^*}$ and $\widehat{\mathfrak{o}}_{\mathfrak{p};l^*}$ of the maps*

$$\mathrm{ev}\colon \mathfrak{M}^{\star}_{k,l;l^*}(B;J,\nu) \longrightarrow X_{k,l} \quad \text{and} \quad \mathrm{ev}\colon \widehat{\mathfrak{M}}^{\star}_{k,l;l^*}(B;J,\nu) \longrightarrow X_{k,l}, \tag{5.12}$$

respectively, with the following properties:

($\mathfrak{o}_{\mathfrak{p}}1$) the restrictions of $\mathfrak{o}_{\mathfrak{p};l^*}$ and $\widehat{\mathfrak{o}}_{\mathfrak{p};l^*}$ to $\mathfrak{M}_{k,l}(B;J,\nu)$ are the same;

($\mathfrak{o}_{\mathfrak{p}}2$) the restrictions of $\mathfrak{o}_{\mathfrak{p};l^*+1}\mathfrak{o}_\omega$ and $\mathfrak{o}^+_{l^*+1}\mathfrak{o}_{\mathfrak{p};l^*}$ to $\mathfrak{M}_{k,l+1}(B;J,\mathfrak{f}^*_{k,l+1;l^*+1}\nu)$ are the same;

($\mathfrak{o}_{\mathfrak{p}}3$) the interchange of two real points x_i and x_j preserves $\mathfrak{o}_{\mathfrak{p};l^*}$;

($\mathfrak{o}_{\mathfrak{p}}4$) if $\mathbf{u}\in\mathfrak{M}_{k,l}(B;J,\nu;\check{X}^\phi)$ and the marked points z_i^+ and z_j^+ are not separated by the fixed locus S^1 of the domain of $\mathbf{u}$, then the interchange of the conjugate pairs (z_i^+,z_i^-) and (z_j^+,z_j^-) preserves $\mathfrak{o}_{\mathfrak{p};l^*}$ at $\mathbf{u}$;

($\mathfrak{o}_{\mathfrak{p}}5$) the interchange of the points in a conjugate pair (z_i^+,z_i^-) with $l^*<i\leqslant l$ preserves $\mathfrak{o}_{\mathfrak{p};l^*}$;

($\mathfrak{o}_{\mathfrak{p}}6$) the interchange of the points in a conjugate pair (z_i^+,z_i^-) with $1^*<i\leqslant l^*$ reverses $\mathfrak{o}_{\mathfrak{p};l^*}$;

($\mathfrak{o}_{\mathfrak{p}}7$) the interchange of the points in the conjugate pair (z_1^+,z_1^-) reverses $\mathfrak{o}_{\mathfrak{p};l^*}$ if and only if

$$\ell_\omega(B) \cong k+2(l-l^*) \mod 4;$$

($\mathfrak{o}_{\mathfrak{p}}$8) *if* $k,l,l^*=1$ *and* $B=0$, $(\mathrm{ev}_1^{\mathbb{R}}\times\mathrm{id}_X,\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\omega})$ *is a Steenrod pseudocycle of degree 1.*

Proposition 5.2. *Suppose* (X,ω,ϕ) *is a real symplectic fourfold,* $\mathfrak{p}$ *is a* Pin^-*-structure on* X^{ϕ}*,* $l\in\mathbb{Z}^+$*,* $l^*\in[l]$*, and* $B\in H_2(X)$ *are such that*

$$k\equiv\ell_{\omega}(B)-2(l-l^*)\geqslant\max(0,3-2l)\,. \tag{5.13}$$

Let $\mathbf{h}\equiv(h_i)_{i\in[l^*]}$ *be a tuple of pseudocycles of codimension 2 in general position. For a generic choice of* $(J,\nu)\in\mathcal{H}_{k,l}^{\omega,\phi}$*, the map (5.10) with the relative orientation* $\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}$ *is a codimension 0 Steenrod pseudocycle. If* $B\neq0$*, then*

$$\deg\big(\mathrm{ev}_{k,l;\mathbf{h}},\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}\big)=\bigg(\prod_{i=1}^{l^*}h_i\cdot_X B\bigg)N_{B,l-l^*}^{\phi;\mathfrak{p}}\,. \tag{5.14}$$

5.2 Decomposition formulas

Let (X,ω,ϕ) be a real symplectic fourfold, $\mathfrak{p}$ be a Pin^--structure on X^{ϕ}, $k,k',l\in\mathbb{Z}^{\geqslant0}$, $l',l^*\in[l]$ with

$$k'\leqslant k\qquad\text{and}\quad k'+2l'\geqslant3, \tag{5.15}$$

$B\in H_2(X)$, and $(J,\nu)\in\mathcal{H}_{k,l}^{\omega,\phi}$. If k and B satisfy (2.1), there is a well-defined forgetful morphism

$$\mathfrak{f}_{k',l'}:\overline{\mathfrak{M}}_{k,l}(B;J,\nu)\longrightarrow\overline{\mathcal{M}}_{k',l'}^{\tau} \tag{5.16}$$

which drops the last $k-k'$ real marked points and the last $l-l'$ conjugate pairs from the nodal marked curve (5.2) associated with each tuple $\mathbf{u}$ as in (5.1) and contracts the unstable irreducible components of the resulting curve. Let $\mathbf{h}$ as in (5.8) be a tuple of smooth maps from oriented manifolds and

$$\mathbf{p}\equiv\big((p_i^{\mathbb{R}})_{i\in[k]},(p_i^+)_{i\in[l]-[l^*]}\big)\in X_{k,l-l^*}\equiv(X^{\phi})^k\times(X-X^{\phi})^{l-l^*}. \tag{5.17}$$

Let $\Gamma\subset\overline{\mathcal{M}}_{k',l'}^{\tau}$ be a primary codimension 2 stratum and $\mathfrak{o}_{\Gamma}^c$ be its canonical co-orientation as in Lemma 4.2. We denote by

$$\mathfrak{M}_{\Gamma;k,l}(B;J,\nu)\subset\mathfrak{f}_{k',l'}^{-1}(\Gamma)\subset\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$$

the subspace consisting of maps from three-component domains. The domain of every element $\mathbf{u}$ of $\mathfrak{M}_{\Gamma;k,l}(B;J,\nu)$ is stable and thus $\mathbf{u}$ is automatically a simple map. Define

$$\mathcal{Z}_{\Gamma;k,l;\mathbf{h}}^{\star}(B;J,\nu)=\big\{\big(\mathbf{u},(y_i)_{i\in[l^*]}\big)\in\mathcal{Z}_{k,l;\mathbf{h}}^{\star}(B;J,\nu)\colon\mathbf{u}\in\mathfrak{M}_{\Gamma;k,l}(B;J,\nu)\big\}.$$

For generic choices of (J,ν) and $\mathbf{h}$,

$$\mathcal{Z}_{\Gamma;k,l;\mathbf{h}}^{\star}(B;J,\nu)\subset\mathcal{Z}_{k,l;\mathbf{h}}^{\star}(B;J,\nu)$$

is a smooth submanifold of a smooth manifold with the normal bundle canonically isomorphic to $\mathfrak{f}_{k',l'}^*\mathcal{N}\Gamma$. We denote by

$$\mathfrak{o}_{\Gamma;\mathfrak{p};\mathbf{h}}\equiv\big(\mathfrak{f}_{k',l'}^*\mathfrak{o}_{\Gamma}^c\big)\big(\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}\big)$$

the relative orientation of the restriction

$$\mathrm{ev}_{\Gamma;\mathbf{h}}\colon \mathcal{Z}^{\star}_{\Gamma;k,l;\mathbf{h}}(B;J,\nu)\longrightarrow X_{k,l-l^*} \tag{5.18}$$

of (5.10) determined by $\mathfrak{f}^*_{k',l'}\mathfrak{o}^c_\Gamma$ and the relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}$ of (5.10), with $\mathfrak{o}_{\mathfrak{p};l^*}$ as in Lemma 5.1; see Section 3.1.

With $l_0(\Gamma), l_{\mathbb{C}}(\Gamma), l_0^*(\Gamma)\in\mathbb{Z}^{\geqslant 0}$ as at the beginning of Section 4.3 and $B\in H_2(X)$, define

$$\langle l^*\rangle_\Gamma = l'-l^*-(l_0(\Gamma)-l_0^*(\Gamma))\in\mathbb{Z}^{\geqslant 0}; \quad \langle B;k\rangle_\Gamma=\begin{cases}1, & \text{if } k'=k=1,\ l_0(\Gamma)=0;\\ 0, & \text{otherwise.}\end{cases}$$

Let $L_0^*(\Gamma)\subset[l^*]$ be the subset indexing the conjugate pairs of marked points (z_i^+,z_i^-) with $i\in[l^*]$ carried by the real component of the curves in Γ. Define

$$L^*_{\mathbb{C}}(\Gamma)=[l^*]-L^*_0(\Gamma), \quad \langle\mathbf{h}\rangle_{l^*;\Gamma}=\begin{cases}h_i\cdot_X h_j, & \text{if } |L^*_{\mathbb{C}}(\Gamma)|=l_{\mathbb{C}}(\Gamma),\ L^*_{\mathbb{C}}(\Gamma)=\{i,j\};\\ 0, & \text{otherwise.}\end{cases}$$

Proposition 5.3. *Suppose (X,ω,ϕ) is a real symplectic fourfold, $\mathfrak{p}$ is a Pin^--structure on X^ϕ, $k',l\in\mathbb{Z}^{\geqslant 0}$, $l'\in[l]$, $l^*\in[l']$, and $B\in H_2(X)$ are such that*

$$k\equiv\ell_\omega(B)-2(l-l^*)-2\geqslant\max(0,3-2l) \tag{5.19}$$

and (5.15) holds. Let $\Gamma\subset\overline{\mathcal{M}}^\tau_{k',l'}$ be a primary codimension 2 stratum, $\mathbf{h}$ as in (5.8) be a tuple of pseudocycles of codimension 2 with $\phi_*[h_i]=-[h_i]$ for every $i\in[l^*]$, and $\mathbf{p}$ be as in (5.17). If the elements of $\mathbf{h}$ and $\mathbf{p}$ are in general position and $(J,\nu)\in\mathcal{H}^{\omega,\phi}_{k,l}$ is generic, then $\mathbf{p}$ is a regular value of (5.18) and the set $\mathrm{ev}^{-1}_{\Gamma;\mathbf{h}}(\mathbf{p})$ is finite. Furthermore,

$$\begin{split}\big|\mathrm{ev}^{-1}_{\Gamma;\mathbf{h}}(\mathbf{p})\big|^{\pm}_{\mathfrak{o}_{\Gamma;\mathfrak{p};\mathbf{h}}} &= 2^{\ell_\omega(B/2)-1-\langle l^*\rangle_\Gamma-l^*}\langle B;k\rangle_\Gamma\bigg(\prod_{i\in[l^*]}h_i\cdot_X B\bigg)\sum_{\substack{B'\in H_2(X)\\ \mathfrak{d}(B')=B}}N^X_{B'}\\ &\quad+\langle\mathbf{h}\rangle_{l^*;\Gamma}\bigg(\prod_{i\in L_0^*(\Gamma)}h_i\cdot_X B\bigg)N^{\phi;\mathfrak{p}}_{B,l-l^*+1}+\sum_{\substack{B_0,B'\in H_2(X)-\{0\}\\ B_0+\mathfrak{d}(B')=B}}2^{\ell_\omega(B')-\langle l^*\rangle_\Gamma}\bigg((B_0\cdot_X B')\\ &\qquad\times\bigg(\prod_{i\in L_0^*(\Gamma)}h_i\cdot_X B_0\bigg)\bigg(\prod_{i\in L^*_{\mathbb{C}}(\Gamma)}h_i\cdot_X B'\bigg)\binom{l-l'}{\ell_\omega(B')-\langle l^*\rangle_\Gamma}N^X_{B'}N^{\phi;\mathfrak{p}}_{B_0,l-l^*-\ell_\omega(B')}\bigg).\end{split} \tag{5.20}$$

Remark 5.4. The domain of (5.18) can be completed to a Steenrod pseudocycle by adding in the codimension 1 strata of its closure. This implies that the set $\mathrm{ev}^{-1}_{\Gamma;\mathbf{h}}(\mathbf{p})$ is finite for a generic choice of $\mathbf{p}$ and $|\mathrm{ev}^{-1}_{\Gamma;\mathbf{h}}(\mathbf{p})|^{\pm}_{\mathfrak{o}_{\Gamma;\mathfrak{p};\mathbf{h}}}$ is the degree of this pseudocycle. The proof of Proposition 5.3 in Section 6.4 instead identifies $\mathrm{ev}^{-1}_{\Gamma;\mathbf{h}}(\mathbf{p})$ with a finite subset of the cross product of two moduli spaces with the signed cardinality given by the right-hand side of (5.20).

Suppose $\mathcal{S}$ is an open codimension 1 stratum of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$. Let

$$K_r(\mathcal{S})\subset[k],\quad L_r(\mathcal{S})\subset[l],\quad L_r^*(\mathcal{S})\subset[l^*],\quad k_r(\mathcal{S}),l_r(\mathcal{S}),l_r^*(\mathcal{S}),\epsilon_{l^*}(\mathcal{S})\in\mathbb{Z},\quad B_r(\mathcal{S})\in H_2(X)$$

be as in Section 5.1 and $\mathcal{S}^*\subset\mathcal{S}$ be the subspace of simple maps. With $M_{\mathbf{h}}$ given by (5.9), define

$$\mathcal{S}_{\mathbf{h}}^*=\big\{\big(\mathbf{u},(y_i)_{i\in[l^*]}\big)\in\mathcal{S}^*\times M_{\mathbf{h}}:\mathrm{ev}_i^+(\mathbf{u})=h_i(y_i)\;\forall\,i\in[l^*]\big\}.$$

The (virtual) normal bundles $\mathcal{N}\mathcal{S}$ of $\mathcal{S}$ in $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ and $\mathcal{N}\mathcal{S}_{\mathbf{h}}^*$ of $\mathcal{S}_{\mathbf{h}}^*$ in

$$\overline{\mathcal{Z}}_{k,l;\mathbf{h}}(B;J,\nu)\equiv\big\{\big(\mathbf{u},(y_i)_{i\in[l^*]}\big)\in\overline{\mathfrak{M}}_{k,l}(B;J,\nu)\times M_{\mathbf{h}}:\mathrm{ev}_i^+(\mathbf{u})=h_i(y_i)\;\forall\,i\in[l^*]\big\}$$

are canonically isomorphic.

If $\mathbf{u}\in\mathcal{S}_{\mathbf{h}}^*$, an orientation $\mathfrak{o}_{\mathcal{S};\mathbf{u}}^c$ of $\mathcal{N}_{\mathbf{u}}\mathcal{S}$ determines a direction of degeneration of elements of the main stratum of $\mathcal{Z}_{k,l;\mathbf{h}}^\star(B;J,\nu)$ to $\mathbf{u}$. The relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}$ of (5.10) limits to a relative orientation $\mathfrak{o}_{\mathfrak{p};\mathbf{h};\mathbf{u}}$ of

$$\mathrm{ev}_{k,l;\mathbf{h}}:\overline{\mathcal{Z}}_{k,l;\mathbf{h}}(B;J,\nu)\longrightarrow X_{k,l-l^*}\tag{5.21}$$

at $\mathbf{u}$ obtained by approaching $\mathbf{u}$ from this direction. Along with $\mathfrak{o}_{\mathcal{S};\mathbf{u}}^c$, $\mathfrak{o}_{\mathfrak{p};\mathbf{h};\mathbf{u}}$ determines a relative orientation $\partial_{\mathfrak{o}_{\mathcal{S};\mathbf{u}}^c}\mathfrak{o}_{\mathfrak{p};\mathbf{h};\mathbf{u}}$ of the restriction

$$\mathrm{ev}_{\mathcal{S};\mathbf{h}}:\mathcal{S}_{\mathbf{h}}^*\longrightarrow X_{k,l-l^*}\tag{5.22}$$

of $\mathrm{ev}_{k,l;\mathbf{h}}$ via the first isomorphism in (3.1).

Lemma 5.5. *Suppose (X,ω,ϕ), $\mathfrak{p},k,l,l^*,B$, and (J,ν) are as in Lemma 5.1 and $\mathbf{h}$ as in (5.8) is a generic tuple of smooth maps from oriented manifolds. If k and B satisfy (2.1), $\mathcal{S}$ is an open codimension 1 stratum of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$, and $\mathbf{u}\in\mathcal{S}_{\mathbf{h}}^*$, then the relative orientation $\partial_{\mathfrak{o}_{\mathcal{S};\mathbf{u}}^c}\mathfrak{o}_{\mathfrak{p};\mathbf{h};\mathbf{u}}$ of (5.22) at $\mathbf{u}$ does not depend on the choice of $\mathfrak{o}_{\mathcal{S};\mathbf{u}}^c$ if and only if $\epsilon_{l^*}(\mathcal{S})\cong 2,3$ mod 4.*

The relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}$ of the restriction of (5.21) to

$$\mathfrak{M}_{k,l;\mathbf{h}}(B;J,\nu)\equiv\big\{\big(\mathbf{u}',(y_i)_{i\in[l^*]}\big)\in\mathfrak{M}_{k,l}(B;J,\nu)\times M_{\mathbf{h}}:\mathrm{ev}_i^+(\mathbf{u}')=h_i(y_i)\;\forall\,i\in[l^*]\big\}$$

extends across $\mathcal{S}_{\mathbf{h}}^*$ if and only if $\partial_{\mathfrak{o}_{\mathcal{S};\mathbf{u}}^c}\mathfrak{o}_{\mathfrak{p};\mathbf{h};\mathbf{u}}$ *depends* on the choice of $\mathfrak{o}_{\mathcal{S};\mathbf{u}}^c$ for every $\mathbf{u}\in\mathcal{S}_{\mathbf{h}}^*$. In particular, the first statement of Lemma 5.1 is an immediate consequence of Lemma 5.5. If $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ is cut along $\overline{\mathcal{S}}$ and $\widehat{\mathcal{S}}^*$ is the double cover of $\mathcal{S}^*$ in the cut, then $\partial_{\mathfrak{o}_{\mathcal{S};\mathbf{u}}^c}\mathfrak{o}_{\mathfrak{p};\mathbf{h};\mathbf{u}}$ is the boundary relative orientation induced by $\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}$ at one of the copies $\widehat{\mathbf{u}}$ of $\mathbf{u}$ in

$$\widehat{\mathcal{S}}_{\mathbf{h}}^*=\big\{\big(\widehat{\mathbf{u}}',(y_i)_{i\in[l^*]}\big)\in\widehat{\mathcal{S}}^*\times M_{\mathbf{h}}:\mathrm{ev}_i^+(\widehat{\mathbf{u}}')=h_i(y_i)\;\forall\,i\in[l^*]\big\};\tag{5.23}$$

we then denote it by $\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h};\widehat{\mathbf{u}}}$. If $\epsilon_{l^*}(\mathcal{S})\cong 2,3$ mod 4, we abbreviate $\partial_{\mathfrak{o}_{\mathcal{S};\mathbf{u}}^c}\mathfrak{o}_{\mathfrak{p};\mathbf{h};\mathbf{u}}$ as $\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h};\mathbf{u}}$.

Remark 5.6. While Lemma 5.5 follows readily from [24, Prop. 5.3], it is also immediately implied by our Lemmas 4.3 and 6.2 (which are also needed to establish Proposition 5.7

below). The terms $w_2(\psi(d''))$ in [24, (17),(18)] appear to be extra (they are omitted in the key invariance argument on [24, p53]). The first equation in [24] with

$$\mu(d'') = \ell_\omega\big(B_2(\mathcal{S})\big), \quad k'' = k_2(\mathcal{S}), \quad l'' = l_2(\mathcal{S}) - l_2^*(\mathcal{S})$$

compares the two possibilities for $\partial_{\mathfrak{o}^c_{\mathcal{S};\mathbf{u}}}\mathfrak{o}_{\mathfrak{p};\mathbf{h};\mathbf{u}}$ when $k=0$ or the real marked point x_1 lies on the same component of $\mathbf{u}$ as the marked points z_1^+; the second equation treats the remaining case. The right-hand side of the latter reduces to the right-hand side of the former if (2.1) holds. The right-hand side of [24, (17)], without the $w_2(\psi(d''))$ term, in turn reduces to

$$\frac{(\mu(d'')-k''-2l'')(\mu(d'')-k''-2l''-1)}{2}+1 \cong \frac{\epsilon_{l^*}(\mathcal{S})(\epsilon_{l^*}(\mathcal{S})-1)}{2}+1 \mod 2;$$

the last expression vanishes (i.e. the two orientations are the same) if and only if $\epsilon_{l^*}(\mathcal{S}) \cong 2, 3$.

The stratum $\mathcal{S}$ satisfies exactly one of the following conditions:

($\mathcal{S}0$) $K_2(\mathcal{S}) \cap [k'] = \varnothing$ and $L_2(\mathcal{S}) \cap [l'] = \varnothing$;

($\mathcal{S}1$) $|K_2(\mathcal{S}) \cap [k']| = 1$ and $L_2(\mathcal{S}) \cap [l'] = \varnothing$;

($\mathcal{S}2$) there exists a codimension 1 stratum $S \subset \overline{\mathcal{M}}^\tau_{k',l'}$ such that $\mathfrak{f}_{k',l'}(\mathcal{S}) \subset S$.

We call a pair $(\mathcal{S}, \Upsilon)$ consisting of $\mathcal{S}$ as above and a (possibly bordered) hypersurface $\Upsilon \subset \overline{\mathcal{M}}^\tau_{k,l}$ admissible if one of the following conditions holds:

($\mathcal{S}1\Upsilon$) $K_2(\mathcal{S}) \cap [k'] = \{i\}$, $L_2(\mathcal{S}) \cap [l'] = \varnothing$, and Υ is regular with respect to $\mathfrak{f}^{\mathbb{R}}_{k',l';i}$;

($\mathcal{S}2\Upsilon$) there exists a codimension 1 stratum $S \subset \overline{\mathcal{M}}^\tau_{k',l'}$ such that $\mathfrak{f}_{k',l'}(\mathcal{S}) \subset S$ and Υ is regular with respect to S.

The notions of Υ being regular with respect to $\mathfrak{f}^{\mathbb{R}}_{k',l';i}$ and S are defined in Section 4.2.

For $\Upsilon \subset \overline{\mathcal{M}}^\tau_{k',l'}$, define

$$f_{\mathbf{p};\Upsilon}\colon \Upsilon \longrightarrow X_{k,l-l^*} \times \overline{\mathcal{M}}^\tau_{k',l'}, \quad f_{\mathbf{p};\Upsilon}(P) = (\mathbf{p}, P), \tag{5.24}$$

$$\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon} = \big\{(\mathbf{u}, P) \in \mathcal{S}^*_{\mathbf{h}} \times \Upsilon\colon \mathrm{ev}_{\mathcal{S};\mathbf{h}}(\mathbf{u}) = \mathbf{p},\, \mathfrak{f}_{k',l'}(\mathbf{u}) = P\big\}. \tag{5.25}$$

If $(\mathcal{S}, \Upsilon)$ is an admissible pair and $\mathfrak{o}^c_\Upsilon$ is a co-orientation on Υ, we denote by $\deg(\mathcal{S}, \mathfrak{o}^c_\Upsilon) \in \mathbb{Z}$ the corresponding degree $\deg^{\mathbb{R}}_i(\Upsilon, \mathfrak{o}^c_\Upsilon)$ or $\deg_S(\Upsilon, \mathfrak{o}^c_\Upsilon)$ defined in Section 4.2.

Proposition 5.7. *Suppose (X, ω, ϕ), $\mathfrak{p}$, $k, k', l, l^*, l', B, \mathbf{p}, \mathbf{h}$ are in as Proposition 5.3,*

$$\mathcal{S} \subset \overline{\mathfrak{M}}_{k,l}(B; J, \nu) \qquad \textit{and} \qquad \Upsilon \subset \overline{\mathcal{M}}^\tau_{k',l'}$$

form an admissible pair, and $\mathfrak{o}^c_\Upsilon$ is a co-orientation on Υ. If $\varepsilon_{l^*}(\mathcal{S}) = 2$, the elements of $\mathbf{h}$ and $\mathbf{p}$ are in general position, and $(J, \nu) \in \mathcal{H}^{\omega,\phi}_{k,l}$ is generic, then

$$\big(\mathrm{ev}_{\mathcal{S};\mathbf{h}}, \mathfrak{f}_{k',l'}\big)\colon \mathcal{S}^*_{\mathbf{h}} \longrightarrow X_{k,l-l^*} \times \overline{\mathcal{M}}^\tau_{k',l'} \quad \textit{and} \quad f_{\mathbf{p};\Upsilon}\colon \Upsilon \longrightarrow X_{k,l-l^*} \times \overline{\mathcal{M}}^\tau_{k',l'}$$

are transverse maps from manifolds of complementary dimensions and the set $\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}$ *is finite. Furthermore,*

$$\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_\Upsilon} = -(-1)^{\dim\Upsilon}\deg(\mathcal{S},\mathfrak{o}^c_\Upsilon)\bigg(\prod_{i\in L_1^*(\mathcal{S})}h_i\cdot_X B_1(\mathcal{S})\bigg)\bigg(\prod_{i\in L_2^*(\mathcal{S})}h_i\cdot_X B_2(\mathcal{S})\bigg) \times N^{\phi;\mathfrak{p}}_{B_1(\mathcal{S}),l_1(\mathcal{S})-l_1^*(\mathcal{S})}N^{\phi;\mathfrak{p}}_{B_2(\mathcal{S}),l_2(\mathcal{S})-l_2^*(\mathcal{S})}. \tag{5.26}$$

Due to the condition $\varepsilon_{l^*}(\mathcal{S})=2$,

$$k_1(\mathcal{S}) = \ell_\omega\big(B_1(\mathcal{S})\big)-2\big(l_1(\mathcal{S})-l_1^*(\mathcal{S})\big) \quad\text{and}\quad k_2(\mathcal{S})+1 = \ell_\omega\big(B_2(\mathcal{S})\big)-2\big(l_2(\mathcal{S})-l_2^*(\mathcal{S})\big),$$

i.e. the second irreducible component of the maps in $\mathcal{S}$ passes through an extra real point.

