Measuring the Irrationality of Abelian Surfaces and Complete Intersections

A Dissertation Presented

by

Nathan Chen

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2021

Copyright by
Nathan Chen
2021

Stony Brook University

The Graduate School

Nathan Chen

We, the dissertation committee for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Robert Lazarsfeld
Distinguished Professor of Mathematics

Samuel Grushevsky
Professor of Mathematics

Olivier Martin
James H. Simons Instructor, Department of Mathematics

Lawrence Ein
LAS Distinguished Professor of Mathematics
University of Illinois at Chicago

This dissertation is accepted by the Graduate School

Eric Wertheimer
Dean of the Graduate School

Abstract of the Dissertation

Measuring the Irrationality of Abelian Surfaces and Complete Intersections

by

Nathan Chen

Doctor of Philosophy

in

Mathematics

Stony Brook University

2021

In this dissertation, we make advances in the study of measures of irrationality for polarized abelian surfaces and codimension two complete intersections, which answer a number of questions posed in [7].

Table of Contents

Acknowledgements **v**

Introduction **1**

1 Measures of irrationality: background and survey **2**

1.1 Measures of irrationality 4
1.2 Beyond hypersurfaces of large degree 7

2 Abelian surfaces **11**

2.1 Examples 12
2.2 Kummer construction and even sections 14
2.3 Constructing maps from sections of $H^0(A, 2L)^+$ 17
2.4 Constructing elliptic fibrations with sections on Kummer K3 surfaces 23

3 Complete intersections **26**

3.1 Covering gonality and separating points 28
3.2 Reduction step 30
3.3 Multiplicative bounds for complete intersection curves 31
3.4 Curves on complete intersections 37
3.5 Multiplicative bounds for codimension two complete intersections 39

Bibliography **43**

Acknowledgements

First and foremost, I would like to express my gratitude to my advisor Rob Lazarsfeld for taking me on as a graduate student several years ago. At the time, I had just finished reading Shafarevich and was beginning to look at Hartshorne. The latter book made me feel as though I was throwing my intuition out the window and driving blind. The walls of algebraic geometry texts and definitions seemed like an endless maze designed to confuse anyone who accidentally wandered into its domain. Under Rob's expert guidance, I was able to find my sense of direction and avoid many of the major traps and pitfalls. Through his encouragement and valuable advice, I learned to appreciate and enjoy algebraic geometry, but this was just a part of the journey. Many of Rob's suggestions could probably be combined into a crash course of several topics:

"how to write a research paper";
"how to give research talks";
"how to be a professional mathematician";
$\vdots$

I hope that I managed to absorb some of his advice over these past few years. In any case, I am extremely thankful that I had the opportunity to work with and learn from Rob.

Thank you to my defense committee: Rob, Sam Grushevsky, Lawrence Ein, and Olivier Martin, for reading through a draft of my thesis.

Next, I want to thank all of my professors, collaborators, colleagues, and friends at Stony Brook, as well as others who have contributed to my journey in grad school. In addition to Rob, I would like to thank Mark de Cataldo, François Greer, Sam Grushevsky, Ljudmila Kamenova, Radu Laza, Olivier Martin, Christian Schnell, John Sheridan, David Stapleton, Jason Starr, Burt Totaro, Dror Varolin, Ruijie Yang, and Aleksey Zinger, from whom I have learned most of what I know mathematically. Furthermore, I would like to acknowledge Michael Albanese, Roberto Albesiano, Tim Alland, Frederik Benirschke, Jack Burkart, Qianyu Chen, Xujia Chen, Jae Ho Cho, Dahye Cho, Prithviraj Chowdhury, Matt Dannenberg, Deya Dasgupta, Ben Dozier, Marlon de Oliveira Gomes, Mohamed El Alami, Silvia Ghinassi, Lisandra Hernandez-Vazquez, Thorsten Herrig, Yoon-Joo Kim, Matt Lam, Kirill Lazebnik, Lisa Marquand, Aleks Milivojevic, Cristian Minoccheri, Jordan Rainone, Tim Ryan, Tobias Shin, Jiasheng Teh, Ying Hong Tham, Sasha Viktorova, Mads Villadsen, Ben Wu, Hang Yuan, and Mu Zhao. It has been amazing to be a part of the tight-knit community of grad students at Stony Brook.

I also want to thank the staff in the math department, including Lynne Barnett, Christine Gathman, Donna McWilliams, Lucille Meci, Pat Tonra, and Diane Williams, for all of their help and assistance throughout my time at Stony Brook.

Finally, I would like to thank my parents and my siblings, Percy and Winnie, for all of the support and encouragement they have provided. My parents in particular have always been welcoming and happy to have me back from grad school, whether it was for holiday breaks or family gatherings. This thesis is dedicated to them.

Introduction

A fundamental idea in algebraic geometry is that it is very useful and rewarding to study algebraic varieties up to birational equivalence. Recall that two irreducible varieties X and Y are said to be *birationally isomorphic* if there are Zariski open subsets $U \subset X$ and $V \subset Y$ such that $U \cong V$. This is a natural concept to consider since many natural geometric constructions – for example, blowing up a subvariety – preserve the birational equivalence class of a variety. In the late 1800s and early 1900s, there were many significant advances in our understanding of birational transformations and various birational invariants of algebraic surfaces, which led to the Enriques-Kodaira classification of compact algebraic surfaces (see for instance [11]). In higher dimension, the past 50 years have witnessed vast progress in birational geometry, culminating in the near completion of the minimal model program ([12], [27]).

From the birational viewpoint, the simplest varieties are those that are rational. By definition, an n-dimensional variety is rational if it is birationally equivalent to $\mathbb{P}^n$. In dimension ≤ 2, the situation is fairly well-understood. For example, an algebraic surface is rational if and only if it is the blow-up of $\mathbb{P}^2$, $\mathbb{P}^1 \times \mathbb{P}^1$, or a Hirzebruch surface. However, the situation in dimension ≥ 3 is already much more subtle (see [30], [3], and [18]), and has been the subject of current active research in the past 20 years. Meanwhile, there has been renewed interest in approaching these rationality questions from a different point of view. Suppose that the nonrationality of a given projective variety X is known (perhaps for simple reasons). This leads to the natural question:

> Can we quantify "how far" X is from being rational?

There are several birational invariants that have been proposed and studied with this question in mind. In this thesis, we contribute to a growing collection of results along these lines. In Chapter 1, we give an exposition of measures of irrationality and discuss the general background. Chapter 2 gives a new approach to the main theorem proved in [15] about the degree of irrationality of abelian surfaces. Finally, we consider complete intersections in Chapter 3 and prove that their measures of irrationality behave multiplicatively in the codimension 2 case.

Chapter 1
Measures of irrationality: background and survey

In this chapter, we will survey known results and give some background to irrationality problems. The story begins with the classical question of determining which varieties are rational. Historically, this began with the study of coarse invariants such as the space of holomorphic forms, the space of m-canonical global sections, and the Lüroth problem for surfaces, which paved the way for the classification of algebraic surfaces. Although questions concerning rationality and the behavior of these invariants in higher dimension are extremely subtle, there are some elementary obstructions to rationality. For a smooth projective variety X, recall that we have:

Definition. The *m-th plurigenus* of X is

$$p_m(X) =_{\text{def}} \dim H^0(X, \mathcal{O}_X(mK_X)) = 0.$$

It is then well-known that:

Theorem 1.1. *The spaces $H^0(X, \mathcal{O}_X(mK_X)$ are birational invariants.*

Since these vanish for projective space, it follows that if X is rational, then $p_m(X) = 0$ holds for all $m > 0$. Unfortunately, all *Fano* varieties (varieties whose anticanonical bundle $-K_X$ is ample) satisfy this condition automatically, so the plurigenera are simply not fine enough to distinguish the rational varieties among all Fanos. In fact, after much work and progress we now know that most Fano varieties are not rational. In order to better understand the rationality question in higher dimension, a whole host of conditions which are weaker than rationality were developed and studied:

Definition. Let X be a complex projective variety of dimension n. We say that X is:

- *unirational* if there is a dominant rational map $\mathbb{P}^n \dashrightarrow X$.
- *uniruled* if there is a dominant rational map $Y \times \mathbb{P}^1 \dashrightarrow X$ for some variety Y of dimension $n-1$.
- *ruled* if it is birational to a variety of the form $Y \times \mathbb{P}^1$.
- *stably rational* if $X \times \mathbb{P}^m$ is rational for some $m \geq 0$.

- *birationally rigid* if there are no rational fibrations $X \dashrightarrow V$ to a lower dimensional variety such that the general fiber has Kodaira dimension $-\infty$, and any birational self-map $X \dashrightarrow X$ extends to an isomorphism (there is a slightly different definition for prime Fano varieties).

It is helpful to survey the picture for hypersurfaces. Suppose

$$X_d \subset \mathbb{P}^{n+1}$$

is a smooth complex hypersurface of dimension n and degree d. By the adjunction formula, the canonical bundle of X is $K_X \cong \mathcal{O}_X(d-n-2)$. From this, we see:

(i) If $d = 2$, then projection from a point shows that X is rational.

(ii) Let us look at the case $d = 3$. If $n = 2$, it is well-known that every smooth cubic has 27 lines and can be realized as the blow-up of $\mathbb{P}^2$ in 6 points. For $n = 3$, Clemens and Griffiths [18] used the intermediate Jacobian to prove the longstanding conjecture that a cubic threefold is not rational. When $n = 4$, cubic fourfolds have attracted a great deal of attention [28], but the rationality question remains open for the general member.

(iii) For $d = 4$ and $n = 3$, Iskovskikh and Manin [30] used the Noether-Fano method to prove that quartic threefolds are birationally (super)rigid. This was later extended to several other classes of Fano varieties, including smooth hypersurfaces with $d = n + 1$ [22]. In particular, it follows that these varieties are not rational since the Cremona group of birational transformations of $\mathbb{P}^n$ contains much more than just linear automorphisms of $\mathbb{P}^n$ for $n \geq 2$.

(iv) When $d \geq n + 2$, it follows that $K_X \cong \mathcal{O}_X(d-n-2)$ is either trivial or very ample, so we have nonvanishing $p_1(X) \neq 0$ of the first plurigenus and X is therefore irrational.

In general, when $3 \leq d \leq n+1$ the canonical bundle of X is negative and thus $p_m(X) = 0$ for all $m > 0$.

In his pioneering work [33], Kollár used specialization to positive characteristic arguments to show:

Theorem 1.2. *If $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a hypersurface of degree $d \geq 2\lceil (n+3)/3 \rceil$, then X is not ruled and therefore not rational.*

This was later improved by Totaro [49] and subsequently Schreieder [45], who proved:

Theorem 1.3. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a hypersurface of degree $d \geq \log_2 n + 2$. Then X_d is not stably rational.*

(Schreieder's theorem actually holds for hypersurfaces over any algebraically closed field of characteristic not equal to 2.) As of now, this is the strongest result regarding nonrationality of hypersurfaces.

1.1. Measures of irrationality

In the last few years, there has been growing interest in studying (non)rationality from a complementary direction:

> Given a projective variety X whose nonrationality is known, can we measure how far it is from being rational?

In dimension 1, recall the:

Definition. The *gonality* of a smooth algebraic curve C, denoted $\operatorname{gon}(C)$, is the minimal degree of a branched covering $C \to \mathbb{P}^1$. For a possibly singular but reduced and irreducible curve C, we define the gonality of C to be that of its normalization.

It follows immediately that if C is a smooth curve, then $\operatorname{gon}(C) = 1$ if and only if $C \cong \mathbb{P}^1$. There are many situations in which the geometry of C is governed by its gonality rather than its genus. Curves of genus $g \geq 1$ with $\operatorname{gon}(C) = 2$ are called *hyperelliptic*; they are the Riemann surfaces associated to equations on $\mathbb{C}^2$ of the form $w^2 = f_{2g+1}(z)$ (where f_{2g+1} is a polynomial of degree $2g+1$) and this fact controls their geometry.

It is thus natural to try to compute the gonality of various classes of curves arising from geometric constructions. For instance, a classical theorem of Noether [43] describes the gonality of plane curves:

Theorem 1.4 (Noether). *Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 4$. Then*

$$\operatorname{gon}(C) = d - 1,$$

and furthermore every map $C \to \mathbb{P}^1$ of degree $d-1$ is given by projection from a point.

It is instructive to recall why this holds. We begin with the simple observation that positivity properties of the canonical bundle of a smooth projective curve C give rise to lower bounds on the gonality of C (see §1 of [7]).

Definition. We say that sections of a line bundle L on a smooth variety X *separate r points on an open set* if there exists a Zariski open subset $U \subset X$ such that for any r distinct points $p_1, \ldots, p_r \in U$, the restriction map

$$H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_{\{p_1,\ldots,p_r\}})$$

is surjective.

Lemma 1.5. *Let C be a smooth projective curve of genus g such that the canonical bundle K_C separates p points on an open set. Then*

$$\operatorname{gon}(C) \geq p + 1.$$

Proof. We will prove the contrapositive; assume that C has gonality $\leq p$. Then C carries a base point free line bundle A of degree $\leq p$ with at least two global sections. Let Z be a general effective divisor with $A \cong \mathcal{O}_C(Z)$, and consider the ideal sheaf sequence of Z twisted by K_C. The long exact sequence on cohomology gives

$$\cdots \to H^0(K_C) \xrightarrow{\alpha} H^0(K_C \otimes \mathcal{O}_Z) \to H^1(K_C \otimes A^{-1}) \to H^1(K_C) \to 0$$

and Serre duality implies that $h^1(K_C) = h^0(\mathcal{O}_C) = 1$ whereas $h^1(K_C \otimes A^{-1}) = h^0(A) \geq 2$. Therefore, the map α cannot be surjective. Since this holds for a general effective Z, the canonical bundle K_C cannot separate p points on an open set. $\square$

Remark 1.6. We may rephrase the proof of this lemma in terms of Geometric Riemann-Roch. Essentially what we are saying is that any divisor A of degree p imposing independent conditions on sections of K_C spans a plane $\overline{\varphi_K(A)}$ of dimension $p-1$ in canonical space. Geometric Riemann-Roch then implies that the dimension of the linear system $|A|$ is equal to

$$\deg(A) - \dim \overline{\varphi_K(A)} - 1.$$

So the divisor A imposing independent conditions on K_C has a linear system of dimension $\dim|A| = 0$.

