On the characterization of rational homotopy types and Chern classes of closed almost complex manifolds

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Abstract of the Dissertation

On the characterization of rational homotopy types and Chern classes of closed almost complex manifolds

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The homotopy theory of rationalized simply connected spaces was shown by Quillen to be encoded algebraically in differential graded Lie algebras in his seminal work on rational homotopy theory. Motivated by this theory and Whitney's treatment of differential forms on arbitrary complexes, Sullivan described a theory of computable algebraic models for rational homotopy types in terms of differential graded algebras of differential forms in his "Infinitesimal Computations in Topology". Following a challenge posed therein, we give a characterization of the possible simply connected rational homotopy types, along with a choice of rational Chern classes and fundamental class, realized by closed almost complex manifolds in complex dimensions three and greater, with a caveat in complex dimensions congruent to 2 mod 4 depending on the first Chern class. As a consequence, beyond demonstrating that rational homotopy types of closed almost complex manifolds are plenty, we observe that the realizability of a simply connected rational homotopy type by a closed almost complex manifold, of complex dimension not equal to 2 mod 4, depends only on its cohomology ring. We conclude with some computations and examples.

To my mother Miriam

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Acknowledgments

Mathematically I would first and foremost thank my advisor Dennis Sullivan for the influence he has had on me. As he once remarked to me, on our planet one can zoom in on virtually any square inch to find a whole world teeming with life and structure, and it is no different in math. My countless hours of discussion with him have taught me that there is great value in revisiting and better understanding the foundations on which mathematical constructions lie.

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Chapter 1

Introduction

1.1 The homotopy types of closed manifolds: background and history, and statement of main theorem

In the 1930's, Hassler Whitney's pioneering work on manifolds, bundles, and cohomology marked the birth of differential topology [MichConf40]. In the same article giving the modern definition of a smooth manifold [Wh36], Whitney showed how every manifold can be embedded in Euclidean space. These embeddings naturally equip manifolds with normal bundles, and Whitney early on saw the need for a general theory of vector bundles beyond the tangent bundle [Wh35]. His investigation of the obstructions to linearly independent sections of vector bundles, a problem concurrently considered on the tangent bundle by Eduard Stiefel in his thesis [St35], initiated the study of characteristic classes.

It was known to Whitney that all vector bundles were pulled back from Grassmannians with their tautological bundles. Lev S. Pontryagin [Po42] studied the homology of these universal spaces, identifying the generators of the integral lattice in rational (co)homology now known as Pontryagin classes. Shiing-Shen Chern conducted a similar study on complex manifolds [Ch46], defining what became known as the Chern classes of the tangent bundle, using the Schubert cell decomposition of the complex Grassmannians; later Wu Wenjun [Wu52] would generalize this notion in his thesis to arbitrary complex vector bundles .

Pontryagin observed that by considering maps of spheres into the one-point compactification of the universal trivial bundle over a point, one can identify the homotopy groups of spheres with equivalence classes of stably framed manifolds up to what is now known as framed cobordism [Po38]. Later, René Thom [Thom54] built on this construction and applied it to all closed smooth manifolds, developing and employing transversality arguments to classify smooth manifolds up to cobordism by calculating the homotopy groups of the one-point compactification of the universal bundle over the Grassmannian.

In the late 1950's and early 1960's, Michel Kervaire and John Milnor introduced surgery, a procedure of removing from a manifold embedded spheres with trivial normal bundle, and used it to determine the finite abelian groups of smooth structures on homotopy spheres [KerMil63], in terms of Bernoulli numbers and homotopy groups of spheres, in dimensions 5 and above. Andrew Wallace [Wal60] independently introduced surgery in the United Kingdom under the name "constructive cobordisms", as applying a surgery to a manifold produces a cobordism to the resulting manifold, and any cobordism can be realized by a finite number of surgeries.

After Stephen Smale proved the generalized Poincaré conjecture in dimensions five and higher by establishing the h–cobordism theorem [Sm62], the work of Kervaire and Milnor could be formulated as classifying the smooth structures on piecewise linear spheres S^n, for any $n \geq 1$. Extending this work, Sergei Novikov in the Soviet Union addressed the problem of classifying smooth structures on simply connected manifolds in dimensions 5 and greater, in terms of vector bundles over their homotopy types and the homotopy groups of the one-point compactification of their normal bundles when embedded in a high-dimensional Euclidean space [Nov64]. William Browder [Br62] in the United States independently did the same, along with characterizing in similar terms as [KerMil63] and [Nov64] which homotopy types were realized by closed smooth manifolds in dimensions 5 and greater. This made use of Spivak's normal spherical fibration [Sp64] characterizing Poincaré duality spaces, a notion earlier identified by Browder in his study of finite complexes admitting a continuous multiplication with unit. Motivated by Hilbert's 5^{th} problem on characterizing Lie groups as locally Euclidean locally compact groups [Hilb02], Browder asked if these complexes with a unital multiplication were realized by smooth manifolds.

Dennis Sullivan in his thesis [Sull65] reformulated the stories of Kervaire–Milnor, Novikov, and Browder without choosing the normal bundle, instead classifying all the simply connected closed manifolds, piecewise–linear or smooth, in a homotopy type via obstruction theory. The obstructions in the piecewise–linear theory lay in a calculable homotopy type with fourfold periodic homotopy groups $0, \mathbb{Z}_2, 0, \mathbb{Z}, 0, \mathbb{Z}_2, 0, \mathbb{Z}, \ldots$. The homotopy groups in the smooth theory are still unknown, though the theory itself can be reduced to stable homotopy using the Adams conjecture, provable using the Frobenius automorphism from algebraic geometry (a possibility first voiced by Daniel Quillen [Q68]). Understanding these results and the utility of localizing homotopy types motivated Sullivan's 1970 MIT notes "Geometric Topology: Localization, Periodicity, and Galois Symmetry" [Sull70] (see also [Sull74]).

Upon tensoring homotopy types and maps by the rationals, the piecewise–linear and smooth obstruction theories become equivalent. The homotopy theory of rationalized simply connected spaces was shown by Quillen to be encoded algebraically in differential graded Lie algebras in his seminal "Rational Homotopy Theory" [Q69]. Motivated by this theory, and influenced by Whitney's treatment of differential forms on arbitrary complexes [Wh57], Sullivan described a theory of computable algebraic models for rational homotopy types in terms of differential graded algebras of differential forms in his "Infinitesimal Computations in Topology" [Sull77].

Here, as our first main goal, we will expound the details and necessary theory to understand and prove a theorem formulated in [Sull77], and accompanied by a sketch proof as an illustration of the developed techniques, on the realization of simply connected rational homotopy types by closed smooth manifolds [Sull77, Theorem 13.2]. The readers of [Sull77] were challenged to formulate and prove an analogous theorem for almost complex manifolds using the enclosed tools. That is our second main goal and theorem: to give a characterization, with a caveat in real dimensions congruent to 4 mod 8, of the possible simply connected rational homotopy types realized by closed almost complex manifolds in dimensions six and greater. The argument is such that the resulting manifold we obtain is *stably almost complex*, i.e. it has an almost complex structure on its stable tangent bundle; this stable almost complex structure is then induced by a genuine almost complex structure on the tangent bundle if (and only if) its top Chern class evaluates to the Euler characteristic:

Theorem 1.1.1. *Let X be a formally n–dimensional simply–connected rational space of finite type satisfying Poincaré duality on its rational cohomology, $n \geq 5$, and let $[X] \in H_n(X;\mathbb{Q})$ be a non-zero element. Furthermore, let $c_i \in H^{2i}(X;\mathbb{Q})$, $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ be cohomology classes. Then we have:*

1. *If n is odd, there is a closed stably almost complex n–manifold M and a rational equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $c_i(TM) = f^*(c_i)$.*

2. *If $n \equiv 2 \bmod 4$, then there is a closed stably almost complex manifold M and a rational equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $c_i(TM) = f^*(c_i)$ if the numbers $\langle c_{i_1} c_{i_2} \cdots c_{i_r}, [X] \rangle$ are integers that satisfy the Stong congruences of a stably almost complex manifold: that is, denoting by σ_i the elementary symmetric polynomials in the variables $e^{x_j} - 1$, where the x_j are given by formally writing $1 + c_1 + c_2 + \cdots = \prod_j (1 + x_j)$, we have*

$$\langle z \cdot \mathrm{Td}(X), [X] \rangle \in \mathbb{Z} \text{ for every } z \in \mathbb{Z}[\sigma_1, \sigma_2, \ldots].$$

   Here $\mathrm{Td}(X)$ denotes the Todd polynomial evaluated on $c_1, c_2, \ldots$.

3. *If $n \equiv 0 \bmod 4$, then there is a closed stably almost complex manifold M and a rational equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $c_i(TM) = f^*(c_i)$ if*

   - *the quadratic form on $H^{\frac{n}{2}}(X;\mathbb{Q})$ given by $q(\alpha, \beta) = \langle \alpha\beta, [X] \rangle$ is equivalent over $\mathbb{Q}$ to one of the form $\sum_i \pm y_i^2$,*

- *if we define* $p_i = (-1)^i \sum_j (-1)^j c_j c_{i-j}$*, then* $\langle L(p_1, \ldots, p_{n/4}), [X] \rangle = \sigma(X)$*, where* L *is Hirzebruch's* L*–polynomial,*
- *the numbers* $\langle c_{i_1} c_{i_2} \cdots c_{i_r}, [X] \rangle$ *are integers that satisfy the Stong congruences of a stably almost complex manifold described above,*
- *if* $c_1 = 0$ *and* $n \equiv 4 \bmod 8$*, the numbers* $\langle p_{i_1} p_{i_2} \cdots p_{i_r}, [X] \rangle$ *are integers that satisfy a further set of Stong congruences: denoting by* σ_i^p *the elementary symmetric polynomials in the variables* $e^{x_j} + e^{-x_j} - 2$*, where the* x_j *are given by formally writing* $1 + p_1 + p_2 + \cdots = \prod_j (1 + x_j^2)$*, we have*

$$\langle z \cdot \hat{A}(X), [X] \rangle \in 2\mathbb{Z} \text{ for every } z \in \mathbb{Z}[\sigma_1^p, \sigma_2^p, \ldots].$$

  Here $\hat{A}(X)$ *denotes the* $\hat{A}$ *polynomial evaluated on* $p_1, p_2, \ldots$*. Note that the above are conditions on* $c_1, c_2, \ldots$*, as they determine* $p_1, p_2, \ldots$*.*

If $c_1 = 0$ *in any of the cases above, and the corresponding conditions in that case are satisfied, then the first Chern class of the resulting stably almost complex manifold* M *vanishes in* integral *cohomology. If* n *is even and* $\langle c_{n/2}, [X] \rangle$ *equals the Euler characteristic of* X*, and the appropriate conditions above are satisfied, then the stable almost complex structure on the obtained manifold* M *is induced by an almost complex structure (in particular, the almost complex structure also has* $f^*(c_i)$ *as its Chern classes).*

I am indebted to Zhixu Su for her elaboration of Sullivan's theorem concerning obtaining a degree one map, along the lines of [Ba76, Théorème 8.2.2], and the clear formulation of the result as given in [Su14], [Su09]. I spend some time below clearing up details surrounding the crucial Main Diagram in [Su09].

In the last chapter we will observe some consequences of Theorem 1.1.1; beyond demonstrating that rational homotopy types of almost complex manifolds are plenty, we observe that the realizability of a simply connected rational homotopy type by an almost complex manifold, of real dimension ≥ 6 not equal to $4 \bmod 8$, depends only on the cohomology ring. We contrast this with the case of rational homotopy types realized by compact complex manifolds satisfying the $\partial\overline{\partial}$–lemma (such as Kähler manifolds), where all the higher multiplications in the associated C_∞ structure on the cohomology necessarily vanish; in this sense one can think of the rational homotopy types of $\partial\overline{\partial}$–manifolds as the free objects on their underlying cohomology algebra. In the almost complex case, for the dimensions not excluded, no further restriction is placed on the higher operations in the associated C_∞ structure beyond the requirement that the cohomology algebra with its binary multiplication satisfies Poincaré duality. One can wonder whether non–$\partial\overline{\partial}$ complex manifolds generally lie somewhere strictly between these two extremes.

Chapter 2

Realization by almost complex manifolds

2.1 Rational spaces, and necessary conditions for realization by a closed almost complex manifold

We say a simply connected space is *rational* if all of its reduced integer homology groups (or equivalently, homotopy groups) are isomorphic as abelian groups to rational vector spaces. A map between two simply connected spaces is a *rational homotopy equivalence* if it induces an isomorphism on homology groups with rational coefficients (or equivalently, on the homotopy groups tensored with the rationals). Spaces X and Y are *rationally homotopy equivalent* if there is a zig-zag of rational homotopy equivalences $X \leftarrow Z_1 \rightarrow Z_2 \leftarrow \cdots \leftarrow Z_k \rightarrow Y$ between them. Note that a rational homotopy equivalence between rational spaces is a homotopy equivalence.

For every simply connected space X, there is a rational space $X_{\mathbb{Q}}$ and a rational homotopy equivalence $X \xrightarrow{f} X_{\mathbb{Q}}$; the space $X_{\mathbb{Q}}$ is unique up to homotopy equivalence; we call f a *rationalization*. To rationalize spheres, one takes the homotopy colimit of the diagram $S^n \xrightarrow{2} S^n \xrightarrow{3} S^n \xrightarrow{4} S^n \xrightarrow{5} \cdots$, i.e. one forms a sequence of cylinders $S^n \times [0,1]$ and glues the appropriate ends via a degree k self-map of the sphere. The inclusion of S^n as, say, the leftmost end is a rationalization.

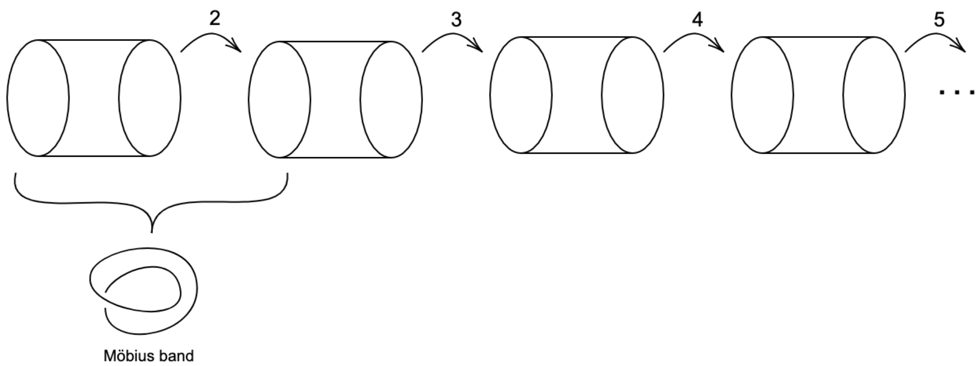


Figure 2.1: A rational circle [GM81, Lemma 7.5]. Note how the circle representing the left end of the leftmost cylinder may be arbitrarily divided in the fundamental group, by pushing it to the right an appropriate number of times (and multiplying if necessary). For example, to divide by three we may push it two cylinders across (seeing how to divide by six) and multiply by two. Hence the fundamental group of this construction is $\mathbb{Q}$. For higher dimensional spheres mapping in, note that by compactness any map will land in a finite stage of the construction, which deformation retracts onto a circle, and is hence nullhomotopic. Hence $\pi_{\geq 2} = 0$.

To rationalize a space (with the homotopy type of a cell complex), we note that we can build the space inductively by starting with a wedge of spheres, and then repeatedly taking the mapping cone of a map from a sphere into the previous stage. We can rationalize the spheres involved, inducing a sequence of mapping cones whose final stage will be the rationalization of our space.

Working with the rationalizations of spaces up to rational homotopy equivalence facilitates computation, as such spaces can be faithfully encoded in nilpotent graded-commutative differential algebras [Sull77]. Algebraic properties and constructions on these nilpotent algebras have corresponding geometric pictures [Sull77, §11]. This is particularly effective when considering smooth manifolds, where this differential algebra capturing the rational homotopy type of the space is, upon tensoring with the reals, a nilpotent replacement of the de

Rham algebra of forms [DGMS75, Corollary 3.4]

2.1.1 The realization problem.

We now aim to describe the simply connected rational homotopy types realizable by closed almost complex manifolds, along with the rational Chern classes they may carry. Precisely, take a simply connected rational space X; we will say a closed n–manifold M realizes X if there is a rational homotopy equivalence $M \xrightarrow{f} X$. Note that the existence of such a map implies that X and M have isomorphic rational cohomology rings, and so X can be equipped with a non-degenerate pairing $H^k(X;\mathbb{Q}) \otimes H^{n-k}(X;\mathbb{Q}) \to \mathbb{Q}$ given by $\alpha \otimes \beta \mapsto \langle \alpha\beta, f_*[M]\rangle$. In particular, the rational cohomology of X vanishes above degree n; we call this integer n the *formal dimension* of the rational space X. Let us now pick any cohomology classes $c_1 \in H^2(X;\mathbb{Q})$, $c_2 \in H^4(X;\mathbb{Q})$, etc. and a nonzero homology class $[X] \in H_n(X;\mathbb{Q})$ which we will think of as Chern classes and the fundamental class of (any closed manifold realizing) this data. If we had a rational homotopy equivalence $M \xrightarrow{f} X$ from an almost complex manifold M with Chern classes c_i, note that X would have a natural choice of such classes, namely $f^{*-1}(c_i)$ and $f_*[M]$. Here $[M]$ denotes the fundamental class corresponding to the orientation on M induced by its almost complex structure. Generally we will say a manifold with a given choice of orientation has an almost complex (or stably almost complex structure) if it admits one that induces the given orientation.

We now state our realization problem as:

Given a simply connected rational space X with prescribed elements $c_i \in H^{2i}(X;\mathbb{Q})$ and $0 \neq [X] \in H_n(X;\mathbb{Q})$, is there a closed almost complex manifold M and a rational equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $f^*(c_i) = c_i(M)$?

We choose to incorporate the fundamental class $[X] \in H_n(X;\mathbb{Q})$ as part of the given data, since this facilitates the calculation of the Chern numbers of the realizing M by

$$\langle c_I, [M]\rangle = \langle f^*(c_I), [M]\rangle = \langle c_I, f_*[M]\rangle = \langle c_I, [X]\rangle.$$

One may scale this fundamental class $[X]$ by any rational to obtain another class in $H_n(X;\mathbb{Q}) \cong \mathbb{Q}$ with respect to which the pairing described above is still nondegenerate.

2.1.2 Necessary conditions for realization.

Let us now consider the necessary implications on $(X, c_i, [X])$ in the case of a positive answer to the above question:

(i) Since a closed manifold has finitely generated homology, we see that $H_*(X;\mathbb{Q})$ must be finite dimensional (we say X is of *finite type*).

