DECOMPOSITION THEOREM FOR SEMISIMPLE LOCAL SYSTEMS

A Dissertation presented

by

Ruijie Yang

to

Stony Brook University

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2021

Stony Brook University

The Graduate School

Ruijie Yang

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Robert Lazarsfeld and Christian Schnell - Dissertation Advisors
Distinguished Professor and Professor, Department of Mathematics

Mark Andrea de Cataldo - Chairperson of Defense
Professor, Department of Mathematics

Lawrence Ein
Professor, Department of Mathematics, University of Illinois at Chicago

This dissertation is accepted by the Graduate School

Eric Wertheimer
Dean of the Graduate School

Abstract of the Dissertation

Decomposition Theorem for Semisimple Local Systems

by

Ruijie Yang

Doctor of Philosophy

in

Mathematics

Stony Brook University

2021

In the first part of this dissertation, I will discuss the Decomposition Theorem for semisimple local systems. In complex algebraic geometry, the decomposition theorem asserts that semisimple geometric objects remain semisimple after taking direct images under proper algebraic maps. This was conjectured by Kashiwara and is proved by Mochizuki and Sabbah in a series of long papers via harmonic analysis and D-modules. My work gives a new geometric proof in the case of semisimple local systems, adapting the method developed by de Cataldo and Migliorini. My main contribution is two-fold: on the one hand, I complement Simpson's theory of weights for local systems by proving a global invariant cycle theorem in the setting of semisimple local systems; on the other hand, I relate Simpson's notion of polarizations on pure twistor structures to Poincare pairings for local systems and give a useful criterion for non-degeneracy of restrictions of Poincare pairings to various subspaces of the cohomology groups of local systems arising from geometry.

In the second part of this dissertation, I will discuss a birational invariant of algebraic varieties measuring how far they are from being stably rational. Then I calculate this invariant for very general hypersurfaces in projective spaces. This section closely follow my paper [57].

To my mother and grandmother

Contents

Acknowledgement **vii**

Publications **ix**

1 Introduction **1**

2 Decomposition Theorem **2**
2.1 Introduction 2
2.1.1 Historic background 2
2.1.2 What is new? 4
2.1.3 Outline of the chapter 4
2.2 Statement of main results 5
2.3 Cohomology of semisimple local systems 11
2.3.1 Statements 11
2.3.2 Cohomology of smooth projective varieties and pure twistor structures 14
2.3.3 Twisted Poincare pairings and polarizations 32
2.3.4 Cohomology of algebraic varieties and mixed twistor structures 37
2.3.5 Weight filtrations 57
2.3.6 Perverse filtrations and twistor structures 62
2.4 Hodge star operators 65
2.4.1 Harmonic bundles 65
2.4.2 Dual harmonic bundles 66
2.4.3 Hodge star operators 67
2.4.4 Properties of Hodge star operators 70
2.4.5 Kahler identities 71
2.5 Preliminary results 71
2.5.1 The cup product with a line bundle 72
2.5.2 Weak-Lefschetz-type results 72
2.5.3 p-splitness and weak Lefschetz 76
2.5.4 Splitting criterion 76
2.6 Proof of Theorem A 77
2.6.1 Set-up 77
2.6.2 Base case 78
2.6.3 Induction step 78
2.7 Proof of Theorem B and Theorem D 79

2.7.1 Theorem B: Hard Lefschetz for Perverse Cohomology Complexes . . . 79
2.7.2 Theorem D: Generalized Hodge-Riemann Bilinear Relations . . . 83
2.7.3 Non-degeneration of intersection forms . . . 88
2.8 Semisimplicity Theorem for $\ell = 0$ . . . 89
2.9 Sabbah's Theorem . . . 92

3 Stable Irrationality of Hypersurfaces **94**
3.1 Introduction . . . 94
3.2 Correspondences . . . 96
3.3 Proofs . . . 97

Bibliography **100**

Acknowledgement

First, I would like to thank Professors Rob Lazarsfeld and Christian Schnell, whose mentorship far exceeded any reasonable expectations for doctoral advisors. Rob lead me into the beautiful world of algebraic geometry through his course on complex geometry at Stony Brook on the fall of my first year in this country and kindly agreed to be my PhD advisor after I was accepted to Stony Brook. I want thank him for showing me how to be a great mathematician not only in terms of math but also about writing, presentation and communication. I want to especially thank Rob for his patience when he tirelessly helped me preparing my talks and papers until they were fully polished. I want to thank Christian for kindly agreeing to be my second PhD advisor three years ago. Many results of this dissertation grew out from our weekly discussions. I want to thank him for many projects we worked on and new math learnt together, for teaching me Hodge theory, D-modules and his curiosity, for his help on technical details and constant encouragement. His precise comments, valuable suggestions and deep insights lead me to math which I never imaged before. My time at Stony Brook is definitely the defining stage of my mathematical career, where I grew up from a college student to a junior mathematician.

I would like to thank Professor Mark de Cataldo for the wonderful math he discovered with Professor Migliorini about Decomposition Theorem and perverse sheaves, a large chunk of this thesis would be impossible without their fantastic works. I want to thank Lawrence Ein for his constant encouragement throughout my time in graduate school ever since we first met at University of Illinois at Chicago many years ago. I am also indebted to other professors at Stony Brook, including: Sam Grushevsky, Radu Laza, Jason Starr and Alex Zinger. I also want to thank Simons Center of Geometry and Physics at Stony Brook for many great conferences and workshops held all these years, which greatly influenced my mathematical tastes.

I would like to thank my friends in Stony Brook, where we grew up together, had a lot of fun time and interesting math discussions, which include Gao Chen, Nathan Chen, Francois Greer, Tim Ryan, John Sheridan, David Stapleton, Yuhan Sun, Zhuang Tao, Letao Zhang, Zili Zhang, Zheng Zhang, Fangyu Zou. Especially thanks to Dingxin Zhang for those inspiring late-night conversations at Math Tower 2-114 and Xiaolei Zhao for his constant guidance.

I would like to thank the staffs in the Math Department for making my graduate life at Stony Brook as smooth as possible: Lynne Barnett, Christine Gathman, Donna McWilliams, Pat Tonra and Diane Williams.

I also would like to thank my friends outside math circles for their friendship which encouraged during the dark time of the graduate school: Qinglin Fan, Liang Lan, Xingyu Yin, Linwei Yu, Yaodong Yu.

Finally I would like to thank my family especially my mom, my uncle and

aunts, and my grandparents for their unconditional supports. Even though I am more than 10000 kilometers from home, their love are still will me wherever I go.

Publications

Research Papers

1. L^2-minimal extensions over Hermitian symmetric domains, available at arxiv: 2006.15193.

2. Hodge modules and singular Hermitian metrics, available at arxiv: 2003.09064, with Christian Schnell, to appear in Mathematische Zeitschrift.

3. Nodal elliptic curves on K3 surfaces, available at arxiv: 2001.05104, with Nathan Chen and Francois Greer.

4. On irrationality of hypersurfaces in $\mathbf{P}^{n+1}$, available at arxiv: 1803.07704. Proceedings of the American Mathematical Society, 147(3):971-976, 2019.

5. Nef cones of nested Hilbert schemes of points on surfaces, available at arxiv: 1708.00903,
with Tim Ryan, International Mathematics Research Notices, 2020(11):3260-3294, 2020.

Fields of Study

Major Field: Mathematics
Specialization: Algebraic Geometry

Chapter 1

Introduction

This dissertation represents several pieces of my research during the graduate school. Besides the material presented here, I have also written additional papers [13], [42], [48], [58].

In the first part of this dissertation, I will discuss the Decomposition Theorem for semisimple local systems. In complex algebraic geometry, the decomposition theorem asserts that semisimple geometric objects remain semisimple after taking direct images under proper algebraic maps. This was conjectured by Kashiwara and is proved by Mochizuki and Sabbah in a series of long papers via harmonic analysis and D-modules. My work gives a new geometric proof in the case of semisimple local systems, adapting the method developed by de Cataldo and Migliorini. My main contribution is two-fold: on the one hand, I complement Simpson's theory of weights for local systems by proving a global invariant cycle theorem in the setting of semisimple local systems; on the other hand, I relate Simpson's notion of polarizations on pure twistor structures to Poincare pairings for local systems and give a useful criterion for non-degeneracy of restrictions of Poincare pairings to various subspaces of the cohomology groups of local systems arising from geometry.

In the second part of this dissertation, I will discuss a birational invariant of algebraic varieties measuring how far they are from being stably rational. Then I calculate this invariant for very general hypersurfaces in projective spaces. This section closely follow my paper [57].

Chapter 2

Decomposition Theorem

2.1 Introduction

The purpose of this chapter is to give a new and geometric proof of the Decomposition theorem for semisimple local systems under proper direct images of algebraic maps, which was originally proved by Sabbah [43].

2.1.1 Historic background

Sabbah's theorem (Theorem A) is a vast generalization of the Hard Lefschetz theorem in classical algebraic geometry. I will start by reviewing this theorem. Let X be a smooth projective variety of dimension n and L be an ample line bundle on X. Classically the theory of harmonic forms puts an extra structure on each singular cohomology group $H^k(X) := H^k(X, \mathbb{C})$, which is the so-called Hodge structure. Moreover, one has the Hard Lefschetz theorem: cup product map with $c_1(L)$ determines an isomorphism

$$\begin{aligned} L^\ell : H^{n-\ell}(X) &\xrightarrow{\sim} H^{n+\ell}(X) \\ \alpha &\mapsto \alpha \wedge c_1(L)^\ell. \end{aligned}$$

The Hard Lefschetz theorem naturally generalizes to the relative setting. Deligne observed that not only there is a relative Hard Lefschetz theorem, but also the cohomology of smooth proper maps behaves like the cohomology of product maps.

Theorem 2.1.1 (Deligne [20, 22])**.** *Let $f : X \to Y$ be a smooth proper map of smooth algebraic varieties. Assume that f has fiber dimension n and that L is an f-ample line bundle on X.*

1. *(Relative Hard Lefschetz) The cup product map with L is an isomorphism:*
$$L^\ell\colon R^{n-\ell} f_* \mathbb{C} \xrightarrow{\sim} R^{n+\ell} f_* \mathbb{C}.$$

2. *(Decomposition Theorem) There exists a (non-canonical) splitting*
$$f_* \mathbb{C} \cong \bigoplus_\ell R^\ell f_* \mathbb{C}[-\ell].$$

Here f_* denotes the total direct image functor.

3. *(Semisimplicity Theorem) $R^\ell f_*\mathbb{C}$ is a semisimple local system for each ℓ.*

Now if one only assumes $f : X \to Y$ is a projective map, even if X and Y are projective, all three statements break down due to the fact that f typically has singularities. The correct statements are formulated using the theory of perverse sheaves introduced by Beilinson-Berstein-Deligne-Gabber [5], which stems from the study of topology of stratified spaces by the work of Goresky-MacPherson [29, 30]. This circle of ideas finally lead to the following *Decomposition Theorem*, which is one of the deepest result concerning the homology of complex algebraic varieties.

Theorem 2.1.2 (Beilinson-Bernstein-Deligne-Gabber [5])**.** *Let $f : X \to Y$ be a proper map between algebraic varieties and let η be an f-ample line bundle on X.*

1. *(Relative Hard Lefschetz Theorem) The cup product map with η is an isomorphism*
$$\eta^\ell : {}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*\mathrm{IC}_X) \xrightarrow{\sim} {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*\mathrm{IC}_X).$$
Here ${}^{\mathfrak{p}}\mathcal{H}^\ell$ denotes the ℓ^{th} perverse cohomology, f_* denotes the total direct image functor. IC_X is the intersection complex of X.

2. *(Decomposition Theorem) There exists a (non-canonical) splitting*
$$f_*\mathrm{IC}_X \cong \bigoplus_\ell {}^{\mathfrak{p}}\mathcal{H}^\ell(f_*\mathrm{IC}_X)[-\ell].$$

3. *(Semisimplicity Theorem) ${}^{\mathfrak{p}}\mathcal{H}^\ell(f_*\mathrm{IC}_X)$ is a semisimple perverse sheaf for each ℓ.*

The original proof of the Decomposition Theorem was arithmetic in nature and used positive characteristic methods. In particular, it ultimately relied on Deligne's work on the Weil conjectures and the theory of weights.

In 2005, de Cataldo and Migliorini [17] found a geometric proof of the Decomposition Theorem. Their proof just used the classical Hodge theory and rested on new Hodge-theoretic results on the cohomology of projective varieties.

On the other hand, inspired by the Decomposition Theorem and Saito's Decomposition Theorem for pure Hodge modules, Kashiwara [35] conjectured that the Decomposition Theorem is true for arbitrary semisimple holonomic D-modules, which are the analogy of the pure complex in [5] and the pure Hodge module in [44]. This is the so-called Kashiwara's conjecture.

Kashiwara's conjecture was proved in the case of semisimple local system by Sabbah [43] using his theory of polarizable twistor D-modules, which combines the ideas of Saito's theory of pure Hodge modules and Simpson's theory of pure twistor structures. Later, Mochizuki [39] proved the full Kashiwara's conjecture. Mochizuki's method is analytic in nature and rests on his generalization of the theory of harmonic bundles on smooth projective varieties to arbitrary algebraic varieties.

Kashiwara's conjecture for semisimple perverse sheaves was also proved by Drinfeld [25], Gaitsgory [28], Böckle-Khare [7] using results from Langlands program. Very roughly speaking, after reduction to positive characteristic, they can relate semisimple perverse sheaves with perverse sheaves of geometric origin and reduce to the Beilinson-Bernstein-Deligne-Gabber Decomposition Theorem.

Recently, El Zein, Lê and Ye [59] give a different geometric proof of the Decomposition Theorem of variation of Hodge structures (which is weaker than Saito's filtered Decomposition Theorem) using the local purity theorem.

2.1.2 What is new?

The original contributions of this part of the dissertation include

1. I set up the theory of Hodge star operators for differential forms on compact Kahler manifolds with harmonic bundle coefficients (§2.4), which is necessary for the discussion of harmonic forms with harmonic bundle coefficients. The latter is briefly mentioned in [52]. This should be well-known to experts but I could not find an adequate reference.

2. I relate Simpson's notion of polarizations of pure twistor structures to twisted Poincaré pairings and give a criterion for non-degeneracy of restrictions of twisted Poincaré pairings (Corollary 2.3.24 and Corollary 2.3.58).

3. I complement Simpson's theory of weights by proving a global invariant cycle theorem for semisimple local systems (Theorem 2.3.99).

4. I obtain new twistor-theoretic results for cohomology of algebraic varieties with semisimple local systems, which include the Hard Lefschetz Theorem for Perverse Cohomology Complexes (Theorem B) and the Generalized Hodge-Riemann Bilinear Relation for cohomology of semisimple local systems (Theorem D). Theorem B actually follows from Mochizuki's results [38], but my proof is more elementary.

5. As the main application, I give a new and geometric proof of the Decomposition Theorem for Semisimple Local systems (Theorem A) adapting the method of de Cataldo-Migliorini.

2.1.3 Outline of the chapter

For the convenience of the reader, here is the general outline of this chapter.

In §2.2 is about the statement of main results and various technical results which will be used in the course of proof of the Decomposition Theorem.

In §2.3 concerns the extra structures on the cohomology of semisimple local systems on algebraic varieties. Since this is the longest section, I would like to give more details. In §2.3.2, I review the definition of polarizable pure twistor structures and discuss Simpson and Sabbah's constructions of natural pure twistor structures on the cohomology of semisimple local systems on smooth projective varieties. An equivalent definition of polarization is introduced, which is related to the twisted Poincaré pairing in §2.3.3. This is the key observation of this thesis where a non-degenerate pairing from topology can be related to a positive-definite pairing. In §2.3.4, Simpson's theory of weights for semisimple local systems is discussed. As a complement, I prove a global invariant cycle theorem for semisimple local systems. In §2.3.5, I set up various definitions concerning weight filtrations for nilpotent operators and polarization of the associated graded spaces on two vector spaces related by a non-degenerate pairing. This is slightly different from the classical setup in Hodge theory, but it is natural from the perspective of polarizations on pure twistor structures. It will finally allow us to prove the Generalized Hodge-Riemann Bilinear Relation for cohomology of semisimple local systems (Theorem D). In §2.3.6, it is proved that perverse filtrations on cohomology of semisimple local systems on smooth projective varieties underlie natural pure twistor structures using the geometric characterization via hyperplanes found by de Cataldo and Migliorini [19].

In §2.4, I set up the theory of Hodge star operators for differential forms with harmonic bundle coefficients. It can be regarded as a replacement of the Weil operator in Hodge theory and therefore is useful in the discussion of polarizations of pure twistor structures on the cohomology of semisimple local systems.

In §2.5, I collect some topological facts about constructible complexes and perverse sheaves, which will be used in the proof of the main results. In particular, the weak Lefschetz hyperplane theorem and the splitting criterion for perverse sheaves are recalled.

In §2.6 - §2.9 , building on the foundation set up in previous sections, I give the proof of Decomposition Theorem for semisimple local systems (Theorem A), Hard Lefschetz Theorem for Perverse Cohomology Complexes (Theorem B) and the Generalized Hodge-Riemann Bilinear Relation for cohomology of semisimple local systems (Theorem D).

2.2 Statement of main results

In this section, I will state my results for projective maps $f : X \to Y$ where X is smooth projective. Using standard reductions (see §2.2), these results remain valid for $f : U \to Y$ and $\mathcal{V}|_U$, where f is a proper map, U is a Zariski open subset of a smooth projective X and $\mathcal{V}$ is a semisimple local system on X. In particular, one recovers Sabbah's Decomposition Theorem [43], which is stated under this setting.

The projective case

Here is the main setup of this chapter.

Set-up 2.2.1. -

- Let $f : X \to Y$ be a map between projective varieties, where X is nonsingular of dimension n and $\dim f(X) = m$.
- Let $\mathcal{V}$ be a semisimple local system on X and denote $K := \mathcal{V}[\dim X]$.
- Let η be an ample line bundle on X and A be an ample line bundle on Y. Denote $L := f^*A$ to be the pull-back line bundle on X.
- Denote $\mathcal{V}^*$ to be the dual local system and $K^* := D_X(K) \cong \mathcal{V}^*[\dim X]$ to be the dual perverse sheaf.

The following statement is proved by Sabbah [43].

Theorem A. *Suppose we are in the Set-up 2.2.1.*

(a) (Relative Hard Lefschetz Theorem) The following cup product map is an isomorphism:

$$\eta^\ell \colon {}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*K) \xrightarrow{\sim} {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K).$$

(b) (Decomposition Theorem) There exists a (non-canonical) splitting in $D(Y)$:

$$f_*K \cong \bigoplus_{\ell} {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)[-\ell].$$

Here f_*K denotes the total direct image functor.

(c) (Semisimplicity Theorem) For each i, ${}^{\mathfrak{p}}\mathcal{H}^i(f_*K)$ is a semisimple perverse sheaf. More precisely, given any stratification for f so that $Y = \amalg_\ell S_\ell$, there is a canonical isomorphism in $\mathrm{Perv}(Y)$*:*

$$ {}^{\mathfrak{p}}\mathcal{H}^i(f_*K) \cong \bigoplus_{\ell=0}^{\dim Y} \mathrm{IC}_{\overline{S_\ell}}(L_{i,\ell})$$

where the local system $L_{i,\ell} := \mathcal{H}^{-\ell}({}^{\mathfrak{p}}\mathcal{H}^i(f_*K))|_{S_\ell}$ on S_ℓ are semisimple.

Remark 2.2.2. The Relative Hard Lefschetz Theorem actually holds for f-ample line bundles, which can be deduced from ample lines as in [17, Remark 5.1.2]. This stronger statement will be used in the inductive proof (see §2.6.3).

Auxiliary results

I would like to state two auxiliary results, which are important for the proof of the Semisimplicity Theorem A(c). Before doing so, I would like to use the following notation adapting from [17].

Notation 2.2.3. Let $f : X \to Y$ be an algebraic map between algebraic varieties and let K be a perverse sheaf on X. We denote the perverse Leray filtration on $\mathbb{H}^b(X, K)$ as follows

$$H^b_{\leq \ell}(X, K) := \operatorname{Im}\{\mathbb{H}^b(Y, {}^{\mathfrak{p}}\tau_{\leq \ell} f_* K) \to \mathbb{H}^b(Y, f_* K)\} \subseteq \mathbb{H}^b(X, K),$$

and

$$H^b_\ell(X, K) := H^b_{\leq \ell}(X, K)/H^b_{\leq \ell-1}(X, K).$$

Remark 2.2.4. It is straight forward to see that the cup product map with η satisfies $\eta\left(H^b_{\leq \ell}(X, K)\right) \subseteq H^{b+2}_{\leq \ell+2}(X, K)$. Together with cup product map with L, we have the following maps

$$\eta : H^b_\ell(X, K) \to H^{b+2}_{\ell+2}(X, K),$$
$$L : H^b_\ell(X, K) \to H^{b+2}_\ell(X, K).$$

Remark 2.2.5. Note the difference between our notations and de Cataldo-Migliorini [17, Definition 4.2.1]. Here we don't shift the cohomology degrees by $[\dim X]$, which has been built into the perverse sheaves.

Remark 2.2.6. Assuming the Decomposition Theorem, one can obtain the following identification map depending on the choice of splitting:

$$H^b_\ell(X, K) \cong \mathbb{H}^{b-\ell}(Y, {}^{\mathfrak{p}}\mathcal{H}^\ell(f_* K)).$$

Theorem B (The Hard Lefschetz Theorem for Perverse Cohomology Complexes)**.** *Suppose we are in the Set-up 2.2.1. Let ℓ and j be any integer. Then the following cup product maps are isomorphisms*

$$L^j \colon H^{\ell-j}_\ell(X, K) \xrightarrow{\sim} H^{\ell+j}_\ell(X, K),$$
$$\eta^\ell \colon H^{j-\ell}_{-\ell}(X, K) \xrightarrow{\sim} H^{j+\ell}_\ell(X, K).$$

Equivalently, assuming the Decomposition Theorem, we have

$$A^j \colon \mathbb{H}^{-j}(Y, {}^{\mathfrak{p}}\mathcal{H}^\ell(f_* K)) \xrightarrow{\sim} \mathbb{H}^j(Y, {}^{\mathfrak{p}}\mathcal{H}^\ell(f_* K)),$$
$$\eta^\ell \colon \mathbb{H}^j(Y, {}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_* K)) \xrightarrow{\sim} \mathbb{H}^j(Y, {}^{\mathfrak{p}}\mathcal{H}^\ell(f_* K)).$$

Consider the following spaces: if $\ell, j \geq 0$, set

$$P^{-j}_{-\ell} := \operatorname{Ker} \eta^{\ell+1} \cap \operatorname{Ker} L^{j+1} \subseteq H^{-\ell-j}_{-\ell}(X, K);$$

if $\ell < 0$ or $j < 0$, one sets $P^{-j}_{-\ell} = 0$. As in the classical Lefschetz decomposition, Theorem B implies that

Corollary C. *There is a double Lefschetz decomposition:*

$$H_{-\ell}^{-\ell-j}(X,K) = \bigoplus_{m,i\geq 0} \eta^m L^i P_{-\ell-2m}^{-j-2i}.$$

Recall that $K^* = \mathcal{V}^*[\dim X]$. Using representatives in

$$H_{\leq -\ell}^{-\ell-j}(X,K) \subseteq \mathbb{H}^{-\ell-j}(X,K) = H^{n-\ell-j}(X,\mathcal{V})$$
$$H_{\leq -\ell}^{-\ell-j}(X,K^*) \subseteq \mathbb{H}^{-\ell-j}(X,K^*) = H^{n-\ell-j}(X,\mathcal{V}^*)$$

one defines a bilinear pairing

$$\begin{aligned} S_{\ell j}^{\eta L} : H_{-\ell}^{-\ell-j}(X,K) \otimes_{\mathbb{C}} H_{-\ell}^{-\ell-j}(X,K^*) &\to \mathbb{C} \\ ([\alpha\otimes e],[\beta\otimes\lambda]) &\mapsto C(n,\ell,j)\int_X \lambda(e)\cdot\eta^\ell\wedge L^j\wedge\alpha\wedge\beta \\ C(n,\ell,j) &= i^{-(n-\ell-j)}(-1)^{(n-\ell-j)(n-\ell-j-1)/2}. \end{aligned}$$

Here α,β are $(n-\ell-j)$-forms on X and $e\in\mathcal{C}^\infty(H)$, $\lambda\in\mathcal{C}^\infty(H^*)$, where $H := \mathcal{V}\otimes_{\mathbb{C}}\mathcal{C}_X^\infty$ is the harmonic bundle associated to $\mathcal{V}$ and H^* is the dual harmonic bundle. By Theorem 2.3.30, we can represent cohomology elements by harmonic forms with coefficients in H and H^*.

Remark 2.2.4. Here $S_{\ell j}^{\eta L}$ is induced by the twisted Poincaré pairing in §2.3.5 for $k = n-i-j$. It will be proved in Corollary 2.7.3 that as a consequence of Theorem B, $S_{\ell j}^{\eta L}$ is non-degenerate.

Theorem D (The Generalized Hodge-Riemann Bilinear Relations)**.** *Suppose we are in the Set-up 2.2.1.*

- *The double Lefschetz decomposition in Corollary C is orthogonal with respect to $S_{\ell j}^{\eta L}$ in the sense that if $(m,i)\neq(m',i')$, then*

  $$S_{\ell j}^{\eta L}(\eta^m L^i P_{-\ell-2m}^{-j-2i}, \eta^{m'} L^{i'} P_{-\ell-2m'}^{*-j-2i'}) = 0.$$

  Here $P_{-\ell}^{*-j}$ denotes the corresponding primitive subspace of $H_{-\ell}^{-\ell-j}(X,K^*)$.

- *Each direct summand $\eta^m L^i P_{-\ell-2m}^{-j-2i}$ underlies a natural pure twistor structure F (Definition 2.3.8) where*

  $$F|_{z=1} = \eta^m L^i P_{-\ell-2m}^{-j-2i}.$$

  There is a canonical map

  $$\phi : \overline{F|_{z=-1}} \xrightarrow{\sim} \eta^m L^i P_{-\ell-2m}^{*-j-2i},$$

  so that F is polarized by the bilinear pairing

  $$S_{\ell j}^{\eta L}(\bullet,\phi(\bullet)) : F|_{z=1}\otimes_{\mathbb{C}}\overline{F|_{z=-1}} \to \mathbb{C}$$

  in the sense of Definition 2.3.21 up to a constant depending on (n,m,ℓ,i,j) (see Remark 2.3.112).

- *As a corollary, $S_{\ell j}^{\eta L}$ restricts to a non-degenerate pairing:*

  $$S_{\ell j}^{\eta L} : \eta^m L^i P_{-\ell-2m}^{-j-2i} \otimes_{\mathbb{C}} \eta^m L^i P_{-\ell-2m}^{*-j-2i} \to \mathbb{C}.$$

The algebraic case

Using the standard reductions in [16, Page 71-74] and [15], we obtain Sabbah's main Theorem in [43]. For the proof, see §2.9.

Theorem E. *Let $f : U \to Y$ be a proper map between quasi-projective varieties where U is the Zariski open subset of a smooth projective variety X. Let $\mathcal{V}$ be a semisimple local system on X. Then Theorem A(b) and Theorem A(c) hold for f and $\mathcal{V}|_U$.*

If in addition, f is projective and η is f-ample, then the Relative Hard Lefschetz Theorem A(a) holds as well.

Remark 2.2.6. In [43], to use D-module theoretic methods, Sabbah assumed that Y is smooth projective. The topological methods used here enables us to deal with any singular base Y. On the other hand, if one assumes the much stronger Decomposition Theorem for semisimple perverse sheaves and projective maps between smooth quasi-projective varieties, then de Cataldo [15] showed that the Decomposition theorem remain valid without the smoothness assumptions.

Technical results: weights and polarizations

In this section we would like to introduce some technical results regarding the structures of cohomology of semisimple local systems on algebraic varieties. Unlike the trivial local system $\mathbb{C}$, these cohomology groups don't necessarily carry mixed Hodge structures. However, Simpson observed that they always underlie mixed twistor structures [51] (see Definition 2.3.4). To prove Theorem B and Theorem D, we need the following two generalizations of classical results in Hodge theory to twistor theory, which play the role of weights and polarizations.

Theorem F (Theorem 2.3.99, Global Invariant Cycle Theorem for Semisimple Local Systems)**.** *Consider the following chain of inclusion maps:*

$$Z \xrightarrow{\alpha} U \xrightarrow{j} X,$$

where X is a smooth projective variety, U is a Zariski open subset of X and Z is a proper subvariety of X contained in U. Let $\mathcal{V}$ be a semisimple local system on X. Then for any integer k, the following two restriction maps have the same image:

$$(j \circ \alpha)^* : H^k(X, \mathcal{V}) \to H^k(Z, (j \circ \alpha)^* \mathcal{V})$$
$$\alpha^* : H^k(U, j^* \mathcal{V}) \to H^k(Z, (j \circ \alpha)^* \mathcal{V}).$$

Remark 2.2.7. This Theorem will be used to prove injectivity of the following cycle map:

$$H^0(Y, \alpha_! \alpha^! {}^{\mathfrak{p}}\mathcal{H}^0(f_* K)) \hookrightarrow H^0(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_* K)).$$

where $\alpha : y \hookrightarrow Y$ is a point in the support of $\mathcal{H}^0({}^{\mathfrak{p}}\mathcal{H}^0(f_* K))$ (see Corollary 2.3.101).

The next result provides a useful criterion for proving non-degeneracy of bilinear pairings under restrictions.

Lemma G (see Corollary 2.3.24)**.** *Let $f : V \to \tilde{V}$ be a map of vector spaces. Assume that*

1. *f underlies the morphism of pure twistor structures $F : E \to \tilde{E}$ (see Definition 2.3.8), i.e. E and $\tilde{E}$ are two holomorphic vector bundles over $\mathbf{P}^1$ with the same slope and there is a commutative diagram*

$$\begin{array}{ccc} E & \xrightarrow{F} & \tilde{E} \\ \Big\downarrow {\scriptstyle \mathrm{ev}_{z=1}} & & \Big\downarrow {\scriptstyle \mathrm{ev}_{z=1}} \\ V = E|_{z=1} & \xrightarrow{f} & \tilde{V} = \tilde{E}|_{z=1}. \end{array}$$

2. *E is polarized by a non-degenerate bilinear pairing (see Definition 2.3.21)*

$$S : E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} \to \mathbb{C}.$$

Then S restricts to a non-degenerate pairing

$$S : \operatorname{Ker} f \otimes_{\mathbb{C}} \overline{\operatorname{Ker} f_{-1}} \to \mathbb{C},$$

where $f_{-1} : E|_{z=-1} \to \tilde{E}|_{z=-1}$ is the evaluation of $F : E \to \tilde{E}$ at $z = -1$.

Lemma H (Lemma 2.3.52)**.** *Let X be a smooth projective variety and $\mathcal{V}$ be a semisimple local system on X. Then for each k, $H^k(X, \mathcal{V})$ underlies a natural pure twistor structure E so that there is an isomorphic canonical map*

$$\phi : \overline{E|_{z=-1}} \xrightarrow{\sim} H^k(X, \mathcal{V}^*).$$

where $\mathcal{V}^*$ is the dual local system.

Remark 2.2.9. This lemma enables us to relate the polarization on the twistor structure E with the twisted Poincaré pairing. See Corollary 2.3.55.

Strategy of the proof

We want to illustrate the main idea of the proof. The overall strategy is borrowed from de Cataldo-Migliorini (see [17, §2.6]). But the main inputs are different.

Step one. By Deligne's Lefschetz splitting criterion [20, Theorem 1.5], the Decomposition Theorem for semisimple local systems A(a) is reduced to the Relative Hard Lefschetz Theorem A(b), which will be proved inductively together with the Semisimplicity Theorem A(c).

Step two. Using the construction of universal hyperplanes in §2.5.2, one can prove the Relative Hard Lefschetz Theorem using the inductive Relative Hard Lefschetz and the inductive Semisimplicity Theorem by induction on the defect of semismallness. This strategy actually already appears in Beilinson-Bernstein-Deligne-Gabber, but the topological inductive feature is explored more systematically by de Cataldo-Migliorini.

Step three. The key part of the proof is to use the Relative Hard Lefschetz to prove the Semisimplicity Theorem. To achieve this, one proves several auxiliary results as stated in §2.2 along with the induction process. In this process, besides the defect of semismallness, one also needs to induct on the dimension of the image of the map.

First, the Relative Hard Lefschetz Theorem implies the Hard Lefschetz Theorem for Perverse Cohomology Complexes (Theorem B), with the help of the inductive Generalized Hodge-Riemann bilinear relation (Theorem D). The fact that perverse filtrations of cohomology of semisimple local system (Lemma 2.3.122) underlie natural pure twistor structures is also used.

Second, the Generalized Hodge-Riemann bilinear relation is proved inductively together with a precise relation between the weight filtrations induced by f^*A and $\lim_{\epsilon\to 0}(f^*A+\epsilon\eta)$, where $f: X\to Y$ is the proper map, A is an ample line bundle on Y and η is an ample line bundle on X. Here is where the setup of weight filtrations on two companion vector spaces is used (§2.3.5). Then one reduces to the classical Hodge-Riemann bilinear relation for semisimple local systems on smooth projective varieties by Simpson and Sabbah (Theorem 2.3.26). This is the most subtle part of the argument.

Last, with these auxiliary results at hand, one proves the Semisimplicity Theorem A(c) in two steps: first splits ${}^{\mathfrak{p}}\mathcal{H}^0(f_*K)$ into a direct sum of intersection complexes of local systems on strata and then show that each local system is semisimple. To verify the splitting criterion of de Cataldo-Migliorini, one uses the global invariant cycle theorem for semisimple local systems (Theorem 2.3.99), the Generalized Hodge-Riemann Bilinear Relation and the fact that bilinear pairings polarizing a pure twistor structure still restricts to non-degeneracy pairings on subspaces underlying sub-twistor structures (see Proposition 2.7.12). Semisimplicity of local systems over strata comes after relating them to direct images of semisimple local systems under smooth projective maps, which are semisimple by Simpson.

2.3 Cohomology of semisimple local systems

In this section, we would like to recall and prove several results of semisimple local systems, which generalize the classical statements in Hodge theory.

2.3.1 Statements

Classical results

Let X be a smooth projective variety of dimension n. Let $\mathcal{V}$ be a semisimple local system on X, which means that the corresponding representation of $\pi_1(X)$ is semisimple. Denote $\mathcal{V}^*$ to be the dual local system. Let η be an ample line bundle on X.

Theorem 2.3.1. -

1. *(Hard Lefschetz Theorem) For each* $j \geq 1$*, the following cup product map is an isomorphism*

$$\eta^j : H^{n-j}(X, \mathcal{V}) \xrightarrow{\sim} H^{n+j}(X, \mathcal{V}).$$

2. *(Primitive Lefschetz decomposition) For each* $k \geq 1$*,* $H^k(X, \mathcal{V})$ *underlies a natural pure twistor structure of weight* k *(Definition 2.3.8). Moreover, for* $j \leq n$*, there is a direct sum decomposition*

$$H^{n-j}(X, \mathcal{V}) \cong \bigoplus_{m \geq 0} \eta^m P^{n-j-2m},$$
$$P^{n-j-2m} := \operatorname{Ker} \eta^{j+1+2m} \subseteq H^{n-j-2m}(X, \mathcal{V})$$

so that each primitive space $\eta^m P^{n-j-2m}$ *underlies a pure sub-twistor structure which is polarizable.*

Remark 2.3.2. -

1. The Hard Lefschetz Theorem for semisimple local systems is proved by Simpson [52, Lemma 2.6].
2. The second statement is proved by Simpson [51, Theorem 4.1] and Sabbah [43, Theorem 2.2.4]. It will be referred later as the Hodge-Simpson Theorem 2.3.26.

Theorem 2.3.3 (Semisimplicity Theorem for smooth projective maps)**.** *Let* $f : X \to Y$ *be a smooth projective morphism between smooth quasi-projective varieties. Let* $\mathcal{V}$ *be a semisimple local system on* X *so that* $\mathcal{V} \otimes_{\mathbb{C}} \mathcal{C}_X^\infty$ *is a harmonic bundle on* X*, then*

$$R^q f_* \mathcal{V}$$

is a semisimple local system on Y *for every* $q \geq 0$*.*

Proof. In [50, Corollary 4.2], Simpson proved the case where X and Y are both smooth projective. The above version actually follows immediately from Theorem [50, Theorem 4.1]. It is because we assume that $\mathcal{V}$ comes from a harmonic bundle on X. Also we use the fact that the local system associated to a harmonic bundle on a smooth variety is always semisimple. □

New results

We will prove the following results in this section.

Theorem 2.3.4 (Corollary 2.3.55, Polarization by twisted Poincaré pairings)**.** *With the notation in Theorem 2.3.1. Let* F *be the natural pure twistor structure on the primitive space* $\eta^m P^{n-j-2m}$*. Denote the corresponding primitive subspace in* $H^{n-j}(X, \mathcal{V}^*)$ *to be*

$$P^{*n-j-2m} := \operatorname{Ker} \eta^{j+1+2m} \subseteq H^{n-j-2m}(X, \mathcal{V}^*).$$

Then there exists a canonical map

$$\phi : \overline{F|_{z=-1}} \xrightarrow{\sim} \eta^m P^{*n-j-2m},$$

so that the twistor structure F is polarized in the sense of Definition 2.3.21 by the bilinear pairing

$$S_X(\bullet, \phi(\bullet)) : F|_{z=1} \otimes_{\mathbb{C}} \overline{F|_{z=-1}} \to \mathbb{C}.$$

Here S_X is the twisted Poincaré pairing (Definition 2.3.48).

Theorem 2.3.5 (Theorem 2.3.99, Global Invariant Cycle Theorem for Semisimple Local Systems)**.** *Consider the following chain of inclusion maps:*

$$Z \xrightarrow{\alpha} U \xrightarrow{j} X,$$

where X is a smooth projective variety, U is a Zariski open subset of X and Z is a proper subvariety of X contained in U. Let $\mathcal{V}$ be a semisimple local system on X. Then for any integer k, the following two restriction maps have the same image:

$$(j \circ \alpha)^* : H^k(X, \mathcal{V}) \to H^k(Z, (j \circ \alpha)^* \mathcal{V})$$
$$\alpha^* : H^k(U, j^* \mathcal{V}) \to H^k(Z, (j \circ \alpha)^* \mathcal{V}).$$

Lemma 2.3.6 (Lemma 2.3.122, Perverse filtrations underlie twistor structures)**.** *For any integers ℓ and b, let E be the natural pure twistor structure on $H^{b+\dim X}(X, \mathcal{V})$ in Theorem 2.3.1. Set $K = \mathcal{V}[\dim X]$, then the subspace induced by the perverse filtration (Notation 2.2.3)*

$$H^b_{\leq \ell}(X, K) \subseteq \mathbb{H}^b(X, K) = H^{b+\dim X}(X, \mathcal{V})$$

underlies a pure sub-twistor structure of E. In particular, the quotient space

$$H^b_\ell(X, K) := H^b_{\leq \ell}(X, K) / H^b_{\leq \ell - 1}(X, K)$$

inherits a natural pure twistor structure of weight $(b + \dim X)$.

Remark 2.3.7. -

1. The relation between polarizations of pure twistor structures and twisted Poincaré pairings is one of the main observations of this paper. It will be discussed in length in §2.3.3.

2. We will prove the Global Invariant Cycle Theorem for Semisimple Local Systems using Simpson's theory of mixed twistor structures [51] in §2.3.4. This should be well-known to experts, but we include a proof here for lack of references.

3. The existence of pure twistor structures on the perverse filtrations is a natural generalization of the Hodge-theoretic results by de Cataldo-Migliorini [19]. It will be proved in §2.3.6. This result will be used in the proof of the Hard Lefschetz Theorem for Perverse Cohomology Complexes B to simplify the original argument in [17], as suggested by [18].