Proof of Noether's theorem. Let $C \subset \mathbb{P}^2$ be a smooth plane curve of degree $d \geq 4$. By the adjunction formula, the canonical bundle is given by $K_C \cong \mathcal{O}_C(d-3)$ so sections of K_C separate $(d-2)$-points. For instance, one can choose lines passing through any subset of $(d-3)$ points but missing the last point. By the lemma, this implies that $\mathrm{gon}(C) \geq d-1$. In fact, since $(d-1)$ points in $\mathbb{P}^2$ fail to impose independent conditions on $\mathcal{O}_{\mathbb{P}^2}(d-3)$ if and only if they are collinear, we see that every map $C \to \mathbb{P}^1$ of degree $d-1$ is given by projection from a point. $\square$

In a different direction, Abramovich [1] has given bounds for the gonality of modular curves. For a congruence subgroup $\Gamma \subset \mathrm{PSL}_2(\mathbb{Z})$, he showed that the gonality of the corresponding modular curve X_Γ is bounded from below by

$$\mathrm{gon}(X_\Gamma) \geq \frac{7}{800} D_\Gamma,$$

where $D_\gamma = [\mathrm{PSL}_2(\mathbb{Z}) : \Gamma]$ is the index. In particular, when $\Gamma = \Gamma_0(N)$ this gives a linear lower bound on the gonality of $X_{\Gamma_0(N)}$ in terms of N. By work of Poonen [44], there are similar results in characteristic p for modular curves which are certain quotients of a special moduli space parameterizing elliptic curves (with some extra data in terms of level structures).

There are several ways of generalizing the notion of gonality to give higher dimensional birational invariants. In this thesis, we will focus on two of these "measures of irrationality":

Definition. Let X be an irreducible complex projective variety. We define the *degree of* *irrationality* of X to be

$$\operatorname{irr}(X) = \min\left\{\delta > 0 \;\middle|\; \begin{array}{c}\exists \text{ degree } \delta \text{ rational covering} \\ X \dashrightarrow \mathbb{P}^{\dim X}\end{array}\right\}.$$

The *covering gonality* of X is defined as

$$\operatorname{cov.gon}(X) = \min\left\{c > 0 \;\middle|\; \begin{array}{c}\text{Given a general point } p \in X, \ \exists \text{ irred.} \\ \text{curve } C \subseteq X \text{ through } p \text{ with } \operatorname{gon}(C) = c\end{array}\right\}.$$

Remark 1.7. It follows from the definitions that:

$$\begin{aligned} \operatorname{irr}(X) = 1 &\iff X \text{ is rational;} \\ \operatorname{cov.gon}(X) = 1 &\iff X \text{ is uniruled.} \end{aligned}$$

By considering the preimage of lines coming from the base of a rational covering $X \dashrightarrow \mathbb{P}^{\dim X}$, we also see that

$$\operatorname{irr}(X) \geq \operatorname{cov.gon}(X).$$

The degree of irrationality was first introduced by Heinzer and Moh in an algebraic setting, in terms of the transcendence degree of the function field of a variety. Yoshihara later computed the degree of irrationality for several types of surfaces ([54], [55], [56]), following classification lines. There were a few sporadic results towards covering gonality as well ([37], [25]). For a thorough summary, see [7] and [40]. Beginning with the work of [4], [6], [7], and [51], the last few years have seen a rejuvenation of activity and progress. We will now summarize existing results for hypersurfaces

$$X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$$

of degree d and dimension $n \geq 2$. Note that projecting from a point on X gives a rational map $\varphi\colon X \dashrightarrow \mathbb{P}^n$ of degree $d-1$, so there is always an upper bound of $\operatorname{irr}(X) \leq d-1$. When the degree of X is large, Bastianelli, De Poi, Ein, Lazarsfeld, and Ullery [7] (building on the work in [6]) proved that this is sharp:

Theorem 1.8 ([7], Theorem C)**.** *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a very general smooth hypersurface of dimension n and degree $d \geq 2n+1$. Then*

$$\operatorname{irr}(X) = d - 1.$$

Furthermore, if $d \geq 2n+2$, then any rational mapping

$$f : X \dashrightarrow \mathbb{P}^n$$

with $\deg(f) = d - 1$ *is birationally equivalent to projection from a point of X.*

In the same paper, they also gave bounds for the covering gonality:

Theorem 1.9 ([7], Theorem A)**.** *If $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a smooth hypersurface of dimension n and degree $d \geq n+2$, then*

$$\operatorname{cov.gon}(X) \geq d - n.$$

One can view these results as a generalization of Noether's theorem (which concerned plane curves). Bastianelli, Ciliberto, Flamini, and Supino [5] improved upon the second theorem by showing that for $d \gg 0$, the covering gonality for very general hypersurfaces behaves like $\operatorname{cov.gon}(X) \approx d - 2\sqrt{n}$.

Many other notions of rationality such as rational connectedness, stable rationality, and unirationality also admit quantitative counterparts. For instance, R. Yang [53] has used similar techniques to those established in [7] to compute the natural analogues of stable rationality and unirationality for very general hypersurfaces $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ of large degree.

A common theme in [7] is that the positivity properties of the canonical bundle yield lower bounds for measures of irrationality. In order to explain this, we recall:

Definition (Mumford's trace map)**.** Let $f \colon X \dashrightarrow Y$ be a dominant rational map between smooth n-dimensional projective varieties X and Y. In [42], Mumford defines a linear *trace map*

$$\operatorname{Tr}_f \colon H^0(X, K_X) \to H^0(Y, K_Y).$$

For $\eta \in H^0(X, K_X)$ and a general $y \in Y$, this is defined by

$$\operatorname{Tr}_f(\eta)(y) =_{\text{def}} \sum_{x \in f^{-1}(y)} \eta(x).$$

Note that the existence of such a map is nontrivial (see [47, §2.3] for a nice exposition). We will now use this to show:

Proposition 1.10. *Assume that sections of K_X separate r points on an open set for some positive integer r. If*

$$f \colon X \dashrightarrow Y$$

is a rational covering to a variety Y with $H^0(Y, K_Y) = 0$, then $\deg(f) > r$.

Proof. Suppose there exists such a map $f \colon X \to Y$ with $\deg(f) \leq r$. Then we may choose a section of $H^0(X, K_X)$ which vanishes at all but one of the points in a general fiber of f. Tracing this section forward to Y gives a nonzero form in $H^0(Y, K_Y)$, which is a contradiction. Therefore, the map f must have degree $> r$. □

1.2. Beyond hypersurfaces of large degree

It is natural to ask what happens for varieties where the canonical bundle is trivial or even negative. For hypersurfaces of small degree, there are limited results. In joint work with Stapleton [16], we gave the first examples of Fano varieties (where $d \leq n+1$) with arbitrarily large degrees of irrationality:

Theorem 1.11. *Let $X_{n,d} \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a very general hypersurface of dimension n and degree d. If $d \geq n+1-\sqrt{n+2}/4$, then*

$$\operatorname{irr}(X_{n,d}) \geq \frac{\sqrt{n+2}}{4}.$$

In fact, we proved a stronger statement: the bound above holds for minimal degree maps to a *ruled variety.* By work of Iskovskih and Manin [30] (using the Noether-Fano method), it is known that a general smooth quartic threefold X has a trivial birational automorphism group, so the degree of irrationality of a general smooth quartic threefold is 3. Our result gives the first examples of rationally connected varieties X with $\operatorname{irr}(X) \geq 4$. The main idea of our argument is to degenerate to positive characteristic in the spirit of [33] and take advantage of the fact that certain cyclic covers in characteristic p carry a lot of $(n-1)$-forms.

Moving on to varieties where the canonical bundle is trivial, it makes sense to consider abelian varieties. As a corollary to estimating the holomorphic length of an abelian variety, Alzati and Pirola [2] showed:

Theorem 1.12. *Let A be an abelian variety of dimension g. Then* $\operatorname{irr}(A) \geq g+1$.

When $g = 2$ and A is an abelian surface, this proves that $\operatorname{irr}(A) \geq 3$; in other words, there are no maps $A \dashrightarrow \mathbb{P}^2$ of degree 2 (the latter statement also follows geometrically from Lemma 2.2). In the last year or so, Colombo, Martin, Naranjo, and Pirola [20] showed that for $g \geq 3$, the degree of irrationality of a very general abelian variety of dimension g is bounded from below by $(3g+1)/2$. This was later improved in the thesis of Martin for abelian varieties which do not admit certain polarizations of low degree:

Theorem 1.13 ([40], Theorem 4.2.11)**.** *If $g \geq 2$, and A is an abelian variety of dimension g which admits no polarizations of degree $d \mid f(g)$ (for some function f), then* $\operatorname{irr}(A) \geq 2n$.

There is a more concrete statement if we specialize to the case $g = 2$ (see [38]):

Theorem 1.14. *Let A be a very general abelian surface with a polarization of type* $(1, d)$*. If $d \nmid 6$, then* $\operatorname{irr}(A) \geq 4$.

On the other hand, for the covering gonality of an abelian variety, Voisin [51] used cycle-theoretic methods to show:

Theorem 1.15. *Let A be a very general abelian variety of dimension g. If $g \geq 2^{k-2}(2k-1)+(2^{k-2}-1)(k-2)$, then* $\operatorname{cov.gon}(A) \geq k+1$.

Another way of interpreting the result above is that the covering gonality of a very general abelian variety A of dimension g is bounded from below by some function $h(g)$, where $h(g)$ grows like $\log(g)$. In particular, the theorem of Voisin settled a conjecture of [7] in the affirmative. In the same paper, Voisin also conjectured a stronger linear bound in g, which was subsequently proved by Martin [39]:

Theorem 1.16. *If A is a very general abelian variety of dimension g, then*

$$\operatorname{cov.gon}(A) \geq \lceil \frac{1}{2} g + 1 \rceil.$$

This summarizes the story about lower bounds for measures of irrationality on abelian varieties.

On the other hand, there has been little progress in finding interesting upper bounds. It is not clear how measures of irrationality depend on the polarization of an abelian variety. For instance, one may ask:

Question 1.17. For a very general abelian variety A_d of dimension g with a polarization L_d of type $(1, \ldots, 1, d)$, how does $\operatorname{irr}(A)$ behave as $d \to \infty$?

In the first part of this thesis, we consider the case of abelian surfaces and prove that the degree of irrationality on a very general $(1, d)$-polarized abelian surface is uniformly bounded from above:

Theorem 1.18. *For an abelian surface $A = A_d$ with Picard number $\rho = 1$, one has*

$$\operatorname{irr}(A) \leq 4.$$

The original argument for this appeared in [15]. In joint work with Stapleton [16], we extended this bound to all complex abelian surfaces. A new proof of this stronger result appears in §2.4 and makes use of elliptic fibrations on Kummer K3 surfaces. Together with the theorem of Martin [38] stated above, this proves that most abelian surfaces have degree of irrationality equal to 4. To prove the theorem above, we construct explicit rational maps $A \dashrightarrow \mathbb{P}^2$ of degree 4. The key ingredient in our proof involves even sections of symmetric line bundles, which vanish to higher order (than one would naively expect) at the two torsion points of A.

Returning to hypersurfaces, a logical next step is to estimate measures of irrationality for complete intersection varieties. For a smooth complete intersection

$$X := X_{a_1} \cap \cdots \cap X_{a_e} \subset \mathbb{P}^{n+e}$$

of dimension n and type $(a_1, \ldots, a_e)$, multiplicative upper bounds are easily realized by projection from linear subspaces of complementary dimension. Similar to what we saw for hypersurfaces, we can apply a generic projection from e general points on X to show that

$$\operatorname{irr}(X) \leq \prod_{i=1}^{e} a_i - e.$$

For lower bounds, the same techniques in [7] give a naive bound of

$$\operatorname{cov.gon}(X) \geq \sum_{i=1}^{e} a_i - n - e + 1.$$

Notice that these lower bounds for the covering gonality are *additive* in the degrees of the defining equations. In positive characteristic, Smith [46] has also given additive bounds for the covering gonality of complete intersections using cycle-theoretic arguments.

However, it has been conjectured [7, §4] that there should be lower bounds on the irrationality invariants of complete intersections which are *multiplicative* in the degrees of the defining equations. For instance, in dimension $n = 1$ there is a theorem of Lazarsfeld [34, Exercise 4.12] which shows precisely this:

Theorem 1.19. *Let $C \subset \mathbb{P}^{e+1}_{\mathbb{C}}$ be a complete intersection curve of type $(a_1, a_2, \ldots, a_e)$ with $2 \leq a_1 \leq \cdots \leq a_e$. Then the gonality of C is bounded from below by*

$$\operatorname{gon}(C) \geq (a_1 - 1)a_2 \cdots a_e.$$

This result turns out to be sharp in some cases. The idea of the proof is to fit the complete intersection curve C into a complete intersection surface S and reach a contradiction by constructing some Bogomolov unstable vector bundle on S. Further refinements due to Hotchkiss, Lau, and Ullery [29] show that when $4 \leq a_1 < a_2 \leq \cdots \leq a_e$ holds, the gonality of the curve C is realized by projection from a suitable linear subspace. For higher dimensional complete intersections, there has been little progress. As a first step, Stapleton [47] gave bounds for codimension two complete intersections that were superlinear (see §3). Stapleton and Ullery [48] then computed the degree of irrationality for codimension two complete intersections of type $(2, d)$ and $(3, d)$.

In the second part of this thesis, we show that both the covering gonality and the degree of irrationality on very general codimension two complete intersections are multiplicative:

Theorem 1.20. *Let $X \subset \mathbb{P}^{n+2}_{\mathbb{C}}$ be a very general smooth complete intersection of type (a, b) and dimension $n \geq 2$. If $a, b \geq 9n$, then*

$$\operatorname{cov.gon}(X) \geq \frac{2}{3(n+1)^2} \cdot ab.$$

We strongly believe that there should be multiplicative bounds for complete intersections of any dimension n and any codimension e.

Some parts of this thesis are contained in two of the author's papers: [15] and [14].