(ii) As we saw above, there is a non-degenerate pairing on the cohomology of X; we say X satisfies rational Poincaré duality. Note that this property does not depend on the choice of non-zero fundamental class in $H_n(X;\mathbb{Q}) \cong \mathbb{Q}$. Given a choice of fundamental class $[X]$, the pairing $H^k(X;\mathbb{Q}) \otimes H^{n-k}(X;\mathbb{Q}) \to \mathbb{Q}$ given by $\alpha \otimes \beta \mapsto \langle \alpha\beta, [X] \rangle$ being nondegenerate is equivalent to the cap product $[X] \cap -$ being an isomorphism $H^k(X;\mathbb{Q}) \to H_{n-k}(X;\mathbb{Q})$ (see [Br72, Proposition I.2.1] for the argument). Furthermore, the formal dimension n must be even (as almost complex manifolds are even-dimensional).

(iii) The Chern numbers $\langle c_I(M), [M] \rangle$ of the realizing manifold must be *integers* that satisfy the congruences of a stably almost complex bordism class. Namely, suppose one has a closed stably almost complex manifold M. Its stable tangent bundle is then classified by a map $M \xrightarrow{\tau_M} BU$, and we can consider the element $\tau_{M*}[M] \in H_*(BU;\mathbb{Q})$. Note that if M is complex bordant to N via W, then $\tau_{M*}[M] = \tau_{N*}[N]$. Indeed, $0 = \tau_{W*}[\partial W] = \tau_{M*}[M] - \tau_{N*}[N]$. Thus we obtain a map from complex bordism Ω^U to the homology of the classifying space $H_*(BU;\mathbb{Q})$. Stong characterized the image of this map in the following way [Stong65a]: a class $\alpha \in H_*(BU;\mathbb{Q})$ is in the image of $\Omega^U \to H_*(BU;\mathbb{Q})$ if and only if $\langle z\,\mathrm{Td}(c_i), \alpha \rangle \in \mathbb{Z}$ for every z in the integer polynomial ring generated by the elementary symmetric polynomials e_i in the variables $e^{x_i} - 1$, where x_i are the

Chern roots of the universal Chern class in BU, i.e. formally we have $c = \prod_i (1 + x_i)$, where c is the total Chern class in $H^*(BU;\mathbb{Z})$. Any considered class α will be of some finite degree and so all sums considered for the elements z are finite. The term $\mathrm{Td}(c_i)$ is the Todd genus, $\mathrm{Td}(c_i) = 1 + \frac{c_1}{2} + \frac{c_1^2 + c_2}{12} + \frac{c_1 c_2}{24} + \cdots$. Now the mentioned congruences among Chern numbers from above, which we will refer to as the Stong congruenes, follow from $\langle c_I(M), [M]\rangle = \langle \tau_M^* c_I, [M]\rangle = \langle c_I, \tau_{M*}[M]\rangle = \langle c_I, \alpha\rangle$. Note that for degree reasons, this reduces to *finitely many conditions*. One can think of these congruences as coming from the Atiyah–Singer index theorem; namely $\int_M \mathrm{ch}(E)\,\mathrm{Td}(M)$ must be an integer for every complex vector bundle $E \to M$.

If our almost complex manifold M has $c_1(M) = 0$ in integral cohomology, and its dimension is congruent to 4 mod 8, then a further set of congruences hold among its Chern numbers, according to Stong's description of the image of the map $\Omega^{SU} \xrightarrow{\tau} H_*(BSU;\mathbb{Q})$. Namely, denoting by σ_i^p the elementary symmetric polynomials in the variables $e^{x_j} + e^{-x_j} - 2$, where the x_j are given by formally writing $1 + p_1 + p_2 + \cdots = \prod_j (1 + x_j^2)$, we have

$$\langle z \cdot \hat{A}(M), [M]\rangle \in 2\mathbb{Z} \text{ for every } z \in \mathbb{Z}[\sigma_1^p, \sigma_2^p, \ldots].$$

Here $\hat{A}(M)$ denotes the $\hat{A}$ polynomial, $\hat{A} = 1 - \frac{p_1}{24} + \frac{1}{5760}(7p_1^2 - 4p_2) + \cdots$. Since one can express Pontryagin classes in terms of Chern classes, these further congruences are also conditions on $c_1, c_2, \ldots$. Together with the previous congruences, this determines the image of $\Omega^{SU} \xrightarrow{\tau} H_*(BSU;\mathbb{Q})$ [Stong65b, Theorem 1]. In dimensions not congruent to 4 mod 8, the congruences in the previous paragraph already describe the image of $\Omega^{SU} \xrightarrow{\tau} H_*(BSU;\mathbb{Q})$.

(iv) If the formal dimension n is furthermore divisible by four, then note that rational Poincaré duality induces a nondegenerate symmetric self-pairing on $H^{\frac{n}{2}}(X;\mathbb{Q})$. This pairing on a realizing manifold is the rationalization of a unimodular integral pairing on $H^{\frac{n}{2}}(M;\mathbb{Z})$, which is a nontrivial condition. From the theory of symmetric bilinear forms [MiHu73, §IV.2.6] this condition is equivalent to the pairing on $H^{\frac{n}{2}}(X;\mathbb{Q})$ being equivalent over $\mathbb{Q}$ to one of the form $y_1^2 + y_2^2 + \cdots + y_r^2 - y_{r+1}^2 - \cdots - y_s^2$.

(v) If the formal dimension is divisible by four, we also have the following: the signature of the realizing manifold M, i.e. the signature of the symmetric pairing on $H^{\frac{n}{2}}(X;\mathbb{Q})$, can be calculated from the Pontryagin numbers of M via Hirzebruch's L–genus $1 + \frac{p_1}{3} + \frac{7p_2 - p_1^2}{45} + \cdots$. Recall that the rational Chern classes of any stable almost complex structure on M determine the Pontryagin classes of M by $p_i = (-1)^i \sum_j (-1)^j c_j c_{i-j}$. So we may speak of the L–genus evaluated on Chern classes, with the understanding that first the Pontryagin classes are to be formed. Now, since the realizing map induces an isomorphism of bilinear pairings on $H^{\frac{n}{2}}(-;\mathbb{Q})$, we must have that the signature of the pairing on $H^{\frac{n}{2}}(X;\mathbb{Q})$ is equal to the L–genus evaluated on $[X]$. Indeed,

$$\begin{aligned}
\langle L(p_1, \ldots, p_{n/4}), [X] \rangle &= \langle L(p_1, \ldots, p_{n/4}), f_*[M] \rangle \\
&= \langle f^* L(p_1, \ldots, p_{n/4}), [M] \rangle \\
&= \langle L(p_1(M), \ldots, p_{n/4}(M)), [M] \rangle \\
&= \sigma(M) = \sigma(X).
\end{aligned}$$

(vi) We must have $\langle c_k(M), [M] \rangle = \chi(M) = \chi(X)$. Indeed, the top Chern class of the tangent bundle (thought of as a complex vector bundle) is the primary and only obstruction to finding a global section, and thus is it precisely the Euler class of the underlying real bundle (finding a section of the underlying real bundle lets one split off a complex line bundle from the tangent bundle by acting on the section with the almost complex structure). This condition translates into $\langle c_k, [X] \rangle = \chi(X)$ on the rational space X.

2.1.3 Sufficiency of the conditions in dimension ≥ 6: ingredients of the proof

Now we will spend the rest of this section confirming that in formal dimensions $n \geq 6$ (i.e. $n \geq 5$ if we only want *stably* almost complex manifolds), these necessary conditions, with a caveat in formal dimensions 4 mod 8 which we will discuss, are in fact sufficient.

The proof proceeds in two stages:

1. We form a simply connected space A with a rational homotopy equivalence $A \xrightarrow{g} X$ to our rational space, such that there is a complex vector bundle ξ on A whose Chern classes are $g^*(\bar{c}_i)$. Here $\bar{c}_i$ denote the cohomology classes determined by $(1 + c_1 + c_2 + \cdots)(1 + \bar{c}_1 + \bar{c}_2 + \cdots) = 1$. We then find a closed manifold M and a map $M \xrightarrow{f} A$ such that $f_*[M] = g_*^{-1}[X]$, and such that the stable normal bundle ν of M is $f^*\xi$. By *the stable normal bundle* we mean the normal bundle to M embedded in a large-dimensional sphere; if the dimension of the sphere is large enough, any two embeddings are isotopic and hence their normal bundles are isomorphic as real vector bundles. The stable normal bundle of M then inherits a complex structure from ξ, giving M the structure of a *stably almost complex manifold.* It is at this stage that make use of property (iii) above, in conjunction with the Pontryagin–Thom construction.

2. Once we have a map $M \to A$ covered by a map from ν to ξ as above, we perform *normal surgery* to obtain a new, simply connected, manifold M' mapping to A satisfying the properties in (1), which is furthermore a rational homotopy equivalence. To achieve this we make use of properties (i), (ii), (iv), (v). One then calculates that the Chern classes of the stable tangent bundle of M' (i.e. the sum of TM' and a trivial real bundle, with its induced almost complex structure) are the pullback of the classes c_i on X by the composition $M' \to A \to X$. We then use property (vi) to conclude that the stable almost complex structure on M' is induced by an almost complex structure.

The purpose of stage (1) is to obtain a space which is for the purposes of rational homotopy equivalence just as good as our original rational space X, but furthermore is equipped with a bundle over it with appropriate rational Chern classes.

2.2 Normal surgery

We will begin with stage (2), as it is here that surgery theory, the main aspect of the proof, comes into play. So, suppose we have a map $M \xrightarrow{f} A$, where A is a simply connected space satisfying Poincaré duality on its rational cohomology, with fundamental class $[A]$, of the same formal dimension as M. Furthermore suppose $f_*[M] = [A]$ and that f is covered by a bundle map $\nu \to \xi$ which is a fiberwise isomorphism, where ν is the stable normal bundle of M, and ξ is a (smooth) real vector bundle over A; we refer to this as a *normal map*.

$$\begin{array}{ccc} \nu & \longrightarrow & \xi \\ \downarrow & & \downarrow \\ M & \xrightarrow{f} & A \end{array}$$

In our setting of interest, ν will be the pullback of ξ, and ξ will have an almost complex structure, giving ν an almost complex structure; this property will be preserved under the process of normal surgery which we now discuss. For now, though, we treat ξ only as a real vector bundle, and we keep in mind for later that there is an operator J on it with $J^2 = -Id$.

By a *surgery* on a manifold M^n we refer to the process of removing the interior of the image of an embedding $S^p \times D^{n-p} \xrightarrow{\varphi} M$, and attaching a $D^{p+1} \times S^{n-p-1}$ identically along the boundary (note that $\partial(S^p \times D^{n-p}) = \partial(D^{p+1} \times S^{n-p-1}) = S^p \times S^{n-p-1}$, obtaining a new manifold M'. Such a process defines a manifold with boundary, the *trace of the surgery* W_φ by taking $M \times [0,1]$ with $D^{p+1} \times D^{n-p}$ attached along its boundary to the boundary of $M \times \{1\}$ with the interior of $S^p \times D^{n-p}$ removed. Note that W_φ is a cobordism between M and M'; as manifold, it has its own stable normal bundle which restricts to that of M and M' on its boundary. Given a normal map $M \xrightarrow{f} A$, we will refer to an extension of this map to a normal map $W_\varphi \xrightarrow{F} A$ as a *normal cobordism*, obtained by performing *normal surgery*. Note that if $M \xrightarrow{f} A$ is degree one, i.e. $f_*[M] = [A]$, then f' is degree one as well, as we have $0 = [\partial W] = [M] - [M']$, so $F_*[M] = F_*[M']$, i.e. $f'_*[M'] = f_*[M] = [A]$.

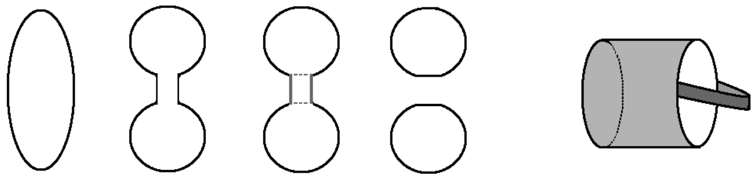


Figure 2.2: Surgery on an embedded $S^0 \times D^1$ in the circle, and its trace. Note how the trace deformation retracts onto a circle with an interval attached.

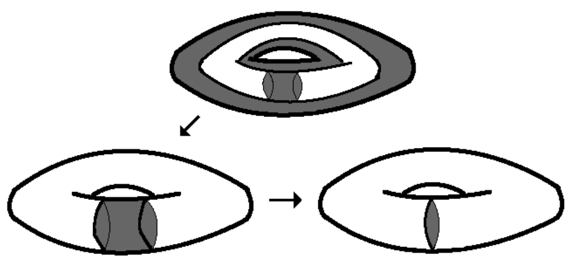


Figure 2.3: The trace of a surgery along an embedded $S^1 \times D^1$ in a torus, deformation retracting to a torus with a D^2 attached.

An important property of the trace is that it deformation retracts onto M with a D^{p+1} attached along the image S^p of the embedding φ; see [Br72, Theorem IV.1.3], and Figure (2) for an illustration. From here it follows that our normal map $M \xrightarrow{f} A$ extends to a normal map $W_\varphi \to A$ if f extends over this attached D^{p+1} and the map of bundles extends to ω restricted to D^{p+1}, where ω is the stable normal bundle of W_φ [Br72, Proposition IV.1.4].

Now, the approach to surgering the normal map $M \xrightarrow{f} A$ to a normal map $M' \to A$ which is a rational homotopy equivalence will be: we consider the exact sequence

$$\pi_{p+1}(M) \otimes \mathbb{Q} \to \pi_{p+1}(A) \otimes \mathbb{Q} \to \pi_{p+1}(f) \otimes \mathbb{Q} \to \pi_p(M) \otimes \mathbb{Q} \to \pi_p(A) \otimes \mathbb{Q} \to \cdots .$$

This is the long exact sequence in homotopy groups of a pair; one often writes $\pi_p(A, M)$ for

$\pi_p(f)$. (Generally $\pi_2(f)$ is not an abelian group, and $\pi_1(f)$ is not a group, so we will first perform a surgery to bring ourselves into a situation where $\pi_2(f)$ is abelian and $\pi_1(f) = 0$.) Elements of $\pi_p(f)$ are represented by maps $S^p \to M$ which extend to a map $D^{p+1} \to A$, i.e. diagrams of the form

$$\begin{array}{ccc} S^p & \longrightarrow & M \\ \big\downarrow & & \big\downarrow f \\ D^{p+1} & \longrightarrow & A \end{array}$$

The idea will be to inductively perform normal surgery on embedded p–dimensional spheres, so that the map $M' \xrightarrow{f'} A$ from the result of the surgery (i.e. the "right end" of the trace) will have smaller-dimensional $\pi_{p+1}(f')$ than the map on the "left end" of the trace, while satisfying that $\pi_{\leq p}(f') \otimes \mathbb{Q} = 0$ if the same was true on the left end of the trace.

Suppose we are given an element in $\pi_{p+1}(f)$, represented by a map $S^p \xrightarrow{\varphi} M$ which extends over $D^{p+1} \to A$. When can we perform a normal surgery on this $S^p \xrightarrow{\varphi} M$? We need the following three conditions to be satisfied:

- S^p must be embedded in M.
- The normal bundle to S^p in M must be trivial, giving us an embedded $S^p \times D^{n-p}$ in M to perform surgery on.
- The normal bundle map from M to A must extend over D^{p+1}.

If the dimension of the sphere p is strictly smaller than half the dimension n of our manifold M, then φ can be modified by a homotopy to an embedding, by Whitney's "weak" embedding theorem [Wh36, III]. This can also be done if $p = \frac{n}{2}$ by Whitney's "strong" embedding theorem [Wh44, 8–12], which we will discuss later. As for the next two bullet points, we have the following: Since we are able to extend f over D^{p+1} (since φ represents an element in $\pi_{p+1}(f)$), then the composition of φ with this extension is a nullhomotopic map to A, and hence the normal bundle ν restricted to S^p is trivial, *with an induced trivialization.* Picturing $M \cup_\varphi D^{p+1}$ as embedded in a large disk (think of Figure (2)), the normal bundle to D^{p+1} is trivial, and hence extending our bundle map is equivalent to extending the trivialization of

ν on S^p to all of D^{p+1} (as, up to homotopy, all of these points will be mapped to a single point in A). That is, our trivialization on S^p gives us a map from S^p to the Stiefel manifold $\mathrm{St}(k, k+n-p)$ of k-frames in $\mathbb{R}^{k+n-p}$, where k is the rank of the stable normal bundle; this map must therefore be nullhomotopic. In fact, this element in $\pi_p(\mathrm{St}(k, k+n-p))$ is the unique obstruction to extending our normal map [Br72, Theorem IV.1.6]. If we can indeed extend our trivialization over D^{p+1}, then the orthogonal complement to the normal bundle of D^{p+1} it determines, when restricted to S^p, gives a trivialization of the normal bundle of S^p in M.

Luckily, when $k \geq 2$ (which will always be the case for us), we have $\pi_{<n-p}(\mathrm{St}(k, k+n-p)) = 0$. This means that for an embedded sphere of dimension p strictly below half the dimension n of our manifold, the above described obstruction will vanish as it lies in a trivial group. In the case of even dimension n, when $p = \frac{n}{2}$, the corresponding homotopy group is $\mathbb{Z}_2$ if p is odd, and $\mathbb{Z}$ if p is even [Br72, Theorem IV.1.12]. It follows from here that the normal bundle of any embedded S^p in M, such that ν restricted to S^p is trivial, is trivial as soon as $p < \frac{n}{2}$. Note that this follows alternatively from the identity $\nu_{S^p,M} \oplus \nu|_{S^p} \cong \nu_{S^p}$, i.e. the sum of the normal bundle to S^p in M with the stable normal bundle of M restricted to M, equals the stable normal bundle of S^p. As S^p is stably parallelizable, the right-hand side is trivial, so the triviality of $\nu_{S^p,M}$ implies the triviality of $\nu|_{S^p}$.

2.2.1 The effect of surgery on homotopy groups.

Now we consider what effect a surgery on a representative φ of $\pi_{p+1}(f)$ has on the homotopy groups of the manifold M and the map f. As the trace W_φ deformation retracts onto $M \cup_\varphi D^{p+1}$, we see that the inclusion $M \hookrightarrow W_\varphi$ induces an isomorphism on $\pi_{<p}$ and the class that φ represents in $\pi_p(M)$ maps to zero. To relate the homotopy groups of M to those of the manifold M' at the other end of the trace, we notice the following symmetry in the surgery process: *since M' is obtained from M by a surgery on an embedded $S^p \times D^{n-p}$, we have that M is obtained from M' by a surgery on an embedded $D^{p+1} \times S^{n-p-1}$*. Furthermore,

the trace of the "backwards" surgery is the same as W_φ; see again Figure (2) for an illustration. From here we have that W_φ deformation retracts onto M' with a D^{n-p} attached. So, looking at the inclusions $M \hookrightarrow W_\varphi \hookleftarrow M'$, some consideration of indices shows that if $p < \frac{n-1}{2}$, then $\pi_{<p}(M') \cong \pi_{<p}(M)$ and that $\pi_p(M')$ is isomorphic to the quotient of $\pi_p(M)$ by the $\pi_1(M)$–module generated by the image of φ in $\pi_p(M)$ [Br72, Theorem IV.1.5].