2.3.2 Cohomology of smooth projective varieties and pure twistor structures

In this section, we review Simpson's notion of polarizable pure twistor structures [51] and the Hodge-Simpson Theorem 2.3.26 which states that the cohomology groups of semisimple local systems on smooth projective varieties underlie natural pure twistor structures.

Pure twistor structures

Definition 2.3.8. -

1. A *twistor structure* is a holomorphic vector bundle E on $\mathbf{P}^1$. A twistor structure E is *pure* of weight w if E is a direct sum of copies of $\mathcal{O}_{\mathbf{P}^1}(w)$.
2. Morphisms of twistor structures are morphisms between holomorphic vector bundle over $\mathbf{P}^1$.

We say a complex vector space V *underlies* a twistor structure if $V \cong E|_{z=1}$ (the fiber over the point $1 \in \mathbf{P}^1$) for some holomorphic bundle E. We also say that E *is a twistor structure on* V.

Remark 2.3.9. By Grothendieck's theorem on holomorphic vector bundles over $\mathbf{P}^1$, the category of pure twistor structures of a fixed weight is equivalent to the category of complex vector spaces.

Identification maps and canonical trivializations

For the purpose of discussing polarizations on pure twistor structures, let us define some canonical structures associated to a pure twistor structure.

Notation 2.3.10. Denote Ω_0 and Ω_∞ to be the standard $\mathbf{A}^1$ neighborhoods of 0 and ∞ in $\mathbf{P}^1$ respectively.

Definition 2.3.11 (Identification map)**.** Let E be a pure twistor structure of weight w. We would like to define an *identification map*

$$\mathrm{Iden} : E|_{z=1} \to E|_{z=-1}.$$

1. If the weight of E is 0, the identification map is defined to be

$$\mathrm{Iden} : E|_{z=1} \xrightarrow{(\mathrm{ev}_{z=1})^{-1}} H^0(\mathbf{P}^1, E) \xrightarrow{\mathrm{ev}_{z=-1}} E|_{z=-1}.$$

   where $\mathrm{ev}_{z=z_0}$ is the evaluation map for global sections, which are isomorphisms because E is a trivial bundle.

2. If the weight of E is w, set $V = H^0(\mathbf{P}^1, E(-w))$ and choose $\mu \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ to be the unique section up to scaling so that $\mu|_{\Omega_0}$ is nowhere zero. Notice that the evaluation of $\mu^{\otimes w}$ at $z = z_0 \in \Omega_0$ gives the isomorphism, which we denote

$$\mathrm{ev}_{z=z_0} : V \to V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}^1}(w)|_{z=z_0} \to E(-w)|_{z=z_0} \otimes \mathcal{O}_{\mathbf{P}^1}(w)|_{z=z_0} = E|_{z=z_0}.$$

The map $V \to E(-w)|_{z=z_0}$ is an isomorphism because $E(-w)$ has weight zero. Then the identification map is defined by the composition map:

$$\mathrm{Iden} : E|_{z=1} \xrightarrow{(\mathrm{ev}_{z=1})^{-1}} V \xrightarrow{\mathrm{ev}_{z=-1}} E|_{z=-1}$$

and it involves no choice of μ.

Definition 2.3.12 (Canonical trivialization)**.** Let $E \cong \oplus \mathcal{O}_{\mathbf{P}^1}(w)$ be a pure twistor structure of weight w. Set $V = H^0(\mathbf{P}^1, E(-w))$, then we have a natural isomorphism

$$E(-w) \cong V \otimes_{\mathbb{C}} \mathcal{O}_{\mathbf{P}^1}.$$

We define *canonical trivializations* of E by the vector space V over two charts Ω_0 and Ω_∞ as follows.

- Over Ω_0: let $\mu \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ be the unique section up to scaling so that $\mu|_{\Omega_0}$ is nowhere zero. Then $\mu^{\otimes w}$ gives a unique trivialization of E over Ω_0 up to scaling:

$$\phi_0 : V \otimes_{\mathbb{C}} \mathcal{O}_{\Omega_0} \xrightarrow{\sim} E(-w)|_{\Omega_0} \xrightarrow{\mu^{\otimes w}} E|_{\Omega_0}.$$

- Over Ω_∞: let $\lambda \in H^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(1))$ be the unique section up to scaling so that $\lambda|_{\Omega_\infty}$ is nowhere zero. Then $\lambda^{\otimes w}$ gives the trivialization over Ω_∞:

$$\phi_\infty : V \otimes_{\mathbb{C}} \mathcal{O}_{\Omega_\infty} \xrightarrow{\sim} E(-w)|_{\Omega_\infty} \xrightarrow{\lambda^{\otimes w}} E|_{\Omega_\infty}.$$

Remark 2.3.13. Set $\mathbf{G}_m = \Omega_0 \cap \Omega_\infty$, we can calculate the transition map $\phi_\infty^{-1} \circ \phi_0$ of the two canonical trivialization maps:

$$\begin{aligned} &V \otimes \mathcal{O}_{\mathbf{G}_m} \to E(-w)|_{\mathbf{G}_m} \xrightarrow{\mu^{\otimes w}} E|_{\mathbf{G}_m} \xrightarrow{(\lambda^{\otimes w})^{-1}} E(-w)|_{\mathbf{G}_m} \to V \otimes \mathcal{O}_{\mathbf{G}_m} \\ &v \otimes 1 \mapsto \mu^{\otimes w}|_{\mathbf{G}_m} \cdot (v \otimes 1) \mapsto \frac{\mu^{\otimes w}}{\lambda^{\otimes w}}|_{\mathbf{G}_m} \cdot v \otimes 1 \end{aligned}$$

Since $\mu(\infty) = 0$ and $\lambda(0) = 0$, we can denote $z := \frac{\lambda}{\mu}$ so that z is the coordinate on Ω_0. Therefore this verifies that the transition map for $E \cong \oplus \mathcal{O}_{\mathbf{P}^1}(w)$ sends $v \in V$ to $z^{-w} v \in V$.

Remark 2.3.14. By the proof of Birkhoff's Theorem or equivalently Grothendieck's theorem for vector bundles over $\mathbf{P}^1$, for any two trivializations of E over Ω_0 and Ω_∞, up to automorphisms over Ω_0 and Ω_∞, the transition map can be written in the form $v \mapsto z^{-w} v$.

Remark 2.3.15. Let E be a pure twistor structure of weight w and suppose there is a trivialization of E over Ω_0

$$\phi_0^W : E|_{\Omega_0} \cong W \otimes \mathcal{O}_{\Omega_0}.$$

This induces an isomorphic evaluation map at $z = z_0$:

$$\begin{aligned} \mathrm{ev}_{z=z_0} : W &\to E|_{z=z_0} \\ w &\mapsto (\phi_0^W)^{-1}(w \otimes 1)(z_0) \end{aligned}$$

Then the identification map can be calculated as

$$E|_{z=1} \xrightarrow{(\mathrm{ev}_{z=1})^{-1}} W \xrightarrow{\mathrm{ev}_{z=-1}} E|_{z=-1}.$$

It is okay to choose any trivialization because one can identify W with V by

$$V \otimes \mathcal{O}_{\Omega_0} \xrightarrow{\phi_0} E|_{\Omega_0} \xrightarrow{(\phi_0^W)^{-1}} W \otimes \mathcal{O}_{\Omega_0}.$$

where ϕ_0 is the canonical trivialization map in Definition 2.3.12.

Polarization and bilinear pairings

In this section, we review Simpson's notion of polarization on pure twistor structures [51, Page 20]. Let σ denote the antipodal involution of $\mathbf{P}^1$ where

$$\sigma(z) = -\frac{1}{\bar{z}}.$$

In particular, it is antilinear, interchanges 0 and ∞ and interchanges 1 and -1.

Construction 2.3.16 (Conjugation over $\mathbf{P}^1$)**.** Let E be a pure twistor structure. We would like to define a locally free sheaf of $\mathcal{O}_{\mathbf{P}^1}$-modules σ^*E as follows. Let $U \subseteq \mathbf{P}^1$ be any open subset. Set

- $(\sigma^*E)(U) := E(\sigma(U))$.
- For $e \in (\sigma^*E)(U)$ and $a \in \mathcal{O}_{\mathbf{P}^1}(U)$, define

$$a \cdot e := \overline{\sigma_*(a)}e.$$

In particular, σ^*E is also a pure twistor structure.

Notation 2.3.17. Let V be a $\mathbb{C}$-vector space. We use the following convention to put a $\mathbb{C}$-vector space structure on $\overline{V}$ by defining

$$\overline{V} := V \otimes_{\mathbb{C}} \overline{\mathbb{C}}$$

where $\overline{\mathbb{C}}$ is viewed as a $\mathbb{C}$-module via the conjugation map:

$$\mathbb{C} \to \overline{\mathbb{C}}, \quad \lambda \mapsto \overline{\lambda}.$$

Remark 2.3.18. On the level of global sections, σ induces a natural isomorphism of $\mathbb{C}$-vector spaces:

$$\sigma : \overline{H^0(\mathbf{P}^1, E)} \xrightarrow{\sim} H^0(\mathbf{P}^1, \sigma^*E).$$

Definition 2.3.19 (Polarization)**.** Let E be a pure twistor structure of weight w. A *polarization* on E is a morphism of pure twistor structures

$$P : E \otimes_{\mathcal{O}_{\mathbf{P}^1}} \sigma^* E \to \mathcal{O}_{\mathbf{P}^1}(2w),$$

which is equivalent to

$$P(-2w) : E(-w) \otimes_{\mathcal{O}_{\mathbf{P}^1}} \sigma^*(E(-w)) \to \mathcal{O}_{\mathbf{P}^1},$$

so that the induced morphism on global sections

$$\begin{aligned} & H^0(\mathbf{P}^1, E(-w)) \otimes_{\mathbb{C}} \overline{H^0(\mathbf{P}^1, E(-w))} \\ \xrightarrow{\mathrm{Id}\otimes\sigma} & H^0(\mathbf{P}^1, E(-w)) \otimes_{\mathbb{C}} H^0(\mathbf{P}^1, \sigma^*(E(-w))) \to \mathbb{C} \end{aligned}$$

is a positive hermitian pairing. Here σ is the isomorphism in Remark 2.3.18.

We say E is *polarizable* if there exists such a morphism P.

For our purpose, it is more convenient to reformulate the notion of pairing in terms of a bilinear pairing between fibers of E, which will be finally related to the twisted Poincaré pairings (see Corollary 2.3.55).

Lemma 2.3.20. *Let E be a pure twistor structure of weight w. Then E is polarizable in the sense of Definition 2.3.19 if and only if there exists a bilinear pairing*

$$S : E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} \to \mathbb{C}$$

so that

$$S(\bullet, \overline{\mathrm{Iden}}(\bullet)) : E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=1}} \to \mathbb{C}$$

is a positive hermitian pairing, where Iden *is the identification map in Definition 2.3.11.*

Proof. First we assume that E is of weight 0. Suppose E is polarizable by $P : E \otimes \sigma^* E \to \mathcal{O}_{\mathbf{P}^1}$. We have the following commutative diagram

$$\begin{array}{ccc} H^0(\mathbf{P}^1, E) \otimes_{\mathbb{C}} H^0(\mathbf{P}^1, \sigma^* E) & \xrightarrow{T} & \mathbb{C} \\ \downarrow{\scriptstyle \mathrm{ev}_{z=1}\otimes \mathrm{ev}_{z=1}} & & \downarrow{\scriptstyle \mathrm{Id}} \\ E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} & \xrightarrow{S} & \mathbb{C} \end{array}$$

Here T and S are two bilinear pairings induced by P on global sections and fiber at $z = 1$ respectively (note that by the construction of σ, the fiber of $\sigma^* E$ at $z = 1$ is $\overline{E|_{z=-1}}$).

Now we would like to show that for any $0 \neq v \in E|_{z=1}$, we have

$$S(v, \overline{\mathrm{Iden}}(\overline{v})) > 0.$$

By the construction of σ, there is a $\mathbb{C}$-linear isomorphism

$$\sigma : \overline{H^0(\mathbf{P}^1, E)} \xrightarrow{\sim} H^0(\mathbf{P}^1, \sigma^* E).$$

The polarization condition on P means that

$$T(\alpha, \sigma(\overline{\alpha})) > 0, \quad \forall 0 \neq \alpha \in H^0(\mathbf{P}^1, E).$$

On the other hand, the conjugation of the evaluation map factors through σ:

$$\overline{\mathrm{ev}_{z=-1}} : \overline{H^0(\mathbf{P}^1, E)} \xrightarrow{\sigma} H^0(\mathbf{P}^1, \sigma^* E) \xrightarrow{\mathrm{ev}_{z=1}} \overline{E|_{z=-1}}.$$

By Definition 2.3.11 the identification map is expressed using the evaluation maps

$$\mathrm{Iden} : E|_{z=1} \xrightarrow{(\mathrm{ev}_{z=1})^{-1}} H^0(\mathbf{P}^1, E) \xrightarrow{\mathrm{ev}_{z=-1}} E|_{z=-1}.$$

Let $0 \neq v \in E|_{z=1}$ and set $0 \neq \alpha := (\mathrm{ev}_{z=1})^{-1}(v) \in H^0(\mathbf{P}^1, E)$. We have

$$\begin{aligned} &S(v, \overline{\mathrm{Iden}}(\overline{v})) = S(v, \overline{\mathrm{ev}_{z=-1} \circ (\mathrm{ev}_{z=1})^{-1}(v)}) = S(v, \overline{\mathrm{ev}_{z=-1}}(\overline{\alpha})) \\ &= S(v, \mathrm{ev}_{z=1}(\sigma(\overline{\alpha}))) = T(\alpha, \sigma(\overline{\alpha})) > 0. \end{aligned}$$

The last equality comes from the commutative diagram above.

Conversely, given a bilinear pairing

$$S : E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} \to \mathbb{C}$$

one can construct a morphism

$$P : E \otimes \sigma^* E \to \mathcal{O}_{\mathbf{P}^1}$$

using the canonical trivializations in Definition 2.3.12 so that if T is the resulting bilinear morphism

$$T : H^0(\mathbf{P}^1, E) \otimes H^0(\mathbf{P}^1, \sigma^* E) \to \mathbb{C},$$

then for any $v \in E|_{z=1}$ and $\alpha = (\mathrm{ev}_{z=1})^{-1}(v)$, we have

$$T(\alpha, \sigma(\overline{\alpha})) = S(v, \overline{\mathrm{Iden}}(\overline{v})).$$

Let E be a pure twistor structure of weight w. It can be reduced to the case of weight 0 using the following commutative diagram

$$\begin{array}{ccc} E(-w)|_{z=1} \otimes \overline{E(-w)|_{z=1}} & \xrightarrow{\mu^{\otimes w} \otimes \mu^{\otimes w}} & E|_{z=1} \otimes \overline{E|_{z=1}} \\ \downarrow{\scriptstyle \mathrm{Id} \otimes \overline{\mathrm{Iden}}} & & \downarrow{\scriptstyle \mathrm{Id} \otimes \overline{\mathrm{Iden}}} \\ E(-w)|_{z=1} \otimes \overline{E(-w)|_{z=-1}} & \xrightarrow{\mu^{\otimes w} \otimes \mu^{\otimes w}} & E|_{z=1} \otimes \overline{E|_{z=-1}} \end{array}$$

Here $\mu \in H^0(\mathbf{P}^1, \mathcal{O}(1))$ is a section only vanishing at ∞. $\square$

As a result of Lemma 2.3.20, we introduce an equivalent definition of the polarization which depends on bilinear pairings of vector spaces.

Definition 2.3.21 (Polarization by a bilinear pairing)**.** Let E be a pure twistor structure. Assume there is a bilinear pairing

$$S : E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} \to \mathbb{C}.$$

We say E *is polarized by* S if

$$S(\bullet, \overline{\mathrm{Iden}}(\bullet)) : E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=1}} \to E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} \to \mathbb{C}$$

is a positive hermitian pairing on $E|_{z=1}$, where Iden is the identification map in Definition 2.3.11.

As in Hodge theory, the following statements show that the bilinear pairing polarizing the twistor structure is always non-degenerate and any sub-twistor structure is automatically polarized.

Lemma 2.3.22. *If S is a bilinear pairing that polarizes a pure twistor structure E , then S is non-degenerate.*

Proof. Use the fact that $\mathrm{Iden} : E|_{z=1} \to E|_{z=-1}$ is an isomorphism. □

Lemma 2.3.23. *Let E be a pure twistor structure and let $G \subseteq E$ be a pure sub-twistor structure. If E is polarized by a bilinear pairing*

$$S_E : E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} \to \mathbb{C},$$

then its restriction

$$S_G : G|_{z=1} \otimes_{\mathbb{C}} \overline{G|_{z=-1}} \to \mathbb{C}$$

polarizes the pure twistor structure G.

Proof. The identification map is compatible with the inclusion $G \hookrightarrow E$:

$$\begin{array}{ccc} G|_{z=1} & \longrightarrow & E|_{z=1} \\ \big\downarrow \mathrm{Iden}_G & & \big\downarrow \mathrm{Iden}_E \\ G|_{z=-1} & \longrightarrow & E|_{z=-1} \end{array}$$

Therefore $S_G(\bullet, \overline{\mathrm{Iden}_G}(\bullet))$ is the restriction of $S_E(\bullet, \overline{\mathrm{Iden}_E}(\bullet))$ to $G|_{z=1} \otimes \overline{G|_{z=1}}$. In particular, S_G polarizes G and it is non-degenerate. □

Lemma 2.3.24. *Let $f : V \to \tilde{V}$ be a map of vector spaces underlying a morphism of pure twistor structures $F : E \to \tilde{E}$. If E is polarized by a bilinear pairing*

$$S : E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} \to \mathbb{C}.$$

Then S restricts to a non-degenerate pairing

$$S : \mathrm{Ker}\, f \otimes_{\mathbb{C}} \overline{\mathrm{Ker}\, F_{-1}} \to \mathbb{C},$$

where $F_{-1} : E|_{z=-1} \to \tilde{E}|_{z=-1}$ is the evaluation of F at $z = -1$.

Proof. Notice that $\mathrm{Ker}\, F \subseteq E$ is pure sub-twistor structure. □

Example: complex Hodge structures

The following statement is taken from [43, §2.1.d]. Let $H = \oplus_{p+q=w} H^{p,q}$ be a vector space equipped with a complex Hodge structure of weight w, polarized by a Hermitian form S in the sense that

$$(-1)^p S(\alpha^{p,q}, \overline{\alpha^{p,q}}) > 0, \quad \forall \alpha^{p,q} \in H^{p,q}.$$

Lemma 2.3.25. *Then H underlies a twistor structure E pure of weight w so that $E|_{z=-1} = H$. The identification map of E*

$$\mathrm{Iden} : E|_{z=1} \to E|_{z=-1}$$

coincides with the map $H^{p,q} \to H^{p,q}$, $\alpha^{p,q} \mapsto (-1)^p \alpha^{p,q}$.

Moreover E is polarized by S viewing as a bilinear pairing

$$S : H \otimes_{\mathbb{C}} \overline{H} = E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} \to \mathbb{C}.$$

Proof. First we construct the twistor structure E as follows. Denote

$$F'^p H = \bigoplus_{r \geq p} H^{r,w-r}, \quad F''^p H = \bigoplus_{r \geq p} H^{w-r,r}.$$

to be the Hodge filtration. On Ω_0, the Rees module associated to $(H, F'^{\bullet} H)$ gives

$$\bigoplus_p F'^p H z^{-p} = \bigoplus_p H^{p,w-p} z^{-p} \mathbb{C}[z].$$

On Ω_∞, the Rees module associated to $(H, F''^{\bullet} H)$ gives

$$\bigoplus_p F''^p H z^p = \bigoplus_p H^{p,w-p} z^{w-p} \mathbb{C}[1/z].$$

The glueing map on the level of sections is defined by

$$\begin{aligned} H^{p,w-p} z^{-p} &\to H^{p,w-p} z^{w-p} \\ \alpha^{p,q} z^{-p} &\mapsto \alpha^{p,q} z^{w-p}. \end{aligned}$$

Therefore these two Rees modules glue to a vector bundle E on $\mathbf{P}^1$ so that

$$E \cong \oplus \mathcal{O}_{\mathbf{P}^1}(w), \quad E|_{z=z_0} = \oplus H^{p,q} z_0^{-p}.$$

Since $H^0(\mathbf{P}^1, E(-w)) = \oplus H^{p,q} z^{-p}$, the identification map (Definition 2.3.11) for E is

$$\begin{aligned} \mathrm{Iden} : E|_{z=1} = \oplus H^{p,q} \to H^0(\mathbf{P}^1, E(-w)) &\to \oplus H^{p,q}(-1)^{-p} = E|_{z=-1}, \\ \alpha^{p,q} \mapsto \alpha^{p,q} z^{-p} &\mapsto \alpha^{p,q}(-1)^{-p} = (-1)^p \alpha^{p,q}. \end{aligned}$$

Therefore, the identification map changes the sign by $(-1)^p$ on $H^{p,q}$.

Now we would like to verify the polarization condition in Definition 2.3.21. Using the identification of $E|_{z=-1}$ with H, we can view S as a bilinear pairing

$$S : E|_{z=1} \otimes_{\mathbb{C}} \overline{E|_{z=-1}} \to \mathbb{C}.$$

Since S polarizes H as a complex Hodge structure, for any $\alpha^{p,q} \in H^{p,q}$, we have

$$S\left(\alpha^{p,q}, \overline{\mathrm{Iden}}(\overline{\alpha^{p,q}})\right) = (-1)^p S(\alpha^{p,q}, \overline{\alpha^{p,q}}) > 0.$$

□

Cohomology of smooth projective varieties with semisimple coefficients

In this section, we review the proof of the following fundamental

Theorem 2.3.26 (Hodge-Simpson). *Let X be a smooth projective variety and let $\mathcal{V}$ be a semisimple local system on X. Let η be an ample line bundle on X.*

1. *$H^k(X, \mathcal{V})$ underlies a natural pure twistor structure of weight k.*
2. *Assume $k \leq \dim X$. The pure sub-twistor structure on*

$$H^k(X, \mathcal{V})_{\mathrm{prim}} := \operatorname{Ker} \eta^{\dim X - k+1} \subseteq H^k(X, \mathcal{V})$$

is polarizable.

Remark 2.3.27. The first statement is proved by Simpson in [51, Theorem 4.1]. The second statement is proved by Sabbah in [43, Theorem 2.2.4].

Remark 2.3.28. In the sequel, I will provide Simpson's construction and Sabbah's construction. They have advantages for different purposes. Simpson's construction is better suited for the theory of weights for semisimple local systems, while Sabbah's construction is more convenient for the application to twisted Poincaré pairings.

Simpson provided two equivalent constructions for the natural pure twistor structure on $H^k(X, \mathcal{V})$. To begin with, we need the following theorems about semisimple local systems.

Theorem 2.3.29 (Corlette [14], Simpson [52]). *Let X be a smooth projective variety and let $\mathcal{V}$ be a semisimple local system on X with a flat connection ∇. Then the $\mathcal{C}^\infty$-bundle $H := \mathcal{V} \otimes_{\mathbb{C}} \mathcal{C}^\infty_X$ admits a harmonic metric h so that there is a decomposition of ∇ into four operators:*

$$\begin{aligned} \nabla &= \partial' + \theta' + \partial'' + \theta'', \\ \partial' &: H \to \mathcal{A}^{1,0}_X \otimes_{\mathcal{C}^\infty_X} H, \\ \theta' &: H \to \mathcal{A}^{1,0}_X \otimes_{\mathcal{C}^\infty_X} H, \\ \partial'' &: H \to \mathcal{A}^{0,1}_X \otimes_{\mathcal{C}^\infty_X} H, \\ \theta'' &: H \to \mathcal{A}^{0,1}_X \otimes_{\mathcal{C}^\infty_X} H. \end{aligned}$$

Here ∂' is a $(1,0)$-connection, ∂'' is a $(0,1)$-connection and $\partial' + \partial''$ is a metric connection of h. θ' is $\mathcal{C}^\infty_X$-linear and θ'' is the adjoint of θ' with respect to h.

Set $D' := \partial' + \theta''$ and $D'' := \partial'' + \theta'$, we have

$$(D')^2 = (D'')^2 = 0.$$

The operator D'' induces a Higgs bundle structure on the holomorphic bundle associated to the complex structure $\partial'' + \theta'$ on H.

Theorem 2.3.30. *With the notation in Theorem 2.3.29. There is an isomorphism*

$$H^k(X,\mathcal{V}) \cong \mathrm{Harm}(X,H) := \{\alpha \in \mathcal{C}^\infty(\mathcal{A}_X^k \otimes_{\mathcal{C}_X^\infty} H)\colon \Delta_\nabla(\alpha) = 0\}.$$

Here Δ_∇ is the Laplacian of ∇ and $\mathcal{A}_X^k$ is the sheaf of $\mathcal{C}^\infty$ k-forms on X. In particular, each cohomology element of $H^k(X,\mathcal{V})$ is uniquely represented by harmonic k-forms with coefficients in H.

Moreover, for any $(a,b) \neq (0,0)$, there is an isomorphism

$$\begin{aligned} H^k(X,\mathcal{A}_X^\bullet(H); aD' + bD'') &\cong \{\alpha \in \mathcal{C}^\infty(\mathcal{A}_X^k \otimes_{\mathcal{C}_X^\infty} H)\colon \Delta_{aD'+bD''}(\alpha) = 0\}. \\ &= \{\alpha \in \mathcal{C}^\infty(\mathcal{A}_X^k \otimes_{\mathcal{C}_X^\infty} H)\colon \Delta_\nabla(\alpha) = 0\}. \end{aligned}$$

where $\mathcal{A}_X^\bullet(H) := \mathcal{A}_X^\bullet \otimes_{\mathcal{C}_X^\infty} H$ denotes the de Rham complex associated to H.

Proof. In [52], Simpson proved a generalized Kahler identity:

$$\Delta_\nabla = 2\Delta_{D'} = 2\Delta_{D''}.$$

We will give a proof sketch in Lemma 2.4.23. Similar calculations will show that

$$\Delta_{aD'+bD''} = \frac{|a|^2 + |b|^2}{2}\Delta_\nabla.$$

□

Now we recall the constructions of the natural pure twistor structures.

Construction 2.3.31 (Differential geometric construction)**.** Let X be a smooth projective variety. Let $\mathcal{V}$ be a semisimple local system on X and let $H = \mathcal{V} \otimes_{\mathbb{C}} \mathcal{C}_X^\infty$ be the associated harmonic bundle on X as in Theorem 2.3.29. Let p_1, p_2 be the projection maps $X \xleftarrow{p_1} X \times \mathbf{P}^1 \xrightarrow{p_2} \mathbf{P}^1$. We consider the following bundles on $X \times \mathbf{P}^1$:

$$\mathcal{F}^i := p_1^*(\mathcal{A}_X^i(H)) \otimes p_2^*(\mathcal{O}_{\mathbf{P}^1}(i)), \quad \forall i \geq 0.$$

Let λ, μ be two sections of $\mathcal{O}_{\mathbf{P}^1}(1)$ which vanish respectively at 0 and ∞. Then we have a natural differential operator

$$\mathbf{d} := \lambda D' + \mu D'' : \mathcal{F}^0 \to \mathcal{F}^1.$$

We can extend $\mathbf{d}$ so that it gives a complex $(\mathcal{F}^\bullet, \mathbf{d})$ on $X \times \mathbf{P}^1$.

The natural twistor structure on $H^k(X,\mathcal{V})$ is defined to be

$$E^k := R^k p_{2,*}(\mathcal{F}^\bullet, \mathbf{d}).$$

For $z = [a,b] \in \mathbf{P}^1$, after identifying $\mathcal{O}_{\mathbf{P}^1}(k)|_{z=z_0}$ with the vector space $\mathbb{C}$, we have the following isomorphism via harmonic representatives using Theorem 2.3.30:

$$E^k|_{z=[a,b]} \cong H^k(X,\mathcal{A}_X^\bullet(H); aD' + bD'') \cong H^k(X,\mathcal{V}),$$

so that combining with the projection formula it implies that

$$E^k \cong H^k(X, \mathcal{V}) \otimes \mathcal{O}_{\mathbf{P}^1}(k).$$

Therefore one conclude that E^k is a pure twistor structure of weight k on $H^k(X, \mathcal{V})$. Moreover for $z_0 \in \mathbf{P}^1 \setminus \{\infty\}$, we have

$$E^k|_{z=z_0} \cong H^k(X, \mathcal{A}_X^\bullet(H); z_0\partial' + \theta' + \partial'' + z_0\theta'').$$

Remark 2.3.32. In [51, Theorem 4.1], Simpson showed that the complex $(\mathcal{F}^\bullet, \mathbf{d})$ is isomorphic to the Rees bundle complex associated to two Hodge filtrations on $\mathcal{A}^i(H)$. Since we only use the Rees construction associated to one filtration in this paper, to avoid confusion, we choose this more explicit construction.

Remark 2.3.33. On the level of harmonic representatives, the identification map for the pure twistor structure E^k in the Differential geometric construction 2.3.31 is an "identity" after identifying fibers of E^k with various cohomology groups of $\mathcal{A}_X^\bullet(H)$ in the following sense. Let

$$\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q}$$

be a harmonic representative of an element in $H^k(X, \mathcal{V})$, where $\alpha^{p,q}$ are (p,q)-forms and $m_{p,q}$ are sections of H. Because choosing the harmonic representative gives the trivialization of E^k over Ω_0, by Remark 2.3.15, the identification map can be calculated as

$$\text{Iden} : E^k|_{z=1} \cong H^k(X, \mathcal{V}) \to H^k(X, \mathcal{A}_X^\bullet(H); -D' + D'') \cong E^k|_{z=-1}$$
$$\left[\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q}\right] \mapsto \left[\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q}\right].$$

In particular, one cannot see the sign change, in contrast to Lemma 2.3.44.

Definition 2.3.34 (Rees bundle)**.** Let $(V, F^\bullet V)$ be a complex vector space equipped with a decreasing filtration. The Rees bundle $\xi(V, F^\bullet)$ is defined as the bundle on $\mathbf{A}^1 = \mathrm{Spec}(\mathbb{C}[z])$ associated to the $\mathbb{C}[z]$-module

$$\bigoplus_\ell F^\ell V \cdot z^{-\ell}.$$

Let X be a complex manifold and let $(E, F^\bullet E)$ be a vector bundle on X with a decreasing filtration $F^\bullet E$. The Rees bundle $\xi(E, F^\bullet)$ is defined to be the bundle on $X \times \mathbf{A}^1$ associated to

$$\bigoplus_\ell F^\ell E \cdot z^{-\ell}.$$

Construction 2.3.35 (Analytic construction)**.** We use the notation in Construction 2.3.31. Denote $\mathcal{E} = p_1^*(H)$, which is a $\mathcal{C}^\infty$-bundle on $X \times \mathbf{P}^1$. Denote Ω_0 and Ω_∞ to be the standard neighborhoods $\mathbf{A}^1$ of 0 and ∞ in $\mathbf{P}^1$ respectively. We define the following triple $(\mathcal{G}, \mathcal{L}, \widetilde{\mathcal{G}})$ associated to $\mathcal{V}$:

1. Let $(\mathcal{G}, \nabla_{\mathcal{G}})$ be the holomorphic bundle on $X \times \Omega_0$ associated to the complex structure $\partial'' + z\theta''$ on the bundle $\mathcal{E}|_{X\times\Omega_0}$, where z is the coordinate on Ω_0. And
$$\nabla_{\mathcal{G}} : \mathcal{G} \to \mathcal{G} \otimes_{\mathcal{O}_{X\times\Omega_0}} \Omega^1_{X\times\Omega_0/\Omega_0}$$
is a z-connection on $X \times \Omega_0/\Omega_0$ induced by $z\partial' + \theta'$, which means that
$$\nabla(f \cdot e) = z \cdot d(f) \otimes e + f\nabla(e)$$
for sections f of $\mathcal{O}_{X\times\Omega_0}$ and e of $\mathcal{G}$.

2. Let $(\widetilde{\mathcal{G}}, \nabla_{\widetilde{\mathcal{G}}})$ be the holomorphic bundle on $\overline{X} \times \Omega_\infty$ associated to the complex structure $\overline{\partial'} + z^{-1}\overline{\theta'}$ on the bundle $\mathcal{E}|_{\overline{X}\times\Omega_\infty}$, where z^{-1} is the coordinate on Ω_∞. And
$$\nabla_{\widetilde{\mathcal{G}}} : \widetilde{\mathcal{G}} \to \widetilde{\mathcal{G}} \otimes_{\mathcal{O}_{\overline{X}\times\Omega_\infty}} \Omega^1_{\overline{X}\times\Omega_\infty/\Omega_\infty}$$
is a z^{-1}-connection induced by $z^{-1}\overline{\partial''} + \overline{\theta''}$. Here $\overline{X}$ denotes the complex conjugated manifold of X (with the same underlying topological space but conjugated structures sheaf) and $\overline{\partial'}, \overline{\partial''}, \overline{\theta'}, \overline{\theta''}$ represent the corresponding operators of the harmonic bundle H on $\overline{X}$. Note

3. Let $\mathcal{L}$ be the local system over $X^{\text{top}} \times \mathbf{G}_m$ associated to the flat connection on $\mathcal{E}|_{X\times\mathbf{G}_m}$:
$$\partial' + z^{-1}\theta' + \partial'' + z\theta''.$$
Here $\mathbf{G}_m := \Omega_0 \cap \Omega_\infty$.

Now we would construct two bundles over Ω_0 and Ω_∞ respectively so that one can glue them over $\mathbf{G}_m = \Omega_0 \cap \Omega_\infty$ to get a bundle over $\mathbf{P}^1$. Abusing notations, we also use p_2 for the second projection map $X \times \Omega_0 \to \Omega_0$, the same for $\mathbf{G}_m$ and Ω_∞.

- Over Ω_0: let $\xi(\Omega^k_X, F^\bullet)$ be the Rees bundle (Definition 2.3.34) on $X \times \Omega_0$ associated to the bundle of holomorphic differential forms with the Hodge filtration $(\Omega^k_X, F^\bullet\Omega^k_X)$ where
$$F^\ell\Omega^k_X = \bigoplus_{p\geq\ell} \Omega^{p,k-p}_X.$$
Then we have the Rees bundle complex $\xi\Omega^\bullet_X(\mathcal{G})$ on $X \times \Omega_0$, with the differential given by the z-connection $\nabla_{\mathcal{G}}$ and
$$\xi\Omega^k_X(\mathcal{G}) := \xi(\Omega^k_X, F^\bullet) \otimes_{\mathcal{O}_{X\times\Omega_0}} \mathcal{G}.$$
Then the bundle over Ω_0 is defined to be
$$M_0 := R^k p_{2,*}(\xi\Omega^\bullet_X(\mathcal{G})).$$
Note that the fiber of M_0 at $z_0 \in \Omega_0$ is isomorphic to
$$H^k(X, \Omega^\bullet_X \otimes \mathcal{G}|_{X\times\{z_0\}}, z_0\partial' + \theta').$$
Here, by abusing notation, ∂', θ' denote the induced operator on the holomorphic bundle $\mathcal{G}|_{X\times\{z_0\}}$.

- Over Ω_∞: let $\xi\Omega^\bullet_{\overline{X}}(\widetilde{\mathcal{G}})$ denote the corresponding Rees bundle complex on $\overline{X}\times\Omega_\infty$ so that

$$\xi\Omega^k_{\overline{X}}(\widetilde{\mathcal{G}}) := \xi(\Omega^k_{\overline{X}}, \overline{F}^\bullet)\otimes_{\mathcal{O}_{\overline{X}\times\Omega_\infty}}\widetilde{\mathcal{G}}.$$

where $\xi(\Omega^k_{\overline{X}}, \overline{F}^\bullet)$ is the Rees bundle on $\overline{X}\times\Omega_\infty$ associated to the filtered bundle $(\Omega^k_{\overline{X}}, \overline{F}^\bullet\Omega^k_{\overline{X}})$ where

$$\overline{F}^\ell\Omega^k_{\overline{X}} = \bigoplus_{p\geq\ell}\Omega^{p,k-p}_{\overline{X}} = \bigoplus_{p\geq\ell}\Omega^{k-p,p}_{X}.$$

The bundle over Ω_∞ is defined to be

$$M_\infty := R^k p_{2,*}(\xi\Omega^\bullet_{\overline{X}}(\widetilde{\mathcal{G}})).$$

Note that the local system $\mathcal{L}$ is the flat bundle associated to $(\mathcal{G}, z^{-1}\nabla_{\mathcal{G}})$ and it is also the flat bundle associated to $(\widetilde{\mathcal{G}}, z\nabla_{\widetilde{\mathcal{G}}})$. Moreover, when calculating $R^k p_{2,*}(\mathcal{L})$ over $\mathbf{G}_m$, one can rescale by z or z^{-1} (for more details, see the rescaling map in Definition 2.3.36) and therefore obtain glueing isomorphisms

$$M_0|_{\mathbf{G}_m}\cong R^k p_{2,*}(\mathcal{L})\cong M_\infty|_{\mathbf{G}_m}.$$

Then M_0 and M_∞ glue to a holomorphic bundle M^k over $\mathbf{P}^1$. By Lemma 2.3.37 below, we know that $M^k\cong\oplus\mathcal{O}_{\mathbf{P}^1}(k)$. Moreover, it follows from the Dolbeault lemma that

$$M^k|_{z=1}\cong H^k(X, \Omega^\bullet_X\otimes\mathcal{G}|_{X\times\{1\}}, \partial'+\theta')\cong H^k(X,\mathcal{V}).$$

Therefore M^k is a pure twistor structure of weight k on $H^k(X,\mathcal{V})$.

Now we want to compare the bundle M^k with the bundle E^k in the Differential geometric construction 2.3.31. Since $\mathcal{G}$ is the holomorphic bundle associated to the complex structure $\partial''+z\theta''$ on $p_1^*(H)|_{X\times\Omega_0}$, we can use a Dolbeault resolution to show that there is a quasi-isomorphism

$$\xi\Omega^\bullet_X(\mathcal{G})\cong\mathcal{F}^\bullet|_{X\times\Omega_0}$$

where $\mathcal{F}^\bullet = p_1^*(\mathcal{A}^\bullet_X(H))\otimes p_2^*(\mathcal{O}_{\mathbf{P}^1}(\bullet))$ is the complex on $X\times\mathbf{P}^1$ in Construction 2.3.31 so that for $z_0\in\Omega_0$ we have a fiberwise isomorphism:

$$H^k(X,\Omega^\bullet_X\otimes\mathcal{G}|_{X\times\{z_0\}}; z_0\partial'+\theta')\cong H^k(X,\mathcal{A}^\bullet_X\otimes H; z_0\partial'+\theta'+\partial''+z_0\theta'').$$

In particular $M^k\cong E^k|_{\Omega_0}$. One can argue similarly over the chart Ω_∞. Since M^k and E^k both have slope k, we conclude that there is a natural isomorphism

$$M^k\cong E^k.$$

Now we would like to recall Sabbah's rescaling map in [43, Theorem 2.2.4] and use it to provide trivializations of the bundle M^k in Simpson's Analytic construction 2.3.35 and prove that the bundle is indeed isomorphic to $\oplus\mathcal{O}_{\mathbf{P}^1}(k)$.