Chapter 2
Abelian surfaces

In this chapter, we will give an alternative proof of the result from [15], namely the degree of irrationality of a very general polarized abelian surface is uniformly bounded from above, independent of the degree of the polarization. Abelian surfaces and K3 surfaces are natural families of varieties to consider because their canonical bundles are trivial, so the techniques of [7] do not apply. As mentioned in Chapter 1, Alzati and Pirola [2] have shown that $\mathrm{irr}(A) \geq n+1$ holds for any abelian variety of dimension n. In the case $n = 2$, Yoshihara [55] proved that $\mathrm{irr}(A) = 3$ for any abelian surface A containing a smooth curve of genus 3 (see Example (2.5)). In particular, this holds for very general $(1,2)$-polarized abelian surfaces.

More generally, let $A = A_d$ be an abelian surface carrying a polarization $L = L_d$ of type $(1,d)$ and assume that $\mathrm{NS}(A) \cong \mathbb{Z}[L]$. In his thesis, Stapleton gave sublinear upper bounds:

Theorem 2.1 ([47], Theorem 5.2)**.** *There is a positive constant C such that*

$$\mathrm{irr}(A) \leq C \cdot \sqrt{d}$$

for $d \gg 0$.

There are similar bounds for very general polarized K3 surfaces of genus g. In [7], it was conjectured that equality holds asymptotically in both cases. For K3 surfaces, as far as we can tell the conjecture seems plausible. However, in a much earlier paper, Keum [32] had shown indirectly that on abelian surfaces this is false by proving that every algebraic Kummer surface is the K3 cover of some Enriques surface S. Since Enriques surfaces have degree of irrationality equal to 2 (more precisely Enriques proved that they are birational to branched double covers of the plane [23]), one can use Keum's result to factor through a series of degree 2 maps

$$A \dashrightarrow A/\iota \dashrightarrow S \dashrightarrow \mathbb{P}^2,$$

which shows that $\mathrm{irr}(A) \leq 8$.

The main result of this chapter is:

Theorem 1.18 ([15], Theorem 1.1)**.** For an abelian surface $A = A_d$ with Picard number $\rho = 1$, one has

$$\mathrm{irr}(A) \leq 4.$$

This bound will turn out to be sharp for most abelian surfaces; degree 2 rational maps from A to $\mathbb{P}^2$ cannot occur, and a recent theorem of Martin [38] proves that for $(1,d)$-polarized abelian surfaces A, there are no maps $A \dashrightarrow \mathbb{P}^2$ of degree 3 as long as $d \nmid 6$. The cases where $d = 1$ (principally polarized abelian surfaces - here the general member is a Jacobian of a genus two curve), $d = 3$, and $d = 6$ remain open.

In §2.1, we collect a few examples and results that are in the literature. In §2.2, we recall the Kummer construction of a K3 surface associated to an abelian surface, and review some properties of its line bundles. We then use these properties in §2.3 to construct rational maps $A \dashrightarrow \mathbb{P}^2$ of degree 4 using sections of $\mathcal{O}_A(2L)$, where L is a polarization of type $(1,d)$ (this is the original argument which appeared in [15]). Finally, in §2.4 we use sections of the line bundle $\mathcal{O}_A(L)$ to exhibit elliptic fibrations on K with sections, which can then be combined with a result of Yoshihara (Proposition 2.4) to give a simplified proof of Theorem 1.18.

2.1. Examples

In this section, we give several examples and lemmas concerning the behavior of rational maps. We first begin with a simple geometric observation (this also follows from [2] for $n = 2$):

Lemma 2.2. *There are no rational dominant maps $A \dashrightarrow \mathbb{P}^2$ of degree 2.*

Proof. Suppose there exists such a map f. We have the following diagram

$$\begin{tikzcd} & & A^{[2]} \arrow[r, "s"] & A \\ A \arrow[r, dashed, "f"] & \mathbb{P}^2 \arrow[ur, dashed, "g"] \arrow[r, dashed, "h"'] & K^{[2]}(A) \arrow[u, hook] & =: s^{-1}(0) \end{tikzcd}$$

where g is the pullback map on 0-cycles, $A^{[2]}$ is the Hilbert scheme of 2 points on A, and s is given by summation composed with the Hilbert-Chow morphism. Since the rational map $s \circ g$ can be extended to a morphism (see [13, Theorem 4.9.4]), it must be constant. This implies that $\overline{\mathrm{Im}(g)}$ is contained in a fiber $s^{-1}(0)$, which is a smooth Kummer K3 surface $K^{[2]}(A)$. Since g is injective, it descends to an injective (and hence birational) map $h\colon \mathbb{P}^2 \dashrightarrow K^{[2]}(A)$, yielding a contradiction. $\square$

Example 2.3. Let $f\colon V \dashrightarrow C$ be a dominant rational map from a smooth variety V of dimension n to a smooth curve C. We will show that $\mathrm{irr}(V) \geq \mathrm{gon}(C)$. Fix a rational map $g\colon V \dashrightarrow \mathbb{P}^n$ of degree $d = \mathrm{irr}(V)$ (i.e. g realizes the degree of irrationality of V). This induces a nonconstant map

$$\mathbb{P}^n \dashrightarrow \mathrm{Sym}^d C \quad \text{defined by} \quad x \mapsto \sum_{p \in \{g^{-1}(x)\}} f(p),$$

and the image will be contained in the fiber of the Abel-Jacobi map $\mathrm{Sym}^d(C) \to \mathrm{Pic}^d(C)$ (since any rational map from projective space to an abelian variety must necessarily be constant). This means that C carries a g^r_d for some $r \geq 1$; in particular $\mathrm{gon}(C) \leq d = \mathrm{irr}(V)$.

Following [55], we will now prove that an abelian surface containing a smooth curve of genus 3 has degree of irrationality 3. This result hinges upon the following

Proposition 2.4 (Yoshihara)**.** *Let $f\colon S \to C$ be a surjective morphism from a smooth surface S to a smooth curve C such that the general fiber F is irreducible. Let $g(F)$ denote the genus of F.*

1. *If $g(F) = 0$, then $\mathrm{irr}(S) = \mathrm{irr}(C)$.*
2. *If $g(F) = 1$ and f has a section, then $\mathrm{irr}(S) \leq 2\,\mathrm{gon}(C)$.*
3. *If $g(F) \geq 2$ and $\mathrm{gon}(F) = 2$, then $\mathrm{irr}(S) \leq 2\,\mathrm{gon}(C)$.*
4. *If $g(F) = 3$, $\mathrm{gon}(F) \neq 2$, and f has a section, then $\mathrm{irr}(S) \leq 3\,\mathrm{gon}(C)$.*

Proof. For part (1), by the hypothesis $g(F) = 0$ we know that S is birational to $C \times \mathbb{P}^1$ and so $\mathrm{irr}(C \times \mathbb{P}^1) \leq \mathrm{gon}(C)$ (consider a product of maps). By Example 2.3, it follows that $\mathrm{irr}(C \times \mathbb{P}^1) = \mathrm{gon}(C)$.

We will now prove parts (2), (3), and (4) simultaneously. The idea is that in each situation, there is a natural low degree map $F \dashrightarrow \mathbb{P}^1$ (either some sort of hyperelliptic map or projection from a point), and the assumptions on F together with the existence of a section Γ allow us to glue the maps across the fibers of f. Let K_S be the canonical bundle on S and let Γ be the section in parts (2) and (4). In each part, we will consider a different sheaf $\mathcal{F}$, which is defined as

$$\underbrace{\mathcal{O}_S(2\Gamma)}_{(2)}, \quad \underbrace{\mathcal{O}_S(K_S + F)}_{(3)}, \quad \underbrace{\mathcal{O}_S(K_S - \Gamma)}_{(4)}.$$

Then $f_*\mathcal{F}$ is a coherent sheaf on C, and there is a rational map

$$g\colon S \dashrightarrow \mathbb{P}(f_*\mathcal{F})$$

which is given by the linear system $H^0(F, \mathcal{F}\big|_F)$ on the general fiber F. It is straightforward to check that the image of the map g is a ruled surface over C (which has degree of irrationality equal to $\mathrm{gon}(C)$). The proposition then follows immediately.

□

Example 2.5. Yoshihara [55] has shown that an abelian surface A which contains a smooth curve C of genus 3 has degree of irrationality 3. Given such a curve C, the adjunction formula shows that $C^2 = 4$ and Nakai-Moishezon together with translations under the group law imply that C is ample. We then observe by Riemann-Roch and Kodaira vanishing that

$$h^0(\mathcal{O}_A(C)) = \chi(\mathcal{O}_A(C)) = \frac{1}{2}C^2 - \chi(\mathcal{O}_A) = 2.$$

The complete linear series $|\mathcal{O}_A(C)|$ gives a rational map $A \dashrightarrow \mathbb{P}^1$ with four base points (corresponding to the fact that $C^2 = 4$). After blowing up these points, we arrive at a morphism $f\colon \tilde{A} \to \mathbb{P}^1$, which has four sections given by the exceptional divisors. We can now apply Proposition 2.4 and Lemma 2.2 to conclude that $\operatorname{irr}(A) = 3$.

The following lemma will be useful later on, as it will give us a simple way of controlling the degree of generically finite rational maps.

Lemma 2.6. *Let X be a smooth projective surface and suppose that $\varphi\colon X \dashrightarrow \mathbb{P}^n$ ($n \geq 2$) is a rational map which is generically finite onto its image $S \subset \mathbb{P}^n$. Let $\mathfrak{d}$ be the linear system corresponding to φ (we may assume that $\mathfrak{d}$ has no base components). For any $D \in \mathfrak{d}$, we have*

$$\deg \varphi \cdot \deg S \leq D^2.$$

Proof. Since X is a surface, the indeterminacy locus of φ is a finite set. After blowing up and resolving the base locus, the self-intersection of the strict transform of divisors in $\mathfrak{d}$ can only decrease. □

2.2. Kummer construction and even sections

The goal of this section is to show how sections of a symmetric line bundle on an abelian surface must vanish with a certain parity at the two-torsion points. There is a parallel point of view that can be taken by working directly with the associated Kummer K3 surface. Many of the details can be found in [8].

Let $A = A_d$ be an abelian surface with $\rho(A) = \operatorname{rank} \mathrm{NS}(A) = 1$. In other words, A is an abelian surface carrying an ample line bundle L of type $(1, d)$ for some $d \geq 1$ and $\mathrm{NS}(A) \cong \mathbb{Z}[L]$. To say that L has type $(1, d)$ means that L is primitive,

$$L^2 = 2d \quad \text{and} \quad h^0(L) = d.$$

Let $\iota\colon A \to A$ be the inverse morphism sending $x \mapsto -x$.

For notation purposes, we recall the Kummer construction, starting with an abelian surface A (a good reference for this is [11] or [8, §1]). The involution ι has exactly 16 fixed points, which we will denote by

$$Z = \{p_1, \dots, p_{16}\}$$

(these are the two-torsion points of A under the group law). After blowing up A along Z to get

$$\hat{A} := \mathrm{Bl}_Z A,$$

one can show that ι lifts to an involution $\tilde{\iota}$ on $\hat{A}$. The Kummer surface K is then defined to be

$$K := \hat{A}/\tilde{\iota},$$

and it is straightforward to check that this is a smooth K3 surface. Alternatively, one can define K to be the minimal desingularization of the quotient A/ι. Both of these viewpoints fit into a commutative diagram:

$$\begin{tikzcd} E_i \subset \hat{A} \arrow[r, "\tilde{\sigma}"] \arrow[d, "\pi"'] & K \supset G_i \arrow[d] \\ A \arrow[r, "\sigma"'] & A/\iota \end{tikzcd}$$

where

- the E_i are disjoint exceptional (-1)-curves;
- A/ι is a singular surface with 16 ordinary double points;
- $\tilde{\sigma}$ is a cover of degree 2 ramified along $E = \sum E_i$;
- the $G_i := \tilde{\sigma}(E_i)$ are smooth rational (-2)-curves (for $i = 1, \ldots, 16$).

Definition. We say that a line bundle $\mathcal{L}$ on an abelian variety A is *symmetric* if there is an isomorphism $\iota^*\mathcal{L} \cong \mathcal{L}$.

Lemma 2.7. *By replacing $\mathcal{L}$ with a suitable translate, we may assume that $\mathcal{L}$ is symmetric.*

Proof. Let $t_x : A \to A$ denote translation by $x \in A$. We have the following identity:

$$\iota \circ t_x = t_{-x} \circ \iota.$$

Now

$$\iota^*\mathcal{L} \cong t_y^*\mathcal{L}$$

for some $y \in A$, so choose an element $x \in A$ such that $2x = y$. It is straightforward to check that $t_x^*\mathcal{L}$ is symmetric. $\square$

Let $\mathcal{L}$ be a symmetric line bundle on an abelian surface. After multiplying by a suitable constant, there is an induced involution $\iota_{\mathbb{L}}$ on the total space $\mathbb{L}$ of the line bundle $\mathcal{L}$:

$$\begin{tikzcd} \mathbb{L} \arrow[r, "\iota_{\mathbb{L}}"] \arrow[d] & \mathbb{L} \arrow[d] \\ A \arrow[r, "\iota"] & A \end{tikzcd}$$

If we require the restriction to the fiber $\mathbb{L}_0$ over the origin to be the identity, there is a unique lift $\iota_{\mathbb{L}}$.

Definition. The restriction of $\iota_{\mathbb{L}}$ to the fiber $\mathbb{L}_p$ over any two-torsion point $p_i \in Z$ is multiplication by ± 1. The two-torsion points p_i corresponding to $+1$ (resp. -1) will be called *even* (resp. *odd*). Define $n^{\pm} = n^{\pm}(\mathcal{L})$ to be the number of even (resp. odd) two-torsion points of $\mathcal{L}$ (see [8, §1] or [13, §4.7]).

Fix an integer $n \geq 1$ and let L be a symmetric line bundle of type $(1, d)$. There is an induced involution

$$\iota^* \colon H^0(A, \mathcal{O}_A(nL)) \to H^0(A, \mathcal{O}_A(nL)),$$

so the only possible eigenvalues of ι^* are ± 1.

Definition. Let $H^0(A, \mathcal{O}_A(nL))^{\pm}$ denote the eigenspaces of the induced involution ι^* on $H^0(A, \mathcal{O}_A(nL))$ corresponding to ± 1. We will call sections of $H^0(A, \mathcal{O}_A(nL))^+$ (respectively $H^0(A, \mathcal{O}_A(nL))^-$) *even* (respectively *odd*).