Let us now apply normal surgery to obtain a normal map $M' \to A$ from a simply connected manifold, so that we may speak freely of tensoring $\pi_k(f) \otimes \mathbb{Q}$. First we achieve connectedness: by the Pontryagin–Thom construction which is used in stage (1) to obtain our starting manifold M, we see that M is a compact subset of some sphere, and hence has finitely many connected components. Pick two points lying in different connected components. Note that this is an embedding $S^0 \hookrightarrow M$. Now by the above discussion we may perform normal surgery on this embedding, and after finitely such surgeries we obtain a connected manifold. The effect of the surgery is forming the connected sum of the two considered connected components along small disks around each point. As for the fundamental group, choose a finite generating set, and represent each element in the set by a smooth embedded loop. As before, we can perform normal surgery on each loop, with the effect that the fundamental group gets smaller after each surgery by the previous paragraph (here we use that our manifold has dimension ≥ 5, though 4 would suffice); after finitely many surgeries we have a simply connected manifold.

Taking an element in $\pi_{p+1}(f)$ represented by an embedding $S^p \xrightarrow{\varphi} M$ with trivial normal bundle in M, such that f restricted to the image of the embedding extends over D^{p+1}, and denoting the extension of f over the trace W_φ by F, we see that $\pi_p(F)$ is the quotient of $\pi_{p+1}(f)$ by the $\pi_1(M)$–module generated by our element in $\pi_{p+1}(f)$ [Br72, Lemma IV.1.14]. In particular, $\dim \pi_{p+1}(F) \otimes \mathbb{Q} < \dim \pi_{p+1}(f) \otimes \mathbb{Q}$.

Now consider the following diagram of long exact sequences,

$$
\begin{array}{ccccccccccccc}
\cdots \longrightarrow & \pi_{p+1}(f) & \longrightarrow & \pi_p(M) & \longrightarrow & \pi_p(A) & \longrightarrow & \pi_p(f) & \longrightarrow & \pi_{p-1}(M) & \longrightarrow & \pi_{p-1}(A) & \longrightarrow \cdots \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & \pi_{p+1}(F) & \longrightarrow & \pi_p(W_\varphi) & \longrightarrow & \pi_p(A) & \longrightarrow & \pi_p(F) & \longrightarrow & \pi_{p-1}(W_\varphi) & \longrightarrow & \pi_{p-1}(A) & \longrightarrow \cdots \\
& \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & \\
\cdots \longrightarrow & \pi_{p+1}(f') & \longrightarrow & \pi_p(M') & \longrightarrow & \pi_p(A) & \longrightarrow & \pi_p(f') & \longrightarrow & \pi_{p-1}(M') & \longrightarrow & \pi_{p-1}(A) & \longrightarrow \cdots
\end{array}
$$

induced by the diagram

$$
\begin{array}{ccc}
M & & \\
\big\downarrow & \overset{f}{\searrow} & \\
W_\varphi & \xrightarrow{F} & A \\
\big\uparrow & \overset{f'}{\nearrow} & \\
M' & &
\end{array}
$$

Since $\pi_{\leq p-1}(M) \to \pi_{\leq p-1}(W_\varphi)$ are isomorphisms and $\pi_p(M) \to \pi_p(W_\varphi)$ is surjective, by the five lemma we have that $\pi_{\leq p}(f) \to \pi_{\leq p}(F)$ are isomorphisms. If furthermore $p < \frac{n-1}{2}$, then recall that W_φ is obtained from M' by performing surgery on a $n-p-1$–dimensional sphere. Since $n-1-p > n-1-\frac{n-1}{2} = \frac{n-1}{2} > p$, we have $n-p-1 \geq p+1$ and so by the previous sentence, replacing p by $n-p-1$, we have in particular that $\pi_{\leq p+1}(f') \to \pi_{\leq p+1}(F)$ are isomorphisms. Tensoring the above ladder of long exact sequences with $\mathbb{Q}$, we have in particular that $\pi_{\leq p}(f') \otimes \mathbb{Q} \cong \pi_{\leq p}(f) \otimes \mathbb{Q}$ and $\dim \pi_{p+1}(f') \otimes \mathbb{Q} < \dim \pi_{p+1}(f) \otimes \mathbb{Q}$. Note that we cannot draw this conclusion if $p \geq \frac{n-1}{2}$.

2.2.2 Surgery below middle degree

Now we proceed inductively. Suppose $M \to A$ is a normal map with M simply connected; then $\pi_1(f)$ is trivial and $\pi_2(f)$ is abelian. Note that $\pi_*(f) \otimes \mathbb{Q}$ is finite dimensional in every degree, since $\pi_*(M)$ and $\pi_*(A)$ are. Hence we may choose a finite basis of $\pi_2(f) \otimes \mathbb{Q}$, and scale each element if necessary so that it is in the image of the rationalization map $\pi_2(f) \to \pi_2(f) \otimes \mathbb{Q}$. Now we can choose representatives of these basis elements, given by maps of S^2 into M that extend over D^3 to A. If $p < \frac{n-1}{2}$, we may choose this map to be a smooth embedding (by first finding a smooth representative of the map, and then using the Whitney embedding theorem). The obstruction to doing normal surgery vanishes, since the appropriate homotopy group of the Stiefel manifold vanishes for such p (recall this also implies the triviality of the normal bundle to S^2, allowing us to perform surgery on an embedded $S^2 \times D^{n-2}$). After applying finitely many such surgeries (one for each basis element), we have obtained a normal map $M' \xrightarrow{f'} A$, where M' is still simply connected, but now $\pi_2(f') \otimes \mathbb{Q} = 0$ as well.

Moving on to $\pi_3(f')$, and so on, the largest p that this procedure works is $p = \lfloor n/2 \rfloor - 1$, where $\lfloor n/2 \rfloor$ denotes the floor function Indeed, if $n = 2m$, then $p < \frac{n-1}{2}$ gives $p \leq m - 1$, i.e. $p \leq \frac{n}{2} - 1$; if $n = 2m+1$, then $p < \frac{n-1}{2}$ gives $p \leq m - 1 = \frac{n}{2} - 1$. So we finally obtain a normal map $M' \xrightarrow{f'} A$ from a simply connected manifold M' such that $\pi_{\leq \lfloor n/2 \rfloor}(f') \otimes \mathbb{Q} = 0$ (recall that at each stage $p + 1$, we have that the vanishing of $\pi_{\leq p}(f) \otimes \mathbb{Q}$ implies the vanishing of $\pi_{\leq p}(f') \otimes \mathbb{Q}$, along with a decrease in dimension of $\pi_{p+1} \otimes \mathbb{Q}$. As we will note later, the homotopy groups of the space A we are working with, though finite dimension after tensoring with $\mathbb{Q}$, are *not finitely generated*, and so we can not hope to kill $\pi_*(f)$ with finitely many surgeries.

2.2.3 Surgery in middle degree, and employing rational Poincaré duality

We may thus assume $M \xrightarrow{f} A$ satisfies $\pi_{\leq \lfloor n/2 \rfloor}(f) \otimes \mathbb{Q}$ (along with M being simply connected); we now must deal with $\pi_{\lfloor n/2 \rfloor + 1}(f)$. The dimension of a representative sphere in an element of this group is half the dimension of our manifold if n is even, and just below $\frac{n}{2}$ if n is odd. The obstruction to performing normal surgery lies in a trivial group if n is odd, and lies in $\mathbb{Z}_2$ or $\mathbb{Z}$ if n is even. It is at this point that we finally make use of rational Poincaré duality on M and A (note that everything above holds without this assumption)

Rational Poincaré duality gives us the following:

- If $\pi_{\leq \lfloor n/2 \rfloor + 1}(f) \otimes \mathbb{Q} = 0$, then the map $M \xrightarrow{f} A$ is a rational homotopy equivalence (so there is no need to consider $\pi_{>\lfloor n/2\rfloor+1}(f)\otimes\mathbb{Q}$, which would be very complicated). Indeed, $\pi_{\leq\lfloor n/2\rfloor+1}(f)\otimes\mathbb{Q}=0$ implies that the induced maps $\pi_{\leq\lfloor n/2\rfloor}(M)\otimes\mathbb{Q}\to\pi_{\leq\lfloor\frac{n}{2}\rfloor}(A)$ are isomorphisms, and hence, since M and A are simply connected, the maps on homology $H_{\leq\lfloor n/2\rfloor}(M;\mathbb{Q})\to H_{\leq\lfloor n/2\rfloor}(M;\mathbb{Q})H_{\leq\lfloor n/2\rfloor}(M;\mathbb{Q})$ are isomorphisms by the Hurewicz theorem (in the context of rational homotopy theory). Now, generally a non-zero degree map of rational Poincaré duality spaces is surjective on homology in all degrees (proof: equivalently, the dual map on cohomology is injective; suppose some $a\in H^a(A;\mathbb{Q})$ is such that $f^*a=0$. Take a' such that $\langle aa',[A]\rangle=1$. Then one the one hand, we must have $\langle f^*(aa'),[M]\rangle=\langle aa',f_*[M]\rangle=\langle aa',\deg(f)[A]\rangle=\deg(f)$, while $\langle f^*(aa'),[M]\rangle=\langle f^*a\,f^*a',[M]\rangle=0$.) Now we see that above half the dimension, in each degree the map $H_*(M;\mathbb{Q})\to H_*(A;\mathbb{Q})$ must be an isomorphism as well, as it is surjection between spaces of equal dimension (by our conclusion up to half the dimension).

- It enables us to study the problem of killing $\pi_{\lfloor n/2\rfloor+1}(f)\otimes\mathbb{Q}$: first observe that if $\pi_1(f)=0$ and $\pi_{\leq\lfloor n/2\rfloor}(f)\otimes\mathbb{Q}=0$, then by the rational version of the relative Hurewicz theorem, we have $\pi_{\lfloor n/2\rfloor+1}(f)\otimes\mathbb{Q}\cong H_{\lfloor n/2\rfloor+1}(f;\mathbb{Q})$. This latter group is isomorphic to the kernel of $H_{\lfloor n/2\rfloor}(M;\mathbb{Q})\to H_{\lfloor n/2\rfloor}(A;\mathbb{Q})$, as seen from the long exact sequence

$$\cdots\to H_{\lfloor n/2\rfloor+1}(A;\mathbb{Q})\to H_{\lfloor n/2\rfloor}(f;\mathbb{Q})\to H_{\lfloor n/2\rfloor}(M;\mathbb{Q})\to H_{\lfloor n/2\rfloor}(A;\mathbb{Q})\to\cdots,$$

  which splits by surjectivity of the maps $H_*(M;\mathbb{Q})\to H_*(A;\mathbb{Q})$ discussed in the previous point.

Hence, with A a rational Poincaré duality space, we may think of this whole surgery procedure as "killing the kernel of f". This viewpoint was not strictly necessary up to this final stage of surgery.

2.2.4 The Whitney embedding theorem in dimension $n/2$

To perform surgery on representatives of elements in $\pi_{\lfloor n/2 \rfloor+1}(f) \otimes \mathbb{Q}$ (that are in the image of $\pi_{\lfloor n/2 \rfloor+1}(f)$), we must first make sure that every map of an $\lfloor n/2 \rfloor$-dimensional sphere to M is homotopic to an embedding. If n is odd, this is guaranteed by the version of the Whitney embedding theorem used before. If n is even, we must use a stronger version of the embedding theorem, proven also by Whitney several years later [Wh36].

First of all, by the “weak" Whitney embedding theorem [Wh36, §III], our map can be approximated by a smooth immersion whose only singular points are transverse double points (i.e. points whose preimage consists of exactly two points). If the sphere is even–dimensional, each such double point carries a sign of ± 1 corresponding to whether the orientation on the tangent space in the ambient manifold obtained from adding the pushforwards of the two tangent planes on the sphere agrees with the ambient orientation or not. Given two double points of opposite sign if the dimension of the sphere is even, or any two double points if it is odd, one can connect these points by two distinct arcs, forming a closed loop. Since our ambient manifold M is simply connected, there is a two–disk whose boundary is this loop, and which intersects the image of the sphere only on its boundary, transversally. To ensure that this disk itself has no self–intersection, we recall our assumption that M is of dimension ≥ 5 and apply the weak Whitney embedding theorem again. *It is at this point that dimension 4 must be omitted from our overarching discussion.* Then, Whitney shows (with an argument now known as the "Whitney trick") that, using this disk, one can find a homotopy of the original map of the sphere *through immersions* to a smooth map without the two double points considered. If the number of double points of the original map was even, and the number of $+1$ double points was equal to the number of -1 double points if the dimension of the sphere is even, then one applies this argument to obtain a homotopy through immersions of the original map to an embedding. Details of this argument can be found in [Whit46, §§8–12]. Now, if the number of double points was not even, or double points of one sign were more numerous, then an additional argument is employed, also due to Whitney. One may choose a small coordinate ball of the domain sphere in which the map from the sphere to M is an embedding, and find a homotopy of the map which is constant outside of the interior of the ball, to one that has one double point in the interior, with sign $+1$ or -1 of our choosing if the sphere is even–dimensional [Whit46, §2]. (Note that this homotopy will *not* be a homotopy through immersions.) One then arranges the number and sign of double points to allow for repeated application of the Whitney trick, to find a homotopy of the original map of the sphere to an embedding.

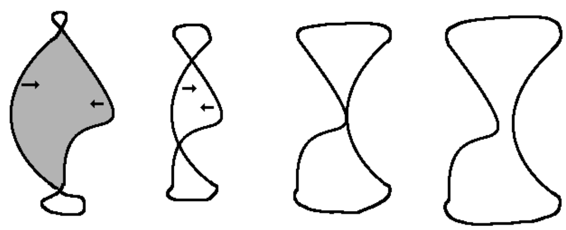


Figure 2.4: The Whitney trick.

Now that we can choose an embedded sphere to represent our element in $\pi_{\lfloor n/2\rfloor+1}(f)\otimes\mathbb{Q}$, we consider the obstruction to performing normal surgery. If n is odd, the corresponding homotopy group of the Stiefel manifold vanishes. If $n \equiv 2 \bmod 4$, the obstruction lies in $\mathbb{Z}_2$. If the obstruction for our choice of map $S^{\frac{n}{2}} \to M$ is non-zero, we precompose it with the degree two self-map $S^{\frac{n}{2}} \xrightarrow{2} S^{\frac{n}{2}}$ of the sphere. We can then find a homotopy of the composition $S^{\frac{n}{2}} \xrightarrow{2} S^{\frac{n}{2}} \to M$ to a smooth embedding, for which the obstruction now vanishes as it is twice the original obstruction class. Here we made use of our aim of achieving only a rational homotopy equivalence; performing surgery on this new element of $\pi_{\lfloor (n-1)/2\rfloor}(f)\otimes\mathbb{Q}$ will regardless decrease the dimension as desired.

If $n \equiv 0 \bmod 4$, denote the homology class represented by our map $S^{\frac{n}{2}} \to M$ (i.e. the pushforward of the fundamental class of the sphere) by x. Then the obstruction to performing normal surgery can be identified with $\langle \mathrm{pd}(x)\mathrm{pd}(x), [M]\rangle$, where $\mathrm{pd}(x)$ denotes the cohomology class Poincaré dual to x; see [Br72, pp. 108–111].

Let us now focus on the case of $n \equiv 0 \bmod 4$, and observe that classes in $H_{\lfloor \frac{n}{2} \rfloor + 1}(f;\mathbb{Q}) \cong \ker(f_*)$ have vanishing surgery obstruction. After this, we will discuss the effect on rational homology of performing surgery on a sphere of dimension $\frac{n}{2}$ if n is even, or $\frac{n-1}{2}$ if n is odd.

2.2.5 The pairing on homology in dimensions $0 \bmod 4$ and the surgery obstruction

First observe that since $A \xrightarrow{h} X$ is a rational equivalence, A satisfies rational Poincaré duality with respect to the fundamental class $[A] = h_*^{-1}[X]$. For a homology class $x \in H_*(A;\mathbb{Q})$, we denote by $\mathrm{pd}(x)$ the unique cohomology class such that $[A] \cap \mathrm{pd}(x) = x$.

We consider the pairing $H_{\frac{n}{2}}(A;\mathbb{Q}) \otimes H_{\frac{n}{2}}(A;\mathbb{Q}) \to \mathbb{Q}$ given by $x \cdot y = \langle \mathrm{pd}(x)\mathrm{pd}(y), [A]\rangle$. Recall, $\mathrm{pd}(x)$ the unique cohomology class such that $[A] \cap \mathrm{pd}(x) = x$. We note that cap product with $[A]$ provides an isometry from the pairing on $H^{\frac{n}{2}}(A;\mathbb{Q})$ to this pairing on $H_{\frac{n}{2}}(A:\mathbb{Q})$: indeed, for cohomology classes $x', y' \in H^{\frac{n}{2}}(A;\mathbb{Q})$ is given by $\langle x'y', [A]\rangle$, while the pairing of $[A] \cap x', [A] \cap y' \in H_{\frac{n}{2}}(A;\mathbb{Q})$ is also given by $\langle \mathrm{pd}([A] \cap x')\mathrm{pd}([A] \cap y'), [A]\rangle = \langle x'y', [A]\rangle$. We note that for homology classes $x, y \in H_{\frac{n}{2}}(A;\mathbb{Q})$, we have $x \cdot y = \langle \mathrm{pd}(x), y\rangle$, since $x \cdot y = \langle \mathrm{pd}(x)\mathrm{pd}(y), [A]\rangle = \langle \mathrm{pd}(x), [A] \cap \mathrm{pd}(y)\rangle = \langle \mathrm{pd}(x), y\rangle$ (see [Br72, Proposition I.1.1]).

2.2.6 Splitting of the pairing

Given our degree one map $M \xrightarrow{f} A$, we will see that the pairing on $H_{\frac{n}{2}}(M;\mathbb{Q})$ splits into a summand isometric to the pairing on $H_{\frac{n}{2}}(A;\mathbb{Q})$ (which we leave alone) along with the kernel, which will consist of summands isometric to the pairing on $H_{\frac{n}{2}}(S^{\frac{n}{2}} \times S^{\frac{n}{2}};\mathbb{Q})$, which has the form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. A representing $S^{\frac{n}{2}}$ in the latter summands will thus have vanishing normal surgery obstruction.