Definition 2.3.36. We define the *rescaling map* to be

$$\iota : \mathbb{C}[z] \otimes_{\mathbb{C}} \mathcal{A}_X^{p,q} \otimes_{\mathcal{C}_X^\infty} H \to z^{-p}\mathbb{C}[z] \otimes_{\mathbb{C}} \mathcal{A}_X^{p,q} \otimes_{\mathcal{C}_X^\infty} H$$
$$\alpha^{p,q} \otimes m \mapsto z^{-p}\alpha^{p,q} \otimes m.$$

Under ι, the differential $z\partial' + \theta' + \partial'' + z\theta''$ is changed into $\partial' + z^{-1}\theta' + \partial'' + z\theta''$. For $0 \neq z_0 \in \Omega_0$, we denote

$$\iota_{z_0} : \mathcal{A}_X^{p,q} \otimes_{\mathcal{C}_X^\infty} H \to \mathcal{A}_X^{p,q} \otimes_{\mathcal{C}_X^\infty} H$$
$$\alpha^{p,q} \otimes m \mapsto z_0^{-p}\alpha^{p,q} \otimes m.$$

Abusing notations, we also denote the resulting map on the cohomology to be

$$\iota_{z_0} : H^k(X, \mathcal{A}_X^\bullet(H); z_0\partial' + \theta' + \partial'' + z_0\theta'') \xrightarrow{\sim} H^k(X, \mathcal{A}_X^\bullet(H); \partial' + z_0^{-1}\theta' + \partial'' + z_0\theta'').$$

Lemma 2.3.37. *With the notation in the Analytic construction 2.3.35, then M_0 and M_∞ glues to a bundle M^k so that*

$$M^k \cong \oplus \mathcal{O}_{\mathbf{P}^1}(k).$$

Proof. Recall $D' = \partial' + \theta''$ and $D'' = \partial'' + \theta'$. Consider the space of harmonic k-forms on X with coefficients in H:

$$\begin{aligned}\mathrm{Harm}(X, H) &= \{\alpha \in \mathcal{C}^\infty(\mathcal{A}_X^k \otimes H) : \Delta_{D'+D''}(\alpha) = 0.\} \\ &= \{\alpha \in \mathcal{C}^\infty(\mathcal{A}_X^k \otimes H) : \Delta_{z_0D'+D''}(\alpha) = 0, z_0 \in \Omega_0.\}\end{aligned}$$

In this proof, we fix the identification of M_0 with the bundle $E^k|_{\Omega_0}$ in the Differential geometric construction 2.3.31 whose fiber at $z = z_0$ is

$$H^k(X, \mathcal{A}_X^\bullet(H), z_0D' + D''),$$

as showed in the end of Analytic construction 2.3.35. By Theorem 2.3.30, there is a trivialization of M_0 via choosing harmonic representatives

$$\mathcal{O}_{\Omega_0} \otimes \mathrm{Harm}(X, H) \cong M_0.$$

By the construction in Definition 2.3.36, we know that under the rescaling map ι, the differential $zD' + D'' = z\partial' + \theta' + \partial'' + z\theta''$ is changed into $\partial' + z^{-1}\theta' + \partial'' + z\theta''$. Therefore the glueing isomorphism

$$M_0|_{\mathbf{G}_m} \cong R^k p_{2,*}(\mathcal{L})$$

can be realized by the rescaling map ι and after precomposing the map of choosing harmonic representatives, it becomes

$$\mathcal{O}_{\mathbf{G}_m} \otimes \mathrm{Harm}(X, H) \to M_0|_{\mathbf{G}_m} \xrightarrow{\iota} R^k p_{2,*}(\mathcal{L})$$
$$1 \otimes (\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q}) \mapsto 1 \otimes (\sum_{p+q=k} z^{-p}\alpha^{p,q} \otimes m_{p,q}).$$

Here $\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q}$ is an element in $\mathrm{Harm}(X, H)$.

Similarly, by identifying M_∞ with $E^k|_{\Omega_\infty}$ with fiber at $z_0^{-1} \in \Omega_\infty$ isomorphic to

$$H^k(X, \mathcal{A}_X^\bullet(H), D' + z_0^{-1} D'').$$

We can trivialize M_∞ over Ω_∞:

$$\mathcal{O}_{\Omega_\infty} \otimes \mathrm{Harm}(X, H) \cong M_\infty.$$

We can define a conjugate rescaling map $\bar\iota$ by

$$\begin{gathered}\bar\iota : \mathbb{C}[1/z] \otimes_{\mathbb{C}} \mathcal{A}_X^{p,q} \otimes_{\mathcal{C}_X^\infty} H \to z^q \mathbb{C}[z] \otimes_{\mathbb{C}} \mathcal{A}_X^{p,q} \otimes_{\mathcal{C}_X^\infty} H \\ \alpha^{p,q} \otimes m \mapsto z^q \alpha^{p,q} \otimes m.\end{gathered}$$

Under $\bar\iota$, the differential $D' + z^{-1} D'' = \partial' + z^{-1}\theta' + z^{-1}\partial'' + \theta''$ is changed into $\partial' + z^{-1}\theta' + \partial'' + z\theta''$. Then $\bar\iota$ provides the glueing isomorphism

$$M_\infty|_{\mathbf{G}_m} \cong R^k p_{2,*}(\mathcal{L})$$

so that we also have

$$\begin{aligned}&\mathcal{O}_{\mathbf{G}_m} \otimes \mathrm{Harm}(X, H) \to M_\infty|_{\mathbf{G}_m} \xrightarrow{\bar\iota} R^k p_{2,*}(\mathcal{L}) \\ &1 \otimes (\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q}) \mapsto 1 \otimes (\sum_{p+q=k} z^q \alpha^{p,q} \otimes m_{p,q}).\end{aligned}$$

Now we can put them together to calculate the transition function of M^k from Ω_0 to Ω_∞, which is

$$\begin{aligned}&\mathcal{O}_{\mathbf{G}_m} \otimes \mathrm{Harm}(X, H) \to M_0|_{\mathbf{G}_m} \xrightarrow{\iota} R^k p_{2,*}(\mathcal{L}) \xrightarrow{\bar\iota^{-1}} M_\infty|_{\mathbf{G}_m} \to \mathcal{O}_{\mathbf{G}_m} \otimes \mathrm{Harm}(X, H) \\ &1 \otimes (\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q}) \mapsto 1 \otimes (\sum_{p+q=k} z^{-p} \alpha^{p,q} \otimes m_{p,q}) \mapsto 1 \otimes (\sum_{p+q=k} z^{-q-p} \alpha^{p,q} \otimes m_{p,q})\end{aligned}$$

Therefore we see that any harmonic representative is multiplied by $z^{-p-q} = z^{-k}$ after the transition map from Ω_0 to Ω_∞. From Remark 2.3.13, we conclude that M^k is isomorphic to direct sum of $\mathcal{O}_{\mathbf{P}^1}(k)$. □

Proof of Hodge-Simpson Theorem 2.3.26. The Differential geometric construction 2.3.31 and Analytic construction 2.3.35 give the same natural pure twistor structure on $H^k(X, \mathcal{V})$. In this proof, we use the notation in the Differential geometric construction 2.3.31, i.e. there is a pure twistor structure E^k so that

$$\begin{aligned}&E^k|_{z=1} \cong H^k(X, \mathcal{V}) \\ &E^k|_{z=-1} \cong H^k(X, \mathcal{A}_X^\bullet(H); -\partial' + \theta' + \partial'' - \theta'').\end{aligned}$$

Now we want to verify the polarization condition on $H^k(X, \mathcal{V})_{\mathrm{prim}}$. By taking the primitive part of the complex in the Differential geometric construction, one can get a bundle $E^k_{\mathrm{prim}} \subseteq E$ which is a pure twistor structure on $H^k(X, \mathcal{V})_{\mathrm{prim}}$ so that

$$E^k_{\mathrm{prim}}|_{z=z_0} \cong H^k(X, \mathcal{A}_X^\bullet(H); z_0\partial' + \theta' + \partial'' + z_0\theta'')_{\mathrm{prim}}, \quad \forall z_0 \in \Omega_0. \tag{2.1}$$

Consider the bilinear map

$$S: H^k(X,\mathcal{V})\otimes_{\mathbb{C}}\overline{H^k(X,\mathcal{A}^\bullet_X(H);\partial'-\theta'+\partial''-\theta'')}\to\mathbb{C}$$
$$[\alpha\otimes m]\otimes\overline{[\beta\otimes n]}\mapsto i^{-k}(-1)^{k(k-1)/2}\int_X h(m,\overline{n})\cdot\eta^{\dim X-k}\wedge\alpha\wedge\overline{\beta}.$$

Recall the rescaling map from Definition 2.3.36

$$\iota_{-1}: H^k(X,\mathcal{A}^\bullet_X(H);-\partial'+\theta'+\partial''-\theta'')\xrightarrow{\sim}H^k(X,\mathcal{A}^\bullet_X(H);\partial'-\theta'+\partial''-\theta'').$$

Then we claim that the bilinear pairing

$$S(\bullet,\overline{\iota_{-1}}(\bullet)): E^k_{\mathrm{prim}}|_{z=1}\otimes_{\mathbb{C}}\overline{E^k_{\mathrm{prim}}|_{z=-1}}\to\mathbb{C}$$

polarize the twistor structure E^k_{prim}. Here one identifies the fibers of E^k_{prim} with the appropriate primitive spaces as in (2.1). By Theorem 2.3.30, we can choose a harmonic representative of an element in $H^k(X,\mathcal{V})_{\mathrm{prim}}$ and write it as

$$\sigma=\sum\alpha^{p,q}\otimes m_{p,q}$$

where $\alpha^{p,q}$ are primitive (p,q)-forms and $m_{p,q}$ are sections of H. By Remark 2.3.33, the identification map for the twistor structure E^k satisfies

$$\mathrm{Iden}(\left[\sum\alpha^{p,q}\otimes m_{p,q}\right])=\left[\sum\alpha^{p,q}\otimes m_{p,q}\right].$$

Then it follows from classical Hodge theory [56, Theorem 6.29] that

$$\begin{aligned}
&S\left([\alpha^{p,q}\otimes m_{p,q}],\overline{\iota_{-1}}\circ\overline{\mathrm{Iden}}(\overline{[\alpha^{p,q}\otimes m_{p,q}]})\right)\\
&S\left([\alpha^{p,q}\otimes m_{p,q}],\overline{\iota_{-1}}(\overline{[\alpha^{p,q}\otimes m_{p,q}]})\right)\\
=&i^{-k}(-1)^{k(k-1)/2}(-1)^p\int_X h(m_{p,q},\overline{m_{p,q}})\cdot\eta^{\dim X-k}\wedge\alpha^{p,q}\wedge\overline{\alpha^{p,q}}\\
=&\int_X h(m_{p,q},\overline{m_{p,q}})\cdot\alpha^{p,q}\wedge\overline{*\alpha^{p,q}}>0.
\end{aligned}$$

where $*$ is the Hodge star operator for forms. In particular, one has

$$S([\sigma],\overline{\iota_{-1}}\circ\overline{\mathrm{Iden}}(\overline{[\sigma]})>0,\quad\forall[\sigma]\in E^k_{\mathrm{prim}}|_{z=1},$$

which is exactly the polarization condition in the sense of Definition 2.3.21. □

It follows immediately from the proof of Theorem 2.3.26 that the restriction maps to smooth closed subvarieties is compatible with the natural twistor structures.

Lemma 2.3.38. *Let X be a smooth projective variety and Z be a smooth subvariety of X. Let $\mathcal{V}$ be a semisimple local system on X. Then the restriction map*

$$f: H^k(X,\mathcal{V})\to H^k(Z,\mathcal{V}|_Z)$$

underlies the morphism of natural pure twistor structures in Theorem 2.3.26.

Proof. We use the notation in the Differential geometric construction 2.3.31. Let $i : Z \to X$ be the inclusion map. The we have a map on $X \times \mathbf{P}^1$:

$$\mathcal{A}_X^k(H) \boxtimes \mathcal{O}_{\mathbf{P}^1}(k) \to \{i_* \mathcal{A}_Z^k(H|_Z)\} \boxtimes \mathcal{O}_{\mathbf{P}^1}(k).$$

where $\boxtimes$ is the box product for $X \times \mathbf{P}^1$. Moreover, it is clear from definition that this map is compatible with the differentials $\mathbf{d}$ for X and Z. □

Sabbah's construction

In [43, Theorem 2.2.4], Sabbah constructs a different but isomorphic pure twistor structure on $H^k(X, \mathcal{V})$. For reader's convenience, we briefly recall his construction here.

Definition 2.3.39. Let $\mathcal{M}$ be a $\mathbb{C}[z]$-module. The conjugation $\overline{\mathcal{M}}$ is defined to be

$$\overline{\mathcal{M}} := \mathcal{M} \otimes_{\mathbb{C}[z]} \mathbb{C}[z^{-1}]$$

where $\mathbb{C}[z^{-1}]$ is viewed as a $\mathbb{C}[z]$-module via the following map

$$\begin{aligned} \mathbb{C}[z] &\to \mathbb{C}[z^{-1}] \\ z &\mapsto -z^{-1} \\ \lambda \in \mathbb{C} &\mapsto \overline{\lambda} \end{aligned}$$

Remark 2.3.40. We can view $\mathcal{M}$ as a $\mathcal{O}$-module over $\mathbf{P}^1 \setminus \{0\}$ and $\overline{\mathcal{M}}$ as a $\mathcal{O}$-module over $\mathbf{P}^1 \setminus \{\infty\}$.

Construction 2.3.41 (Sabbah's construction)**.** We use the notation in Construction 2.3.31. Denote Ω_0 and Ω_∞ to be the standard $\mathbf{A}^1$-neighborhoods of 0 and ∞ in $\mathbf{P}^1$. Assume $k \leq \dim X$. Denote the projections of $X \times \Omega_0$ to be $X \xleftarrow{p_1} X \times \Omega_0 \xrightarrow{p_2} \Omega_0$ and define $\mathcal{A}^m_{X \times \Omega_0}$ to be the m-th wedge product of

$$\mathcal{A}^1_{X \times \Omega_0} := z^{-1} \mathcal{A}^1_{X \times \Omega_0 / \Omega_0}.$$

Let $\mathcal{M}$ be the $\mathbb{C}[z]$-module defined by

$$\mathcal{M} := H^k(X, \mathcal{A}^\bullet_{X \times \Omega_0} \otimes p_1^*(H), \partial' + z^{-1}\theta' + \partial'' + z\theta'')$$

so that

$$\mathcal{M}|_{z=0} = H^k\left(X, (\Omega_X^\bullet \otimes_{\mathcal{O}_X} F, \theta'_F)\right)$$

where $\Omega_X^\bullet$ is the complex of holomorphic forms on X and (F, θ'_F) is the holomorphic Higgs bundle associated to the semisimple local system $\mathcal{V}$. Sabbah showed that $\mathcal{M}$ is a free $\mathbb{C}[z]$-module.

Let $\mathbf{S}$ to be the unit disk in $\mathbf{P}^1$ and $\mathcal{M}|_{\mathbf{S}}$ is viewed as a $\mathcal{O}_{\Omega_0}$-module restricting to $\mathbf{S}$. Denote $\mathcal{O}_{\mathbf{S}} := \mathcal{O}_{\Omega_0}|_{\mathbf{S}}$. Consider the following sesquilinear pairing

$$\mathcal{M}|_{\mathbf{S}} \otimes_{\mathcal{O}_{\mathbf{S}}} \overline{\mathcal{M}}|_{\mathbf{S}} \to \mathcal{O}_{\mathbf{S}}$$

induced by

$$[\alpha \otimes m] \otimes \overline{[\beta \otimes n]} \mapsto \int_X h_{\mathbf{S}}(m, \overline{n}) \cdot \eta^{\dim X - k} \wedge \alpha \wedge \overline{\beta}.$$

This pairing induced an $\mathcal{O}_{\mathbf{S}}$-linear isomorphism

$$\mathcal{M}|_{\mathbf{S}} \xrightarrow{\sim} (\overline{\mathcal{M}|_{\mathbf{S}}})^{\vee} := \overline{\mathrm{Hom}_{\mathcal{O}_{\Omega_0}}(\mathcal{M}, \mathcal{O}_{\Omega_0})|_{\mathbf{S}}}.$$

Sabbah showed that $\mathcal{M}$ and $\overline{\mathcal{M}}$ glue to a holomorphic bundle F^k over $\mathbf{P}^1$ with slope k so that $F^k|_{\Omega_0} \cong \mathcal{M}$ and $F^k|_{\Omega_\infty} \cong \overline{\mathcal{M}^{\vee}}$. Moreover,

$$F^k|_{z=1} \cong H^k(X, \mathcal{V}).$$

Remark 2.3.42. Our presentation is slightly different from Sabbah's original construction. First, we omit the sign before the sesquilinear pairing, which is okay since we don't discuss the polarization here. Second, Sabbah used the glueing

$$\overline{\mathcal{M}|_{\mathbf{S}}} \xrightarrow{\sim} (\mathcal{M}|_{\mathbf{S}})^{\vee},$$

so that the resulting bundle over Ω_∞ is $\overline{\mathcal{M}}$. We adjust it so that the fiber $E^k|_{z=1}$ is isomorphic to $H^k(X, \mathcal{V})$.

Remark 2.3.43. The twistor structures in Sabbah's construction and Simpson's construction are not identically the same because of the following reason. For $z_0 \in \mathbf{G}_m$, the fiber at z_0 of the twistor structure in Simpson's construction is naturally isomorphic to

$$H^k(X, \mathcal{A}_X^{\bullet}(H); z_0\partial' + \theta' + \partial'' + z_0\theta''),$$

while the fiber at z_0 of the twistor structure in Sabbah's construction is naturally isomorphic to

$$H^k(X, \mathcal{A}_X^{\bullet}(H); \partial' + z_0^{-1}\theta' + \partial'' + z_0\theta'').$$

These two spaces are isomorphic via the rescaling map ι_{z_0} (Definition 2.3.36).

Lemma 2.3.44. *With the notation in Sabbah's construction 2.3.41. Then on the level of harmonic representatives, the identification map for F^k is*

$$\mathrm{Iden}: F^k|_{z=1} \cong H^k(X, \mathcal{V}) \to H^k(X, \mathcal{A}_X^{\bullet}(H); \partial' - \theta' + \partial'' - \theta'') \cong F^k|_{z=-1}$$
$$\left[\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q}\right] \mapsto \left[\sum_{p+q=k} (-1)^p \alpha^{p,q} \otimes m_{p,q}\right].$$

Here $\sum \alpha^{p,q} \otimes m_{p,q}$ is a harmonic representative of an element in $H^k(X, \mathcal{V})$ where $\alpha^{p,q}$ are (p,q)-forms and $m_{p,q}$ are sections of H.

Proof. Denote $\mathrm{Harm}(X, H)$ to be the space of harmonic representatives. Let $A \in H^k(X, \mathcal{V})$ be an element and let

$$\Phi = \sum \alpha^{p,q} \otimes m_{p,q}$$

be a harmonic representative of A. The rescaling map ι in Definition 2.3.36 gives the trivialization map $F^k|_{\Omega_0} \cong \mathrm{Harm}(X, H) \otimes_{\mathbb{C}} \mathcal{O}_{\Omega_0}$ with

$$\begin{aligned} \mathrm{Harm}(X, H) &\xrightarrow{\sim} H^k(X, \mathcal{A}_X^\bullet(H); \partial' + z_0^{-1}\theta' + \partial'' + z_0\theta'') \\ \Phi &\mapsto [\iota_{z_0}(\Phi)] \end{aligned}$$

Then by Remark 2.3.15, the identification map for F^k is

$$\begin{aligned} \mathrm{Iden} : H^k(X, \mathcal{V}) \cong F^k|_{z=1} \to F^k(\Omega_0) \to F^k|_{z=-1} &\cong H^k(X, \mathcal{A}_X^\bullet(H); \partial' - \theta' + \partial'' - \theta'') \\ [\Phi] \mapsto \Phi \otimes 1 \mapsto [\iota_{-1}(\Phi)]. \end{aligned}$$

In particular, on the level of harmonic forms, we see that

$$\mathrm{Iden} : \left[\sum \alpha^{p,q} \otimes m_{p,q}\right] \mapsto \left[\sum (-1)^p \alpha^{p,q} \otimes m_{p,q}\right].$$

□

Remark 2.3.45. A more conceptual way of understanding Sabbah's construction can be described as follows: over $\mathbf{G}_m$, there is a bundle F with fiber at $z = z_0$ identified with

$$H^k(X, \mathcal{A}_X^\bullet(H), \partial' + z_0^{-1}\theta' + \partial'' + z_0\theta'').$$

Using the rescaling map, there is an isomorphism

$$F \cong M_0|_{\mathbf{G}_m}$$

where M_0 is the holomorphic bundle over Ω_0 in Simpson's Analytic construction 2.3.35 with fiber at $z = z_0$ identified with

$$H^k(X, \mathcal{A}_X^\bullet(H), z_0\partial' + \theta' + \partial'' + z_0\theta'').$$

In particular, the fiber of M_0 is well defined for $z_0 = 0$ and therefore F extends to M_0 over $\{0\}$ under the rescaling map. Similarly, F extends over $\{\infty\}$ using the conjugate construction. Finally, one use the glueing map to show that F is isomorphic to $\oplus\mathcal{O}_{\mathbf{P}^1}(k)$.

Remark 2.3.46. The fiberwise description of Sabbah's construction has the following advantage over Simpson's construction. Suppose $T \subset X$ is a smooth subvariety, the operator

$$\partial' + z_0^{-1}\theta' + \partial'' + z_0\theta''$$

restricts to T and defined the corresponding cohomology group.

On the other hand, if one identifies the fiber of Simpson's construction with harmonic forms on X with coefficient in H, they don't necessarily restrict to harmonic forms on T.

2.3.3 Twisted Poincare pairings and polarizations

In this section, we will relate the twisted Poincaré pairings with polarizations of the natural twistor structure on $H^k(X, \mathcal{V})$ and use this relation to show non-degeneracy of certain restricted twisted Poincaré pairings. We will work in the following Set-up from Theorem 2.3.26 and Theorem 2.3.29.

Set-up 2.3.47. -

- X is a smooth projective variety of dimension n and $\mathcal{V}$ is a semisimple local system on X with a flat connection ∇. η is an ample line bundle on X.
- (H, h) is the harmonic bundle on X associated to $\mathcal{V}$ and

  $$\nabla = \partial' + \theta' + \partial'' + \theta''$$

  where $\partial'' + \theta'$ is the Higgs structure. Here we view h as a $\mathcal{C}^\infty_X$-linear morphism

  $$h : H \otimes_{\mathcal{C}^\infty_X} \overline{H} \to \mathcal{C}^\infty_X.$$

- E^k is the natural twistor structure on $H^k(X, \mathcal{V})$ so that

  $$E^k|_{z=1} \cong H^k(X, \mathcal{V})$$

  and

  $$E^k|_{z=-1} \cong H^k(X, \mathcal{A}^\bullet_X(H); -\partial' + \theta' + \partial'' - \theta''),$$

  where $\mathcal{A}^\bullet_X(H)$ is the de Rham complex associated to H.

Definition 2.3.48. Assume $k \leq n$. Denote $\mathcal{V}^*$ to be the dual local system of $\mathcal{V}$. We define the *twisted Poincaré pairing* S_X determined by η to be the bilinear pairing

$$\begin{aligned} S_X : H^k(X, \mathcal{V}) \otimes_{\mathbb{C}} H^k(X, \mathcal{V}^*) &\to \mathbb{C} \\ [\alpha \otimes e] \otimes [\beta \otimes \lambda] &\mapsto i^{-k}(-1)^{k(k-1)/2} \int_X \lambda(e) \cdot \eta^{n-k} \wedge \alpha \wedge \beta. \end{aligned}$$

Here α, β are k-forms on X and e, λ are global sections of $\mathcal{V}$ and $\mathcal{V}^*$ respectively.

Remark 2.3.49. There are two ways to understand the wedge product between cohomology elements. On the one hand, it can be defined to be the descent of wedge product of the forms $(\mathcal{A}^k \otimes_{\mathbb{C}} \mathcal{V}) \wedge (\mathcal{A}^k \otimes_{\mathbb{C}} \mathcal{V}^*)$, where a local section of $\mathcal{V}$ is paired with a local section of $\mathcal{V}^*$. On the other hand, one can represent the cohomology elements in $H^k(X, \mathcal{V})$ by harmonic k-forms with coefficients in $\mathcal{V}$ using Theorem 2.3.30. Same for $\mathcal{V}^*$, whose associated harmonic bundle is H^* as explained in Construction 2.3.50.

Construction 2.3.50 (Dual harmonic bundle)**.** Assume we are in the Set-up 2.3.47. Let H^* be the dual bundle of H. In [52], Simpson showed that H^* is equipped with the following harmonic structure so that it is the harmonic bundle associated to $\mathcal{V}^*$.

1. (Dual metric) For two sections $\lambda, \mu \in \mathcal{C}^\infty(H^*)$, h^* is defined by
$$h^*(\lambda, \overline{\mu}) := \lambda(e),$$
where $e \in \mathcal{C}^\infty(H)$ satisfying $\mu(\bullet) = h(\bullet, \overline{e})$.

2. (Dual Higgs structure) D'' is defined by
$$(D''\lambda)(e) + \lambda[(\partial'' + \theta')e] = \overline{\partial}(\lambda(e)),$$
where $\lambda \in \mathcal{C}^\infty(H^*), e \in \mathcal{C}^\infty(H)$.

3. (Dual connection) ∇ is defined by
$$\nabla := \partial' + \theta' + \partial'' + \theta''.$$
where $\partial' + \partial''$ is a metric connection for h^* and θ'' is the adjoint of θ' with respect to h^*.

Remark 2.3.51. Here we use the same notation D'' and ∇ for relevant structures on dual harmonic bundles. It should not cause confusions.

Lemma 2.3.52. *Assume we are in the Set-up 2.3.47, then there is an isomorphism*
$$h : \overline{H^k(X, \mathcal{A}_X^\bullet(H); \partial' - \theta' + \partial'' - \theta'')} \to H^k(X, \mathcal{V}^*),$$
which is induced by the $\mathbb{C}$*-linear isomorphism between* $\mathcal{C}_X^\infty$*-bundles:*
$$h : \overline{H} \to H^*, \quad \overline{e} \mapsto h(\bullet, \overline{e}),$$
where H^* *is the dual harmonic bundle in Construction 2.3.50.*

Proof. Recall that ∇ is the connection on H^* associated to $\mathcal{V}^*$. Then we claim there is a commutative diagram
$$\begin{array}{ccc} \overline{H} & \xrightarrow{h} & H^* \\ \Big\downarrow{\scriptstyle h^*(\nabla)} & & \Big\downarrow{\scriptstyle \nabla} \\ \overline{H} \otimes \mathcal{A}_X^1 & \xrightarrow{h \otimes \mathrm{Id}} & H^* \otimes \mathcal{A}_X^1 \end{array}$$
where
$$h^*(\nabla) := \overline{\partial''} - \overline{\theta''} + \overline{\partial'} - \overline{\theta'}. \tag{2.2}$$
Granting this claim, we see that h induces the isomorphism on the cohomology, which is what we want. To prove the claim, we calculate the pull back of each component of $\nabla = \partial' + \theta' + \partial'' + \theta''$ under h.

1. $\theta' \mapsto -\overline{\theta''}$. The definition of θ' on H^* is defined by
$$(\theta'\lambda)(e) + \lambda(\theta' e) = 0.$$
where $\lambda \in \mathcal{C}^\infty(H^*)$ and $e \in \mathcal{C}^\infty(H)$. Since we want to calculate the pull back of θ', we can test on the dual section $\lambda = h(\bullet, \overline{e_2})$. Then
$$\theta' h(\bullet, \overline{e_2})(e_1) = -h(\theta' e_1, \overline{e_2}) = h(e_1, \overline{-\theta'' e_2}).$$
Therefore $\theta' h(\bullet, \overline{e_2}) = h(\bullet, \overline{-\theta'' e_2})$, which means that θ' corresponds to $-\overline{\theta''}$.

2. $\partial'' \mapsto \overline{\partial'}$. The definition of ∂'' on H^* is

$$(\partial''\lambda)(e) + \lambda(\partial'' e) = \bar{\partial}\lambda(e).$$

Let us test on the dual section $\lambda = h(\bullet, \overline{e_2})$:

$$\partial'' h(\bullet, \overline{e_2})(e_1) = \bar{\partial} h(e_1, \overline{e_2}) - h(\partial'' e_1, \overline{e_2}) = h(e_1, \overline{\partial' e_2}).$$

The last equality comes from the fact that $\partial' + \partial''$ is the metric connection with respect to h. In particular $\partial'' h(\bullet, \overline{e_2}) = h(\bullet, \overline{\partial' e_2})$ and this means that ∂'' corresponds to $\overline{\partial'}$.

3. $\theta'' \mapsto -\overline{\theta'}$ and $\partial' \mapsto \overline{\partial''}$ can be verified using their definition through the dual harmonic metric h^*.

□

In this paper, the map in Lemma 2.3.52 is the key to relate the topological intersection pairings with the polarization on twistor structures.

Definition 2.3.53. Assume we are in the Set-up 2.3.47. The *canonical map* is defined to be the map

$$\begin{aligned} \phi = h \circ \overline{\iota_{-1}} : \overline{E^k|_{z=-1}} \cong & \overline{H^k(X, \mathcal{A}_X^\bullet(H); -\partial' + \theta' + \partial'' - \theta'')} \\ & \xrightarrow{\overline{\iota_{-1}}} \overline{H^k(X, \mathcal{A}_X^\bullet(H); \partial' - \theta' + \partial'' - \theta'')} \xrightarrow{h} H^k(X, \mathcal{V}^*). \end{aligned}$$

where ι_{-1} is the rescaling map in Definition 2.3.36.

Remark 2.3.54. Notice that ϕ depends on the choice of harmonic metric on $H = \mathcal{V} \otimes_{\mathbb{C}} \mathcal{C}_X^\infty$.

Corollary 2.3.55. *Assume $k \le n$. Denote $E^k_{\mathrm{prim}} \subseteq E^k$ to be the sub-twistor structure of the natural twistor structure on $H^k(X, \mathcal{V})$:*

$$H^k(X, \mathcal{V})_{\mathrm{prim}} := \operatorname{Ker} \eta^{n-k+1} \subseteq H^k(X, \mathcal{V}).$$

Then E^k_{prim} is polarized by the bilinear form

$$S_X(\bullet, \phi(\bullet)) : E^k_{\mathrm{prim}}|_{z=1} \otimes \overline{E^k_{\mathrm{prim}}|_{z=-1}} \xrightarrow{\mathrm{Id}\otimes\phi} H^k(X, \mathcal{V})_{\mathrm{prim}} \otimes H^k(X, \mathcal{V}^*)_{\mathrm{prim}} \xrightarrow{S_X} \mathbb{C}.$$

where ϕ is the restriction of the canonical map in Definition 2.3.53. The same statement holds for any other direct summand in the Lefschetz decomposition of $H^k(X, \mathcal{V})$ with respect to η.

Proof. We will only deal with the case of $H^k(X, \mathcal{V})_{\mathrm{prim}}$ and leave other cases to the reader. The proof is quite similar to the proof of Hodge-Simpson Theorem 2.3.26. For any element in $H^k(X, \mathcal{V})$, consider the harmonic representative

$$\sum_{p+q=k} \alpha^{p,q} \otimes m_{p,q},$$

where $\alpha^{p,q}$ are (p,q)-forms and $m_{p,q}$ are sections of the harmonic bundle H. Then by Remark 2.3.33 and the construction of ϕ, we see that

$$\phi(\overline{\mathrm{Iden}[\alpha^{p,q}\otimes m_{p,q}]}) = \phi(\overline{[\alpha^{p,q}\otimes m_{p,q}]}) = (-1)^p\overline{\alpha^{p,q}}\otimes m^{\vee}_{p,q}$$

where $m^{\vee}_{p,q}$ is the section of the dual harmonic bundle H^* satisfying

$$m^{\vee}_{p,q}(\bullet) = h(\bullet, \overline{m_{p,q}}).$$

Therefore

$$\begin{aligned}
&S_X\left(\left[\sum \alpha^{p,q}\otimes m_{p,q}\right], \phi\circ\overline{\mathrm{Iden}\left[\sum \alpha^{p,q}\otimes m_{p,q}\right]}\right)\\
&= \sum_{p+q=k} S_X\left([\alpha^{p,q}\otimes m_{p,q}], [(-1)^p\overline{\alpha^{p,q}}\otimes m^{\vee}_{p,q}]\right)\\
&= \sum_{p+q=k} i^{-k}(-1)^{k(k-1)/2}\int_X h(m_{p,q},\overline{m_{p,q}})\cdot\eta^{n-k}\wedge\alpha^{p,q}\wedge\overline{\alpha^{p,q}} > 0.
\end{aligned}$$

The positivity follows from the classical calculation of the Hodge star operators for primitive forms [56, Theorem 6.29]. □

Remark 2.3.56. The construction of the canonical map actually works for harmonic bundles over any smooth algebraic varieties. In particular, if $T\subseteq X$ is a smooth open subvariety, by Lemma 2.3.84 (whose proof is independent of this section), there exists a vector bundle E^k_T so that $E^k_T|_{z=1}\cong H^k(T,\mathcal{V}|_T)$ and there is an isomorphic canonical map

$$\phi : \overline{E^k_T|_{z=-1}} \xrightarrow{\sim} H^k(T,\mathcal{V}^*|_T).$$

Lemma 2.3.57. *Assume we are in the Set-up 2.3.47. Denote ϕ_X to be the canonical map in Definition 2.3.53 associated to $(X,\mathcal{V})$.*

- *Let η be an ample line bundle on X. Then there is a commutative diagram:*

$$\begin{array}{ccc}
\overline{E^k|_{z=-1}} & \xrightarrow{\overline{F|_{z=-1}}} & \overline{E^{k+2\ell}|_{z=-1}}\\
\downarrow{\scriptstyle\phi_X} & & \downarrow{\scriptstyle\phi_X}\\
H^k(X,\mathcal{V}^*) & \xrightarrow{\eta^\ell} & H^{k+2\ell}(X,\mathcal{V}^*)
\end{array}$$

  where $F: E^k\to E^{k+2\ell}$ is the morphism of twistor structures induced by the cup product with η^ℓ.

- *Let T be a smooth (open or closed) subvariety of X and let E^k_T be the natural mixed twistor structures on $H^k(T,\mathcal{V}|_T)$ as in Lemma 2.3.84. Let ϕ_T be the canonical map as in Remark 2.3.56. Then there is a commutative diagram:*

$$\begin{array}{ccc}
\overline{E^k|_{z=-1}} & \xrightarrow{\overline{F|_{z=-1}}} & \overline{E^k_T|_{z=-1}}\\
\downarrow{\scriptstyle\phi_X} & & \downarrow{\scriptstyle\phi_T}\\
H^k(X,\mathcal{V}^*) & \xrightarrow{R^*} & H^k(T,\mathcal{V}^*|_T)
\end{array}$$

where $F : E^k \to E^k_T$ is the morphism of twistor structures induced by the restriction map and R^* is the restriction map.

Proof. The commutativity with cup product is clear and we focus on the restriction to subvarieties. Recall that (H, h) is the harmonic bundle on X associated to $\mathcal{V}$. Then denote $(H_T, h_T) := (H|_T, h|_T)$ to be the harmonic bundle on T associated to $\mathcal{V}|_T$. We have the following commutative diagram

$$\begin{array}{ccc} \overline{(H, -\partial' + \theta' + \partial'' - \theta'')} & \longrightarrow & \overline{(H_T, -\partial' + \theta' + \partial'' - \theta''|_T)} \\ \big\downarrow \overline{\iota_{-1}} & & \big\downarrow \overline{\iota_{-1}} \\ \overline{(H, \partial' - \theta' + \partial'' - \theta'')} & \longrightarrow & \overline{(H_T, \partial' - \theta' + \partial'' - \theta''|_T)} \\ \big\downarrow h_X & & \big\downarrow h_T \\ (H^*, \nabla) & \longrightarrow & (H^*_T, \nabla_T) \end{array}$$

Since T is open or closed , it would follow from Lemma 2.3.84 or Lemma 2.3.26 that

$$E^k_T|_{z=-1} \cong H^k(T, \mathcal{A}^\bullet_T(H_T); -\partial' + \theta' + \partial'' - \theta''|_T).$$

Taking the cohomology of this diagram, we obtain the desired commutativity. □

Corollary 2.3.58. *Assume we are in the Set-up 2.3.47. Let η be an ample line bundle on X. Let $T \subseteq X$ be a smooth (open or closed) subvariety. Consider the restriction maps*

$$R : H^k(X, \mathcal{V})_{\text{prim}} \to H^k(T, \mathcal{V}|_T),$$
$$R^* : H^k(X, \mathcal{V}^*)_{\text{prim}} \to H^k(T, \mathcal{V}^*|_T).$$

Then the twisted Poincaré pairing S_X (Definition 2.3.48) restricts to a non-degenerate pairing

$$S_X : \operatorname{Ker} R \otimes_{\mathbb{C}} \operatorname{Ker} R^* \to \mathbb{C}.$$

Proof. It follows from Corollary 2.3.55, Lemma 2.3.57 and Corollary 2.3.24. More precisely, one uses the following commutative diagram:

$$\begin{array}{ccc} \overline{E^k|_{z=1}} & \longrightarrow & \overline{E^k_T|_{z=1}} \\ \big\downarrow \text{Iden} & & \big\downarrow \text{Iden} \\ \overline{E^k|_{z=-1}} & \longrightarrow & \overline{E^k_T|_{z=-1}} \\ \big\downarrow \phi_X & & \big\downarrow \phi_T \\ H^k(X, \mathcal{V}^*) & \xrightarrow{R^*} & H^k(T, \mathcal{V}^*|_T) \end{array}$$

□

Remark 2.3.59. Even though we only use the case of smooth subvarieties, the statement is actually true for arbitrary subvariety $Z \subseteq X$. One can reduce to the smooth case by considering a resolution map $\pi : \tilde{Z} \to Z$ and proving a similar lemma in Hodge theory that

$$W_k H^k(Z, \mathcal{V}|_Z) \hookrightarrow H^k(\tilde{Z}, \pi^*(\mathcal{V}|_Z)).$$

where $W_\bullet$ is the weight filtration of the mixed twistor structure on $H^k(Z, \mathcal{V}|_Z)$.

Lemma 2.3.60. *Let E be the natural pure twistor structure on the space $H^k(X, \mathcal{V})_{\text{prim}}$ as in Theorem 2.3.26. Let $*$ be the Hodge star operator (see Definition 2.4.13). For integers $k \le n$, consider the following composition map*

$$(L^{n-k})^{-1} \circ * : H^k(X, \mathcal{V})_{\text{prim}} \to H^{2n-k}(X, \overline{\mathcal{V}}^*)_{\text{prim}} \to H^k(X, \overline{\mathcal{V}}^*)_{\text{prim}}.$$

Then its complex conjugation can be identified up to a scaling constant with the composition map

$$\phi \circ \overline{\text{Iden}} : \overline{H^k(X, \mathcal{V})_{\text{prim}}} \cong \overline{E|_{z=1}} \to \overline{E|_{z=-1}} \to H^k(X, \mathcal{V}^*)_{\text{prim}}.$$

Here Iden *is the identification map in Definition 2.3.11 and ϕ is the canonical map in Definition 2.3.53.*

Proof. Let $\alpha^{p,q}$ be a primitive harmonic (p, q)-form and e is a global section of $H := \mathcal{V} \otimes_{\mathbb{C}} \mathcal{C}^\infty_X$. By Lemma 2.4.16 we have

$$(L^{n-k})^{-1}(*(\alpha^{p,q} \otimes e)) = \frac{(-1)^{k(k+1)/2} i^{p-q}}{(n-k)!} \alpha^{p,q} \otimes \overline{e^\vee} = \frac{(-1)^{k(k+1)/2} i^{-k}}{(n-k)!} \cdot (-1)^p \alpha^{p,q} \otimes \overline{e^\vee}.$$

Here $e^\vee$ is the section of H^* so that

$$e^\vee(\bullet) = h(\bullet, \overline{e}).$$

On the other hand,

$$\phi(\overline{\text{Iden}}(\overline{\alpha^{p,q} \otimes e})) = (-1)^p \overline{\alpha^{p,q}} \otimes e^\vee.$$

□

2.3.4 Cohomology of algebraic varieties and mixed twistor structures

In this section, we will prove the Global Invariant Cycle Theorem for Semisimple Local Systems (c.f. Theorem 2.3.99). To do this, we will review the notion of mixed twistor structures and Simpson's theory of weights for cohomology groups of semisimple local systems on algebraic varieties.