Remark 5.8. The crucial property of $(\mathcal{S},\Upsilon)$ used in the proof is that the condition $f_{k',l'}(\mathbf{u})\in\Upsilon$ in (5.25) factors through

$$\begin{aligned}\mathcal{S}&\longrightarrow \overline{\mathfrak{M}}_{k_1(\mathcal{S})+1,l_1(\mathcal{S})}\big(B_1(\mathcal{S});J,\nu_1\big)\times\overline{\mathfrak{M}}_{k_2(\mathcal{S})+1,l_2(\mathcal{S})}\big(B_2(\mathcal{S});J,\nu_2\big)\\ &\longrightarrow \overline{\mathfrak{M}}_{k_1(\mathcal{S})+1,l_1(\mathcal{S})}\big(B_1(\mathcal{S});J,\nu_1\big)\end{aligned}$$

for a good choice of ν. Thus, Lemma 3.4 applies.

5.3 Proof of Theorem 1.1

Fix (X,ω,ϕ), $\mathfrak{p}$, and B as in Theorem 1.1, take k as in (1.3) and $l^*\in\{2,3\}$, and choose generic tuples

$$\mathbf{h}\equiv\big(h_i\colon H_i\longrightarrow X\big)_{i\in[l^*]} \qquad\text{and}\qquad \mathbf{p}\equiv\big((p_i^{\mathbb{R}})_{i\in[k]},(p_i^+)_{i\in[l+l^*-1]-[l^*]}\big)\in X_{k,l-1}$$

so that each h_i is a codimension 2 pseudocycle,

$$\ell_\omega(B) = 2l+k, \qquad\text{and}\qquad h_1\cdot_X h_3 = 0. \tag{5.27}$$

We deduce the three relations of Theorem 1.1, with $N^{\phi;\mathfrak{p}}_{B,l}(X^\phi)$ replaced by $N^{\phi;\mathfrak{p}}_{B,l}$ and the left-hand sides multiplied by $h_1\cdot_X h_2$, from Propositions 5.3 and 5.7 and several lemmas stated so far.

For $(k',l')=(1,2),(0,3)$ such that $k'\leqslant k$ and $l'\leqslant l+l^*-1$, we denote by

$$\mathfrak{f}_{k',l'}\colon\widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J,\nu)\longrightarrow\overline{\mathcal{M}}^\tau_{k',l'}$$

the composition of (5.16) and the quotient map q in (5.4) with l replaced by $l+l^*-1$. For a stratum $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l+l^*-1}(B;J,\nu)$, let

$$\widehat{\mathcal{S}}^* = q^{-1}(\mathcal{S}^*)\subset\widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J,\nu).$$

With the notation as in (5.5), let

$$\begin{aligned}M_{\mathbf{h}} &= H_1\times\ldots\times H_{l^*},\\ \widehat{\mathcal{Z}}^\star_{k,l+l^*-1;\mathbf{h}}(B;J,\nu) &= \big\{(\mathbf{u},y_1,\ldots,y_{l^*})\in\widehat{\mathfrak{M}}^\star_{k,l+l^*-1;l^*}(B;J,\nu)\times M_{\mathbf{h}}\colon \mathrm{ev}_i^+(\mathbf{u})=h_i(y_i)\ \forall\, i\in[l^*]\big\}.\end{aligned}$$

For $(J,\nu)\in\mathcal{H}^{\omega,\phi}_{k,l+l^*-1}$ generic, the relative orientation $\widehat{\mathfrak{o}}_{\mathfrak{p};l^*}$ of Lemma 5.1 and the orientation $\mathfrak{o}_{\mathbf{h}}$ of $M_{\mathbf{h}}$ determine a relative orientation $\widehat{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}}$ of the map

$$\mathrm{ev}_{k,l+l^*-1;\mathbf{h}}\colon \widehat{\mathcal{Z}}^{\star}_{k,l+l^*-1;\mathbf{h}}\big(B;J,\nu\big)\longrightarrow X_{k,l-1}$$

induced by (5.6) with l replaced by $l+l^*-1$.

Below we take $\Upsilon\subset\overline{\mathcal{M}}^{\tau}_{k',l'}$ to be the bordered compact hypersurfaces of Lemmas 4.4 and 4.5 with their co-orientations $\mathfrak{o}^c_{\Upsilon}$. For a stratum $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l+l^*-1}(B;J,\nu)$, let

$$\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\subset\mathcal{S}^*\times M_{\mathbf{h}}\times\Upsilon \quad\text{and}\quad \widehat{\mathcal{S}}^*_{\mathbf{h}}\equiv\widehat{\mathcal{Z}}^{\star}_{k,l+l^*-1;\mathbf{h}}\big(B;J,\nu\big)\cap\big(\widehat{\mathcal{S}}^*\times M_{\mathbf{h}}\big)$$

be as in (5.25) and (5.23), respectively, and

$$\widehat{\mathcal{S}}^*_{\mathbf{h},\mathbf{p};\Upsilon}=\big\{\big(\widehat{\mathbf{u}},P\big)\in\widehat{\mathcal{S}}^*_{\mathbf{h}}\times\Upsilon\colon \mathrm{ev}_{k,l+l^*-1;\mathbf{h}}(\widehat{\mathbf{u}})=\mathbf{p},\ \mathfrak{f}_{k',l'}(\widehat{\mathbf{u}})=P\big\}\,.$$

We establish the next two statements at the end of this section.

Lemma 5.9. *For a generic choice of $(J,\nu)\in\mathcal{H}^{\omega,\phi}_{k,l+l^*-1}$, the map*

$$\big(\mathrm{ev}_{k,l+l^*-1;\mathbf{h}},\mathfrak{f}_{k',l'}\big)\colon \widehat{\mathcal{Z}}^{\star}_{k,l+l^*-1;\mathbf{h}}(B;J,\nu)\longrightarrow X_{k,l-1}\times\overline{\mathcal{M}}^{\tau}_{k',l'} \tag{5.28}$$

is a bordered $\mathbb{Z}_2$-pseudocycle of dimension $4l+2k-2$ transverse to (5.24).

Corollary 5.10. *For a generic choice of $(J,\nu)\in\mathcal{H}^{\omega,\phi}_{k,l+l^*-1}$,*

$$M_{(\mathrm{ev}_{k,l+l^*-1;\mathbf{h}},\mathfrak{f}_{k',l'}),f_{\mathbf{p};\Upsilon}}\subset\widehat{\mathcal{Z}}^{\star}_{k,l+l^*-1;\mathbf{h}}(B;J,\nu)\times\Upsilon$$

is a compact one-dimensional manifold with boundary (2.4) and

$$\big(\partial\widehat{\mathcal{Z}}^{\star}_{k,l+l^*-1;\mathbf{h}}(B;J,\nu)\big)_{(\mathrm{ev}_{k,l+l^*-1;\mathbf{h}},\mathfrak{f}_{k',l'})}\times_{f_{\mathbf{p};\Upsilon}}\Upsilon=\bigsqcup_{\epsilon_{l^*}(\mathcal{S})=2}\widehat{\mathcal{S}}^*_{\mathbf{h},\mathbf{p};\Upsilon}, \tag{5.29}$$

with the union taken over the codimension 1 strata $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l+l^*-1}(B;J,\nu)$ that satisfy either ($\mathcal{S}1$) or ($\mathcal{S}2$) above Proposition 5.7.

In our case, $\Upsilon\cap\overline{S}_1=\varnothing$. If $\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\neq\varnothing$ and $\mathcal{S}$ satisfies ($\mathcal{S}2$), then $S\neq S_1$. Combined with the assumption that (k',l') is either $(1,2)$ or $(0,3)$, this implies that the pair $(\mathcal{S},\Upsilon)$ is admissible in the sense defined above Proposition 5.7 whenever $\mathcal{S}$ contributes to the right-hand side of (5.29).

The identity (2.5) follows from Lemma 3.5. We use Corollary 5.10 and Proposition 5.7 to express the right-hand side of this identity, i.e. the signed cardinality of (5.29), in terms of the real invariants $N^{\phi,\mathfrak{p}}_{B',l'}$. We use Lemma 3.3(1) and Proposition 5.3 to express the left-hand side of (2.5) in terms of the real invariants $N^{\phi,\mathfrak{p}}_{B',l'}$ and the complex invariants $N^X_{B'}$. Setting the two expressions equal and dividing by 2, we obtain the two identities of Theorem 1.1.

Proof of (ℝWDVV1). We take $l^*=2$. Since $k,l\geqslant 1$ in this case, the morphism

$$\mathfrak{f}_{1,2}\colon \widehat{\mathcal{Z}}^{\star}_{k,l+1;\mathbf{h}}(B;J,\nu)\longrightarrow \overline{\mathcal{M}}^{\tau}_{1,2}$$

is well-defined. Let $P^{\pm}\in\overline{\mathcal{M}}^{\tau}_{1,2}$, $\Upsilon\subset\overline{\mathcal{M}}^{\tau}_{1,2}$, $\mathfrak{o}^c_{P^\pm}$, and $\mathfrak{o}^c_{\Upsilon}$ be as in the statement of Lemma 4.4. Let $\mathcal{A}_1^{\mathbb{R}}$ (resp. $\mathcal{A}_2$) be the collection of the codimension 1 strata $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l+1}(B;J,\nu)$ with $\epsilon_2(\mathcal{S})=2$ such that the irreducible component $\mathbb{P}^1_1$ carrying (z_1^+,z_1^-) (resp. the other component $\mathbb{P}^1_2$) of the maps in $\mathcal{S}$ carries the conjugate pair (z_2^+,z_2^-), but not the real marked point x_1. Each such stratum is doubly covered by a stratum $\widehat{\mathcal{S}}$ of $\partial\widehat{\mathfrak{M}}^{\star}_{k,l+1;2}(B;J,\nu)$.

By Corollary 5.10 and Proposition 5.7, *half* of the right-hand side of (2.5), *not* including the sign in front, equals

$$\begin{split}\sum_{\mathcal{S}\in\mathcal{A}_1^{\mathbb{R}}}\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}+\sum_{\mathcal{S}\in\mathcal{A}_2}\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}&=\sum_{\mathcal{S}\in\mathcal{A}_1^{\mathbb{R}}}\big(h_1\cdot_X B_1(\mathcal{S})\big)\big(h_2\cdot_X B_1(\mathcal{S})\big)N^{\phi;\mathfrak{p}}_{B_1(\mathcal{S}),l_1(\mathcal{S})-2}N^{\phi;\mathfrak{p}}_{B_2(\mathcal{S}),l_2(\mathcal{S})}\\&\quad-\sum_{\mathcal{S}\in\mathcal{A}_2}\big(h_1\cdot_X B_1(\mathcal{S})\big)\big(h_2\cdot_X B_2(\mathcal{S})\big)N^{\phi;\mathfrak{p}}_{B_1(\mathcal{S}),l_1(\mathcal{S})-1}N^{\phi;\mathfrak{p}}_{B_2(\mathcal{S}),l_2(\mathcal{S})-1}.\end{split}$$

Summing over all splittings of $B\in H_2(X)$ into B_1 and B_2, of $l-1$ conjugate pairs of points into sets of cardinalities l_1 and l_2, and of $k-1$ real points into sets of cardinalities $\ell_\omega(B_i)-2l_i$, we obtain

$$\begin{split}&\frac{1}{2}\big|\big(\partial\widehat{\mathcal{Z}}^{\star}_{k,l+1;\mathbf{h}}(B;J)\big)\,{}_{(\mathrm{ev}_{k,l+1;\mathbf{h}},\mathfrak{f}_{k',l'})}\!\times_{f_{\mathbf{p}};\Upsilon}\Upsilon\big|^{\pm}_{\partial\widehat{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}\\&\qquad=\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\B_1+B_2=B\\l_1+l_2=l-1,\,l_1,l_2\geqslant0}}\big(h_1\cdot_X B_1\big)\big(h_2\cdot_X B_1\big)\binom{l-1}{l_1}\binom{\ell_\omega(B)-2l-1}{\ell_\omega(B_1)-2l_1}N^{\phi;\mathfrak{p}}_{B_1,l_1}N^{\phi;\mathfrak{p}}_{B_2,l_2}\\&\qquad\quad-\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\B_1+B_2=B\\l_1+l_2=l-1,\,l_1,l_2\geqslant0}}\big(h_1\cdot_X B_1\big)\big(h_2\cdot_X B_2\big)\binom{l-1}{l_1}\binom{\ell_\omega(B)-2l-1}{\ell_\omega(B_1)-2l_1-1}N^{\phi;\mathfrak{p}}_{B_1,l_1}N^{\phi;\mathfrak{p}}_{B_2,l_2}.\end{split}\tag{5.30}$$

We note that $l_1\equiv l_1(\mathcal{S})-2$ in the $\mathcal{A}_1^{\mathbb{R}}$ sum above and $l_1\equiv l_1(\mathcal{S})-1$ in the $\mathcal{A}_2$ sum, because the subtractions from $l_1(\mathcal{S})$ correspond to the insertions of the divisors H_1,H_2; the meaning of l_2 is analogous.

By Lemma 3.3(1) and Proposition 5.3, *half* of the left-hand side of (2.5) equals

$$\begin{split}\big|\mathrm{ev}^{-1}_{P^+;\mathbf{h}}(\mathbf{p})\big|^{\pm}_{\mathfrak{o}_{P^+;\mathfrak{p};\mathbf{h}}}&=\big(h_1\cdot_X h_2\big)N^{\phi;\mathfrak{p}}_{B,l}+2^{\ell_\omega(B/2)-3}\langle B\rangle_l\big(h_1\cdot_X B\big)\big(h_2\cdot_X B\big)\sum_{\substack{B'\in H_2(X)\\\mathfrak{d}(B')=B}}N^X_{B'}\\&\quad+\sum_{\substack{B_0,B'\in H_2(X)-\{0\}\\B_0+\mathfrak{d}(B')=B}}2^{\ell_\omega(B')}\big(B_0\cdot_X B'\big)\big(h_1\cdot_X B'\big)\big(h_2\cdot_X B'\big)\binom{l-1}{\ell_\omega(B')}N^X_{B'}N^{\phi;\mathfrak{p}}_{B_0,l-1-\ell_\omega(B')}.\end{split}$$

Equating this expression with the negative of (5.30), as dictated by (2.5), we obtain the first identity in Theorem 1.1 with the left-hand side multiplied by $h_1\cdot_X h_2$. □

Proof of ($\mathbb{R}$WDVV2). We again take $l^*=2$. Since $l\geqslant 2$ in this case, the morphism

$$\mathfrak{f}_{0,3}\colon \widehat{\mathcal{Z}}^{\star}_{k,l+1;\mathbf{h}}(B;J,\nu)\longrightarrow \overline{\mathcal{M}}^{\tau}_{0,3}$$

is well-defined. Let $\Gamma_i^{\pm},\Upsilon,\gamma'\subset\overline{\mathcal{M}}^{\tau}_{0,3}$, $\mathfrak{o}^c_{\Gamma_i^{\pm}}$, $\mathfrak{o}^c_{\Upsilon}$, and $\mathfrak{o}^c_{\gamma'}$ be as in the statement of Lemma 4.5. Let $\mathcal{A}_2$ (resp. $\mathcal{A}_3$) be the collection of the codimension 1 strata $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l+1}(B;J,\nu)$ with $\epsilon_2(\mathcal{S})=2$ such that $\mathbb{P}^1_1$ (resp. $\mathbb{P}^1_2$) carries the conjugate pair (z_3^+,z_3^-), but not (z_2^+,z_2^-).

By Corollary 5.10 and Proposition 5.7, *half* of the right-hand side of (2.5) equals

$$\begin{aligned}\sum_{\mathcal{S}\in\mathcal{A}_2}\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}+\sum_{\mathcal{S}\in\mathcal{A}_3}\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}} &= \sum_{\mathcal{S}\in\mathcal{A}_3}\big(h_1\cdot_X B_1(\mathcal{S})\big)\big(h_2\cdot_X B_1(\mathcal{S})\big)N^{\phi;\mathfrak{p}}_{B_1(\mathcal{S}),l_1(\mathcal{S})-2}N^{\phi;\mathfrak{p}}_{B_2(\mathcal{S}),l_2(\mathcal{S})}\\ &\quad-\sum_{\mathcal{S}\in\mathcal{A}_2}\big(h_1\cdot_X B_1(\mathcal{S})\big)\big(h_2\cdot_X B_2(\mathcal{S})\big)N^{\phi;\mathfrak{p}}_{B_1(\mathcal{S}),l_1(\mathcal{S})-1}N^{\phi;\mathfrak{p}}_{B_2(\mathcal{S}),l_2(\mathcal{S})-1}.\end{aligned}$$

Summing over all splittings of $B\in H_2(X)$ into B_1 and B_2, of l–2 conjugate pairs of points into sets of cardinalities l_1 and l_2, and of k real points into sets of the appropriate cardinalities, we obtain

$$\begin{aligned}&\frac{1}{2}\big|\big(\partial\widehat{\mathcal{Z}}^{\star}_{k,l+1;\mathbf{h}}(B;J)\big)_{(\mathrm{ev}_{k,l+1;\mathbf{h}},\mathfrak{f}_{k',l'})}\times_{f_{\mathbf{p}};\Upsilon}\Upsilon\big|^{\pm}_{\partial\widehat{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}\\ &\quad=\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\ B_1+B_2=B\\ l_1+l_2=l-2,\,l_1,l_2\geqslant 0}}\big(h_1\cdot_X B_1\big)\big(h_2\cdot_X B_1\big)\binom{l-2}{l_1}\binom{\ell_\omega(B)-2l}{\ell_\omega(B_1)-2l_1}N^{\phi;\mathfrak{p}}_{B_1,l_1}N^{\phi;\mathfrak{p}}_{B_2,l_2+1}\\ &\qquad-\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\ B_1+B_2=B\\ l_1+l_2=l-2,\,l_1,l_2\geqslant 0}}\big(h_1\cdot_X B_1\big)\big(h_2\cdot_X B_2\big)\binom{l-2}{l_1}\binom{\ell_\omega(B)-2l}{\ell_\omega(B_1)-2l_1-2}N^{\phi;\mathfrak{p}}_{B_1,l_1+1}N^{\phi;\mathfrak{p}}_{B_2,l_2}.\end{aligned}\tag{5.31}$$

We note that $l_1\equiv l_1(\mathcal{S})-2$ in both sums above, because $l_1(\mathcal{S})-1$ includes the conjugate pair (z_3^+,z_3^-) in the $\mathcal{A}_2$ sum; this pair is included into $l_2(\mathcal{S})$ in the $\mathcal{A}_3$ sum.

By Lemma 3.3(1) and Proposition 5.3, *half* of the *negative* of the left-hand side of (2.5) equals

$$\begin{aligned}&\big|\mathrm{ev}^{-1}_{\Gamma_3^+;\mathbf{h}}(\mathbf{p})\big|^{\pm}_{\mathfrak{o}_{\Gamma_3^+;\mathfrak{p};\mathbf{h}}}-\big|\mathrm{ev}^{-1}_{\Gamma_2^+;\mathbf{h}}(\mathbf{p})\big|^{\pm}_{\mathfrak{o}_{\Gamma_2^+;\mathfrak{p};\mathbf{h}}}=\big(h_1\cdot_X h_2\big)N^{\phi;\mathfrak{p}}_{B,l}\\ &\qquad+\sum_{\substack{B_0,B'\in H_2(X)-\{0\}\\ B_0+\mathfrak{d}(B')=B}}2^{\ell_\omega(B')}\big(B_0\cdot_X B'\big)\big(h_1\cdot_X B'\big)\big(h_2\cdot_X B'\big)\binom{l-2}{\ell_\omega(B')}N^X_{B'}N^{\phi;\mathfrak{p}}_{B_0,l-1-\ell_\omega(B')}\\ &\qquad-\sum_{\substack{B_0,B'\in H_2(X)-\{0\}\\ B_0+\mathfrak{d}(B')=B}}2^{\ell_\omega(B')-1}\big(B_0\cdot_X B'\big)\big(h_1\cdot_X B'\big)\big(h_2\cdot_X B_0\big)\binom{l-2}{\ell_\omega(B')-1}N^X_{B'}N^{\phi;\mathfrak{p}}_{B_0,l-1-\ell_\omega(B')}.\end{aligned}$$

Equating this expression with the negative of (5.31), we obtain the second identity in Theorem 1.1 with the left-hand side multiplied by $h_1\cdot_X h_2$. $\square$

Proof of ($\mathbb{R}$WDVV3). We now take $l^*=3$. Since $l\geqslant 1$ in this case, the morphism

$$\mathfrak{f}_{0,3}\colon \widehat{\mathcal{Z}}^{\star}_{k,l+2;\mathbf{h}}(B;J,\nu)\longrightarrow \overline{\mathcal{M}}^{\tau}_{0,3}$$

is well-defined. Let $\Gamma_i^{\pm},\Upsilon,\gamma'\subset\overline{\mathcal{M}}^{\tau}_{0,3}$, $\mathfrak{o}^c_{\Gamma_i^{\pm}}$, $\mathfrak{o}^c_{\Upsilon}$, and $\mathfrak{o}^c_{\gamma'}$ be as in the statement of Lemma 4.5. Let $\mathcal{A}_2$ (resp. $\mathcal{A}_3$) be the collection of the codimension 1 strata $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l+2}(B;J,\nu)$ with $\epsilon_3(\mathcal{S})=2$ such that $\mathbb{P}^1_1$ (resp. $\mathbb{P}^1_2$) carries the conjugate pair (z_3^+,z_3^-), but not (z_2^+,z_2^-).

By Corollary 5.10 and Proposition 5.7, *half* of the right-hand side of (2.5) equals

$$\begin{aligned}\sum_{\mathcal{S}\in\mathcal{A}_2}&\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathbf{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}+\sum_{\mathcal{S}\in\mathcal{A}_3}\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathbf{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}\\
&=\sum_{\mathcal{S}\in\mathcal{A}_3}\big(h_1\cdot_X B_1(\mathcal{S})\big)\big(h_2\cdot_X B_1(\mathcal{S})\big)\big(h_3\cdot_X B_2(\mathcal{S})\big)N^{\phi;\mathfrak{p}}_{B_1(\mathcal{S}),l_1(\mathcal{S})-2}N^{\phi;\mathfrak{p}}_{B_2(\mathcal{S}),l_2(\mathcal{S})-1}\\
&\qquad-\sum_{\mathcal{S}\in\mathcal{A}_2}\big(h_1\cdot_X B_1(\mathcal{S})\big)\big(h_3\cdot_X B_1(\mathcal{S})\big)\big(h_2\cdot_X B_2(\mathcal{S})\big)N^{\phi;\mathfrak{p}}_{B_1(\mathcal{S}),l_1(\mathcal{S})-2}N^{\phi;\mathfrak{p}}_{B_2(\mathcal{S}),l_2(\mathcal{S})-1}.\end{aligned}$$

Summing over all splittings of $B\in H_2(X)$ into B_1 and B_2, of l–1 conjugate pairs of points into sets of cardinalities l_1 and l_2, and of k real points into sets of the appropriate cardinalities, we obtain

$$\begin{aligned}\frac12\Big|\big(\partial\widehat{\mathcal{Z}}^{\star}_{k,l+2;\mathbf{h}}(B;J)\big)_{(\mathrm{ev}_{k,l+2;\mathbf{h}},\mathfrak{f}_{k',l'})}&\times_{f_{\mathbf{p};\Upsilon}}\Upsilon\Big|^{\pm}_{\partial\widehat{\mathfrak{o}}_{\mathbf{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}\\
=\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\ B_1+B_2=B\\ l_1+l_2=l-1,\,l_1,l_2\geqslant0}}&\big(h_1\cdot_X B_1\big)\big(h_2\cdot_X B_1\big)\big(h_3\cdot_X B_2\big)\binom{l-1}{l_1}\binom{\ell_\omega(B)-2l}{\ell_\omega(B_1)-2l_1}N^{\phi;\mathfrak{p}}_{B_1,l_1}N^{\phi;\mathfrak{p}}_{B_2,l_2}\\
-\sum_{\substack{B_1,B_2\in H_2(X)-\{0\}\\ B_1+B_2=B\\ l_1+l_2=l-1,\,l_1,l_2\geqslant0}}&\big(h_1\cdot_X B_1\big)\big(h_3\cdot_X B_1\big)\big(h_2\cdot_X B_2\big)\binom{l-1}{l_1}\binom{\ell_\omega(B)-2l}{\ell_\omega(B_1)-2l_1}N^{\phi;\mathfrak{p}}_{B_1,l_1}N^{\phi;\mathfrak{p}}_{B_2,l_2}.\end{aligned}\tag{5.32}$$

We note that $l_1\equiv l_1(\mathcal{S})-2$ and $l_2\equiv l_2(\mathcal{S})-1$ in both sums above, because the subtractions from $l_1(\mathcal{S})$ and $l_2(\mathcal{S})$ correspond to the insertions of the divisors H_1,H_2,H_3.