It follows that the space of sections of $\mathcal{O}_A(nL)$ decomposes as

$$H^0(A, \mathcal{O}_A(nL) = H^0(A, \mathcal{O}_A(nL))^+ \oplus H^0(A, \mathcal{O}_A(nL))^-.$$

The dimensions of these vector spaces can be computed (see [8, Theorem 3.1] and [13, Corollary 4.6.6]):

Proposition 2.8. *Let L be a symmetric line bundle of type $(1, d)$. The space of even (resp. odd) sections of $\mathcal{O}_A(nL)$ has dimension*

$$h^0(A, \mathcal{O}_A(nL))^{\pm} = 2 + \frac{n^2 d}{2} - \frac{n^{\mp}(\mathcal{O}_A(nL))}{4}.$$

Remark 2.9. If L is symmetric of type $(1, d)$ and $n = 2k$ is even, then all two-torsion points of $\mathcal{O}_A(nL)$ are even so $n^+(\mathcal{O}_A(nL)) = 16$ and $n^-(\mathcal{O}_A(nL)) = 0$. For the space of even sections, the formula above reduces to

$$h^0(A, \mathcal{O}_A(2kL))^+ = 2 + 2dk^2.$$

In §2.4, we will need the following special case of [13, Prop. 4.7.5]:

Proposition 2.10. *Let L be a symmetric line bundle on A of type $(1, d)$. If d is odd, then*

$$n^+ = 10, 6, \text{ or } 8.$$

If d is even, then

$$n^+ = 12, 4, \text{ or } 8.$$

Corollary 2.11. *For a symmetric line bundle L of type $(1, d)$, since $n^+ + n^- = 16$, we see that $n^+ \geq 4$ and $n^- \geq 4$. In other words, there are at least four even and four odd two-torsion points.*

Let D be the divisor corresponding to an even (resp. odd) section $s \in H^0(A, L)^+$ (resp. $s \in H^0(A, L)^-$). The divisor D will have a certain multiplicity m_i at any two-torsion p_i. The magical fact that we will take advantage of is that the multiplicities of these even/odd sections at the two-torsion points satisfy a certain parity according to the following chart (see Proposition 1.2 of [8]):

	p_i even	p_i odd
D even	m_i even	m_i odd
D odd	m_i odd	m_i even

Remark 2.12. Let $H^0(A, L)^+$ be the space of even sections of a symmetric line bundle L, and let $p \in Z$ be an even two-torsion point. It is at most

$$1 + 3 + \cdots + (2m - 1) = m^2$$

conditions for an even section to vanish to order $2m$ at a fixed point $p \in Z$. We can see this in local coordinates (x, y) around p. It takes at most one linear condition to have an even section s vanish at p. Then s will automatically vanish to order at least two. We can then require

$$\frac{\partial^2 s}{\partial x^2}, \quad \frac{\partial^2 s}{\partial x \partial y}, \quad \text{and} \quad \frac{\partial^2 s}{\partial y^2}$$

to vanish, which will impose at most three conditions on the space of sections. By definition, s then vanishes to order at least 3. By the fact above, s must have multiplicity at least 4 at p, and so on (also see [8] and the Appendix to [9] for more details).

2.3. Constructing maps from sections of $H^0(A, 2L)^+$

Construction 2.13. Let L be a symmetric line bundle of type $(1, d)$. Then $\mathcal{O}_A(2L)$ is a line bundle of type $(2, 2d)$ and Riemann-Roch together with Kodaira vanishing imply that

$$h^0(\mathcal{O}_A(2L)) = \chi(\mathcal{O}_A(2L)) = \frac{1}{2}(2L)^2 + \chi(\mathcal{O}_A) = 4d.$$

By Remark 2.9, we see that the space of *even* sections $H^0(A, \mathcal{O}_A(2L))^+$ of the line bundle $\mathcal{O}_A(2L)$ has dimension

$$h^0(A, \mathcal{O}_A(2L))^+ = 2d + 2.$$

By Prop 2.10, all two-torsion points for $\mathcal{O}_A(2L)$ are even and therefore an even section of $\mathcal{O}_A(2L)$ vanishes to even order at any $p \in Z$. For a fixed $p \in Z$, it takes at most

$$1 + 3 + \cdots + (2m-1) = m^2$$

conditions for an even section of $\mathcal{O}_A(2L)$ to vanish to order $2m$.

Now fix any integer solutions $a_1, \ldots, a_{16} \geq 0$ to the equation

$$\sum_{i=1}^{16} a_i^2 = 2d - 2,$$

with $a_{15} = 0 = a_{16}$ (this last assumption will be useful in Corollary 2.18). Such a solution always exists by Lagrange's four-squares theorem. Let $V \subset H^0(A, \mathcal{O}_A(2L))^+$ be the space of even sections vanishing to order at least $2a_i$ at each point p_i, so that

$$\dim V \geq 2d + 2 - \sum_{i=1}^{16} a_i^2 \geq 4.$$

Projectivizing via subspaces, let $\mathfrak{d} = \mathbb{P}_{\text{sub}}(V) \subseteq |2L|^+$ be the corresponding linear system of divisors, whose dimension is $N := \dim \mathfrak{d} \geq 3$. Write

$$d_i := \operatorname{mult}_{p_i} D$$

for a general divisor $D \in \mathfrak{d}$, so that $d_i \geq 2a_i$.

Remark 2.14. From [13, Section 4.8], it follows that sections of V are pulled back from the singular Kummer surface A/ι, so any divisor $D \in \mathfrak{d}$ is symmetric, i.e. $\iota(D) = D$.

Let $\varphi\colon A \dashrightarrow \mathbb{P}^N$ be the rational map given by the linear system $\mathfrak{d}$ above (if $\mathfrak{d}$ has a fixed component F, take $\mathfrak{d} - F$), and write $S := \overline{\operatorname{Im}(\varphi)}$ for the image of φ. Regardless of whether or not $\mathfrak{d}$ has a fixed component, we find that:

Proposition 2.15. *$S \subset \mathbb{P}^N$ is an irreducible and nondegenerate surface.*

Proof. Suppose for the sake of contradiction that $\overline{\operatorname{Im}(\varphi)}$ is a nondegenerate curve C. Then $\deg C \geq 3$ since $N \geq 3$, and a hyperplane section of $C \subset \mathbb{P}^N$ pulls back to a divisor with at least three irreducible components. This contradicts the fact that any divisor $D(\sim_{lin} 2L) \in \mathfrak{d}$ has at most two irreducible components since $\operatorname{NS}(A) \cong \mathbb{Z}[L]$. So the image of φ is a surface. $\square$

We will now study the numerical properties of the linear series $\mathfrak{d}$ constructed earlier. There are two possibilities for $\mathfrak{d}$; either (i) $\mathfrak{d}$ has no fixed component, or (ii) $\mathfrak{d}$ has a fixed component, denoted by $F \neq 0$. In fact, with a little more work one can show that the second case does not actually occur; see Remark 2.19.

In the second case, let $\mathfrak{b}$ be the movable component of $\mathfrak{d}$, so that we may write every divisor $D \in \mathfrak{d}$ as

$$D = F + M \quad \text{where} \quad M \in \mathfrak{b}.$$

By definition, $\dim \mathfrak{d} = \dim \mathfrak{b}$. Since $\mathrm{NS}(A) \cong \mathbb{Z}[L]$, $D \sim_{lin} 2L$ implies $F, M \sim_{alg} L$ and are irreducible effective divisors for all $M \in \mathfrak{b}$. Choose a general divisor $M \in \mathfrak{b}$ and write

$$m_i := \operatorname{mult}_{p_i} M \quad \text{and} \quad f_i := \operatorname{mult}_{p_i} F,$$

so that $d_i = m_i + f_i \geq 2a_i$ for all i. We claim that F must be symmetric as a divisor. If not, then

$$\iota(M) + \iota(F) = \iota(D) = D = M + F \quad \text{for all} \quad D \in \mathfrak{d}.$$

This implies that $M = \iota(F)$ and $F = \iota(M)$ for all $M \in \mathfrak{b}$, which would mean that M must also be fixed, leading to a contradiction. Hence, F must be symmetric, and likewise for all $M \in \mathfrak{b}$.

We first need an intermediate estimate:

Proposition 2.16. *Assume $\mathfrak{d}$ has a fixed component $F \neq 0$. With the same notation as above, we see that*

$$\sum_{i=1}^{16} m_i^2 \geq 2d - 8.$$

Proof. The idea here is to use the Kummer construction to push our fixed curve F onto a K3 surface and apply Riemann-Roch. This is analagous to an argument of Bauer's in [10, Theorem 6.1]. With the notation from §2.2, we have:

$$\begin{array}{ccc} E \subset \hat{A} & \xrightarrow{\tilde{\sigma}} & K \\ \big\downarrow{\scriptstyle \pi} & & \\ Z \subset A & & \end{array}$$

where $E = \sum_{i=1}^{16} E_i$ is the total exceptional divisor of π. Since F is symmetric, its strict transform

$$\hat{F} = \pi^* F - \sum_{i=1}^{16} f_i E_i$$

descends to an irreducible curve $\bar{F} \subset K$. We claim that

$$h^0(K, \mathcal{O}_K(\bar{F})) = 1.$$

In fact, if the linear system $|\mathcal{O}_K(\bar{F})|$ were to contain a pencil, then this would give us a pencil of symmetric curves in $|\mathcal{O}_A(F)|$ with the same multiplicities at the two-torsion points, which contradicts F being a fixed component of $\mathfrak{d}$.

On the other hand, it is well-known that an irreducible curve $\bar{F}$ on a K3 surface with $h^0(K, \bar{F}) = 1$ satisfies $(\bar{F})^2 = -2$, so

$$-4 = 2(\bar{F})^2 = (\gamma^* \bar{F})^2 = (\hat{F})^2 = F^2 - \sum_{i=1}^{16} f_i^2 = 2d - \sum_{i=1}^{16} f_i^2. \tag{2.1}$$

By definition $d_i = f_i + m_i$ and the area of a rectangle with fixed perimeter is maximized when it is a square, so

$$\sum_{i=1}^{16} f_i m_i \leq \sum_{i=1}^{16} (\frac{d_i}{2})^2.$$

Combining these two expressions yields

$$\sum_{i=1}^{16} d_i^2 = \sum_{i=1}^{16} (f_i^2 + m_i^2 + 2f_i m_i) \leq 2d + 4 + \sum_{i=1}^{16} m_i^2 + \frac{1}{2} \sum_{i=1}^{16} d_i^2.$$

After rearranging the terms, we find that

$$\sum_{i=1}^{16} m_i^2 \geq -2d - 4 + \frac{1}{2} \sum_{i=1}^{16} d_i^2 \geq -2d - 4 + 2 \sum_{i=1}^{16} a_i^2 = 2d - 8 \tag{2.2}$$

for a general divisor $D = F + M \in \mathfrak{d}$, which is the desired inequality. □

As an immediate consequence:

Theorem 2.17. *Keeping the notation from earlier, let $\varphi \colon A \dashrightarrow \mathbb{P}^N$ be the rational map corresponding to $\mathfrak{d}$ (or $\mathfrak{b}$ if $F \neq 0$), with image S. Then*

$$\deg \varphi \cdot \deg S \leq 8. \tag{2.3}$$

Proof. By applying Proposition 2.15 and blowing-up A along the collection of two-torsion points Z to resolve some of the base points of $\mathfrak{d}$, we arrive at the diagram

$$\begin{tikzcd} \hat{A} := \mathrm{Bl}_Z A \arrow[d, "\pi"'] \arrow[dr, dashed, "\psi"] & \\ A \arrow[r, dashed, "\varphi"'] & S \subset \mathbb{P}^N. \end{tikzcd}$$

(i) If the linear system $\mathfrak{d}$ has no fixed components, the divisors corresponding to ψ are of the form

$$\hat{D} \sim_{lin} \pi^* D - \sum_{i=1}^{16} d_i E_i,$$

where $\hat{D}$ denotes the strict transform of D. By Lemma 2.6 applied to ψ, we have

$$\deg \varphi \cdot \deg S = \deg \psi \cdot \deg S \leq \hat{D}^2 = 4L^2 - \sum_{i=1}^{16} d_i^2 \leq 4\left(2d - \sum_{i=1}^{16} a_i^2\right) = 8.$$

(ii) If the linear system $\mathfrak{d}$ has a fixed component $F \neq 0$, replace $\hat{D}$ and d_i in the equation above with $\hat{M}$ and m_i, respectively. Proposition 2.16 then gives an analogous bound. $\square$

Corollary 2.18. *There exists a rational dominant map $\varphi\colon A \dashrightarrow \mathbb{P}^2$ of degree 4.*

Proof. From Remark 2.14, it follows that $\varphi\colon A \dashrightarrow S \subset \mathbb{P}^N$ factors through the quotient $A \to A/\iota$, so $\deg \varphi$ must be even. In addition, $\deg S \geq 2$ since S is nondegenerate. By Lemma 2.2 there are no rational maps of degree 2 from A to a rational surface. Therefore, the possibility $\{\deg \varphi = 2, \deg S = 2, 3\}$ is ruled out by the classification of quadric and cubic surfaces (using the fact that $\rho(A) = 1$).

Together with the upper bound $\deg \varphi \cdot \deg S \leq 8$ given by Theorem 2.17, there are two possibilities:

$$\{\deg \varphi = 4,\ \deg S = 2\} \quad \text{or} \quad \{\deg \varphi = 2,\ \deg S = 4\}.$$

Either of these cases imply that we have equality in (2.3), which means that the map φ becomes a morphism ψ when passing to the blow-up $\hat{A}$. By the discussion in §2.4, one can actually show that the map ψ factors through K. All of this fits into the diagram:

$$\begin{tikzcd}
\hat{A} \arrow[r] \arrow[d, "\pi"'] \arrow[drr, bend left, "\psi"] & K \arrow[d] \arrow[dr, "\alpha"] & \\
A \arrow[r] \arrow[rr, dashed, bend right, "\varphi"'] & A/\iota \arrow[r, dashed] & S \subset \mathbb{P}^N
\end{tikzcd}$$

In the first case where $\deg \varphi = 4$ and $\deg S = 2$, note that S is rational so we get our degree 4 map.