Mixed twistor structures

As explained in [51], the notion of mixed twistor structures is a natural generalization of mixed Hodge structures so that the passage from "Hodge" to "Twistor" is simply forgetting to have an action of $\mathbf{G}_m$.

Definition 2.3.61. -

1. A *mixed twistor structure* is a twistor structure E (see Definition 2.3.8) so that E is filtered by an increasing sequence of strict subbundles W_iE such that
$$\mathrm{Gr}_i^W(E) = W_iE/W_{i-1}E$$
is a pure twistor structure of weight i for all $i \in \mathbb{Z}$.

2. A morphism of mixed twistor structures is defined to be a morphism of filtered bundles on $\mathbf{P}^1$ preserving the filtration.

Remark 2.3.62. Since there is no morphism from a semistable vector bundle of weight k to a semistable vector bundle of weight $< k$, the morphisms of mixed twistor structures are automatically strict with respect to the weight filtrations. In particular, Simpson [51, Proposition 1.2] proved that the category of mixed twistor structures is abelian.

Remark 2.3.63. For comparison with Hodge theory, Simpson [51, Lemma 1.3] also proved the category of $\mathbf{G}_m$-equivariant mixed twistor structures is naturally equivalent to the category of complex mixed Hodge structures.

The theory of weights for semisimple local systems

In Hodge theory, generalizing the classical theorem about Hodge structures on cohomology groups on smooth projective varieties, Deligne [22, 23] showed that there are other situations that one can get (mixed) Hodge structures.

Definition 2.3.64. A mixed Hodge structure $(V, W_\bullet V)$ is of weight q if $\mathrm{Gr}_i^W V$ is a pure Hodge structure of weight $q + i$.

Theorem 2.3.65 (Deligne's yoga of weights [23])**.** *Let X be a complex algebraic variety and $\mathcal{V}$ be a local system on X underlies a polarizable variation of Hodge structures of weight w. Then $H^k(X, \mathcal{V})$ carries a mixed Hodge structure of weight $k + w$. Moreover,*

- *If X is proper, then $\mathrm{Gr}_i^W H^k(X, \mathcal{V}) = 0$ for $i > 0$.*
- *If X is smooth, then $\mathrm{Gr}_i^W H^k(X, \mathcal{V}) = 0$ for $i < 0$.*
- *If X is smooth and projective, then $\mathrm{Gr}_i^W H^k(X, \mathcal{V}) = 0$ if $i \neq 0$.*

In [51, Theorem 5.2], Simpson generalized Deligne's results of the existence of mixed Hodge structures to mixed twistor structures.

Theorem 2.3.66 (Simpson)**.** *Let X be a complex quasi-projective algebraic variety. Let $\mathcal{V}$ be a local system on X coming from the restriction of a semisimple local system on a smooth projective compactification $\overline{X} \supseteq X$. Then $H^k(X, \mathcal{V})$ carries a natural mixed twistor structure, which is functorial in X.*

In the remark under [51, Theorem 5.2], Simpson says that the same yoga of weights hold as in [23, Theorème 8.2.4]. In particular, we have the following statement.

Theorem 2.3.67 (Simpson's yoga of weights)**.** *With the assumption in Theorem 2.3.66. Denote $W_\bullet$ to be the weight filtration on $H^k(X, \mathcal{V})$ induced by the natural mixed twistor structures.*

- *If X is proper, then $\mathrm{Gr}_i^W H^k(X, \mathcal{V}) = 0$ for $i > k$.*
- *If X is smooth, then $\mathrm{Gr}_i^W H^k(X, \mathcal{V}) = 0$ for $i < k$.*
- *If X is smooth and proper, then $\mathrm{Gr}_i^W H^k(X, \mathcal{V}) = 0$ for $i \neq k$.*

Remark 2.3.68. The shift of weights by k arises from different conventions between twistor structures and Hodge structures. On the one hand, as in Definition 2.3.61, Simpson defines

$$\text{weight } \mathrm{Gr}_i^W H^k(X, \mathcal{V}) = i.$$

On the other hand, in Hodge theory, if $\mathcal{V}$ is a polarized VHS of weight 0, then $H^k(X, \mathcal{V})$ is a mixed Hodge structure of weight k, therefore

$$\text{weight } \mathrm{Gr}_i^W H^k(X, \mathcal{V}) = i + k.$$

Remark 2.3.69. In [51], Simpson only proved the last statement of Theorem 2.3.67, which is the Hodge-Simpson Theorem 2.3.26. We provide a proof for open varieties (Corollary 2.3.87) and projective varieties (Corollary 2.3.97).

Mixed Twistor Complexes

In this section, we will recall Simpson's notion of mixed twistor complexes, which is the generalization of mixed Hodge complex in the sense of Deligne [22]. It will be used to construct mixed twistor structures.

Definition 2.3.70. A *mixed twistor complex* is a filtered complex $(M^\bullet, W_\bullet^{\mathrm{pre}})$ of sheaves of $\mathcal{O}_{\mathbf{P}^1}$-modules on $\mathbf{P}^1$ such that

$$\mathcal{H}^i(\mathrm{Gr}_\ell^{W^{\mathrm{pre}}}(M^\bullet))$$

is a locally free sheaf of $\mathcal{O}_{\mathbf{P}^1}$-modules of finite rank, pure of weight $\ell + i$.

A standard argument using the strictness between morphisms of $\mathcal{O}_{\mathbf{P}^1}$ bundles yields the following

Lemma 2.3.71. *([51, Lemma 5.3]) Let $(M^\bullet, W^{\mathrm{pre}}_\bullet)$ be a mixed twistor complex. Then the spectral sequence for a filtered complex which calculates $\mathcal{H}^i(M^\bullet)$ degenerates at E_3. In particular*

$$E_1^{p,q} = \mathrm{Gr}_{-p}^{W^{\mathrm{pre}}}(M^{p+q}), \quad E_3^{p,q} = \mathrm{Gr}_{-p}^{W^{\mathrm{pre}}}\mathcal{H}^{p+q}(M^\bullet).$$

Moreover,

$$W_n\mathcal{H}^i(M^\bullet) := W^{\mathrm{pre}}_{n-i}\mathcal{H}^i(M^\bullet)$$

is the weight filtration for a mixed twistor structure on $\mathcal{H}^i(M^\bullet)$.

Remark 2.3.72. In Hodge theory, we always say the spectral sequence associated to the weight filtration degenerate at E_2-page. Simpson's convention is slightly different because he starts one page later so that his E_3-page is the E_2-page in Hodge theory.

Remark 2.3.73. The shift of the weights in this Lemma is the reason Simpson called this the *pre-weight filtration* and he used the superfix W^{pre}.

Degeneracy at E_3-page makes the explicit description of the lowest piece of the weight filtration possible.

Corollary 2.3.74. *Let $(M^\bullet, W^{\mathrm{pre}}_\bullet)$ be a mixed twistor complex. Suppose the lowest weight of $W^{\mathrm{pre}}M^\bullet$ is ℓ, then*

$$W^{\mathrm{pre}}_\ell\mathcal{H}^k(M^\bullet) = \mathrm{Im}\left\{\mathcal{H}^k(W^{\mathrm{pre}}_\ell M^\bullet) \to \mathcal{H}^k(M^\bullet)\right\},$$

the latter is induced by the inclusion map $W^{\mathrm{pre}}_\ell M^\bullet \to M^\bullet$.

Proof. Let p, q be integers so that $k = p + q$ and $\ell = -p$. Since the spectral sequence associated to the weight filtration degenerate at E_3-page and $-p$ is the lowest weight, we know that

$$W^{\mathrm{pre}}_{-p}\mathcal{H}^{p+q}(M^\bullet) = \mathrm{Gr}^{W^{\mathrm{pre}}}_{-p}\mathcal{H}^{p+q}(M^\bullet) = E_3^{p,q}$$

By definition, we have $E_1^{p,q} = \mathrm{Gr}^{W^{\mathrm{pre}}}_{-p}(M^{p+q})$ and $E_2^{p,q} = \mathcal{H}^{p+q}(\mathrm{Gr}^{W^{\mathrm{pre}}}_{-p}(M^\bullet))$. Then $E_3^{p,q}$ is the cohomology of

$$\mathcal{H}^{p+q-1}(\mathrm{Gr}^{W^{\mathrm{pre}}}_{-p+1}(M^\bullet)) \xrightarrow{\delta} \mathcal{H}^{p+q}(\mathrm{Gr}^{W^{\mathrm{pre}}}_{-p}(M^\bullet)) \to \mathcal{H}^{p+q+1}(\mathrm{Gr}^{W^{\mathrm{pre}}}_{-p-1}(M^\bullet)) = 0.$$

The last equality comes from the assumption that $-p$ is the lowest weight.

By [56, Lemma 8.24], δ is the connecting map of the long exact sequence associated to the short exact sequence of complexes

$$0 \to W^{\mathrm{pre}}_{-p}M^\bullet \to W^{\mathrm{pre}}_{-p+1}M^\bullet \to \mathrm{Gr}^{W^{\mathrm{pre}}}_{-p+1}(M^\bullet) \to 0.$$

Therefore

$$\begin{aligned} W^{\mathrm{pre}}_{-p}\mathcal{H}^{p+q}(M^\bullet) &= \mathrm{coker}(\delta) \\ &= \mathrm{Im}\left\{\mathcal{H}^{p+q}(W^{\mathrm{pre}}_{-p}(M^\bullet)) \to \mathcal{H}^{p+q}(W^{\mathrm{pre}}_{-p+1}(M^\bullet))\right\} \\ &= \mathrm{Im}\left\{\mathcal{H}^{p+q}(W^{\mathrm{pre}}_{-p}(M^\bullet)) \to \mathcal{H}^{p+q}(M^\bullet)\right\}. \end{aligned}$$

□

Glueing construction

In previous section, we have seen how to glue two vector bundles over coordinate charts of $\mathbf{P}^1$ to construct a twistor structure on $H^k(X, \mathcal{V})$. Since the natural mixed twistor structure on $H^k(U, \mathcal{V}|_U)$ will be obtained via a mixed twistor complex, we would like to review Simpson's glueing construction for complexes.

Set-up 2.3.75. -

1. Let $M^\bullet$ and $N^\bullet$ be filtered complexes of sheaves of $\mathcal{O}$-modules (with the pre-weight filtrations denoted by $W^{\text{pre}}(-)$), respectively over Ω_0 and Ω_∞, which are the standard neighborhoods $\mathbf{A}^1$ of 0 and ∞ in $\mathbf{P}^1$.

2. Let $P^\bullet$ be a filtered complex of sheaves of $\mathcal{O}$-modules on $\mathbf{G}_m = \Omega_0 \cap \Omega_\infty$.

3. There are filtered quasi-isomorphisms
$$M^\bullet|_{\mathbf{G}_m} \xleftarrow{f} P^\bullet \xrightarrow{g} N^\bullet|_{\mathbf{G}_m}.$$

Notation 2.3.76. -

- Let i denote one of the three inclusions $\Omega_0 \hookrightarrow \mathbf{P}^1$ or $\Omega_\infty \hookrightarrow \mathbf{P}^1$ or $\mathbf{G}_m \hookrightarrow \mathbf{P}^1$. Denote $Ri_{\text{ex}} := i_* \circ \text{Go}$, the composition of direct image with the Godement resolution, and require that there is a fixed functorial quasi-isomorphism
$$\mathcal{F}^\bullet \to i^* Ri_{\text{ex}}(\mathcal{F}^\bullet).$$

- Denote $M^\bullet_{\text{ex}} := Ri_{\text{ex}} M^\bullet$, $P^\bullet_{\text{ex}} := Ri_{\text{ex}} P^\bullet$ and $N^\bullet_{\text{ex}} := Ri_{\text{ex}} N^\bullet$, with i being the appropriate inclusions.

- If $f : A^\bullet \to B^\bullet$ is a map of filtered complexes, then there is a filtered complex $\text{Cone}(A \to B)$ defined as
$$\text{Cone}(A \to B)^k := A^{k+1} \oplus B^k$$
with differential equal to $d_A + d_B + f$.

Definition 2.3.77. With the notation above, we define
$$\text{Patch}(M \leftarrow P \to N) := \text{Cone}(P^\bullet_{\text{ex}} \xrightarrow{(f,-g)} M^\bullet_{\text{ex}} \oplus N^\bullet_{\text{ex}}).$$

Remark 2.3.78. If the third condition in Set-up 2.3.75 becomes
$$M^\bullet|_{\mathbf{G}_m} \xrightarrow{f} P^\bullet \xleftarrow{g} N^\bullet|_{\mathbf{G}_m},$$
then one define
$$\text{Patch}(M \to P \leftarrow N) := \text{Cone}(M^\bullet_{\text{ex}} \oplus N^\bullet_{\text{ex}} \xrightarrow{(f,-g)} P^\bullet_{\text{ex}})[-1].$$

Definition 2.3.79. Suppose we have filtered quasi-isomorphisms

$$M^\bullet|_{\mathbf{G}_m} \leftarrow P^\bullet \rightarrow Q^\bullet \leftarrow R^\bullet \rightarrow N^\bullet|_{\mathbf{G}_m}$$

where $P^\bullet, Q^\bullet, R^\bullet$ are filtered complexes over $\mathbf{G}_m$, then we define

$$\mathrm{Patch}(M, P, Q, R, N) := \mathrm{Patch}(M \leftarrow \mathrm{Cone}(P \oplus R \rightarrow Q)[-1] \rightarrow N).$$

Lemma 2.3.80. *Suppose we have filtered quasi-isomorphisms*

$$M^\bullet|_{\mathbf{G}_m} \xrightarrow{f} P^\bullet \xleftarrow{g} N^\bullet|_{\mathbf{G}_m}.$$

Then there is a natural filtered quasi-isomorphism

$$\mathrm{Patch}(M \rightarrow P \leftarrow N) \cong \mathrm{Patch}(M, M|_{\mathbf{G}_m}, P, N|_{\mathbf{G}_m}, N),$$

where the latter is induced by

$$M^\bullet|_{\mathbf{G}_m} \xleftarrow{\mathrm{Id}} M^\bullet|_{\mathbf{G}_m} \rightarrow P^\bullet \leftarrow N^\bullet|_{\mathbf{G}_m} \xrightarrow{\mathrm{Id}} N^\bullet|_{\mathbf{G}_m}.$$

Proof. Denote $T = \mathrm{Cone}(M^\bullet|_{\mathbf{G}_m} \oplus N^\bullet|_{\mathbf{G}_m} \rightarrow P)[-1]$. By definition, we have natural morphisms

$$T_{\mathrm{ex}} \rightarrow \mathrm{Cone}(M^\bullet_{\mathrm{ex}} \oplus N^\bullet_{\mathrm{ex}} \rightarrow P^\bullet_{\mathrm{ex}})[-1] = \mathrm{Patch}(M \rightarrow P \leftarrow N).$$

$$T_{\mathrm{ex}} \rightarrow (M^\bullet|_{\mathbf{G}_m})_{\mathrm{ex}} \rightarrow \mathrm{Cone}(T_{\mathrm{ex}} \rightarrow M^\bullet_{\mathrm{ex}} \oplus N^\bullet_{\mathrm{ex}}) = \mathrm{Patch}(M, M|_{\mathbf{G}_m}, P, N|_{\mathbf{G}_m}, N).$$

$$T_{\mathrm{ex}} \rightarrow (N^\bullet|_{\mathbf{G}_m})_{\mathrm{ex}} \rightarrow \mathrm{Cone}(T_{\mathrm{ex}} \rightarrow M^\bullet_{\mathrm{ex}} \oplus N^\bullet_{\mathrm{ex}}) = \mathrm{Patch}(M, M|_{\mathbf{G}_m}, P, N|_{\mathbf{G}_m}, N).$$

The first one is a filtered quasi-isomorphism because it is determined by its restriction to $\mathbf{G}_m$, which is an identity map. The second one and the third one are also filtered quasi-isomorphism by the fact that if A, B, C are three filtered complexes, $A \rightarrow B$ is a filtered quasi-isomorphism between two complexes and $A \rightarrow C$ is any morphism, then the natural morphism

$$C \rightarrow \mathrm{Cone}(A \rightarrow B \oplus C)$$

is a filtered quasi-isomorphism. □

Simpson proved that the glueing construction has the effect of glueing corresponding cohomology sheaves in §2.3.2.

Lemma 2.3.81. *The sheaf of $\mathcal{O}_{\mathbf{P}^1}$-modules*

$$\mathcal{H}^k(\mathrm{Gr}^{W^{\mathrm{pre}}}\mathrm{Patch}(M \leftarrow P \rightarrow N))$$

is the sheaf of $\mathcal{O}_{\mathbf{P}^1}$-modules obtained by glueing together $\mathcal{H}^k(\mathrm{Gr}^{W^{\mathrm{pre}}}M^\bullet)$ over Ω_0 with $\mathcal{H}^k(\mathrm{Gr}^{W^{\mathrm{pre}}}N^\bullet)$ over Ω_∞, via the isomorphism of cohomology sheaves induced by the filtered quasi-isomorphisms

$$M^\bullet|_{\mathbf{G}_m} \xleftarrow{f} P^\bullet \xrightarrow{g} N^\bullet|_{\mathbf{G}_m}.$$

Let X be a smooth projective variety and let $\mathcal{V}$ be a semisimple local system on X. We would like to review the construction of the natural pure twistor structure on $H^k(X, \mathcal{V})$ (c.f. Theorem 2.3.26) from the point of view of the glueing construction. Recall from Construction 2.3.35, there is a triple $(\mathcal{G}, \mathcal{L}, \widetilde{\mathcal{G}})$ associated to the local system $\mathcal{V}$:

- $(\mathcal{G}, \nabla_{\mathcal{G}})$ is a holomorphic bundle on $X \times \Omega_0$ with a z-connection.
- $\mathcal{L} = \{L_{z_0}\}_{z_0 \in \mathbf{G}_m}$ is a family of local systems so that L_{z_0} is the local system over X^{top} consisting of flat sections of $(\mathcal{G}|_{X \times \{z_0\}}, z_0^{-1} \nabla_{\mathcal{G}}|_{X \times \{z_0\}})$. [Note that L_t has no reference to the holomorphic structure of X which is important for the patching].
- $(\widetilde{\mathcal{G}}, \nabla_{\widetilde{\mathcal{G}}})$ is a holomorphic bundle with a z^{-1}-connection on $\overline{X} \times \Omega_\infty$.

Let $p : X \times \mathbf{P}^1 \to \mathbf{P}^1$ be the second projection.

Lemma 2.3.82. *Consider the complex of sheaves of $\mathcal{O}_{\mathbf{P}^1}$-modules*

$$\operatorname{Patch}\left(Rp_*(\xi \Omega^\bullet_X(\mathcal{G})) \to Rp_*(\mathcal{L}) \leftarrow Rp_*(\xi \Omega^\bullet_{\overline{X}}(\widetilde{\mathcal{G}})) \right).$$

Then the k-th cohomology sheave of this complex is the natural pure twistor structure on $H^k(X, \mathcal{V})$ in Theorem 2.3.26.

Proof. By Lemma 2.3.81, the k-th cohomology sheaf is obtained by glueing $R^k p_*(\xi \Omega^\bullet_X(\mathcal{G}))$ and $R^k p_*(\xi \Omega^\bullet_{\overline{X}}(\widetilde{\mathcal{G}}))$ over $\mathbf{G}_m$, which is exactly the natural pure twistor structure on $H^k(X, \mathcal{V})$ by the construction in Theorem 2.3.26. □

Mixed twistor structures on open varieties

Let X be a smooth projective variety and $U \subseteq X$ be a Zariski-open subvariety where $D := X \setminus U$ is a normal crossing divisor. Let $\mathcal{V}$ be a semisimple local system on X. In this section, we will review the construction of the natural mixed twistor structure on $H^k(U, \mathcal{V}|_U)$ (c.f. Theorem 2.3.66) using the glueing construction.

First, we would like to review some basic properties of the complex of holomorphic logarithmic differentials $\Omega^\bullet_X(\log D)$. Denote $j : U \hookrightarrow X$ to be the open embedding, it is a standard fact that we have a quasi-isomorphism

$$Rj_* \mathbb{C}_U \xrightarrow{\sim} \Omega^\bullet_X(\log D)$$

so that

$$H^k(U, \mathcal{V}|_U) \cong \mathbb{H}^k(X, \Omega^\bullet_X(\log D) \otimes_{\mathbb{C}} \mathcal{V}).$$

There are two filtrations on $\Omega^\bullet_X(\log D)$.

1. The weight filtration $W^{\text{pre}}_\bullet$:

$$W^{\text{pre}}_\ell \Omega^\bullet_X(\log D) := \wedge^\ell \Omega^1_X(\log D) \wedge \Omega^{\bullet - \ell}_X.$$

2. The filtration τ:

$$\tau_\ell \Omega_X^\bullet(\log D) := \tau_{\leq \ell}\Omega_X^\bullet(\log D)$$

which is the truncation functor associated to a complex so that for a complex $(K^\bullet, d)$,

$$(\tau_{\leq \ell} K^\bullet)^a = \begin{cases} K^a & \text{if } a < \ell, \\ \mathrm{Ker}\, d & \text{if } a = \ell, \\ 0, & \text{if } a > \ell. \end{cases}$$

Notice that

$$W_\ell^{\mathrm{pre}} \Omega_X^\bullet(\log D)^k = \Omega_X^\bullet(\log D)^k, \quad \forall k \leq \ell.$$

Therefore we can define a natural filtered morphism

$$(\Omega_X^\bullet(\log D), W_\bullet^{\mathrm{pre}}) \leftarrow (\Omega_X^\bullet(\log D), \tau). \tag{$*$}$$

Moreover, Deligne [22, Proposition 3.1.8] showed that this is a filtered quasi-isomorphism.

Now, consider the following complexes on subsets of the topological space $Y := X^{\mathrm{top}} \times \mathbf{P}^1$. In the following construction, p denotes the second projection map to the $\mathbf{P}^1$-direction and $j : U \to X$ denotes the open embedding. Recall there is a triple $(\mathcal{G}, \mathcal{L}, \widetilde{\mathcal{G}})$ associated to $\mathcal{V}$ from Construction 2.3.35.

1. On $X \times \Omega_0$, set

$$M^\bullet := (\xi\Omega_X^\bullet(\log D) \otimes_{\mathcal{O}_{X \times \Omega_0}} \mathcal{G}, W_\bullet^{\mathrm{pre}})$$

where $\xi\Omega_X^\bullet(\log D)$ is the Rees bundle complex associated to the Hodge filtration on $\Omega_X^\bullet(\log D)$ (as in Construction 2.3.35) and $W_\bullet^{\mathrm{pre}}$ is the induced filtrations from the first projection map.

2. On $X \times \mathbf{G}_m$, set

$$\begin{aligned} P^\bullet &:= (\xi\Omega_X^\bullet(\log D) \otimes_{\mathcal{O}_{X \times \mathbf{G}_m}} \mathcal{G}, \tau) \\ Q^\bullet &:= (\xi j_* \mathcal{A}_U^\bullet \otimes_{p^{-1}\mathcal{O}_{\mathbf{G}_m}} \mathcal{L}, \tau) \\ R^\bullet &:= (\xi\Omega_{\overline{X}}^\bullet(\log D) \otimes_{\mathcal{O}_{\overline{X} \times \mathbf{G}_m}} \widetilde{\mathcal{G}}, \tau) \end{aligned}$$

Here τ denotes the induced filtration from the projection map to X.

3. On $\overline{X} \times \Omega_\infty$, set

$$N^\bullet := (\xi\Omega_{\overline{X}}^\bullet(\log D) \otimes_{\mathcal{O}_{\overline{X} \times \Omega_\infty}} \widetilde{\mathcal{G}}, W_\bullet^{\mathrm{pre}}).$$

Here are the filtered morphisms between these complexes.

- The morphism $(*)$ induces the filtered quasi-isomorphism

$$M^\bullet|_{X \times \mathbf{G}_m} \leftarrow P^\bullet$$

- The morphism $P^\bullet \to Q^\bullet$ is defined by

$$P^\bullet = (\xi\Omega_X^\bullet(\log D) \otimes_{\mathcal{O}_{X\times\mathbf{G}_m}} \mathcal{G}, \tau) \to (\xi j_*\Omega_U^\bullet \otimes_{\mathcal{O}_{X\times\mathbf{G}_m}} \mathcal{G}, \tau)$$
$$\to (\xi j_*\mathcal{A}_U^\bullet \otimes_{\mathcal{O}_{X\times\mathbf{G}_m}} \mathcal{G}, \tau) = (\xi j_*\mathcal{A}_U^\bullet \otimes_{p^{-1}\mathcal{O}_{\mathbf{G}_m}} \mathcal{L}, \tau)$$

  where τ all denotes the filtration induces by truncations. The last equality comes from the fact that $\mathcal{G}|_{X\times\{t\}} = \mathcal{O}_{X\times\{t\}} \otimes_{\mathbb{C}} \mathcal{L}|_{X\times\{t\}}$ (because $\mathcal{L}|_{X\times\{t\}}$ is the local system of flat sections of $(\mathcal{G}|_{X\times\{t\}}, t^{-1}\nabla_{\mathcal{G}}|_{X\times\{t\}})$.

- One defines similar filtered quasi-isomorphisms

$$Q^\bullet \leftarrow R^\bullet \to N^\bullet|_{\overline{X}\times\mathbf{G}_m}$$

  with X and $\mathcal{G}$ replaced by $\overline{X}$ and $\widetilde{\mathcal{G}}$.

At the end of day, we have the diagram of filtered quasi-isomorphisms of complexes on $\mathbf{G}_m$:

$$Rp_*M^\bullet|_{\mathbf{G}_m} \leftarrow Rp_*P^\bullet \to Rp_*Q^\bullet \leftarrow Rp_*R^\bullet \to Rp_*N^\bullet|_{\mathbf{G}_m}.$$

with filtration defined by $W_\bullet^{\mathrm{pre}}Rp_*K := Rp_*(W_\bullet^{\mathrm{pre}}K)$, where $(K, W^{\mathrm{pre}}K)$ is any one of the five complexes.

Construction 2.3.83. The complex associated to the open set U and the local system $\mathcal{V}$ is defined by

$$\mathrm{MTC}(\mathcal{V}) := \mathrm{Patch}(Rp_*M, Rp_*P, Rp_*Q, Rp_*R, Rp_*N)$$

as in Definition 2.3.79.

Lemma 2.3.84 (Simpson)**.** *Let X be a smooth projective variety and $\mathcal{V}$ be a semisimple local system on X. Let $U \subseteq X$ be a Zariski open subset. Then the complex $\mathrm{MTC}(\mathcal{V})$ is a mixed twistor complex in the sense of Definition 2.3.70. In particular, for any integer k, $H^k(U, \mathcal{V}|_U)$ underlies a natural mixed twistor structure, which is $\mathcal{H}^k\mathrm{MTC}(\mathcal{V})$.*

Remark 2.3.85. This gives an outline of the construction for the natural mixed twistor structures on smooth algebraic varieties in Theorem 2.3.66.

Remark 2.3.86. It is actually enough to only use three complexes $P^\bullet, Q^\bullet, R^\bullet$ and

$$\mathrm{Patch}(Rp_*P \to Rp_*Q \leftarrow Rp_*R)$$

to construct the mixed twistor structure on $H^k(U, \mathcal{V}|_U)$.

Corollary 2.3.87. *Let X be a smooth projective variety and $\mathcal{V}$ be a semisimple local system on X. Let $U \subseteq X$ be a Zariski open subset. Then $H^k(U, \mathcal{V}|_U)$ underlies a natural mixed twistor structure so that*

$$\mathrm{Gr}_\ell^W H^k(U, \mathcal{V}|_U) = 0, \quad \forall \ell < k.$$

Proof. Let $\pi : \tilde{X} \to X$ be a log resolution of $(X, X \setminus U)$ so that $\pi^{-1}(X \setminus U)$ is a normal crossing divisor in $\tilde{X}$ and $\tilde{U} := \pi^{-1}(U) \to U$ is the identity map. Since $\mathcal{V}$ is associated to a harmonic bundle, the local system $\pi^*\mathcal{V}$ is also semisimple. Then Lemma 2.3.84 says that

$$H^k(U, \mathcal{V}|_U) \cong H^k(\tilde{U}, \pi^*\mathcal{V}|_{\tilde{U}})$$

underlies a natural mixed twistor structure. Moreover, it follows from the construction that the weight is greater or equal to k. □

Lowest weight filtration on cohomology of open varieties

As an analogy with the classical yoga of weights, there should be a result saying that the lowest piece of weight filtration of Hodge structure on the cohomology of an Zariski open subset coincides with the restriction from the ambient smooth projective variety. Since Simpson didn't mention about it explicitly either, we decide to include the proof of this statement here, which will be the key step for proving the Global Invariant Cycle Theorem for Semisimple Local Systems 2.3.99.

Lemma 2.3.88. *Let $j : U \to X$ be the inclusion of a Zariski open subset U of a smooth projective variety X. Let $\mathcal{V}$ be a semisimple local system on X, then*

$$W_k H^k(U, \mathcal{V}|_U) = j^* H^k(X, \mathcal{V}),$$

where $W_\bullet$ is the weight filtration of the natural mixed twistor structure on $H^k(U, \mathcal{V}|_U)$.

Proof. **Step 1**: it suffices to assume that $X \setminus U$ is a normal crossing divisor because one can show that the image of restriction map $j^*H^k(X, \mathcal{V})$ on $H^k(U, \mathcal{V}|_U)$ is independent of the compactification of U. More concretely, consider the following commutative diagram

$$\begin{array}{ccc} U & \xrightarrow{\tilde{j}} & \tilde{X} \\ {\scriptstyle \mathrm{id}}\downarrow & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow[j]{} & X \end{array}$$

Here $\tilde{X} \setminus U$ is a normal crossing divisor. If $\mathcal{V}$ is semisimple, then $\pi^*\mathcal{V}$ is also a semisimple local system on $\tilde{X}$. Then

$$W_k H^k(U, j^*\mathcal{V}) = W_k H^k(U, \tilde{j}^*\pi^*\mathcal{V}) = \tilde{j}^* H^k(\tilde{X}, \pi^*\mathcal{V}).$$

We just need to show that

$$\tilde{j}^* H^k(\tilde{X}, \pi^*\mathcal{V}) = \tilde{j}^*\pi^* H^k(X, \mathcal{V}).$$

Note that in general

$$H^k(\tilde{X}, \pi^*\mathcal{V}) \neq \pi^* H^k(X, \mathcal{V}).$$

But they become equal after restricting to U because the extra terms live on $X \setminus U$ via Leray spectral sequence for the map $\pi : \tilde{X} \to X$.

$$E_2^{p,q} = H^p(X, \mathcal{V} \otimes R^q\pi_*\mathbb{C}) = H^p(X, R^q\pi_*(\pi^*\mathcal{V})) \Longrightarrow H^{p+q}(\tilde{X}, \pi^*\mathcal{V}).$$

Note here that $R^q\pi_*\mathbb{C}$ supports on $X \setminus U$ when $q > 0$.

Step 2: We will prove the statement

$$W_k H^k(U, \mathcal{V}|_U) = j^* H^k(X, \mathcal{V})$$

on the level of twistor structures. Let E^k be the natural twistor structure on $H^k(X, \mathcal{V})$ as in Theorem 2.3.26. By Lemma 2.3.82 and Lemma 2.3.80

$$\begin{aligned} E^k &\cong \mathcal{H}^k\mathrm{Patch}\left(Rp_*(\xi\Omega_X^\bullet(\mathcal{G})) \to Rp_*(\mathcal{L}) \leftarrow Rp_*(\xi\Omega_{\overline{X}}^\bullet(\widetilde{\mathcal{G}}))\right) \\ &\cong \mathcal{H}^k\mathrm{Patch}\left(Rp_*(\xi\Omega_X^\bullet(\mathcal{G})), Rp_*(\xi\Omega_X^\bullet(\mathcal{G}))|_{\mathbf{G}_m}, Rp_*(\mathcal{L}), Rp_*(\xi\Omega_{\overline{X}}^\bullet(\widetilde{\mathcal{G}}))|_{\mathbf{G}_m}, Rp_*(\xi\Omega_{\overline{X}}^\bullet(\widetilde{\mathcal{G}}))\right) \end{aligned}$$

By Lemma 2.3.84, the map $j^* : H^k(X, \mathcal{V}) \to H^k(U, \mathcal{V}|_U)$ can be lifted as a morphism of mixed twistor structures induced by the following morphisms of complexes.

- $\xi\Omega_X^\bullet \otimes \mathcal{G} \to \xi\Omega_X^\bullet(\log D) \otimes \mathcal{G}$.
- $\mathcal{L} \to \xi j_*\Omega_U^\bullet \otimes \mathcal{L}$.

By Corollary 2.3.74, $W_k H^k(U, \mathcal{V}|_U)$ underlies the glueing of the images of the inclusion maps

1. $R^k p_*(W_0^{\mathrm{pre}}\xi\Omega_X^\bullet(\log D) \otimes \mathcal{G}) \to R^k p_*(\xi\Omega_X^\bullet(\log D) \otimes \mathcal{G})$.
2. $R^k p_*(W_0^{\tau}\xi\Omega_X^\bullet(\log D) \otimes \mathcal{G}) \to R^k p_*(\xi\Omega_X^\bullet(\log D) \otimes \mathcal{G})$.
3. $R^k p_*(W_0^{\tau}\xi j_*\Omega_U^\bullet \otimes \mathcal{L}) \to R^k p_*(\xi j_*\Omega_U^\bullet \otimes \mathcal{L})$.

By the residue map, we have

$$W_0^{\mathrm{pre}}\xi\Omega_X^\bullet(\log D) \otimes \mathcal{G} \cong \xi\Omega_X^\bullet \otimes \mathcal{G}.$$

Topological calculations from [22, 3.1.8.1] show that

$$W_0^{\tau}\xi\Omega_X^\bullet(\log D) \otimes \mathcal{G} = \mathcal{H}^0(\xi\Omega_X^\bullet(\log D) \otimes \mathcal{G}) \cong \xi\Omega_X^\bullet \otimes \mathcal{G}$$

and by definition of the truncation functor

$$W_0^{\tau}\xi j_*\Omega_U^\bullet \otimes \mathcal{L} = \mathcal{H}^0(\xi j_*\Omega_U^\bullet \otimes \mathcal{L}) \cong \mathcal{L}.$$

In particular, on each corresponding subset of $\mathbf{P}^1$, the lowest weight filtration coincides with the image of restriction from X. Since the glueing construction preserves the quasi-isomorphism, this is what we want. □

Mixed twistor structures on projective varieties

In this section, we will construct natural mixed twistor structures on projective varieties, which is enough for the purpose of proving the Global Invariant Cycle Theorem for semisimple local systems. One can construct mixed twistor structures on arbitrary proper varieties as described in [51] using simplicial methods.

Instead of simplicial methods, we will adapt the method of El Zein [26], where we first construct mixed twistor structures on normal crossing varieties as in Griffiths-Schmid [31] and then use resolution of singularities to deal with arbitrary projective varieties. For construction of mixed Hodge structures on the cohomology of projective varieties, see [27].

We use the following convention for spectral sequences.

Theorem 2.3.89 ([56], Theorem 8.21)**.** *Let* $(A^\bullet, F^pA^\bullet)$ *be a filtered complex in an abelian category with a decreasing filtration. Assume that there exists an integer* ℓ *so that*

$$F^\ell A^k = 0, \quad \forall k.$$

Then there is a spectral sequence

$$(E_r^{p,q}, d_r), \quad d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

so that

$$E_0^{p,q} = \mathrm{Gr}_p^F A^{p+q}, \quad E_\infty^{p,q} = \mathrm{Gr}_p^F H^{p+q}(A^\bullet).$$

Corollary 2.3.90. *Let* $(A^{\bullet,\bullet}, d, \delta)$ *be a double complex in an abelian category with*

$$\begin{aligned} d &: A^{p,q} \to A^{p+1,q} \\ \delta &: A^{p,q} \to A^{p,q+1} \\ d \circ \delta &= \delta \circ d. \end{aligned}$$

Let $(A^\bullet, D)$ *be the total complex associated to the double complex with* $D = d + (-1)^p\delta$ *on* $A^{p,q}$*. Consider an increasing filtration* W *so that*

$$W_pA^n = \bigoplus_{r \le p} A^{r,n-r}.$$

Then there is a spectral sequence

$$(E_r^{p,q}, d_r), \quad d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

so that

$$\begin{aligned} E_0^{p,q} &= A^{q,p}, \quad d_0 = d : A^{q,p} \to A^{q+1,p} \\ E_1^{p,q} &= H_d^q(A^{\bullet,p}), \quad d_1 = (-1)^q\delta : H_d^q(A^{\bullet,p}) \to H_d^q(A^{\bullet,p+1}) \\ E_\infty^{p,q} &= \mathrm{Gr}_q^W H_D^{p+q}(A^\bullet). \end{aligned}$$

Proof. We consider a decreasing filtration $\tilde{W}^\bullet$ on $A^\bullet$ where

$$\tilde{W}^p A^n := W_{n-p} A^n = \bigoplus_{s \geq p} A^{n-s,s}.$$

By Theorem 2.3.89, there is a spectral sequence

$$(E_r^{p,q}, d_r), \quad d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

so that

$$E_0^{p,q} = \mathrm{Gr}_p^{\tilde{W}} A^{p+q} = A^{q,p}, \quad d_0 = d.$$

Therefore $E_1^{p,q}$ is the q-th cohomology of $A^{\bullet,p}$. Finally,

$$E_\infty^{p,q} = \frac{\tilde{W}^p H_D^{p+q}(A^\bullet)}{\tilde{W}^{p+1} H_D^{p+q}(A^\bullet)} = \frac{W_{(p+q)-p} H_D^{p+q}(A^\bullet)}{W_{(p+q)-(p+1)} H_D^{p+q}(A^\bullet)} = \mathrm{Gr}_q^W H_D^{p+q}(A^\bullet).$$

□

Remark 2.3.91. Because of the way the weight filtration $W_\bullet$ is defined, we have to use a decreasing filtration different from the one in [56, Proposition 8.25].

Proposition 2.3.92. *Let X be a smooth projective variety and $\mathcal{V}$ be a semisimple local system on X. Let $Z = D_1 \cup \ldots \cup D_N$ be a normal crossing divisor of X. Then $H^k(Z, \mathcal{V}|_Z)$ underlies a natural mixed twistor structure so that*

$$\mathrm{Gr}_\ell^W H^k(Z, \mathcal{V}|_Z) = 0, \quad \forall \ell > k.$$

Moreover, the restriction map

$$H^k(X, \mathcal{V}) \to H^k(Z, \mathcal{V}|_Z)$$

underlies a morphism between natural mixed twistor structures.

Proof. We follow the construction in [31] but with the convention in Corollary 2.3.90.

Step 1. For $q \geq 1$, we set

$$D^{[q]} = \bigsqcup_{|I|=q} D_I.$$

Here for an index set $I = \{i_1, \ldots, i_q\} \subset \{1, \ldots, N\}$ with $|I| = q$, we set $D_I := D_{i_1} \cap \ldots \cap D_{i_q}$. For each q, there is a natural morphism $\pi^q : D^{[q]} \to X$ and we denote

$$\mathcal{V}_{D^{[q]}} := (\pi^q)^* \mathcal{V}.$$

Now we define a double complex

$$(A^{p,q}, d, \delta)$$

with $p \geq 0, q \geq 1$ in the following way.

1. For $q \geq 1$, $A^{p,q} := H^0(D^{[q]}, \mathcal{A}^p_{D^{[q]}} \otimes \mathcal{V}_{D^{[q]}})$ is the space of $\mathcal{C}^\infty$ p-forms on the smooth projective manifold $D^{[q]}$ with coefficient in the pull-back local system.