To see that the pairing on $H_{\frac{n}{2}}(M;\mathbb{Q})$ splits, we note that the map $H_{\frac{n}{2}}(M;\mathbb{Q}) \xrightarrow{f_*} H_{\frac{n}{2}}(A;\mathbb{Q})$ admits a section $H_{\frac{n}{2}}(A;\mathbb{Q}) \xrightarrow{\alpha_*} H_{\frac{n}{2}}(M;\mathbb{Q})$. The computations to follow are contained in [Ba76, pp.477–478]; we include them here for the convenience of the reader. We define α_* by $\alpha_*(a) = [M] \cap (f^*\mathrm{pd}(a))$; it is indeed a section of f_* since

$$f_*\alpha_*a = f_*([M] \cap (f^*\mathrm{pd}(a))) = (f_*[M]) \cap \mathrm{pd}(a) = [A] \cap \mathrm{pd}(a) = a.$$

Now, this section provides an isometry from $H_{\frac{n}{2}}(A;\mathbb{Q})$ onto its image in $H_{\frac{n}{2}}(M;\mathbb{Q})$ (which is generally a proper subspace of $H_{\frac{n}{2}}(M;\mathbb{Q})$). For ease of reading we will use the notation $\langle -, - \rangle$ for the pairing on homology of either space, and for evaluation of cohomology classes on homology classes in either space; to differentiate these pairings we include a subscript (so for example $x \cdot y$ will now be denoted $\langle x, y \rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})}$), whereby a subscript of M or A will denote the evaluation of cohomology on homology:

$$\begin{aligned}\langle \alpha_*a, \alpha_*a' \rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} &= \langle \mathrm{pd}(\alpha_*a), \alpha_*a' \rangle_M = \langle \mathrm{pd}([M] \cap (f^*\mathrm{pd}(a))), \alpha_*a' \rangle_M \\ &= \langle f^*\mathrm{pd}(a), \alpha_*a' \rangle_M = \langle \mathrm{pd}(a), f_*\alpha_*a' \rangle_A = \langle \mathrm{pd}(a), a' \rangle_A \\ &= \langle a, a' \rangle_{H_{\frac{n}{2}}(A;\mathbb{Q})}.\end{aligned}$$

Now we will see that image of this splitting α_* is the orthogonal complement in $H_{\frac{n}{2}}(M;\mathbb{Q})$ to $\ker_{f_*}$ in degree $\frac{n}{2}$. Indeed, for $a \in H_{\frac{n}{2}}(A;\mathbb{Q})$ and $x \in \ker f_* \cap H_{\frac{n}{2}}(M;\mathbb{Q})$ we have

$$\langle \alpha_*a, x \rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} = \langle \mathrm{pd}(\alpha_*a), x \rangle_M = \langle f^*\mathrm{pd}(a), x \rangle_M = \langle \mathrm{pd}(a), f_*x \rangle_A = 0,$$

where we used in the second equality that $\mathrm{pd}(\alpha_*a) = f^*\mathrm{pd}(a)$ from the definition of α_*. Conversely, suppose $x \in H_{\frac{n}{2}}(M;\mathbb{Q})$ is such that $\langle x, y \rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} = 0$ for all $y' \in \ker f_*$. First we note that for any two classes z, z', we have

$$\begin{aligned}\langle \alpha_*f_*z, z' \rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} &= \langle \mathrm{pd}(\alpha_*f_*z), z' \rangle_M = \langle f^*pdf_*z, z' \rangle_M \\ &= \langle pdf_*x, f_*y \rangle_A = \langle f_*x, f_*y \rangle_{H_{\frac{n}{2}}(A;\mathbb{Q})}.\end{aligned}$$

Now, for x orthogonal to $\ker f_*$, note that we have $\langle x, y - \alpha_*f_*y \rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})}$ for all y, since $f_*(y - \alpha_*f_*y) = f_*y - f_*y = 0$. This equation now gives us $\langle x, y \rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} - \langle x, \alpha_*f_*y \rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} =$

0, which by the previous observation yields $\langle x, y\rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} - \langle f_*x, f_*y\rangle_{H_{\frac{n}{2}}(A;\mathbb{Q})}$. Now we observe that for any y,

$$\begin{aligned}\langle x - \alpha_* f_* x, y\rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} &= \langle x, y\rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} - \langle \alpha_* f_* x, y\rangle_{H_{\frac{n}{2}}(M;\mathbb{Q})} \\ &= \langle f_*x, f_*y\rangle_{H_{\frac{n}{2}}(A;\mathbb{Q})} - \langle f_*x, f_*y\rangle_{H_{\frac{n}{2}}(A;\mathbb{Q})} = 0\end{aligned}$$

and so $x = \alpha_*(f_*x)$.

Since $\ker f_*$ is the orthogonal complement to the image of α_* in $H_{\frac{n}{2}}(M;\mathbb{Q})$, we see that the pairing on $H_{\frac{n}{2}}(M;\mathbb{Q})$ restricted to $\ker f_*$ is nondegenerate as well [Ba76, Corollaire 2.4.4].

2.2.7 Signature of M

Recall that our realization problem started with a simply connected rational space X with a choice of rational cohomology classes $c_i(X)$. In the first stage of the construction, to be discussed later, we find a simply connected space A with a rational homotopy equivalence $A \xrightarrow{g} X$ such that A has a complex vector bundle over it with Chern classes $g^*\overline{c_i(X)}$; it is this vector bundle with respect to which we have been performing our normal surgery. Here $\overline{c}_i(X)$ denotes the unique classes solving the equation $(1+c_1(X)+c_2(X)+\cdots)(1+\overline{c}_1(X)+\overline{c}_2(X)+\cdots)$. The classes $g^*\overline{c}_i(X)$ pull back to be the Chern classes of the almost complex structure on the stable normal bundle to M, while the classes $g^*c_i(X)$ pull back to those of the induced almost complex structure on the stable tangent bundle. The Pontryagin classes $p_i(M)$ of M are determined by these Chern classes, by the universal equation $1 - p_1 + p_2 - \cdots = (1 - c_1 + c_2 - \cdots)(1 + c_1 + c_2 + \cdots)$. From here we see that the rational "Pontryagin classes" of X, so formed from the classes $c_i(X)$ on X, pull back via the composition $M \xrightarrow{f} A \xrightarrow{g} X$ to the Pontryagin classes of the (stable) tangent bundle of M. By construction, $f_*[M] = [A] = h_*^{-1}[X]$, i.e. $g_*f_*[M] = [X]$. Denoting by L Hirzebruch's L–polynomial, we have $\langle L(p_i(M)), [M]\rangle_M = \langle f^*L(p_i(X)), [M]\rangle_M = \langle L(p_i(X)), [X]\rangle_X$, where the latter quantity is the signature of the pairing on $H^{\frac{n}{2}}(X;\mathbb{Q})$ (and hence of the pairing on $H_{\frac{n}{2}}(X;\mathbb{Q})$), and $\langle L(p_i(M)), [M]\rangle_M$ is the signature of the pairing on $H^{\frac{n}{2}}(M;\mathbb{Q})$ (and hence of the pairing on $H_{\frac{n}{2}}(M;\mathbb{Q})$) by the Hirzebruch signature theorem. Note that the pairing on $H^{\frac{n}{2}}(A;\mathbb{Q})$ is equivalent to the pairing on $H^{\frac{n}{2}}(X;\mathbb{Q})$, with the isometry given by g^*. Since α_* is an isometry onto a direct summand of the form on $H_{\frac{n}{2}}(M;\mathbb{Q})$, we conclude that the signature of the pairing on $H_{\frac{n}{2}}(M;\mathbb{Q})$ is equal to the signature of the pairing on $H_{\frac{n}{2}}(X;\mathbb{Q})$ plus the signature of the pairing on $\ker f_*$; combined with the previous calculation this yields that the signature of the pairing on $\ker f_*$ is zero.

2.2.8 The kernel pairing is equivalent to a sum of hyperbolic forms: the Witt cancellation theorem

We now determine the form of this pairing on $\ker f_*$, using the following form of the Witt cancellation theorem [MiHu73 §I.4.4]:

Suppose B_1,B_2,B_3 are nondegenerate symmetric bilinear forms over $\mathbb{Q}$ (or any field of characteristic not equal to 2). If the form $B_1 \oplus B_2$ is equivalent to the form $B_1 \oplus B_3$, then B_2 is equivalent to B_3.

We apply the Witt cancellation theorem in the following way: First of all, note that by the necessary condition (iv), the form on $H_{\frac{n}{2}(X;\mathbb{Q})}$ is equivalent to $\sum_{i=1}^{r} \pm y_i^2$ for some r. Denote the isometric image of this form under α_* by B_1. Let B_2 be the bilinear form on $\ker f_*$; denote the dimension of $\ker f_*$ by s. Now, we know by previous considerations that $B_1 \oplus B_2$ is equivalent to the pairing on $H_{\frac{n}{2}}(M;\mathbb{Q})$, which is equivalent to one of the form $\sum_{i=1}^{r+s} \pm y_i^2$ since, M being a closed manifold, it is induced by a unimodular pairing over the integers [MiHu73, §IV.2.6]. Let B_3 be the form $\sum_{i=r+1}^{r+s} \pm y_i^2$, i.e. the last s summands of the pairing on M. Then $B_1 \oplus B_2$ is equivalent to $B_1 \oplus B_3$, and so B_3, the pairing on $\ker f_*$, is equivalent to $\sum_{i=r+1}^{r+s} \pm y_i^2$. Since the signature of B_3 is zero, we see that s is even, and we may relabel the basis elements so that B_3 is of the form

$$(z_1^2 - z_2^2) + (z_3^2 - z_4^2) + \cdots = (z_1 - z_2)(z_1 + z_2) + (z_3 - z_4)(z_3 + z_4) + \cdots$$

Notice that each $(z_i - z_{i+1})(z_i + z_{i+1})$, in new basis elements $\tilde{z_i} = \frac{1}{2}(z_i - z_{i+1}), \tilde{z_{i+1}} = z_i + z_{i+1}$, is represented by the matrix $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

We conclude that B_3 is equivalent to a pairing of the form $\bigoplus_{i=1}^{s/2} \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. We may thus represent (multiples of) homology classes x in the kernel of f_* in degree $\frac{n}{2}$ by embedded spheres for which $x \cdot x = 0$, i.e. for which the obstruction to performing normal surgery vanishes. Next we will see that each such surgery in fact gets rid of two homology classes, removing one summand of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ from the pairing on the kernel.

Remark 2.2.1. *Above we used the result [MiHu73, §IV.2.6] characterizing the rationalization of a non-degenerate symmetric unimodular form over the integers. We briefly review how this result is obtained. For a given ring R, it is useful to consider the Witt group $W(R)$ of symmetric bilinear forms over R, where two forms are identified if they are equivalent upon adding a split form to each. A split form over a local ring or a principal ideal domain (the only cases we will need) is one that is equivalent to $\left(\begin{smallmatrix} 0 & I \\ I & A \end{smallmatrix}\right)$ for some A; if 2 is invertible in R then this is further equivalent to $\left(\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right)$ (which is in turn equivalent to a direct sum of $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$). (Compare the Witt group with the K–group of vector bundles, with split forms playing the role of trivial bundles.) A map of rings induces a map of Witt groups, and we would like to describe the image of the map $W(\mathbb{Z}) \to W(\mathbb{Q})$. We will determine the image by considering for every prime p the local ring $\mathbb{Z}_{(p)}$ consisting of rational numbers of the form $\frac{a}{b}$, where b is not divisible by p. The maximal ideal of non-units in $\mathbb{Z}_{(p)}$ consists of numbers of the form $\frac{pa}{b}$ (where b is not divisible by p). Note that the inclusion $\mathbb{Z} \to \mathbb{Q}$ factors through $\mathbb{Z}_{(p)}$ for every p, so we have the induced factorization of the map of Witt groups $W(\mathbb{Z}) \to W(\mathbb{Z}_{(p)}) \to W(\mathbb{Q})$. Now, note that $\mathbb{Z}_{(p)}$ modulo its maximal ideal of non-units is isomorphic to the field $\mathbb{F}_p$ of order p. There is an additive map $W(\mathbb{Q}) \xrightarrow{\partial} W(\mathbb{F}_p)$ defined in the following way [MiHu73 Lemma IV.1.2]: given a one-by-one bilinear form (u), write $u = p^i \frac{a}{b}$, where a and b are not divisible by p. Then $\partial(u) = 0$ if i is even, and equal to $(\frac{a}{b}) \in W(\mathbb{F}_p)$ if i is odd. (Note that the form (u) is equivalent to the form $(\alpha^2 u)$ for any non-zero rational number α, corresponding to the form x^2 becoming $(\alpha x')^2$ under change of basis. The parity of i used in the definition of ∂ is invariant under such change of repre-*

sentative.) In general, a form over $\mathbb{Q}$ is equivalent to a direct sum of one-by-one forms (u) where u is a rational unit (see [MiHu73 Corollary I.3.4]), so this defines a map ∂ on any rational bilinear form, which one checks induces a map on the Witt group. The fact that a form over $\mathbb{Q}$ decomposes in such a manner is a consequence of the general fact that if there is a subspace on which a non-degenerate form is non-degenerate (i.e. the determinant of any matrix representative is a unit), then the form splits into the induced form on the subspace direct sum with its orthogonal complement [MiHu73 Theorem I.3.2].

We now observe that the composition $W(\mathbb{Z}_{(p)}) \to W(\mathbb{Q}) \xrightarrow{\partial} W(\mathbb{F}_p)$ is zero: Take a (symmetric bilinear, as always) form over $\mathbb{Z}_{(p)}$. If we can find an x such that x paired with itself is a unit u in $\mathbb{Z}_{(p)}$, then our form decomposes into (u) plus its orthogonal complement. Since $u = \frac{a}{b}$ is a unit in $\mathbb{Z}_{(p)}$, neither a nor b are divisible by p and so ∂ sends (u) to 0 in $W(\mathbb{F}_p)$. We continue doing this until we cannot find an x which pairs with itself to give a unit in $\mathbb{Z}_{(p)}$. Now take any x, pairing with itself to a non-unit α. By non-degeneracy we can find a y such that x pairs with y to give 1, and by assumption (as all of the elements we are now considering do not pair with themselves to give units) y likewise pairs with itself to give a non-unit β. Consider the corresponding matrix $\left(\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right)$. Note that its determinant $\alpha\beta - 1$ is a unit, since $\alpha\beta$ is a non-unit and 1 is a unit, and the non-units form an ideal by locality of the ring. Therefore our original form decomposes into a sum of one-by-one forms (u) (which ∂ sends to 0) and these two-by-two forms $\left(\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right)$. If $\alpha = 0$ such a two-by-two form is split and hence represents 0 in the Witt group, which is sent to 0 under ∂. If $\alpha \neq 0$, we see that the form $\alpha x^2 + 2xy + \beta y^2$ it represents (over $\mathbb{Q}$) can be rewritten as $\alpha x'^2 + \alpha(\alpha\beta - 1)y'^2$ in the new basis $x' = x + \frac{1}{\alpha}y$, $y' = \frac{1}{\alpha}y$. So, $\left(\begin{smallmatrix} \alpha & 1 \\ 1 & \beta \end{smallmatrix}\right)$ is equivalent to $\left(\begin{smallmatrix} \alpha & 0 \\ 0 & \alpha(\alpha\beta - 1) \end{smallmatrix}\right) = (\alpha) \oplus (\alpha(\alpha\beta - 1))$ in $W(\mathbb{Q})$ (note, not in $\mathbb{Z}_{(p)}$). Since $\alpha\beta - 1$ is a unit, the highest power of p in $\alpha(\alpha\beta - 1)$ is the highest power of p in α. If this power is even, ∂ sends both (α) and $(\alpha(\alpha\beta - 1))$ to zero. Otherwise, since $\alpha\beta - 1$ is congruent to -1 modulo the non-units in $\mathbb{Z}_{(p)}$, the image of $(\alpha(\alpha\beta - 1))$ under ∂ is $(-\alpha)$. Observe that $(\alpha) \oplus (-\alpha) = 0$ in $W(\mathbb{F}_p)$. Indeed, if $p \neq 2$, then $\alpha x^2 - \alpha y^2 = \alpha(x - y)(x + y)$ and so this form is equivalent to the split form $\begin{pmatrix} 0 & \frac{\alpha}{2} \\ \frac{\alpha}{2} & 0 \end{pmatrix}$.

If $p = 2$, then taking the new basis x and $x + y$, we see that our this form is equivalent to the split form $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. We conclude that the composition $W(\mathbb{Z}) \to W(\mathbb{Q}) \to W(\mathbb{F}_p)$ is zero for every prime p. Thus, if we look at the image of a unimodular integral form in $W(\mathbb{Q})$, we can first decompose it into a direct sum of one-by-one forms (u), and conclude that each such (u) must map to 0 in $W(\mathbb{F}_p)$ for every p. It follows that every prime p shows up with an even exponent in u, i.e. $u = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_n^{\alpha_n}$ with each $\alpha_i \in \mathbb{Z}$ even, and so (u) is equivalent to (1), (-1), or (0). By unimodularity we know that (0) cannot show up, and so we conclude that the image of a unimodular integral form in $W(\mathbb{Q})$ consists of sums of (1) and (-1), i.e. forms $\sum_i x_i^2 - \sum_j y_j^2$.