Let $G_i := \tilde{\sigma}(E_i)$ be the smooth rational (-2) curves on K which are the images of the exceptional divisors of π. In the second case where $\deg \varphi = 2$ and $\deg S = 4$, recall from the construction at the beginning of this section that we chose the multiplicities a_i so that $a_{15} = 0 = a_{16}$. Thus, equality in (2.3) forces either $d_{15} = 0 = d_{16}$ or $m_{15} = 0 = m_{16}$. This implies that the curves G_{15}, G_{16} are contracted and their images q_{15}, q_{16} under the map $\alpha\colon K \to S$ are double points on S (since α is a birational morphism). Projection from a general $(N-3)$-plane containing one but not both of the q_i defines a rational map $A \dashrightarrow \mathbb{P}^2$ of degree 2 (if q_{15} is a cone point of S, pick a general plane passing through q_{16}, and vice versa), which contradicts Lemma 2.2. $\square$

This immediately implies Theorem 1.18.

Remark 2.19. The case when $\mathfrak{d}$ has a fixed component $F \neq 0$ cannot occur. To see this, suppose $F \neq 0$ and note that the two cases given in Corollary 2.18 imply that equality must hold throughout the proof of Proposition 2.16. In particular, $d_i = m_i + f_i$ and $\sum_{i=1}^{16} f_i m_i = \sum_{i=1}^{16} (\frac{d_i}{2})^2$ implies $f_i = m_i$ for all i. Combining this with (2.1) and (2.2) gives

$$2d + 4 = \sum_{i=1}^{16} f_i^2 = \sum_{i=1}^{16} m_i^2 = 2d - 8,$$

which is a contradiction.

In fact, one may relax the hypothesis that $\mathrm{NS}(A) = \mathbb{Z}[L]$. It suffices to have:

Proposition 2.20. *If A is an abelian surface not containing a smooth elliptic curve E, then all of the theorems from this section carry through.*

Proof. If $C \subset A$ is a curve, then $p_g(C) \geq 1$. By the assumption that A does not contain any smooth elliptic curves,

$$p_a(C) \neq 1 \implies p_a(C) \geq 2 \implies C^2 = 2p_a(C) - 2 \geq 2.$$

This implies that every effective divisor $D \neq 0$ on A has positive self-intersection and is therefore ample (one can translate D and apply the Nakai-Moishezon criterion).

Choose L to be a line bundle (polarization) of minimal degree, i.e. $L^2 = 2d > 0$ and no other polarizations have strictly smaller degree (it is possible that there are multiple polarizations with self-intersection equal to $2d$). Then L is of type $(1, d)$. After translating, we may assume that L is symmetric.

If the linear system $\mathfrak{d} \subset |2L|^+$ in Construction 2.13 has no fixed components, then we are done. Let us now assume that $\mathfrak{d}$ has a fixed component $F \neq 0$. Then $D = M + F \sim_{lin} 2L$. Since M and F are both effective and not smooth elliptic curves, they must have positive self-intersection. By our choice of L, it follows that

$$M^2, F^2 \geq L^2.$$

The Hodge Index theorem tells us that

$$M^2 \cdot F^2 \leq (M \cdot F)^2 \implies (M \cdot F) \geq L^2$$

(we can apply HI since M is ample). So

$$4L^2 = D^2 = (M + F)^2 = M^2 + F^2 + 2(M \cdot F) \geq 4L^2$$

and we have equality throughout. In particular, $L^2 = M^2 = F^2$ and both M, F are of type $(1, d)$ by the assumption that L is the minimal degree polarization. $\square$

2.4. Constructing elliptic fibrations with sections on Kummer K3 surfaces

The purpose of this section is to give an alternative proof of Theorem 1.18 by working directly on the associated Kummer K3 surface. We will actually prove:

Theorem 2.21. *Every Kummer K3 surface admits an elliptic fibration with at least four sections.*

This will indirectly allow us to construct rational maps $A \dashrightarrow \mathbb{P}^2$ of degree 4 by applying Prop 2.4 to the Kummer K3 surface K and then composing with the degree 2 rational cover from A to K. Note that the approach above allows us to generalize Theorem 1.18 to all complex abelian surfaces (which was already established in joint work with Stapleton [16, Corollary D]).

Let L be a symmetric line bundle on an abelian surface A. In [8, Prop. 1.1], Bauer proved that the direct image sheaf $M := \tilde{\sigma}_*(\pi^*L)$ on K is locally free of rank 2, and it admits a decomposition

$$M = M^+ \oplus M^-$$

into line bundles M^+ and M^-. On the level of global sections, there are isomorphisms

$$H^0(K, M^\pm) \cong H^0(A, L)^\pm$$

(compare with §2.2). Let E^+ (resp. E^-) denote the union of the exceptional divisors E_i on the blow-up $\hat{A}$ corresponding to even (resp. odd) two-torsion points p_i of L. Note that $E^+ + E^- = \sum_{i=1}^{16} E_i$. The next result describes the relationship between $M^\pm$ and L.

Proposition 2.22 ([8], Proposition 1.3)**.** *With the notation above, $\tilde{\sigma}^* M^\pm = \pi^* L - E^\mp$.*

Let G_i denote the smooth rational (-2)-curves on K. Then we have:

Proposition 2.23 ([8], Proposition 1.4)**.** *It follows that*

$$M^+ \cdot G_i = \begin{cases} 0 & \textit{if } p_i \textit{ is even,} \\ 1 & \textit{if } p_i \textit{ is odd.} \end{cases}$$

We are now ready to construct our elliptic fibrations on K.

Proof of Theorem 2.21. Let L be a primitive line bundle on A of minimal degree $= L^2 > 0$. We may assume that $L^2 = 2d$ and L is a symmetric line bundle of type $(1, d)$ on A. By Proposition 2.8, we know that the space of global sections of $M^\pm$ has dimension

$$h^0(M^\pm) = h^0(A, L)^\pm = \frac{d}{2} + 2 - \frac{n^\mp}{4}.$$

From Proposition 2.22 we know that $\tilde{\sigma}^*M^{\pm} = \pi^*L - E^{\mp}$. The projection formula applied to $\sigma\colon \hat{A} \to K$ then gives

$$2d - n^{\mp} = (\pi^*L - E^{\mp})^2 = (\tilde{\sigma}^*M^{\pm})^2 = 2 \cdot (M^{\pm})^2,$$

and therefore the self-intersections of the line bundles M^+, M^- on K are given by

$$(M^{\pm})^2 = d - \frac{n^{\mp}}{2}.$$

By Corollary 2.11, L has at least four even and four odd two-torsion points. The divisors $D \in H^0(A, L)^+$ are even, so

$$\operatorname{mult}_{p_i} D = \begin{cases} \text{even} & \text{if } p_i \text{ is even,} \\ \text{odd} & \text{if } p_i \text{ is odd.} \end{cases}$$

Without loss of generality let $p_1, \dots, p_4$ be four of the even two-torsion points. Then we may impose multiplicity $2a_i$ at these points p_i (for $i = 1, \dots, 4$) with the condition that

$$\sum_{i=1}^{4} a_i^2 = \frac{d}{2} - \frac{n^-}{4}.$$

Again, an integer solution with $a_i \geq 0$ exists by Lagrange's four squares theorem. Note that the right hand side will always be an integer by Proposition 2.10. This gives us a space of global sections $V \subset H^0(A, L)^+$ of dimension ≥ 2. We would like to point out that sections of V correspond to sections of

$$H^0\Big(K, M^+ - \sum_{i=1}^{4} a_i G_i\Big),$$

so there is a completely parallel story on K.

Given a general divisor $D \in |V|$ on A with the imposed multiplicities, let C be the image of its strict transform on $\hat{A}$ to K. By part (a) of Prop. 1.5 in [8], we see that

$$2p_a(C) - 2 = C^2 = \frac{1}{2}D^2 - \frac{1}{2}\sum d_i^2$$

for some integers

$$d_i \geq \begin{cases} 2a_i & \text{for even } p_i \in Z, \\ 1 & \text{for odd } p_i \in Z. \end{cases}$$

This gives

$$\begin{aligned}C^2 &\le d - \frac{1}{2}(4 \cdot \sum_{p_i \text{ even}} a_i^2 + \sum_{p_i \text{ odd}} d_i^2) \\ &\le d - \frac{1}{2}(2d - n^- + n^-) = 0.\end{aligned}$$

On the other hand, C is an irreducible curve on K generating a linear system of dimension at least 1 with no fixed component (since L is primitive), so this actually implies $C^2 = 0$ since base points would only decrease the self-intersection. Therefore the complete linear system $|C|$ must have dimension 1 with no base points, $p_a(C) = 1$, and Bertini's theorem implies that the generic fiber is a smooth elliptic curve. In particular, the inequalities above must all be equalities. For any odd two-torsion point p_i, it follows from Proposition 2.23 that

$$C \cdot G_i = M^+ \cdot G_i = 1,$$

i.e. the smooth (-2)-curves G_i corresponding to the odd two-torsion points are all sections of the elliptic fibration $|C|$. In total, there are at least $n^- \ge 4$ of these sections. □

Remark 2.24. By using sections of a primitive line bundle L rather than multiples of L, we avoid having to deal with the possibility that our linear system $|V|$ might have a fixed component. Note that L is a primitive polarization of minimal degree $L^2 = 2d > 0$ on an abelian surface. Suppose for the sake of contradiction that $|V| \subset |L|^+$ has a fixed component; then we may write

$$L = M + F$$

for some fixed F. If $M^2 > 0$, then by minimality of L we know that $M^2 \ge L^2 = 2d$. From the computation

$$L^2 = (M + F)^2 = M^2 + 2(M \cdot F) + F^2 = 2d + F^2$$

we see that $M^2 = 2d$, $M \cdot F = 0$, and $F^2 = 0$. But M and F are effective, so $M \cdot F = 0$ implies that M and F are translates of each other, which is a contradiction.

Therefore, we must have $M^2 = 0 = F^2$ and $(M \cdot F) = d$. By the adjunction formula, this says that M and F must both have arithmetic genus 1. Since we are working on an abelian surface, this implies that M and F are both sums of smooth elliptic curves and $h^0(A, M) = 2$. We also know that they must both be symmetric. On the other hand, the general $M \in |V| - F$ must have the same imposed multiplicities at the two-torsion points since the multiplicities are equal to the intersections of the corresponding divisors with the G_i on K by the proof of Theorem 2.21. This contradicts the fact that $M^2 = 0$.

Chapter 3
Complete intersections

For complex hypersurfaces of large degree in projective space, the covering gonality comes in two flavors. Recall from §1 that the covering gonality of a smooth hypersurface is always bounded from below when d is large enough:

Theorem 1.9 ([7], Theorem A)**.** If $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ is a smooth hypersurface of dimension n and degree $d \geq n+2$, then

$$\operatorname{cov.gon}(X) \geq d - n.$$

We will review in §3.1 how this follows from Proposition 3.4. In the generic situation, Bastianelli, Ciliberto, Flamini, and Supino [5] have shown more precisely:

Theorem 3.1. *Let $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ be a very general smooth hypersurface of degree $d \geq 2n+2$. Then*

$$\operatorname{cov.gon}(X) = d - \left\lfloor \frac{\sqrt{16n+1}-1}{2} \right\rfloor,$$

apart from the cases $n \in \{4\alpha^2 + 3\alpha, 4\alpha^2 + 5\alpha + 1 \mid \alpha \in \mathbb{N}\}$ where the covering gonality may drop by one.

It is natural to explore the same question for complete intersections

$$X := X_{a_1} \cap \cdots \cap X_{a_e} \subset \mathbb{P}^{n+e}$$

of dimension n and type $(a_1, \ldots, a_e)$. Namely, how do the covering gonality and the degree of irrationality behave for complete intersections? As mentioned in the introduction, projection from a linear subspace of dimension $e-1$ defines a map $X \dashrightarrow \mathbb{P}^n$ of degree $\approx a_1 \cdots a_e$ (up to a constant depending on the extent to which the linear subspace is secant to X). So we have

$$\operatorname{cov.gon}(X) \leq \operatorname{irr}(X) \leq a_1 \cdots a_e.$$

The existing lower bounds up until now seem to fall short of this by a wide margin. From the adjunction formula we have

$$K_X \cong \mathcal{O}_X(a_1 + \cdots + a_e - n - e - 1),$$

and one can apply Proposition 3.4 to show that

$$\operatorname{cov.gon}(X) \geq a_1 + \cdots + a_e - n - e + 1.$$

Note that the numerics here are additive in $a_1, \ldots, a_e$ whereas the upper bound is multiplicative. However, there is some evidence to suggest that measures of irrationality on complete intersections *should* behave multiplicatively. For instance, Lazarsfeld [34, Exercise 4.12] proved that this is the case in dimension $n = 1$:

Theorem 1.19. Let $C \subset \mathbb{P}^{e+1}_{\mathbb{C}}$ be a complete intersection curve of type $(a_1, a_2, \ldots, a_e)$ with $2 \leq a_1 \leq \cdots \leq a_e$. Then the gonality of C is bounded from below by

$$\operatorname{gon}(C) \geq (a_1 - 1)a_2 \cdots a_e.$$

One can show that equality holds in some special cases (see Example 3.6). The idea of the proof is to fit the complete intersection curve C into a complete intersection surface S. The existence of a low degree map $C \to \mathbb{P}^1$ allows one to construct some Bogomolov unstable vector bundle on S and use the numerics of the vector bundle to reach a contradiction. Further refinements due to Hotchkiss, Lau, and Ullery [29] show that when $4 \leq a_1 < a_2 \leq \cdots \leq a_e$ holds, the gonality of the curve C is realized by projection from a suitable linear subspace.

For higher dimensional complete intersections, Stapleton [47] showed that the covering gonality of a complete intersection of two hypersurfaces in $\mathbb{P}^{n+1}$ grows superlinearly:

Theorem 3.2. *Let $X \subset \mathbb{P}^{n+2}_{\mathbb{C}}$ be a very general smooth complete intersection of type (a, b). Then*

$$\operatorname{irr}(X) \geq \frac{b\lfloor \sqrt[n+1]{a} \rfloor}{n+1}.$$

More precisely, if

$$\frac{b\lfloor \sqrt[n+1]{a} \rfloor}{n+1} \geq p + 1,$$

then K_X separates p general points on Z.