2. The differential $d : A^{p,q} \to A^{p+1,q}$ is the composition of the exterior derivative on $D^{[q]}$ with the pull-back connection on $\mathcal{V}_{D^{[q]}}$.

3. A form $\phi \in A^{p,q}$ can be written as

$$\phi = \sum_{|I|=q} \phi_I.$$

The Cech differential $\delta : A^{p,q} \to A^{p,q+1}$ is defined by

$$(\delta\phi)_{(j_1,\dots,j_{q+1})} = \sum_{\ell=1}^{q} (-1)^\ell \phi_{(j_1,\dots,\hat{j}_\ell,\dots,j_{q+1})}|_{D_{(j_1,\dots,j_{q+1})}}.$$

Note that δ is well-defined because $\mathcal{V}_{D^{[q]}}$ restricts to $\mathcal{V}_{D^{[q+1]}}$ via

$$D^{[q+1]} \hookrightarrow D^{[q]}.$$

4. It is easy to verify that δ commutes with d.

Step 2. Define the weight filtration on the associated total complex $(A^\bullet, D)$ to be

$$W_p A^n = \bigoplus_{r \leq p} A^{r,n-r}.$$

By Corollary 2.3.90, there is a spectral sequence

$$(E_r^{p,q}, d_r), \quad d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}$$

so that

$$E_1^{p,q} = H_d^q(A^{\bullet,p}) = H^q(D^{[p]}, \mathcal{V}_{D^{[p]}}), \quad d_1 = (-1)^q \delta : H_d^q(A^{\bullet,p}) \to H_d^q(A^{\bullet,p+1}).$$
$$E_\infty^{p,q} = \mathrm{Gr}_q^W H_D^{p+q}(A^\bullet).$$

Step 3. We claim that the spectral sequence degenerates at E_2-page. This is similar to the proof of Lemma 2.3.71. To do this, we consider a double complex

$$(B^{p,q}, \mathbf{d}, \delta)$$

with values in holomorphic bundles over $\mathbf{P}^1$ whose restriction at $z = 1$ recoverd the double complex $A^{p,q}$. The same applies to the spectral sequence associated to $B^{p,q}$ with the same weight filtration on the total complex. Abusing notations, we use the same notation for the spectral sequence.

From the previous step, we see that $E_1^{p,q}$ is a pure twistor structure of weight q so that $E_1^{p,q}|_{z=1} = H^q(D^{[p]}, \mathcal{V}_{D^{[p]}})$ and the differential

$$d_1 = (-1)^q \delta : E_1^{p,q} \to E^{p+1,q}$$

is a morphism of pure twistor structures. Hence $E_2^{p,q}$ is a pure twistor structure of weight q. Now for $r \geq 2$ we have

$$d_r : E_r^{p,q} \to E_r^{p+r,q-r+1}.$$

$E_r^{p,q}$ is a pure twistor structure of weight q but $E_r^{p+r,q-r+1}$ is a pure twistor structure of weight $q-r+1 < q$. Therefore we conclude that $d_r = 0$ and

$$\mathrm{Gr}_q^W H_D^{p+q}(B^\bullet) = E_\infty^{p,q} = E_2^{p,q}$$

is a pure twistor structure of weight q. Therefore $(H_D^{p+q}(B^\bullet), W_\bullet)$ is a mixed twistor structure where the weight filtration is the one from the spectral sequence.

Step 4. By a similar argument in [31, Lemma 4.6], we conclude that

$$H_D^k(B^\bullet)|_{z=1} \cong H^k(Z, \mathcal{V}|_Z).$$

Therefore Step 3 implies that $H^k(Z, \mathcal{V}|_Z)$ underlies a natural mixed twistor structure so that

$$W_\ell H^k(Z, \mathcal{V}|_Z) = W_\ell H_D^k(B^\bullet)|_{z=1}.$$

Moreover by the construction, both indices of the double complex $A^{p,q}$ and $B^{p,q}$ are non-negative, we conclude that

$$\mathrm{Gr}_\ell^W H^k(Z, \mathcal{V}|_Z) = E_\infty^{k-\ell,\ell}|_{z=1} = 0, \quad \forall \ell > k.$$

□

Remark 2.3.93. The degeneracy statement in Step 3 can also be proved by showing that the total complex $(B^\bullet, W_\bullet^{\mathrm{pre}})$ with the pre-weight filtration (modified from $W_\bullet$ using the convention in Lemma 2.3.71) is a mixed twistor complex. Therefore the weight spectral sequence degenerates at E_3-page in the notation of Lemma 2.3.71.

Lemma 2.3.94 (Trace map)**.** *Let $\pi : X' \to X$ be a proper morphism of complex smooth varieties of the same dimension. Let $\mathcal{V}$ be a local system on X and denote $\mathcal{V}' = \pi^*\mathcal{V}$. Then there is a trace map*

$$\mathrm{Tr}_\pi : H^k(X', \mathcal{V}') \to H^k(X, \mathcal{V})$$

so that $\mathrm{Tr}_\pi \circ \pi^* = \mathrm{Id}$.

Moreover, if $Y \subseteq X$ is a close subvariety and denote $Y' = \pi^{-1}Y$, then the trace map restricts to

$$\mathrm{Tr}_{\pi/Y} : H^k(Y', \mathcal{V}'|_{Y'}) \to H^k(Y, \mathcal{V}|_Y)$$

satisfying $\mathrm{Tr}_{\pi/Y} \circ \pi_Y^* = \mathrm{Id}$. *The induced trace map commutes with the restriction map.*

Proof. There is trace map defined by Verdier [54] in the derived category

$$\mathrm{Tr}_\pi : R\pi_*\mathbb{Z}_{X'} \to \mathbb{Z}_X$$

so that the composition map is the identity:

$$\mathbb{Z}_X \xrightarrow{\pi^*} R\pi_*\mathbb{Z}_{X'} \xrightarrow{\mathrm{Tr}_\pi} \mathbb{Z}_X.$$

Then for the local system $\mathcal{V}$, the trace map induces

$$\mathrm{Tr}_\pi : R\pi_*(\mathbb{Z}_{X'} \otimes_{\mathbb{C}} \mathcal{V}') \xrightarrow{\sim} R\pi_*\mathbb{Z}_{X'} \otimes_{\mathbb{C}} \mathcal{V} \to \mathbb{Z}_X \otimes_{\mathbb{C}} \mathcal{V},$$

whose cohomology gives the trace map we want.

Let $Y \subseteq X$ be a subvariety and denote $Y' = \pi^{-1}(Y)$. El Zein [26, Part I, Proposition] showed that the Trace map restricts to

$$\mathrm{Tr}_{\pi/Y} : R\pi_*\mathbb{Z}_{Y'} \to \mathbb{Z}_Y.$$

The trace map over Y is defined by taking the cohomology of the morphism above twisted with $\mathcal{V}|_Y$. □

Remark 2.3.95. By the Differential geometric construction 2.3.31, the trace map is compatible with the natural pure twistor structure on $H^k(X', \mathcal{V}')$ and $H^k(X, \mathcal{V})$.

Proposition 2.3.96. *Let X be a smooth projective variety and $\mathcal{V}$ be a semisimple local system on X. Let $i : Z \hookrightarrow X$ be a closed embedding. Let*

$$\pi : (X', Z') \to (X, Z)$$

be a log resolution of (X, Z) without modifying $X \setminus Z$. Denote $\mathcal{V}' = \pi^*\mathcal{V}$, then there is a short exact sequence

$$0 \to H^k(X', \mathcal{V}') \xrightarrow{i'^* - \mathrm{Tr}_\pi} H^k(Z', \mathcal{V}'|_{Z'}) \oplus H^k(X, \mathcal{V}) \xrightarrow{\mathrm{Tr}_{\pi/Z} + i^*} H^k(Z, \mathcal{V}|_Z) \to 0.$$

Proof. Denote $j : U := X \setminus Z \hookrightarrow X$ to be the open embedding. The distinguished triangle

$$j_!\mathbb{C}_U \otimes_{\mathbb{C}} \mathcal{V} \to \mathcal{V} \to i_*\mathbb{C}_Z \otimes_{\mathbb{C}} \mathcal{V}$$

and the corresponding one on X' give to the following commutative diagram of long exact sequences

$$\begin{array}{ccccccc}
H^k_c(U, \mathcal{V}) & \xrightarrow{j_*} & H^k(X, \mathcal{V}) & \xrightarrow{i^*} & H^k(Z, \mathcal{V}|_Z) & \xrightarrow{\delta} & H^{k+1}_c(U, \mathcal{V}) \\
\pi_U^* \downarrow \cong & & \pi^* \downarrow & & \pi_Z^* \downarrow & & \pi_U^* \downarrow \cong \\
H^k_c(U', \mathcal{V}') & \xrightarrow{j'_*} & H^k(X', \mathcal{V}') & \xrightarrow{i'^*} & H^k(Z', \mathcal{V}'|'_Z) & \xrightarrow{\delta'} & H^{k+1}_c(U', \mathcal{V}')
\end{array}$$

Here $U' := X' \backslash Z'$ and $H^k_c(U, \mathcal{V}) := R^k\Gamma(X, j_!\mathbb{C}_U \otimes_{\mathbb{C}} \mathcal{V})$, the same for $H^k_c(U', \mathcal{V}')$. δ and δ' denote the connecting map.

By Lemma 2.3.94 and the fact that the trace map for forms on smooth manifolds commutes with the exterior product, we have another commuting diagram with respect to trace maps

$$\begin{array}{ccccccc}
H_c^k(U,\mathcal{V}) & \xrightarrow{j_*} & H^k(X,\mathcal{V}) & \xrightarrow{i^*} & H^k(Z,\mathcal{V}|_Z) & \xrightarrow{\delta} & H_c^{k+1}(U,\mathcal{V}) \\
{\scriptstyle \mathrm{Tr}_{\pi/U}}\big\uparrow & & {\scriptstyle \mathrm{Tr}_{\pi}}\big\uparrow & & {\scriptstyle \mathrm{Tr}_{\pi/Z}}\big\uparrow & & {\scriptstyle \mathrm{Tr}_{\pi/U}}\big\uparrow \\
H_c^k(U',\mathcal{V}') & \xrightarrow{j'_*} & H^k(X',\mathcal{V}') & \xrightarrow{i'^*} & H^k(Z',\mathcal{V}'|'_Z) & \xrightarrow{\delta'} & H_c^{k+1}(U',\mathcal{V}')
\end{array}$$

Now, we would like to prove the proposition.

1. The map $i'^* - \mathrm{Tr}_\pi$ is injective.
   Let α' be a cohomology class in $H^k(X',\mathcal{V}')$ so that

   $$(i'^*\alpha', -\mathrm{Tr}_\pi(\alpha')) = (0,0).$$

   The first vanishing implies that there is an element $\beta' \in H_c^k(U',\mathcal{V}')$ so that

   $$\alpha' = j'_*\beta'.$$

   Since π_U^* is an isomorphism, there is an element $\beta \in H_c^k(U,\mathcal{V})$ so that

   $$\beta' = \pi_U^*\beta.$$

   Then by the commutativity,

   $$\alpha' = j'_*\beta' = j'_*\pi_U^*\beta = \pi^* j_*\beta.$$

   Now because

   $$\mathrm{Tr}_\pi(\alpha') = 0, \quad \mathrm{Tr}_\pi \circ \pi^* = \mathrm{Id},$$

   We conclude that $\alpha' = 0$.

2. $\mathrm{Im}\{i'^* - \mathrm{Tr}_\pi\} = \mathrm{Ker}\{\mathrm{Tr}_{\pi/Z} + i^*\}$.
   First, since $\mathrm{Tr}_{\pi/Z} \circ i'^* = i^* \circ \mathrm{Tr}_\pi$, we have

   $$(\mathrm{Tr}_{\pi/Z} + i^*) \circ (i'^* - \mathrm{Tr}_\pi) = 0.$$

   Now let $(\alpha'_Z, \beta) \in H^k(Z',\mathcal{V}'|_Z) \oplus H^k(X,\mathcal{V})$ be a tuple so that

   $$\mathrm{Tr}_{\pi/Z}(\alpha'_Z) + i^*\beta = 0.$$

   Using the commutativitys, we see that

   $$\mathrm{Tr}_{\pi/U}(\delta'\alpha'_Z) = \delta\mathrm{Tr}_{\pi/Z}(\alpha'_Z) = -\delta i^*\beta = 0.$$

   Since $\mathrm{Tr}_{\pi/U}$ is an isomorphism, we have

   $$\delta'\alpha'_Z = 0.$$

   Therefore there is an element $\alpha' \in H^k(X',\mathcal{V}')$ so that

   $$\alpha'_Z = i'^*\alpha'. \tag{2.3}$$

Moreover,

$$i^*\mathrm{Tr}_\pi(\alpha') = \mathrm{Tr}_{\pi/Z}(i'^*\alpha') = \mathrm{Tr}_{\pi/Z}(\alpha'_Z) = -i^*\beta.$$

This implies that $i^*(\mathrm{Tr}_\pi(\alpha') + \beta) = 0$ and there is an element $\gamma \in H^k_c(U, \mathcal{V}|_U)$ so that

$$\mathrm{Tr}_\pi(\alpha') + \beta = j_*\gamma. \tag{2.4}$$

Now we claim that $\alpha' - j'_*\pi^*_U\gamma$ is the element in $H^k(X', \mathcal{V}')$ satisfying

$$(\alpha'_Z, \beta) = \left(i'^*(\alpha' - j'_*\pi^*_U\gamma), -\mathrm{Tr}_\pi(\alpha' - j'_*\pi^*_U\gamma)\right).$$

This is because

$$i'^*(\alpha' - j'_*\pi^*_U\gamma) = i'^*\alpha' \overset{2.3}{=} \alpha'_Z,$$

and

$$-\mathrm{Tr}_\pi(\alpha' - j'_*\pi^*_U\gamma) = -\mathrm{Tr}_\pi(\alpha') + j_*\mathrm{Tr}_{\pi/U}\pi^*_U\gamma = -\mathrm{Tr}_\pi(\alpha') + j_*\gamma \overset{2.4}{=} \beta.$$

3. The map $\mathrm{Tr}_{\pi/Z} + i^*$ is surjective.
Since $\mathrm{Tr}_{\pi/Z} \circ \pi^*_Z = \mathrm{Id}$, the map $\mathrm{Tr}_{\pi/Z}$ is surjective. In particular, the surjectivity of $\mathrm{Tr}_{\pi/Z} + i^*$ follows.

□

Corollary 2.3.97. *Let X be a smooth projective variety and $\mathcal{V}$ be a semisimple local system on X. Let $Z \subseteq X$ be a closed subvariety. Then $H^k(Z, \mathcal{V}|_Z)$ underlies a natural mixed twistor structure so that*

$$\mathrm{Gr}^W_\ell H^k(Z, \mathcal{V}|_Z) = 0, \quad \forall \ell > k.$$

Proof. Choose a log resolution of (X, Z) as in Proposition 2.3.96. By Proposition 2.3.92 and Remark 2.3.95, the map $i'^* - \mathrm{Tr}_\pi$ underlies a morphism of the natural mixed twistor structures on $H^k(X', \mathcal{V}')$ and $H^k(Z', \mathcal{V}'|_{Z'}) \oplus H^k(X, \mathcal{V})$. Since the category of the mixed twistor structure is abelian (Remark 2.3.62), we define the natural mixed twistor structure on $H^k(Z, \mathcal{V}|_Z)$ to be the induced quotient mixed twistor structure. It follows from Proposition 2.3.92 that

$$\mathrm{Gr}^W_\ell H^k(Z, \mathcal{V}|_Z) = 0, \quad \forall \ell > k$$

since this is true for X', X and Z' with the appropriate local system coefficients.

The proof that the resulting mixed twistor structure on $H^k(Z, \mathcal{V}|_Z)$ is the same as Simpson's construction in [51, Theorem 5.2] is similar to the proofs in [26, 27] where they showed that the mixed Hodge structure on $H^k(Z, \mathbb{C})$ constructed this way is the same as Deligne's mixed Hodge structure. □

Remark 2.3.98. An alternative construction of $H^k(Z, \mathcal{V}|_Z)$ is to use the natural quasi-isomorphism in the proof of [26, Theorem III.1.1']

$$i_*\mathbf{Q}_Z \xrightarrow{\pi^*_Z} \mathrm{Cone}\left(R\pi_*\mathbf{Q}_{X'} \xrightarrow{i'^* - \mathrm{Tr}_\pi} i_*R\pi_{Z,*}\mathbf{Q}_{Z'} \oplus \mathbf{Q}_X\right)$$

to produce the corresponding quasi-isomorphism for $\mathcal{V}$ and prove that the cone gives rise to a mixed twistor complex and finally the natural mixed twistor structure on $H^k(Z, \mathcal{V}|_Z)$. We find our presentation is notationally less heavier.

Global Invariant Cycle Theorem for Semisimple Local Systems

Now we are ready to prove the Global Invariant Cycle Theorem for semisimple local systems which are the key ingredients for the Semisimplicity Theorem. It follows immediately from the yoga of weights discussed above. Because of lack of reference, we give a proof sketch.

Theorem 2.3.99. *Consider the following chain of inclusion maps:*

$$Z \xrightarrow{\alpha} U \xrightarrow{j} X,$$

where X is a smooth projective variety, U is a Zariski open subset of X and Z is a proper subvariety of X contained in U. Let $\mathcal{V}$ be a semisimple local system on X. Then for any integer k, the following two restriction maps have the same image:

$$(j \circ \alpha)^* : H^k(X, \mathcal{V}) \to H^k(Z, (j \circ \alpha)^* \mathcal{V})$$
$$\alpha^* : H^k(U, j^* \mathcal{V}) \to H^k(Z, (j \circ \alpha)^* \mathcal{V}).$$

Proof. Since $\mathcal{V}$ is a semisimple local system, by the theory of weights (c.f. Theorem 2.3.67, Corollary 2.3.87 and Corollary 2.3.97), we know that

- if $i > k$, then $\mathrm{Gr}_i^W H^k(Z, (j \circ \alpha)^* \mathcal{V}) = 0$ [Z is proper],
- if $i < k$, then $\mathrm{Gr}_i^W H^k(U, j^* \mathcal{V}) = 0$ [U is smooth].

Since both pull-back maps are morphisms of mixed twistor structures, we have

$$\begin{aligned} \mathrm{Im}\ \alpha^* &= \mathrm{Im}\ \alpha^* \cap W_k H^k(Z, (j \circ \alpha)^* \mathcal{V}) \\ &= \alpha^*(W_k H^k(U, j^* \mathcal{V})). \end{aligned}$$

The last equality follows from the strictness of morphisms between mixed twistor structures. Now Lemma 2.3.88 says that

$$W_k H^k(U, j^* \mathcal{V}) = j^* H^k(X, \mathcal{V}).$$

Putting these two equality together, we conclude that

$$\mathrm{Im}\ \alpha^* = \alpha^* j^* (H^k(X, \mathcal{V})) = \mathrm{Im}(j \circ \alpha)^*.$$

□

Surjectivity of restriction maps

In this section, as the consequence of Global Invariant Cycle Theorem for Semisimple Local Systems (Theorem 2.3.99), we prove two surjectivity statements about restriction maps of cohomology groups of perverse complexes assuming the Decomposition Theorem. They will play an important role in the proof of non-degeneracy of intersection forms (c.f. Proposition 2.7.12).

Lemma 2.3.100. *Suppose we are in the Set-up 2.2.1 and assume there exists an isomorphism*

$$f_*K \cong \bigoplus {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)[-\ell].$$

Then for each point $i : \{y\} \hookrightarrow Y$ *in the support of the sheaf* $\mathcal{H}^0({}^{\mathfrak{p}}\mathcal{H}^0(f_*K))$ *on* Y*, the restriction map*

$$\mathbb{H}^0(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \to \mathbb{H}^0(Y, i_*i^{*\mathfrak{p}}\mathcal{H}^0(f_*K))$$

is surjective.

Proof. Now we have the Global Invariant Cycle Theorem for Semisimple Local Systems (c.f. Theorem 2.3.99), the proof follows the same line as in the proof of [17, Proposition 6.2.2]. □

Corollary 2.3.101. *With the assumption in Lemma 2.3.100, the cycle map below is injective*

$$\mathbb{H}^0(Y, i_!i^{!\mathfrak{p}}\mathcal{H}^0(f_*K)) \hookrightarrow \mathbb{H}^0(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)).$$

Proof. Lemma 2.3.100 also holds for $K^* \cong \mathcal{V}^*[\dim X]$. Hence we have a surjective restriction map

$$\mathbb{H}^0(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*K^*)) \twoheadrightarrow \mathbb{H}^0(Y, i_*i^{*\mathfrak{p}}\mathcal{H}^0(f_*K^*)).$$

Therefore the cycle map, being the dual of the restriction, is injective. □

We need a relative version of Lemma 2.3.100.

Lemma 2.3.102. *Consider the following diagram*

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{\Phi} & \mathcal{Y} \\ \downarrow{\scriptstyle F} & & \downarrow{\scriptstyle \pi} \\ T & \xrightarrow{=} & T \end{array}$$

and $\theta : T \to \mathcal{Y}$ *is a section of* π*. Suppose*

1. $\mathcal{X}$ *is nonsingular of dimension* n*,* T *is nonsingular of dimension* ℓ*;*

2. F *is surjective and smooth projective of relative dimension* $n - \ell$*;*

3. *The map* Φ *is stratified and the strata of* $\mathcal{Y}$ *map smoothly and surjectively onto* T*;*

4. $\theta(T)$ *is a stratum of* $\mathcal{Y}$*.*

5. *Decomposition holds for* $\Phi_*\mathcal{V}[\dim \mathcal{X}]$*, where* $\mathcal{V}$ *is a semisimple local system on* $\mathcal{X}$*.*

Then there is a surjective map of local systems on T*:*

$$R^{-\ell}F_*(\mathcal{V}[\dim \mathcal{X}]) \to \mathcal{H}^{-\ell}(\theta^{*\mathfrak{p}}\mathcal{H}^0(\Phi_*\mathcal{V}[\dim \mathcal{X}])).$$

In particular, the latter local system on T *is semisimple.*

Proof. The proof is basically identical to [17, Lemma 6.4.1]: by working stalkwise, one can reduce to Lemma 2.3.100, which is the local system version of [17, Proposition 6.2.2]. One also need the Semisimplicity Theorem for smooth projective maps (c.f. Theorem 2.3.3).

□

2.3.5 Weight filtrations

In this section, we would like to set up weight filtrations in the category of polarized pure twistor structures. The outline is pretty close to [17, §4.5]. By Remark 2.3.9, all results in [17, §4.5] hold automatically in the category of pure twistor structures. But for our purpose, it is important to set everything up so that they work for bilinear pairings between two different vector spaces in view of Definition 2.3.21.

One weight filtration

Let H be a finite dimensional vector space with an increasing weight filtration W. We denote the associated graded space to be

$$\mathrm{Gr}_i^W H := W_i/W_{i-1}.$$

Here is a standard lemma from linear algebra [47, Lemma 6.4].

Lemma 2.3.103. *Given a finite dimensional vector space H and a nilpotent endomorphism N, there is a unique filtration W with the properties that*

1. $NW_i \subseteq W_{i-2}$.

2. *By abusing notations we denote N to be the induced map on grade spaces, then*
$$N^i : \mathrm{Gr}_i^W H \cong \mathrm{Gr}_{-i}^W H.$$

3. *There is a Lefschetz decomposotion*
$$\mathrm{Gr}_i^W = \bigoplus_{\ell\in\mathbb{Z}} N^{-i+\ell} P^{i-2\ell}, \quad i \in \mathbb{Z}.$$
where $P^{-i} := \mathrm{Ker}\, N^{i+1} \subseteq \mathrm{Gr}_i^W$ *and* $P^{-i} = 0$ *for* $i < 0$.

Definition 2.3.104. Let N be a nilpotent endomorphism on a finite dimensional vector space H. We denote W^N to be the unique filtration in Lemma 2.3.103 and call it the *weight filtration* of N. For ease of notation, we also denote

$$\mathrm{Gr}_i^N H := \mathrm{Gr}_i^W H.$$

Two vector spaces

Definition 2.3.105. Suppose we have two vector spaces (H, N) and (H^*, N^*) equipped with nilpotent endomorphisms so that $\dim H = \dim H^*$ and the orders of N and N^* as linear operators are the same. We say that (N, N^*) are *infinitesimal automorphisms* of (H, H^*, S) if there is a non-degenerate bilinear pairing

$$S : H \otimes H^* \to \mathbb{C}$$

so that

$$S(Na, b^*) + S(a, N^*b^*) = 0, \quad \forall a \in H, b^* \in H^*.$$

Definition 2.3.106. Let $S : V \otimes W \to \mathbb{C}$ be a non-degenerate pairing between two vector spaces. Let $V_1 \subseteq V$ be a subspace, we define the *orthogonal complement* of V_1 in W to be

$$(V_1)^\perp := \{w \in W | S(v, w) = 0, \forall v \in V_1\}.$$

Lemma 2.3.107. *If (N, N^*) are infinitesimal automorphisms of (H, H^*, S). Let W^N and W^{N^*} be the associated weight filtrations. Then the weight filtrations are orthogonal to each other, i.e.*

$$(W_i^N)^\perp = W_{-i-1}^{N^*}, \quad \forall i \in \mathbb{Z}.$$

Proof. This can be proved by induction on the order of N as in the construction of weight filtration. To demonstrate the idea, let us check the simplest situation:

$$N^2 = N^{*2} = 0.$$

Then one can write down the weight filtration:

$$W_{-1}^N = \mathrm{Im}(N), W_0^N = \mathrm{Ker}\, N, W_1^N = H.$$
$$W_{-1}^{N^*} = \mathrm{Im}(N^*), W_0^{N^*} = \mathrm{Ker}\, N^*, W_1^{N^*} = H.$$

Suppose we want to verify

$$\mathrm{Im}(N)^\perp = (W_{-1}^N)^\perp = W_0^{N^*} = \mathrm{Ker}\, N^*.$$

Note that if $b^* \in H^*$ satisfies

$$0 = S(Na, b^*) = -S(a, N^*b^*)$$

for all $a \in H$ is equivalent to $b^* \in \mathrm{Ker}\, N^*$ due to the non-degeneracy of S. □

Corollary 2.3.108. *With the assumption in Lemma 2.3.107. Then S descends to a non-degenerate bilinear pairing*

$$S_i : \mathrm{Gr}_i^N H \otimes \mathrm{Gr}_i^{N^*} H^* \to \mathbb{C}$$

where

$$S_i([a], [b^*]) := S(a, N^{*i}b^*), \quad \text{for } i \geq 0.$$

And one require that N^i and N^{*i} preserve the pairing between S_i and S_{-i} in the sense that

$$S_{-i}(N^i[a], N^{*i}[b^*]) = S_i([a], [b^*])$$

for $[a] \in \mathrm{Gr}_i^N H$ and $[b^*] \in \mathrm{Gr}_i^{N^*} H^*$.

Remark 2.3.109. The Lefschetz decomposition is orthogonal with respect to S_i in the following sense. Recall that

$$\mathrm{Gr}_i^N = \bigoplus_{\ell\in\mathbb{Z}} N^{-i+\ell}P^{i-2\ell}, \quad i \in \mathbb{Z}.$$
$$\mathrm{Gr}_i^{N^*} = \bigoplus_{\ell\in\mathbb{Z}} N^{*-i+\ell}P^{*i-2\ell}, \quad i \in \mathbb{Z}.$$

Then

$$S_i(N^{-i+\ell}P^{i-2\ell}, N^{*-i+\ell'}P^{*i-2\ell'}) = 0, \quad \ell \neq \ell'.$$

Two weight filtrations

Let H and H^* be two vector spaces with a non-degenerate bilinear pairing

$$S : H \otimes H^* \to \mathbb{C}.$$

Assume

1. N, M are two commuting nilpotent operator on H and N^*, M^* are two commuting nilpotent operator on H^*.
2. (N, N^*) and (M, M^*) are infinitesimal automorphisms of (H, H^*, S) in the sense of Definition 2.3.105.
3. The shifted weight filtration $W^M[j]$ (with $W^M[j]_i = W^M_{j+i}$) is the weight-j filtration of M relative to W^N on H for every $j \in \mathbb{Z}$ (see [17, §4.5]) in the sense that
$$M(W^M_{j+i}) \subseteq W^M_{j+i-2}, \quad W^M[j](\mathrm{Gr}_j^N H) = (W^N)^{\mathrm{Gr}_j M}.$$
Here $(W^N)^{\mathrm{Gr}_j M}$ denotes the weight filtration on $\mathrm{Gr}_j^N H$ induced by
$$\mathrm{Gr}_j M : \mathrm{Gr}_j^N H \to \mathrm{Gr}_j^N H,$$
since M is a nilpotent endomorphism of H preserving W^N. Same assumption for M^* and N^*.

In this setting, we have

$$M^i : \mathrm{Gr}^M_{j+i}\mathrm{Gr}^N_j H \cong \mathrm{Gr}^M_{j-i}\mathrm{Gr}^N_j H, \quad i \geq 0.$$
$$M^{*i} : \mathrm{Gr}^{M^*}_{j+i}\mathrm{Gr}^{N^*}_j H^* \cong \mathrm{Gr}^{M^*}_{j-i}\mathrm{Gr}^{N^*}_j H^*, \quad i \geq 0.$$

For $i, j \geq 0$, define

$$P^{-j}_{-i} := \mathrm{Ker}\, M^{i+1} \cap \mathrm{Ker}\, N^{j+1} \subseteq \mathrm{Gr}^M_{j+i}\mathrm{Gr}^N_j H,$$

otherwise set $P^{-j}_{-i} = 0$. Define P^{*-j}_{-i} to be the corresponding spaces for M^* and N^*. Then we have the double Lefschetz decomposition

$$\mathrm{Gr}^M_{j+i}\mathrm{Gr}^N_j H \cong \bigoplus_{\ell,m\in\mathbb{Z}} M^{-i+\ell}N^{-j+m}P^{j-2m}_{i-2\ell}, \quad i, j \in \mathbb{Z}$$
$$\mathrm{Gr}^{M^*}_{j+i}\mathrm{Gr}^{N^*}_j H^* \cong \bigoplus_{\ell,m\in\mathbb{Z}} M^{*-i+\ell}N^{*-j+m}P^{*j-2m}_{i-2\ell}, \quad i, j \in \mathbb{Z}.$$

Moreover, the following lemma can be proved using similar arguments as in the previous section.

Lemma 2.3.110. *With the notation above, the nondegenerate pairing* S_i : $\mathrm{Gr}_i^N \otimes \mathrm{Gr}_i^{N^*} \to \mathbb{C}$ *in Corollary 2.3.108 descends to a non-degenerate pairing*

$$S_{ij} : \mathrm{Gr}^M_{j+i}\mathrm{Gr}^N_j H \otimes \mathrm{Gr}^{M^*}_{j+i}\mathrm{Gr}^{N^*}_j H^* \to \mathbb{C},$$

by defining

$$S_{ij}([a],[b^*]) = S(a, M^{*i}N^{*j}b^*), \quad i,j \geq 0.$$

Moreover, the double Lefschetz decomposition is S_{ij}*-orthogonal in the sense*

$$S_{ij}(M^{-i+\ell}N^{-j+m}P^{j-2m}_{i-2\ell}, M^{*-i+\ell'}N^{*-j+m'}P^{*j-2m'}_{i-2\ell'}) = 0, \quad \ell \neq \ell' \text{ or } m \neq m'.$$

Remark 2.3.111. Fix $n \in \mathbb{Z}$ and assume that the spaces $\mathrm{Gr}^M_{j+i}\mathrm{Gr}^N_j H$ and $\mathrm{Gr}^{M^*}_{j+i}\mathrm{Gr}^{N^*}_j H^*$ underlie pure twistor structures so that they are identified as fiber at $z = 1$ and $z = -1$ respectively, and that the maps

$$N : \mathrm{Gr}^M_{j+i}\mathrm{Gr}^N_j H \to \mathrm{Gr}^M_{j-2+i}\mathrm{Gr}^N_{j-2}H$$
$$M : \mathrm{Gr}^M_{j+i}\mathrm{Gr}^N_j H \to \mathrm{Gr}^M_{j+i-2}\mathrm{Gr}^N_j H$$

underlie morphisms of twistor structures of weight 2 so that the fiber at $z = -1$ identifies with the corresponding map for N^* and M^*. Then the double Lefschetz decomposition underlies a direct sum of pure sub-twistor structures.

Remark 2.3.112. If in addition the pairing S_{ij} polarizes the pure twistor structure on P^{-j}_{-i} for every pair $(i,j) \neq (0,0)$ such that $i,j \geq 0$, then one can show that there is a constant C depending on (n,ℓ,m,i,j) so that $C \cdot S_{ij}$ polarizes the pure twistor structure on the summands $M^{-i+\ell}N^{-j+m}P^{j-2m}_{i-\ell}$ except possibly for P^0_0. For the signs in the setting of Hodge theory, see Remark [17, Remark 4.5.2]. Note that there the polarization of Hodge structures of weight k means $(-1)^k i^{p-q} = (-1)^p i^k$ on $H^{p,q}$, here we use $(-1)^p$ on $H^{p,q}$. One can translate the signs from there to here.

Filtrations on $H^*(X, \mathcal{V})$

Assume we are in the Set-up 2.2.1. Denote

$$H^*(X,\mathcal{V}) := \bigoplus_j H^j(X,\mathcal{V}).$$

The two complex vector spaces we start with are

$$H := H^*(X,\mathcal{V}), \quad H^* := H^*(X,\mathcal{V}^*).$$

We will construct a pairing between H and H^* and put two weight filtrations on each space.

1. Denote η and L to be the cup product maps with the corresponding line bundles on H. In particular, η and L are two commuting nilpotent operators on H. They induce weight filtrations W^η, W^L on H.

2. To keep track of the filtrations on H^*, denote η^* and L^* be the corresponding operators on H^* respectively. Same for $W^{\eta *}$ and W^{L*} on H^*.

3. The twisted Poincaré pairing

$$S : H \otimes H^* \to \mathbb{C}$$

is defined by

$$S(\sum e_k \otimes \alpha_k, \sum \lambda_k \otimes \beta_k) := \sum_k C(k) \int_X \lambda_{2n-k}(e_k) \cdot \alpha_k \wedge \beta_{2n-k}$$
$$C(k) := i^{-k}(-1)^{k(k-1)/2}$$

where $e_k \in \mathcal{C}^\infty(\mathcal{V})$, $\lambda_k \in \mathcal{C}^\infty(\mathcal{V}^*)$ and α_k, β_k are k-forms on X. This is compatible with the twisted Poincaré pairing in Definition 2.3.48.

Remark 2.3.113. It follows immediately from definitions that (η, η^*) and (L, L^*) are infinitesimal automorphisms for (H, H^*, S).

By the Hard Lefschetz Theorem for semisimple local systems 2.3.1, the weight filtrations of W^η can be described explicitly

$$W_i^\eta = \bigoplus_{\ell \geq n-i} H^\ell(X, \mathcal{V}) = \bigoplus_{\ell \geq -i} H^\ell(X, K).$$

One also consider the total filtration on $H = \oplus_\ell H^\ell(X, K)$

$$W_j^{\text{tot}} := \bigoplus_{b \in \mathbb{Z}} H^b_{\leq b+j}(X, K).$$

Denote $W^{\eta *}$ and $W^{\text{tot}*}$ to be the corresponding filtrations on H^*.

Remark 2.3.114. Here we switch to K and use the notation on perverse filtrations from Notation 2.2.3.

Then we have

$$H_{-i}^{-i-j}(X, K) = \text{Gr}^\eta_{j+i} \text{Gr}^{\text{tot}}_j H, \quad i, j \in \mathbb{Z}$$
$$H_{-i}^{-i-j}(X, K^*) = \text{Gr}^{\eta *}_{j+i} \text{Gr}^{\text{tot}*}_j H^*, \quad i, j \in \mathbb{Z}.$$

Remark 2.3.115. As in [17], we will show that $W^L = W^{\text{tot}}$ and $W^{L*} = W^{\text{tot}*}$ as the consequence of Hard Lefschetz Theorem for Perverse Complexes (Theorem B). The pairing S_{ij} will then be identified with the twisted Poincaré pairing in Theorem D. See Corollary 2.7.3.

Remark 2.3.116. Assuming $W^L = W^{\text{tot}}$ and $W^{L*} = W^{\text{tot}*}$, then by Lemma 2.3.110, there is a non-degenerate bilinear pairing

$$S_{ij} : H_{-i}^{-i-j}(X, K) \otimes_\mathbb{C} H_{-i}^{-i-j}(X, K^*) \to \mathbb{C}.$$

2.3.6 Perverse filtrations and twistor structures

In this section, we use the Set-up 2.2.1. We would like to prove that perverse filtrations are twistor-theoretic using the geometric description found by de Cataldo and Migliorini [19].

Perverse filtrations via flag filtrations

Let us recall the main results of [19]. Suppose that $\dim Y = k$ and let $Y \subseteq \mathbf{P}^N$ be a fixed *affine* embedding.

Definition 2.3.117. A linear k-flag $\mathfrak{F}$ on $\mathbf{P}^N$ is defined to be

$$\mathfrak{F} := \{\mathbf{P}^N = \Lambda_0 \supseteq \Lambda_{-1} \supseteq \cdots \supseteq \Lambda_{-k}\}$$

where $\Lambda_{-\ell}$ is a codimension ℓ linear subspace. A linear k-flag $\mathfrak{F}$ is *general* if it belongs to a suitable Zariski dense open subset of the corresponding flag variety parametrizing all such k-flags.

Let $\mathfrak{F}^1 = \{\Lambda^1_*\}, \mathfrak{F}^2 = \{\Lambda^2_*\}$ be two, possibly identical, linear k-flags on $\mathbf{P}^N$. They induce two pre-image flags X_*, Z_* on X:

$$X = X_0 \supseteq X_{-1} \supseteq \cdots \supseteq X_{-k} \supseteq X_{-k-1} = \varnothing, \quad \text{with } X_\ell := f^{-1}(\Lambda^1_\ell \cap Y).$$
$$X = Z_0 \supseteq Z_{-1} \supseteq \cdots \supseteq Z_{-k} \supseteq Z_{-k-1} = \varnothing, \quad \text{with } Z_\ell := f^{-1}(\Lambda^2_\ell \cap Y)$$

Notation 2.3.118. Let $i\colon W \subseteq X$ be a locally closed embedding. We denote

$$(-)_W := i_! i^*(-)$$

to be the complex compactly supported on W. We also denote $H^*_W(-)$ to be the local cohomology group with support in W.

Notation 2.3.119. (c.f. Notation 2.2.3) The perverse Leray filtration on $\mathbb{H}^\ell(X, K)$ is defined to be

$$H^b_{\leq \ell}(X, K) = \mathrm{Im}\{\mathbb{H}^b(Y, {}^{\mathfrak{p}}\tau_{\leq \ell} f_* K) \to \mathbb{H}^b(Y, f_* K)\} \subseteq \mathbb{H}^b(X, K).$$

Theorem 2.3.120 (Theorem 4.2.1 [19])**.** *With the notation above. Assume the pair of flags $\mathfrak{F}^1, \mathfrak{F}^2$ are general. Then there is a natural isomorphism*

$$H^b_{\leq \ell}(X, K) \cong \mathrm{Im}\left\{ \bigoplus_{j+i=b+\ell} \mathbb{H}^b_{Z_{-j}}(X, K_{X\setminus X_i}) \to \mathbb{H}^b(X, K) \right\}.$$

where X_i and Z_{-j} are subvarieties of X determined by the flags.

Remark 2.3.121. In [19] the authors use the decreasing perverse filtrations. We transform their results to increasing filtrations, which is consistent with the notation in this paper.