2.2.9 The effect of middle–degree surgery on homology

Now we will see the effect that surgery on embedded $\lfloor \frac{n}{2} \rfloor$–spheres with trivial normal bundle, $\lfloor \frac{n}{2} \rfloor \geq 2$, has on homology. For ease of notation let us denote $\ell = \lfloor \frac{n}{2} \rfloor$. The following is an overview of [Br72, pp. 97–99] adapted to the rational setting. Let $S^\ell \times D^{n-\ell} \xrightarrow{\varphi} M$ be an embedding, where $\varphi|_{S^\ell}$ represents a homology class x that is nonzero in $H_\ell(M; \mathbb{Q})$, and denote by M_0 the manifold with boundary obtained by removing from M the interior of the image of φ. The result of the surgery M' will be M_0 with a $D^{\ell+1} \times S^{n-\ell-1}$ attached along the boundary. By excision, the map $(S^\ell \times D^{n-\ell}, \partial(S^\ell \times D^{n-\ell})) \to (M, M_0)$ induced by φ is a homology isomorphism; furthermore, from the long exact sequence in homology of a pair, we see that $H_*(S^\ell \times D^{n-\ell}, \partial(S^\ell \times D^{n-\ell}); \mathbb{Z}) \cong H_*(S^{n-\ell}; \mathbb{Z})$. The homology product with the rational class x obtained by restricting φ to S^ℓ induces an isomorphism $H_{n-\ell}(S^\ell \times D^{n-\ell}, \partial(S^\ell \times D^{n-\ell}); \mathbb{Q}) \xrightarrow{x \cdot} \mathbb{Q}$ (by the Thom isomorphism theorem). Combining this isomorphism and the excision isomorphism, the long exact sequence in homology for (M, M_0),

$$\cdots \to H_{n-\ell}(M_0; \mathbb{Q}) \to H_{n-\ell}(M; \mathbb{Q}) \to H_{n-\ell}(M, M_0; \mathbb{Q}) \to H_{n-\ell-1}(M_0; \mathbb{Q}) \to H_{n-\ell-1}(M; \mathbb{Q}) \to \cdots$$

becomes [Br72, Corollary IV.2.11]

$$0 \to H_{n-\ell}(M_0; \mathbb{Q}) \to H_{n-\ell}(M; \mathbb{Q}) \xrightarrow{x \cdot -} \mathbb{Q} \to H_{n-\ell-1}(M_0; \mathbb{Q}) \to H_{n-\ell-1}(M; \mathbb{Q}) \to 0.$$

Similarly, looking at the pair (M', M_0), which has the homology of $S^{\ell+1}$, we obtain the sequence

$$0 \to H_{\ell+1}(M_0;\mathbb{Q}) \to H_{\ell+1}(M';\mathbb{Q}) \xrightarrow{y\cdot -} \mathbb{Q} \to H_\ell(M_0;\mathbb{Q}) \to H_\ell(M';\mathbb{Q}) \to 0,$$

where y denotes the rational homology class of $S^{n-\ell-1}$ in the embedding $D^{\ell+1}\times S^{n-\ell-1} \hookrightarrow M'$. Now, if we consider the commutative diagram

$$\begin{array}{ccccccccccc} 0 & \longrightarrow & H_{\ell+1}(M_0;\mathbb{Q}) & \longrightarrow & H_{\ell+1}(M';\mathbb{Q}) & \xrightarrow{y\cdot -} & \mathbb{Q} & \longrightarrow & H_\ell(M_0;\mathbb{Q}) & \longrightarrow & H_\ell(M';\mathbb{Q}) & \longrightarrow & 0 \\ & & & & & & & \searrow & \downarrow & & & & \\ & & & & & & & & H_\ell(M;\mathbb{Q}) & & & & \end{array}$$

where the map $H_\ell(M_0;\mathbb{Q}) \to H_\ell(M;\mathbb{Q})$ is induced by inclusion, then the composition $\mathbb{Q} \to H_\ell(M;\mathbb{Q})$ contains x in its image [Br72, Lemma IV.2.12]. This means the diagonal map $\mathbb{Q} \to H_\ell(M;\mathbb{Q})$ is injective, and so the long exact sequence in homology for (M', M_0) splits into

$$0 \to H_{\ell+1}(M_0;\mathbb{Q}) \to H_{\ell+1}(M';\mathbb{Q}) \to 0$$

and

$$0 \to \mathbb{Q} \to H_\ell(M_0;\mathbb{Q}) \to H_\ell(M';\mathbb{Q}) \to 0.$$

In particular, we see that y is zero in $H_{n-\ell-1}(M';\mathbb{Q})$. (This allays our concern that by surgering the sphere representing x, we might introduce a new non-zero class in the same degree or one below). As for the pair (M, M_0), since x is rationally non-zero by assumption, by Poincaré duality there is some class x' such that $x \cdot x' \neq 0$. Therefore, the map $H_{\ell+1}(M;\mathbb{Q}) \xrightarrow{x\cdot -} \mathbb{Q}$ is surjective, splitting the long exact sequence of the pair (M, M_0) into

$$0 \to H_{\ell+1}(M_0;\mathbb{Q}) \to H_{\ell+1}(M;\mathbb{Q}) \to \mathbb{Q} \to 0$$

and

$$0 \to H_\ell(M_0;\mathbb{Q}) \to H_\ell(M;\mathbb{Q}) \to 0.$$

So, if $\ell = n - \ell$, i.e. we are in the case of performing middle–dimensional surgery for $n = 2\ell$, we conclude that $H_\ell(M;\mathbb{Q}) \cong H_\ell(M_0;\mathbb{Q}) \oplus \mathbb{Q}$ and

$$H_i(M;\mathbb{Q}) \cong H_i(M_0;\mathbb{Q}) \text{ for } i \neq \ell,$$

Figure 2.5: Surgery in degree $\lfloor \frac{n}{2} \rfloor$ on a non-zero element in rational homology kills the homology class and its dual under the homology pairing.

along with $H_\ell(M_0;\mathbb{Q}) \cong H_\ell(M';\mathbb{Q}) \oplus \mathbb{Q}$ and $H_i(M';\mathbb{Q}) \cong H_i(M_0;\mathbb{Q})$ for $i \neq \ell$. It follows that $\dim H_\ell(M';\mathbb{Q}) = \dim H_\ell(M;\mathbb{Q}) - 2$ and $\dim H_i(M';\mathbb{Q}) = \dim H_i(M;\mathbb{Q})$ for $i \neq \ell$.

If $\ell = n - \ell - 1$, i.e. $n = 2\ell + 1$ and we are performing surgery just below half the dimension, we conclude that $H_{\ell+1}(M;\mathbb{Q}) \cong H_{\ell+1}(M_0;\mathbb{Q}) \oplus \mathbb{Q}$ and $H_i(M;\mathbb{Q}) \cong H_i(M_0;\mathbb{Q})$ for $i \neq \ell + 1$, along with $H_\ell(M_0;\mathbb{Q}) \cong H_\ell(M';\mathbb{Q}) \oplus \mathbb{Q}$ and $H_i(M';\mathbb{Q}) \cong H_i(M_0;\mathbb{Q})$ for $i \neq \ell$. From here we see

$$\dim H_\ell(M';\mathbb{Q}) = \dim H_\ell(M_0;\mathbb{Q}) - 1 = \dim H_\ell(M;\mathbb{Q}) - 1$$

and

$$\dim H_{\ell+1}(M';\mathbb{Q}) = \dim H_{\ell+1}(M_0;\mathbb{Q}) = \dim H_{\ell+1}(M;\mathbb{Q}) - 1.$$

In conclusion, if n is even, middle–dimensional surgery lowers the rank of the homology by two (and leaves the other ranks unchanged), while if n is odd, the homology right above and below the middle decreases in rank by one.

2.2.10 Conclusion of stage 2: obtaining a rational equivalence

Using stage (1) of the proof, discussed below, we first find a closed n–manifold with a degree one normal map to A. Applying normal surgery below dimension ℓ, where $n = 2\ell$ or $n = 2\ell + 1$, we can then find a simply connected manifold M with a degree one normal map $M \xrightarrow{f} A$ such that $\pi_{\leq \ell}(f) \otimes \mathbb{Q} = 0$. Then applying the above discussion on middle–dimensional surgery, we find a manifold M' and a degree one normal map $M' \xrightarrow{f'} A$ such that $\pi_{\leq \ell+1}(f) \otimes \mathbb{Q} = 0$. Indeed, since $H_i(M'; \mathbb{Q}) \cong H_i(M; \mathbb{Q})$ for $i \neq \ell$ the map f' still satisfies $\pi_{\leq \ell}(f) \otimes \mathbb{Q} = 0$ since it is surjective on rational homology as we saw, and through a sequence of surgeries we achieve that the kernel on $H_k(-; \mathbb{Q})$ is trivial; this is enough to conclude that f' is a rational equivalence.

Remark 2.2.2. *In [Br62], one will see the discussion of surgering a normal map to a homotopy equivalence as a discussion of killing the kernel on homology $M \to A$. We followed [Br72] and decided to postpone equating killing the kernel on homology with surgering to a homotopy equivalence, since this equivalence requires the assumption that A is a Poincaré duality complex. However, one can see that even without having A satisfy Poincaré duality, much of the story applies: we can still surgery our map $M \to A$ to be a (rational) homotopy equivalence up to right below the middle degree. Applying this to the map classifying the stable normal bundle of a manifold with some additional structure (such as spin, string, almost complex) we obtain statements such as: in large enough dimension, every spin manifold is spin cobordant to a 3–connected one (since BSpin is 3–connected), every string manifold is string cobordant to a 7–connected one (since BString is 7–connected), etc.*

2.3 The first stage of the proof

Recall, given a simply connected rational space X satisfying rational Poincaré duality, with fundamental class $[X]$, and cohomology classes $c_i(X)$, our goal is to obtain a closed (simply connected) almost complex manifold M with a rational homotopy equivalence $M \xrightarrow{f} X$ such that $f_*[M] = [X]$ and $f^* c_i(X) = c_i(M)$.

First, we find an intermediate simply connected space A, with a rational homotopy equivalence $A \xrightarrow{g} X$, such that A comes equipped with a complex vector bundle ξ over it whose Chern classes $c_i(\xi)$ satisfy $(1 + g^*c_1(X) + g^*c_2(X) + \cdots)(1 + c_1(\xi) + c_2(\xi) + \cdots) = 1$. To do this, we recall that one obtains complex vector bundles, of an arbitrary complex rank N, by mapping to the Grassmannian of complex N–planes in $\mathbb{C}^\infty$ and pulling back the tautological bundle γ via this map. All of our considerations are homotopy theoretic, so for ease of notation we will replace this Grassmannian by the space $BU(N)$ it deformation retracts onto, i.e. the classifying space of the unitary group $U(N)$. The integral cohomology ring of $BU(N)$ is given by $\mathbb{Z}[c_1, c_2, \ldots, c_N]$, where the c_i are the Chern classes of the tautological bundle γ over $BU(N)$. Hence the rational cohomology ring of $BU(N)$ is $\mathbb{Q}[c_1, c_2, \ldots, c_N]$. Since a rational cohomology class of degree $2i$ is determined by (the homotopy classes of) a map to $K(\mathbb{Q}, 2i)$ (analogously to the above, by pulling back a natural generator of $H^{2i}(K(\mathbb{Q}, 2i); \mathbb{Q})$), we have a map $BU(N) \to K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots K(\mathbb{Q}, 2N)$ given by the cohomology class $(c_1, c_2, \ldots, c_N)$. We will also denote the corresponding generators of the cohomology ring of $K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots K(\mathbb{Q}, 2N)$ by $c_1, c_2, \ldots, c_N$.

The rational cohomology ring of $K(\mathbb{Q}, 2i)$ is the polynomial algebra on one generator in degree $2i$ (see e.g. [GM81, p.55], using that $K(\mathbb{Q}, 2i)$ is the rationalization of $K(\mathbb{Z}, 2i)$). From here we see that our map $BU(N) \to K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots K(\mathbb{Q}, 2N)$ induces an isomorphism on rational cohomology, and hence on rational homology; since both spaces are simply connected, this is a rationalization map. From now on we write $BU(N)_{\mathbb{Q}}$ for $K(\mathbb{Q}, 2) \times K(\mathbb{Q}, 4) \times \cdots K(\mathbb{Q}, 2N)$.

One can consider the classes $\overline{c}_i$ on $BU(N)$ determined by the equation

$$(1 + c_1 + c_2 + \cdots + c_N)(1 + \overline{c}_1 + \overline{c}_2 + \cdots).$$

There will be non-zero $\overline{c}_i$ of arbitrarily large degree, but notice, by solving equations inductively by degree, that $\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_N$ generate the cohomology of $BU(N)$. The terms $\overline{c}_{\geq N+1}$ will be polynomials in the $\overline{c}_{\leq N}$. Hence the map $BU(N) \xrightarrow{v} BU(N)_{\mathbb{Q}}$ given by $(\overline{c}_1, \overline{c}_2, \ldots, \overline{c}_N)$ is a rationalization as well.

Now we consider the map $X \xrightarrow{c^X} BU(N)_{\mathbb{Q}}$ given by $(c_1(X), c_2(X), \ldots)$. (Here we assume that N is greater than the formal dimension; for the surgery step, stage 2, we needed N, which will be the rank of the stable normal bundle, to be much larger than the formal dimension. From now on we take this to be the case.) Consider the homotopy fiber product of the maps c^X and v:

$$\begin{array}{ccc} A & \xrightarrow{u} & BU(N) \\ \Big\downarrow{\scriptstyle g} & & \Big\downarrow{\scriptstyle v} \\ X & \xrightarrow{c^X} & BU(N)_{\mathbb{Q}} \end{array}$$

This diagram is commutative up to homotopy. It is the space A with the complex vector bundle $\xi = u^*\gamma$ which we wish to use in our discussion in stage (2). We list the properties we require of A and the above diagram:

- A is a simply connected space.
- The map g is a rational homotopy equivalence.
- There is a degree 1 map from some closed manifold M to A such that the stable normal bundle of M is the pullback of ξ.

Note that the third point only makes sense after we have verified the second; the fundamental class of A we take will be $h_*^{-1}[X]$.

2.3.1 Fundamental group of A

An issue we face now is that, as constructed, A need not be simply connected. Indeed, denote the homotopy fiber of $BU(N) \xrightarrow{v} BU(N)_{\mathbb{Q}}$ by F. The long exact sequence in homotopy groups tells us the following sequence is exact:

$$\pi_3(BU(N)_{\mathbb{Q}}) \to \pi_2(F) \to \pi_2(BU(N)) \to \pi_2(BU(N)_{\mathbb{Q}}) \to \pi_1(F) \to \pi_1(BU),$$

i.e. since $N \geq 2$ (and hence the listed homotopy groups of $BU(N)$ are stable),

$$0 \to \pi_2(F) \to \mathbb{Z} \to \mathbb{Q} \to \pi_1(F) \to 0$$

is exact. The map $\mathbb{Z} \to \mathbb{Q}$ is injective since it is induced by rationalization, so we conclude that $\pi_1(F)$ is the abelian group $\mathbb{Q}/\mathbb{Z}$. Now consider the induced map of long exact sequences in homotopy groups associated to the above homotopy fiber product diagram:

$$\begin{array}{ccccc} \pi_2(A) & \longrightarrow & \pi_2(X) & \longrightarrow & \pi_1(F) \\ \downarrow & & \downarrow & & \downarrow \cong \\ \pi_2(BU(N)) & \longrightarrow & \pi_2(BU(N)_{\mathbb{Q}}) & \longrightarrow & \pi_1(F) \end{array}$$

Since the map $\pi_2(BU(N)_{\mathbb{Q}}) \to \pi_1(F)$ is surjective (since $\pi_1(BU(N)) = 0$), we see that $\pi_2(X) \to \pi_1(F)$ is surjective if and only if $\pi_2(X) \to \pi_2(BU(N)_{\mathbb{Q}}) \cong \mathbb{Q}$ is surjective. By the Hurewicz theorem this is equivalent to the map $H_2(X;\mathbb{Z}) \to H_2(BU(N)_{\mathbb{Q}};\mathbb{Z})$ being surjective. Since both spaces are rational and $H_2(BU(N)_{\mathbb{Q}};\mathbb{Z}) \cong \mathbb{Q}$, this is equivalent to $H^2(BU(N)_{\mathbb{Q}};\mathbb{Q}) \xrightarrow{(c^X)^*} H^2(X;\mathbb{Q})$ being non-zero; i.e. to $c_1(X)$ being a non-zero element in $H^2(X;\mathbb{Q})$. Since $\pi_1(X) = 0$, this is furthermore equivalent to $\pi_1(A) = 0$.

So, if $c_1(X) \neq 0$, we have ensured the first point above. If $c_1(X) = 0$, we will have to make a modification to our setup in order to proceed. Recall that complex rank N vector bundles with vanishing first integral Chern class are classified by maps to $BSU(N)$, where $SU(N)$ is the special unitary group. The integral cohomology of $BSU(N)$ is given by $H^*(BSU(N);\mathbb{Z}) \cong \mathbb{Z}[c_2, c_3, \ldots, c_N]$, and so $K(\mathbb{Q},4) \times K(\mathbb{Q},6) \times \cdots \times K(\mathbb{Q},2N)$ is a rationalization of $BSU(N)$, which we denote by $BSU(N)_{\mathbb{Q}}$. As above, we have a map $BSU(N) \xrightarrow{v} BSU(N)_{\mathbb{Q}}$ (where now $\bar{c}_1 = 0$), and we can consider the homotopy fiber product

$$\begin{array}{ccc} A & \xrightarrow{u} & BSU(N) \\ \downarrow g & & \downarrow v \\ X & \xrightarrow{c^X} & BSU(N)_{\mathbb{Q}} \end{array}$$

where $c^X = (c_2, c_3, \ldots)$. Since $\pi_2(BSU(N)) = 0$, the homotopy fiber of $BSU(N) \xrightarrow{v} BSU(N)_{\mathbb{Q}}$ is simply connected, and so we have $\pi_1(A) = 0$.

In either case, since the homotopy fiber of v has trivial rational homotopy groups, the map $A \xrightarrow{g} X$ is a map of simply connected spaces inducing an isomorphism on rational homotopy groups, i.e. it is a rational homotopy equivalence, so the second point above is satisfied.

2.3.2 Finding a degree one normal map

As for the third point above, i.e. finding a degree 1 map from some closed manifold M to A (where A has fundamental class $h_*^{-1}[X]$)) such that the stable normal bundle of M is the pullback of ξ, we will have to take into consideration the two distinct cases of $c_1(X) \neq 0$ and $c_1(X) = 0$. What follows is an adaptation of the argument used in the smooth case ([Sull77, Theorem 13.2], expanded on in [Su09], see also [Su14]; we expand on details of some arguments adapted from there).

Consider now the tautological complex rank N bundle γ over $BU(N)$, or over $BSU(N)$ if $c_1 = 0$. Denote by $\xi = g^*\gamma$ the pullback bundle over A. We consider the Thom spaces $\mathrm{Thom}(\gamma)$ and $\mathrm{Thom}(\xi)$ of these bundles, i.e. we consider the underlying real vector bundle, choose a metric on the fibers, take the unit disc bundle, and collapse the boundary to a point. Equivalently, we can obtain the Thom space by taking the mapping cone of the projection from the sphere bundle of our vector bundle to the base space. Any map $S^{n+2N} \to \mathrm{Thom}(\xi)$ is homotopic to one whose preimage of $A \subset \mathrm{Thom}(\xi)$ is a smooth n–dimensional submanifold M of S^{n+2N}, see [Br72, p.33]; the normal bundle of M in S^{n+2N}, i.e. the stable normal bundle of M, is the pullback of ξ by this map.