By Lemma 3.3(1), Proposition 5.3, and the last equation in (5.27), *half* of the *negative* of the left-hand side of (2.5) equals

$$\begin{aligned}\big|\mathrm{ev}^{-1}_{\Gamma_3^+;\mathbf{h}}(\mathbf{p})\big|^{\pm}_{\mathfrak{o}_{\Gamma_3^+;\mathbf{p};\mathbf{h}}}-\big|\mathrm{ev}^{-1}_{\Gamma_2^+;\mathbf{h}}(\mathbf{p})\big|^{\pm}_{\mathfrak{o}_{\Gamma_2^+;\mathbf{p};\mathbf{h}}}&=\big(h_1\cdot_X h_2\big)\big(h_3\cdot_X B\big)N^{\phi;\mathfrak{p}}_{B,l}\\
+\sum_{\substack{B_0,B'\in H_2(X)-\{0\}\\ B_0+\mathfrak{d}(B')=B}}&2^{\ell_\omega(B')}\big(B_0\cdot_X B'\big)\big(h_1\cdot_X B'\big)\big(h_2\cdot_X B'\big)\big(h_3\cdot_X B_0\big)\binom{l-1}{\ell_\omega(B')}N^X_{B'}N^{\phi;\mathfrak{p}}_{B_0,l-1-\ell_\omega(B')}\\
-\sum_{\substack{B_0,B'\in H_2(X)-\{0\}\\ B_0+\mathfrak{d}(B')=B}}&2^{\ell_\omega(B')}\big(B_0\cdot_X B'\big)\big(h_1\cdot_X B'\big)\big(h_3\cdot_X B'\big)\big(h_2\cdot_X B_0\big)\binom{l-1}{\ell_\omega(B')}N^X_{B'}N^{\phi;\mathfrak{p}}_{B_0,l-1-\ell_\omega(B')}.\end{aligned}$$

Equating this expression with the negative of (5.31), we obtain the last identity in Theorem 1.1 with the left-hand side multiplied by $h_1\cdot_X h_2$. $\Box$

Proof of Lemma 5.9. It is sufficient to show that

$$\big(\mathrm{ev},\mathfrak{f}_{k',l'}\big)\colon \widehat{\mathfrak{M}}^{\star}_{k,l+l^*-1;l^*}(B;J,\nu)\longrightarrow X_{k,l+l^*-1}\times\overline{\mathcal{M}}^{\tau}_{k',l'} \tag{5.33}$$

is a bordered $\mathbb{Z}_2$-pseudocycle of dimension $4l+2k+2(l^*-1)$ transverse to

$$\begin{gathered}f_{\mathbf{h};\mathbf{p};\Upsilon}\colon M_{\mathbf{h}}\times\Upsilon\longrightarrow X_{k,l+l^*-1}\times\overline{\mathcal{M}}^{\tau}_{k',l'},\\ f_{\mathbf{h};\mathbf{p};\Upsilon}(y_1,\ldots,y_{l^*},P)=\big(h_1(y_1),\ldots,h_{l^*}(y_{l^*}),\mathbf{p},P\big).\end{gathered} \tag{5.34}$$

Since $\dim_{\mathbb{R}}X=4$, $\langle c_1(X,\omega),B'\rangle\geqslant 1$ for every $B'\in H_2(X)-\{0\}$ which can be represented by a J-holomorphic map $u\colon\mathbb{P}^1\longrightarrow X$ for a generic $J\in\mathcal{J}^{\phi}_{\omega}$.

For a stratum $\mathcal{S}$ of $\widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J,\nu)$, we denote by $\mathcal{S}^*\subset\mathcal{S}$ the subspace of simple maps and by $\mathfrak{c}(\mathcal{S}^*)$ the number of nodes of maps in $\mathcal{S}$. The image of $\mathcal{S}$ under $\mathfrak{f}_{k',l'}$ is contained in a stratum $\mathcal{S}^\vee$ of $\overline{\mathcal{M}}^{\tau}_{k',l'}$ with $\mathfrak{c}(\mathcal{S}^\vee)\leqslant\mathfrak{c}(\mathcal{S})$. For a generic choice of (J,ν), $\mathcal{S}^*\subset\mathcal{S}$ is a smooth manifold of dimension

$$\dim\mathcal{S}^*=\ell_\omega(B)+2(l+l^*-1)+k-\mathfrak{c}(\mathcal{S})=4l+2k+2(l^*-1)-\mathfrak{c}(\mathcal{S})\,. \tag{5.35}$$

The image of $\mathcal{S}-\mathcal{S}^*$ under

$$\big(\mathrm{ev},\mathfrak{f}_{k',l'}\big)\colon \widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J,\nu)\longrightarrow X_{k,l+l^*-1}\times\overline{\mathcal{M}}^{\tau}_{k',l'} \tag{5.36}$$

is covered by smooth maps from manifolds $\mathcal{S}'$ with

$$\dim\mathcal{S}'\leqslant\ell_\omega(B)+2(l+l^*-1)+k-2-\mathfrak{c}(\mathcal{S}^\vee)=4l+2k+2(l^*-2)-\mathfrak{c}(\mathcal{S}^\vee); \tag{5.37}$$

see [23, Section 3] and [33, Section 3.4].

The space $\widehat{\mathfrak{M}}^{\star}_{k,l+l^*-1;l^*}(B;J,\nu)$ consists of the main stratum $\mathfrak{M}_{k,l+l^*-1;l^*}(B;J,\nu)$ and the subspaces $\mathcal{S}^*$ of the strata $\mathcal{S}$ with either one real node only or one conjugate pair of nodes only. Such strata have disjoint open neighborhoods in $\widehat{\mathfrak{M}}^{\star}_{k,l+l^*-1;l^*}(B;J,\nu)$. Thus, the gluing maps as in [20] for these strata can be chosen so that their images do not overlap. Along with the smooth structure of $\mathfrak{M}_{k,l+l^*-1;l^*}(B;J,\nu)$, these maps then determine a smooth structure on $\widehat{\mathfrak{M}}^{\star}_{k,l+l^*-1;l^*}(B;J,\nu)$ with respect to which the map (5.33) is smooth.

Since the space $\widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J,\nu)$ is compact,

$$\Omega\big((\mathrm{ev},\mathfrak{f}_{k',l'})|_{\widehat{\mathfrak{M}}^{\star}_{k,l+l^*-1;l^*}(B;J,\nu)}\big)\subset\big\{(\mathrm{ev},\mathfrak{f}_{k',l'})\big\}\big(\widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J,\nu)-\widehat{\mathfrak{M}}^{\star}_{k,l+l^*-1;l^*}(B;J,\nu)\big). \tag{5.38}$$

The complement on the right-hand side above consists of the subspaces $\mathcal{S}^*$ of the strata $\mathcal{S}$ with $\mathfrak{c}(\mathcal{S})\geqslant 2$ nodes and of the subspaces $\mathcal{S}-\mathcal{S}^*$ with $\mathfrak{c}(\mathcal{S})\geqslant 1$. Combining this with (5.35) and (5.37), we conclude that the left-hand side of (5.38) is covered by smooth maps from manifolds of dimension at most $4l+2k+2(l^*-2)$. Thus, (5.33) is a bordered $\mathbb{Z}_2$-pseudocycle of dimension $4l+2k+2(l^*-1)$.

For a generic (J,ν), the restriction

$$\big(\mathrm{ev},\mathfrak{f}_{k',l'}\big)\colon\mathcal{S}^*\longrightarrow X_{k,l+l^*-1}\times\mathcal{S}^\vee \tag{5.39}$$

of (5.36) to $\mathcal{S}^*$ is transverse (in the target above) to $f_{\mathbf{h};\mathbf{p};\Upsilon'}$ for every given submanifold $\Upsilon' \subset \mathcal{S}^\vee$. Along with the smoothings of the nodes, this implies that (5.33) is transverse to $f_{\mathbf{h};\mathbf{p};\Upsilon'}$. Since

$$\mathfrak{f}_{k',l'}\big(\partial\widehat{\mathfrak{M}}^\star_{k,l+l^*-1;l^*}(B;J,\nu)\big) \cap \partial\Upsilon = \varnothing$$

with our choices of Υ, we conclude that the restriction of (5.33) to $\partial\widehat{\mathfrak{M}}^\star_{k,l+l^*-1;l^*}(B;J,\nu)$ is transverse to (5.34) and to the restriction of (5.34) to the boundary $M_{\mathbf{h}}\times\partial\Upsilon$ of its domain.

Since the image of $\mathfrak{f}_{k',l'}$ is disjoint from $\partial\overline{\mathcal{M}}^\tau_{k',l}$, it is also disjoint from the limit set

$$\Omega(\Upsilon) \equiv \overline{\Upsilon}-\Upsilon$$

of Υ ($\Omega(\Upsilon)$ is empty in the case of Lemma 4.4 and consists of 8 points in $\partial\overline{\mathcal{M}}^\tau_{0,3}$ in the case of Lemma 4.5). It remains to show that smooth maps from manifolds of dimensions at most $4l+2k+2(l^*-2)$ covering the right-hand side of (5.38) can be chosen so that they are transverse to (5.34) and to the restriction of (5.34) to $M_{\mathbf{h}}\times\partial\Upsilon$. If $\mathfrak{c}(\mathcal{S})\geqslant 2$ and $\mathcal{S}$ is not a stratum of $\widehat{\mathfrak{M}}^\star_{k,l+l^*-1;l^*}(B;J,\nu)$, i.e. the maps in $\mathcal{S}$ do not just contain a conjugate pair of nodes, then the transversality of Υ to every stratum of $\overline{\mathcal{M}}^\tau_{k',l'}-\partial\Upsilon$, the transversality of (5.39) to $f_{\mathbf{h};\mathbf{p};\Upsilon'}$ for every given submanifold $\Upsilon'\subset\mathcal{S}^\vee$, and (5.35) imply that

$$\big\{(\mathrm{ev},\mathfrak{f}_{k',l'})\big\}(\mathcal{S}^*) \cap f_{\mathbf{h};\mathbf{p};\Upsilon}\big(M_{\mathbf{h}}\times\Upsilon\big) = \varnothing. \tag{5.40}$$

For any stratum $\mathcal{S}$ of $\widehat{\mathfrak{M}}_{k,l+l^*-1;l^*}(B;J,\nu)$, the image of $\mathcal{S}-\mathcal{S}^*$ under (5.33) is covered by smooth maps

$$h_{\mathcal{S}'}\colon \mathcal{S}' \longrightarrow X_{k,l+l^*-1}\times\mathcal{S}^\vee$$

satisfying (5.37); these maps are transverse (in the target above) to $f_{\mathbf{h};\mathbf{p};\Upsilon'}$ for every given submanifold $\Upsilon'\subset\mathcal{S}^\vee$ for a generic (J,ν). This implies that (5.40) holds with $\mathcal{S}^*$ replaced by $\mathcal{S}-\mathcal{S}^*$. Thus, the bordered $\mathbb{Z}_2$-pseudocycle (5.33) is transverse to (5.34). ☐

Proof of Corollary 5.10. For a codimension 1 stratum $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l+l^*-1}(B;J,\nu)$ and $r=1,2$, let

$$k_r \equiv k_r(\mathcal{S}), \qquad l_r \equiv l_r(\mathcal{S}), \qquad l^*_r \equiv l^*_r(\mathcal{S}), \qquad B_r \equiv B_r(\mathcal{S}) \tag{5.41}$$

be as in Section 5.1 . Suppose that $\widehat{\mathcal{S}}^*$ is a stratum of $\partial\widehat{\mathfrak{M}}^\star_{k,l+l^*-1;l^*}(B;J,\nu)$, i.e.

$$\epsilon_{l^*}(\mathcal{S}) \equiv \big\langle c_1(X,\omega),B_2\big\rangle-2(l_2-l^*_2)-k_2$$

is congruent to 2 or 3 modulo 4.

Since $\Upsilon\cap S_1=\varnothing$, $(l_1,k_1)\neq(1,0)$ if $\widehat{\mathcal{S}}^*_{\mathbf{h},\mathbf{p};\Upsilon}\neq\varnothing$. If $B_2=0$, $l_2,l^*_2=1$, and $k_2=0$, then $\epsilon_{l^*}(\mathcal{S})=0$, contrary to the assumption on $\mathcal{S}$ above. Suppose $B_2=0$, $l_2=1$, and $l^*_2,k_2=0$. For good choices of ν (still sufficiently generic), the restriction to $\widehat{\mathcal{S}}^*$ of (5.7) with l replaced by $l+l^*-1$ then factors as

$$\widehat{\mathcal{S}}^* \longrightarrow \mathfrak{M}_{k+1,l+l^*-2}\big(B;J,\nu_1\big)\times\mathfrak{M}_{1,1}(0;J,0) \longrightarrow X_{k,l+l^*-2}\times X^\phi \longrightarrow X_{k,l+l^*-1}\,.$$

Thus, $\widehat{\mathcal{S}}^*_{\mathbf{h},\mathbf{p};\Upsilon}=\varnothing$ for generic choices of $\mathbf{h}$ and $\mathbf{p}$. Suppose $B_2=0$, $l_2=0$, and $k_2=2$. For good choices of ν, the restriction to $\widehat{\mathcal{S}}^*$ of (5.7) with l replaced by $l+l^*-1$ then factors as

$$\widehat{\mathcal{S}}^* \longrightarrow \mathfrak{M}_{k-1,l+l^*-1}\big(B;J,\nu_1\big)\times\mathfrak{M}_{3,0}(0;J,0) \longrightarrow X_{k-2,l+l^*-1}\times\Delta^\phi_X \longrightarrow X_{k,l+l^*-1}\,,$$

where $\Delta_X^\phi \subset (X^\phi)^2$ is the diagonal. Thus, $\widehat{\mathcal{S}}^*_{\mathbf{h},\mathbf{p};\Upsilon}=\varnothing$ for generic choices of $\mathbf{h}$ and $\mathbf{p}$.

We can thus assume that either $B_r\neq 0$ or $2l_r+k_r\geqslant 3$ for $r=1,2$. For good choices of ν, the restriction to $\widehat{\mathcal{S}}^*$ of (5.7) with l replaced by $l+l^*-1$ then factors as

$$\begin{aligned}\widehat{\mathcal{S}}^* &\longrightarrow \mathfrak{M}_{k_1+1,l_1}\big(B_1;J,\nu_1\big)\times\mathfrak{M}_{k_2+1,l_2}\big(B_2;J,\nu_2\big)\\ &\longrightarrow \mathfrak{M}_{k_1,l_1}\big(B_1;J,\nu_1'\big)\times\mathfrak{M}_{k_2,l_2}\big(B_2;J,\nu_2'\big)\longrightarrow X_{k_1,l_1-l_1^*}\times X_{k_2,l_2-l_2^*}\longrightarrow X_{k,l-1}\,.\end{aligned}$$

Thus, $\widehat{\mathcal{S}}^*_{\mathbf{h},\mathbf{p};\Upsilon}=\varnothing$ for generic choices of $\mathbf{h}$, $\mathbf{p}$, and (J,ν) unless

$$\ell_\omega\big(B_r\big)+2\big(l_r-l_r^*\big)+k_r \geqslant 4\big(l_r-l_r^*\big)+2k_r \qquad \forall\, r=1,2.$$

Since

$$\ell_\omega(B_1)+\ell_\omega(B_2)=\ell_\omega(B)-1=2l+k-1,\quad k_1+k_2=k,\quad l_1+l_2=l+l^*-1,\quad l_1^*+l_2^*=l^*,$$

and $\epsilon_{l^*}(\mathcal{S})\cong 2,3 \bmod 4$, it follows that $\epsilon_{l^*}(\mathcal{S})=2$ and

$$\ell_\omega(B_1)=2\big(l_1-l_1^*\big)+k_1. \tag{5.42}$$

If $\mathcal{S}$ satisfies ($\mathcal{S}0$) above Proposition 5.7, the restriction to $\widehat{\mathcal{S}}^*$ of the composition of (5.33) with the projection to the product $X_{k_1,l_1}\times\overline{\mathcal{M}}^\tau_{k',l'}$ factors as

$$\widehat{\mathcal{S}}^*\longrightarrow\mathfrak{M}_{k_1+1,l_1}\big(B_1;J,\nu_1\big)\times\mathfrak{M}_{k_2+1,l_2}\big(B_2;J,\nu_2\big)\longrightarrow\mathfrak{M}_{k_1,l_1}\big(B_1;J,\nu_1'\big)\longrightarrow X_{k_1,l_1}\times\overline{\mathcal{M}}^\tau_{k',l'}.$$

Since the restriction of (5.33) to $\widehat{\mathcal{S}}^*$ is transverse to (5.34) and Υ is a real hypersurface, (5.42) then implies that $\widehat{\mathcal{S}}^*_{\mathbf{h},\mathbf{p};\Upsilon}=\varnothing$. □

6 Proofs of structural statements

6.1 Orienting the linearized $\bar\partial$-operator

For $\mathbf{u}$ as in (5.1), let

$$\begin{aligned}D^\phi_{J,\nu;\mathbf{u}}\colon \Gamma(\mathbf{u}) &\equiv\big\{\xi\in\Gamma(\Sigma;u^*TX)\colon\ \xi\circ\sigma=\mathrm{d}\phi\circ\xi\big\}\\ &\longrightarrow\Gamma^{0,1}(\mathbf{u})\equiv\big\{\zeta\in\Gamma(\Sigma;(T^*\Sigma,\mathfrak{j})^{0,1}\otimes_{\mathbb{C}}u^*(TX,J))\colon\ \zeta\circ\mathrm{d}\sigma=\mathrm{d}\phi\circ\zeta\big\}\end{aligned}$$

be the linearization of the $\{\bar\partial_J-\nu\}$-operator on the space of real maps from (Σ,σ) with its complex structure $\mathfrak{j}$. We define

$$\begin{aligned}\lambda^{\mathbb{C}}_{\mathbf{u}}(X)&=\bigotimes_{i=1}^{l}\lambda\big(T_{u(z_i^+)}X\big), &\lambda^{\mathbb{R}}_{\mathbf{u}}(X)&=\lambda\bigg(\bigoplus_{i=1}^{k}T_{u(x_i)}X^\phi\bigg)=\bigotimes_{i=1}^{k}\lambda\big(T_{u(x_i)}X^\phi\big),\\ \lambda_{\mathbf{u}}\big(D^\phi_{J,\nu}\big)&=\det D^\phi_{J,\nu;\mathbf{u}}\,, &\widetilde\lambda_{\mathbf{u}}\big(D^\phi_{J,\nu},X\big)&=\lambda^{\mathbb{R}}_{\mathbf{u}}(X)^*\otimes\lambda^{\mathbb{C}}_{\mathbf{u}}(X)^*\otimes\lambda_{\mathbf{u}}\big(D^\phi_{J,\nu}\big);\end{aligned}$$

the summands and the factors in the definition of $\lambda_{\mathbf{u}}^{\mathbb{R}}(X)$ are *not* ordered. By [11, Appendix], the projection

$$\widetilde{\lambda}\big(D_{J,\nu}^{\phi},X\big)\equiv\bigcup_{\mathbf{u}\in\overline{\mathfrak{M}}_{k,l}(B;J,\nu)}\{\mathbf{u}\}\times\widetilde{\lambda}_{\mathbf{u}}\big(D_{J,\nu}^{\phi},X\big)\longrightarrow\overline{\mathfrak{M}}_{k,l}(B;J,\nu) \tag{6.1}$$

is a line orbi-bundle with respect to a natural topology on its domain.

For $i\in[k]$ and $\mathbf{u}\in\mathfrak{M}_{k,l}(B;J,\nu)$ with the associated marked curve $\mathcal{C}$ as in (5.2), let

$$j_i(\mathbf{u})=j_i(\mathcal{C})\in[k]$$

be as in Section 4.1. The next statement is a consequence of the orienting construction of [24, Prop. 3.1], a more systematic perspective of which appears in the proof of [8, Thm. 7.1].

Lemma 6.1. *Suppose (X,ω,ϕ) is a real symplectic fourfold, $l\in\mathbb{Z}^+$, $k\in\mathbb{Z}^{\geqslant 0}$ with $k+2l\geqslant 3$, $B\in H_2(X)$, and $(J,\nu)\in\mathcal{H}_{k,l}^{\omega,\phi}$. If k and B satisfy (2.1), then a Pin^--structure $\mathfrak{p}$ on X^{ϕ} determines an orientation $\mathfrak{o}_{\mathfrak{p}}^{D}$ on the restriction of (6.1) to $\mathfrak{M}_{k,l}(B;J,\nu)$ with the following properties:*

($\mathfrak{o}_{\mathfrak{p}}^{D}1$) *the interchange of two real points x_i and x_j preserves $\mathfrak{o}_{\mathfrak{p}}^{D}$;*

($\mathfrak{o}_{\mathfrak{p}}^{D}2$) *if $\mathbf{u}\in\mathfrak{M}_{k,l}(B;J,\nu)$, the interchange of the real points x_1 and $x_{j_i(\mathbf{u})}$ with $2\leqslant i\leqslant k$ preserves $\mathfrak{o}_{\mathfrak{p}}^{D}$ at $\mathbf{u}$ if and only if $(k-1)(i-1)\in 2\mathbb{Z}$;*

($\mathfrak{o}_{\mathfrak{p}}^{D}3$) *if $\mathbf{u}\in\mathfrak{M}_{k,l}(B;J,\nu;\check{X}^{\phi})$ and the marked points z_i^+ and z_j^+ are not separated by the fixed locus S^1 of the domain of $\mathbf{u}$, then the interchange of the conjugate pairs (z_i^+,z_i^-) and (z_j^+,z_j^-) preserves $\mathfrak{o}_{\mathfrak{p}}^{D}$ at $\mathbf{u}$;*

($\mathfrak{o}_{\mathfrak{p}}^{D}4$) *the interchange of the points in a conjugate pair (z_i^+,z_i^-) with $1<i\leqslant l$ preserves $\mathfrak{o}_{\mathfrak{p}}^{D}$;*

($\mathfrak{o}_{\mathfrak{p}}^{D}5$) *the interchange of the points in the conjugate pair (z_1^+,z_1^-) preserves $\mathfrak{o}_{\mathfrak{p}}^{D}$ if and only if*

$$k\neq 0\ \ \textit{and}\ \ \ell_\omega(B)\cong 2,3 \bmod 4\qquad \textit{or}\qquad k=0\ \ \textit{and}\ \ \ell_\omega(B)\cong 0 \bmod 4;$$

($\mathfrak{o}_{\mathfrak{p}}^{D}6$) *if $k,l,l^*=1$, $B=0$, and ν is small, then $\mathfrak{o}_{\mathfrak{p}}^{D}$ is the orientation induced by the evaluation at x_1.*

Proof. Let $\mathbf{u}$ be as in (5.1). For the purposes of applying [8, Thm. 7.1], we take the distinguished half-surface $\mathbb{D}^2\subset\mathbb{P}^1$ to be the disk so that $\partial\mathbb{D}^2$ is the fixed locus S^1 of τ and $z_1^+\in\mathbb{D}^2$. A Pin^--structure $\mathfrak{p}$ on X^{ϕ} then determines an orientation $\mathfrak{o}_{\mathfrak{p};\mathbb{R}}^{D}$ on the line $\lambda_{\mathbf{u}}^{\mathbb{R}}(X)^*\otimes\lambda_{\mathbf{u}}(D_{J,\nu}^{\phi})$ varying continuously with $\mathbf{u}$. Since $\mathfrak{o}_{\mathfrak{p};\mathbb{R}}^{D}$ does not depend on the conjugate pairs of marked points, except for $z_1^{\pm}$ which determines $\mathbb{D}^2$, $\mathfrak{o}_{\mathfrak{p};\mathbb{R}}^{D}$ satisfies ($\mathfrak{o}_{\mathfrak{p}}^{D}3$) and ($\mathfrak{o}_{\mathfrak{p}}^{D}4$). By the CROrient 1$\mathfrak{p}$ property in [8, Section 7.2], $\mathfrak{o}_{\mathfrak{p};\mathbb{R}}^{D}$ satisfies ($\mathfrak{o}_{\mathfrak{p}}^{D}1$), ($\mathfrak{o}_{\mathfrak{p}}^{D}2$), and ($\mathfrak{o}_{\mathfrak{p}}^{D}5$). By the CROrient 5b and 6b properties in [8, Section 7.2], it also satisfies (($\mathfrak{o}_{\mathfrak{p}}^{D}6$)). Along with the symplectic orientations of $T_{u(z_i^+)}X$, $\mathfrak{o}_{\mathfrak{p};\mathbb{R}}^{D}$ determines an orientation $\mathfrak{o}_{\mathfrak{p}}^{D}$ on $\widetilde{\lambda}_{\mathbf{u}}(D_{J,\nu}^{\phi},X)$ varying continuously with $\mathbf{u}$. Since the complex dimension of X is even, $\mathfrak{o}_{\mathfrak{p}}^{D}$ also satisfies all six properties. $\square$

Suppose now that $l \in \mathbb{Z}^+$ and $\mathcal{S}$ is an open codimension 1 stratum of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$. Define

$$r(\mathcal{S}) = \begin{cases} 1, & \text{if } k=0 \text{ or } 1 \in K_1(\mathcal{S}); \\ 2, & \text{if } 1 \in K_2(\mathcal{S}). \end{cases}$$

An orientation $\mathfrak{o}^c_{\mathcal{S};\mathbf{u}}$ of $\mathcal{N}_{\mathbf{u}}\mathcal{S}$ determines a direction of degeneration of elements of $\mathfrak{M}_{k,l}(B;J,\nu)$ to $\mathbf{u}$. The orientation $\mathfrak{o}^D_{\mathfrak{p}}$ on (6.1) limits to an orientation $\mathfrak{o}^D_{\mathfrak{p};\mathbf{u}}$ of $\widetilde{\lambda}_{\mathbf{u}}(D^{\phi}_{J,\nu}, X)$ by approaching $\mathbf{u}$ from this direction. The orientation $\mathfrak{o}^D_{\mathfrak{p};\mathbf{u}}$ is called the limiting orientation induced by $\mathfrak{p}$ and $\mathfrak{o}^c_{\mathcal{S};\mathbf{u}}$ in [8, Section 7.3]. If in addition $l^*_2(\mathcal{S}) \geqslant 1$, the possible orientations $\mathfrak{o}^{c;\pm}_{\mathcal{S};\mathbf{u}}$ of $\mathcal{N}_{\mathbf{u}}\mathcal{S}$ are distinguished as above Lemma 4.3. We denote by $\mathfrak{o}^{D;\pm}_{\mathfrak{p};\mathbf{u}}$ the limiting orientation of $\widetilde{\lambda}_{\mathbf{u}}(D^{\phi}_{J,\nu}, X)$ induced by $\mathfrak{p}$ and $\mathfrak{o}^{c;\pm}_{\mathcal{S};\mathbf{u}}$.