Perverse filtrations are twistor-theoretic

Lemma 2.3.122. *For any integers ℓ and b,*

$$H^b_{\leq \ell}(X,K) \subseteq \mathbb{H}^b(X,K)$$

underlies a pure sub-twistor structure of weight $(b+\dim X)$ inherited from the natural pure twistor structure on $\mathbb{H}^b(X,K)$. In particular, the quotient space

$$H^b_{\ell}(X,K) := H^b_{\leq \ell}(X,K)/H^b_{\leq \ell-1}(X,K)$$

inherits a natural pure twistor structure of weight $(b+\dim X)$.

Moreover, if we assume the Decomposition Theorem for f and let F be the natural twistor structure on $H^b_\ell(X,K)$, then the canonical map ϕ in Definition 2.3.53 restricts to

$$\phi : \overline{F|_{z=-1}} \to H^b_\ell(X,K^*).$$

Proof. We study the image space in two steps.

Step 1: we would like to show that for each i, the image space

$$\operatorname{Im}\left\{\mathbb{H}^b(X,K_{X\setminus X_i}) \to \mathbb{H}^b(X,K)\right\}$$

underlies a pure twistor structure. Consider the distinguished triangle in $D^b_c(X)$ associated to the closed and open embeddings $X_i \xrightarrow{\alpha} X \xleftarrow{\beta} X \setminus X_i$:

$$K_{X\setminus X_i} = \beta_!\beta^*K = \beta_!\beta^!K \to K \to \alpha_*\alpha^*K \xrightarrow{[1]}$$

We have the associated long exact sequence:

$$\cdots \to \mathbb{H}^b(X,K_{X\setminus X_i}) \to \mathbb{H}^b(X,K) \to H^b(X,\alpha_*\alpha^*K) \to \cdots$$

One can choose a general $\mathfrak{F}^1$ so that X_i is smooth projective. It follows from Lemma 2.3.38 that $\operatorname{Im}\left\{\mathbb{H}^b(X,K_{X\setminus X_i}) \to \mathbb{H}^b(X,K)\right\}$ underlies a pure twistor structure F. Since $X \setminus X_i$ is smooth, it also follows from Lemma 2.3.57 that the canonical map ϕ in Definition 2.3.53 restricts to

$$\overline{F|_{z=-1}} \to \operatorname{Im}\left\{\mathbb{H}^b(X,K^*_{X\setminus X_i}) \to \mathbb{H}^b(X,K^*)\right\}$$

Step 2: For each j, the local cohomology group $\mathbb{H}^b_{Z_{-j}}(X,K)$ fits into the long exact sequence associated to the closed and open embeddings $Z_{-j} \xrightarrow{A} X \xleftarrow{B} X \setminus Z_{-j}$:

$$\cdots \to \mathbb{H}^b_{Z_{-j}}(X,K) \to \mathbb{H}^b(X,K) \to H^b(X,B_*B^*K) \to \cdots$$

Therefore the image space $\operatorname{Im}\left\{\mathbb{H}^b_{Z_{-j}}(X,K) \to \mathbb{H}^b(X,K)\right\}$ admits a pure twistor structure as in Step 1. Similarly, we have a diagram of short exact sequence

$$\begin{array}{ccccc}
\mathbb{H}^b_{Z_{-j}}(X,K_{X\setminus X_i}) & \longrightarrow & \mathbb{H}^b_{Z_{-j}}(X,K) & \longrightarrow & \mathbb{H}^b_{Z_{-j}}(X,\alpha_*\alpha^*K) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^b(X,K_{X\setminus X_i}) & \longrightarrow & \mathbb{H}^b(X,K) & \longrightarrow & \mathbb{H}^b(X,\alpha_*\alpha^*K) \\
\downarrow & & \downarrow & & \downarrow \\
\mathbb{H}^b(X,B_*B^*K_{X\setminus X_i}) & \longrightarrow & \mathbb{H}^b(X,B_*B^*K) & \longrightarrow & \mathbb{H}^b(X,\alpha_*\alpha^*(B_*B^*K))
\end{array}$$

where the vertical maps come from local cohomology and the horizontal maps come from Step 1. One concludes that the image space

$$\operatorname{Im}\left\{\mathbb{H}^b_{Z_{-j}}(X, K_{X\setminus X_i}) \to \mathbb{H}^b(X, K)\right\}$$

admits a natural pure sub-twistor structure. And the statement for the canonical map ϕ can be argued in a similar fashion. □

Now we can prove a strengthening of Lemma 2.3.38 and Lemma 2.3.57 so that the statement also works for perverse filtrations.

Corollary 2.3.123. *Assume that we are in the Set-up 2.2.1. Assume the Decomposition Theorem holds for f. Let $T \subseteq X$ be any smooth subvariety of X and let F and F_T be the natural pure twistor structures on $H^b_\ell(X, K)$ and $H^b_\ell(T, K_T)$ as in Lemma 2.3.122, where $K_T := K[\dim T - \dim X]|_T$ is a perverse sheaf on T. Then we have*

1. *The restriction map*

$$R : H^b_\ell(X, K) \to H^b_\ell(T, K_T)$$

underlies a morphism $G : F \to F_T$ between the natural pure twistor structures of weight $b + \dim X$.

2. *There is a commutative diagram:*

$$\begin{array}{ccc} \overline{F|_{z=-1}} & \xrightarrow{\overline{G|_{z=-1}}} & \overline{F_T|_{z=-1}} \\ \Big\downarrow \phi_X & & \Big\downarrow \phi_T \\ H^b_\ell(X, K^*) & \xrightarrow{R} & H^b_\ell(T, K^*_T) \end{array}$$

where ϕ_X is the canonical map in Lemma 2.3.122.

Proof. The proof of Lemma 2.3.122 works for the smooth variety T and one obtains a natural mixed twistor structure on $H^b_\ell(T, K_T)$. Moreover, by Lemma 2.3.57, there is a compatible canonical map ϕ_T making the diagram commutative. The other parts follow from Theorem 2.3.120. □

Corollary 2.3.124. *With the notation above. Recall from Corollary C that*

$$P^{-j}_{-\ell}(X, K) = \operatorname{Ker} \eta^{\ell+1} \cap L^{j+1} \subseteq H^{-\ell-j}_{-\ell}(X, K).$$

Consider the restriction maps

$$\begin{aligned} R &: P^{-j}_{-\ell}(X, K) \to P^{-j}_{-\ell}(T, K_T), \\ R^* &: P^{*-j}_{-\ell}(X, K) \to P^{*-j}_{-\ell}(T, K_T). \end{aligned}$$

Then these restriction maps underly morphisms of pure twistor structures and the twisted Poincaré pairing S_X in Theorem D restricts to a non-degenerate pairing

$$S_X : \operatorname{Ker} R \otimes \operatorname{Ker} R^* \to \mathbb{C}.$$

2.4 Hodge star operators

In this section, for the lack of appropriate references in the literature, we briefly set up the theory of Hodge star operators for differential forms with Harmonic bundle coefficients. It is used to help understand the natural pure twistor structures on $H^k(X, \mathcal{V})$ as in Theorem 2.3.26.

2.4.1 Harmonic bundles

Let X be a complex manifold. Let (H, h) be a harmonic bundle on X where the flat connection has the decomposition

$$\nabla = \partial' + \theta' + \partial'' + \theta''.$$

where θ' is the Higgs field satisfying $\theta' \wedge \theta' = 0$ and θ'' is the adjoint of θ' with respect to the harmonic metric h. We denote

$$D'' = \partial'' + \theta', \quad D' = \partial' + \theta''$$

to be the Higgs operator so that

$$\nabla = D'' + D'.$$

Remark 2.4.1. Since there are calculations involve conjugation, to avoid confusions, we use ∂' and ∂'' to represent the usual notation ∂_E and $\bar{\partial}_E$ and we use θ' and θ'' instead of θ and $\bar{\theta}$ to represent the Higgs field and its dual.

Definition 2.4.2. The Higgs operator D'' can be defined inductively for all k-forms:

$$D'' : H \otimes_{\mathcal{C}^\infty_X} \mathcal{A}^k_X \to H \otimes_{\mathcal{C}^\infty_X} \mathcal{A}^{k+1}_X$$

where $\mathcal{A}^k_X$ is the sheaf of $\mathcal{C}^\infty$ k-forms on X so that

$$D''(e \otimes \alpha) = D''(e) \wedge \alpha + e \otimes \bar{\partial}(\alpha).$$

Similarly we can define

$$D' : H \otimes_{\mathcal{C}^\infty_X} \mathcal{A}^k_X \to H \otimes_{\mathcal{C}^\infty_X} \mathcal{A}^{k+1}_X$$

so that it satisfies

$$D'(e \otimes \alpha) = D'(e) \wedge \alpha + e \otimes \partial(\alpha).$$

Lemma 2.4.3. *As in the usual exterior calculus, one have the Leibniz rule for higher order differential forms:*

$$D''(e \otimes \alpha \wedge \beta) = D''(e \otimes \alpha) \wedge \beta + (-1)^k (e \otimes \alpha) \wedge \bar{\partial}(\beta),$$

where $e \otimes \alpha \in H \otimes \mathcal{A}^k_X$ *and* $\beta \in \mathcal{A}^m_X$. *Similarly*

$$D'(e \otimes \alpha \wedge \beta) = D'(e \otimes \alpha) \wedge \beta + (-1)^k (e \otimes \alpha) \wedge \partial(\beta).$$

Proof. We will only prove it for D''. This follows from the usual Leibniz rule for differential forms.

$$\begin{aligned} LHS &= D''(e \otimes (\alpha \wedge \beta)) \\ &= D''(e) \otimes (\alpha \wedge \beta) + e \otimes \bar{\partial}(\alpha \wedge \beta) \quad \text{[Higgs definition for } \alpha \wedge \beta] \\ &= (D''(e) \otimes \alpha) \wedge \beta + e \otimes \bar{\partial}(\alpha) \wedge \beta + (-1)^k (e \otimes \alpha) \wedge \bar{\partial}(\beta) \quad \text{[Leibniz]} \\ &= D''(e \otimes \alpha) \wedge \beta + (-1)^k (e \otimes \alpha) \wedge \bar{\partial}(\beta) \quad \text{[Higgs definition for } \alpha] \\ &= RHS. \end{aligned}$$

□

2.4.2 Dual harmonic bundles

Let (H^*, h^*) be the dual harmonic bundle from Construction 2.3.50. Recall that D'' and ∇ are the corresponding operators on H^*.

Remark 2.4.4. As in the previous section, we can extend the Higgs field D'' inductively to all k-forms:

$$D'' : H^* \otimes \mathcal{A}_X^k \to H^* \otimes \mathcal{A}_X^{k+1}$$

where

$$D''(e \otimes \alpha) = D''(e) \wedge \alpha + e \otimes \bar{\partial}(\alpha).$$

Definition 2.4.5 (Wedge product)**.** We define the following wedge product map

$$(H \otimes \mathcal{A}_X^k) \otimes (H^* \otimes \mathcal{A}_X^m) \to \mathcal{A}_X^{k+m}$$

by

$$(e \otimes \alpha, \lambda \otimes \beta) \mapsto \lambda(e) \cdot \alpha \wedge \beta.$$

Lemma 2.4.6. *We have the following Leibniz rule between harmonic bundles and their duals:*

$$\bar{\partial}(A \wedge B) = D''(A) \wedge B + (-1)^k A \wedge D''(B)$$

where A is a k-form with coefficient in H and B is a form with coefficient in H^*. Similarly, we have

$$\partial(A \wedge B) = D'(A) \wedge B + (-1)^k A \wedge D'(B).$$

Proof. We only prove the formula for D''. Write $A = e \otimes \alpha$ with $e \in \mathcal{C}^\infty(H)$ and α being a k-form. Write $B = \lambda \otimes \beta$ with $\lambda \in \mathcal{C}^\infty(H)$ and β being a form.

Then

$$
\begin{aligned}
LHS &= \bar{\partial}(\lambda(e)\cdot\alpha\wedge\beta)\\
&= \bar{\partial}(\lambda(e)\cdot\alpha)\wedge\beta + (-1)^k\lambda(e)\alpha\wedge\bar{\partial}(\beta)\\
&= \bar{\partial}\lambda(e)\wedge\alpha\wedge\beta + \lambda(e)\bar{\partial}(\alpha)\wedge\beta + (-1)^k\lambda(e)\alpha\wedge\bar{\partial}(\beta)\\
&= [\lambda(D''(e)) + (D''\lambda)(e)]\wedge\alpha\wedge\beta + \lambda(e)\bar{\partial}(\alpha)\wedge\beta + (-1)^k\lambda(e)\alpha\wedge\bar{\partial}(\beta)\\
RHS &= D''(e\otimes\alpha)\wedge(\lambda\otimes\beta) + (-1)^k(e\otimes\alpha)\wedge D''(\lambda\otimes\beta)\\
&= [D''(e)\wedge\alpha + e\otimes\bar{\partial}(\alpha)]\wedge(\lambda\otimes\beta) + (-1)^k(e\otimes\alpha)\wedge D''(\lambda\otimes\beta)\\
&= \lambda(D''(e))\wedge\alpha\wedge\beta + \lambda(e)\bar{\partial}(\alpha)\wedge\beta + (-1)^k(e\otimes\alpha)\wedge[D''(\lambda)\wedge\beta + \lambda\otimes\bar{\partial}(\beta)]\\
&= \lambda(D''(e))\wedge\alpha\wedge\beta + (-1)^k(e\otimes\alpha)\wedge D''(\lambda)\wedge\beta + \lambda(e)\bar{\partial}(\alpha)\wedge\beta + (-1)^k\lambda(e)\alpha\wedge\bar{\partial}(\beta)
\end{aligned}
$$

Now notice that the second term is

$$(-1)^k(e\otimes\alpha)\wedge D''(\lambda)\wedge\beta = e\otimes D''(\lambda)\wedge\alpha\wedge\beta = (D''\lambda)(e)\wedge\alpha\wedge\beta.$$

Hence LHS=RHS. □

2.4.3 Hodge star operators

Assume X is a compact complex manifold of complex dimension n. Assume X admits a metric so that there are induced Hermitian pairings on the sheaf of sections $\mathcal{A}_X^k$. Let H be a harmonic bundle on X with harmonic metric h. For any point $x \in X$, we have a product Hermitian pairing on every fiber

$$H_x \otimes \mathcal{A}_{X,x}^k.$$

There are two ways of defining a Hodge star operator for bundle valued forms.

Anti-ℂ-linear Hodge star operator

Definition 2.4.7 (Anti-ℂ-linear)**.** The $*$-operator is defined to be the following anti-ℂ-linear map

$$*: H_x\otimes\mathcal{A}_{X,x}^k \to \mathrm{Hom}(H_x\otimes\mathcal{A}_{X,x}^k, \mathbb{C}) \to H_x\otimes\mathcal{A}_{X,x}^{2n-k},$$

where

$$(\alpha_x, \beta_x)\mathrm{Vol} = \alpha_x\wedge *\beta_x,$$

here $\alpha, \beta \in H\otimes\mathcal{A}_X^k$. In particular,

$$(\alpha,\beta) = \int_X \alpha\wedge *\beta.$$

where (α,β) is defined to be $\int_X(\alpha_x,\beta_x)\mathrm{Vol}$.

Remark 2.4.8. The L^2 norm on the space of forms with bundle coefficients depend on the global geometry of X. For example, the definition makes sense since we assume that X is compact. Otherwise we need to work with compactly supported forms.

Lemma 2.4.9. *We have*

$$*: H \otimes \mathcal{A}_X^{p,q} \to H^* \otimes \mathcal{A}_X^{n-p,n-q}.$$

Remark 2.4.10. The advantage of this definition is that we don't need to consider operations on the conjugate bundle $\bar{H}$ and this is what Voisin did in [56]. The disadvantage is that it is different from the classical definition

$$*: \mathcal{A}_X^{p,q} \to \mathcal{A}_X^{n-q,n-p}$$

for Hodge star operators on forms, so that to verify the formula of primitive forms, one need to check each step again.

Let e be a $\mathcal{C}^\infty$ section of H. One construct a section $e^\vee \in \mathcal{C}^\infty(H^*)$ via

$$e^\vee(\bullet) := h(\bullet, \overline{e}),$$

where h is the harmonic metric.

Lemma 2.4.11. *The $*$-operator for H can be expressed by*

$$*(e \otimes \alpha) = e^\vee \otimes (*\alpha),$$

where $e \otimes \alpha \in H \otimes \mathcal{A}_X^k$ and $*\alpha$ is the $*$-operator on $\mathcal{A}_X^k$ so that

$$(\alpha, \beta) = \int_X \alpha \wedge *\beta.$$

Lemma 2.4.12. *The formal adjoint of ∇, D'' and D' are*

$$\begin{aligned}
\nabla^* &= - * \nabla *, \\
(D'')^* &= - * D'' *, \\
(D')^* &= - * D' * .
\end{aligned}$$

Proof. Let us verify the second property. Suppose A is a k-form with coefficient in H and B is a $(k+1)$-form with coefficient in H.

$$\begin{aligned}
(D''(A), B) &= \int_X D''(A) \wedge (*B) \\
&= \int_X \bar{\partial}(A \wedge *B) - (-1)^k \int_X A \wedge D''(*B) \quad \text{[Lemma 2.4.6]} \\
&= -(-1)^k \int_X A \wedge D''(*B) \quad [A \wedge *B \text{ is a } (2n-1)\text{-form}] \\
&= \int_X A \wedge *(- * D'' * B) \quad [*^2 = (-1)^k] \\
&= (A, - * D'' * B).
\end{aligned}$$

Therefore $(D'')^* = - * D'' *$. □

$\mathbb{C}$-linear Hodge star operators

Definition 2.4.13 ($\mathbb{C}$-linear)**.** The $*$-operator is defined to be the following $\mathbb{C}$-linear map

$$*: H_x \otimes \mathcal{A}^k_{X,x} \to \mathrm{Hom}(H_x \otimes \mathcal{A}^k_{X,x}, \mathbb{C}) \to \overline{H^*_x \otimes \mathcal{A}^{2n-k}_{X,x}},$$

where

$$(\alpha_x, \beta_x)\mathrm{Vol} = \alpha_x \wedge \overline{*\beta_x},$$

here $\alpha, \beta \in H \otimes \mathcal{A}^k_X$. In particular,

$$(\alpha, \beta) = \int_X \alpha \wedge \overline{*\beta}.$$

where (α, β) is defined to be $\int_X (\alpha_x, \beta_x)\mathrm{Vol}$.

Lemma 2.4.14. *We have*

$$*: H \otimes \mathcal{A}^{p,q}_X \to \overline{H^* \otimes \mathcal{A}^{n-p,n-q}_X} = \overline{H^*} \otimes \mathcal{A}^{n-q,n-p}_X.$$

Let e be a $\mathcal{C}^\infty$ section of H. One construct a section $\overline{e^\vee} \in \mathcal{C}^\infty(\overline{H^*})$ via

$$e^\vee(\bullet) := h(\bullet, \overline{e}),$$

where h is the harmonic metric. Or equivalently,

$$\overline{e^\vee}(\bullet) = h(e, \bullet).$$

Lemma 2.4.15. *The $*$-operator for H can be expressed by*

$$*(e \otimes \alpha) = \overline{e^\vee} \otimes (*\alpha),$$

where $e \otimes \alpha \in H \otimes \mathcal{A}^k_X$ and $*\alpha$ is the $*$-operator on $\mathcal{A}^k_X$ so that

$$(\alpha, \beta) = \int_X \alpha \wedge *\overline{\beta}.$$

This lemma gives the calculation on primitive forms with bundle coefficients.

Lemma 2.4.16. *Let L be an ample line bundle on X. Let $e \otimes \alpha$ be a primitive (p,q)-form with coefficient in H so that $p + q = k$. Then for $k \leq n$ we have*

$$*(e \otimes \alpha) = C\overline{e^\vee} \otimes L^{n-k} \wedge \alpha,$$

where $C = \frac{(-1)^{k(k+1)/2} i^{p-q}}{(n-k)!}$ is a constant depending on n and k. For $k \geq n$ we have

$$*(e \otimes \alpha) = C\overline{e^\vee} \otimes (L^{k-n})^{-1}(\alpha),$$

here $(L^{k-n})^{-1}$ represents the inverse of

$$L^{k-n}: \mathcal{A}^{2n-k}_X \to \mathcal{A}^k_X.$$

Lemma 2.4.17. *The formal adjoint of* ∇, D'' *and* D' *are*

$$\nabla^* = -*\overline{\nabla}*,$$
$$(D'')^* = -*\overline{D''}*,$$
$$(D')^* = -*\overline{D'}*\,.$$

Proof. Let us verify the second property. Suppose A is a k-form with coefficient in H and B is a $(k+1)$-form with coefficient in H.

$$\begin{aligned}(D''(A), B) &= \int_X D''(A) \wedge \overline{*B} \\ &= \int_X \bar{\partial}(A \wedge \overline{*B}) - (-1)^k \int_X A \wedge D''(\overline{*B}) \quad \text{[Lemma 2.4.6]} \\ &= \int_X \bar{\partial}(A \wedge \overline{*B}) - (-1)^k \int_X A \wedge \overline{\overline{D''} * B} \quad \text{[Definition of } \overline{D''}\text{]} \\ &= -(-1)^k \int_X A \wedge \overline{\overline{D''} * B} \quad [A \wedge \overline{*B} \text{ is a } (2n-1)\text{-form}] \\ &= \int_X A \wedge \overline{*(-*\overline{D''} * B)} \quad [*^2 = (-1)^k] \\ &= (A, -*\overline{D''} * B).\end{aligned}$$

Therefore $(D'')^* = -*\overline{D''}*$. □

2.4.4 Properties of Hodge star operators

Let (H, ∇) be a harmonic bundle and the Laplacian is defined by

$$\Delta_\nabla := \nabla^* \circ \nabla + \nabla \circ \nabla^*,$$

where $\nabla^* : H \otimes \mathcal{A}_X^k \to H \otimes \mathcal{A}_X^{k-1}$ is the formal adjoint of ∇ via the harmonic metric on H and the Kahler metric on X. It follows immediately from the theory of second order elliptic operators that

Theorem 2.4.18 (Simpson [52])**.** *Let* $\mathcal{V}$ *be the semisimple local system associated to the harmonic bundle* H*. There is an isomorphism*

$$H^k(X, \mathcal{V}) \cong \mathrm{Harm}(X, H),$$

where $\mathrm{Harm}(X,H) := \mathrm{Ker}\,\Delta_\nabla \subseteq \mathcal{C}^\infty(H \otimes \mathcal{A}_X^k)$ *is the space of harmonic* k*-forms with coefficient in* H*.*

Corollary 2.4.19. *With the same assumption above, denote* $*$ *to be the* $\mathbb{C}$*-linear Hodge star operator for harmonic bundles*

$$* : H \otimes \mathcal{A}_X^k \to \overline{H^*} \otimes \mathcal{A}_X^{2n-k}.$$

Then we have an isomorphism of complex vector spaces

$$* : H^k(X, \mathcal{V}) \to H^{2n-k}(X, \overline{\mathcal{V}}^*).$$

Proof. One represents a cohomology element by a harmonic form with coefficient in H and apply the Hodge star operator. □

Lemma 2.4.20. *Denote* $*$ *to be the* $\mathbb{C}$*-linear Hodge star operator for harmonic bundles*

$$* : H \otimes \mathcal{A}_X^k \to \overline{H^*} \otimes \mathcal{A}_X^{2n-k}.$$

Then we have the following identity

$$* \circ \Delta_\nabla = \Delta_{\overline{\nabla}} \circ *.$$

Corollary 2.4.21. *The Poincaré pairing with a semisimple local system coefficient*

$$H^k(X, \mathcal{V}) \otimes H^{2n-k}(X, \mathcal{V}^*) \to \mathbb{C}$$

is non-degenerate.

Proof. One can represent a nonzero cohomology class in $H^k(X, \mathcal{V})$ by a harmonic form $A = \sum e_i \otimes \alpha_i$. Then we know that $\Delta_{\overline{\nabla}}(*A) = *\Delta_\nabla(A) = 0$. In particular, $*A$ is a closed form, thus represents a cohomology class on $H^{2n-k}(X, \overline{\mathcal{V}^*})$. Hence $\overline{*A}$ represents a cohomology class on $H^{2n-k}(X, \mathcal{V}^*)$ and

$$\int_X A \wedge \overline{*A} > 0.$$

□

2.4.5 Kahler identities

No matter which definition one choose to define Hodge star operators, they always satisfy the Kahler identity.

Lemma 2.4.22 (Simpson)**.** *We have identities*

$$[\Lambda, D''] = -i(D')^*, [\Lambda, D'] = i(D'')^*.$$

Proof. By Lemma 2.4.11, we know the expression of $*$. Therefore one can reduce this to the standard Kahler identities. □

Lemma 2.4.23 (Simpson)**.** *As a consequence, there are second order Kahler identities*

$$\Delta_\nabla = 2\Delta_{D''} = 2\Delta_{D'}.$$

2.5 Preliminary results

In this section, we collect some topological facts about constructible complexes and perverse sheaves, which will be used in the proof of the main results.

2.5.1 The cup product with a line bundle

Let X be a projective variety and let η be a line bundle on X. Let $D(X)$ be the derived category of constructible complexes of $\mathbb{C}$-vector spaces on X and let $K \in D(X)$ be an object. Denote D_X to be the Verdier dual functor on X. The first Chern class of η corresponds to an element in $\mathrm{Hom}_{D(X)}(\mathbf{Q}_X, \mathbf{Q}_X[2])$ via isomorphism with $H^2(X, \mathbf{Q})$.

Definition 2.5.1 (Cup product)**.** The cup product map

$$\eta : K \to K[2]$$

is defined by $K \cong K \otimes \mathbf{Q}_X \to K \otimes \mathbf{Q}_X[2] \cong K[2]$.

Lemma 2.5.2. *As a map in $D(X)$ we have*

$$D_X(K \to K[2]) = (D_X(K) \to D_X(K)[2])[-2].$$

Proof. Let D be a divisor corresponds to η so that $i : D \to X$ defines a normally nonsingular codimension one inclusion. By [17, Remark 4.4.1], the map $\eta : K \to K[2]$ can be identified with the composition of the following maps:

$$K \to i_* i^* K \cong i_! i^! K[2] \to K[2].$$

Taking Verdier dual, we get

$$D_X(K)[-2] \to D_X(i_! i^! K[-2]) \cong D_X(i_* i^* K) \to D_X(K).$$

The compatibility between those functors implies that

$$D_X \circ i_! \circ i^! \cong i_* \circ i^* \circ D_X,$$

and

$$D_X \circ i_* \circ i^* \cong i_! \circ i^! \circ D_X.$$

Hence the dual of $\eta : K \to K[2]$ isomorphic to

$$D_X(K)[-2] i_* i^* D_X(K)[-2] \cong i_! i^! D_X(K)[-2][2] \to D_X(K)[-2][2],$$

which is exactly the shifted cupping product map $(\eta : D_X(K) \to D_X(K)[2])[-2]$. □

Corollary 2.5.3. *Suppose we are in the Set-up 2.2.1, then*

$$D_Y \left\{ \eta^\ell \colon {}^{\mathfrak{p}}\mathcal{H}^{-\ell} f_* K \to {}^{\mathfrak{p}}\mathcal{H}^{-\ell} f_* K \right\} = \left\{ \eta^\ell \colon {}^{\mathfrak{p}}\mathcal{H}^{-\ell} f_* D_X(K) \to {}^{\mathfrak{p}}\mathcal{H}^{-\ell} f_* D_X(K) \right\}.$$

2.5.2 Weak-Lefschetz-type results

Suppose we are in the Set-up 2.2.1. The central idea of the induction proof of Theorem A is to reduce the complexity of f via "cutting with hyperplanes". There are several ways and each case has the corresponding weak Lefschetz theorem. These results are obtained in [17, §4.7, §5.2, §5.3]. We would like to review them in the category of *pure twistor structures* (see Definition 2.3.8).

A general weak Lefschetz Theorem

Lemma 2.5.4 (Lemma 4.7.6 [17])**.** *Let*

$$\begin{array}{ccccc} X' & \xrightarrow{i} & X & \xleftarrow{j} & X \setminus X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle u} \\ Y & \xrightarrow{\mathrm{Id}} & Y & \xleftarrow{\mathrm{Id}} & Y \end{array}$$

be a comutative diagram of algebraic varieties with i a closed embedding, f proper, u affine and let $K \in \mathrm{Perv}(X)$. Then

1. *the restriction map ${}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*K) \to {}^{\mathfrak{p}}\mathcal{H}^{-\ell}(g_*i^*K)$ is iso for $\ell \geq 2$ and mono for $\ell = 1$,*
2. *the Gysin map ${}^{\mathfrak{p}}\mathcal{H}^{\ell}(g_*i^*K) \to {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)$ is iso for $\ell \geq 2$ and epi for $\ell = 1$.*

Universal hyperplane

Let us recall the construction of universal hyperplane map associated to a map.

Construction 2.5.5. Fix a projective embedding $X \subseteq \mathbf{P}$, consider the following commutative diagram (not a Cartesian square!):

$$\begin{array}{ccc} X & \xleftarrow{p_X} & \mathcal{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Y & \xleftarrow{p_Y} & \mathcal{Y} \end{array}$$

Here $\mathcal{Y} := Y \times \mathbf{P}^\vee$ and $\mathcal{X}$ is defined to be

$$\mathcal{X} := \{(x, s)\colon s(x) = 0\} \subseteq X \times \mathbf{P}^\vee.$$

The map g is defined to be the restriction of $f \times \mathrm{Id} : X \times \mathbf{P}^\vee \to Y \times \mathbf{P}^\vee$ to $\mathcal{X}$ so that

$$g(x, h) = (f(x), h).$$

Two horizontal maps p_X and p_Y are natural projections.

Definition 2.5.6. The *defect of semismallness* of the map f is defined to be

$$r(f) := \max_k\{\dim Y_k + 2k - \dim X \colon Y_k \neq \varnothing\}.$$

where $Y_k = \{y \in Y\colon \dim f^{-1}(y) = k\}$. If $r(f) = 0$, we say f is *semismall.*

Here are several facts from [17, §4.7].

Proposition 2.5.7 (The Relative Weak Lefschetz Theorem)**.** *Let $d = \dim \mathbf{P}^\vee$ and $M := p_X^*K[d-1]$.*

1. *If $r(f) > 0$, then $r(g) < r(f)$.*

2. The restriction map

$$p_Y^{*}\,{}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*K)[d] \to {}^{\mathfrak{p}}\mathcal{H}^{-\ell+1}(g_*M)$$

is iso for $\ell \geq 2$ and mono for $\ell = 1$,

3. The Gysin map

$${}^{\mathfrak{p}}\mathcal{H}^{\ell-1}(g_*M) \to p_Y^{*}\,{}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)[d]$$

is iso for $\ell \geq 2$ and epi for $\ell = 1$.

4. $p_Y^{*}\,{}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*K)[d]$ is the biggest perverse subsheaf of ${}^{\mathfrak{p}}\mathcal{H}^{0}(g_*M)$ coming from Y, and $p_Y^{*}\,{}^{\mathfrak{p}}\mathcal{H}^{1}(f_*K)[d]$ is the biggest quotient perverse sheaf of ${}^{\mathfrak{p}}\mathcal{H}^{0}(g_*M)$ coming from Y.

Restriction to hyperplanes of X

The following results come from [17, Proposition 4.7.5, Lemma 4.7.6].

Proposition 2.5.8. *Suppose we are in the Set-up 2.2.1. There exists m_0 so that for any $m \geq m_0$, the following statements hold: let $i : X^1 \to X$ be a general hyperplane section of $|\eta^{\otimes m}|$ and $f^1 : X^1 \to Y$ be the restriction map. Set $K^1 := i^*K[-1]$. Then*

1. $r(f^1) \leq \max\{r(f) - 1, 0\}$.

2. The natural restriction map

$${}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*K) \to {}^{\mathfrak{p}}\mathcal{H}^{-\ell+1}(f^1_*K^1)$$

is an isomorphism for $\ell \geq 2$ and a monomorphism for $\ell = 1$.

3. The natural Gysin map

$${}^{\mathfrak{p}}\mathcal{H}^{\ell-1}(f^1_*K^1) \to {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)$$

is an isomorphism for $\ell \geq 2$ and an epimorphism for $\ell = 1$.

Corollary 2.5.9. *With the notation above and assume the Decomposition Theorem holds for f.*

1. The natural restriction map

$$i^*\colon H^{-\ell-j}_{-\ell}(X, K) \to H^{-\ell+1-j}_{-\ell+1}(X^1, K^1), \quad j \geq 0$$

is an isomorphism for $\ell \geq 2$ and an injective map for $\ell = 1$.

2. There is an injective morphism of pure twistor structures

$$i^* : P^{-j}_{-\ell}(X, K) \to P^{-j}_{-\ell+1}(X^1, K^1), \quad \ell \geq 1, j \geq 0.$$

It is an equality when $\ell \geq 2$.

Moreover, all morphisms underlie morphisms of pure twistor structures.

Proof. (1) follows from previous Proposition and the identification

$$H^{-\ell-j}_{-\ell}(X,K) = \mathbb{H}^{-j}(Y, {}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*K)).$$

For (2), recall that

$$P^{-j}_{-\ell}(X,K) = \operatorname{Ker}\eta^{\ell+1} \cap L^{j+1} \subseteq H^{-\ell-j}_{-\ell}(X,K).$$

The generality of $X^1 \in |\eta|$ induces the following diagram

$$\begin{array}{ccc} {}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*K) & \longrightarrow & {}^{\mathfrak{p}}\mathcal{H}^{-\ell+1}(f^1_*K^1) \\ \downarrow{\scriptstyle \eta^{\ell+1}} & & \downarrow{\scriptstyle (\eta|_{X^1})^{\ell}} \\ {}^{\mathfrak{p}}\mathcal{H}^{\ell+2}(f_*K) & \longleftarrow & {}^{\mathfrak{p}}\mathcal{H}^{\ell+1}(f^1_*K^1) \end{array}$$

By Proposition 2.5.8, the bottom row is always an isomorphism for $\ell \geq 1$. Therefore we have the inclusion

$$i^*(P^{-j}_{-\ell}(X,K)) \subseteq P^{-j}_{-\ell+1}(X^1,K^1), \quad \ell \geq 1, j \geq 0.$$

By Proposition 2.5.8, the top row is an isomorphism for $\ell \geq 2$ and the corresponding inclusion is an equality.

Lastly, by Corollary 2.3.123 and Corollary 2.3.124, all morphisms underlie morphisms of pure twistor structures. □

Restriction to hyperplanes of Y

The following result follows from applying Lemma 2.5.4 and taking hypercohomology.

Proposition 2.5.10. *Suppose Y_1 is a general section in $|A|$, transversal to the strata to Y. Set $X_1 = f^{-1}(Y_1)$, $f_1 : X_1 \to Y_1$ the restriction, and $K_1 := K[-1]|_{X_1}$. Then*

1. $r(f_1) = r(f)$.
2. *The natural restriction map*
$$\mathbb{H}^{-j}(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \to \mathbb{H}^{-j+1}(Y_1, {}^{\mathfrak{p}}\mathcal{H}^0(f_{1*}K_1))$$
is an isomorphism for $j \geq 2$ and an injection for $j = 1$.
3. *The Gysin pushforward*
$$\mathbb{H}^{j-1}(Y_1, {}^{\mathfrak{p}}\mathcal{H}^0(f_{1*}K_1)) \to \mathbb{H}^{-j}(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*K))$$
is an isomorphism for $j \geq 2$ and a surjection for $j = 1$.
4. *Assuming the Decomposition Theorem A(b), there is an injective morphism*
$$i^*(P^{-j}_0(X,K)) \subseteq P^{-j+1}_0(X_1,K_1), \quad j \geq 1.$$
It is an equality when $j \geq 2$.

Moreover, all morphisms underlie morphisms of pure twistor structures.

2.5.3 p-splitness and weak Lefschetz

We need the following result from [17, Lemma 4.3.8].

Lemma 2.5.11. *Suppose we are in the Set-up 2.2.1. Let*

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{v} & X \\ \downarrow{\scriptstyle \tilde{f}} & & \downarrow{\scriptstyle f} \\ \tilde{Y} & \xrightarrow{u} & Y \end{array}$$

be a Cartesian diagram of maps of algebraic varieties and f is projective. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be a stratification for f. Assume that f_*K satisfies the Decomposition Theorem and that u is either smooth or it is a normally nonsingular inclusion. Then we have the following base-change formula

$$^{\mathfrak{p}}\mathcal{H}^\ell(\tilde{f}_* v^* K) \cong u^{*\mathfrak{p}}\mathcal{H}^\ell(f_* K).$$

Equivalently, set $\tilde{K} := v^* K[\dim \tilde{X} - \dim X]$,

$$^{\mathfrak{p}}\mathcal{H}^\ell(\tilde{f}_* v^* \tilde{K}) \cong u^{*\mathfrak{p}}\mathcal{H}^\ell(f_* K)[\dim \tilde{X} - \dim X].$$

2.5.4 Splitting criterion

In this section, we would like to recall two splitting criterions for perverse sheaves from [17, 37]. For MacPherson and Vilonen's description of the category of perverse sheaves [37], we follow the presentation in [18, §5.7].

de Cataldo-Migliorini's criterion

Let Y be an algebraic variety and d be an integer. Suppose that there is a stratification $\mathfrak{Y}$ byof Y with

$$Y = U \cup S, \quad U = \bigcup_{d' > d} S_{d'}, \quad S = S_d.$$

Here $S_{d'}$ denotes a d'-dimensional stratum. Let P be a perverse sheaf on Y which is constructible with respect to this stratification. Let $S \xrightarrow{\alpha} Y \xleftarrow{\beta} U$ be the closed and open embedding.

Lemma 2.5.12 (Lemma 4.1.3 in [17])**.** *Assume* $\dim \mathcal{H}^{-d}(\alpha_! \alpha^! P)_y = \dim \mathcal{H}^{-d}(\alpha_* \alpha^* P)_y$ *for any $y \in S$. Then the following are equivalent:*

1. $P \cong \beta_{!*}\beta^* P \oplus \mathcal{H}^{-d}(P)[d]$.
2. $\mathcal{H}^{-d}(\alpha_! \alpha^! P) \to \mathcal{H}^{-d}(P)$ *is an isomorphism.*

MacPherson-Vilonen's criterion

Let S be a closed and *contractible* d-dimensional stratum of Y. Denote by $S \xrightarrow{\alpha} Y \xleftarrow{\beta} Y \setminus S$.

Theorem 2.5.13 (Theorem 5.7.2 in [18]). *There is a one-to-one correspondence between a perverse sheaf P on Y which is constructible with respect to $\mathfrak{Y}$ and β^*P with the exact sequence*

$$\mathcal{H}^{-d-1}(\alpha^*\beta_*\beta^*P) \to \mathcal{H}^{-d}(\alpha^!P) \to \mathcal{H}^{-d}(\alpha^*P) \to \mathcal{H}^{-d}(\alpha^*\beta_*\beta^*P).$$

Corollary 2.5.14. *With the notation above, if there is an isomorphism*

$$\mathcal{H}^{-d}(Y, \alpha_!\alpha^!P) \cong \mathcal{H}^{-d}(Y, \alpha_*\alpha^*P),$$

then we have

$$P \cong \beta_{!*}\beta^*P \oplus \mathcal{H}^{-d}(P)[d].$$

2.6 Proof of Theorem A

In this section, we will prove Theorem A, Theorem B and Theorem D by double induction on the defect of semismallness $r = r(f)$ (see §2.5.2) and $m = \dim f(X)$. In particular, we will prove that if these statements are true for $(r' < r, m')$ and $(r' = r, m' < m)$, then they are true for (r, m). This strategy is borrowed from de Cataldo-Migliorini [17, Remark 2.6.3].