Remark 2.3.1. *One will see that [Br72, p.33] assumes the analogue of our space A to be a finite complex; if this is satisfied, we embed this finite complex into some Euclidean space and thicken it to a manifold [Br62]. Our A will not be a finite complex, but we can do the following: first, find a cell complex A' with a weak homotopy equivalence $A' \to A$, and pull ξ back via this map. Then we consider maps of spheres into the Thom space of this bundle. We choose a cell decomposition of the Thom space that extends that of A'; then our given map of a sphere into the Thom space, being compact, intersects only finite many cells of A' (if the map misses A' completely, it is nullhomotopic, and hence homotopic to a constant map landing in A'). The Thom space of our bundle restricted to A' naturally sits inside the Thom space of the bundle over A'.*

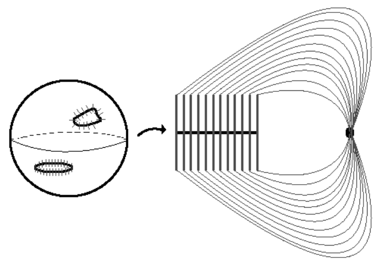
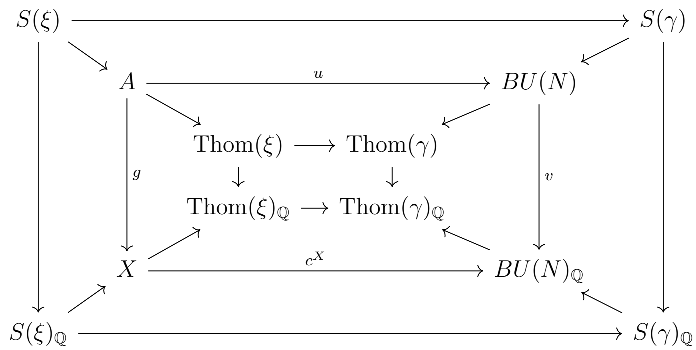


Figure 2.6: The Pontryagin–Thom construction.

We thus obtain a normal map $M \xrightarrow{f} A$. However, the degree of this map remains unknown to us at this point. The class $f_*[M]$ in integer homology is obtained by taking the image of the Hurewicz homomorphism applied to the homotopy class of $S^{n+2N} \to \mathrm{Thom}(\xi)$, followed by cap product with the Thom class of ξ, see [Br72, p.39]. We compose this map further with the rationalization on homology, and refer to this composition $\pi_{n+2N}(\mathrm{Thom}(\xi)) \xrightarrow{ht_\xi} H_(A;\mathbb{Q})$ as the *Hurewicz–Thom map*. Hence, our goal is to show that $[A] = g_*^{-1}[X]$ is in the image of this map, since this will provide us with a degree one normal map $M \to A$.

To show this, we first note that rationalizing the sphere bundle $S(\xi) \to A$ gives a fiber bundle over $A_{\mathbb{Q}} = X$ whose fibers are rational spheres. Denote this bundle by $S(\xi)_{\mathbb{Q}} \to X$. We can do the same for $S(\gamma) \to BU(N)$ (or $S(\gamma) \to BSU(N)$; we will write $BU(N)$ for simplicity of notation from now on), and we can form the "Thom spaces" of these rational sphere bundles by taking the respective mapping cones; the induced map of long exact sequences in homology, together with the five lemma, shows that the induced map of the Thom space to the "Thom space" of the rational sphere bundles is a rationalization (these spaces are all simply connected).

We can now consider the following diagram (cf. [Su09, p.21]):

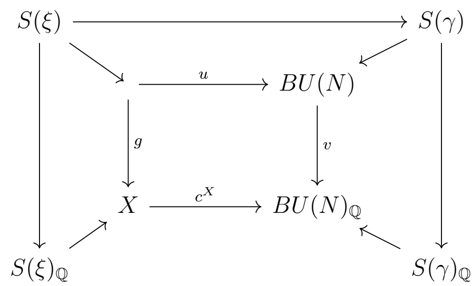


(2.3.1)

We explain what the maps are in this diagram. We remark that we will construct this diagram starting from the central square, which we have implicitly replaced with a strictly commutative square (by converting the right vertical map into a fibration and forming the pullback square). While constructing the diagram, we will note that there exist arrows, unique up to homotopy, that make certain squares commute. At each such moment, we replace the total diagram we have up to that point with a homotopy equivalent strictly commutative diagram; indeed, each such diagram will be easily seen to be homotopy coherent and hence will admit a lift to a strictly commutative diagram [DKS89]. For ease of notation we will assume this process to be implicit and will not rename our objects and maps.

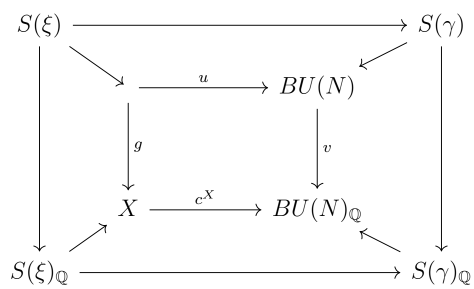
Now, let us recall a basic lemma in rational homotopy theory: given a map $Y \xrightarrow{f} Z_{\mathbb{Q}}$, where $Z_{\mathbb{Q}}$, and a rationalization $Y \xrightarrow{\rho} Y_{\mathbb{Q}}$, there is a map $Y_{\mathbb{Q}} \xrightarrow{f_{\mathbb{Q}}} Z_{\mathbb{Q}}$, unique up to homotopy, such that $f = f_{\mathbb{Q}}\rho$ up to homotopy. This follows from obstruction theory: the obstructions to extending f over the map ρ lie in $H^*(Y_{\mathbb{Q}}, Y; \pi_{*-1}(Z_{\mathbb{Q}}))$, where $(Y_{\mathbb{Q}}, Y)$ denotes the mapping cone of ρ (i.e. we convert ρ into a inclusion and consider the corresponding pair of spaces). Since ρ is a rationalization, the pair $(Y_{\mathbb{Q}}, Y)$ has only torsion in its homotopy groups. Since $Z_{\mathbb{Q}}$ is rational, the groups $H^*(Y_{\mathbb{Q}}, Y; \pi_{*-1}(Z_{\mathbb{Q}}))$ vanish. Likewise, the obstructions to uniqueness of the extension, which lie in $H^*(Y_{\mathbb{Q}}, Y; \pi_*(Z_{\mathbb{Q}}))$ vanish.

Now, choosing a rationalization $S(\xi) \to S(\xi)_{\mathbb{Q}}$, we have that the map $S(\xi) \to A \to X$ factors through $S(\xi)_{\mathbb{Q}}$ by the above; likewise for $S(\gamma) \to BU \to BU_{\mathbb{Q}}$. For now this gives us the following (homotopy) commutative diagram:



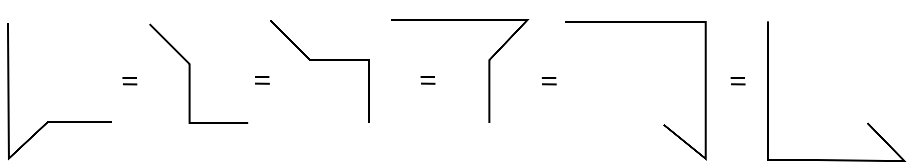
(2.3.2)

Now consider the composition $S(\xi) \to S(\gamma) \to S(\gamma)_{\mathbb{Q}}$. By the above lemma, we have factorization $S(\xi)_{\mathbb{Q}} \to S(\gamma)_{\mathbb{Q}}$ through $S(\xi) \to S(\xi)_{\mathbb{Q}}$.



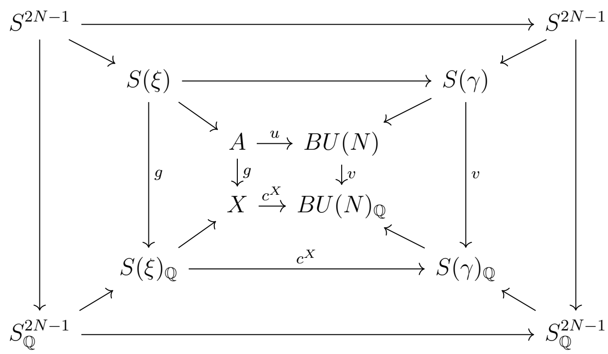
(2.3.3)

We now check that the bottom square is homotopy commutative. Consider the map from $S(\xi)$ to $BU(N)_{\mathbb{Q}}$ obtained by following any arrows from $S(\xi)$ to $BU(N)_{\mathbb{Q}}$ except for the bottom-most arrow. Some diagram-chasing shows that the compositions $S(\xi) \to S(\xi)_{\mathbb{Q}} \to X \to BU(N)_{\mathbb{Q}}$ and $S(\xi) \to S(\xi)_{\mathbb{Q}} \to S(\gamma)_{\mathbb{Q}} \to BU(N)_{\mathbb{Q}}$ are homotopic; diagramatically we have:



where we use the commutativity of the diagram omitting the bottom-most arrow, and in the last equality the fact that the bottom-most arrow makes the outer square commute, by construction. By the uniqueness property discussed above, we conclude that the bottom square commutes up to homotopy.

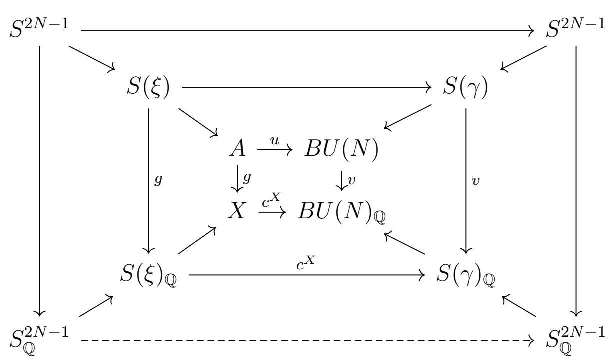
Now, considering the homotopy fibers of these (rational) sphere bundles, we similarly have the following commutative diagram:



(2.3.4)

Here, we take the left-most and right-most vertical arrows, along with the top-most horizontal arrow, to be the induced map of homotopy fibers in the appropriate map of fibrations. Now notice that the left-most and right-most vertical arrows are rationalizations, as seen from the induced map of long exact sequences in homotopy groups (tensored with $\mathbb{Q}$; recall $\mathbb{Q}$ is a flat $\mathbb{Z}$–module).

Thus the composition $S^{2N-1} \to S^{2N-1} \to S^{2N-1}_{\mathbb{Q}}$ from the top-left to the bottom-right corner factors through the rationalization $S^{2N-1} \to S^{2N-1}_{\mathbb{Q}}$, giving us



(2.3.5)

We check that the bottom square, with this induced map indicated by the dashed arrow, commutes. This will imply, by the uniqueness (up to homotopy) of the map between homotopy fibers, that the dashed arrow is homotopic to the map $S^{2N-1}_{\mathbb{Q}} \to S^{2N-1}_{\mathbb{Q}}$ induced between the homotopy fibers of the bottom two fibrations. The same diagrammatic reasoning as before, applied to the outer two squares, shows that the bottom square above commutes, by uniquess of the factorization of $S^{2N-1} \to S(\gamma)_{\mathbb{Q}}$ through $S^{2N-1}_{\mathbb{Q}}$.

Now, consider the outer-most square,

$$\begin{array}{ccc} S^{2N-1} & \longrightarrow & S^{2N-1} \\ \downarrow & & \downarrow \\ S^{2N-1}_{\mathbb{Q}} & \longrightarrow & S^{2N-1}_{\mathbb{Q}} \end{array} \tag{2.3.6}$$

Since ξ is the pullback of γ, the upper horizontal arrow is a homotopy equivalence. The vertical arrows are isomorphisms on rational homology, and hence by commutativity the lower horizontal arrow is an isomorphism on rational homology. That is, it is a rational homotopy equivalence of rational spaces, and hence a homotopy equivalence. Therefore the homotopy fibers of the upper and lower horizontal arrows are contractible, and so the induced map between them is a homotopy equivalence. We conclude that this diagram is a homotopy pullback square.

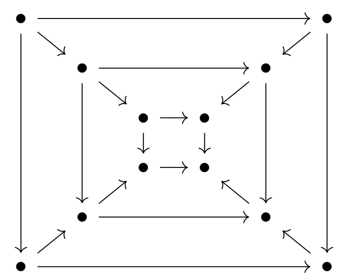
Now we will use that this square, and our original square involving A, are homotopy pullback squares, to conclude that the diagram

$$\begin{array}{ccc} S(\xi) & \longrightarrow & S(\gamma) \\ \downarrow & & \downarrow \\ S(\xi)_{\mathbb{Q}} & \longrightarrow & S(\gamma)_{\mathbb{Q}} \end{array} \tag{2.3.7}$$

is a homotopy pullback square. Indeed, consider again the diagram 2.3.5. The diagonal sequences are fibrations, and two of the three "main" squares involved are homotopy pullbacks.

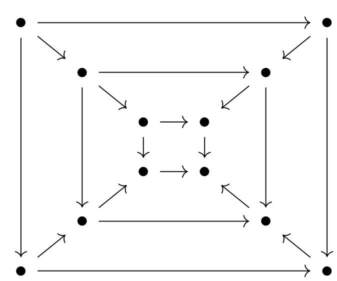
We have the following two lemmas; the first is an immediate adaptation of [Su09, Lemma 3.2.4], with the same proof (note only a graphical typo in the direction of some arrows in the proof therein, which has no bearing on the reasoning), and the second is the corresponding statement for fibrations instead of cofibrations (with the proof proceeding by replacing the long exact sequence in homology for a cofibration by the long exact sequence in homotopy for a fibration):

Lemma 2.3.2. *([Su09, Lemma 3.2.3]) Suppose we have a diagram*



where the diagonals are cofibrations, and two of the three central squares are homotopy pushouts. Then the third central square is a homotopy pushout.

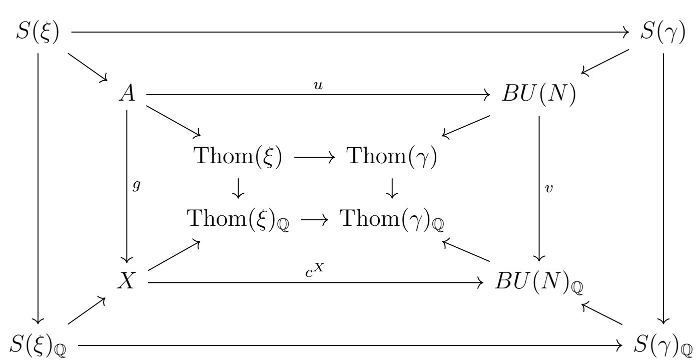
Lemma 2.3.3. *Suppose we have a diagram*



where the diagonals are fibrations, and two of the three central squares are homotopy pullbacks. Then the third central square is a homotopy pullback.

From Lemma 2.3.3 we have that the diagram 2.3.7 showing up in diagram 2.3.5 is a homotopy pullback.

Now recall diagram 2.3.2:



The Thom spaces are obtained as the homotopy cofibers of the outer diagonal arrows, with the maps between them being the induced maps between homotopy cofibers. After choosing particular homotopies making the homotopy commutative squares commute, the inner square commutes up to homotopy by the uniqueness of the induced map between homotopy cofibers from the top-left to the bottom-right [1]. We want to conclude that this inner central square of Thom spaces is a homotopy pullback. To do so, we will use Lemma 2.3.2 above combined with the following lemma (see e.g. [TayWil79, Lemma 6.1], and [Su09, Lemma 2.5.1] for a proof):

Lemma 2.3.4. *Suppose we have a commutative square of simply connected spaces, where the vertical arrows are rationalizations. Then the square is a homotopy pullback square if and only if it is a homotopy pushout square.*

Now by Lemma 2.3.2, our square of Thom spaces is a homotopy pushout, and hence by the above it is a homotopy pullback.

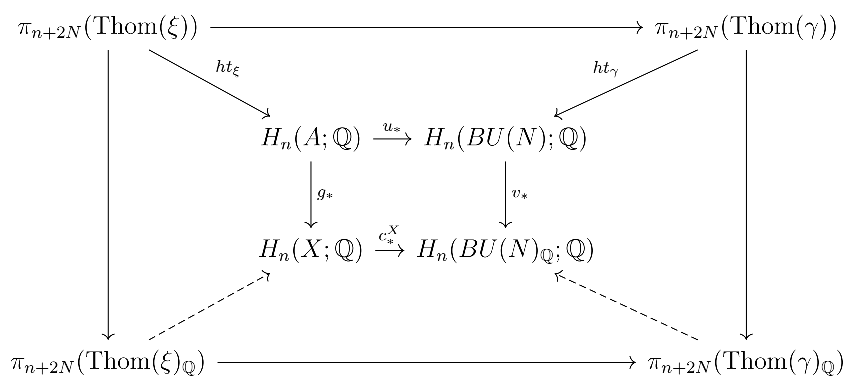
2.3.3 Verifying that the prescribed fundamental class is hit

With this in hand, we now consider the diagram

$$
\begin{array}{ccccc}
\pi_{n+2N}(\mathrm{Thom}(\xi)) & & \longrightarrow & & \pi_{n+2N}(\mathrm{Thom}(\gamma)) \\
\downarrow & \searrow^{ht_\xi} & & \swarrow^{ht_\gamma} & \downarrow \\
\downarrow & H_n(A;\mathbb{Q}) & \xrightarrow{u_*} & H_n(BU(N);\mathbb{Q}) & \downarrow \\
\downarrow & \downarrow g_* & & \downarrow v_* & \downarrow \\
\downarrow & H_n(X;\mathbb{Q}) & \xrightarrow{c_*^X} & H_n(BU(N)_{\mathbb{Q}};\mathbb{Q}) & \downarrow \\
\downarrow & \nearrow_{ht_\xi^{\mathbb{Q}}} & & \nwarrow_{ht_\gamma^{\mathbb{Q}}} & \downarrow \\
\pi_{n+2N}(\mathrm{Thom}(\xi)_{\mathbb{Q}}) & & \longrightarrow & & \pi_{n+2N}(\mathrm{Thom}(\gamma)_{\mathbb{Q}})
\end{array} \tag{2.3.8}
$$

[1] see Proposition 4.1 in `https://www.math.uni-bielefeld.de/~tcutler/pdf/Week%209%20-%20Homotopy%20Pushouts%20II.pdf`, and [MP11, §1.2] for a discussion.

where the upper diagonals are the corresponding Hurewicz–Thom maps, and the lower diagonals the induced maps on rationalizations. More precisely, we first have the following diagram of abelian groups:



The dashed arrows are the unique maps making the left square and right square commute, respectively (a map from a finitely generated abelian group G to a rational vector space factors uniquely through a given rationalization $G \to G \otimes \mathbb{Q}$). Again using the same diagrammatic reasoning as before, together with this uniqueness property of the factorization through a rationalization on the level of abelian groups, we conclude that the bottom square commutes.