The result above relies on lower bounds for the Seshadri constant of a very general hypersurface, which are proved in [31] (see Remark 3.8 for a short discussion on this). Stapleton and Ullery [48] later computed the degree of irrationality for complete intersections of type $(2, d)$ and $(3, d)$.

In this chapter, we establish lower bounds for the covering gonality (and hence the degree of irrationality) of certain types of complete intersection varieties which are multiplicative. Our first result is:

Theorem 3.3. *Let $X \subset \mathbb{P}^{1+e}_{\mathbb{C}}$ be a very general smooth complete intersection curve of type $(a_1, \ldots, a_e)$ where $d \geq 4$. Then*

$$\operatorname{gon}(X) \geq \frac{1}{8} a_1 \cdots a_e.$$

Although this is a weaker inequality than what Lazarsfeld [34] has proven, it highlights some of the key ideas that can be generalized to higher dimension. Our main theorem gives new lower bounds for codimension two complete intersections:

Theorem 1.20. Let $X \subset \mathbb{P}^{n+2}_{\mathbb{C}}$ be a very general smooth complete intersection of type (a, b) and dimension $n \geq 2$. If $a, b \geq 9n$, then

$$\operatorname{cov.gon}(X) \geq \frac{2}{3(n+1)^2} \cdot ab.$$

The idea behind both of these results is to reduce to showing nefness of a particular family of curves on a complete intersection $Y \supset X$ whose dimension is one larger than the dimension of X. In §3.1, we will collect some examples and known results. In §3.2, we will then present the reduction step mentioned just now, which first appeared in the thesis of Stapleton [47, §5.2]. In §3.3, we will prove Theorem 3.3. Numerical bounds for curves in generic complete intersections will appear in §3.4, and we will use these in §3.5 to prove Theorem 1.20.

3.1. Covering gonality and separating points

Recall that sections of a line bundle L on a smooth variety X *separate r points on an open set* if there exists a Zariski open subset $U \subset X$ such that for any r distinct points $p_1, \ldots, p_r \in U$, the restriction map

$$H^0(X, L) \to H^0(X, L \otimes \mathcal{O}_{\{p_1,\ldots,p_r\}})$$

is surjective. In Lemma 1.5, we saw how these sorts of positivity properties for the canonical bundle of a smooth projective curve C gave lower bounds for the gonality of C. In higher dimension, we can use similar ideas to bound the covering gonality:

Proposition 3.4 ([7], Theorem 1.10)**.** *Let X be a smooth projective variety and suppose that there is an integer r such that the canonical bundle K_X separates r points on an open set. Then*

$$\operatorname{cov.gon}(X) \geq r + 1.$$

Proof. A covering family of curves of gonality c fits into the diagram:

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"'] & \mathcal{X} \\ T & \end{tikzcd}$$

and we may assume without loss of generality that $\mathcal{C}$ and T are smooth, f is generically finite, and the restriction of the map f to the general fiber of π is birational onto its image. Then

$$K_{\mathcal{C}} \cong f^* K_X + E$$

where E is the ramification divisor of f. By generic smoothness,

$$K_{C_t} \cong K_{\mathcal{C}}\big|_{C_t}$$

for the smooth fiber C_t over a general point $t \in T$, and C_t meets the effective divisor E properly. We may also assume that the image of C_t will not be contained in the complement of the open set coming from the definition of K_X separating r points on an open set. Therefore, K_{C_t} also separates r points on an open set and Lemma 1.5 implies that $c \geq r+1$. $\square$

Proof of Theorem 3.1. For a hypersurface $X \subset \mathbb{P}^{n+1}_{\mathbb{C}}$ of degree d, the adjunction formula gives $K_X \cong \mathcal{O}_X(d-n-2)$. The canonical bundle K_X then separates $d-n-1$ points on an open set, so Proposition 3.4 implies that $\operatorname{cov.gon}(X) \geq d-n$. $\square$

Example 3.5. Let us summarize the picture for measures of irrationality on surfaces

$$V_d \subset \mathbb{P}^3_{\mathbb{C}}$$

of degree d. For $d=4$, V_4 is a quartic K3 surface and it is well-known that there is a 1-parameter family of elliptic curves covering V (see [41, Appendix]), so $\operatorname{cov.gon}(V_4) = 2$. The degree of irrationality of V_4 is equal to 2 or 3, depending on whether V_4 has a birational involution. If $d \geq 5$, then Theorem 1.8 and Theorem 1.9 together imply that

$$d-2 \leq \operatorname{cov.gon}(V_d) \leq \operatorname{irr}(V_d) = d-1.$$

We claim that $\operatorname{cov.gon}(V_d) = d-2$. For a general point $p \in V_d$, we may consider the tangent plane T_pV_d to the surface V_d at p. The general hyperplane section

$$H_p =_{\text{def}} V_d \cap T_pV_d \subset T_pV_d \cong \mathbb{P}^2$$

will then be a curve of degree d with a single node, and projecting from the singular point shows that $\operatorname{gon}(H_p) \leq d-2$. Therefore, $\operatorname{cov.gon}(V_d) = d-2$.

Finally, we return to complete intersection curves and show how the theorem of Lazarsfeld is sharp for special complete intersection curves:

Example 3.6. Let

$$C := W_{a_1} \cap \cdots \cap W_{a_e} \subset \mathbb{P}^{1+e}$$

be smooth complete intersection curve of type $(a_1, \ldots, a_e)$ with $2 \leq a_1 \leq \cdots \leq a_e$ such that the smooth hypersurface W_{a_1} contains an $(e-1)$-plane Λ. Such a hypersurface exists by a dimension count. By choosing the other W_{a_i} carefully, we get a smooth complete intersection curve C which is not contained in Λ. But $\Lambda \subset W_{a_1}$ so

$$\underbrace{\Lambda \cap W_{a_2} \cap \cdots \cap W_{a_e}}_{a_2 \cdots a_e \text{ points}} \subset C$$

and therefore $\Lambda \cap C$ consists of $a_2 \cdots a_e$ points. Projecting the curve C away from Λ defines a map $C \to \mathbb{P}^1$ of degree

$$a_1 \cdots a_e - a_2 \cdots a_e = (a_1 - 1) a_2 \cdots a_e,$$

which matches the lower bound in Theorem 1.19.

3.2. Reduction step

In this section, we will show how Theorem 3.3 and Theorem 1.20 essentially follow from the nefness of a particular family of line bundles. Consider the inclusions $X \subset Y \subset \mathbb{P}^{n+e}$ where

- Y is a complete intersection of dimension $n+1$ type $(a_1, \ldots, a_{e-1})$;
- $X \in |\mathcal{O}_Y(a_e)|$ is a complete intersection of dimension n and type $(a_1, \ldots, a_e)$.

The strategy is to apply Proposition 3.4 from the previous section. For this, we will need to show that the canonical bundle of X separates any collection of r distinct points $p_1, \ldots, p_r \in X$ which are supported in some open set of X. We claim that the following statement is sufficient to give upper bounds on the covering gonality of X:

Proposition 3.7. *Suppose that there exists a positive integer r such that on the blow-up $\mu\colon \tilde{Y} \to Y$ along any points $p_1, \ldots, p_r \in X$ with exceptional divisors $E_1, \ldots, E_r$, the line bundle*

$$L := \mu^* \mathcal{O}_Y(a_e) - \sum_{i=1}^{r} (n+1) E_i$$

is nef and big. Then $\operatorname{cov.gon}(X) \geq r+1$.

Proof. Since L is both nef and big, and we have the vanishing of

$$\begin{aligned} &H^1(Y, (K_Y + \mathcal{O}_Y(a_e)) \otimes \mathcal{I}_{\{p_1, \ldots, p_r\}}) \\ &= H^1\left(\tilde{Y}, \mu^*(K_Y + \mathcal{O}_Y(a_e)) - \sum_{i=1}^{r} E_i\right) \\ &= H^1(\tilde{Y}, K_{\tilde{Y}} + L) = 0. \end{aligned}$$

Here, we use the fact that $K_{\tilde{Y}} \cong \mu^* K_Y + nE$ since Y has dimension $n+1$. This implies that there is a surjection

$$H^0(Y, K_Y + \mathcal{O}_Y(a_e)) \twoheadrightarrow H^0\left(Y, (K_Y + \mathcal{O}_Y(a_e)) \otimes \mathcal{O}_{\{p_1, \ldots, p_r\}}\right).$$

In other words, sections of $K_Y + \mathcal{O}_Y(a_e)$ separate any finite set of r distinct points in X. Since $X \in |\mathcal{O}_Y(a_e)|$, the adjunction formula tells us that

$$K_X \cong (K_Y + \mathcal{O}_Y(a_e))\big|_X,$$

and hence sections of K_X separate any finite set of r distinct points in X. By Proposition 3.4, it follows that

$$\operatorname{cov.gon}(X) \geq r+1.$$

□

In practice, once we prove nefness of L, it will follow numerically that L is big (see [35, Theorem 2.2.16]).

Remark 3.8. We would like to point out that the hypothesis in Proposition 3.7 about nefness of L is very similar to the type of statements that come from multi-point Seshadri constants. Indeed, Stapleton [47] applied the work of Ito [31] on Seshadri constants for very general hypersurfaces to prove that there are stronger than additive bounds for the covering gonality of codimension two complete intersections. One can obtain a nef line bundle involving twists of a sum of exceptional divisors by adding several nef line bundles which are produced from Ito's results about single point Seshadri constants. In our situation, we are able to show that certain families of line bundles (twisted by a sum of exceptional divisors) with larger slopes are nef by blowing up multiple points at once in X and using the definition of nefness. The key difference between our results and those coming from multi-point Seshadri constants is that we consider *arbitrary* collections of points $p_1, \ldots, p_r \in X$, whereas in most of the literature on multi-point Seshadri constants, the points are in (very) general position with respect to each other. Here, it is crucial to consider any collection of points in X (or at least in an open subset of X) rather than in general position because of the necessary assumptions in Theorem 3.4.

Our goal for the rest of the chapter is to prove nefness of L in two separate settings:

1. when $n = 1$ and e is arbitrary (complete intersection curves), and
2. when $n \geq 2$ and $e = 2$ (codimension two complete intersections).

Proceeding by contradiction, the failure of L to be nef means that there exists a curve $\tilde{C}$ on $\tilde{Y}$ which intersects negatively against L. We analyze the multiplicities m_i of the image curve $C := \mu(\tilde{C})$ at the points p_i to reach a contradiction. Along the way, we will need lower bounds for the geometric genus of C.

3.3. Multiplicative bounds for complete intersection curves

In this section, we will prove nefness of line bundles on blow-ups of a complete intersection surface along points contained in a complete intersection curve. Ultimately this will be applied to Proposition 3.7 in order to prove Theorem 3.3.

Set-up: Let $Y \subset \mathbb{P}^{1+e}$ be a very general complete intersection surface of type

$$(a_1, \ldots, a_{e-1}).$$

By the Noether-Lefschetz theorem [36], we may assume that $\mathrm{Pic}(Y) = \mathbb{Z}[\mathcal{O}_Y(1)]$ as long as Y is not a complete intersection of two quadric 3-folds in $\mathbb{P}^4$ or a surface in $\mathbb{P}^3$ of degree ≤ 3. Let $X \in |\mathcal{O}_Y(a_e)|$ be any smooth curve. Without loss of generality we may arrange for

$$a_1 \leq a_2 \leq \cdots \leq a_e.$$

Theorem 3.9. *With the set-up above, fix an integer r such that*

$$r \leq \frac{1}{8} a_1 \cdot a_2 \cdots a_e$$

and let $p_1, \ldots, p_r \in X$ be any collection of distinct points. Let

$$\mu\colon \tilde{Y} =_{\mathrm{def}} \mathrm{Bl}_{\{p_1,\ldots,p_r\}} \to Y$$

be the blow-up with exceptional divisors E_i over p_i and $H = \mu^* \mathcal{O}_Y(1)$. Then the line bundle

$$L := a_e H - \sum_{i=1}^{r} 2E_i \quad \textit{is nef.}$$

Proof. To simplify notation, fix $\gamma := a_1 \cdots a_{e-1}$. The proof will proceed by induction on r. For the base case $r = 1$, the theorem is trivial since H is very ample and $a_e \geq 2$. By induction, we may assume that the theorem holds for $r = s$ where s is an integer satisfying

$$1 \leq s \leq \frac{1}{8} a_1 a_2 \cdots a_e - 1.$$

We want to show that the theorem holds for $r = s + 1$.

Step 1: Suppose for the sake of contradiction the theorem fails when $r = s + 1$, i.e. there exists a collection of points $p_1, \ldots, p_{s+1} \in X$ such that on the corresponding blow-up $\tilde{Y}$, the line bundle

$$L = a_e H - \sum_{i=1}^{s+1} 2E_i \quad \text{is not nef.}$$

By definition, there exists an integral curve $\tilde{C} \subset \tilde{Y}$ such that

$$L \cdot \tilde{C} < 0.$$

We assumed that $\mathrm{Pic}(Y) = \mathbb{Z}[\mathcal{O}_Y(1)]$ so $\mathrm{Pic}(\tilde{Y}) = \mathbb{Z}[H, E_1, \ldots, E_r]$ with intersection pairing given by

$$\begin{aligned} H^2 &= \gamma, \\ H \cdot E_i &= 0, \end{aligned}$$

$$E_i \cdot E_j = \begin{cases} -1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases}$$

for all $1 \leq i, j \leq s+1$.

The curve $\tilde{C}$ is not equal to any of the exceptional divisors because the intersection $L \cdot \tilde{C}$ is negative, so we may write

$$\tilde{C} = kH - \sum_{i=1}^{s+1} m_i E_i$$

for some integers $k \geq 1$ and $m_i \geq 0$. If we think of the curve $\tilde{C}$ as the strict transform of

$$C := \mu(\tilde{C}) \subset Y$$

under the blow-up, then $C \in |\mathcal{O}_Y(k)|$, $\deg(C) = k \cdot \gamma$, and the m_i are the multiplicities of C at p_i.