To illustrate the idea, we will omit m. Suppose we know the statement is true for r and all $r' < r$, we would like to prove the statement for $r'' > r$. The induction will be carried out in the following way:

$$\begin{aligned}
(\S 2.7.1) \quad & \text{Thm A(b)}_r + \text{Thm B}_{r'<r} + \text{Thm C}_{r'<r} \Longrightarrow \text{Thm B}_r,\\
(\S 2.7.2) \quad & \text{Thm A(b)}_r + \text{Thm B}_r + \text{Thm C}_{r'<r} \Longrightarrow \text{Thm C}_r,\\
(\S 2.8) \quad & \text{Thm A(b)}_r + \text{Thm C}_r + \text{Thm A(c)}_{r'<r} \Longrightarrow \text{Thm A(c)}_r,\\
(\S 2.6.3 - \S 2.6.3) \quad & \text{Thm A(c)}_r \Longrightarrow \text{Thm A(a)}_{r''>r} \Longrightarrow \text{Thm A(b)}_{r''>r}
\end{aligned}$$

Remark 2.6.1. The basic reason for double induction is that in the proof of Theorem B and Theorem D, one needs two different ways of cutting hyperplanes. Cutting on X gives $r' < r$ and cutting on Y gives $r' = r$ and $m' < m$.

2.6.1 Set-up

We will work with the Set-up 2.2.1. We fix two finite algebraic Whitney stratifications $\mathfrak{X}$ on X and $\mathfrak{Y}$ on Y adapting to the map f. All perverse sheaf we work with are constructible with respect to these stratifications. For precise definitions, the reader can consult [17, §6.1].

2.6.2 Base case

We start from the case $m = 0$. In this case, all three Theorems follow from Simpson's Hard Lefschetz Theorem for semisimple local systems (c.f. Theorem 2.3.1) and the fact about polarization of primitive parts by twisted Poincaré pairings (c.f Corollary 2.3.55).

2.6.3 Induction step

Suppose all three Theorems are true for $(r' < r, m')$ or $(r' = r, m' < m)$. We will prove them for (r, m). There are two cases.

Case I: $r(f) = 0$. Then f is semismall. By [34, Prop 8.2.30], f_*K is a perverse sheaf. In particular, Theorem A(a) and Theorem A(b) automatically hold. The reader can directly proceeds to §2.7 for the proof of Theorem B and Theorem D. After that, Theorem A(c) will be proved in §2.8.

Case II: $r(f) > 0$. Fix a projective embedding $X \subseteq \mathbf{P}$. Consider the following diagram

$$\begin{array}{ccc} X & \xleftarrow{p_X} & \mathcal{X} \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} \\ Y & \xleftarrow{p_Y} & \mathcal{Y} \end{array}$$

where g is the universal hyperplane map associated to f (c.f. §2.5.2). Since we know that $r(g) < r(f)$, we can apply the inductive hypothesis. In the next three subsections, we will explain the proof of Theorem A.

Theorem A(a): Relative Hard Lefschetz Theorem

Let $d = \dim \mathbf{P}^\vee, M := p_X^* K[d-1]$. Since the functor $p_Y^*[d]$ is fully-faithful, it suffices to show that

$$p_Y^*(\eta^\ell)[d] \colon p_Y^* {}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*K)[d] \to p_Y^* {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)[d]$$

is an isomorphism. There are two cases.

Case I: $\ell \geq 2$. Consider the following diagram

$$\begin{array}{ccc} p_Y^* {}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*K)[d] & \xrightarrow{\cong} & {}^{\mathfrak{p}}\mathcal{H}^{-\ell+1}(g_*M) \\ \downarrow{\scriptstyle p_Y^*(\eta^\ell)[d]} & & \downarrow{\scriptstyle p_X^*(\eta)^{(\ell-1)}} \\ p_Y^* {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)[d] & \xleftarrow{\cong} & {}^{\mathfrak{p}}\mathcal{H}^{\ell-1}(g_*M) \end{array}$$

The Relative Weak Lefschetz Theorem 2.5.7 implies that both horizontal maps are isomorphic. Then the Relative Hard Lefschetz Theorem A(a) follows from the inductive hypothesis applying to the g-ample line bundle $p_X^*(\eta)$.

Remark 2.6.2. Note here we actually need the Relative Hard Lefschetz Theorem A(a) for f-ample line bundles, which can be deduced from the ample case as in [17, Remark 5.1.2]: Let $\tilde{\eta}$ be a f-ample line bundle, choose $m \gg 0$ so that $\tilde{\eta} + mL$ is ample on X. Then we have

$${}^{\mathfrak{p}}\mathcal{H}^{\ell}(\tilde{\eta}) = {}^{\mathfrak{p}}\mathcal{H}^{\ell}(\tilde{\eta} + mL).$$

Case II: $\ell = 1$. The cup product can be factored as follows:

$$p_Y^*(\eta)[d] \colon p_Y^* {}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*K)[d] \to {}^{\mathfrak{p}}\mathcal{H}^0(g_*M) \to p_Y^* {}^{\mathfrak{p}}\mathcal{H}^1(f_*K)[d].$$

Since ${}^{\mathfrak{p}}\mathcal{H}^0(g_*M)$ is semisimple by the inductive hypothesis, combining with Theorem 2.5.7(4), a standard argument as in the proof of [17, Lemma 5.1.1] implies that

$$\eta \colon {}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*K) \to {}^{\mathfrak{p}}\mathcal{H}^1(f_*K) \tag{Cup}$$

is a monomorphism. Since $K^* := D_X(K) \cong \mathcal{V}^*[\dim X]$ and $\mathcal{V}^*$ is also a semisimple local system, we know that

$$\eta \colon {}^{\mathfrak{p}}\mathcal{H}^{-1}(f_*K^*) \to {}^{\mathfrak{p}}\mathcal{H}^1(f_*K^*) \tag{Cup*}$$

is a monomorphism. By Corollary 2.5.3, the Verdier dual of (Cup*) can be identified with the morphism (Cup). Hence the morphism (Cup) is also an epimorphism. This finishes the inductive proof of Relative Hard Lefschetz.

Theorem A(b): Decomposition Theorem

By Deligne's Lefschetz splitting criterion [20, Theorem 1.5], the Relative Hard Lefschetz Theorem A(a) implies the Decomposition Theorem A(b).

Theorem A(c): Semisimplicity Theorem

There are two cases.

Case I: $\ell \neq 0$. The semisimplicity of ${}^{\mathfrak{p}}\mathcal{H}^\ell(f_*K)$ follows from the Relative Weak Lefschetz Theorem 2.5.7 and the inductive semisimplicity of ${}^{\mathfrak{p}}\mathcal{H}^{\ell+1}(g_*M)$ and ${}^{\mathfrak{p}}\mathcal{H}^{\ell-1}(g_*M)$.

Case II: $\ell = 0$. Since we have proven the Decomposition Theorem A(b), the semisimplicity of ${}^{\mathfrak{p}}\mathcal{H}^0(f_*K)$ will follow from Proposition 2.8.1, whose proof is postponed to the end of this chapter.

2.7 Proof of Theorem B and Theorem D

In order to prove the Semisimplicity Theorem in the case of $\ell = 0$, we need to prove two auxiliary results Theorem B and Theorem D.

2.7.1 Theorem B: Hard Lefschetz for Perverse Cohomology Complexes

Proposition 2.7.1. *Assume the Theorem A(a) and Theorem A(b) hold for f. Assume Theorem B and Theorem D hold for all $f' : X' \to Y'$ so that*

$$r(f') < r(f) \quad \text{or} \quad r(f') = r(f), \quad \dim Y' < \dim Y.$$

Then Theorem B holds for f, i.e.

$$A^j \colon \mathbb{H}^{-j}(Y, {}^{\mathfrak{p}}\mathcal{H}^\ell(f_*K)) \cong \mathbb{H}^j(Y, {}^{\mathfrak{p}}\mathcal{H}^\ell(f_*K)),$$
$$\eta^\ell \colon \mathbb{H}^j(Y, {}^{\mathfrak{p}}\mathcal{H}^{-\ell}(f_*K)) \cong \mathbb{H}^j(Y, {}^{\mathfrak{p}}\mathcal{H}^\ell(f_*K)).$$

Proof. The statement for η follows from the Relative Hard Lefschetz Theorem A(a) for f. To prove the statement for A, the plan is to cut X by hyperplane sections in $|\eta|$ or $|L|$ and use the corresponding weak Lefschetz theorem.

Case I: $\ell \neq 0$. By the Relative Hard Lefschetz Theorem A(a), we can assume $\ell < 0$. Choose a general hyperplane section $X^1 \in |\eta|$. Set

$$f^1 : X^1 \to Y, \quad K^1 := K[-1]|_{X^1}.$$

Since it suffices to prove Theorem B for any tensor powers of η, we can replace η by $\eta^{\otimes m}$ for some integer m using Proposition 2.5.8 so that

$$r(f^1) \leq r(f) - 1 \quad \text{or} \quad r(f^1) = r(f) = 0, \quad \dim f^1(X^1) < \dim f(X).$$

Now consider the diagram

$$\begin{array}{ccc} \mathbb{H}^{-j}(Y, {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)) & \xrightarrow{i^*} & \mathbb{H}^{-j}(Y, {}^{\mathfrak{p}}\mathcal{H}^{\ell+1}(f^1_*K^1)) \\ \downarrow{\scriptstyle A^j} & & \downarrow{\scriptstyle A^j} \\ \mathbb{H}^{j}(Y, {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)) & \xrightarrow{i^*} & \mathbb{H}^{j}(Y, {}^{\mathfrak{p}}\mathcal{H}^{\ell+1}(f^1_*K^1)) \end{array}$$

This diagram is commutative because cup products with η and $L = f^*A$ are commutative. By Proposition 2.5.8, the pull-back map i^* and the Gysin push-forward i_* are both injective. Then the injectivity of A^j follows from inductive hypothesis. The surjectivity follows from a dual argument as in the Case II of §2.6.3.

Case II: $\ell = 0$ and $j \geq 2$. Consider the commutative diagram

$$\begin{array}{ccc} X_1 := f^{-1}(Y_1) & \longrightarrow & X \\ \downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f} \\ Y_1 & \longrightarrow & Y \end{array}$$

where Y_1 is sufficiently general hyperplane section in $|A|$ so that $f^{-1}(Y_1)$ is nonsingular and Y_1 is transversal to all strata of Y. Then we have

$$r(f_1) = r(f), \quad \dim f_1(X_1) < \dim f(X).$$

This leads to the following commutative diagram of the restriction and Gysin maps, where $K_1 := K[-1]|_{X_1}$:

$$\begin{array}{ccc} \mathbb{H}^{-j}(Y, {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)) & \xrightarrow{i^*} & \mathbb{H}^{-j+1}(Y_1, {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_{1*}K_1)) \\ \downarrow{\scriptstyle A^j} & & \downarrow{\scriptstyle (A|_{Y_1})^{j-1}} \\ \mathbb{H}^{j}(Y, {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_*K)) & \xleftarrow{i_*} & \mathbb{H}^{j-1}(Y_1, {}^{\mathfrak{p}}\mathcal{H}^{\ell}(f_{1*}K_1)) \end{array}$$

By Proposition 2.5.10, both horizontal maps are isomorphic, therefore the bijectivity of A^j follows from the inductive Theorem B for $(A|_{Y_1})^{j-1}$.

Case III: $\ell = 0$ and $j = 1$. This step we need to use the polarization. We use the same choice as in Case II, where

$$Y_1 \in |A|, \quad X_1 := f^{-1}(Y_1).$$

The cup product with $L = f^*A$ can be decomposed into

$$L\colon H^{n-1}(X,\mathcal{V}) \xrightarrow{Q} H^{n-1}(X_1,\mathcal{V}|_{X_1}) \xrightarrow{G} H^{n+1}(X,\mathcal{V})$$

where Q denotes the restriction map and G denotes the Gysin pushforward. Similarly, for $\mathcal{V}^*$ we have

$$L\colon H^{n-1}(X,\mathcal{V}^*) \xrightarrow{Q^*} H^{n-1}(X_1,\mathcal{V}^*|_{X_1}) \xrightarrow{G^*} H^{n+1}(X,\mathcal{V}^*).$$

Remark 2.7.2. Here Q^* and G^* denote the corresponding restriction and Gysin push-forward. Note that G^* can be identified with the dual of Q and Q^* is the dual of G.

Since we already have the Decomposition Theorem for f and $\mathcal{V}$, these maps give decomposition of cup product with A:

$$A\colon \mathbb{H}^{-1}(Y,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \xrightarrow{Q} \mathbb{H}^0(Y_1,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{Y_1}) \xrightarrow{G} \mathbb{H}^1(Y,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K)).$$

Dually we also have

$$A\colon \mathbb{H}^{-1}(Y,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K^*)) \xrightarrow{Q^*} \mathbb{H}^0(Y_1,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K^*)|_{Y_1}) \xrightarrow{G^*} \mathbb{H}^1(Y,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K^*)).$$

To prove that the cup product map A is bijective, we will produce a non-degenerate pairing in several steps:

$$S : \operatorname{Ker} G \otimes \operatorname{Ker} G^* \to \mathbb{C}.$$

Here G and G^* are the maps for Y_1 and Y.

Step 1: Consider the twisted Poincairé pairing as in §2.3.5:

$$S : H^*(X_1,\mathcal{V}|_{X_1}) \otimes H^*(X_1,\mathcal{V}^*|_{X_1}) \to \mathbb{C}.$$

By definition, the pull back map Q and the Gysin morphism G^* are adjoint to each other with respect to S:

$$S(Q(\alpha),\beta^*) = S(\alpha, G^*(\beta^*)).$$

for $\alpha \in H^{n-1}(X,\mathcal{V})$ and $\beta^* \in H^{n-1}(X_1,\mathcal{V}^*|_{X_1})$.

Step 2: By the inductive Generalized Hodge-Riemann Bilinear relation (c.f. Theorem D) for $f_1 : X_1 \to Y_1$, S restricts to a non-degenerate pairing

$$S : \mathbb{H}^0(Y_1,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{Y_1}) \otimes \mathbb{H}^0(Y_1,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K^*)|_{Y_1}) \to \mathbb{C}$$

where we represent elements in $\mathbb{H}^0(Y_1,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{Y_1})$ by elements in $H^{n-1}(X_1,\mathcal{V}|_{X_1})$.

Step 3. By Corollary 2.3.123 and the fact that G is the dual of Q^*, the Gysin map

$$G : \mathbb{H}^0(Y_1,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{Y_1}) \to \mathbb{H}^1(Y,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K))$$

underlies a morphism of pure twistor structures. On the other hand, using the inductive Theorem D, we see that the vector space

$$\operatorname{Ker} A|_{Y_1} \subseteq \mathbb{H}^0(Y_1,{}^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{Y_1})$$

can be decomposed into primitive subspaces $\eta^m L^i P^{-2i}_{-2m}$ with each of them being polarized by $S^{\eta L}_{\ell j}(\bullet, \phi(\bullet))$, where ϕ is the restriction of the canonical map in Definition 2.3.53 to the pure twistor structure on $\eta^m L^i P^{*-2i}_{-2m}$. In particular, one can put all polarization together and show that S restricts to a non-degenerate pairing

$$S : \operatorname{Ker} A|_{Y_1} \otimes \operatorname{Ker} A|_{Y_1} \to \mathbb{C}$$

note that the second kernel is a subspace of $\mathbb{H}^0(Y_1, {}^{\mathfrak{p}}\mathcal{H}^0(f_* K^*)|_{Y_1})$.

By the property of Gysin morphism, we have a natural inclusion map

$$\operatorname{Ker} G \subseteq \operatorname{Ker} A|_{Y_1},$$

which also underlies a morphism of pure twistor structure. Moreover, because this inclusion map is compatible with the double Lefschetz decomposition, S is non-degenerate when restricting to

$$S : \operatorname{Ker} G \otimes \operatorname{Ker} G^* \to \mathbb{C}$$

using Corollary 2.3.24 and Corollary 2.3.123.

Step 4. Finally we can prove the injectivity of A. Suppose by contradiction that there is a nonzero $\alpha \in \operatorname{Ker} A = \operatorname{Ker} G \circ Q$, then $Q(\alpha) \in \operatorname{Ker} G$. Step 3 says that

$$S : \operatorname{Ker} G \otimes \operatorname{Ker} G^* \to \mathbb{C}$$

is non-degenerate, hence we can find an element $\beta^* \in \operatorname{Ker} G^*$ so that

$$S(Q(\alpha), \beta^*) \neq 0.$$

On the other hand,

$$S(Q(\alpha), \beta^*) = S(\alpha, G^*(\beta^*)) = S(\alpha, 0) = 0.$$

This is a contradiction! Therefore $\operatorname{Ker} A = \{0\}$. The surjectivity of A follows from the injectivity of A for the dual local system $\mathcal{V}^*$. □

Corollary 2.7.3. *Let W^L_i and W^η_i be the weight filtrations on $H^*(X, K) := \bigoplus_b H^b(X, K)$ induced by cup products with L and η (see §2.3.5). Then*

1. $W^\eta_i = \bigoplus_{b \geq -i} H^b(X, K), \quad \mathrm{Gr}^\eta_i = H^{-i}(X, K).$
2. $W^L_i = \bigoplus_{b \in \mathbb{Z}} H^b_{\leq b+i}(X, K), \quad \mathrm{Gr}^L_i = \bigoplus_{b \in \mathbb{Z}} H^b_{b+i}(X, K).$
3. $\mathrm{Gr}^\eta_{j+\ell} \mathrm{Gr}^L_j H^*(X, K) = H^{-\ell-j}_{-\ell}(X, K)$ *and the twisted Poincaré pairings $S^{\eta L}_{\ell j}$ in §2.3.5 are therefore defined on*
$$S^{\eta L}_{\ell j} : H^{-\ell-j}_{-\ell}(X, K) \otimes_{\mathbb{C}} H^{-\ell-j}_{-\ell}(X, K^*) \to \mathbb{C}.$$
4. *The filtration $W^\eta[j]$ is the weight-j filtration of η relative to W^L and*
$$H^{-\ell-j}_{-\ell}(X, K) = \bigoplus_{m, i \geq 0} \eta^m L^i P^{-j-2i}_{-\ell-2m},$$
where $P^{-j}_{-\ell} = \operatorname{Ker} \eta^{\ell+1} \cap \operatorname{Ker} L^{j+1}$.

5. *Each direct summand* $\eta^m L^i P^{-j-2i}_{-\ell-2m} \subseteq H^{-\ell-j}(X, K)$ *underlies a pure sub-twistor structure* F *so that the canonical map* ϕ *in Definition 2.3.53 restricts to*
$$\phi : \overline{F|_{z=-1}} \to \eta^m L^i P^{*-j-2i}_{-\ell-2m},$$
where $P^{*-j}_{-\ell} := \operatorname{Ker} \eta^{\ell+1} \cap \operatorname{Ker} L^{j+1} \subseteq H^{-\ell-j}_{-\ell}(X, K^*)$.

Proof. The description of weight filtration of η follows from the Hard Lefschetz theorem for Semisimple Local Systems (c.f. Theorem 2.3.1). For the weight filtration of L, we need to show that

- $L(\bigoplus_{b\in\mathbb{Z}} H^b_{\leq b+i}(X, K)) \subseteq \bigoplus_{b\in\mathbb{Z}} H^b_{\leq b+i+2}(X, K)$.
- $L^i : \bigoplus_{b\in\mathbb{Z}} H^b_{b+i}(X, K) \cong \bigoplus_{b\in\mathbb{Z}} H^{b+2i}_{b+i}(X, K)$.

The first and second claims follow from repeated use of definition of $L : K \to K[2]$ and Theorem B. The rest follows from Lemma 2.3.110 and Lemma 2.3.57 (the canonical map is compatible with cup products). □

2.7.2 Theorem D: Generalized Hodge-Riemann Bilinear Relations

Proposition 2.7.4. *Suppose we are in the Set-up 2.2.1. Assume the Theorem A(b) and Theorem B hold for* f*. Then Theorem D holds for* f*: Let* F *be the pure twistor structure on* $\eta^m L^i P^{-j-2i}_{-\ell-2m}$ *and* ϕ *be the canonical map from Corollary 2.7.3, we have*

- F *is polarized by* $S^{\eta L}_{\ell j}(\bullet, \phi(\bullet))$ *in the sense of Definition 2.3.21 up to a constant depending on* (n, ℓ, m, i, j).
- *In particular,* $S^{\eta L}_{\ell j}$ *restricts to a non-degenerate pairing*
$$S^{\eta L}_{\ell j} : \eta^m L^i P^{-j-2i}_{-\ell-2m} \otimes \eta^m L^i P^{*-j-2i}_{-\ell-2m} \to \mathbb{C}.$$

Proof. The double Lefschetz decomposition and the existence of pure twistor structure on each direct summand is proved in Corollary 2.7.3. We will focus on the part about polarizations, whose proof is divided into two cases.

Case I: Consider all
$$\eta^m L^i P^{-j-2i}_{-\ell-2m} \neq P^0_0.$$
It suffices to deal with the case $\ell, j \geq 0$ by definition and Remark 2.3.112. We can also assume $i = m = 0$. Now there are two subcases.

Case I.1: $P^{-j}_{-\ell}$ for $\ell \geq 1$. Cut X by general hyperplane in $|\eta|$ and apply Corollary 2.5.9, Corollary 2.3.123 and the compatibility of twisted Poincaré paring between X and its hyperplanes up to a constant depending on $\dim X$.

Case I.2: P^{-j}_0 for $j \geq 1$. Cut X by general hyperplane in $|L|$ and apply Proposition 2.5.10, Corollary 2.3.123 and the compatibility of twisted Poincaré paring between X and its hyperplanes up to a constant depending on $\dim X$.

Case II: P_0^0. We follow the same strategy as in [17, §5.4] by investigating the relations between $P_0^0 = \operatorname{Ker}\eta \cap \operatorname{Ker} L$ and $\lim_{\epsilon\to 0}\operatorname{Ker}(L+\epsilon\eta)$. For $\epsilon > 0$, consider

$$\Lambda_\epsilon := \operatorname{Ker}(L+\epsilon\eta) \subseteq \mathbb{H}^0(X,K) = H^n(X,\mathcal{V}).$$
$$\Lambda_\epsilon^* := \operatorname{Ker}(L+\epsilon\eta) \subseteq \mathbb{H}^0(X,K^*) = H^n(X,\mathcal{V}^*).$$

By the Hodge-Simpson Theorem 2.3.26, each Λ_ϵ underlies a pure twistor structure and has the same dimension $b = \dim \mathbb{H}^0(X,K) - \dim \mathbb{H}^2(X,K)$. Let E be the pure twistor structure on $\mathbb{H}^0(X,K)$. By Definition 2.3.53, there is a canonical map ϕ so that

$$\phi : \overline{E|_{z=-1}} \cong H^n(X,\mathcal{V}^*) = \mathbb{H}^0(X,K^*).$$

Lemma 2.7.5. *Consider the limiting spaces in the Grassmannians of b-dimensional subspaces of $H^0(X,K)$ and $H^0(X,K^*)$:*

$$\Lambda := \lim_{\epsilon\to 0}\Lambda_\epsilon \in G(b,\mathbb{H}^0(X,K)).$$
$$\Lambda^* := \lim_{\epsilon\to 0}\Lambda_\epsilon^* \in G(b,\mathbb{H}^0(X,K^*)).$$

Then Λ underlies a pure sub-twistor structure $F \subseteq E$. And the composition of the identification map (see Definition 2.3.11) and the restriction of ϕ

$$\phi\circ\overline{\mathrm{Iden}} : \overline{\Lambda} = \overline{F|_{z=1}} \to \overline{F|_{z=-1}} \to \Lambda^*$$

is the restriction of the conjugated Hodge star operator

$$C : \overline{H^n(X,\mathcal{V})} \to H^n(X,\mathcal{V}^*).$$

Notation 2.7.6. In this section, we denote the conjugated Hodge star operator by C.

Proof. It suffices to prove there is a natural map

$$\overline{\Lambda} \to \Lambda^*,$$

which can be used to construct the pure sub-twistor structure on Λ.

Suppose $F_\epsilon \subseteq E$ is the pure sub-twistor structure on Λ_ϵ. Then by Lemma 2.3.60, the composition

$$\phi\circ\overline{\mathrm{Iden}} : \overline{\Lambda_\epsilon} = \overline{F_\epsilon|_{z=1}} \to \overline{F_\epsilon|_{z=-1}} \to \Lambda_\epsilon^*$$

is the restriction of the conjugated Hodge star operator

$$C : \overline{H^n(X,\mathcal{V})} \to H^n(X,\mathcal{V}^*).$$

In particular, because we work with middle cohomology and the Hodge star operator doesn't depend on the choice of ample line bundles, there $\overline{\Lambda_\epsilon} \to \Lambda_\epsilon^*$ is the restriction of the same map for all ϵ. Therefore, the limiting map is the restriction of C as well. □

To remember the cohomological degrees, we set

$$L_r^k \colon \mathbb{H}^{-r}(X,K) \to \mathbb{H}^{-r+2k}(X,K).$$
$$L_r^{*k} \colon \mathbb{H}^{-r}(X,K^*) \to \mathbb{H}^{-r+2k}(X,K^*).$$

Notation 2.7.7. To differentiate between subspaces of $\mathbb{H}^{-r}(X,K)$ and $\mathbb{H}^{r}(X,K^*)$, we denote L^* and η^* to the cup product maps for $\mathbb{H}^r(X,K^*)$.

Lemma 2.7.8. *With the notation above, we have*

$$\eta \operatorname{Ker} L_2^1 \cap (\eta^* \operatorname{Ker} L_2^{*1})^\perp \cap \cdots \cap (\eta^{*i} \operatorname{Ker} L_{2i}^{*i})^\perp = \{0\} \in \mathbb{H}^0(X,K), \quad i \gg 0.$$

Proof. We will prove the following more precise equality by induction on i:

$$\eta \operatorname{Ker} L_2^1 \cap (\eta^* \operatorname{Ker} L_2^{*1})^\perp \cap \cdots \cap (\eta^{*i} \operatorname{Ker} L_{2i}^{*i})^\perp = \eta \operatorname{Ker} L_2^1 \cap W_{-i}^L, \forall i \geq 0.$$

where $W_\bullet^L$ denotes the weight filtration on $\mathbb{H}^k(X,K)$ with respect to L.
Base case: $i=0$. We need to show that

$$\eta \operatorname{Ker} L_2^1 = \eta \operatorname{Ker} L_2^1 \cap W_0^L.$$

First, the hard Lefschetz for weight filtration $L^k : \mathrm{gr}_k^{W^L} \cong \mathrm{gr}_{-k}^{W^L}$ implies that

$$\operatorname{Ker} L^k \subseteq W_{k-1}^L.$$

In particular, it follows from the commutativity of η and L that

$$\eta \operatorname{Ker} L_2^1 \subseteq \eta W_0^L \subseteq W_0^L.$$

Inductive case. Suppose it holds for i, we need to show that

$$\eta \operatorname{Ker} L_2^1 \cap W_{-i-1}^L = \eta \operatorname{Ker} L_2^1 \cap W_{-i}^L \cap (\eta^{*i+1} \operatorname{Ker} L_{2i+2}^{*i+1})^\perp.$$

Here the orthogonal decomposition is taken with respect to S as in the Definition 2.3.106.

For the inclusion "$\subseteq$": as in the base case, we have

$$\operatorname{Ker} L_{2i+2}^{*i+1} \subseteq W_i^{L^*}.$$

Therefore
$$\eta^{*i+1} \operatorname{Ker} L_{2i+2}^{*i+1} \subseteq \eta^{*i+1} W_i^{L^*} \subseteq W_i^{L^*} = (W_{-i-1}^L)^\perp.$$

The last equality follows from the self-duality in the sense of Lemma 2.3.107.

For the inclusion "$\supseteq$", we will use the non-degeneracy of $S_{i+2,i}^{\eta L}$. As in the proof of [17, Lemma 5.4.1], one can show that

$$\alpha \in \eta \operatorname{Ker} L_2^1 \cap W_{-i}^L \Rightarrow \alpha = \eta L_{2i+2}^i \alpha', \quad \alpha' \in \operatorname{Ker} L_{2i+2}^{i+1}.$$
$$\beta^* \in \eta^{*i+1} \operatorname{Ker} L_{2i+2}^{*i+1} \Rightarrow \beta^* = \eta^{*i+1} \beta^{*'}, \quad \beta^{*'} \in \operatorname{Ker} L_{2i+2}^{*i+1}.$$

Then by the definition of $S_{i+2,i}^{\eta L}$ we have

$$S_{i+2,i}^{\eta L}([\alpha'],[\beta^{*'}]) = S(\eta L^i \alpha', \eta^{*i+1} \beta^{*'}) = S(\alpha, \beta^*),$$

where $[\alpha']$ and $[\beta^{*'}]$ are viewed as classes in the associated graded space of W^L and W^{L*}. If we have

$$\alpha \in \eta \operatorname{Ker} L_2^1 \cap W_{-i}^L \cap (\eta^{*i+1} \operatorname{Ker} L_{2i+2}^{*i+1})^\perp,$$

then

$$S_{i+2,i}^{\eta L}([\alpha'], [\beta^{*'}]) = 0, \quad \forall \beta^{*'} \in \operatorname{Ker} L^{*i+1}.$$

Since $\operatorname{Ker} L^{i+1} \subseteq H_{-i-2}^{-2i-2}(X, K)$, it follows from Theorem D and Lemma 2.3.23 that $S_{i+2,i}^{\eta L}$ restricts to a non-degenerate pairing on

$$S_{i+2,i}^{\eta L} : \operatorname{Ker} L^{i+1} \otimes \operatorname{Ker} L^{*i+1} \to \mathbb{C}.$$

This implies the class of α' in $H_{-i-2}^{-2i-2}(X, K)$ is zero. In particular,

$$\alpha' \in W_{i-1}^L.$$

Therefore

$$\alpha = \eta L^i \alpha' \in \eta W_{-i-1}^L \subseteq W_{-i-1}^L.$$

□

Lemma 2.7.9. *We have*

$$\Lambda = \operatorname{Ker} L_0^1 \cap \left(\bigcap_{i \geq 1} (\eta^{*i} \operatorname{Ker} L_{2i}^{*i})^\perp \right) \subseteq H_{\leq 0}^0(X, K)$$

we have a direct sum decomposition

$$\operatorname{Ker} L_0^1 = \Lambda \oplus \eta \operatorname{Ker} L_2^1.$$

and we have an "orthogonal decomposition" in the sense that

$$\Lambda = (\eta^* \operatorname{Ker} L_2^{*1})^\perp \cap \operatorname{Ker} L_0^1.$$

Proof. First, we show that

$$\Lambda_\epsilon \subseteq (\eta^{*i} \operatorname{Ker} L_{2i}^{*i})^\perp, \quad \forall i \geq 1, \epsilon > 0.$$

Let $\alpha_\epsilon \in \Lambda_\epsilon = \operatorname{Ker}(L + \epsilon\eta)$, then

$$\eta \alpha_\epsilon = \frac{L\alpha_\epsilon}{-\epsilon} \Rightarrow \eta^i \alpha_\epsilon = \frac{L^i \alpha_\epsilon}{(-\epsilon)^i}.$$

On the one hand, for $\beta^* \in \operatorname{Ker} L_{2i}^{*i}$, we have

$$S(\alpha_\epsilon, \eta^{*i}\beta^*) = S(\eta^i \alpha_\epsilon, \beta^*) = (-\epsilon)^{-i} S(L^i \alpha_\epsilon, \beta^*) = (-\epsilon)^{-i} S(\alpha_\epsilon, L^{*i}\beta^*) = 0.$$

Therefore, by taking limits we have

$$\Lambda \subseteq \bigcap_{i \geq 1} (\eta^{*i} \operatorname{Ker} L_{2i}^{*i})^\perp.$$

On the other hand, let $\alpha = \lim_{\epsilon\to 0}\alpha_\epsilon \in \Lambda$. Then

$$L\alpha = \lim_{\epsilon\to 0} L\alpha_\epsilon = \lim_{\epsilon\to 0}(-\epsilon)\eta\alpha_\epsilon = 0.$$

Hence

$$\Lambda \subseteq \operatorname{Ker} L_0^1.$$

From the previous lemma we know that,

$$\eta \operatorname{Ker} L_2^1 \cap \bigcap_{i\geq 1}(\eta^{*i}\operatorname{Ker} L_{2i}^{*i})^\perp = \{0\}.$$

Therefore

$$\eta \operatorname{Ker} L_2^1 \cap \Lambda = \{0\}.$$

Now counting dimensions, we conclude that

$$\operatorname{Ker} L_0^1 = \Lambda \oplus \eta \operatorname{Ker} L_2^1,$$
$$\Lambda = \operatorname{Ker} L_0^1 \cap \left(\bigcap_{i\geq 1}(\eta^{*i}\operatorname{Ker} L_{2i}^{*i})^\perp\right)$$

For the orthogonality, because the twisted Poincaré pairing is non-degenerate, using the direct sum decomposition we see that

$$\begin{aligned}\dim(\eta^* \operatorname{Ker} L_2^{*1})^\perp \cap \operatorname{Ker} L_0^1 &= \dim \operatorname{Ker} L_0^1 - \dim \eta^* \operatorname{Ker} L_2^{*1} \\ &= \dim \operatorname{Ker} L_0^1 - \dim \eta \operatorname{Ker} L_2^1 = \dim \Lambda.\end{aligned}$$

In particular,

$$(\eta^* \operatorname{Ker} L_2^{*1})^\perp \cap \operatorname{Ker} L_0^1 = \Lambda.$$

□

Lemma 2.7.10. *Define* $\Lambda_0 := \Lambda/(\Lambda \cap H^0_{\leq -1}(X,K))$. *Then* Λ_0 *underlies a pure twistor structure, which is polarized by the form* $S_{00}^{\eta L}(\bullet, \phi(\bullet))$.

Proof. Let F be the pure twistor structure on Λ. Lemma 2.7.5 implies that the pairing

$$S_{00}^{\eta L}(\bullet, C(\bullet)) : \Lambda \otimes \overline{\Lambda} \xrightarrow{\mathrm{Id}\otimes C} \Lambda \otimes \Lambda^* \xrightarrow{S_{00}^{\eta L}} \mathbb{C}$$

is semipositive definite, where $C = \phi \circ \overline{\mathrm{Iden}}$ is the restriction of the conjugated Hodge star operator $C : \overline{\mathbb{H}^0(X,K)} \to \mathbb{H}^0(X,K^*)$.

On the other hand, since $H^0_{\leq -1}(X,K)$ underlies a pure twistor structure, then Λ_0 is a pure subtwistor structure F_0 with the descent canonical map

$$\phi : \overline{F_0|_{z=-1}} \to \Lambda_0^* := \Lambda^*/(\Lambda^* \cap H^0_{\leq -1}(X,K^*)).$$

By Lemma 2.3.107 we have $(W_0^{L*})^\perp = W_{-1}^L$, therefore the descent pairing

$$S_{00}^{\eta L} : \Lambda_0 \otimes \Lambda_0^* \to \mathbb{C}$$

is non-degenerate. In particular $S_{00}^{\eta L}(\bullet, C\bullet)$ descends to a non-degenerate pairing

$$S_{00}^{\eta L}(\bullet, C\bullet) : \Lambda_0 \otimes \overline{\Lambda_0} \to \mathbb{C}.$$

Therefore $S_{00}^{\eta L}(\bullet, C\bullet)$ is positive definite being semipositive definite and non-degenerate at the same time. In particular, Λ_0 is polarized by $S_{00}^{\eta L}(\bullet, \phi(\bullet))$. □

Remark 2.7.11. Note here that, in contrast to [17, Lemma 5.4.1], we don't multiply $S_{00}^{\eta L}$ by $(-1)^n$ because the sign is built into the Hodge star operator C.

Now we can finish the proof the Generalized Hodge-Riemann Bilinear Relation for P_0^0. By definition

$$P_0^0 = \operatorname{Ker} \eta \cap \operatorname{Ker} L_0^1 \cap H_0^0(X, K).$$

Here $H_0^0(X, K)$ is a quotient space of $H^*(X, K)$ and the third intersection means the descend of the subspace on the quotient space.

By the characterization of perverse filtration in terms of the weight filtration, we have $H_0^0(K) \subseteq \operatorname{Gr}_0^L$. Therefore

$$P_0^0 \subseteq \frac{\operatorname{Ker} \eta \cap \operatorname{Ker} L_0^1 \cap W_0^L}{\operatorname{Ker} \eta \cap \operatorname{Ker} L_0^1 \cap W_{-1}^L} = \frac{\operatorname{Ker} \eta \cap \operatorname{Ker} L_0^1}{\operatorname{Ker} \eta \cap \operatorname{Ker} L_0^1 \cap W_{-1}^L}.$$

Using $\operatorname{Ker} \eta \subseteq (\eta^* \operatorname{Ker} L_2^{*1})^\perp$ and Lemma 2.7.9 we have

$$P_0^0 \subseteq \frac{(\eta^* \operatorname{Ker} L_2^{*1})^\perp \cap \operatorname{Ker} L_0^1}{(\eta^* \operatorname{Ker} L_2^{*1})^\perp \cap \operatorname{Ker} L_0^1 \cap W_{-1}^L} = \frac{\Lambda}{\Lambda \cap W_{-1}^L} = \Lambda_0.$$

Moreover, since the inclusion $P_0^0 \subseteq \Lambda_0$ underlies a morphism of pure twistor structures and Λ_0 is polarized by $S_{00}^{\eta L}(\bullet, \phi(\bullet))$ by Lemma 2.7.10, we conclude by Lemma 2.3.23 that P_0^0 is polarized by the restriction of $S_{00}^{\eta L}(\bullet, \phi(\bullet))$. $\square$

2.7.3 Non-degeneration of intersection forms

In this section, we use the Generalized Hodge-Riemann Bilinear Relation Theorem D to verify the splitting criterion needed for the proof of Semisimplicity Theorem.

Proposition 2.7.12. *Suppose we are in the Set-up 2.2.1. Assume Theorem A(b) and Theorem D hold for f. Let $i : y \hookrightarrow Y$ be a point in the support of $\mathcal{H}^0({}^{\mathfrak{p}}\mathcal{H}^0(f_*K))$. Then the following natural map is an isomorphism*

$$H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \cong H^0(Y, i_* i^* {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)).$$

Proof. Consider the adjunction map induced by i:

$$i_! i^! f_* K \to f_* K \to i_* i^* f_* K.$$

The Decomposition Theorem A(b) gives $f_*K \cong \bigoplus_\ell {}^{\mathfrak{p}}\mathcal{H}^\ell(f_*K)[-\ell]$ and induces

$$i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K) \to {}^{\mathfrak{p}}\mathcal{H}^0(f_*K) \to i_* i^* {}^{\mathfrak{p}}\mathcal{H}^0(f_*K).$$

The cohomology of this map gives the desired natural map

$$H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \xrightarrow{\mathrm{cl}} H^0(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \to H^0(Y, i_* i^* {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)). \quad (2.5)$$

where cl is the natural cycle map.