These lower diagonals, $ht_\xi^\mathbb{Q}$ and $ht_\gamma^\mathbb{Q}$ are isomorphisms. Indeed, since we are taking N to be large with respect to n, the rationalized Hurewicz map $\pi_{n+2N}(\mathrm{Thom}(\xi))\otimes \to H_{n+2N}(\mathrm{Thom}(\xi);\mathbb{Q})$ is an isomorphism; this follows from the fact that $\mathrm{Thom}(\xi)$ is a simply connected space whose first non-trivial rational homology group is in degree $2N$ (by the Thom isomorphism theorem) and the rational Hurewicz theorem. A direct way to see this would be through employing minimal models: the minimal model of $\mathrm{Thom}(\xi)$ has no generators below degree $2N$, and so any non-trivial elements between degrees $2N$ and $4N-1$ must be a linear combination of generators. Since the differential contains no linear terms, elements in degree $\leq 4N-2$ must be closed; in particular, $H^*(\mathrm{Thom}(\xi);\mathbb{Q})$ is spanned by closed generators of the minimal model, which is equivalent to the (dual) rationalized Hurewicz homomorphism being an isomorphism. (Note that elements in degree $4N-1$ must be linear in the generators, but may not be closed; this gives the surjectivity part of the rational Hurewicz theorem). Then, the map $H_{n+2N}(\mathrm{Thom}(\xi);\mathbb{Q}) \to H_n(A;\mathbb{Q})$ is an isomorphism by the Thom isomorphism theorem, and hence the composition $\pi_{n+2N}(\mathrm{Thom}(\xi))\otimes \to H_{n+2N}(\mathrm{Thom}(\xi);\mathbb{Q}) \to H_n(A;\mathbb{Q})$ is an isomorphism, giving that $ht_\xi^{\mathbb{Q}}$ is an isomorphism (by tensoring the left-most square in diagram 2.3.8 with $\mathbb{Q}$); likewise for $ht_\gamma^{\mathbb{Q}}$.

Our goal is to show that $[A] \in H_n(A;\mathbb{Q})$ is in the image of ht_ξ, i.e. that $[X] \in H_n(X;\mathbb{Q})$ is in the image of $g_* ht_\xi$. Recall, that will imply the existence of a manifold M in the above discussion with a normal map to A such that $f_*[M] = [X]$. So, consider the element $c_*^X[X] \in H_n(BU(N)_\mathbb{Q};\mathbb{Q})$. We first show that this class is in the image of the map $v_* ht_\gamma$. For clarity, let us denote the elements $c_i \in H^{2i}(X;\mathbb{Q})$ by $c_i(X)$, and Chern numbers by c_α (i.e. $c_\alpha = c_{i_1}\cdots c_{i_r}$ for some $i_1,\ldots,i_r$).

As a first case, we show that $[X]$ is in the image of $g_* ht_\xi$ in the case of n odd. In this case $c_*^X[X] = 0$ since $H^{\mathrm{odd}}(BU(N)_\mathbb{Q};\mathbb{Q}) = 0$. Then $0 \in \pi_{n+2N}\mathrm{Thom}(\gamma)$ will map to $c_*^X[X]$ under $v_* ht_\gamma$. Since $ht_\gamma^{\mathbb{Q}}$ is an isomorphism, it follows that $(ht_\xi^{\mathbb{Q}})^{-1}[X] \in \pi_{n+2N}\mathrm{Thom}(\xi)_\mathbb{Q}$ and $0 \in \pi_{n+2N}\mathrm{Thom}(\gamma)$ map to the same element in $\pi_{n+2N}\mathrm{Thom}(\gamma)_\mathbb{Q}$ (namely $(ht_\gamma^{\mathbb{Q}})^{-1} c_*^X[X]$).

Now, as discussed, the diagram of Thom spaces is a homotopy pullback square, and so we have an induced Mayer–Vietoris long exact sequence in homotopy groups,

$$\cdots \xrightarrow{\partial} \pi_*(\mathrm{Thom}(\xi)) \xrightarrow{(\hat{u}_*,\hat{g}_*)} \pi_*(\mathrm{Thom}(\gamma)) \oplus \pi_*(\mathrm{Thom}(\xi)_\mathbb{Q}) \xrightarrow{\hat{v}_* - \widehat{c^X}_*} \pi_*(\mathrm{Thom}(\gamma)_\mathbb{Q}) \xrightarrow{\partial} \pi_{*-1}(\mathrm{Thom}(\xi)) \to \cdots$$

where $\hat{u}, \hat{g}, \hat{v}, \widehat{c^X}$ denote the induced maps on Thom spaces. From here it follows that there is an element $\beta \in \pi_{n+2N}(\mathrm{Thom}(\xi))$ that maps to $(ht_\xi^{\mathbb{Q}})^{-1}[X]$ and 0 respectively. Then $g_* ht_\xi(\beta) = [X]$ as desired.

Now suppose that $[X]$ is even, so $c_*^X[X]$ is not necessary zero. We now take into consideration condition (iii) from the beginning, namely that the "Chern numbers" on X are integers satisfying the Stong congruences. If we are in the case of $c_1(X) = 0$, then the Stong congruences are strictly stronger in dimensions $n \equiv 4 \bmod 8$ (i.e. the description of the image of $\Omega^{SU} \xrightarrow{\tau} H_*(BSU;\mathbb{Q})$ involves more congruences than those describing the image of $\Omega^U \xrightarrow{\tau} H_*(BU;\mathbb{Q})$). In either case, suppose the Stong congruences are satisfied. This means there is some stably almost complex manifold Y (with $c_1 = 0$ integrally if we are in the $c_1(X) = 0$ case) such that $\langle c_{i_1}(X)c_{i_2}(X)\cdots c_{i_r}(X), [X]\rangle = \langle c_{i_1}(Y)c_{i_2}(Y)\cdots c_{i_r}(Y), [Y]\rangle$ for all tuples $(i_1, i_2, \ldots, i_r)$ whose total degree is n. For simplicity, let us denote e.g. $c_{i_1}(X)c_{i_2}(X)\cdots c_{i_r}(X)$ by $c_\alpha(X)$.

Note,

$$\langle c_\alpha(X), [X]\rangle = \langle (c^X)^*(c_\alpha), [X]\rangle = \langle c_\alpha, (c^X)_*[X]\rangle.$$

On the other hand, consider the map $Y \xrightarrow{\nu_Y} BU$ (or, to BSU), classifying the stable normal bundle of Y, i.e. $\nu_Y^*\gamma$ is the stable normal bundle of Y (with its complex structure). By the Pontryagin–Thom construction, there is an element $\beta_Y \in \pi_{n+2N}(\mathrm{Thom}(\gamma))$ such that $ht_\gamma(\beta_Y) = \nu_*[Y]$ (namely, Y is constructed by taking the preimage of BU or BSU under a suitable representative of the homotopy element, and as a consequence the induced map from Y to BU or BSU classifies the stable normal bundle Y). Here we consider $\nu_*[Y]$ as an element in rational homology. Since ν_Y pulls back the universal Chern classes c_i to the Chern classes of the stable normal bundle of Y, it follows that ν_Y pulls back the classes $\overline{c}_i$ to the Chern classes of (the stable tangent bundle of) Y. So, we have

$$\langle c_\alpha(Y), [Y]\rangle = \langle \nu^*\overline{c}_\alpha, [Y]\rangle = \langle \overline{c}_\alpha, (\nu_Y)_*[Y]\rangle = \langle \overline{c}_\alpha, ht_\gamma(\beta_Y)\rangle.$$

Furthermore, since we are only considering Chern classes up to degree n, we have $\overline{c}_i = v^*c_i$, and hence

$$\langle \overline{c}_\alpha, ht_\gamma(\beta_Y)\rangle = \langle v^*(c_\alpha), ht_\gamma(\beta_Y)\rangle = \langle c_\alpha, v_*(ht_\gamma(\beta_Y))\rangle.$$

In conclusion, $\langle c_\alpha, (c^X)_*[X]\rangle = \langle c_\alpha, v_*(ht_\gamma(\beta_Y))\rangle$ for all α, and since the c_α span the vector space $H^n(BU_\mathbb{Q};\mathbb{Q}) \cong \mathrm{Hom}(H_n(BU_\mathbb{Q};\mathbb{Q}),\mathbb{Q})$, we conclude that $c^X_*[X] = v_*(ht_\gamma(\beta_Y))$. Hence, there is an element $\beta \in \pi_{n+2N}(\mathrm{Thom}(\xi))$ that maps to $(ht_\xi^\mathbb{Q})^{-1}[X]$, and $g_*ht_\xi(\beta) = [X]$.

2.3.4 Stable almost complex structure on the resulting manifold and its Chern classes

Now that we have obtained a manifold and a degree one normal map to A, we go through the second stage of the proof to obtain a degree one normal map $M \xrightarrow{f} A$ which is a also a rational homotopy equivalence, and thus the composition $M \xrightarrow{f} A \xrightarrow{g} X$ is a rational homotopy equivalence.

We pull back the complex structure from the vector bundle γ to ξ and then to the stable normal bundle ν_M (a normal map gives a real bundle isomorphism between ν_M and $f^*\xi$, and so we can transport the complex structure from $f^*\xi$ to ν_M.) By construction, the fundamental class $[M]$ is determined by the orientation of the stable normal bundle (as a real bundle) in the Pontryagin–Thom construction (see e.g. [Br62, Lemma 2]), and hence the complex structure we are equipping the stable normal bundle with induces this same orientation. This can also be seen tautologically from the diagram

$$M \to A \to BU(N) \to BSO(2N),$$

where the map $BU(N) \to BSO(2N)$ classifies the real vector bundle underlying the tautological bundle γ. The stable normal bundle to M, as a real vector bundle, is by the Pontryagin–Thom construction classified by this composition, and hence it lifts to a complex vector bundle by looking at $M \to A \to BU(N)$.

As a complex structure on the stable normal bundle of a manifold determines a complex structure on the stable tangent bundle, we have a stably almost complex structure on M. Indeed, the stable normal bundle as a complex vector bundle is classified by a map $M \to Gr_{\mathbb{C}}(N, N')$ of complex N-planes in $\mathbb{C}^{N'}$, for some large N'. With the standard hermitian inner product on $\mathbb{C}^{N'}$, we have a diffeomorphism $Gr_{\mathbb{C}}(N, N') \xrightarrow{\perp} Gr_{\mathbb{C}}(N'-N, N')$ sending a plane to its orthogonal complement. The composition $M \to Gr_{\mathbb{C}}(N, N') \xrightarrow{\perp} Gr_{\mathbb{C}}(N'-N, N')$ gives a complex structure on the stable tangent bundle to M, as seen from the commutative diagram

$$\begin{array}{ccc}
Gr_{\mathbb{C}}(N, N') & \xrightarrow{\perp} & Gr_{\mathbb{C}}(N'-N, N') \\
\downarrow & & \downarrow \\
Gr_{\mathbb{R}}(2N, 2N') & \xrightarrow{\perp} & Gr_{\mathbb{R}}(2N'-2N, 2N')
\end{array}$$

where the map $\perp$ between real Grassmannians sends a real plane to its orthogonal complement with respect to the standard euclidean inner product on $\mathbb{R}^{2N'}$. We see that the total Chern classes of the stable normal bundle to M and the stable tangent bundle (with this complex structure) multiply to the trivial class 1.

Now we calculate the Chern classes of this stable almost complex structure. Recall the following diagram:

$$\begin{array}{ccccc}
\nu_M & \longrightarrow & \xi & \longrightarrow & \gamma \\
\downarrow & & \downarrow & & \downarrow \\
M & \xrightarrow{f} & A & \xrightarrow{u} & BU(N) \\
 & & \downarrow g & & \downarrow v \\
 & & X & \xrightarrow{c^X} & BU(N)_{\mathbb{Q}}
\end{array}$$

The Chern classes of this complex structure on the stable tangent bundle of M satisfy

$$\begin{aligned}
c_i(M) = \overline{c}_i(\nu_M) &= f^*\overline{c_i(\xi)} = f^*\overline{c_i(u^*\gamma)} = f^*u^*\overline{c_i(\gamma)} = (uf)^*\overline{c_i(\gamma)} \\
&= (uf)^*v^*c_i = (vuf)^*c_i = (c^X uf)^*c_i = f^*g^*(c^X)^*c_i = (gf)^*c_i(X),
\end{aligned}$$

where we used that $i \leq \frac{n}{2}$ in order to have $\overline{c_i(\gamma)} = v^*c_i$.

Now recall our necessary condition (vi) in order for this stable almost complex structure to be induced by an almost complex structure: we must have $\langle c_n(M), [M]\rangle = \chi(M)$, by the obstruction-theoretic definition of the top Chern class. Note, since

$$\langle c_n(M), [M]\rangle = \chi(M) = \langle (gf)^*c_n(X), [M]\rangle = \langle c_n(X), (gf)_*[M]\rangle = \langle c_n(X), [X]\rangle,$$

this is equivalent to $\langle c_n(X), [X]\rangle = \chi(X)$, since M and X have the same Euler characteristic. Conversely, if $\langle c_n(X), [X]\rangle = \chi(X)$, then the stable almost complex structure on M is induced by an almost complex structure; the top Chern class evaluating to the Euler characteristic is a necessary *and sufficient* condition for reducing to a genuine almost complex structure, see e.g. [Kahn69, Corollary 3].

This concludes the proof of Theorem 1.1.1.

Remark 2.3.5. • *Consider again the diagram*

$$\begin{array}{ccc} \pi_{n+2N}(\mathrm{Thom}(\xi)) & \longrightarrow & \pi_{n+2N}(\mathrm{Thom}(\gamma)) \\ {\scriptstyle ht_\xi}\searrow & & \swarrow{\scriptstyle ht_\gamma} \\ H_n(A;\mathbb{Q}) & \xrightarrow{u_*} & H_n(BU(N);\mathbb{Q}) \end{array}$$

Choosing an element of $\pi_{n+2N}\mathrm{Thom}(\xi)$ *gives us a fundamental class* $[A]$ *in* $H_n(A;\mathbb{Q})$ *by looking at its image under the Hurewicz–Thom map, and this fundamental class will be mapped to by a stably almost complex manifold by applying the Pontryagin–Thom construction to our chosen homotopy element in the Thom space. Indeed, by construction (i.e. commutativity of the diagram) the image of* $[A]$ *in the homology of* $BU(N)$ *will land in the lattice described by Stong. The issue here is that we do not have control of what exactly the Chern numbers of our manifold will be, so we do not know what the top Chern class will evaluate to, and, in the case of dimension divisible by four, whether the signature will be computed correctly.*

- *Let us comment now on the case of* $c_1 = 0$ *(if the dimension* n *is not congruent to 4 mod 8, suppose we do not replace* BU *by* BSU*). The homotopy pullback of* $X \xrightarrow{c^X} BU(N)_{\mathbb{Q}}$ *and* $BU(N) \xrightarrow{v} BU(N)_{\mathbb{Q}}$ *gives us the map* $A \xrightarrow{g} X$*. From the long exact sequence in homotopy groups, since the homotopy fiber of* v *has trivial rational homotopy groups, we see that* $A \xrightarrow{g} X$ *is an isomorphism on rational homotopy groups.*

  The fundamental group of A *is* $\mathbb{Q}/\mathbb{Z}$*. Now, as before, we can obtain a stably almost complex manifold* M *with a degree one normal map to* A*. Since* A *is not simply connected as it was before, we cannot simply surger* M *down to a simply connected* *manifold. However, noticing that all commutators $\pi_1(M)$ become trivial when mapped over to A, we can surger M down to a manifold with abelian fundamental group. Since this is a finitely generated abelian group, we can identify the infinite cyclic summands in the group; the generators of these groups, mapped to A, become torsion, and hence some multiple of the generator in each of the infinite cyclic summands can be surgered out. We end up with M whose fundamental group is finite and abelian; that is, the map on fundamental groups to A is a rational isomorphism. Hence $\pi_1(f) \otimes \mathbb{Q} = 0$ (where f denotes the normal map from the new manifold). We can perform surgery, getting rid of $\pi_*(f) \otimes \mathbb{Q}$ up to the middle degree; however, it is in middle degree that the following difficulty arises: we have no guarantee that the relative Hurewicz map will give an isomorphism between the homotopy group of f and the homology of the kernel, an identification that was crucial earlier. One could hope that relative Hurewicz would hold (rationally) if the pair (A, M) were nilpotent, but whether this is the case is not at all clear. We believe this issue of $c_1 = 0$ is not merely a bug in the technique of the proof, but rather indicative of some interesting phenomenon at play.*

Chapter 3

Consequences, examples, and computations

3.1 Consequences of the proof of the main theorem

We notice that all the necessary conditions (i)-(vi) for realization were cohomological. Since in dimensions ≥ 6 not congruent to 4 mod 8, these were also sufficient, we have:

Corollary 3.1.1. *The realizability of a simply connected rational homotopy type by a simply connected closed almost complex manifold of dimension* $n \not\equiv 4 \bmod 8$*, depends only on its cohomology ring.*

The case of dimensions congruent to 4 mod 8 is somewhat different, since there might exist an almost complex manifold with $c_1 = 0$ rationally not satisfying the stronger set of SU congruences, whose rational homotopy type would then not be constructible using only Theorem 1.1.1 as stated.

A simply connected rational homotopy type is determined, up to homotopy equivalence, by a minimal C_∞-algebra structure on its cohomology (extending the given multiplication), up to isomorphism [Kad08]. That is, for a given graded-commutative algebra (H, m_2) (where m_2 is the multiplication), we have

$$\{\text{rational spaces } X \text{ with } H^*X \cong (H, m_2)\}/_\sim \equiv \{C_\infty \text{ structures } (H, m_2, m_3, m_4, \ldots)\}/_\sim.$$

The realizability of a rational space, in dimensions not 4 mod 8, is insensitive to the higher operations $m_{\geq 3}$. Contrast this with the case of compact complex manifolds which satisfy the $\partial\overline{\partial}$-lemma, where among all rational homotopy types realizing a given cohomology algebra, at most one of them (the formal one) is realized by such a manifold.

In particular, for every simply connected almost complex manifold, there is a formal almost complex manifold with the same rational cohomology ring (as a definition of formality we can take that some representative of the associated C_∞-algebra structure on the cohomology has trivial $m_{\geq 3}$; equivalently, the minimal model of the space is isomorphic to the minimal model of its cohomology (where the cohomology algebra is equipped with the trivial differnetial). It is perhaps the other direction that is more interesting: knowing that a formal rational homotopy type can be realized by a closed almost complex manifold implies that any rational homotopy type with the same cohomology ring can also be realized.

An easy consequence of our main theorem that demonstrates the abundance of rational homotopy types of closed almost complex manifolds is the following:

Corollary 3.1.2. *Any simply connected rational homotopy type satisfying Poincaré duality with Euler characteristic and signature zero (if the dimension is divisible by four) is realized by a closed almost complex manifold.*

Proof. Since the Euler characteristic and signature vanish, one can choose all rational Chern classes to be trivial. (Note that the signature without a choice of fundamental class is only well-defined up to sign, so is well-defined in the signature zero case.) □

This includes, for example, the rational homotopy types of the product of any odd–dimensional simply connected manifold and a sphere of odd dimension ≥ 3.

3.2 Chern number congruences in dimensions ≤ 10

We list the congruences among Chern numbers for stably almost complex manifolds of dimension ≤ 10. The congruences in dimension ≤ 8 were listed by Hirzebruch in [Hirz60].