The domain of each element $\mathbf{u} \in \mathcal{S}$ consists of an irreducible component $\mathbb{P}^1_1$ carrying the marked points $z^{\pm}_1$ with fixed locus S^1_1 and another irreducible component $\mathbb{P}^1_2$ with fixed locus S^1_2. The fixed locus S^1_r splits $\mathbb{P}^1_r$ into two disks. Let $\mathcal{S}_* \subset \mathcal{S}$ be the subspace of all maps with fixed distributions of the marked points z^+_i with $i \in [l]$ between the four disks and with fixed orderings of the marked points x_i with $i \in [k]$ and the nodal points on the two fixed loci. We call such a subspace a substratum of $\mathcal{S}$. If $k_2(\mathcal{S})+2l_2(\mathcal{S}) \geqslant 2$, i.e. the marked domain (5.2) of every element $\mathbf{u} \in \mathcal{S}$ is stable, then the image of $\mathcal{S}$ under the forgetful morphism

$$\mathfrak{f}_{k,l} : \overline{\mathfrak{M}}_{k,l}(B;J,\nu) \longrightarrow \overline{\mathcal{M}}^{\tau}_{k,l}$$

is contained in a codimension 1 stratum $\mathcal{S}^{\vee}$. In such a case, a substratum $\mathcal{S}_*$ of $\mathcal{S}$ is given by

$$\mathcal{S}_* = \mathcal{S} \cap f^{-1}_{k,l}(\mathcal{S}^{\vee}_*)$$

for some topological component $\mathcal{S}^{\vee}_*$ of $\mathcal{S}^{\vee}$.

For good choices of ν, there are a natural embedding

$$\mathcal{S}_* \longleftrightarrow \mathfrak{M}_1 \times \mathfrak{M}_2 \subset \mathfrak{M}_{k_1(\mathcal{S})+1, l_1(\mathcal{S})}\big(B_1(\mathcal{S});J,\nu_1\big) \times \mathfrak{M}_{k_2(\mathcal{S})+1, l_2(\mathcal{S})}\big(B_2(\mathcal{S});J,\nu_2\big) \tag{6.2}$$

for some unions $\mathfrak{M}_1$ and $\mathfrak{M}_2$ of topological components of the moduli spaces on the right-hand side above and forgetful morphisms

$$\begin{aligned} \mathfrak{f}_{\mathrm{nd}} &: \mathfrak{M}_{k_1(\mathcal{S})+1, l_1(\mathcal{S})}\big(B_1(\mathcal{S});J,\nu_1\big) \longrightarrow \mathfrak{M}_{k_1(\mathcal{S}), l_1(\mathcal{S})}\big(B_1(\mathcal{S});J,\nu'_1\big), \\ \mathfrak{f}_{\mathrm{nd}} &: \mathfrak{M}_{k_2(\mathcal{S})+1, l_2(\mathcal{S})}\big(B_2(\mathcal{S});J,\nu_2\big) \longrightarrow \mathfrak{M}_{k_2(\mathcal{S}), l_2(\mathcal{S})}\big(B_2(\mathcal{S});J,\nu'_2\big) \end{aligned} \tag{6.3}$$

dropping the real marked points corresponding to the nodal points nd on the two components. We choose the embedding in (6.2) so that it satisfies ($\mathfrak{o}_S1$) and ($\mathfrak{o}_S2$) in Section 4.3. For an element $\mathbf{u} \in \mathcal{S}$, we denote by

$$\mathbf{u}_1 \in \mathfrak{M}_{k_1(\mathcal{S})+1, l_1(\mathcal{S})}\big(B_1(\mathcal{S});J,\nu_1\big) \quad \text{and} \quad \mathbf{u}_2 \in \mathfrak{M}_{k_2(\mathcal{S})+1, l_2(\mathcal{S})}\big(B_2(\mathcal{S});J,\nu_2\big)$$

the pair of maps corresponding to $\mathbf{u}$ via (6.2). Let

$$\mathbf{u}'_1 \in \mathfrak{M}_{k_1(\mathcal{S}), l_1(\mathcal{S})}\big(B_1(\mathcal{S});J,\nu'_1\big) \quad \text{and} \quad \mathbf{u}'_2 \in \mathfrak{M}_{k_2(\mathcal{S}), l_2(\mathcal{S})}\big(B_2(\mathcal{S});J,\nu'_2\big)$$

be the images of $\mathbf{u}_1$ and $\mathbf{u}_2$ under the forgetful morphisms in (6.3).

Suppose k and B satisfy (2.1), $l_2^*(\mathcal{S})\geqslant 1$, and $i^*\in L_2(\mathcal{S}_*)$ is as above Lemma 4.3. For each $\mathbf{u}\in\mathcal{S}_*$, the exact sequences

$$\begin{aligned} &0\longrightarrow D^{\phi}_{J,\nu;\mathbf{u}}\longrightarrow D^{\phi}_{J,\nu_1';\mathbf{u}_1'}\oplus D^{\phi}_{J,\nu_2;\mathbf{u}_2}\longrightarrow T_{u(\mathrm{nd})}X^{\phi}\longrightarrow 0, \quad \big(\xi_1,\xi_2\big)\longrightarrow \xi_2(\mathrm{nd})-\xi_1(\mathrm{nd}), \quad (6.4)\\ &0\longrightarrow D^{\phi}_{J,\nu;\mathbf{u}}\longrightarrow D^{\phi}_{J,\nu_1;\mathbf{u}_1}\oplus D^{\phi}_{J,\nu_2;\mathbf{u}_2'}\longrightarrow T_{u(\mathrm{nd})}X^{\phi}\longrightarrow 0, \quad \big(\xi_1,\xi_2\big)\longrightarrow \xi_2(\mathrm{nd})-\xi_1(\mathrm{nd}), \end{aligned}$$

of Fredholm operators determine isomorphisms

$$\begin{aligned} \lambda_{\mathbf{u}}\big(D^{\phi}_{J,\nu}\big)\otimes\lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big) &\approx \lambda_{\mathbf{u}_1'}\big(D^{\phi}_{J,\nu_1'}\big)\otimes\lambda_{\mathbf{u}_2}\big(D^{\phi}_{J,\nu_2}\big)\,,\\ \lambda_{\mathbf{u}}\big(D^{\phi}_{J,\nu}\big)\otimes\lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big) &\approx \lambda_{\mathbf{u}_1}\big(D^{\phi}_{J,\nu_1}\big)\otimes\lambda_{\mathbf{u}_2'}\big(D^{\phi}_{J,\nu_2'}\big). \end{aligned} \qquad (6.5)$$

If $\epsilon_{l^*}(\mathcal{S})\in 2\mathbb{Z}$ (for any $l^*\in[l]$), a Pin^--structure $\mathfrak{p}$ on X^{ϕ} determines homotopy classes of isomorphisms

$$\lambda_{\mathbf{u}_1'}\big(D^{\phi}_{J,\nu_1'}\big)\longrightarrow \lambda^{\mathbb{R}}_{\mathbf{u}_1'}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_1'}(X) \quad\text{and}\quad \lambda_{\mathbf{u}_2}\big(D^{\phi}_{J,\nu_2}\big)\longrightarrow \lambda^{\mathbb{R}}_{\mathbf{u}_2}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_2}(X);$$

see Lemma 6.1. Combining these isomorphisms with the first isomorphism in (6.5), we obtain a homotopy class of isomorphisms

$$\begin{aligned} \lambda_{\mathbf{u}}\big(D^{\phi}_{J,\nu}\big)\otimes\lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big) &\approx \lambda^{\mathbb{R}}_{\mathbf{u}_1'}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_1'}(X)\otimes\lambda^{\mathbb{R}}_{\mathbf{u}_2}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_2}(X)\\ &\approx \lambda^{\mathbb{R}}_{\mathbf{u}_1'}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_1'}(X)\otimes\lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big)\otimes\lambda^{\mathbb{R}}_{\mathbf{u}_2'}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_2'}(X)\\ &\approx \lambda^{\mathbb{R}}_{\mathbf{u}}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}}(X)\otimes\lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big)\,. \end{aligned} \qquad (6.6)$$

If $\epsilon_{l^*}(\mathcal{S})\notin 2\mathbb{Z}$, a Pin^--structure $\mathfrak{p}$ on X^{ϕ} similarly determines a homotopy class of isomorphisms

$$\begin{aligned} \lambda_{\mathbf{u}}\big(D^{\phi}_{J,\nu}\big)\otimes\lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big) &\approx \lambda^{\mathbb{R}}_{\mathbf{u}_1}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_1}(X)\otimes\lambda^{\mathbb{R}}_{\mathbf{u}_2'}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_2'}(X)\\ &\approx \lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big)\otimes\lambda^{\mathbb{R}}_{\mathbf{u}_1'}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_1'}(X)\otimes\lambda^{\mathbb{R}}_{\mathbf{u}_2'}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_2'}(X)\\ &\approx \lambda^{\mathbb{R}}_{\mathbf{u}}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}}(X)\otimes\lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big). \end{aligned}$$

In either case, we denote the associated orientation on $\widetilde{\lambda}_{\mathbf{u}}(D^{\phi}_{J,\nu},X)$ by $\mathfrak{o}^{D}_{\mathfrak{p};\mathbf{u}}$.

If $l_2^*(\mathcal{S})\geqslant 1$, we choose the embedding (6.2) so that the real marked points of the tuples of $\mathbf{u}_1$ and $\mathbf{u}_2$ corresponding to $\mathbf{u}\in\mathcal{S}_*$ are ordered by their position on $S^1_1\subset\mathbb{P}^1_1$ and $S^1_2\subset\mathbb{P}^1_2$, respectively, starting from the node in the counterclockwise direction with respect to $z_1^+\in\mathbb{P}^1_1$ and $z_{i^*}^+\in\mathbb{P}^1_2$. We define $\delta^{\pm}_{\mathbb{R}}(\mathcal{S}_*)\in\mathbb{Z}$ as above Lemma 4.3 and set

$$\delta^+_D(\mathcal{S})=k_2(\mathcal{S})\langle w_2(X),B_1(\mathcal{S})\rangle, \quad \delta^-_D(\mathcal{S})=\delta^+_D(\mathcal{S})+\frac{(\ell_\omega(B_2(\mathcal{S}))-k_2(\mathcal{S}))(\ell_\omega(B_2(\mathcal{S}))-k_2(\mathcal{S})+1)}{2}\,.$$

Lemma 6.2. *Suppose (X,ω,ϕ), $\mathfrak{p}$, k,l,B, and (J,ν) are as in Lemma 6.1, the pair (k,B) satisfies (2.1), and $\mathcal{S}_*$ is a substratum of a codimension 1 stratum $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ with $l_2^*(\mathcal{S})\geqslant 1$. The orientations $\mathfrak{o}^{D;\pm}_{\mathfrak{p}}$ and $\mathfrak{o}^{D}_{\mathfrak{p}}$ on $\widetilde{\lambda}(D^{\phi}_{J,\nu},X)|_{\mathcal{S}_*}$ are the same if and only if $\delta^{\pm}_D(\mathcal{S})\cong\delta^{\pm}_{\mathbb{R}}(\mathcal{S}_*)$ mod 2.*

Proof. Let $\mathbf{u}\in\mathcal{S}_*$. We define $r_\epsilon(\mathcal{S})$ to be 1 if $\epsilon_{l^*}(\mathcal{S})\in 2\mathbb{Z}$ and 2 if $\epsilon_{l^*}(\mathcal{S})\notin 2\mathbb{Z}$. Let $j'_\epsilon(\mathbf{u})\in\mathbb{Z}^{\geqslant 0}$ be the number of real marked points that lie on the oriented arc of $S^1_{r_\epsilon(\mathcal{S})}$ between the node and the real marked point $x_i\in S^1_{r_\epsilon(\mathcal{S})}$ with the smallest value of i; if $k_{r_\epsilon(\mathcal{S})}(\mathcal{S})=0$, we take $j'_\epsilon(\mathbf{u})=0$. The marked points $z_1^+\in\mathbb{P}^1_1$ and $z_{i^*}^+\in\mathbb{P}^1_2$ determine the distinguished disks as in the proof of Lemma 6.1. By Lemma 6.1($\mathfrak{o}_{\mathfrak{p}}^D 2$), the orientation $\mathfrak{o}_{\mathfrak{p}}^D$ at $\mathbf{u}$ agrees with the split orientation of [8, Section 7.4] if and only if $(k_{r_\epsilon(\mathcal{S})}-1)j'_\epsilon(\mathbf{u})$ is even. Thus, [8, Cor. 7.5] implies the claim for $\lambda^{\mathbb{R}}_{\mathbf{u}}(X)^*\otimes\lambda_{\mathbf{u}}(D^\phi_{J,\nu})$. Since the conjugate pairs of marked points have the same effect on $\mathfrak{o}_{\mathfrak{p}}^{D;\pm}$ and $\mathfrak{o}_{\mathfrak{p}}^D$, the claim follows. □

6.2 Proofs of Lemmas 5.1 and 5.5 and Proposition 5.2

Suppose (X,ω,ϕ), $\mathfrak{p},k,l,l^*,B$, and (J,ν) are as in Lemma 5.1, the pair (k,B) satisfies (2.1), and (J,ν) is generic. The exact sequences

$$0\longrightarrow \ker D^\phi_{J,\nu;\mathbf{u}}\longrightarrow T_{\mathbf{u}}\mathfrak{M}^*_{k,l}(B;J,\nu)\longrightarrow T_{\mathfrak{f}_{k,l}(\mathbf{u})}\mathcal{M}^\tau_{k,l}\longrightarrow 0$$

with $\mathbf{u}\in\mathfrak{M}^*_{k,l}(B;J,\nu)$ induced by the forgetful morphism $\mathfrak{f}_{k,l}$ determine an isomorphism

$$\begin{aligned}\lambda\big(\mathrm{ev}|_{\mathfrak{M}^*_{k,l}(B;J,\nu)}\big)&\equiv \mathrm{ev}^*\lambda^{\mathbb{R}}(X)^*\otimes\mathrm{ev}^*\lambda^{\mathbb{C}}(X)^*\otimes\lambda\big(\mathfrak{M}^*_{k,l}(B;J,\nu)\big)\\ &\approx\widetilde{\lambda}\big(D^\phi_{J,\nu},X\big)\otimes\mathfrak{f}^*_{k,l}\lambda\big(\mathcal{M}^\tau_{k,l}\big)\end{aligned}\tag{6.7}$$

of line bundles over $\mathfrak{M}^*_{k,l}(B;J,\nu)$. By Lemma 6.1, the Pin^--structure $\mathfrak{p}$ on X^ϕ induces an orientation $\mathfrak{o}_{\mathfrak{p}}^D$ on the first factor on the right-hand side above. Along with the orientation $\mathfrak{o}_{k,l;l^*}$ on the second factor defined in Section 4.1, it determines a relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}$ on the restrictions of (5.12) to $\mathfrak{M}^*_{k,l}(B;J,\nu)$ via (6.7).

Proofs of Lemmas 5.1 and 5.5. By Lemmas 4.1 and 6.1, the relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}$ above satisfies all properties listed in Lemma 5.1 wherever it is defined. Every (continuous) extension of $\mathfrak{o}_{\mathfrak{p};l^*}$ to subspaces of the domains of the maps in (5.12) satisfies the same properties. The relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}$ automatically extends over all strata of codimension 2 and higher. By Lemma 5.5, it extends over the codimension 1 strata of the two domains as well. Lemma 5.5 in turn follows immediately from Lemmas 4.3 and 6.2. □

The next observation, which is used in the proof of Proposition 5.2, is straightforward.

Lemma 6.3. *Suppose A_{ij} with $i,j\in[3]$ are oriented finite-dimensional vector spaces, the rows and columns in the diagram in Figure 8 are exact sequences of vector-space homomorphisms, and this diagram commutes. The total number of rows and columns in this diagram which (do not) respect the orientations is congruent to* $\dim(A_{13})\dim(A_{31})$ *mod 2.*

Proof of Proposition 5.2. We continue with the notation in the proof of Lemma 5.9, but apply it to the strata $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$. Let

$$\mathrm{ev}_{[l^*]}=\prod_{i=1}^{l^*}\mathrm{ev}_i^+:\overline{\mathfrak{M}}_{k,l}(B;J,\nu)\longrightarrow X^{l^*}\tag{6.8}$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{11} & \longrightarrow & A_{12} & \longrightarrow & A_{13} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{21} & \longrightarrow & A_{22} & \longrightarrow & A_{23} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A_{31} & \longrightarrow & A_{32} & \longrightarrow & A_{33} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Figure 8: Commutative square of vector spaces with exact rows and columns for the statement of Lemma 6.3

and $h\colon Z\longrightarrow X^{l^*}$ be a smooth map from a manifold of dimension $2l^*-2$ that covers $\Omega(f_{\mathbf{h}})$. Let

$$
\begin{aligned}
\mathrm{ev}_{k,l;\mathbf{h}}\colon \overline{\mathcal{Z}}_{k,l;\mathbf{h}}(B;J,\nu)&\equiv\big\{(\mathbf{u},y)\in\overline{\mathfrak{M}}_{k,l}(B;J,\nu)\times M_{\mathbf{h}}\colon \mathrm{ev}_{[l^*]}(\mathbf{u})=f_{\mathbf{h}}(y)\big\}\longrightarrow X_{k,l-l^*}, \qquad (6.9)\\
\mathrm{ev}_{k,l;h}\colon \overline{\mathcal{Z}}_{k,l;h}(B;J,\nu)&\equiv\big\{(\mathbf{u},z)\in\overline{\mathfrak{M}}_{k,l}(B;J,\nu)\times Z\colon \mathrm{ev}_{[l^*]}(\mathbf{u})=h(z)\big\}\longrightarrow X_{k,l-l^*}
\end{aligned}
$$

be the maps induced by (5.6).

For each stratum $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$, define

$$
\mathcal{S}_{\mathbf{h}}=\overline{\mathcal{Z}}_{k,l;\mathbf{h}}(B;J,\nu)\cap\big(\mathcal{S}\times M_{\mathbf{h}}\big),\quad \mathcal{S}^*_{\mathbf{h}}=\overline{\mathcal{Z}}_{k,l;\mathbf{h}}(B;J,\nu)\cap\big(\mathcal{S}^*\times M_{\mathbf{h}}\big).
$$

For a generic (J,ν), the subspace $\mathcal{S}^*$ of simple maps in $\mathcal{S}$ is a smooth manifold of dimension

$$
\dim\mathcal{S}^*=\ell_\omega(B)+2l+k-\mathfrak{c}(\mathcal{S})=4l-2l^*+2k-\mathfrak{c}(\mathcal{S}) \qquad (6.10)
$$

and the restriction of (6.8) to $\mathcal{S}^*$ is transverse to $f_{\mathbf{h}}$ and to h. Along with (6.10), the first transversality property implies that $\mathcal{S}^*_{\mathbf{h}}$ is a smooth manifold of dimension

$$
\dim\mathcal{S}^*_{\mathbf{h}}=\dim\mathcal{S}^*-2l^*=4(l-l^*)+2k-\mathfrak{c}(\mathcal{S})\,. \qquad (6.11)
$$

For every stratum $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$, there is a smooth manifold $\mathcal{S}'$ and smooth maps

$$
\mathrm{ev}_{[l^*]}\colon\mathcal{S}'\longrightarrow X^{l^*}\qquad\text{and}\qquad \mathrm{ev}_{k,l;l^*}\colon\mathcal{S}'\longrightarrow X_{k,l-l^*}
$$

such that $\mathrm{ev}_{[l^*]}$ is transverse to $f_{\mathbf{h}}$ and to h,

$$
\mathrm{ev}_{[l^*]}(\mathcal{S}-\mathcal{S}^*)\subset\mathrm{ev}_{[l^*]}(\mathcal{S}'),\quad\text{and}\quad \dim\mathcal{S}'\leqslant\ell_\omega(B)+2l+k-2=4l-2l^*+2k-2\,; \qquad (6.12)
$$

see [23, Section 3] and [33, Section 3.4]. In particular, the map

$$
\mathrm{ev}_{k,l;\mathbf{h}}\colon\mathcal{S}'_{\mathbf{h}}\equiv\big\{(\mathbf{u},y)\in\mathcal{S}'\times M_{\mathbf{h}}\colon\mathrm{ev}_{[l^*]}(\mathbf{u})=f_{\mathbf{h}}(y)\big\}\longrightarrow X_{k,l-l^*}
$$

induced by $\mathrm{ev}_{k,l;l^*}$ is smooth,

$$\mathrm{ev}_{k,l;\mathbf{h}}\big(\mathcal{S}_{\mathbf{h}}-\mathcal{S}_{\mathbf{h}}^*\big)\subset \mathrm{ev}_{k,l;\mathbf{h}}(\mathcal{S}_{\mathbf{h}}'), \quad \text{and} \quad \dim \mathcal{S}_{\mathbf{h}}' \leqslant 4(l-l^*)+2k-2\,. \tag{6.13}$$

By the reasoning in the proof of Lemma 5.9 applied to the space $\mathfrak{M}^{\star}_{k,l;l^*}(B;J,\nu)$ instead of $\widehat{\mathfrak{M}}^{\star}_{k,l+l^*-1;l^*}(B;J,\nu)$, $\mathfrak{M}^{\star}_{k,l;l^*}(B;J,\nu)$ is a smooth manifold. Along with the first transversality property after (6.10), this implies that (5.10) is a smooth map between smooth manifolds of the same dimension. The relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}$ of the first map in (5.12) and the orientation $\mathfrak{o}_{\mathbf{h}}$ determine a relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}$ of (5.10).