Let $j : U = Y \setminus \{y\} \hookrightarrow Y$ be the smooth subvariety. The distinguished triangle associated to $\{y\} \xrightarrow{i} Y \xleftarrow{j} U$ gives rise to the following short exact sequence:

$$H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \xrightarrow{\mathrm{cl}} H^0(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \to H^0(Y, j_* j^* {}^{\mathfrak{p}}\mathcal{H}^0(f_*K))$$

Since the map cl is injective by Corollary 2.3.101, we can use Corollary 2.3.123 to conclude that $H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K))$ underlies a pure sub-twistor structure F so that the canonical map ϕ in Corollary 2.7.3 restricts to

$$\phi : \overline{F|_{z=-1}} \to H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K^*)).$$

Notice that we have an inclusion

$$\mathrm{Im}(\mathrm{cl}) \subseteq \mathrm{Ker}\, L \subseteq H^0(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) = H^0_0(X, K).$$

One can decompose this inclusion with respect to Lefschetz decomposition of η so that each direct summand of $H^0_0(X, K)$ is polarized by $S^{\eta L}_{00}(\bullet, \phi(\bullet))$ by Theorem D. Using Lemma 2.3.23 and the compatibility of ϕ with restrictions, we conclude that the twisted Poincaré pairing $S^{\eta L}_{00}$ restricts to a non-degenerate pairing

$$\begin{aligned} S^{\eta L}_{00} : H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \otimes H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K^*)) &\to \mathbb{C} \\ [\alpha] \otimes [\beta] &\mapsto \int_X \alpha \wedge \beta \end{aligned}$$

where we represent a class in $H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \hookrightarrow H^0(Y, {}^{\mathfrak{p}}\mathcal{H}^0(f_*K))$ by a class in $\mathbb{H}^0(X, K)$. Using Verdier duality, the twisted Poincaré pairing gives rise to the map

$$H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)) \xrightarrow{S^{\eta L}_{00}} H^0(Y, i_! i^! {}^{\mathfrak{p}}\mathcal{H}^0(f_*K^*))^\vee \xrightarrow{\mathrm{VD}} H^0(Y, i_* i^* {}^{\mathfrak{p}}\mathcal{H}^0(f_*K)),$$

which can be identified with the map (2.5). Therefore, the non-degeneracy of the twisted Poincaré pairing implies that the map (2.5) is an isomorphism. □

2.8 Semisimplicity Theorem for $\ell = 0$

Proposition 2.8.1. *Suppose we are in the Set-up 2.2.1. Assume Theorem A(b) and Theorem D for f. Assume Theorem A(c) hold for all $f' : X' \to Y'$ so that*

$$r(f') < r(f) \quad or \quad r(f') = r(f) = 0, m' < m.$$

Then Theorem A(c) holds for f, i.e. ${}^{\mathfrak{p}}\mathcal{H}^0(f_*K)$ is semisimple.

We first show that ${}^{\mathfrak{p}}\mathcal{H}^0(f_*K)$ decomposes into the direct sum of intersection complexes of local systems over the strata of Y.

Proposition 2.8.2. *There is a canonical morphism in* Perv(Y)*:*

$$^{\mathfrak{p}}\mathcal{H}^0(f_*K) \cong \bigoplus_{d=\dim S_d} \mathrm{IC}_{\overline{S_d}}(\mathcal{H}^{-d}(^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{S_d})).$$

Proof. Let S_d be a d-dimensional stratum on Y and

$$U_d = \bigcup_{d' \geq d} S_{d'}.$$

We denote two embeddings to be $S_d \xrightarrow{\alpha} U_d \xleftarrow{\beta} U_{d+1}$. Since the intersection complex can be defined using iterated truncations, it suffices to show that

$$^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{U_d} \cong \beta_{*!}(^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{U_{d+1}}) \oplus \mathcal{H}^{-d}(^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{U_d})[d].$$

Here $\beta_{*!} \cong \tau_{\leq -d-1}\beta_*$. There are two cases.

Case I. If $d \geq 1$, it can be handled by induction on d. By Lemma 2.5.12, it suffices to show

$$\mathcal{H}^{-d}(\alpha_!\alpha^{!\,\mathfrak{p}}\mathcal{H}^0(f_*K)) \to \mathcal{H}^{-d}(^{\mathfrak{p}}\mathcal{H}^0(f_*K))$$

is an isomorphism. Notice that this is a local condition so that we can check it stalkwise. Let $y \in S_d$ be any point, choose a generic d-dimensional complete intersection $Y_d \subseteq Y$ which contains y and is transversal to all strata of Y. Then we have the following Cartesian diagram

$$\begin{array}{ccc} X_d & \xrightarrow{i_d} & X \\ \downarrow{\scriptstyle f_d} & & \downarrow{\scriptstyle f} \\ Y_d & \longrightarrow & Y \end{array}$$

Notice that $\dim Y_d < \dim Y$ and $r(f_d) \leq \max\{r(f) - d, 0\}$ by repeated use of Proposition 2.5.10. Set $K_d = i_d^*K[-d]$, then one can apply the inductive hypothesis to conclude that

$$^{\mathfrak{p}}\mathcal{H}^0(f_{d*}K_d)$$

is semisimple. Denote the inclusion to be $i : y \hookrightarrow Y_d$. We have the following commutative diagram

$$\begin{array}{ccc} \mathcal{H}^0(i_!i^{!\,\mathfrak{p}}\mathcal{H}^0(f_{d*}K_d)) & \longrightarrow & \mathcal{H}^0(^{\mathfrak{p}}\mathcal{H}^0(f_{d*}K_d))_y \\ \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ \mathcal{H}^{-d}(\alpha_!\alpha^{!\,\mathfrak{p}}\mathcal{H}^0(f_*K))_y & \longrightarrow & \mathcal{H}^{-d}(^{\mathfrak{p}}\mathcal{H}^0(f_*K))_y \end{array}$$

The isomorphisms in the vertical direction follows from Lemma 2.5.11. Inductive assumption on the decomposition of $^{\mathfrak{p}}\mathcal{H}^0(f_{d*}K_d)$ coupled with Lemma 2.5.12 imply that the first horizontal map is an isomorphism, so is the second one.

Case II. If $d = 0$, we apply MacPherson and Vilonen's criterion (c.f. Corollary 2.5.14) and use Proposition 2.7.12.

□

Proof of Proposition 2.8.1. By Proposition 2.8.2, it suffices to show the local system

$$\mathcal{H}^{-d}({}^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{S_d})$$

is semisimple over S_d for each $d = \dim S_d$. Let $i : y \hookrightarrow Y$ be a point lying in S_d, the fiber of the local system at y is

$$\mathbb{H}^{-d}(Y, i_* i^{*\mathfrak{p}}\mathcal{H}^0(f_*K)) \subseteq \mathbb{H}^{-d}(f^{-1}(y), K|_{f^{-1}(y)}) = H^{n-d}(f^{-1}(y), \mathcal{V}|_{f^{-1}(y)}).$$

Case I. $d = 0$ or $d = m$. If $\dim S_d = 0$, the semisimplicity is trivial. If $d = m = \dim f(X)$, it follows from the Semisimplicity Theorem for smooth projective maps (c.f. Theorem 2.3.3) that $R^{n-d}f_*\mathcal{V}|_{S_d}$ is semisimple, so is the sub-local system $\mathcal{H}^{-d}({}^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{S_d})$.

Cast II. $1 \leq d \leq m-1$. Since $f| \colon f^{-1}(S_d) \to S_d$ may not be smooth projective, we cannot apply Theorem 2.3.3 directly. Instead, we use a geometric construction from [17, §6.4], which roughly means "change a point to its complete intersection normal slice in Y". Concretely, choose A to be an very ample line bundle on Y, one can get to the following situation:

1. $T \subseteq S_d$ is a Zariski open subset.
2. There is a commutative diagram

$$\begin{array}{ccc} \mathcal{X}_T & \xrightarrow{p_X} & X \\ \big\downarrow{\scriptstyle \Phi} & & \big\downarrow{\scriptstyle f} \\ \mathcal{Y}_T & \xrightarrow{p_Y} & Y \\ \big\downarrow{\scriptstyle \pi} & & \\ T & & \end{array}$$

   Here $\mathcal{Y}_T \to T$ admits a section θ, p_X and p_Y are smooth projective maps of relative dimension zero and Φ inherits a stratification from the one on f. Moreover all strata on $\mathcal{Y}_T$ map surjectively and smoothly onto T.

3. For every $t \in T$, $Y_t := \pi^{-1}(t)$ is a complete intersection d hyperplanes passing through $t \in T \subseteq Y$, meeting all strata of Y transversally and such that $X_t := (\pi \circ \Phi)^{-1}(t)$ is a smooth projective variety of dimension $n - d$.

Let $F = \pi \circ \Phi$. Since p_X is smooth projective and $\mathcal{V}$ is a semisimple local system on X, we have $p_X^*\mathcal{V}$ is a semisimple local system on $\mathcal{X}_T$. Apply Lemma 2.3.102 to $p_X^*\mathcal{V}$ and the diagram

$$\mathcal{X}_T \xrightarrow{\Phi} \mathcal{Y}_T \xrightarrow{\pi} T \xrightarrow{\theta} \mathcal{Y}_T,$$

we obtain that $\mathcal{H}^{-d}(\theta^{*\mathfrak{p}}\mathcal{H}^0(\Phi_* p_X^* K))$ is a semisimple local system on T. Now the standard base change formula implies that

$$\begin{aligned} \mathcal{H}^{-d}(\theta^{*\mathfrak{p}}\mathcal{H}^0(\Phi_* p_X^* K)) &\cong \mathcal{H}^{-d}(\theta^{*\mathfrak{p}}\mathcal{H}^0(p_Y^* f_* K)) \\ &\cong \mathcal{H}^{-d}(\theta^* p_Y^{*\,\mathfrak{p}}\mathcal{H}^0(f_*K)) \quad [p_Y^* \text{ is } t\text{-exact}] \\ &\cong \mathcal{H}^{-d}({}^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_T) \end{aligned}$$

Since T is Zariski-dense in S_d, we conclude that $\mathcal{H}^{-d}({}^{\mathfrak{p}}\mathcal{H}^0(f_*K)|_{S_d})$ is a semisimple local system on S_d. □

2.9 Sabbah's Theorem

In this section, we prove Sabbah's main Theorem E in [43] using the standard reductions in [16, Page 71-74] and [15] to Theorem A.

Theorem 2.9.1. *Let $f : U \to Y$ be a proper map between quasi-projective varieties where U is the Zariski open subset of a smooth projective variety X. Let $\mathcal{V}$ be a semisimple local system on X. Then Theorem A(b) and Theorem A(c) hold for f and $\mathcal{V}|_U$.*

If in addition, f is projective and η is f-ample, then the Relative Hard Lefschetz Theorem A(a) holds as well.

Remark 2.9.2. Let Y be a normal variety and let $Y_0 \subseteq Y$ be a Zariski dense open subset, then a local system $\mathcal{V}$ on Y is semisimple if and only if $\mathcal{V}|_{Y_0}$ is semisimple. This is because the natural map

$$\pi_1(Y_0, y) \to \pi_1(Y, y)$$

is surjective for any $y \in Y_0$.

Proof. We will prove the theorem by progressively relaxing conditions on $(f, X, Y, \mathcal{V})$. The hypothesis in Theorem A is denoted by

$$(f_{\text{proj}}, X_p^{\text{sm}}, Y_p, \mathcal{V})$$

which means that f is projective, X is smooth projective, Y is projective and $\mathcal{V}$ is a semisimple local system on X.

1. Theorem A holds for $(f_{\text{proj}}, X_p^{\text{sm}}, Y_{qp}, \mathcal{V})$.
   Choose a closed embedding $g : Y \to \mathbf{P}^N$ and apply Theorem A to $h := g \circ f$. One needs to identify perverse sheaves on Y with perverse sheaves on $\mathbf{P}^N$ with support on Y.

2. Theorem A holds for $(f_{\text{proj}}, U, Y_{qp}, \mathcal{V}|_U)$, where U is a Zariski open subset of a smooth projective variety X and $\mathcal{V}$ is a semisimple local system on X.
   Let Y' be a projective compactification of Y so that Y is a Zariski dense open subset of Y', then f can be viewed as a rational map

   $$f : X \dashrightarrow Y'.$$

   Then we can use Hironaka's theorem to resolve the indeterminacy of f by blowing up smooth subvarieties of X disjoint from U so that we have a following commutative diagram

   $$\begin{array}{ccc} U & \xrightarrow{j} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \longrightarrow & Y' \end{array}$$

Here X' is a smooth projective obtained by blowing up X, U can be viewed as a Zariski dense open subset of X' and $f'|_U = f$. One can pull back the semisimple local system $\mathcal{V}$ on X to get a semisimple local system $\mathcal{V}'$ on X' (this is because the harmonic metric on $\mathcal{V} \otimes \mathcal{C}^\infty_X$ pulls back to a harmonic metric on $\mathcal{V}' \otimes \mathcal{C}^\infty_{X'}$). Moreover, $\mathcal{V}|_U$ is the restriction of $\mathcal{V}'$ to U. Hence we have

$$ {}^{\mathfrak{p}}\mathcal{H}^\ell(f_*\mathcal{V}|_U) = {}^{\mathfrak{p}}\mathcal{H}^\ell(f'_*\mathcal{V}')|_Y. $$

we can apply Theorem A to $(f', X', Y', \mathcal{V}')$ and Remark 2.9.2 to conclude that ${}^{\mathfrak{p}}\mathcal{H}^0(f_*\mathcal{V}|_U)$ is semisimple (semisimplicity can be inherited when restricting to Zariski-dense subset over each smooth stratum). The rest of the proof of Theorem A can be carried out in the same way.

3. Theorem A(b) and Theorem A(c) hold for $(f_{\text{proper}}, U, Y_{qp}, \mathcal{V}|_U)$, where U is a Zariski open subset of a smooth projective variety X and $\mathcal{V}$ is a semisimple local system on X.
   Let $g : U' \to U$ be a Chow envelope for U so that g is projective and birational and $f' := f \circ g$ is projective. Notice that we have

$$ Rg_*(g^*\mathcal{V}|_U) = \mathcal{V}|_U \oplus R $$

   for some $R \in \mathrm{Ob}(D(X))$. Then one apply Theorem A(b) and Theorem A(c) to $(g, U', Y, g^*\mathcal{V}|_U)$ and argue as in [16, Theorem 10.0.6].

□

Chapter 3

Stable Irrationality of Hypersurfaces

The purpose of this chapter is to study various measures of irrationality for hypersurfaces in projective spaces which were proposed recently by [4],[1]. In particular, we answer the question raised by Bastianelli that if $X \subseteq \mathbf{P}^{n+1}$ is a very general smooth hypersurface of dimension n and degree $d \geq 2n+2$, then $\text{stab.irr}(X) = \text{uni.irr}(X) = d-1$. As a corollary, we prove that $\text{irr}(X \times \mathbf{P}^m) = \text{irr}(X)$ for any integer $m \geq 1$.

3.1 Introduction

There has been recent interest in studying measures of irrationality for algebraic varieties [4],[1]. For example, given an irreducible projective variety X of dimension n, the *degree of irrationality* of X is defined as

$$\text{irr}(X) := \min \{\delta > 0 \mid \exists \text{ degree } \delta \text{ rational covering } X \dashrightarrow \mathbf{P}^n\}.$$

Therefore $\text{irr}(X) = 1$ if and only if X is rational. It was established in [3],[4] that if $X \subseteq \mathbf{P}^{n+1}$ is a very general smooth hypersurface of dimension n and degree $d \geq 2n+1$, then $\text{irr}(X) = d-1$.

By analogy with notions of stable rationality and unirationality, Bastianelli [1] introduced two birational invariants measuring the failure of a projective variety to be stably rational or unirational:

$$\text{stab.irr}(X) := \min \{ \text{ irr}(X \times \mathbf{P}^m) \mid m \in \mathbf{N}\};$$

$$\text{uni.irr}(X) := \min \{ \text{ irr}(T) \mid \exists \text{ a rational covering } T \dashrightarrow X\}.$$

Thus

$$\begin{aligned} \text{stab.irr}(X) = 1 \quad &\Longleftrightarrow \quad \text{X is stably rational,} \\ \text{uni.irr}(X) = 1 \quad &\Longleftrightarrow \quad \text{X is unirational,} \end{aligned}$$

and in general one has the inequalities

$$\text{uni.irr}(X) \leq \text{stab.irr}(X) \leq \text{irr}(X).$$

It was established by Bastianelli in [1] that if X is a very general surface of degree $d \geq 5$, then

$$\text{stab.irr}(X) = \text{uni.irr}(X) = d - 1,$$

and Bastianelli also classified the exceptional cases. Here we extend the computation to hypersurfaces of all dimensions.

In fact we will consider more generally correspondences on $\mathbf{P}^n \times X$. We consider the following birational invariant:

$$\text{corr}(X) := \min \{ \deg(\pi_1) \mid \exists \text{ a correspondence } \Gamma \subseteq \mathbf{P}^n \times X \}.$$

where π_1 is the first projection map from Γ to $\mathbf{P}^n$ and Γ is any subvariety of $\mathbf{P}^n \times X$ that both dominates $\mathbf{P}^n$ and X.

Our first results concern corr(X):

Theorem A. *Let $X \subseteq \mathbf{P}^{n+1}$ be a very general smooth hypersurface of degree $d \geq 2n + 2$. Then*

$$\text{corr}(X) = d - 1.$$

Lopez and Pirola [36, Theorem 1.3] classified correspondences with null trace (see Def 3.2.2) of minimum degree on smooth hypersurfaces in $\mathbf{P}^3$. Our results can be seen as a partial generalization to higher dimensions: if we restrict ourselves to null trace correspondences on $\mathbf{P}^n \times X$, we can compute their minimal degree.

As in [1], we notice that the study of uni.irr(X) is equivalent to the study of correspondences on $\mathbf{P}^n \times X$. In particular, we will show that corr(X) = uni.irr(X) (cf. Lemma 3.3.2). From this we deduce our second result, which answers the question of [1]:

Theorem B. *Let $X \subseteq \mathbf{P}^{n+1}$ be a very general smooth hypersurface of degree $d \geq 2n + 2$. Then*

$$\text{stab.irr}(X) = \text{uni.irr}(X) = d - 1.$$

In particular, we have the following

Corollary C. *Let $X \subseteq \mathbf{P}^{n+1}$ be a very general smooth hypersurface of degree $d \geq 2n + 2$. Then*

$$\text{irr}(X \times \mathbf{P}^m) = \text{irr}(X),$$

for any integer $m \geq 1$.

Totaro [53] showed that a very general hypersurface $X \subseteq \mathbf{P}^{n+1}$ of degree $d \geq 2\lceil \frac{n+2}{3} \rceil$ is not stably rational. A couple of years later, Schreieder [49] found a very striking lower bound where $d \geq \log_2 n + 2$, which means that most of hypersurfaces are not stably rational. Therefore, one has stab.irr$(X) > 1$ if $d \geq \log_2 n + 2$. It's interesting to ask further what is the stable irrationality of a degree d hypersurface in this range.

On the other hand, Bastianelli, Ciliberto, Flamini and Supino [2, Section 5.2] conjectured that

$$\text{conn.gon}(X) \leq d - \left\lfloor \frac{\sqrt{8n+9}-1}{2} \right\rfloor < d - 1 = \text{uni.irr}(X).$$

This means that even though it's very hard to determine whether rationally connected varieties are unirational (equivalently whether $\text{conn.gon}(X) = 1$ implies $\text{uni.irr}(X) = 1$), when d is large these two invariants should capture very different phenomena.

For the proof of Theorem A, we first show that if the degree of a correspondence is less or equal than $d-2$, then one can find on X a relatively large subvariety with bounded covering gonality; this is impossible for very general hypersurface. The method is essentially the same as [4] but the difference is that we work directly on the correspondence instead of passing to the Grassmannian.

In §3.2 we discuss some properties of correspondences with null trace and §3.3 is devoted to the proof of the main theorems.

3.2 Correspondences

In this section, we sketch some basic properties of correspondences following [3].

Let X and Y be smooth irreducible complex projective varieties of dimension n.

Definition 3.2.1. An correspondence of Y-degree m on $Y \times X$ is a reduced pure n-dimensional subvariety $\Gamma \subseteq Y \times X$ such that the projections $\pi_1 : \Gamma \to Y$, $\pi_2 : \Gamma \to X$ are generically finite dominant morphisms with $\deg(\pi_1) = m$.

Recall that for any correspondence $\Gamma \subseteq Y \times X$, one has *Mumford's trace map* (cf.[3]):

$$\text{Tr}_{X/Y} : H^0(X, K_X) \to H^0(Y, K_Y).$$

In brief, $\text{Tr}_{X/Y}(\omega) = \text{Tr}_{\Gamma/Y}(\pi_2^*\omega)$, where $\text{Tr}_{\Gamma/Y}$ is the trace map associated to the generically finite morphism $\Gamma \to Y$.

Definition 3.2.2. A correspondence $\Gamma \subseteq Y \times X$ has *null trace* to Y if the associated trace map is identically zero.

Using the Cayley-Bacharach properties, correspondences with null trace on a smooth hypersurface are analyzed by Bastianelli, Cortini and De Poi in [3, Theorem 2.5]. Their result is

Theorem 3.2.3. *Let $X \subseteq \mathbf{P}^{n+1}$ be a smooth hypersurface of degree $d \geq n+3$ and let*

$$\Gamma \subseteq Y \times X$$

be a correspondence of Y-degree m with null trace to Y. Let $y \in Y$ be a point such that dim $\pi_1^{-1}(y) = 0$ and let $\pi_1^{-1}(y) = \{(y, x_i) \in \Gamma \mid i = 1, \ldots, m\}$ where the x_i are distinct points. Then

$$m \geq d - n,$$

and if $m \leq 2d - 2n - 3$, then the 0-cycle $Z_y = \sum_{i=1}^{m} x_i$ lies on a line in $\mathbf{P}^{n+1}$.

We will work with the following

Set-up 3.2.4. Denote by $X \subseteq \mathbf{P}^{n+1}$ a very general smooth hypersurface of degree d, and suppose given a correspondence $\Gamma \subseteq \mathbf{P}^n \times X$ of $\mathbf{P}^n$-degree m. We assume that

$$d \geq 2n + 2 \text{ and } m \leq d - 2.$$

Corollary 3.2.5. *Assume that we are in the situation of 3.2.4. For general $y \in \mathbf{P}^n$, define Z_y as in the previous theorem. Then we have*

1. *$m \geq d - n$.*
2. *Z_y lies on a line $l_y \subseteq \mathbf{P}^{n+1}$.*

Proof. Notice that Γ has null trace to $\mathbf{P}^n$ because $H^0(\mathbf{P}^n, K_{\mathbf{P}^n}) = \{0\}$. Moreover the pair (d, m) satisfies the condition $m \leq 2d - 2n - 3$. Therefore Theorem 3.2.3 applies. □

3.3 Proofs

In this section, we give the proof of main theorems in the introduction. We will establish Theorem A first.

We assume until the end of the proof of Theorem A that we are in the situation of 3.2.4. Notice that any rational covering $X \dashrightarrow \mathbf{P}^n$ of degree δ gives rise to a correspondence of $\mathbf{P}^n$-degree δ on $\mathbf{P}^n \times X$. Hence by [4, Theorem C] we have

$$\mathrm{corr}(X) \leq \mathrm{irr}(X) = d - 1.$$

Therefore it suffices to show that $\mathrm{corr}(X) \geq d - 1$ and we will argue by contradiction.

Since we are in the situation of 3.2.4, by Corollary 3.2.5 one has a classifying map:

$$\phi : U \to \mathbf{G} = \mathbf{G}(1, n + 1).$$

Here U is the Zariski-open subset of $\mathbf{P}^n$ where the fiber $Z_y = \pi_1^{-1}(y)$ consists of m distinct points. Note that U being open in $\mathbf{P}^n$ is a rational variety itself. Another observation is that ϕ is a generically finite map onto its image because $\pi_2 : \Gamma \to X$ is generically finite.

Now we have the following diagram:

$$\begin{array}{ccccc} W' & \xrightarrow{\phi'} & W & \xrightarrow{\mu} & \mathbf{P}^{n+1} \\ {\scriptstyle \pi'}\big\downarrow & & {\scriptstyle \pi}\big\downarrow & & \\ U & \xrightarrow{\phi} & \mathbf{G} & & \end{array}$$

Here $\pi : W \to \mathbf{G}$ is the tautological $\mathbf{P}^1$-bundle on $\mathbf{G}$, $\mu : W \to \mathbf{P}^{n+1}$ is the evaluation map and

$$W' =_{\text{def}} \phi^* W$$

is the pullback of W via the classifying map ϕ.

Claim 3.3.1. *W' is an irreducible $n+1$-dimensional variety and $\psi =_{def} \mu \circ \phi'$ is dominant onto $\mathbf{P}^{n+1}$.*

Proof. Notice that $\pi' : W' \to U$ is a $\mathbf{P}^1$-bundle and U is irreducible, so W' must be irreducible. Since dim $\psi(W') \leq n+1$, it suffices to show that ψ is dominant. We prove this by contradiction. Suppose dim $\psi(W') \leq n$. Since $\Gamma \to X$ is dominant and an open subset of Γ is contained in W' by Corollary 3.2.5, this would imply that X contains $\psi(W')$ as an open subset. Therefore X is uniruled, but this is impossible since $\deg(X)$ is greater than $n+1$. $\square$

Proof of Theorem A. Recall that we are in the situation of 3.2.4, where $\Gamma \subseteq \mathbf{P}^n \times X$ is a correspondence of $\mathbf{P}^n$-degree $m \leq d-2$ by contradiction. Define Γ' to be the restriction of Γ to $U \times \mathbf{P}^n$. By Corollary 3.2.5, Γ' is a divisor in W' of relative degree m over U. Let X' be the full pre-image of X in W' so that X' is a divisor in W' of relative degree d over U. We can write

$$X' = \Gamma' + F,$$

where F is a divisor of relative degree $d - m \geq 2$ over U. Now fix any irreducible component $R \subseteq F$ that dominates U and view R as a reduced irreducible variety of dimension n. Thus R sits in a diagram

$$\begin{array}{ccccc} X & \longleftarrow & R & & \\ \big\downarrow & & \big\downarrow & & \\ \mathbf{P}^{n+1} & \xleftarrow[\psi=\mu\circ\phi']{} & W' & \longrightarrow & U \end{array} \tag{3.1}$$

and we have

$$0 < e =_{\text{def}} \deg(R \to U) \leq d - m. \tag{3.2}$$

Put

$$S =_{\text{def}} \psi(R) \subseteq X, \tag{3.3}$$

and let $s = \dim S$. Suppose first that $s = 0$, i.e. S consists of a single point $p \in X$. But this would imply $\deg(\Gamma \to U) = d-1$, which contradicts with our assumption. Therefore we may assume that $1 \leq s \leq n-1$.

Note next that cov.gon$(S) \leq e$. In fact, one can choose a rational subvariety $L \subseteq U$ of dimension s with the property that an irreducible component $R^* \subseteq R$ of the inverse image of L in R is generically finite over S. Since $\deg(R^* \to L) \leq e$, and since L is rational, we see that cov.gon$(R^*) \leq e$. Hence [4, lemma 1.9] applies to show that cov.gon$(S) \leq e$.

Now denote by $K_{W'/\mathbf{P}^{n+1}}$ the relative canonical bundle of ψ, and consider a general fiber $l = l_y$ of $(W' \to U)$. We assert: [1] [2]

$$l \cdot K_{W'/\mathbf{P}^{n+1}} = n. \tag{3.4}$$

Grant this for now. Since dim $\psi(R) = s$, by [4, Corollary A.6] we have

$$\mathrm{ord}_R(K_{W'/\mathbf{P}^{n+1}}) \geq n - s. \tag{3.5}$$

Hence we must have

$$n = l \cdot K_{W'/\mathbf{P}^{n+1}} \geq \mathrm{ord}_R(K_{W'/\mathbf{P}^{n+1}}) \cdot l \cdot R \geq (n-s) \cdot \deg(R \to U) \geq (n-s)e.$$

Now recall that we assume $s \geq 1$. Then it follows from the computations of Ein and Voisin [4, Proposition 3.8] that

$$e \geq \mathrm{cov.gon}(S) \geq d - 2n + s.$$

One finds that

$$d \leq 2n - s + e \leq 2n - s + \frac{n}{n-s} \leq 2n + 1.$$

which is impossible since $d \geq 2n + 2$.

It remains to prove (3.4). We consider the restriction of the tangent map $T_{W'} \to \psi^* T_{\mathbf{P}}$ to $l \cong \mathbf{P}^1$. By the Euler sequence, one has

$$\psi^* T_{\mathbf{P}}|_l \cong \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n} \oplus \mathcal{O}_{\mathbf{P}^1}(2).$$

For $T_{W'}|_l$, we have the following exact sequence:

$$0 \to T_{W'/U}|_l \to T_{W'}|_l \to \pi'^* T_U|_l \to 0.$$

The first term is isomorphic to $T_l \cong \mathcal{O}_{\mathbf{P}^1}(2)$, and the third term is isomorphic to $\mathcal{O}_{\mathbf{P}^1}^{\oplus n}$. Notice that this exact sequence of vector bundles splits because

$$\mathrm{Ext}_{\mathbf{P}^1}(\mathcal{O}_{\mathbf{P}^1}^{\oplus n}, \mathcal{O}_{\mathbf{P}^1}(2)) \cong H^1(\mathbf{P}^1, \mathcal{O}(2))^{\oplus n} = \{0\}.$$

Hence we have

$$T_{W'}|_l \cong \mathcal{O}_{\mathbf{P}^1}^{\oplus n} \oplus \mathcal{O}_{\mathbf{P}^1}(2).$$

Therefore the restriction of the tangent map to $l \cong \mathbf{P}^1$ becomes

$$\mathcal{O}_{\mathbf{P}^1}^{\oplus n} \oplus \mathcal{O}_{\mathbf{P}^1}(2) \to \mathcal{O}_{\mathbf{P}^1}(1)^{\oplus n} \oplus \mathcal{O}_{\mathbf{P}^1}(2),$$

[1] Notice that even though we are working on an open variety, this intersection product still makes sense because we are intersecting a divisor with the fiber of a proper map.

[2] Bastianelli pointed out to me that it is possible to avoid this assertion by passing to the Grassmannian and argue as in [4].

whose degeneracy locus is thus given by a linear form of degree n on $\mathbf{P}^1$. Since a general fiber doesn't lie in the ramification locus, we must have

$$l \cdot K_{W'/\mathbf{P}^{n+1}} = n.$$

□

Now we turn to the proof of Theorem B. We first establish a lemma connecting $\mathrm{corr}(X)$ and $\mathrm{uni.irr}(X)$.

Lemma 3.3.2. *Let X be an irreducible smooth projective variety, then*

$$\mathrm{uni.irr}(X) = \mathrm{corr}(X).$$

Proof. Let T be a smooth n-dimensional variety with two dominant rational maps

$$f : T \dashrightarrow \mathbf{P}^n, g : T \dashrightarrow X.$$

By considering the closure of the graph of f and g, we see that T maps onto a correspondence $\Gamma \subseteq \mathbf{P}^n \times X$ and $\deg(f)$ is a multiple of $\deg(\Gamma \to \mathbf{P}^n)$. Hence $\mathrm{uni.irr}(X) \geq \mathrm{corr}(X)$. The other inequality is obvious. □

Proof of Theorem B. By Lemma 3.3.2 and Theorem A, one has

$$\mathrm{uni.irr}(X) = \mathrm{corr}(X) = d - 1.$$

On the other hand, by [4, Theorem C] and [1, Lemma 2.2]

$$d - 1 = \mathrm{irr}(X) \geq \mathrm{stab.irr}(X) \geq \mathrm{uni.irr}(X),$$

we conclude that $\mathrm{stab.irr}(X) = \mathrm{uni.irr}(X) = d - 1$. □

Bibliography

[1] F. Bastianelli. On irrationality of surfaces in $\mathbb{P}^3$. *J. Algebra*, 488:349–361, 2017.

[2] F. Bastianelli, C. Ciliberto, F. Flamini, and P. Supino. Gonality of curves on general hypersufaces. *ArXiv e-prints*, July 2017.

[3] F. Bastianelli, R. Cortini, and P. De Poi. The gonality theorem of Noether for hypersurfaces. *J. Algebraic Geom.*, 23(2):313–339, 2014.

[4] F. Bastianelli, P. De Poi, L. Ein, R. Lazarsfeld, and B. Ullery. Measures of irrationality for hypersurfaces of large degree. *Compos. Math.*, 153(11):2368–2393, 2017.

[5] A. A. Beĭlinson, J. Bernstein, and P. Deligne. Faisceaux pervers. In *Analysis and topology on singular spaces, I (Luminy, 1981)*, volume 100 of *Astérisque*, pages 5–171. Soc. Math. France, Paris, 1982.

[6] Z. Błocki. Suita conjecture and the Ohsawa-Takegoshi extension theorem. *Inventiones mathematicae*, 193(1):149–158, 2013.

[7] G. Böckle and C. Khare. Mod l representations of arithmetic fundamental groups. II. A conjecture of A. J. de Jong. *Compos. Math.*, 142(2):271–294, 2006.

[8] Y. Brunebarbe. Semi-positivity from Higgs bundles. *arXiv:1707.08495*, 2017.

[9] N. Budur and M. Saito. Multiplier ideals, V-filtration, and spectrum. *arXiv preprint math/0305118*, 2003.

[10] J. Cao and M. Păun. Kodaira dimension of algebraic fiber spaces over abelian varieties. *Inventiones mathematicae*, 207(1):345–387, 2017.

[11] E. Cattani and A. Kaplan. Polarized mixed Hodge structures and the local monodromy of a variation of Hodge structure. *Invent. Math.*, 67(1):101–115, 1982.

[12] E. Cattani, A. Kaplan, and W. Schmid. Degeneration of Hodge structures. *Annals of Mathematics*, 123(3):457–535, 1986.

[13] N. Chen, F. Greer, and R. Yang. Nodal elliptic curves on K3 surfaces. *arXiv:2001.05104*, 2020.

[14] K. Corlette. Flat G-bundles with canonical metrics. *J. Differential Geom.*, 28(3):361–382, 1988.

[15] M. A. A. de Cataldo. Decomposition theorem for semi-simples. *J. Singul.*, 14:194–197, 2016.

[16] M. A. A. de Cataldo and L. Migliorini. The Hodge theory of algebraic maps. *arXiv preprint math/0306030*, 2003.

[17] M. A. A. de Cataldo and L. Migliorini. The Hodge theory of algebraic maps. *Ann. Sci. École Norm. Sup. (4)*, 38(5):693–750, 2005.

[18] M. A. A. de Cataldo and L. Migliorini. The decomposition theorem, perverse sheaves and the topology of algebraic maps. *Bull. Amer. Math. Soc. (N.S.)*, 46(4):535–633, 2009.

[19] M. A. A. de Cataldo and L. Migliorini. The perverse filtration and the Lefschetz hyperplane theorem. *Ann. of Math. (2)*, 171(3):2089–2113, 2010.

[20] P. Deligne. Théorème de Lefschetz et critères de dégénérescence de suites spectrales. *Inst. Hautes Études Sci. Publ. Math.*, (35):259–278, 1968.

[21] P. Deligne. *Équations différentielles à points singuliers réguliers*, volume 163. Springer, 1970.

[22] P. Deligne. Théorie de Hodge. II. *Inst. Hautes Études Sci. Publ. Math.*, (40):5–57, 1971.

[23] P. Deligne. Théorie de Hodge. III. *Inst. Hautes Études Sci. Publ. Math.*, (44):5–77, 1974.

[24] J.-P. Demailly. Complex analytic and differential geometry, 2007.

[25] V. Drinfeld. On a conjecture of Kashiwara. *Math. Res. Lett.*, 8(5-6):713–728, 2001.

[26] F. El Zein. Mixed Hodge structures. *Trans. Amer. Math. Soc.*, 275(1):71–106, 1983.

[27] F. El Zein and L. D. ung Tráng. Mixed Hodge structures. In *Hodge theory*, volume 49 of *Math. Notes*, pages 123–216. Princeton Univ. Press, Princeton, NJ, 2014.

[28] D. Gaitsgory. On de Jong's conjecture. *Israel J. Math.*, 157:155–191, 2007.

[29] M. Goresky and R. MacPherson. Intersection homology theory. *Topology*, 19(2):135–162, 1980.

[30] M. Goresky and R. MacPherson. Intersection homology. II. *Invent. Math.*, 72(1):77–129, 1983.

[31] P. Griffiths and W. Schmid. Recent developments in Hodge theory: a discussion of techniques and results. In *Discrete subgroups of Lie groups and applicatons to moduli (Internat. Colloq., Bombay, 1973)*, pages 31–127. 1975.

[32] Q. Guan and X. Zhou. A solution of an L2 extension problem with an optimal estimate and applications. *Annals of Mathematics*, pages 1139–1208, 2015.

[33] C. Hacon, M. Popa, and C. Schnell. Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun. In *Local and global methods in algebraic geometry*, volume 712 of *Contemp. Math.*, pages 143–195. Amer. Math. Soc., Providence, RI, 2018.

[34] R. Hotta, K. Takeuchi, and T. Tanisaki. *D-modules, perverse sheaves, and representation theory*, volume 236 of *Progress in Mathematics*. Birkhäuser Boston, Inc., Boston, MA, 2008. Translated from the 1995 Japanese edition by Takeuchi.

[35] M. Kashiwara. Semisimple holonomic D-modules. In *Topological field theory, primitive forms and related topics (Kyoto, 1996)*, volume 160 of *Progr. Math.*, pages 267–271. Birkhäuser Boston, Boston, MA, 1998.

[36] A. F. Lopez and G. P. Pirola. On the curves through a general point of a smooth surface in $\mathbf{P}^3$. *Math. Z.*, 219(1):93–106, 1995.

[37] R. MacPherson and K. Vilonen. Elementary construction of perverse sheaves. *Invent. Math.*, 84(2):403–435, 1986.

[38] T. Mochizuki. Asymptotic behaviour of tame harmonic bundles and an application to pure twistor D-modules. I. *Mem. Amer. Math. Soc.*, 185(869):xii+324, 2007.

[39] T. Mochizuki. Wild harmonic bundles and wild pure twistor D-modules. *Astérisque*, (340):x+607, 2011.

[40] M. Mustaţă and M. Popa. Hodge ideals. *Mem. Amer. Math. Soc.*, 262(1268):v+80, 2019.

[41] M. Păun and S. Takayama. Positivity of twisted relative pluricanonical bundles and their direct images. *Journal of Algebraic Geometry*, 27(2):211–272, 2018.

[42] T. Ryan and R. Yang. Nef cones of nested Hilbert schemes of points on surfaces. *International Mathematics Research Notices*, 2020(11):3260–3294, 2020.

[43] C. Sabbah. Polarizable Twistor D-modules. *Astérisque*, 300, 2005.

[44] M. Saito. Modules de Hodge polarisables. *Publications of the Research Institute for Mathematical Sciences*, 24(6):849–995, 1988.

[45] M. Saito. Mixed Hodge modules. *Publications of the Research Institute for Mathematical Sciences*, 26(2):221–333, 1990.

[46] M. Saito. *On Kollár's conjecture.* Kyoto University, Research Institute for Mathematical Sciences, 1990.

[47] W. Schmid. Variation of Hodge structure: the singularities of the period mapping. *Inventiones mathematicae*, 22(3):211–319, 1973.

[48] C. Schnell and R. Yang. Hodge modules and singular Hermitian metrics. *arXiv:2003.09064*, 2020.

[49] S. Schreieder. Stably irrational hypersurfaces of small slopes. *Journal of the American Mathematical Society*, 32(4):1171–1199, 2019.

[50] C. Simpson. Some families of local systems over smooth projective varieties. *Ann. of Math. (2)*, 138(2):337–425, 1993.

[51] C. Simpson. Mixed twistor structures. *arXiv preprint alg-geom/9705006*, 1997.

[52] C. T. Simpson. Higgs bundles and local systems. *Inst. Hautes Études Sci. Publ. Math.*, (75):5–95, 1992.

[53] B. Totaro. Hypersurfaces that are not stably rational. *J. Amer. Math. Soc.*, 29(3):883–891, 2016.

[54] J.-L. Verdier. Dualité dans la cohomologie des espaces localement compacts. *Séminaire Bourbaki*, 9:337–349, 1965.

[55] E. Viehweg et al. *Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces.* North-Holland Publ., 1983.

[56] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.

[57] R. Yang. On irrationality of hypersurfaces in $\mathbf{P}^{n+1}$. *Proceedings of the American Mathematical Society*, 147(3):971–976, 2019.

[58] R. Yang. L^2-minimal extensions over Hermitian symmetric domains. *arXiv:2006.15193*, 2020.

[59] F. E. Zein, D. T. Lê, and X. Ye. Decomposition, purity and fibrations by normal crossing divisors, 2018.

[60] S. Zucker. Hodge theory with degenerating coefficients: L2 cohomology in the Poincaré metric. *Annals of Mathematics*, pages 415–476, 1979.