Step 2: It is straightforward to check that

$$\begin{aligned} L \cdot \tilde{C} < 0 &\iff ka_e\gamma - \sum_{i=1}^{s+1} 2m_i < 0 \\ &\iff \sum_{i=1}^{s+1} m_i > \frac{1}{2} ka_e\gamma. \end{aligned}$$

For a fixed quantity $\sum_{i=1}^{s+1} m_i$ with $m_i \geq 0$, the expression

$$\sum_{i=1}^{s+1} m_i^2$$

is *minimized* when all m_i are the same. If we set

$$N := \frac{1}{2} ka_e\gamma,$$

then

$$\sum_{i=1}^{s+1} m_i^2 > (s+1) \cdot \left(\frac{N}{(s+1)}\right)^2 = \frac{N^2}{(s+1)} = \frac{1}{4(s+1)} k^2 a_e^2 \gamma^2.$$

Since $s + 1 \leq \frac{1}{8}\gamma a_e$, this becomes

$$\sum_{i=1}^{s+1} m_i^2 > 2k^2 a_e \gamma \tag{3.1}$$

Our goal is to reach a contradiction by bounding $\sum_{i=1}^{s+1} m_i^2$ from above and then comparing.

Step 3: The induction hypothesis implies that the theorem holds for $r = s$ and any collection of r points in X. So the line bundle

$$L_I := a_e H - \sum_{i \in I} 2E_i$$

is nef for every subset $I \subset \{1, 2, \dots, s+1\}$ with $|I| = s$. We can sum over all possible I to get a nef line bundle on $\tilde{Y}$:

$$\begin{aligned}
\sum_{I\colon |I|=s} L_I &= \sum_{I\colon |I|=s} \left[a_e H - \sum_{i \in I} 2E_i \right] \\
&= (s+1)a_e H - \sum_{I\colon |I|=s} \sum_{i \in I} 2E_i \\
&= (s+1)a_e H - \sum_{i=1}^{s+1} 2sE_i.
\end{aligned}$$

The intersection of this nef line bundle with $\tilde{C}$ is ≥ 0 by definition:

$$\left(kH - \sum_{i=1}^{s+1} m_i E_i \right) \cdot \left((s+1)a_e H - \sum_{i \in I} 2sE_i \right) \geq 0$$

implies that

$$\sum_{i=1}^{s+1} m_i \leq \frac{(s+1)}{2s} k a_e \gamma. \tag{3.2}$$

Step 4: In our situation, we have an integral curve C contained in a smooth surface Y. We claim that for each i,

$$\frac{m_i(m_i - 1)}{2} \leq \delta_{p_i}$$

where δ_{p_i} is the *delta-invariant* of the point $p_i \in C$. In other words, we have

$$\sum_{i=1}^{s+1} \frac{m_i(m_i-1)}{2} \leq p_a(C) - p_g(C).$$

This basically follows from the adjunction formula for a curve on a smooth surface. Note that

$$K_{\tilde{Y}} = \mu^* K_Y + \sum_{i=1}^{s+1} E_i$$

and

$$\tilde{C} = \mu^* C - \sum_{i=1}^{s+1} m_i E_i$$

so

$$\begin{aligned}
2p_a(\tilde{C}) - 2 &= \tilde{C} \cdot (\tilde{C} + K_{\tilde{Y}}) \\
&= (\mu^* C - \sum_{i=1}^{s+1} m_i E_i) \cdot (\mu^* C + \mu^* K_Y - \sum_{i=1}^{s+1} (m_i - 1) E_i) \\
&= C \cdot (C + K_Y) - \sum_{i=1}^{s+1} m_i (m_i - 1) \\
&= 2p_a(C) - 2 - \sum_{i=1}^{s+1} m_i (m_i - 1),
\end{aligned}$$

which proves the claim.

For a complete intersection curve $C \subset Y$ of degree k, the adjunction formula allows us to compute its arithmetic genus:

$$\begin{aligned}
p_a(C) &= 1 + \frac{1}{2} C \cdot (C + K_Y) \\
&= 1 + \frac{1}{2} \mathcal{O}_Y(k) \cdot \mathcal{O}_Y(k + \sum_{j=1}^{e-1} a_j - e - 2) \\
&= 1 + \frac{1}{2} k (k + \sum_{j=1}^{e-1} a_j - e - 2) \gamma.
\end{aligned}$$

Now Y is a very general surface of type $(a_1, \ldots, a_{e-1})$ and $C \in |\mathcal{O}_Y(k)|$ is a curve, so Proposition 3.10 (see §3.4) tells us that

$$p_g(C) \geq \frac{1}{2} (\sum_{j=1}^{e-1} a_i - 4 - (e-1)) k \gamma + 1.$$

So

$$\begin{aligned}
\sum_{i=1}^{s+1} m_i (m_i - 1) &\leq 2p_a(C) - 2p_g(C) \\
&\leq k(k + \sum_{j=1}^{e-1} a_j - e - 2)\gamma - (\sum_{j=1}^{e-1} a_j - 4 - (e-1)) k \gamma \\
&\leq (k+1) k \gamma. \qquad (3.3)
\end{aligned}$$

Step 5: Let us combine the inequalities from the previous steps. We have

$$\sum_{i=1}^{s+1} m_i^2 = \sum_{i=1}^{s+1} m_i + \sum_{i=1}^{s+1} m_i (m_i - 1)$$

$$
\begin{aligned}
&\leq \frac{(s+1)}{2s} k a_e \gamma + (k+1)k\gamma && \text{by (3.2) and (3.3),} \\
&\leq (a_e + k + 1)k\gamma && \text{since } s \geq 1 \text{ by the base case.}
\end{aligned}
$$

From (3.1) we see that

$$2k^2 a_e \gamma < \sum_{i=1}^{s+1} m_i^2 \leq (a_e + k + 1)k\gamma$$

This simplifies to

$$2ka_e < a_e + k + 1,$$

and solving for k gives

$$k < \frac{a_e + 1}{2a_e - 1},$$

which is a contradiction since $a_e \geq 2$ and k is a positive integer. $\square$

As a last step, we will verify that the line bundle L in Theorem 3.9 is big. Since

$$r \leq \frac{1}{8} a_1 \cdots a_e,$$

we see that

$$
\begin{aligned}
L^2 &= \left(a_e H - \sum_{i=1}^{r} 2E_i \right) \\
&= a_e^2 a_1 \cdots a_{e-1} - 4r \\
&\geq a_e^2 a_1 \cdots a_{e-1} - \frac{1}{2} a_1 \cdots a_e \\
&= \left(a_e - \frac{1}{2} \right) a_1 \cdots a_e > 0.
\end{aligned}
$$

Therefore L is big and nef. Now set

$$r = \left\lfloor \frac{1}{8} a_1 \cdots a_e \right\rfloor.$$

By Proposition 3.7 applied to L and r, we see that

$$\operatorname{gon}(C) \geq r + 1 \geq \frac{1}{8} a_1 \cdots a_e.$$

This completes the proof of Theorem 3.3.

3.4. Curves on complete intersections

In this section, we begin by giving lower bounds for the geometric genus of curves on generic complete intersections. The starting point is a paper of Clemens [17], which dealt with curves on generic hypersurfaces. Later, Ein [24] gave much stronger results which applied to subvarieties of generic complete intersections, and subsequently these bounds were improved by Voisin [50] (similar results have been proved in [52], [19], [21], etc). As a corollary of the estimates of Ein and Voisin (for comparison, see [7, proof of Proposition 3.8]), we have:

Proposition 3.10. *Let $X \subset \mathbb{P}^{n+e}$ be a very general complete intersection of dimension $n \geq 2$ and type $(d_1, \ldots, d_e)$. For any integral curve $C \subset X$, we have*

$$p_g(C) \geq 1 + \frac{1}{2}(\sum_{i=1}^{e} d_i - 2n - e) \cdot \deg_{\mathbb{P}^{n+e}}(C).$$

Proof. Let $V^{d_i} := H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^{n+k}}(d_i))$ for $d_i \geq 2$ and let $V = \prod_i V^{d_i}$. Consider the universal complete intersection $\mathcal{X} \subseteq V \times \mathbb{P}^{n+k}$ of type $(d_1, \ldots, d_e)$ with the two projections

$$pr_1 \colon \mathcal{X} \longrightarrow V \quad \text{and} \quad pr_2 \colon \mathcal{X} \longrightarrow \mathbb{P}^{n+e}.$$

Let $v = \dim V$ and suppose that a very general complete intersection of type $(d_1, \ldots, d_e)$ in $\mathbb{P}^{n+e}$ contains an irreducible curve of geometric genus g. By standard arguments, there is a diagram:

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{f} & \mathcal{X} \\ \pi \downarrow & & \downarrow pr_1 \\ T & \xrightarrow{\rho} & V \end{array}$$

where

- $\pi \colon \mathcal{C} \to T$ is a family of curves of geometric genus g whose general member $\mathcal{C}_t = \pi^{-1}(t)$ is smooth;
- ρ is étale;
- $f_t \colon \mathcal{C}_t \to X_{\rho(t)}$ is birational onto its image.

In this setting, Ein and Voisin show that if $t \in T$ is a general point, then

$$\Omega_{\mathcal{C}}^{v+1} \otimes \left((pr_2 \circ f)^* \mathcal{O}_{\mathbb{P}^{n+e}}(2(n+e) - \sum_{i=1}^{e} d_i - e) \right) \Bigg|_{\mathcal{C}_t}$$

is generically generated by its global sections. This implies that the canonical bundle of the general curve $\mathcal{C}_t$ is of the form

$$K_{\mathcal{C}_t} \cong (\sum_{i=1}^{e} d_i - 2n - e)H_{\mathcal{C}_t} + (\text{effective divisor}),$$

where $H_{\mathcal{C}_t}$ is the pull-back of the hyperplane bundle from $\mathbb{P}^{n+e}$. Comparing degrees on both sides, we arrive at the desired result:

$$2g(C_t) - 2 \geq (\sum_{i=1}^{e} d_i - 2n - e) \cdot \deg(f(C_t)). \qquad \square$$

Note that this proof also gives a lower bound for the gonality of the curve C_t.

We will also need the following:

Lemma 3.11. *Let $C \subset \mathbb{P}^N$ (for $N \geq 3$) be a reduced and irreducible curve of degree k with a finite collection of points p_i ($i = 1, \ldots, \ell$) which have multiplicity m_i. After a generic projection of $\varphi\colon C \to C' \subset \mathbb{P}^2$, the multiplicities of the image points $\varphi(p_i)$ in C' remain the same, so we obtain an estimate:*

$$\sum_{i=1}^{\ell} \frac{m_i(m_i-1)}{2} \leq p_a(C') - p_g(C') = \frac{(k-1)(k-2)}{2} - p_g(C).$$

This essentially follows from the observation that the tangent cone of a singular point on C will be disjoint to a general $(N-3)$-plane in $\mathbb{P}^N$.

Remark 3.12. Given a smooth variety X an a curve $C \subset X$, the multiplicity of C at a point p is equal to the intersection of the strict transform $\tilde{C}$ against the exceptional divisor E_p of the blow-up $\mu\colon \tilde{X} \to X$ at p (see [26, pg. 79]).

Now consider a smooth hypersurface $Y \subset \mathbb{P}^N$ (for $N \geq 4$) of degree a and let $\mu\colon \tilde{Y} \to Y$ be the blow-up of Y at a finite collection of points p_i with exceptional divisors E_i ($i = 1, \ldots, r$). Let H_Y denote the pullback via μ of the hyperplane class on Y, and suppose that $\tilde{C} \subset \tilde{Y}$ is an integral curve which is not entirely contained in one of the exceptional divisors. By the Lefschetz hyperplane theorem and Poincaré duality, $\tilde{C}$ is numerically a $\mathbb{Q}$-linear combination of terms involving H_Y^{N-2} and E_i^{N-2}. Note that the mixed terms involving $H \cdot E_i$ must vanish because we have blown up points. Furthermore, we have

$$H_Y^{N-1} = a \quad \text{and} \quad (-E_i)^{N-1} = -1,$$

and the intersection numbers $\tilde{C} \cdot H \geq 1$ and $\tilde{C} \cdot E_i \geq 0$ must be integers. Therefore, we can write down the numerical class of the curve $\tilde{C}$:

Lemma 3.13. *With the set-up above, we can write*

$$\tilde{C} \equiv_{num} \frac{k}{a} H_Y^{N-2} + (-1)^n \sum_{i=1}^{r} m_i E_i^{N-2}$$

for some integers $k \geq 1$ and $m_i \geq 0$.

3.5. Multiplicative bounds for codimension two complete intersections

Recall that we have the inclusions $X \subset Y \subset \mathbb{P}^{n+2}$, where $Y \in |\mathcal{O}_{\mathbb{P}^{n+2}}(a)|$ is a very general hypersurface and $X \in |\mathcal{O}_Y(b)|$ is a very general complete intersection of dimension n such that $b \geq a \geq 9n$. The reader will notice similarities between the proof of Theorem 3.9 and the argument below.

Theorem 3.14. *With the set-up above, fix an integer r such that*

$$r \leq \frac{2}{3(n+1)^2} ab$$

and let $p_1, \ldots, p_r \in X$ be any collection of distinct points. Let

$$\mu \colon \tilde{Y} =_{\text{def}} \text{Bl}_{\{p_1,\ldots,p_r\}} \to Y$$

be the blow-up with exceptional divisors E_i over p_i and let $H_Y = \mu^* \mathcal{O}_Y(1)$. Then the line bundle

$$L := bH_Y - \sum_{i=1}^{r} (n+1) E_i \quad \textit{is nef.}$$

Proof. We will prove this by induction on r. For the base case $r \leq 2$, the statement is trivial as soon as $b \geq 2(n+1)$ since H_Y is very ample. By induction, we may assume that the theorem holds for $r = s$, where

$$2 \leq s \leq \frac{2}{3(n+1)^2} ab - 1.$$

We want to prove that the theorem holds for $r = s+1$.

Suppose for the sake of contradiction that the theorem fails when $r = s+1$. Then there exists a collection of points $p_1, \ldots, p_{s+1} \in X$ such that the corresponding divisor

$$L := bH_Y - \sum_{i=1}^{s+1} (n+1) E_i$$

on the blow-up $\tilde{Y}$ is not nef. By definition, this means that there is an integral curve $\tilde{C} \subset \tilde{Y}$ such that $L \cdot \tilde{C} < 0$.