Since the space $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ is compact,

$$\begin{aligned}\Omega\big(\mathrm{ev}_{k,l;\mathbf{h}}\big|_{\mathcal{Z}^{\star}_{k,l;\mathbf{h}}(B;J,\nu)}\big)\subset \mathrm{ev}_{k,l;\mathbf{h}}\big(\overline{\mathcal{Z}}_{k,l;\mathbf{h}}(B;J,\nu)-\mathcal{Z}^{\star}_{k,l;\mathbf{h}}(B;J,\nu)\big)&\\ \cup \mathrm{ev}_{k,l;h}\big(\overline{\mathcal{Z}}_{k,l;h}(B;J,\nu)\big)\,.&\end{aligned} \tag{6.14}$$

By (6.10), (6.12), and the transversality of the maps $\mathrm{ev}_{[l^*]}$ on $\mathcal{S}^*$ and $\mathcal{S}'$ to h, the last set above is covered by smooth maps from finitely many manifolds of dimension at most

$$\dim \mathcal{Z}^{\star}_{k,l;\mathbf{h}}(B;J,\nu)-2 = 4(l-l^*)+2k-2. \tag{6.15}$$

The set

$$\overline{\mathcal{Z}}_{k,l;\mathbf{h}}(B;J,\nu)-\mathcal{Z}^{\star}_{k,l;\mathbf{h}}(B;J,\nu)\subset \overline{\mathfrak{M}}_{k,l}(B;J,\nu)\times M_{\mathbf{h}}$$

consists of the subspaces $\mathcal{S}_{\mathbf{h}}^*$ corresponding to the strata $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ with either $\mathfrak{c}(\mathcal{S})\geqslant 2$ nodes or $\epsilon_{l^*}(\mathcal{S})\cong 2,3 \bmod 2$ and of the subspaces $\mathcal{S}_{\mathbf{h}}-\mathcal{S}_{\mathbf{h}}^*$ with $\mathfrak{c}(\mathcal{S})\geqslant 1$. By (6.11) and (6.13), a smooth map from manifold of dimension (6.15) covers $\mathrm{ev}_{k,l;\mathbf{h}}(\mathcal{S}_{\mathbf{h}}^*)$ if $\mathfrak{c}(\mathcal{S})\geqslant 2$ and $\mathrm{ev}_{k,l;\mathbf{h}}(\mathcal{S}_{\mathbf{h}}-\mathcal{S}_{\mathbf{h}}^*)$ for any stratum $\mathcal{S}$ of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$. We show in the next two paragraphs that a smooth map from manifold of dimension (6.15) also covers $\mathrm{ev}_{k,l;\mathbf{h}}(\mathcal{S}_{\mathbf{h}}^*)$ if $\mathcal{S}$ is a stratum of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ with $\mathfrak{c}(\mathcal{S})=1$ and $\epsilon_{l^*}(\mathcal{S})\cong 2,3 \bmod 4$. This will conclude the proof of the first claim of the proposition.

Suppose $\mathcal{S}$ is a stratum of $\overline{\mathfrak{M}}_{k,l}(B;J,\nu)$ with $\mathfrak{c}(\mathcal{S})=1$ and k_i,l_i,l_i^*,B_i are as in (5.41). Let

$$f_{\mathbf{h}_1}\colon M_{\mathbf{h}_1}\longrightarrow X^{l_1^*} \qquad \text{and} \qquad f_{\mathbf{h}_2}\colon M_{\mathbf{h}_2}\longrightarrow X^{l_2^*}$$

be the pseudocycles determined by the maps $h_1,\ldots,h_{l^*}$ corresponding to the conjugate pairs of marked points indexed by $i\in[l^*]$ that are carried by the first and second components of the maps in $\mathcal{S}$, respectively. If

$$\epsilon_{l^*}(\mathcal{S})\equiv \big\langle c_1(X,\omega),B_2\big\rangle-2(l_2-l_2^*)-k_2\equiv \big(\ell_\omega(B_2)-2(l_2-l_2^*)-k_2\big)+1$$

is congruent to 2 or 3 modulo 4, then

$$(B_1,l_1,k_1)\neq(0,1,0) \qquad \text{and} \qquad (B_2,l_2,l_2^*,k_2)\neq(0,1,1,0).$$

Suppose $B_2=0$, $l_2=1$, and $l_2^*,k_2=0$. For good choices of ν (still sufficiently generic), the restriction of (6.9) to $\mathcal{S}_{\mathbf{h}}^*$ then factors as

$$\mathcal{S}_{\mathbf{h}}^*\longrightarrow \mathcal{Z}^{\star}_{k+1,l-1;\mathbf{h}}(B;J,\nu_1)\times\mathfrak{M}_{1,1}(0;J,0)\longrightarrow X_{k,l-1-l^*}\times X^{\phi}\longrightarrow X_{k,l-l^*}\,.$$

Thus, $\mathrm{ev}_{k,l;\mathbf{h}}(\mathcal{S}^*_{\mathbf{h}})$ is contained in a smooth manifold of dimension $4(l-l^*)+2k-2$. Suppose $B_2=0$, $l_2,l_2^*=0$, and $k_2=2$. For good choices of ν, the restriction of (6.9) to $\mathcal{S}^*_{\mathbf{h}}$ then factors as

$$\mathcal{S}^*_{\mathbf{h}} \longrightarrow \mathcal{Z}^{\star}_{k-1,l;\mathbf{h}}(B;J,\nu_1)\times\mathfrak{M}_{3,0}(0;J,0) \longrightarrow X_{k-2,l-l^*}\times\Delta_X^{\phi} \longrightarrow X_{k,l-l^*}\,,$$

where $\Delta_X^{\phi}\subset(X^{\phi})^2$ is the diagonal. Thus, $\mathrm{ev}_{k,l;\mathbf{h}}(\mathcal{S}^*_{\mathbf{h}})$ is again contained in a smooth manifold of dimension $4(l-l^*)+2k-2$.

We can thus assume that either $B_i\neq 0$ or $2l_i+k_i\geqslant 3$ for $i=1,2$. For good choices of ν, the restriction of (6.9) to $\mathcal{S}^*_{\mathbf{h}}$ then factors as

$$\begin{array}{ccccc}
\mathcal{S}^*_{\mathbf{h}} \longrightarrow & \mathcal{Z}^{\star}_{k_1,l_1;\mathbf{h}_1}(B_1;J,\nu_1') & \times & \mathcal{Z}^{\star}_{k_2,l_2;\mathbf{h}_2}(B_2;J,\nu_2') & \\
& \Big\downarrow{\scriptstyle \mathrm{ev}_{k_1,l_1;\mathbf{h}_1}} & & \Big\downarrow{\scriptstyle \mathrm{ev}_{k_2,l_2;\mathbf{h}_2}} & \\
& X_{k_1,l_1-l_1^*} & \times & X_{k_2,l_2-l_2^*} \longrightarrow & X_{k,l-l^*}.
\end{array}$$

Thus, $\mathrm{ev}_{k,l;\mathbf{h}}(\mathcal{S}^*_{\mathbf{h}})$ is covered by a smooth map from a manifold of dimension

$$\begin{aligned}
\dim\mathcal{Z}^{\star}_{k_i,l_i;\mathbf{h}_i}(B_i;J,\nu_i)+\dim X_{k_{3-i},l_{3-i}-l^*_{3-i}} &= \ell_\omega(B_i)+2(l_i-l_i^*)+k_i+4(l_{3-i}-l^*_{3-i})+2k_{3-i}\\
&= 4(l-l^*)+2k+\big(\ell_\omega(B_i)-2(l_i-l_i^*)-k_i\big)
\end{aligned}$$

for $i=1,2$. Since

$$\big(\ell_\omega(B_1)-2(l_1-l_1^*)-k_1\big)+\big(\ell_\omega(B_2)-2(l_2-l_2^*)-k_2\big)=\ell_\omega(B)-1-2(l-l^*)-k=-1,$$

it follows that $\mathrm{ev}_{k,l;\mathbf{h}}(\mathcal{S}^*_{\mathbf{h}})$ is covered by a smooth map from a manifold of dimension (6.15) unless $\epsilon_{l^*}(\mathcal{S})$ is either 0 or 1. Along with the previous paragraph, this confirms the claim at the end of the paragraph containing (6.15).

It remains to establish (5.14). We can assume that $B\neq 0$ and can be represented by a J-holomorphic map; thus, $\langle\omega,B\rangle\neq 0$. Let $H\in H^2(X;\mathbb{Z})$ be such $\phi^*H=-H$ and $\langle H,B\rangle\neq 0$; such a class H can be obtained by slightly deforming ω so that it represents a rational class, taking a multiple of the deformed class that represents an integral class, and then taking the anti-invariant part of the multiple. Let h_1 and h_2 be two pseudocycles as in the statement of the proposition representing the Poincare dual of H. By definition,

$$N^{\phi;\mathfrak{p}}_{B,l-l^*}=\frac{1}{\langle H,B\rangle^2}\deg\big(\mathrm{ev}_{k,l-l^*+2;(h_1,h_2)},\mathfrak{o}_{\mathfrak{p};2}\mathfrak{o}_{(h_1,h_2)}\big)\,.$$

An implicit implication of a similar definition in [24, Section 4] is that $N^{\phi;\mathfrak{p}}_{B,l-l^*}$ does not depend on the choices of H, h_1, and h_2. This follows from (6.16) below, which also implies (5.14).

Let $k,l,l^*,B,\mathbf{h}$ be as in the statement of the proposition and $h'\colon H'\longrightarrow X$ be another codimension 2 pseudocycle in general position. We denote by $\mathbf{h}h'$ the tuple $(h_1,\ldots,h_{l^*},h')$ and show below that

$$\deg\big(\mathrm{ev}_{k,l+1;\mathbf{h}h'},\mathfrak{o}_{\mathfrak{p};l^*+1}\mathfrak{o}_{\mathbf{h}h'}\big)=\big(h'\cdot_X B\big)\deg\big(\mathrm{ev}_{k,l;\mathbf{h}},\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}\big),\tag{6.16}$$

with $\mathrm{ev}_{k,l;\mathbf{h}}$ as in (5.10) and

$$\mathrm{ev}_{k,l+1;\mathbf{h}h'}\colon \mathcal{Z}^{\bigstar}_{k,l+1;\mathbf{h}h'}\big(B;J,\mathfrak{f}^*_{k,l+1;l^*+1}\nu\big) \longrightarrow X_{k,(l+1)-(l^*+1)} = X_{k,l-l^*}\,.$$

The second forgetful morphism in (5.3) with (l,i) replaced by $(l+1,l^*+1)$ induces a morphism

$$\mathfrak{f}\colon \mathcal{Z}^{\bigstar}_{k,l+1;\mathbf{h}h'}\big(B;J,\mathfrak{f}^*_{k,l+1;l^*+1}\nu\big) \longrightarrow \mathcal{Z}^{\bigstar}_{k,l;\mathbf{h}}\big(B;J,\nu\big)$$

so that $\mathrm{ev}_{k,l+1;\mathbf{h}h'} = \mathrm{ev}_{k,l;\mathbf{h}}\circ\mathfrak{f}$. The relative orientations $\mathfrak{o}_{\mathfrak{p};l^*+1}\mathfrak{o}_{\mathbf{h}h'}$ of $\mathrm{ev}_{k,l+1;\mathbf{h}h'}$ and $\mathfrak{o}_{\mathfrak{p};l^*}\mathfrak{o}_{\mathbf{h}}$ of $\mathrm{ev}_{k,l;\mathbf{h}}$ determine a relative orientation $\mathfrak{o}_{\mathfrak{p}}$ of $\mathfrak{f}$. The number of the preimages

$$\widetilde{\mathbf{u}} \equiv \big(\mathbf{u},(z^+_{l^*+1},z^-_{l^*+1}),y,y'\big)$$

of a generic point

$$(\mathbf{u},y) \in \mathcal{Z}^{\bigstar}_{k,l;\mathbf{h}}(B;J,\nu) \cap \big(\mathfrak{M}_{k,l}(B;J,\nu)\times M_{\mathbf{h}}\big)$$

under $\mathfrak{f}$ is finite. For such a preimage $\widetilde{\mathbf{u}}$, $\mathrm{d}_{\widetilde{\mathbf{u}}}\mathfrak{f}$ is an isomorphism. With $\mathbf{u}$ as in (5.1), the homomorphism

$$T_{z^+_{l^*+1}}\mathbb{P}^1 \oplus T_{y'}H' \longrightarrow T_{u(z^+_{l^*+1})}X = T_{h'(y')}X, \quad (v,w) \longrightarrow \mathrm{d}_{z^+_{l^*+1}}u(v) + \mathrm{d}_{y'}h'(w), \qquad (6.17)$$

is an isomorphism. Its domain and target are oriented by the complex orientation of $\mathbb{P}^1$ (i.e. the vertical orientation $\mathfrak{o}^+_{l^*+1}$ in the notation of Lemma 5.1($\mathfrak{o}_{\mathfrak{p}}$2)), the given orientation $\mathfrak{o}_{h'}$ of H', and the symplectic orientation $\mathfrak{o}_\omega$ of X. We set $\mathfrak{s}_{\widetilde{\mathbf{u}}}$ to be $+1$ if this isomorphism is orientation-preserving and to be -1 if it is orientation-reversing. We show below that $\mathfrak{s}_{\widetilde{\mathbf{u}}}(\mathfrak{o}_{\mathfrak{p}}) = \mathfrak{s}_{\widetilde{\mathbf{u}}}$. Since

$$\sum_{\widetilde{\mathbf{u}}\in\mathfrak{f}^{-1}(\mathbf{u},y)} \mathfrak{s}_{\widetilde{\mathbf{u}}} = h'\cdot_X B, \qquad (6.18)$$

the desired identity (6.16) then follows from (3.6).

Let $\Delta \subset X^2$ and $\Delta^{l^*} \subset (X^{l^*})^2$ denote the diagonals. The orientation $\mathfrak{o}_\omega$ of X induces an orientation $\mathfrak{o}_\Delta$ on the normal bundle $\mathcal{N}\Delta$ of Δ and an orientation $\mathfrak{o}^{l^*}_\Delta$ on the normal bundle $\mathcal{N}\Delta^{l^*}$ of Δ^{l^*}. Define

$$\begin{aligned} \mathcal{Z}_{\mathbf{h}} &= \mathcal{Z}^{\bigstar}_{k,l;\mathbf{h}}\big(B;J,\nu\big), & \mathfrak{M}_l &= \mathfrak{M}_{k,l}\big(B;J,\nu\big),\\ \mathcal{Z}_{\mathbf{h}h'} &= \mathcal{Z}^{\bigstar}_{k,l+1;\mathbf{h}h'}\big(B;J,\mathfrak{f}^*_{k,l+1;l^*+1}\nu\big), & \mathfrak{M}_{l+1} &= \mathfrak{M}_{k,l+1}\big(B;J,\mathfrak{f}^*_{k,l+1;l^*+1}\nu\big). \end{aligned}$$

Let $(\mathbf{u},y)$ and $\widetilde{\mathbf{u}}$ be as above. Fix an orientation $\mathfrak{o}$ on $T_{\mathrm{ev}(\mathbf{u})}X_{k,l}$. The differentials of the obvious maps induce a commutative square in Figure 9 with exact rows and columns. Since the dimensions of X and H' are even, the sign $\mathfrak{s}_{\widetilde{\mathbf{u}}}$ of (6.17) is the sign of the isomorphism in the left column with respect to the orientations $\mathfrak{o}^+_{l^*+1}$, $\mathfrak{o}'_h$, and $\mathfrak{o}_\Delta$. Along with the relative orientation $\mathfrak{o}_{\mathfrak{p};l^*}$ (resp. $\mathfrak{o}_{\mathfrak{p};l^*+1}$) and the orientation $\mathfrak{o}_{\mathbf{h}}$ (resp. $\mathfrak{o}_{\mathbf{h}h'}$), $\mathfrak{o}$ induces an orientation $\mathfrak{o}_{(\mathbf{u},y)}$ on $T_{(\mathbf{u},y)}(\mathfrak{M}_l\times M_{\mathbf{h}})$ (resp. $\mathfrak{o}_{\widetilde{\mathbf{u}}}$ on $T_{\widetilde{\mathbf{u}}}(\mathfrak{M}_{l+1}\times M_{\mathbf{h}h'})$). Since the dimension of H' is even, Lemma 5.1($\mathfrak{o}_{\mathfrak{p}}$2) implies that the middle row respects the orientations. Along with the orientation $\mathfrak{o}^{l^*}_\Delta$ on $\mathcal{N}\Delta^{l^*}$ (resp. $\mathfrak{o}^{l^*+1}_\Delta$ on $\mathcal{N}\Delta^{l^*+1}$), $\mathfrak{o}_{(\mathbf{u},y)}$ (resp. $\mathfrak{o}_{\widetilde{\mathbf{u}}}$) induces an orientation $\mathfrak{o}'_{(\mathbf{u},y)}$ on $T_{(\mathbf{u},y)}\mathcal{Z}_{\mathbf{h}}$ (resp. $\mathfrak{o}'_{\widetilde{\mathbf{u}}}$ on $T_{\widetilde{\mathbf{u}}}\mathcal{Z}_{\mathbf{h}h'}$) so that the right (resp. middle) column of the diagram respects the orientations. The bottom row respects the orientations. Lemma 6.3 then implies that $\mathrm{d}_{\widetilde{u}}\mathfrak{f}$ is orientation-preserving with respect to $\mathfrak{o}'_{\widetilde{\mathbf{u}}}$ and $\mathfrak{o}'_{(\mathbf{u},y)}$ if and only if the isomorphism in the left column is. The latter is the case if and only if $\mathfrak{s}_{\widetilde{\mathbf{u}}} = +1$. These two statements imply that $\mathfrak{s}_{\widetilde{\mathbf{u}}}(\mathfrak{o}_{\mathfrak{p}}) = \mathfrak{s}_{\widetilde{\mathbf{u}}}$. □

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & T_{\tilde{\mathbf{u}}}\mathcal{Z}_{\mathbf{h}h'} & \xrightarrow{\mathrm{d}_{\tilde{u}}\mathfrak{f}} & T_{(\mathbf{u},y)}\mathcal{Z}_{\mathbf{h}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{z^+_{l^*+1}}\mathbb{P}^1 \oplus T_{y'}H' & \longrightarrow & T_{\tilde{\mathbf{u}}}(\mathfrak{M}_{l+1}\times M_{\mathbf{h}h'}) & \longrightarrow & T_{(\mathbf{u},y)}(\mathfrak{M}_l \times M_{\mathbf{h}}) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathcal{N}\Delta & \longrightarrow & \mathcal{N}\Delta^{l^*+1} & \longrightarrow & \mathcal{N}\Delta^{l^*} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

Figure 9: Commutative square of vector spaces with exact rows and columns for the proof of (5.14).

6.3 Proof of Proposition 5.7

For $k', l' \in \mathbb{Z}^{\geqslant 0}$ with $k'+2l' \leqslant 2$, we denote by $\mathcal{H}^{\omega,\phi}_{k',l'}$ the set of pairs $(J,0)$ with $J \in \mathcal{J}^{\phi}_{\omega}$. We continue with the notation in the statement of this proposition and just above. Let k_r, l_r, l^*_r, B_r be as in (5.41) and

$$\mathfrak{M}^{\star} = \mathfrak{M}^{\star}_{k,l;l^*}(B; J, \nu).$$

Since $(\mathcal{S}, \Upsilon)$ is admissible, $k_1+2l_1 \geqslant 3$ and either $k_2 \geqslant 1$ or $l_2 \geqslant 1$. We assume that there exist $\nu'_1 \in \mathcal{H}^{\omega,\phi}_{k_1,l_1}$ and $\nu_2 \in \mathcal{H}^{\omega,\phi}_{k_2+1,l_2}$ so that every substratum $\mathcal{S}_* \subset \mathcal{S}$ admits an embedding as in (6.2) with $\nu_1 = \mathfrak{f}^*_{k_1+1,l_1;\mathrm{nd}}\nu'_1$ subject to the conditions specified below (6.3) and above Lemma 6.2.

We first assume that $l^*_2 \neq 0$ and take $i^* \in L^*_2(\mathcal{S})$ to be the smallest element of this set. By this assumption, the image of $\mathcal{S}$ under the forgetful morphism $\mathfrak{f}_{k,l}$ is contained in a codimension 1 stratum $\mathcal{S}^{\vee}$ of $\overline{\mathcal{M}}^{\tau}_{k,l}$. By Lemma 5.5, we can assume that the orientation $\mathfrak{o}^c_{\mathcal{S}}$ of $\mathcal{N}\mathcal{S}$ used to define the relative orientation $\partial\mathfrak{o}_{\mathfrak{p};l^*} \equiv \partial_{\mathfrak{o}^c_{\mathcal{S}}}\mathfrak{o}_{\mathfrak{p};l^*}$ of (5.22) is $\mathfrak{o}^{c;+}_{\mathcal{S}}$ in the notation of Lemma 6.2.

For $\mathbf{u} \in \mathcal{S}$, let

$$\begin{gathered}
\mathbf{u}_1 \in \mathfrak{M}_1 \equiv \mathfrak{M}_{k_1+1,l_1}\big(B_1; J, \nu_1\big), \qquad \mathbf{u}'_1 \in \mathfrak{M}'_1 \equiv \mathfrak{M}_{k_1,l_1}\big(B_1; J, \nu'_1\big), \\
\mathbf{u}_2 \in \mathfrak{M}_2 \equiv \mathfrak{M}_{k_2+1,l_2}\big(B_2; J, \nu_2\big), \qquad \mathrm{nd} \in \mathbb{P}^1_1, \mathbb{P}^1_2, \quad S^1_1 \subset \mathbb{P}^1_1, \\
D^{\phi}_{\mathbf{u}} = D^{\phi}_{J,\nu;\mathbf{u}}, \qquad D^{\phi}_{\mathbf{u}_1} = D^{\phi}_{J,\nu_1;\mathbf{u}_1} = D^{\phi}_{J,\nu'_1;\mathbf{u}'_1}, \qquad D^{\phi}_{\mathbf{u}_2} = D^{\phi}_{J,\nu_2;\mathbf{u}_2}
\end{gathered}$$

be as above Lemma 6.2 and in Section 4.2. We denote by

$$\begin{gathered}
\mathcal{C} \equiv \mathfrak{f}_{k,l}(\mathbf{u}) \in \mathcal{S}^{\vee} \subset \overline{\mathcal{M}} \equiv \overline{\mathcal{M}}^{\tau}_{k,l}, \qquad \mathcal{C}_1 \equiv \mathfrak{f}_{k_1+1,l_1}(\mathbf{u}_1) \in \mathcal{M}_1 \equiv \mathcal{M}^{\tau}_{k_1+1,l_1}, \\
\mathcal{C}'_1 \equiv \mathfrak{f}_{k_1,l_1}(\mathbf{u}'_1) \in \mathcal{M}'_1 \equiv \mathcal{M}^{\tau}_{k_1,l_1}, \qquad \mathcal{C}_2 \equiv \mathfrak{f}_{k_2+1,l_2}(\mathbf{u}_2) \in \mathcal{M}_2 \equiv \mathcal{M}^{\tau}_{k_2+1,l_2}
\end{gathered}$$

the marked domains of the maps $\mathbf{u}$, $\mathbf{u}_1$, $\mathbf{u}'_1$, and $\mathbf{u}_2$, respectively.

The exact sequence

$$0 \longrightarrow T_{\mathbf{u}}\mathcal{S} \longrightarrow T_{\mathbf{u}_1}\mathfrak{M}_1 \oplus T_{\mathbf{u}_2}\mathfrak{M}_2 \longrightarrow T_{u(\mathrm{nd})}X^{\phi} \longrightarrow 0, \quad \big(\xi_1,\xi_2\big) \longrightarrow \xi_2(\mathrm{nd}) - \xi_1(\mathrm{nd}), \tag{6.19}$$

of vector spaces determines an isomorphism

$$\lambda_{\mathbf{u}}(\mathcal{S}) \otimes \lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big) \approx \lambda_{\mathbf{u}_1}(\mathfrak{M}_1) \otimes \lambda_{\mathbf{u}_2}(\mathfrak{M}_2). \tag{6.20}$$

Since $\epsilon_{l^*}(\mathcal{S}) \in 2\mathbb{Z}$, the Pin^--structure $\mathfrak{p}$ on X^{ϕ} determines homotopy classes $\mathfrak{o}_{\mathfrak{p};l_1^*}$ and $\mathfrak{o}_{\mathfrak{p};l_2^*}$ of isomorphisms

$$\lambda_{\mathbf{u}_1'}(\mathfrak{M}_1') \longrightarrow \lambda^{\mathbb{R}}_{\mathbf{u}_1'}(X) \otimes \lambda^{\mathbb{C}}_{\mathbf{u}_1'}(X) \quad \text{and} \quad \lambda_{\mathbf{u}_2}(\mathfrak{M}_2) \longrightarrow \lambda^{\mathbb{R}}_{\mathbf{u}_2}(X) \otimes \lambda^{\mathbb{C}}_{\mathbf{u}_2}(X), \tag{6.21}$$

respectively; see Lemma 5.1. Combining the first homotopy class of isomorphisms above with the first S^1-fibration in (6.3) and the orientation $\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}$ on its vertical tangent bundle $T_{\mathbf{u}_1}\mathfrak{M}_1^v = T_{\mathrm{nd}}S^1_1$, we obtain a homotopy class $\widetilde{\mathfrak{o}}_{\mathfrak{p};l_1^*} \equiv \mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}\mathfrak{o}_{\mathfrak{p};l_1^*}$ of isomorphisms

$$\lambda_{\mathbf{u}_1}(\mathfrak{M}_1) \approx \lambda_{\mathbf{u}_1'}(\mathfrak{M}_1') \otimes T_{\mathrm{nd}}S^1_1 \approx \lambda^{\mathbb{R}}_{\mathbf{u}_1'}(X) \otimes \lambda^{\mathbb{C}}_{\mathbf{u}_1'}(X)\,. \tag{6.22}$$

Along with (6.20) and the second homotopy class of isomorphisms in (6.21), it determines a homotopy class of isomorphisms

$$\begin{aligned} \lambda_{\mathbf{u}}(\mathcal{S}) \otimes \lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big) &\approx \lambda^{\mathbb{R}}_{\mathbf{u}_1'}(X) \otimes \lambda^{\mathbb{C}}_{\mathbf{u}_1'}(X) \otimes \lambda^{\mathbb{R}}_{\mathbf{u}_2}(X) \otimes \lambda^{\mathbb{C}}_{\mathbf{u}_2}(X) \\ &\approx \lambda^{\mathbb{R}}_{\mathbf{u}_1'}(X) \otimes \lambda^{\mathbb{C}}_{\mathbf{u}_1'}(X) \otimes \lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big) \otimes \lambda^{\mathbb{R}}_{\mathbf{u}_2'}(X) \otimes \lambda^{\mathbb{C}}_{\mathbf{u}_2'}(X) \\ &\approx \lambda^{\mathbb{R}}_{\mathbf{u}}(X) \otimes \lambda^{\mathbb{C}}_{\mathbf{u}}(X) \otimes \lambda\big(T_{u(\mathrm{nd})}X^{\phi}\big)\,. \end{aligned} \tag{6.23}$$

We denote by $\mathfrak{o}^{\mathcal{S}}_{\mathfrak{p};l^*;\mathbf{u}}$ the homotopy class of isomorphisms

$$\lambda_{\mathbf{u}}(\mathcal{S}) \longrightarrow \lambda^{\mathbb{R}}_{\mathbf{u}}(X) \otimes \lambda^{\mathbb{C}}_{\mathbf{u}}(X)$$

determined by (6.23). The next lemma is deduced from Lemmas 4.3 and 6.2 at the end of this section.