Claim: $\tilde{C}$ cannot be contained in some exceptional divisor E_j. If this were the case, then

$$\mathcal{O}_{E_j}(E_j) \cong \mathcal{O}_{E_j}(-1) \implies \tilde{C} \cdot E_j = \deg \mathcal{O}_{\tilde{C}}(E_j) < 0.$$

We would also have $\tilde{C} \cdot H_Y = 0$ so $\tilde{C} \cdot L > 0$, which is false.

By Lemma 3.13 we may write

$$\tilde{C} \equiv_{\text{num}} \frac{k}{a} H_Y^n + (-1)^n \sum_{i=1}^{s+1} m_i E_i^n,$$

where $k \geq 1$ is the degree of the image curve $C \subset \mathbb{P}^{n+2}$ and $m_i \geq 0$ are the multiplicities of C at p_i. Note that

$$L \cdot \tilde{C} < 0 \implies \sum_{i=1}^{s+1} m_i > \frac{1}{n+1} bk.$$

For a fixed quantity $\sum_{i=1}^{s+1} m_i$, the expression

$$\sum m_i^2$$

is minimized when all m_i are the same. If we set

$$N = \frac{1}{n+1} kb,$$

then it follows that

$$\sum_{i=1}^{s+1} m_i^2 > \left(\frac{N}{s+1} \right)^2 \cdot (s+1) = \frac{k^2 b^2}{(n+1)^2 (s+1)}.$$

From our induction set-up, $s + 1 \leq \frac{2}{3(n+1)^2} ab$ so

$$\sum_{i=1}^{s+1} m_i^2 > \frac{3b}{2a} k^2. \tag{3.4}$$

On the other hand, our induction hypothesis implies the theorem holds for $r = s$ (and any collection of s points in X). Hence, the divisor

$$L_I := bH_Y - \sum_{i \in I} (n+1) E_i$$

is nef for any subset $I \subset \{1, 2, \ldots, s+1\}$ with $\#I = s$. Averaging over all I shows that

$$L_{s+1} := \frac{s+1}{s} bH_Y - \sum_{i=1}^{s+1} (n+1) E_i$$

is nef. This implies that $L_{s+1} \cdot \tilde{C} \geq 0$, and hence

$$\sum_{i=1}^{s+1} m_i \leq \frac{3}{2(n+1)} bk. \tag{3.5}$$

By Lemma 3.11 and Proposition 3.10 applied to $C \subset Y \subset \mathbb{P}^{n+2}$, we have

$$\begin{aligned} \sum_{i=1}^{r} \frac{m_i(m_i-1)}{2} &\leq \frac{1}{2}(k-1)(k-2) - p_g(K) \\ &\leq \frac{1}{2}(k-1)(k-2) - \frac{1}{2}(a-2n-3)k - 1 \\ &= \frac{1}{2}(k-a+2n)k. \end{aligned} \tag{3.6}$$

Next, we combine this bound with the inequalities in (3.4) and (3.5):

$$\begin{aligned} \frac{3b}{2a}k^2 < \sum_{i=1}^{r} m_i^2 = \sum_{i=1}^{r} m_i(m_i-1) + \sum_{i=1}^{r} m_i \\ \leq (k-a+2n)k + \frac{3}{2(n+1)}bk \\ \leq k^2 + \left(2n - a + \frac{1}{2}b\right) k, \end{aligned}$$

where the last inequality follows from the fact that $n \geq 2$. After solving for k, we can further simplify using $b \geq a \geq 1$ and $b - 2a \leq b - \frac{2}{3}a$:

$$\begin{aligned} k &< \frac{a(4n+b-2a)}{3b-2a} \\ &= \frac{4an}{3b-2a} + \frac{a(b-2a)}{3(b-2a/3)} \\ &\leq 4n + \frac{1}{3}a. \end{aligned} \tag{3.7}$$

Since all $m_i \geq 0$, the inequality in (3.6) also gives

$$\frac{1}{2}(k-1)(k-2) - \frac{1}{2}k(a-2n-3) - 1 \geq \sum_{i=1}^{r} \frac{m_i(m_i-1)}{2} \geq 0 \implies k \geq a - 2n,$$

which is a contradiction of (3.7) as soon as $a \geq 9n$. □

Finally, let us verify that the line bundle L in Theorem 3.14 is big. Fix $n \geq 2$, choose $b, a \geq 9n$, and set

$$r = \left\lfloor \frac{2}{3(n+1)^2} \cdot ab \right\rfloor.$$

So far, we have shown that for any tuple of r distinct points $p_1, \ldots, p_r \in X$, the divisor

$$L := bH - \sum_{i=1}^{r} (n+1) E_i$$

on the blow-up $\mu \colon \tilde{Y} \to Y$ is nef. It is straightforward to check that $(L^{n+1}) > 0$ on $\tilde{Y}$:

$$\begin{aligned} &\left(bH - \sum_{i=1}^{r} (n+1) E_i \right)^{n+1} \\ &= ab^{n+1} - r \cdot (n+1)^{n+1} \\ &\geq ab^{n+1} - \frac{2}{3}(n+1)^{n-1} ab > 0 \end{aligned}$$

holds as long as $b \geq a \geq n+1$. Therefore, L is both nef and big. By Proposition 3.7, it follows that

$$\operatorname{cov.gon}(X) \geq r + 1 \geq \frac{2}{3(n+1)^2} \cdot ab.$$

This completes the proof of Theorem 1.20.

Bibliography

[1] Dan Abramovich, *A linear lower bound on the gonality of modular curves*, International Mathematics Research Notices **1996** (1996), no. 20, 1005–1011.

[2] Alberto Alzati and Gian Pietro Pirola, *On the holomorphic length of a complex projective variety*, Archiv der Mathematik **59** (1992), no. 4, 398–402.

[3] Michael Artin and David Mumford, *Some elementary examples of unirational varieties which are not rational*, Proceedings of the London Mathematical Society **3** (1972), no. 1, 75–95.

[4] Francesco Bastianelli, *On symmetric products of curves*, Transactions of the American Mathematical Society **364** (2012), no. 5, 2493–2519.

[5] Francesco Bastianelli, Ciro Ciliberto, Flaminio Flamini, and Paola Supino, *A note on gonality of curves on general hypersurfaces*, Bollettino dell'Unione Matematica Italiana **11** (2018), no. 1, 31–38.

[6] Francesco Bastianelli, Renza Cortini, and Pietro De Poi, *The gonality theorem of Noether for hypersurfaces*, Journal of Algebraic Geometry **23** (2014), no. 2, 313–339.

[7] Francesco Bastianelli, Pietro De Poi, Lawrence Ein, Robert Lazarsfeld, and Brooke Ullery, *Measures of irrationality for hypersurfaces of large degree*, Compositio Mathematica **153** (2017), no. 11, 2368–2393.

[8] Thomas Bauer, *Projective images of Kummer surfaces*, Mathematische Annalen **299** (1994), no. 1, 155–170.

[9] ———, *Seshadri constants and periods of polarized abelian varieties*, Mathematische Annalen **312** (1998), no. 4, 607–623, With an appendix by the author and Tomasz Szemberg.

[10] ———, *Seshadri constants on algebraic surfaces*, Mathematische Annalen **313** (1999), no. 3, 547–583.

[11] Arnaud Beauville, *Complex algebraic surfaces*, no. 34, Cambridge University Press, 1996.

[12] Caucher Birkar, Paolo Cascini, Christopher D Hacon, and James McKernan, *Existence of minimal models for varieties of log general type*, Journal of the American Mathematical Society **23** (2010), no. 2, 405–468.

[13] Christina Birkenhake and Herbert Lange, *Complex abelian varieties*, vol. 302, Springer Science & Business Media, 2004.

[14] Nathan Chen, *Irrationality of complete intersections is multiplicative*, in preparation.

[15] ———, *Degree of irrationality of very general abelian surfaces*, Algebra & Number Theory **13** (2019), no. 9, 2191–2198.

[16] Nathan Chen and David Stapleton, *Fano hypersurfaces with arbitrarily large degrees of irrationality*, Forum of Mathematics, Sigma **8** (2020), e.24, 12pp.

[17] Herbert Clemens, *Curves on generic hypersurfaces*, Annales scientifiques de l'École Normale Supérieure **19** (1986), no. 4, 629–636.

[18] Herbert Clemens and Phillip A Griffiths, *The intermediate Jacobian of the cubic threefold*, Annals of Mathematics **95** (1972), no. 2, 281–356.

[19] Herbert Clemens and Ziv Ran, *Twisted genus bounds for subvarieties of generic hypersurfaces*, American journal of mathematics **126** (2004), no. 1, 89–120.

[20] Elisabetta Colombo, Olivier Martin, Juan Carlos Naranjo, and Gian Pietro Pirola, *Degree of irrationality of a very general abelian variety*, arXiv preprint arXiv:1906.11309 (2019).

[21] Izzet Coskun and Eric Riedl, *Algebraic hyperbolicity of the very general quintic surface in* $\mathbb{P}^3$, Advances in Mathematics **350** (2019), 1314–1323.

[22] Tommaso de Fernex, *Birationally rigid hypersurfaces*, Inventiones mathematicae **192** (2013), no. 3, 533–566.

[23] Igor V. Dolgachev, *A brief introduction to Enriques surfaces*, Mathematical Society of Japan, 2016, pp. 1–32.

[24] Lawrence Ein, *Subvarieties of generic complete intersections*, Inventiones mathematicae **94** (1988), no. 1, 163–169.

[25] Najmuddin Fakhruddin, *Zero cycles on generic hypersurfaces of large degree*, arXiv preprint math/0208179 (2002).

[26] William Fulton, *Intersection theory*, second ed., Ergebnisse der Math. und ihrer Grenzgebiete (3), vol. 2, Springer-Verlag, Berlin, 1998.

[27] Christopher D Hacon and James McKernan, *Existence of minimal models for varieties of log general type ii*, Journal of the American Mathematical Society **23** (2010), no. 2, 469–490.

[28] Brendan Hassett, *Cubic fourfolds, K3 surfaces, and rationality questions*, Rationality Problems in Algebraic Geometry, Springer, 2016, pp. 29–66.

[29] James Hotchkiss, Chung Ching Lau, and Brooke Ullery, *The gonality of complete intersection curves*, Journal of Algebra **560** (2020), 579–608.

[30] V. A. Iskovskih and Ju. I. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sb. (N.S.) **86(128)** (1971), 140–166. MR 0291172

[31] Atsushi Ito, *Seshadri constants via toric degenerations*, J. Reine Angew. Math. **695** (2014), 151–174.

[32] Jong Hae Keum, *Every algebraic Kummer surface is the K3-cover of an Enriques surface*, Nagoya Mathematical Journal **118** (1990), 99–110.

[33] János Kollár, *Nonrational hypersurfaces*, Journal of the American Mathematical Society **8** (1995), no. 1, 241–249.

[34] Robert Lazarsfeld, *Lectures on linear series, with the assistance of Guillermo Fernández del Busto*, IAS/Park City Math. Ser **3** (1997), 161–219.

[35] ———, *Positivity in algebraic geometry I: Classical setting: Line bundles and linear series*, vol. 48, Springer, 2004.

[36] Solomon Lefschetz, *On certain numerical invariants of algebraic varieties with application to abelian varieties*, Transactions of the American mathematical Society **22** (1921), no. 3, 327–406.

[37] Angelo Felice Lopez and Gian Pietro Pirola, *On the curves through a general point of a smooth surface in* $\mathbb{P}^3$, Mathematische Zeitschrift **219** (1995), no. 1, 93–106.

[38] Olivier Martin, *The degree of irrationality of most abelian surfaces is 4*, to appear in Ann. Sci. Éc. Norm. Supér.

[39] ———, *On a conjecture of voisin on the gonality of very general abelian varieties*, Advances in Mathematics **369** (2020), 107173.

[40] ———, *Zero-cycles and measures of irrationality for abelian varieties*, Ph.D. thesis, The University of Chicago, 2020.

[41] Shigefumi Mori and Shigeru Mukai, *The uniruledness of the moduli space of curves of genus 11*, Algebraic Geometry (Michel Raynaud and Tetsuji Shioda, eds.), Springer Berlin Heidelberg, 1983, pp. 334–353.

[42] David Mumford et al., *Rational equivalence of 0-cycles on surfaces*, Journal of mathematics of Kyoto University **9** (1969), no. 2, 195–204.

[43] Max Noether, *Zur Grundlegung der Theorie der algebraischen Raumcurven*, Konig. Akad. d. Wiss., Berlin (1883).

[44] Bjorn Poonen, *Gonality of modular curves in characteristic p*, Mathematical Research Letters **14** (2007), no. 4, 691–701.

[45] Stefan Schreieder, *Stably irrational hypersurfaces of small slopes*, Journal of the American Mathematical Society **32** (2019), no. 4, 1171–1199.

[46] Geoffrey Smith, *Covering gonalities of complete intersections in positive characteristic*, arXiv:2005.08878 (2020).

[47] David Stapleton, *The degree of irrationality of very general hypersurfaces in some homogeneous spaces*, Ph.D. thesis, PhD thesis, Stony Brook University, 2017.

[48] David Stapleton and Brooke Ullery, *The degree of irrationality of hypersurfaces in various Fano varieties*, manuscripta mathematica **161** (2020), no. 3, 377–408.

[49] Burt Totaro, *Hypersurfaces that are not stably rational*, Journal of the American Mathematical Society **29** (2016), no. 3, 883–891.

[50] Claire Voisin, *On a conjecture of Clemens on rational curves on hypersurfaces*, J. Diff. Geom **44** (1996), no. 1, 200–213.

[51] ______, *Chow rings and gonality of general abelian varieties*, Annales Henri Lebesgue **1** (2018), 313–332.

[52] Geng Xu, *Subvarieties of general hypersurfaces in projective space*, Journal of Differential Geometry **39** (1994), no. 1, 139–172.

[53] Ruijie Yang, *On irrationality of hypersurfaces in* $\mathbb{P}^{n+1}$, Proceedings of the American Mathematical Society **147** (2019), no. 3, 971–976.

[54] Hisao Yoshihara, *Degree of irrationality of an algebraic surface*, Journal of Algebra **167** (1994), no. 3, 634–640.

[55] ______, *Degree of irrationality of a product of two elliptic curves*, Proceedings of the American Mathematical Society **124** (1996), no. 5, 1371–1375.

[56] ______, *Degree of irrationality of hyperelliptic surfaces*, Algebra Colloquium, vol. 7, Springer, 2000, pp. 319–328.