Lemma 6.4. *The orientations $\partial\mathfrak{o}_{\mathfrak{p};l^*}$ and $\mathfrak{o}^{\mathcal{S}}_{\mathfrak{p};l^*}$ of $\lambda(\mathrm{ev}|_{\mathcal{S}})$ are opposite.*

We take $\mathbf{h}_1$ and $\mathbf{h}_2$ to be the components of $\mathbf{h}$ as in the proof of Proposition 5.2 and

$$\mathbf{p}_1 \in X_{k_1,l_1-l_1^*} \qquad \text{and} \qquad \mathbf{p}_2 \in X_{k_2,l_2-l_2^*}$$

to be the components of $\mathbf{p} \in X_{k,l-l^*}$ defined analogously. Let

$$\begin{gathered} \mathcal{Z}_1 = \mathcal{Z}^{\star}_{k_1+1,l_1;\mathbf{h}_1}(B_1; J,\nu_1) \cap \big(\mathfrak{M}_1 \times M_{\mathbf{h}_1}\big), \quad \mathcal{Z}_1' = \mathcal{Z}^{\star}_{k_1,l_1;\mathbf{h}_1}(B_1; J,\nu_1') \cap \big(\mathfrak{M}_1' \times M_{\mathbf{h}_1}\big), \\ \mathcal{Z}_2 = \mathcal{Z}^{\star}_{k_2+1,l_2;\mathbf{h}_2}(B_2; J,\nu_2) \cap \big(\mathfrak{M}_2 \times M_{\mathbf{h}_2}\big). \end{gathered}$$

The first forgetful morphism in (6.3) induces a fibration $\mathfrak{f}_{\mathcal{Z}_1}$ so that the diagram

$$\begin{array}{ccc} \mathcal{Z}_1 & \xrightarrow{\ \mathfrak{f}_{\mathcal{Z}_1}\ } & \mathcal{Z}_1' \\ {\scriptstyle \pi_{\mathcal{Z}}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\mathcal{Z}'}} \\ \mathfrak{M}_1 & \xrightarrow{\ \mathfrak{f}_{\mathrm{nd}}\ } & \mathfrak{M}_1' \end{array}$$

commutes. Since $\pi_{\mathcal{Z}}$ induces an isomorphism between the vertical tangent bundles $T\mathcal{Z}_1^v$ of $\mathfrak{f}_{\mathcal{Z}_1}$ and $T\mathfrak{M}_1^v$ of $\mathfrak{f}_{\rm nd}$, it pulls back $\mathfrak{o}_{\rm nd}^{\mathbb{R}}$ to an orientation $\mathfrak{o}_{\mathcal{Z}_1}^v$ on the fibers of $\mathfrak{f}_{\mathcal{Z}_1}$. The relative orientations $\mathfrak{o}_{\rm nd}^{\mathbb{R}}\mathfrak{o}_{\mathfrak{p};l_1^*}$, $\mathfrak{o}_{\mathfrak{p};l_1^*}$, and $\mathfrak{o}_{\mathfrak{p};l_2^*}$ on

$$\mathrm{ev}\colon \mathfrak{M}_1 \longrightarrow X_{k_1,l_1}, \quad \mathrm{ev}'\colon \mathfrak{M}_1' \longrightarrow X_{k_1,l_1}, \quad \text{and} \quad \mathrm{ev}\colon \mathfrak{M}_2 \longrightarrow X_{k_2+1,l_2},$$

respectively, the orientations $\mathfrak{o}_{h_i}$ of H_i, and the symplectic orientation $\mathfrak{o}_\omega$ on X determine relative orientations $\widetilde{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}_1}$, $\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1}$, and $\mathfrak{o}_{\mathfrak{p};\mathbf{h}_2}$ of

$$\mathrm{ev}_{\mathbf{h}_1}\colon \mathcal{Z}_1 \longrightarrow X_{k_1,l_1-l_1^*}, \quad \mathrm{ev}_{\mathbf{h}_1}'\colon \mathcal{Z}_1' \longrightarrow X_{k_1,l_1-l_1^*}, \quad \text{and} \quad \mathrm{ev}_{\mathbf{h}_2}\colon \mathcal{Z}_2 \longrightarrow X_{k_2+1,l_2-l_2^*},$$

respectively. Since the dimensions of X and H_i are even,

$$\widetilde{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}_1} = \mathfrak{o}_{\mathcal{Z}_1}^v\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1} \equiv \big(\pi_{\mathcal{Z}}^*\mathfrak{o}_{\rm nd}^{\mathbb{R}}\big)\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1}. \tag{6.24}$$

For $\widetilde{\mathbf{u}}\in\mathcal{S}_{\mathbf{h}}^*$, we denote by

$$\widetilde{\mathbf{u}}_1\in\mathcal{Z}_1, \qquad \widetilde{\mathbf{u}}_1'\in\mathcal{Z}_1', \qquad \widetilde{\mathbf{u}}_2\in\mathcal{Z}_2$$

the images of $\widetilde{\mathbf{u}}$ under the projections induced by the embedding (6.2), the first forgetful morphism in (6.3), and the decomposition

$$M_{\mathbf{h}} \approx M_{\mathbf{h}_1}\times M_{\mathbf{h}_2}.$$

The exact sequence

$$0 \longrightarrow T_{\widetilde{\mathbf{u}}}\mathcal{S}_{\mathbf{h}}^* \longrightarrow T_{\widetilde{\mathbf{u}}_1}\mathcal{Z}_1\oplus T_{\widetilde{\mathbf{u}}_2}\mathcal{Z}_2 \longrightarrow T_{u(\rm nd)}X^\phi \longrightarrow 0, \quad \big(\xi_1,\xi_2\big) \longrightarrow \xi_2(\mathrm{nd})-\xi_1(\mathrm{nd}),$$

of vector spaces determines an isomorphism

$$\lambda_{\widetilde{\mathbf{u}}}(\mathcal{S}_{\mathbf{h}}^*)\otimes\lambda\big(T_{u(\rm nd)}X^\phi\big) \approx \lambda_{\widetilde{\mathbf{u}}_1}(\mathcal{Z}_1)\otimes\lambda_{\widetilde{\mathbf{u}}_2}(\mathcal{Z}_2).$$

Along with the relative orientations $\widetilde{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}_1}$ and $\mathfrak{o}_{\mathfrak{p};\mathbf{h}_2}$ above, this isomorphism determines a homotopy class of isomorphisms

$$\begin{aligned}\lambda_{\widetilde{\mathbf{u}}}(\mathcal{S}_{\mathbf{h}}^*)\otimes\lambda\big(T_{u(\rm nd)}X^\phi\big) &\approx \lambda_{\widetilde{\mathbf{u}}_1'}^{\mathbb{R}}(X)\otimes\lambda_{\widetilde{\mathbf{u}}_1'}^{\mathbb{C}}(X)\otimes\lambda_{\widetilde{\mathbf{u}}_2}^{\mathbb{R}}(X)\otimes\lambda_{\widetilde{\mathbf{u}}_2}^{\mathbb{C}}(X)\\ &\approx \lambda_{\widetilde{\mathbf{u}}_1'}^{\mathbb{R}}(X)\otimes\lambda_{\widetilde{\mathbf{u}}_1'}^{\mathbb{C}}(X)\otimes\lambda\big(T_{u(\rm nd)}X^\phi\big)\otimes\lambda_{\widetilde{\mathbf{u}}_2'}^{\mathbb{R}}(X)\otimes\lambda_{\widetilde{\mathbf{u}}_2'}^{\mathbb{C}}(X)\\ &\approx \lambda_{\widetilde{\mathbf{u}}}^{\mathbb{R}}(X)\otimes\lambda_{\widetilde{\mathbf{u}}}^{\mathbb{C}}(X)\otimes\lambda\big(T_{u(\rm nd)}X^\phi\big).\end{aligned}$$

We denote the associated relative orientation of (5.22) by

$$\mathfrak{o}_{\mathcal{S};\mathbf{h}} \equiv (\widetilde{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}_1})_{\rm nd}\times_{\rm nd}\mathfrak{o}_{\mathfrak{p};\mathbf{h}_2}. \tag{6.25}$$

Since the dimensions of X and H_i are even, Lemma 6.4 implies that

$$\big|\mathcal{S}_{\mathbf{h},\mathbf{p};\Upsilon}^*\big|_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}_\Upsilon^c}^{\pm} = -\big|\mathcal{S}_{\mathbf{h},\mathbf{p};\Upsilon}^*\big|_{\mathfrak{o}_{\mathcal{S};\mathbf{h}},\mathfrak{o}_\Upsilon^c}^{\pm}. \tag{6.26}$$

If $\mathcal{S}$ and Υ satisfy ($\mathcal{S}1\Upsilon$) above Proposition 5.7 with $i\in[k']$ as in ($\mathcal{S}1\Upsilon$), let

$$k_1' = k'-1, \quad l_1' = l', \quad k_2' = 1, \quad l_2' = 0, \quad \Upsilon_1 = \Upsilon, \quad \mathrm{nd} = i.$$

If $\mathcal{S}$ and Υ satisfy ($\mathcal{S}2\Upsilon$) and $S\subset\overline{\mathcal{M}}^{\tau}_{k',l'}$ as in ($\mathcal{S}2\Upsilon$), let

$$k_1'=k_1(S),\quad l_1'=l_1(S),\quad k_2'=k_2(S),\quad l_2'=l_2(S)$$

and denote by

$$\pi_1\colon S\approx\overline{\mathcal{M}}^{\tau}_{k_1'+1,l_1'}\times\overline{\mathcal{M}}^{\tau}_{k_2'+1,l_2'}\longrightarrow\overline{\mathcal{M}}^{\tau}_{k_1'+1,l_1'}$$

the projection to the first component in the second identification in (4.7). In this case,

$$\Upsilon\cap\overline{S}\approx\Upsilon_1\times\overline{\mathcal{M}}^{\tau}_{k_2'+1,l_2'}\subset\overline{\mathcal{M}}^{\tau}_{k_1'+1,l_1'}\times\overline{\mathcal{M}}^{\tau}_{k_2'+1,l_2'}$$

for some $\Upsilon_1\subset\overline{\mathcal{M}}^{\tau;\bigstar}_{k_1'+1,l_1';\mathrm{nd}}$. The co-orientation $\mathfrak{o}^c_{\Upsilon\cap S}$ on $\Upsilon\cap\overline{S}$ in $\overline{S}$ induced by $\mathfrak{o}^c_{\Upsilon}$ is the pullback by π_1 of a co-orientation $\mathfrak{o}^c_{\Upsilon_1}$ on Υ_1 in $\overline{\mathcal{M}}^{\tau}_{k_1'+1,l_1'}$. Let

$$\mathfrak{f}_{k_1'+1,l_1'}=\pi_1\circ\mathfrak{f}_{k',l'}\colon\mathcal{S}^*_{\mathbf{h}}\longrightarrow S\longrightarrow\overline{\mathcal{M}}^{\tau}_{k_1'+1,l_1'}\,.$$

In both cases,

$$\dim\Upsilon_1=\dim\Upsilon+1-k_2'-2l_2' \tag{6.27}$$

and the forgetful morphism $\mathfrak{f}_{k_1'+1,l_1'}$ factors as

$$\mathcal{S}^*_{\mathbf{h}}\hookrightarrow\mathcal{Z}_1\times\mathcal{Z}_2\longrightarrow\mathcal{Z}_1\xrightarrow{\mathfrak{f}_{k_1'+1,l_1'}}\overline{\mathcal{M}}^{\tau}_{k_1'+1,l_1'}\,.$$

We define

$$\mathfrak{f}_{\mathcal{M}}=\mathfrak{f}_{k_1'+1,l_1';\mathrm{nd}}\colon\overline{\mathcal{M}}^{\tau}_{k_1'+1,l_1'}\longrightarrow\overline{\mathcal{M}}^{\tau}_{k_1',l_1'}\,.$$

If $\mathcal{S}$ and Υ satisfy ($\mathcal{S}2\Upsilon$), (2) and (3) in Lemma 3.3 give

$$\begin{aligned}\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\mathfrak{o}_{\mathcal{S};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}&=-\big|M_{(\mathrm{ev}_{\mathcal{S};\mathbf{h}},\mathfrak{f}_{k',l'}),f_{\mathbf{p}};\Upsilon|_{\Upsilon\cap\overline{S}}}\big|^{\pm}_{\mathfrak{o}_{\mathcal{S};\mathbf{h}},\pi_1^*\mathfrak{o}^c_{\Upsilon_1}|_{\Upsilon\cap\overline{S}}}\\&=-(-1)^{k_2'}\big|M_{(\mathrm{ev}_{\mathcal{S};\mathbf{h}},\mathfrak{f}_{k_1'+1,l_1'}),f_{\mathbf{p}};\Upsilon_1}\big|^{\pm}_{\mathfrak{o}_{\mathcal{S};\mathbf{h}},\mathfrak{o}^c_{\Upsilon_1}};\end{aligned} \tag{6.28}$$

the signed fiber products in the second and third expressions above are taken with respect to $X_{k,l-l^*}\times S$ and $X_{k,l-l^*}\times\overline{\mathcal{M}}^{\tau}_{k_1'+1,l_1'}$, respectively. The first and last expressions in (6.28) are the same if $\mathcal{S}$ and Υ satisfy ($\mathcal{S}1\Upsilon$). By (6.25) and Lemma 3.4,

$$\big|M_{(\mathrm{ev}_{\mathcal{S};\mathbf{h}},\mathfrak{f}_{k_1'+1,l_1'}),f_{\mathbf{p}};\Upsilon_1}\big|^{\pm}_{\mathfrak{o}_{\mathcal{S};\mathbf{h}},\mathfrak{o}^c_{\Upsilon_1}}=\big|M_{(\mathrm{ev}_{\mathbf{h}_1},\mathfrak{f}_{k_1'+1,l_1'}),f_{\mathbf{p}_1};\Upsilon_1}\big|^{\pm}_{\widetilde{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}_1},\mathfrak{o}^c_{\Upsilon_1}}\deg\big(\mathrm{ev}_{\mathbf{h}_2},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_2}\big). \tag{6.29}$$

By Lemma 3.3(1),

$$\big|M_{(\mathrm{ev}_{\mathbf{h}_1},\mathfrak{f}_{k_1'+1,l_1'}),f_{\mathbf{p}_1};\Upsilon_1}\big|^{\pm}_{\widetilde{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}_1},\mathfrak{o}^c_{\Upsilon_1}}=(-1)^{\dim\Upsilon_1}\deg\big(\mathrm{ev}_{\mathbf{h}_1}|_{\mathfrak{f}^{-1}_{k_1'+1,l_1'}(\Upsilon_1)},(\mathfrak{f}^*_{k_1'+1,l_1'}\mathfrak{o}^c_{\Upsilon_1})\widetilde{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}_1}\big)\,.$$

By the first identity in (6.24) and (3.6),

$$\deg\big(\mathrm{ev}_{\mathbf{h}_1}|_{\mathfrak{f}^{-1}_{k_1'+1,l_1'}(\Upsilon_1)},(\mathfrak{f}^*_{k_1'+1,l_1'}\mathfrak{o}^c_{\Upsilon_1})\widetilde{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}_1}\big)=\deg\big(\mathfrak{f}_{\mathcal{Z}_1}|_{\mathfrak{f}^{-1}_{k_1'+1,l_1'}(\Upsilon_1)},(\mathfrak{f}^*_{k_1'+1,l_1'}\mathfrak{o}^c_{\Upsilon_1})\mathfrak{o}^v_{\mathcal{Z}_1}\big)\deg\big(\mathrm{ev}'_{\mathbf{h}_1},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1}\big).$$

By the second identity in (6.24) and Lemma 3.2,

$$\mathfrak{s}_{\tilde{\mathbf{u}}}\big((\mathfrak{f}^*_{k_1'+1,l_1'}\mathfrak{o}^c_{\Upsilon_1})\mathfrak{o}^v_{\mathcal{Z}_1}\big)=\mathfrak{s}_{\tilde{\mathbf{u}}}\big(\mathfrak{f}_{k_1'+1,l_1'},\pi^*_{\mathcal{Z}}\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}},\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}\big)\mathfrak{s}_{\mathfrak{f}_{k_1'+1,l_1'}(\tilde{\mathbf{u}})}\big(\mathfrak{o}^c_{\Upsilon_1}\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}\big)=\mathfrak{s}_{\mathfrak{f}_{k_1'+1,l_1'}(\tilde{\mathbf{u}})}\big(\mathfrak{o}^c_{\Upsilon_1}\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}\big)$$

for a generic $\tilde{\mathbf{u}}\in\mathfrak{f}^{-1}_{k_1'+1,l_1'}(\Upsilon_1)$.

Combining the last three equations with (6.27) and Lemma 3.1(2), we obtain

$$\begin{aligned}\big|M_{(\mathrm{ev}_{\mathbf{h}_1},\mathfrak{f}_{k_1'+1,l_1'}),f_{\mathbf{p}_1};\Upsilon_1}\big|^{\pm}_{\tilde{\mathfrak{o}}_{\mathfrak{p};\mathbf{h}_1},\mathfrak{o}^c_{\Upsilon_1}}&=-(-1)^{\dim\Upsilon+k_2'}\deg\big(\mathfrak{f}_{\mathcal{M}}|\Upsilon_1,\mathfrak{o}^c_{\Upsilon_1}\mathfrak{o}^{\mathbb{R}}_{\mathrm{nd}}\big)\deg\big(\mathrm{ev}'_{\mathbf{h}_1},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1}\big)\\&=-(-1)^{\dim\Upsilon+k_2'}\deg_S\big(\Upsilon,\mathfrak{o}^c_{\Upsilon}\big)\deg\big(\mathrm{ev}'_{\mathbf{h}_1},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1}\big).\end{aligned}$$

Along with (6.28) and (6.29), this gives

$$\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\mathfrak{o}_{\mathcal{S};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}=(-1)^{\dim\Upsilon}\deg_S\big(\Upsilon,\mathfrak{o}^c_{\Upsilon}\big)\deg\big(\mathrm{ev}'_{\mathbf{h}_1},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1}\big)\deg\big(\mathrm{ev}_{\mathbf{h}_2},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_2}\big)\,.$$

Combining this equation with (6.26) and (5.14), we obtain (5.26).

Suppose $l_2^*=0$. Choose another codimension 2 pseudocycle $h':H'\longrightarrow X$ in general position with $h'\cdot_X B_2\neq0$. Let

$$\mathcal{S}^*_{\mathbf{h}h'}\subset\mathcal{Z}^{\star}_{k,l+1;\mathbf{h}h'}\big(B;J,\mathfrak{f}^*_{k,l+1;l^*+1}\nu\big)$$

be the codimension 1 stratum so that $\mathfrak{f}_{k,l+1;l^*+1}(\mathcal{S}^*_{\mathbf{h}h'})=\mathcal{S}^*_{\mathbf{h}}$ and the irreducible component $\mathbb{P}^1_2$ of the maps in this stratum carries the marked points $z^{\pm}_{l^*+1}$. The vertical tangent bundle of the projection

$$\mathfrak{f}_{k,l+1;l^*+1}\colon\mathcal{Z}^{\star}_{k,l+1;\mathbf{h}}\big(B;J,\mathfrak{f}^*_{k,l+1;l^*+1}\nu\big)\longrightarrow\mathcal{Z}^{\star}_{k,l;\mathbf{h}}\big(B;J,\nu\big)$$

is oriented by the position of $z^{\pm}_{l^*+1}$; we denote this orientation by $\mathfrak{o}^+_{l^*}$.

By the $l_2^*\neq0$ case above,

$$\big|\mathcal{S}^*_{\mathbf{h}h',\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}h'},\mathfrak{o}^c_{\Upsilon}}=-(-1)^{\dim\Upsilon}\deg_S\big(\Upsilon,\mathfrak{o}^c_{\Upsilon}\big)\deg\big(\mathrm{ev}'_{\mathbf{h}_1},\mathfrak{o}_{\mathfrak{p};\mathbf{h}_1}\big)\big(h'\cdot_X B_2\big)N^{\phi;\mathfrak{p}}_{B_2,l_2}\,.\tag{6.30}$$

Since the dimensions of X and H' are even,

$$\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}h'}=\partial\big(\mathfrak{o}^+_{l^*}\mathfrak{o}_{\mathfrak{p};\mathbf{h}}\mathfrak{o}_{h'}\big)=\mathfrak{o}^+_{l^*}\big(\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}}\big)\mathfrak{o}_{h'}\,.$$

By the reasoning in the proof of (5.14), this implies that

$$\big|\mathcal{S}^*_{\mathbf{h}h',\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}h'},\mathfrak{o}^c_{\Upsilon}}=\big(h'\cdot_X B_2\big)\big|\mathcal{S}^*_{\mathbf{h},\mathbf{p};\Upsilon}\big|^{\pm}_{\partial\mathfrak{o}_{\mathfrak{p};\mathbf{h}},\mathfrak{o}^c_{\Upsilon}}.\tag{6.31}$$

Combining (6.30) and (6.31) with (5.14), we again obtain (5.26).

Proof of Lemma 6.4. Fix orientations of $T_{u(\mathrm{nd})}X^{\phi}$ and $T_{u(x_i)}X^{\phi}$ for all $i\in[k]$. We can then view all relevant relative orientations as orientations in the usual sense. Let $\mathcal{S}_*\subset\mathcal{S}$ be the substratum containing $\mathbf{u}$.

The differential of the forgetful morphism $\mathfrak{f}_{k,l}$ induces the first exact square of Figure 10. The two spaces in the bottom row are oriented by $\mathfrak{o}^{c;+}_{\mathcal{S}}$ and $\mathfrak{o}^{c;+}_{\mathcal{S}^\vee}$ with the isomorphism between them being orientation-preserving. These orientations and the orientations $\mathfrak{o}_{\mathfrak{p}}^{D}$, $\mathfrak{o}_{\mathfrak{p};l^*}$, and $\mathfrak{o}_{k,l;l^*}$ determine the limiting orientations $\mathfrak{o}_{\mathfrak{p}}^{D;+}$ on $\ker D_{\mathbf{u}}^{\phi}$, $\mathfrak{o}_{\mathfrak{p};l^*}^{+}$ on $T_{\mathbf{u}}\mathfrak{M}^{\star}$, and $\mathfrak{o}_{k,l;l^*}^{+}$ on $T_{\mathcal{C}}\overline{\mathcal{M}}$, respectively. By (6.7), the middle row respects these orientations. The middle (resp. right) column respects the orientations $\partial\mathfrak{o}_{\mathfrak{p};l^*}$ on $T_{\mathbf{u}}\mathcal{S}$, $\mathfrak{o}_{\mathfrak{p};l^*}^{+}$ on $T_{\mathbf{u}}\mathfrak{M}^{\star}$, and $\mathfrak{o}_{\mathcal{S}}^{c;+}$ on $\mathcal{N}_{\mathbf{u}}\mathcal{S}$ (resp. $\mathfrak{o}_{\mathcal{S}^\vee;l^*}^{+}$ on $T_{\mathbf{u}}\mathcal{S}^\vee$, $\mathfrak{o}_{k,l;l^*}^{+}$ on $T_{\mathcal{C}}\overline{\mathcal{M}}$, and $\mathfrak{o}_{\mathcal{S}^\vee}^{c;+}$ on $\mathcal{N}_{\mathbf{u}}\mathcal{S}^\vee$). Lemma 6.3 then implies that the top row in the first exact square of Figure 10 *respects* the orientations $\mathfrak{o}_{\mathfrak{p}}^{D;+}$ on $\ker D_{\mathbf{u}}^{\phi}$, $\partial\mathfrak{o}_{\mathfrak{p};l^*}$ on $T_{\mathbf{u}}\mathcal{S}$, and $\mathfrak{o}_{\mathcal{S}^\vee;l^*}^{+}$ on $T_{\mathbf{u}}\mathcal{S}^\vee$.

The differentials of forgetful morphisms induce the second exact square of Figure 10. The two spaces in the top row are oriented by $\mathfrak{o}_{\mathrm{nd}}^{\mathbb{R}}$ as in Section 4.2 with the isomorphism between them being orientation-preserving. The first real marked point of $\mathbf{u}_1$ is the node, while the second one (if $k_1\neq 0$) is the next marked point on the fixed locus $S_1^1\subset\mathbb{P}_1^1$ in the counterclockwise direction with respect to z_1^+. By ($\mathfrak{o}_{\mathcal{M}}1$) and ($\mathfrak{o}_{\mathcal{M}}4$) in Lemma 4.1, the right column thus does not respect the orientations $\mathfrak{o}_{\mathrm{nd}}^{v}$ on $T_{\mathrm{nd}}S_1^1$, $\mathfrak{o}_{k_1+1,l_1;l_1^*}$ on $T_{\mathcal{C}_1}\mathcal{M}_1$, and $\mathfrak{o}_{k_1,l_1;l_1^*}$ on $T_{\mathcal{C}_1'}\mathcal{M}_1'$ because

$$k_1+\dim\mathcal{M}_1' = 2k_1+2l_1-3\notin 2\mathbb{Z}.$$

By (6.7), the bottom row respects the orientations $\mathfrak{o}_{\mathfrak{p}}^{D}$ on $\ker D_{\mathbf{u}_1'}^{\phi}$, $\mathfrak{o}_{\mathfrak{p};l_1^*}$ on $T_{\mathbf{u}_1'}\mathfrak{M}_1'$, and $\mathfrak{o}_{k_1,l_1;l_1^*}$ on $T_{\mathcal{C}_1'}\mathcal{M}_1'$. By (6.22), the middle column respects the orientations $\mathfrak{o}_{\mathrm{nd}}^{v}$ on $T_{\mathrm{nd}}S_1^1$, $\widetilde{\mathfrak{o}}_{\mathfrak{p};l_1^*}$ on $T_{\mathbf{u}_1}\mathfrak{M}_1$, and $\mathfrak{o}_{\mathfrak{p};l_1^*}$ on $T_{\mathbf{u}_1'}\mathfrak{M}_1'$ because

$$\dim\mathfrak{M}_1' = \ell_\omega(B_1)+2l_1+k_1$$

is even by the assumption that $\epsilon_{l^*}(\mathcal{S})=2$. Lemma 6.3 then implies that the middle row respects the orientations $\mathfrak{o}_{\mathfrak{p}}^{D}$ on $\ker D_{\mathbf{u}_1}^{\phi}$, $\widetilde{\mathfrak{o}}_{\mathfrak{p};l_1^*}$ on $T_{\mathbf{u}_1}\mathfrak{M}_1$, and $\mathfrak{o}_{k_1+1,l_1;l_1^*}$ on $T_{\mathcal{C}_1}\mathcal{M}_1$ if and only if

$$1+\dim\ker D_{\mathbf{u}_1}^{\phi} = 1+2+\langle c_1(X,\omega),B_1\rangle$$

is even. Since $\epsilon_{l^*}(\mathcal{S})=2$, we conclude that the middle row in the second exact square of Figure 10 respects the orientations $\mathfrak{o}_{\mathfrak{p}}^{D}$ on $\ker D_{\mathbf{u}_1}^{\phi}$, $\widetilde{\mathfrak{o}}_{\mathfrak{p};l_1^*}$ on $T_{\mathbf{u}_1}\mathfrak{M}_1$, and $\mathfrak{o}_{k_1+1,l_1;l_1^*}$ on $T_{\mathcal{C}_1}\mathcal{M}_1$ if and only if k_1 is even.

The short exact sequences (6.4) and (6.19) and the differential of the forgetful morphism $\mathfrak{f}_{k,l}$ induce the third exact square of Figure 10. By (6.7), the short exact sequence of the second summands in the middle row respects the orientations $\mathfrak{o}_{\mathfrak{p}}^{D}$ on $\ker D_{\mathbf{u}_2}^{\phi}$, $\mathfrak{o}_{\mathfrak{p};l_2^*}$ on $T_{\mathbf{u}_2}\mathfrak{M}_2$, and $\mathfrak{o}_{k_2+1,l_2;l_2^*}$ on $T_{\mathcal{C}_2}\mathcal{M}_2$. Along with the conclusion of the previous paragraph and Lemma 6.3, this implies that the middle row respects the orientations $\mathfrak{o}_{\mathfrak{p}}^{D}\oplus\mathfrak{o}_{\mathfrak{p}}^{D}$, $\widetilde{\mathfrak{o}}_{\mathfrak{p};l_1^*}\oplus\mathfrak{o}_{\mathfrak{p};l_2^*}$, and $\mathfrak{o}_{k_1+1,l_1;l_1^*}\oplus\mathfrak{o}_{k_2+1,l_2;l_2^*}$ if and only if

$$k_1+\big(\dim\ker D_{\mathbf{u}_2}^{\phi}\big)\big(\dim\mathcal{M}_1\big) = k_1+\big(2+\langle c_1(X,\omega),B_2\rangle\big)\big(k_1+2l_1-2\big)$$

is even. Since $\epsilon_{l^*}(\mathcal{S})=2$, this is the case if and only if $k_1+k_1k_2\in 2\mathbb{Z}$. By Lemma 6.2, the left column respects the orientations $\mathfrak{o}_{\mathfrak{p}}^{D;+}$, $\mathfrak{o}_{\mathfrak{p}}^{D}\oplus\mathfrak{o}_{\mathfrak{p}}^{D}$, and the chosen orientation $\mathfrak{o}_{\mathrm{nd}}^{X}$ on $T_{u(\mathrm{nd})}X^{\phi}$ if and only if $\delta_D^+(\mathcal{S})\cong\delta_{\mathbb{R}}^+(\mathcal{S}_*)$ mod 2. Since $\epsilon_{l^*}(\mathcal{S})=2$, this is the case if and only if the number $k_2+k_1k_2+\delta_{\mathbb{R}}^+(\mathcal{S}_*)$ is even. By Lemma 4.3, the non-trivial isomorphism in the right column respects the orientations $\mathfrak{o}_{\mathcal{S}^\vee;l^*}^{+}$ and $\mathfrak{o}_{k_1+1,l_1;l_1^*}\oplus\mathfrak{o}_{k_2+1,l_2;l_2^*}$ if and only if $\delta_{\mathbb{R}}^+(\mathcal{S}_*)\cong k+1$ mod 2. By (6.23), the middle column respects the orientations $\mathfrak{o}_{\mathfrak{p};l^*}^{\mathcal{S}}$, $\widetilde{\mathfrak{o}}_{\mathfrak{p};l_1^*}\oplus\mathfrak{o}_{\mathfrak{p};l_2^*}$, and $\mathfrak{o}_{\mathrm{nd}}^{X}$.

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker D^{\phi}_{\mathbf{u}} & \longrightarrow & T_{\mathbf{u}}\mathcal{S} & \longrightarrow & T_{\mathcal{C}}\mathcal{S}^{\vee} & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker D^{\phi}_{\mathbf{u}} & \longrightarrow & T_{\mathbf{u}}\mathfrak{M}^{\star} & \longrightarrow & T_{\mathcal{C}}\overline{\mathcal{M}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & \mathcal{N}_{\mathbf{u}}\mathcal{S} & \longrightarrow & \mathcal{N}_{\mathcal{C}}\mathcal{S}^{\vee} & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & 0 & \longrightarrow & T_{\mathrm{nd}}S^1_1 & \longrightarrow & T_{\mathrm{nd}}S^1_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker D^{\phi}_{\mathbf{u}_1} & \longrightarrow & T_{\mathbf{u}_1}\mathfrak{M}_1 & \longrightarrow & T_{\mathcal{C}_1}\mathcal{M}_1 & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker D^{\phi}_{\mathbf{u}_1} & \longrightarrow & T_{\mathbf{u}'_1}\mathfrak{M}'_1 & \longrightarrow & T_{\mathcal{C}'_1}\mathcal{M}'_1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker D^{\phi}_{\mathbf{u}} & \longrightarrow & T_{\mathbf{u}}\mathcal{S} & \longrightarrow & T_{\mathcal{C}}\mathcal{S}^{\vee} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker D^{\phi}_{\mathbf{u}_1} \oplus \ker D^{\phi}_{\mathbf{u}_2} & \longrightarrow & T_{\mathbf{u}_1}\mathfrak{M}_1 \oplus T_{\mathbf{u}_2}\mathfrak{M}_2 & \longrightarrow & T_{\mathcal{C}_1}\mathcal{M}_1 \oplus T_{\mathcal{C}_2}\mathcal{M}_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & T_{u(\mathrm{nd})}X^{\phi} & = & T_{u(\mathrm{nd})}X^{\phi} & \longrightarrow & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

Figure 10: Commutative squares of vector spaces with exact rows and columns for the proof of Lemma 6.4

Combining these statements with Lemma 6.3, we conclude that the top row does *not* respect the orientations $\mathfrak{o}_{\mathfrak{p}}^{D;+}$, $\mathfrak{o}_{\mathfrak{p};l*}^{\mathcal{S}}$, and $\mathfrak{o}_{\mathcal{S}^\vee;l*}^{+}$ because

$$(k_1+k_1k_2)+\big(k_2+k_1k_2+\delta_{\mathbb{R}}^+(\mathcal{S}_*)\big)+\big(k+1+\delta_{\mathbb{R}}^+(\mathcal{S}_*)\big)+\big(\dim\mathcal{S}^\vee\big)\big(\dim X^\phi\big)=1 \mod 2.$$

Comparing this conclusion with the conclusion concerning the top row in the first exact square of Figure 10 above, we obtain the claim. □

6.4 Proof of Proposition 5.3

We denote by $\mathcal{H}_{0,m}^{\omega}$ the space of pairs (J,ν') consisting of $J\in\mathcal{J}_\omega$ and a Ruan-Tian perturbation ν' of the $\bar\partial_J$-equation if $m\geqslant 3$ and take $\mathcal{H}_{0,2}^{\omega}$ to be the set of pairs $(J,0)$ with $J\in\mathcal{J}_\omega$. For $B'\in H_2(X)$ and $\nu'\in\mathcal{H}_{0,m}^{\omega}$, we denote by $\mathfrak{M}_m^{\mathbb{C}}(B';J,\nu')$ the moduli space of (complex) genus 0 degree B' (J,ν')-holomorphic maps from smooth domains with m marked points and by

$$\mathrm{ev}_i:\mathfrak{M}_m^{\mathbb{C}}(B';J,\nu')\longrightarrow X, \qquad i\in[m],$$

the evaluation maps at the marked points. For $I\subset[m]$, let $\mathfrak{o}_{\mathbb{C};I}$ be the orientation of $\mathfrak{M}_m^{\mathbb{C}}(B';J,\nu')$ obtained by twisting the standard complex orientation by $(-1)^{|I|}$. Define

$$\Theta_i^I:X\longrightarrow X, \qquad \Theta_i^I=\begin{cases}\mathrm{id}_X, & \text{if } i\notin I;\\ \phi, & \text{if } i\in I;\end{cases}$$
$$\mathrm{ev}^I:\mathfrak{M}_m^{\mathbb{C}}(B';J,\nu')\longrightarrow X^m, \quad \mathrm{ev}^I(\mathbf{u})=\big(\big(\Theta_i^I(\mathrm{ev}_i(\mathbf{u}))\big)_{i\in[m]}\big).$$

We continue with the notation in the statement of Proposition 5.3 and just above. The co-orientation $\mathfrak{o}_\Gamma^c$ of Γ in $\overline{\mathcal{M}}_{k',l'}^{\tau}$ and the relative orientation $\mathfrak{o}_{\mathfrak{p};l*}$ of Lemma 5.1 induce a relative orientation $(\mathfrak{f}_{k',l'}^*\mathfrak{o}_\Gamma^c)\mathfrak{o}_{\mathfrak{p};l*}$ of the restriction

$$\mathrm{ev}_\Gamma:\mathfrak{M}_{\Gamma;k,l}(B;J,\nu)\longrightarrow X_{k,l}$$

of (5.11).

Fix a stratum $\mathcal{S}\subset\mathfrak{M}_{\Gamma;k,l}(B;J,\nu)$. Let $B_{\mathbb{R}}$ be the degree of the restrictions of the maps in $\mathcal{S}$ to the real component $\mathbb{P}_0^1$ of the domain and $B_{\mathbb{C}}$ be the degree of their restrictions to the component $\mathbb{P}_+^1$ of the domain carrying the marked point z_1^+. Denote by $L_0,L_{\mathbb{C}}\subset[l]$ the subsets indexing the conjugate pairs of marked points carried by $\mathbb{P}_0^1$ and $\mathbb{P}_+^1$, respectively. Let $L_-^*\subset L_{\mathbb{C}}^*$ be the subset indexing the conjugate pairs of marked points (z_i^+,z_i^-) of curves in Γ so that z_i^- lies on $\mathbb{P}_+^1$. Define

$$L_0^*=L_0^*(\Gamma), \quad L_{\mathbb{C}}^*=L_{\mathbb{C}}^*(\Gamma), \quad l_0=|L_0|, \quad l_{\mathbb{C}}=|L_{\mathbb{C}}|, \quad l_0^*=l_0^*(\Gamma), \quad l_{\mathbb{C}}^*\equiv l^*-l_0^*.$$

By (5.19),

$$(l_0-l_0^*)+(l_{\mathbb{C}}-l_{\mathbb{C}}^*)=l-l^*, \quad \big(\ell_\omega(B_{\mathbb{R}})-(k+2(l_0-l_0^*))\big)+2\big(\ell_\omega(B_{\mathbb{C}})-(l_{\mathbb{C}}-l_{\mathbb{C}}^*)\big)=0. \tag{6.32}$$

For a good choice of ν, there exist $\nu_{\mathbb{R}}\in\mathcal{H}_{k,l_0+1}^{\omega,\phi}$, $\nu_{\mathbb{C}}\in\mathcal{H}_{l_{\mathbb{C}}+1}^{\omega}$, and a natural embedding

$$\iota_{\mathcal{S}}:\mathcal{S}\longrightarrow\mathfrak{M}_{\mathbb{R}}\times\mathfrak{M}_{\mathbb{C}}\equiv\mathfrak{M}_{k,l_0+1}\big(B_{\mathbb{R}};J,\nu_{\mathbb{R}}\big)\times\mathfrak{M}_{l_{\mathbb{C}}+1}^{\mathbb{C}}\big(B_{\mathbb{C}};J,\nu_{\mathbb{C}}\big) \tag{6.33}$$

satisfying $(\mathfrak{o}_\Gamma 1)$ - $(\mathfrak{o}_\Gamma 3)$ in Section 4.3. If $B_{\mathbb{R}}\neq 0$, we also assume that there exists $\nu'_{\mathbb{R}}\in\mathcal{H}^{\omega,\phi}_{k,l_0}$ so that the forgetful morphism

$$\mathfrak{f}_{\mathrm{nd}}:\mathfrak{M}_{\mathbb{R}}\longrightarrow\mathfrak{M}'_{\mathbb{R}}\equiv\mathfrak{M}_{k,l_0}\big(B_{\mathbb{R}};J,\nu'_{\mathbb{R}}\big) \tag{6.34}$$

dropping the conjugate pair corresponding to the node nd (i.e. the first one) is defined. If $B_{\mathbb{C}}\neq 0$, we similarly assume that there exists $\nu'_{\mathbb{C}}\in\mathcal{H}^{\omega}_{l_{\mathbb{C}}}$ so that the analogous forgetful morphism

$$\mathfrak{f}_{\mathrm{nd}}:\mathfrak{M}_{\mathbb{C}}\longrightarrow\mathfrak{M}'_{\mathbb{C}}\equiv\mathfrak{M}^{\mathbb{C}}_{l_{\mathbb{C}}}\big(B_{\mathbb{C}};J,\nu'_{\mathbb{C}}\big) \tag{6.35}$$

is defined. Denote by $I\subset[l_{\mathbb{C}}+1]$ (resp. $I'\subset[l_{\mathbb{C}}]$) the subset indexing the marked points of a map in $\mathfrak{M}_{\mathbb{C}}$ (resp. $\mathfrak{M}'_{\mathbb{C}}$) corresponding to the marked points on the left-hand side of (6.33) indexed by L^*_- under (6.33) (resp. (6.33) and (6.35)).

For an element $\mathbf{u}\in\mathcal{S}$, we denote by $\mathbf{u}_0\in\mathfrak{M}_{\mathbb{R}}$ and $\mathbf{u}_+\in\mathfrak{M}_{\mathbb{C}}$ the pair of maps corresponding to $\mathbf{u}$ via (6.33). Let $\mathbf{u}'_0\in\mathfrak{M}'_{\mathbb{R}}$ and $\mathbf{u}'_+\in\mathfrak{M}'_{\mathbb{C}}$ be the image of $\mathbf{u}_0$ under (6.34) if $B_{\mathbb{R}}\neq 0$ and the image of $\mathbf{u}_+$ under (6.35) if $B_{\mathbb{C}}\neq 0$, respectively. The exact sequence

$$0\longrightarrow T_{\mathbf{u}}\mathcal{S}\longrightarrow T_{\mathbf{u}_0}\mathfrak{M}_{\mathbb{R}}\oplus T_{\mathbf{u}_+}\mathfrak{M}_{\mathbb{C}}\longrightarrow T_{u(\mathrm{nd})}X\longrightarrow 0,\quad \big(\xi_1,\xi_2\big)\longrightarrow\xi_2(\mathrm{nd})-\xi_1(\mathrm{nd}),$$

of vector spaces determines an isomorphism

$$\lambda_{\mathbf{u}}(\mathcal{S})\otimes\lambda\big(T_{u(\mathrm{nd})}X\big)\approx\lambda_{\mathbf{u}_0}(\mathfrak{M}_{\mathbb{R}})\otimes\lambda_{\mathbf{u}_+}(\mathfrak{M}_{\mathbb{C}}).$$

The Pin^--structure $\mathfrak{p}$ on X^ϕ determines a homotopy class $\mathfrak{o}_{\mathfrak{p};l_0^*}$ of isomorphisms

$$\lambda_{\mathbf{u}_0}(\mathfrak{M}_{\mathbb{R}})\longrightarrow\lambda^{\mathbb{R}}_{\mathbf{u}_0}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}_0}(X);$$

see Lemma 5.1. Combining the above two homotopy classes of isomorphisms with the complex orientations of $\lambda(T_{u(\mathrm{nd})}X)$ and $\lambda^{\mathbb{C}}_{\mathbf{u}_+}(X)$ and the orientation $\mathfrak{o}_{\mathbb{C};I}$ of $\lambda_{\mathbf{u}_+}(\mathfrak{M}_{\mathbb{C}})$, we obtain a homotopy class $\mathfrak{o}^{\Gamma}_{\mathfrak{p};l_0^*;\mathbf{u}}$ of isomorphisms

$$\lambda_{\mathbf{u}}(\mathcal{S})\longrightarrow\lambda^{\mathbb{R}}_{\mathbf{u}}(X)\otimes\lambda^{\mathbb{C}}_{\mathbf{u}}(X).$$

Lemma 6.5. *The relative orientations* $(\mathfrak{f}^*_{k',l'}\mathfrak{o}^c_\Gamma)\mathfrak{o}_{\mathfrak{p};l^*}$ *and* $\mathfrak{o}^{\Gamma}_{\mathfrak{p};l_0^*}$ *of* $\lambda(\mathrm{ev}_\Gamma)$ *are the same.*

Proof. The proof of Lemma 6.4 readily adapts using Lemma 4.2. The relevant analogue of Lemma 6.2 follows readily from [8, Cor. 7.3]. In light of Lemma 4.2, the $(k',l')=(0,3)$ case of Lemma 6.5 also follows from Lemma 5.2 and Remark 5.3 in [10]; the proof in [10] extends to arbitrary (k',l'). □

We denote by

$$\mathbf{h}_{\mathbb{R}}:M_{\mathbf{h}_{\mathbb{R}}}\longrightarrow X^{l_0^*},\quad \mathbf{h}_{\mathbb{C}}:M_{\mathbf{h}_{\mathbb{C}}}\longrightarrow X^{l_{\mathbb{C}}^*},\quad \mathbf{p}_{\mathbb{R}}\in X_{k,l_0-l_0^*},\quad\text{and}\quad \mathbf{p}_{\mathbb{C}}\in X^{l_{\mathbb{C}}-l_{\mathbb{C}}^*}$$

the components of $\mathbf{h}$ and $\mathbf{p}$ corresponding to the marked points on $\mathbb{P}^1_0$ and $\mathbb{P}^1_\pm$ for the maps $\mathbf{u}$ in $\mathcal{S}$. With the notation as in (5.9), let

$$\mathcal{S}_{\mathbf{h}}=\mathcal{S}_{\mathrm{ev}}\times_{f_{\mathbf{h}}}M_{\mathbf{h}},\quad \mathcal{Z}_{\mathbb{R}}=(\mathfrak{M}_{\mathbb{R}})_{\mathrm{ev}}\times_{f_{\mathbf{h}_{\mathbb{R}}}}M_{\mathbf{h}_{\mathbb{R}}},\quad \mathcal{Z}_{\mathbb{C}}=(\mathfrak{M}_{\mathbb{C}})_{\mathrm{ev}^I}\times_{f_{\mathbf{h}_{\mathbb{C}}}}M_{\mathbf{h}_{\mathbb{C}}}$$

be the corresponding spaces cut out by $\mathbf{h}$ and

$$\mathrm{ev}_{\mathbb{C}}\colon \mathcal{Z}_{\mathbb{C}} \longrightarrow X^{l_{\mathbb{C}}-l_{\mathbb{C}}^*}$$

be the evaluation map induced by ev^I. The relative orientations $\mathfrak{o}_{\mathfrak{p};l_0^*+1}$ and $\mathfrak{o}_{\mathbb{C};I}$ of

$$\mathrm{ev}\colon \mathfrak{M}_{\mathbb{R}} \longrightarrow X_{k,l_0+1} \qquad \text{and} \qquad \mathrm{ev}^I\colon \mathfrak{M}_{\mathbb{C}} \longrightarrow X^{l_{\mathbb{C}}+1},$$

respectively, the orientations $\mathfrak{o}_{h_i}$ of H_i, and the symplectic orientation $\mathfrak{o}_\omega$ on X determine relative orientations $\mathfrak{o}_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}}$ and $\mathfrak{o}_{\mathbf{h}_{\mathbb{C}};I}$ of the induced maps

$$\mathrm{ev}_{\mathbb{R}}\colon \mathcal{Z}_{\mathbb{R}} \longrightarrow X_{k,l_0-l_0^*} \qquad \text{and} \qquad \mathrm{ev}_{\mathbb{C}}\colon \mathcal{Z}_{\mathbb{C}} \longrightarrow X^{l_{\mathbb{C}}-l_{\mathbb{C}}^*}$$

respectively.

For $\tilde{\mathbf{u}}\in\mathcal{S}_{\mathbf{h}}$, let $\tilde{\mathbf{u}}_0\in\mathcal{Z}_{\mathbb{R}}$ and $\tilde{\mathbf{u}}_+\in\mathcal{Z}_{\mathbb{C}}$ be the components of $\tilde{\mathbf{u}}$ in the corresponding spaces. The exact sequence

$$0 \longrightarrow T_{\tilde{\mathbf{u}}}\mathcal{S}_{\mathbf{h}} \longrightarrow T_{\tilde{\mathbf{u}}_0}\mathcal{Z}_{\mathbb{R}}\oplus T_{\tilde{\mathbf{u}}_+}\mathcal{Z}_{\mathbb{C}} \longrightarrow T_{u(\mathrm{nd})}X \longrightarrow 0 \tag{6.36}$$

of vector spaces determines an isomorphism

$$\lambda_{\tilde{\mathbf{u}}}(\mathcal{S}_{\mathbf{h}})\otimes\lambda\big(T_{u(\mathrm{nd})}X\big) \approx \lambda_{\tilde{\mathbf{u}}_0}(\mathcal{Z}_{\mathbb{R}})\otimes\lambda_{\tilde{\mathbf{u}}_+}(\mathcal{Z}_{\mathbb{C}}).$$

Along with $\mathfrak{o}_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}}$ and $\mathfrak{o}_{\mathbf{h}_{\mathbb{C}};I}$, these isomorphisms determine a relative orientation

$$\mathfrak{o}_{\mathfrak{p};\mathbf{h}}^{\Gamma} \equiv \big(\mathfrak{o}_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}}\big)_{\mathrm{nd}}\times_{\mathrm{nd}}\mathfrak{o}_{\mathbf{h}_{\mathbb{C}};I} \tag{6.37}$$

of the restriction of (5.18) to $\mathcal{S}_{\mathbf{h}}$. Since the dimensions of H_i and X are even, Lemma 6.5 implies that

$$\big|\mathrm{ev}_{\Gamma;\mathbf{h}}^{-1}(\mathbf{p})\cap\mathcal{S}_{\mathbf{h}}\big|^{\pm}_{\mathfrak{o}_{\Gamma;\mathfrak{p};\mathbf{h}}} = \big|\mathrm{ev}_{\Gamma;\mathbf{h}}^{-1}(\mathbf{p})\cap\mathcal{S}_{\mathbf{h}}\big|^{\pm}_{\mathfrak{o}_{\mathfrak{p};\mathbf{h}}^{\Gamma}}. \tag{6.38}$$

We denote by $\mathfrak{o}_{\mathcal{S}}^c$ the co-orientation of $\mathcal{S}$ in $\mathcal{Z}_{\mathbb{R}}\times\mathcal{Z}_{\mathbb{C}}$ determined by the symplectic orientation $\mathfrak{o}_\omega$ of X via (6.36).

If $B_{\mathbb{R}}\neq 0$ (resp. $B_{\mathbb{C}}\neq 0$), we also define

$$\mathcal{Z}_{\mathbb{R}}' = (\mathfrak{M}_{\mathbb{R}}')_{\mathrm{ev}}\times_{\mathbf{h}_{\mathbb{R}}} M_{\mathbf{h}_{\mathbb{R}}} \qquad \big(\text{resp. } \mathcal{Z}_{\mathbb{C}}' = (\mathfrak{M}_{\mathbb{C}}')_{\mathrm{ev}^{I'}}\times_{\mathbf{h}_{\mathbb{C}}} M_{\mathbf{h}_{\mathbb{C}}}\big)$$

and denote by $\tilde{\mathbf{u}}_0'\in\mathcal{Z}_{\mathbb{R}}'$ (resp. $\tilde{\mathbf{u}}_+'\in\mathcal{Z}_{\mathbb{C}}'$) the image of $\tilde{\mathbf{u}}_0$ (resp. $\tilde{\mathbf{u}}_+$) under the forgetful morphism

$$\mathfrak{f}_{\mathbb{R}}\colon \mathcal{Z}_{\mathbb{R}} \longrightarrow \mathcal{Z}_{\mathbb{R}}' \qquad \big(\text{resp. } \mathfrak{f}_{\mathbb{C}}\colon \mathcal{Z}_{\mathbb{C}} \longrightarrow \mathcal{Z}_{\mathbb{C}}'\big) \tag{6.39}$$

dropping the marked points corresponding to the nodes. If $B_{\mathbb{R}}\neq 0$ and $l_0^*\neq 0$, $\mathfrak{o}_{\mathfrak{p};l_0^*}$ determines a relative orientation $\mathfrak{o}_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}}'$ of the evaluation map

$$\mathrm{ev}_{\mathbb{R}}'\colon \mathcal{Z}_{\mathbb{R}}' \longrightarrow X_{k,l_0-l_0^*} \tag{6.40}$$

induced by ev. If $B_{\mathbb{C}}\neq 0$, $\mathfrak{o}_{\mathbb{C};I}$ determines a relative orientation $\mathfrak{o}_{\mathbf{h}_{\mathbb{C}};I'}'$ of the evaluation map

$$\mathrm{ev}_{\mathbb{C}}'\colon \mathcal{Z}_{\mathbb{C}}' \longrightarrow X^{l_{\mathbb{C}}-l_{\mathbb{C}}^*} \tag{6.41}$$

induced by $\mathrm{ev}^{I'}$.

Since the projections $\pi_{\mathbb{R}}$ and $\pi_{\mathbb{C}}$ in the commutative diagrams

$$\begin{array}{ccc} \mathcal{Z}_{\mathbb{R}} & \xrightarrow{\mathfrak{f}_{\mathbb{R}}} & \mathcal{Z}'_{\mathbb{R}} \\ \pi_{\mathbb{R}}\downarrow & & \downarrow\pi'_{\mathbb{R}} \\ \mathfrak{M}_{\mathbb{R}} & \xrightarrow{\mathfrak{f}_{\mathrm{nd}}} & \mathfrak{M}'_{\mathbb{R}} \end{array} \qquad \begin{array}{ccc} \mathcal{Z}_{\mathbb{C}} & \xrightarrow{\mathfrak{f}_{\mathbb{C}}} & \mathcal{Z}'_{\mathbb{C}} \\ \pi_{\mathbb{C}}\downarrow & & \downarrow\pi'_{\mathbb{C}} \\ \mathfrak{M}_{\mathbb{C}} & \xrightarrow{\mathfrak{f}_{\mathrm{nd}}} & \mathfrak{M}'_{\mathbb{C}} \end{array}$$

induce isomorphisms between the vertical tangent bundles of $\mathfrak{f}_{\mathbb{R}}$, $\mathfrak{f}_{\mathbb{C}}$, and $\mathfrak{f}_{\mathrm{nd}}$, they pull back the relative orientations $\mathfrak{o}_{\mathrm{nd}}^{+}$ of (6.34) and (6.35), respectively, to relative orientations $\mathfrak{o}_{\mathrm{nd}}^{+}$ of (6.39). Since the dimensions of X and H_i are even,

$$\mathfrak{o}_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}} = \mathfrak{o}_{\mathbb{R}}^{+}\mathfrak{o}'_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}} \equiv \big(\pi_{\mathbb{R}}^{*}\mathfrak{o}_{\mathrm{nd}}^{+}\big)\mathfrak{o}'_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}} \quad \text{and} \quad \mathfrak{o}_{\mathbf{h}_{\mathbb{C}};I} = \mathfrak{o}_{\mathbb{C}}^{+}\mathfrak{o}'_{\mathbf{h}_{\mathbb{C}};I'} \equiv \big(\pi_{\mathbb{C}}^{*}\mathfrak{o}_{\mathrm{nd}}^{+}\big)\mathfrak{o}'_{\mathbf{h}_{\mathbb{C}};I'}\,, \tag{6.42}$$

whenever $B_{\mathbb{R}}\neq 0$ and $B_{\mathbb{C}}\neq 0$, respectively.

Suppose $B_{\mathbb{R}}, B_{\mathbb{C}}\neq 0$. By (6.32), we can assume that

$$\ell_{\omega}(B_{\mathbb{R}}) = k+2(l_0-l_0^*) \qquad \text{and} \qquad \ell_{\omega}(B_{\mathbb{C}}) = l_{\mathbb{C}}-l_{\mathbb{C}}^*; \tag{6.43}$$

otherwise, either $\mathrm{ev}_{\mathbb{R}}^{-1}(\mathbf{p}_{\mathbb{R}}) = \varnothing$ or $\mathrm{ev}_{\mathbb{C}}^{-1}(\mathbf{p}_{\mathbb{C}}) = \varnothing$ for a generic $\mathbf{p}\in X_{k,l-l^*}$. By the first statement above and Lemma 5.1($\mathfrak{o}_{\mathfrak{p}}$7), the interchange of the marked points of the elements in $\mathcal{Z}_{\mathbb{R}}$ reverses the orientation $\mathfrak{o}_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}}$. This interchange also reverses the vertical orientation $\mathfrak{o}_{\mathbb{R}}^{+}$ of the fibration $\mathfrak{f}_{\mathbb{R}}$. Thus, the orientation $\mathfrak{o}'_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}}$ on $\mathcal{Z}'_{\mathbb{R}}$ can be defined by the first equation in (6.42) if $l_0^*=0$. By the second equation in (6.43) and the assumption that $\phi_*[h_i]=-[h_i]$ for every $i\in[l^*]$,

$$\big|\mathrm{ev}'^{-1}_{\mathbb{C}}(\mathbf{p}_{\mathbb{C}})\big|^{\pm}_{\mathfrak{o}_{\mathbf{h}_{\mathbb{C}}}} = \big|(\mathfrak{M}'_{\mathbb{C}})_{\mathrm{ev}^I}\times_{(\mathbf{h}_{\mathbb{C}},\mathbf{p}_{\mathbb{C}})} M_{\mathbf{h}_{\mathbb{C}}}\big|^{\pm}_{\mathfrak{o}_{\mathbb{C};I},\mathfrak{o}_{\mathbf{h}_{\mathbb{C}}}} = \prod_{i\in L_{\mathbb{C}}^*}(h_i\cdot_X B_{\mathbb{C}})N^X_{B_{\mathbb{C}}}\,; \tag{6.44}$$

see [10, Prop. 4.3].

Since the real dimension of X is 4, the homomorphism

$$\mathrm{d}_{\mathrm{nd}}u_{+}-\mathrm{d}_{\mathrm{nd}}u_0\colon T_{\mathrm{nd}}\mathbb{P}^1_0\oplus T_{\mathrm{nd}}\mathbb{P}^1_{+} \longrightarrow T_{u(\mathrm{nd})}X = \mathcal{N}_{\tilde{\mathbf{u}}}\mathcal{S}_{\mathbf{h}}$$

in the commutative diagram of Figure 11 is an isomorphism for a generic element $\mathbf{u}\in\mathcal{S}_{\mathbf{h}}$. So is the bottom row in this diagram. The number of the preimages $(\tilde{\mathbf{u}}_0,\tilde{\mathbf{u}}_{+})$ of a generic point $(\mathbf{u}_0,\mathbf{u}_{+})$ of $\{\mathcal{Z}'_{\mathbb{R}}\times\mathcal{Z}'_{\mathbb{C}}\}\circ\iota_{\mathcal{S}}$ under $\mathfrak{f}_{\mathbb{R}}\times\mathfrak{f}_{\mathbb{C}}$ is finite. Since the dimensions of $T_{\mathrm{nd}}\mathbb{P}^1_0$ and $T_{\mathrm{nd}}\mathbb{P}^1_{+}$ are even,

$$\mathfrak{o}^{v}_{\mathbb{R}\mathbb{C}} \equiv \mathfrak{o}_{\mathbb{R}}^{+}\oplus\mathfrak{o}_{\mathbb{C}}^{+}$$

is the vertical orientation of the fibration $\mathfrak{f}_{\mathbb{R}}\times\mathfrak{f}_{\mathbb{C}}$. By Lemma 3.1(1) and the reasoning in the proof of (5.14) at the end of Section 6.2,

$$\mathfrak{s}_{(\tilde{\mathbf{u}}_0,\tilde{\mathbf{u}}_{+})}(\mathfrak{o}^{c}_{\mathcal{S}}\mathfrak{o}^{v}_{\mathbb{R}\mathbb{C}}) \in \big\{\pm 1\big\}$$

is the sign of the intersection of u_0 and u_1 at nd and the number of such preimages counted with sign is $B_{\mathbb{R}}\cdot_X B_{\mathbb{C}}$. Along with the commutativity of the square in Figure 11 and (3.6), this implies that

$$\big|\mathrm{ev}^{-1}_{\Gamma;\mathbf{h}}(\mathbf{p})\cap\mathcal{S}_{\mathbf{h}}\big|^{\pm}_{\mathfrak{o}^{\Gamma}_{\mathfrak{p};\mathbf{h}}} = \big(B_{\mathbb{R}}\cdot_X B_{\mathbb{C}}\big)\big|\mathrm{ev}'^{-1}_{\mathbb{R}}(\mathbf{p}_{\mathbb{R}})\big|^{\pm}_{\mathfrak{o}'_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}}}\big|\mathrm{ev}'^{-1}_{\mathbb{C}}(\mathbf{p}_{\mathbb{C}})\big|^{\pm}_{\mathfrak{o}'_{\mathbf{h}_{\mathbb{C}};I'}}\,.$$

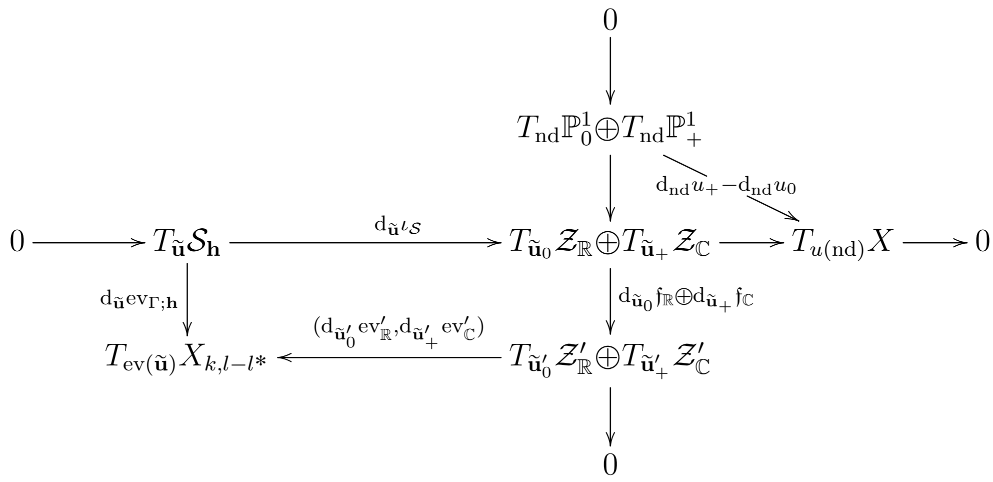

Figure 11: Computation of (6.38) in the $B_{\mathbb{R}}, B_{\mathbb{C}} \neq 0$ case

Combining this statement with (6.38), (5.14), and (6.44), we conclude that

$$\big|\mathrm{ev}_{\Gamma;\mathbf{h}}^{-1}(\mathbf{p}) \cap \mathcal{S}_{\mathbf{h}}\big|^{\pm}_{\mathfrak{o}_{\Gamma;\mathfrak{p};\mathbf{h}}} = \big(B_{\mathbb{R}} \cdot_X B_{\mathbb{C}}\big)\bigg(\prod_{i \in L_0^*} h_i \cdot_X B_{\mathbb{R}}\bigg)\bigg(\prod_{i \in L_{\mathbb{C}}^*} h_i \cdot_X B_{\mathbb{C}}\bigg) N^{\phi;\mathfrak{p}}_{B_{\mathbb{R}}, l_0 - l_0^*} N^X_{B_{\mathbb{C}}}.$$

Summing this over all possibilities for $\mathcal{S}$ with $B_{\mathbb{R}}, B_{\mathbb{C}} \neq 0$ that satisfy (6.43), we obtain the (B_0, B') sum in (5.20).

Suppose $B_{\mathbb{C}} = 0$ and thus $B_{\mathbb{R}} = B$. We can assume that $l^*_{\mathbb{C}} = l_{\mathbb{C}} = 2$; otherwise, $\mathcal{Z}_{\mathbb{C}} = \varnothing$ for generic $\mathbf{h}$ and $\mathbf{p}$. Thus, $\mathcal{Z}_{\mathbb{C}}$ is a finite collection of points, while the dimensions of $\mathcal{Z}_{\mathbb{R}}$ and $X \times X_{k,l-l^*}$ are the same by (6.32) and (5.19). By (6.37) and Lemma 3.4 with $\Upsilon, \mathcal{M} = \mathrm{pt}$ applied to the left diagram in Figure 12,

$$\big|\mathrm{ev}_{\Gamma;\mathbf{h}}^{-1}(\mathbf{p}) \cap \mathcal{S}_{\mathbf{h}}\big|^{\pm}_{\mathfrak{o}^{\Gamma}_{\mathfrak{p};\mathbf{h}}} = \big|\mathcal{Z}_{\mathbb{C}}\big|^{\pm}_{\mathfrak{o}_{\mathbb{C};I}} \big|\big\{\mathrm{ev}_{\mathrm{nd}} \times \mathrm{ev}_{\mathbb{R}}\big\}^{-1}(\mathrm{pt}, \mathbf{p})\big|^{\pm}_{\mathfrak{o}_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}} \mathfrak{o}_{\omega}}. \tag{6.45}$$

By the proof of (5.14),

$$\big|\big\{\mathrm{ev}_{\mathrm{nd}} \times \mathrm{ev}_{\mathbb{R}}\big\}^{-1}(\mathrm{pt}, \mathbf{p})\big|^{\pm}_{\mathfrak{o}_{\mathfrak{p};\mathbf{h}_{\mathbb{R}}} \mathfrak{o}_{\omega}} = \bigg(\prod_{i \in L_0^*} h_i \cdot_X B\bigg) N^{\phi;\mathfrak{p}}_{B, l_0 - l_0^* + 1}$$

Combining this statement with (6.38) and (6.45), we obtain the second term on the right-hand side of (5.20).

Suppose $B_{\mathbb{R}} = 0$ and thus $\mathfrak{d}(B_{\mathbb{C}}) = B$. We can assume that $k', k = 1$ and $l_0 = 0$; otherwise, $\mathcal{Z}_{\mathbb{R}} = \varnothing$ for generic $\mathbf{h}$ and $\mathbf{p}$. Thus, $\langle B; k \rangle_{\Gamma} = 1$, the dimensions of $\mathcal{Z}_{\mathbb{R}}$ and X^{ϕ} are the same, and so are the dimensions of $\mathcal{Z}_{\mathbb{C}}$ and $X \times X_{l-l^*}$ by (6.32) and (5.19). By (6.37) and Lemma 3.4 with $\Upsilon, \mathcal{M} = \mathrm{pt}$ applied to the right diagram in Figure 12,

$$\big|\mathrm{ev}_{\Gamma;\mathbf{h}}^{-1}(\mathbf{p}) \cap \mathcal{S}_{\mathbf{h}}\big|^{\pm}_{\mathfrak{o}^{\Gamma}_{\mathfrak{p};\mathbf{h}}} = \big|\mathrm{ev}_{\mathbb{R}}^{-1}(p_1^{\mathbb{R}})\big|^{\pm}_{\mathfrak{o}_{\mathfrak{p};1}} \big|\big\{\mathrm{ev}_{\mathrm{nd}} \times \mathrm{ev}_{\mathbb{C}}\big\}^{-1}(\mathrm{pt}, \mathbf{p}_{\mathbb{C}})\big|^{\pm}_{\mathfrak{o}_{\mathbb{C};I} \mathfrak{o}_{\omega}}.$$

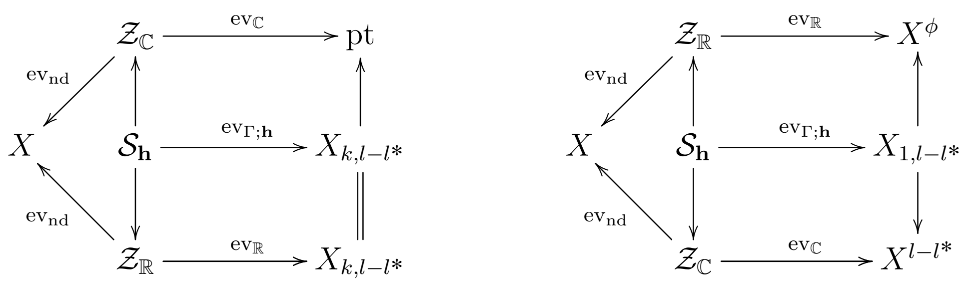

Figure 12: Computation of (6.38) in the $B_{\mathbb{C}}=0$ and $B_{\mathbb{R}}=0$ cases

Combining this statement with (6.38), Lemma 5.1($\mathfrak{o}_{\mathfrak{p}}$8), and (6.44), we conclude that

$$\left|\mathrm{ev}_{\Gamma;\mathbf{h}}^{-1}(\mathbf{p})\cap\mathcal{S}_{\mathbf{h}}\right|^{\pm}_{\mathfrak{o}_{\Gamma;\mathbf{p};\mathbf{h}}}=\bigg(\prod_{i\in[l^*]}h_i\cdot_X B_{\mathbb{C}}\bigg)N^X_{B_{\mathbb{C}}}=2^{-l^*}\bigg(\prod_{i\in[l^*]}h_i\cdot_X B\bigg)N^X_{B_{\mathbb{C}}}\,.$$

Summing this over all possibilities for $\mathcal{S}$ with $B_{\mathbb{R}}=0$, we obtain the first term on the right-hand side of (5.20).

References

[1] A. Alcolado, *Extended Frobenius manifolds and the open WDVV equations*, PhD thesis, McGill, 2017

[2] A. Arroyo, E. Brugallé, and L. López de Medrano, *Recursive formulas for Welschinger invariants of the projective plane*, Int. Math. Res. Not. 2011, no. 5, 1107—1134

[3] E. Brugallé, *Floor diagrams relative to a conic, and GW-W invariants of del Pezzo surfaces*, Adv. Math. 279 (2015), 438—500

[4] E. Brugallé, *On the invariance of Welschinger invariants*, math/1811.06891

[5] E. Brugallé and N. Puignau, *Behavior of Welschinger invariants under Morse simplifications*, Rend. Semin. Mat. Univ. Padova 130 (2013), 147—153

[6] X. Chen and A. Zinger, *WDVV-Type Relations for Welschinger's Invariants: Applications*, math/1809.08938

[7] X. Chen and A. Zinger, *WDVV-type relations for disk Gromov-Witten invariants in dimension 6*, Math. Ann. 379 (2021), 1231–1313

[8] X. Chen and A. Zinger, *Spin/Pin Structures and Real Enumerative Geometry*, math/1905.11316

[9] P. Georgieva, *Open Gromov-Witten invariants in the presence of an anti-symplectic involution*, Adv. Math. 301 (2016), 116-–160

[10] P. Georgieva and A. Zinger, *Enumeration of real curves in $\mathbb{CP}^{2n-1}$ and a WDVV relation for real Gromov-Witten invariants*, Duke Math. J. 166 (2017), no. 17, 3291-–3347

[11] P. Georgieva and A. Zinger, *Real Gromov-Witten theory in all genera and real enumerative geometry: construction*, Ann. of Math. 188 (2018), no. 3, 685-–752

[12] P. Georgieva and A. Zinger, *Real Gromov-Witten theory in all genera and real enumerative geometry: properties*, math/1507.06633v2

[13] A. Givental, *Semisimple Frobenius structures at higher genus*, IMNR 2001, no. 23, 1265—1286

[14] A. Horev and J. Solomon, *The open Gromov-Witten-Welschinger theory of blowups of the projective plane*, math/1210.4034

[15] I. Itenberg, V. Kharlamov, and E. Shustin, *A Caporaso-Harris type formula for Welschinger invariants of real toric del Pezzo surfaces*, Comment. Math. Helv. 84 (2009), no. 1, 87—126

[16] I. Itenberg, V. Kharlamov, and E. Shustin, *Welschinger invariants of small non-toric Del Pezzo surfaces*, JEMS 15 (2013), no. 2, 539—594

[17] I. Itenberg, V. Kharlamov, and E. Shustin, *Welschinger invariants of real Del Pezzo surfaces of degree* $\geqslant 3$, Math. Ann. 355 (2013), no. 3, 849–878

[18] I. Itenberg, V. Kharlamov, and E. Shustin, *Welschinger invariants of real del Pezzo surfaces of degree* $\geqslant 2$, Internat. J. Math. 26 (2015), no. 8, 1550060, 63pp

[19] M. Kontsevich and Y. Manin, *Gromov-Witten classes, quantum cohomology, and enumerative geometry*, Comm. Math. Phys. 164 (1994), no. 3, 525–562

[20] J. Li and G. Tian, *Virtual moduli cycles and Gromov-Witten invariants of general symplectic manifolds*, Topics in Symplectic 4-Manifolds, 47-83, First Int. Press Lect. Ser., I, Internat. Press, 1998

[21] D. McDuff and D. Salamon, *J-holomorphic Curves and Symplectic Topology*, Colloquium Publications 52, AMS, 2012

[22] Y. Ruan and G. Tian, *A mathematical theory of quantum cohomology*, J. Differential Geom. 42 (1995), no. 2, 259–367

[23] Y. Ruan and G. Tian, *Higher genus symplectic invariants and sigma models coupled with gravity*, Invent. Math. 130 (1997), no. 3, 455–516

[24] J. Solomon, *Intersection theory on the moduli space of holomorphic curves with Lagrangian boundary conditions*, math/0606429

[25] J. Solomon, *A differential equation for the open Gromov-Witten potential*, note 2007

[26] N. Steenrod, *Homology with local coefficients*, Ann. of Math. 44 (1943), no. 4, 610-–627

[27] C. Teleman, *The structure of 2D semi-simple field theories*, Invent. Math. 188 (2012), no. 3, 525—588

[28] F. Warner, *Foundations of Differentiable Manifolds and Lie Groups*, GTM 94, Springer-Verlag, 1983

[29] J.-Y. Welschinger, *Invariants of real symplectic 4-manifolds and lower bounds in real enumerative geometry*, Invent. Math. 162 (2005), no. 1, 195—234

[30] J.-Y. Welschinger, *Spinor states of real rational curves in real algebraic convex 3-manifolds and enumerative invariants*, Duke Math. J. 127 (2005), no. 1, 89—121

[31] J.-Y. Welschinger, *Optimalité, congruences et calculs d'invariants des variértés symplectiques réelles de dimension quatre*, math/0707.4317

[32] A. Zinger, *Pseudocycles and integral homology*, Trans. Amer. Math. Soc. 360 (2008), no. 5, 2741–2765

[33] A. Zinger, *Real Ruan-Tian perturbations*, math/1701.01420