We will omit the integral sign, with the understanding that the given class is to be paired with the fundamental class of the stably almost complex manifold.

- Dimension 2: $c_1 \in 2\mathbb{Z}$
- Dimension 4: $c_1^2 + c_2 \in 12\mathbb{Z}$
- Dimension 6: $c_1^3 \in 2\mathbb{Z}$, $c_3 \in 2\mathbb{Z}$, $c_1c_2 \in 24\mathbb{Z}$
- Dimension 8:

$$\begin{aligned} 2c_1^4 + c_1^2c_2 &\in 12\mathbb{Z}, \\ c_1c_3 - 2c_4 &\in 4\mathbb{Z}, \\ -c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4 &\in 720\mathbb{Z} \end{aligned}$$

- Dimension 10:

$$\begin{aligned} c_1c_4 + c_5 &\in 12\mathbb{Z}, \\ 4c_1^3c_2 + 8c_1^2c_3 + c_1c_4 + 9c_5 &\in 24\mathbb{Z}, \\ 15c_1^5 - 5c_1^3c_2 + 12c_1c_2^2 + 8c_1^2c_3 - 8c_1c_4 &\in 24\mathbb{Z}, \\ c_1^5 + c_1^3c_2 + 6c_1^2c_3 &\in 12\mathbb{Z}, \\ -c_1^3c_2 + 3c_1c_2^2 + c_1^2c_3 - c_1c_4 &\in 1440\mathbb{Z}. \end{aligned}$$

3.3 Further remarks on realization by almost complex manifolds

From the previous example, we see that any simply connected rational homotopy satisfying Poincaré duality in dimension 6 is realized by an almost complex manifold. Indeed, we can choose $c_1 = 0$, $c_2 = 0$, and the fundamental class and c_3 so that c_3 evaluates to the Euler characteristic. The congruences in dimension 6 require c_3 to be even, but this will be automatically satisfied as the Euler characteristic of a $4k+2$–dimensional Poincaré duality algebra is even. A simply connected rational space satisfying rational Poincaré duality is formal, so we see that for degree reasons any such rational homotopy type will be of the form $M\#N$, where M is a simply connected 6–manifold with $b_3 = 0$, and N is a connected sum of some number of copies of $S^3 \times S^3$. We remark that even for a small value of b_2 there are many rational homotopy types with $b_3 = 0$ and this b_2; for example the real homotopy type of $\mathbb{CP}^3\#\mathbb{CP}^3$ contains infinitely many rational homotopy types (see [Mar90, Example 3.5]). (For $b_2 \leq 1$ and $b_3 = 0$ there is only one rational homotopy type.)

Since a simply-connected rational Poincaré duality space of formal dimension 6 is formal [Mill79], its rational homotopy type is determined by its cohomology algebra. Denoting the cohomology by H^*, Poincaré duality gives us an isomorphism $H^2 \cong (H^4)^\vee$, and so the product $H^2 \otimes H^2 \to H^4$ is given by a symmetric trilinear form $H^2 \otimes H^2 \otimes H^2 \to \mathbb{Q}$. This trilinear form determines the cohomology algebra of our space, and hence its rational homotopy type. In the case of $\dim H^2 = 3$, such trilinear forms correspond to (rational) cubic plane curves; the abundance and structure of such curves, paired with choices of rational Chern classes, suggests this may be an interesting line of further study.

In all even dimensions $n \geq 8$ there are examples of simply connected rational homotopy types not realized by almost complex manifolds; indeed one can take the rationalized spheres $S^n_{\mathbb{Q}}$ (see [AM19, Theorem 2.2], adapting a famous observation of Borel and Serre to the rational setting).

In dimension 10, we see from the congruences in Section 3.2 that any simply connected rational homotopy type satisfying Poincaré duality of dimension 10, with Euler characteristic divisible by 24, is realized by an almost complex manifold (by setting the lower Chern classes to be 0). In all dimensions of the form $4k+2$, we see that the only obstruction to realizability is a finite divisibility constraint on the Euler characteristic.

Corollary 3.3.1. *In odd complex dimensions, realizability is guaranteed if the Euler characteristic is divisible by a certain positive integer $d(n)$ depending on the dimension (e.g.*

$d(6) = 2, d(10) = 24$, see Section 3.2)).

One can also ask about realizability of *real* homotopy types by closed almost complex manifolds. Unfortunately, an immediate problem presents itself in this case as $H^*(K(\mathbb{R}, n); \mathbb{R})$ is *not* the free graded-commutative algebra on one generator (see Oprea's review Zbl 0865.55009 of Brown and Szczarba's "Real and rational homotopy theory"). In fact, $H_*(K(\mathbb{R}, n); \mathbb{R})$ has uncountable dimension. It is the case that $H^*(K(\mathbb{R}, n); \mathbb{R})$ is the free graded-commutative algebra on one generator if one interprets the former in the context of *continuous cohomology* (see Brown–Szczarba), but we will not pursue this here.

Remark 3.3.2. *In Sullivan's original formulation of the realization theorem for closed smooth manifolds [Sull77, Theorem 13.2], one sees that the Stong congruences (for BSO; they are nontrivial only in dimensions of the form $n = 4k$) are not mentioned in the signature 0 case. If the quadratic form on $H^{\frac{n}{2}}(X; \mathbb{Q})$ given by $\alpha \otimes \beta \mapsto \langle \alpha\beta, [X] \rangle$ is equivalent over $\mathbb{Q}$ to one of the form $\sum_i \pm x_i^2$ for some choice of fundamental class $[X] \in H_n(X; \mathbb{Q})$, then it will be of this form for any other non-zero choice of $[X]' \in H_n(X; \mathbb{Q})$. Indeed, since the signature is zero, by assumption we can write the intersection form with respect to $[X]$ as $\sum_i x_i^2 - y_i^2$. Scaling the fundamental class by a rational changes this into $\sum_i \frac{p}{q} x_i^2 - \frac{p}{q} y_i^2$, which is the same as $\sum_i ((1 + \frac{p}{4q})x + (1 - \frac{p}{4q})y)^2 - ((1 - \frac{p}{4q})x + (1 + \frac{p}{4q})y)^2$. In particular, we may scale the fundamental class until all of the Stong congruences are satisfied. We cannot do the same in the almost complex realization problem in the signature 0 case, as our choice of top Chern class is tethered to the fundamental class by the requirement $\langle c_{2n}, [X] \rangle = \chi(X)$.*

3.4 Rational connected sums of quaternionic projective planes

Using the results of [GeM00], one calculates that $k\mathbb{HP}^2 \# \ell\overline{\mathbb{HP}^2}$ (with its standard smooth structure) admits an almost complex structure if and only if $(k, l) = (4n + 3, 2n + 1)$ for some n. Let us see what happens in the rational case; i.e. we consider 8–manifolds M with $H^*(M;\mathbb{Q}) = 0$ except for $* \in \{0,4,8\}$, with intersection form given by $k\langle 1\rangle \oplus \ell\langle -1\rangle$. Let us denote such a manifold by $\mathbb{Q}(k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2})$, following notation suggested by Z. Su.

We will use the Chern number congruences for stably almost complex 8–manifolds in what follows (see Section 3.2) :

$$\begin{aligned} -c_4 + c_1c_3 + 3c_2^2 + 4c_1^2c_2 - c_1^4 &\in 720\mathbb{Z}, \\ c_1^2c_2 + 2c_1^4 &\in 12\mathbb{Z}, \\ -2c_4 + c_1c_3 &\in 4\mathbb{Z}, \end{aligned}$$

which in our case trivially becomes

$$-c_4 + 3c_2^2 \in 720\mathbb{Z} \quad \text{and} \quad c_4 \text{ is even.}$$

Now, suppose we have a $\mathbb{Q}(k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2})$ that admits an almost complex structure. Then $\sigma = k - \ell$ and $\chi = 2 + k + \ell$, so from Hirzebruch's relation $\sigma \equiv \chi \bmod 4$ in dimension 8 [Hirz87, p.777], we have $k - \ell \equiv 2 + k + \ell \bmod 4$, i.e. $2\ell \equiv 2 \bmod 4$, i.e. ℓ is odd. Since $k + \ell + 2 = \chi = c_4$ must be even, we conclude that k is odd as well.

We now show by example that the above observation on when $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$ admits almost complex structures does not carry over to the rational case. Let us consider $\mathbb{Q}(23\mathbb{HP}^2\#23\overline{\mathbb{HP}^2})$. Then $c_4 = \chi = 48$ and $\sigma = 0$. We can write c_2 as $c_2 = \sum_{i=1}^{23} x_i + \sum_{i=1}^{23} y_i^2$, where the x_i are classes such that $\langle x_i^2, \mu\rangle = 1$ and $\langle y_i^2, \mu\rangle = -1$ for some chosen fundamental class μ, and all pairwise products of these elements are zero. The signature equation in terms of Chern classes is $\frac{1}{45}(3c_2^2 + 14c_4) = 0$, i.e. $3c_2^2 = -14\cdot 48$, which has a solution $c_2^2 = -224$. We see that the congruences are satisfied for this c_2^2 and c_4; indeed, c_4 is even and $-c_4 + 3c_2^2 = 0$. It only remains to check that one can solve for c_2. By Lagrange's four–squares theorem, 224 can be written as a sum of four integer squares, so we choose four y_i's and take these integers to be the coefficients of c_2 along these y_i's (and set the remaining coefficients to be zero). By the almost complex realization theorem we conclude that there is an almost complex manifold realizing this data. It is possible that the underlying homotopy type is that of $k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2}$, equipped with a non-standard smooth structure.

We now observe that the above fits into a more general solution. Let us consider the general case of $\mathbb{Q}(k\mathbb{HP}^2\#\ell\overline{HP^2})$. The signature is $k-\ell$ and the Euler characteristic is $2+k+\ell$. Besides the Euler characteristic being even, we must have $3c_2^2+14c_4=45(k-\ell)$ and $-c_4+3c_2^2 \in 720\mathbb{Z}$. Let us write this as

$$3c_2^2 = 31k - 59\ell - 28,$$

$$3c_2^2 = 720m + k + \ell + 2.$$

From $31k-59\ell-28 = 720m+k+\ell+2$ we have $k = 2\ell+1+24m$. In particular, k is odd, so we write $k = 2n+1$. Then $\ell = n-12m$, and $c_2^2 = 236m+n+1$. Since $2+k+\ell$ must be even (by the Euler characteristic requirement), we have that ℓ is odd as well, i.e. n is odd; we write $n = 2u+1$.

So, the solutions are $(k, \ell, c_2^2) = (4u+3, 2u+1-12m, 2u+2+236m)$; since we require $k, \ell \geq 0$, we have $u \geq 0$ and $2u+1 \geq 12m$. Fixing k and ℓ, i.e. u and m, we see that the problem of realizing $\mathbb{Q}(k\mathbb{HP}^2\#\ell\overline{\mathbb{HP}^2})$ comes down to finding a class c_2 such that $c_2^2 = 2u+2+236m$. We use the same notation and method as in the case of $\mathbb{Q}(23\mathbb{HP}^2\#23\overline{\mathbb{HP}^2})$ considered above. If $m=0$, and $u=0$, we have $(k,\ell,c_2^2) = (3,1,2)$, and we can take $c_2 = x_1+x_2$, which satisfies $c_2^2 = 2$. If $m = 0$ and $u > 0$, then $k \geq 4$, and we may solve for c_2 using Lagrange's four–square theorem since $c_2^2 > 0$. If $m > 0$, then $u \geq 6$ and so $k \geq 4$, and we may again apply Lagrange's four–square theorem to solve for c_2 since $c_2^2 > 0$. If $m < 0$, then $\ell \geq 4$, and so if $c_2^2 \leq 0$ we can solve for c_2. If $m < 0$ and $c_2^2 > 0$, then $2u+2 > -236m$, so in particular $k = 4u+3 \geq 4$, and we can solve for c_2 again.

In conclusion, we have that there is a $\mathbb{Q}(k\mathbb{HP}^2\#\ell\overline{HP^2})$ if and only if $(k,l) = (4u+3, 2u+1+12m)$ with $k,l \geq 0$. Note the similarity with the integral case, where there was no summand of $12m$ involved. The observation above with $\mathbb{Q}(23\mathbb{HP}^2\#23\overline{\mathbb{HP}^2})$ follows for $u=5$, $m=1$.

Remark 3.4.1. *Above we used Hirzebruch's relation that on a closed almost complex $4n$–manifold, we have $\chi \equiv (-1)^n\sigma \bmod 4$. Since every even-dimensional stably almost complex manifold is complex cobordant to an almost complex manifold [Kahn69, Corollary 5], and the Chern numbers and signature are complex cobordism invariants, this shows us that Hirzebruch's relation is the restriction to almost complex manifolds of a general congruence $c_{2n} \equiv (-1)^n\sigma \bmod 4$ for stably almost complex manifolds. Since this is a relation between Chern numbers, it must be implied by the Stong congruences.*

For example, in the case of almost complex 4–manifolds, we have $\chi+\sigma \equiv 0 \bmod 0$. This follows from the integrality of the Todd genus, $c_1^2+c_2 \in 12\mathbb{Z}$, combined with $3\sigma = p_1 = c_1^2-2c_2$. Indeed, expressing c_1^2 in two ways gives us $12k - c_2 = 3\sigma + 2c_2$ for some integer k, i.e. $3(\sigma + c_2) = 12k$, whence $\sigma + c_2 \equiv 0 \bmod 4$.

3.5 The weak form of Hirzebruch's prize problem

In [HiBeJu92], Hirzebruch asked for an example of a closed smooth spin 24–manifold X with the following properties: $p_1(TX) = 0$, $w_2(TX) = 0$, $\int \hat{A}(TX) = 1$, $\int \hat{A}(TX)ch(TX \otimes \mathbb{C}) = 0$. The interest in such a manifold is the observation that the $\hat{A}$–genus of certain linear combinations of symmetric powers of the complexified tangent bundle computes the dimensions of the irreducible representations of the Monster group. For this property, one need only require that $p_1(TX) = 0$ rationally.

For this, we will use a version of the realization theorem for spin manifolds; as the proof is analogous, we only state it here (see also [CN20, §3.5]):

Theorem 3.5.1. *(Realization for spin manifolds) Let X be a formally n–dimensional simply–connected rational space of finite type satisfying Poincaré duality on its rational cohomology, $n \geq 5$, and let $[X] \in H_n(X;\mathbb{Q})$ be a non-zero element. Furthermore, let $p_i \in H^{4i}(X;\mathbb{Q})$, $1 \leq i \leq \left\lfloor \frac{n}{4} \right\rfloor$ be cohomology classes. Then we have:*

1. *If n is not divisible by 4, there is a closed spin n–manifold M and a rational equivalence* $M \xrightarrow{f} X$ *such that* $f_*[M] = [X]$ *and* $p_i(TM) = f^*(p_i)$.

2. *If n is divisible by 4, then there is a closed spin manifold M and a rational equivalence* $M \xrightarrow{f} X$ *such that* $f_*[M] = [X]$ *and* $p_i(TM) = f^*(p_i)$ *if*

   - *the numbers* $\langle p_{i_1} p_{i_2} \cdots p_{i_r}, [X] \rangle$ *satisfy the Stong congruences of a spin manifold ([Stong65b], discussed below),*
   - *the quadratic form on* $H^{\frac{n}{2}}(X; \mathbb{Q})$ *given by* $q(\alpha, \beta) = \langle \alpha\beta, [X] \rangle$ *is equivalent over* $\mathbb{Q}$ *to one of the form* $\sum_i \pm y_i^2$,
   - *we have* $\langle L(p_1, \ldots, p_{n/4}), [X] \rangle = \sigma(X)$, *where L is Hirzebruch's L–polynomial.*

Remark 3.5.2. *The proof of this realization result starts with the observation that the map BSpin → BSO induced by the twofold covering Spin → SO is a rational equivalence. Note that admitting a spin structure is a stable property of a bundle, and as in the almost complex case, such a structure on the stable normal bundle canonically induces one on the stable tangent bundle.*

Using this realization result, we will construct a smooth closed simply connected 24–manifold with $w_2 = 0$ $p_1 = 0$ (*rationally*), $\hat{A} = 1$, $\hat{A}(M, T_{\mathbb{C}}) = 0$.

First we discuss the Stong congruences for spin manifolds [Stong65b]. Given a closed smooth spin manifold M, consider the formal splitting $p = \prod_{j=1}^{6}(1 + x_j^2)$, where x_j^2 are the Pontryagin roots (so $\deg(x_j^2) = 4$). Then consider a new set of variables given by $e^{x_j} + e^{-x_j} - 2$. Considering terms only up to degree 24, we have

$$e^{x_j} + e^{-x_j} - 2 = x_j^2 + \frac{x_j^4}{12} + \frac{x_j^6}{360} + \frac{x_j^8}{\frac{8!}{2}} + \frac{x_j^{10}}{\frac{10!}{2}} + \frac{x_j^{12}}{\frac{12!}{2}}.$$

Now form the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_6$ in these six variables $e^{x_j} + e^{-x_j} - 2$, and express them in terms of the Pontryagin classes p_j (which are the elementary symmetric polynomials in x_j^2). For these and other calculations we use the Macaulay2 package "SymmetricPolynomials". Suppose $p_1 = p_3 = p_5 = 0$ rationally. Then we obtain the following modulo torsion:

$$\sigma_1 = \tfrac{-1}{119750400}p_2^3 + \tfrac{1}{39916800}p_2p_4 - \tfrac{1}{39916800}p_6 + \tfrac{1}{10080}p_2^2 - \tfrac{1}{5040}p_4 - \tfrac{1}{6}p_2$$

$$\sigma_2 = \tfrac{1}{1814400}p_2^3 - \tfrac{11}{604800}p_2p_4 + \tfrac{31}{604800}p_6 + \tfrac{1}{720}p_2^2 + \tfrac{1}{40}p_4 + p_2$$

$$\sigma_3 = \tfrac{-1}{7560}p_2p_4 - \tfrac{4}{945}p_6 - \tfrac{1}{3}p_4$$

$$\sigma_4 = \tfrac{1}{720}p_2p_4 + \tfrac{19}{240}p_6 + p_4$$

$$\sigma_5 = \tfrac{-1}{2}p_6$$

$$\sigma_6 = p_6.$$

Now, [Stong65b] tells us that the following congruences must be satisfied: First of all, the Pontryagin numbers must be integers (in the stably almost complex case, the integrality of Chern numbers was implied by the other congruences). Furthermore, any polynomial in the above σ_i with integer coefficients multiplied by the $\hat{A}$-genus must be an integer. Here is the $\hat{A}$-genus up to degree 24, modulo terms involving p_1, p_3, p_5:

$$\hat{A}_0 = 1$$

$$\hat{A}_1 = 0$$

$$\hat{A}_2 = \tfrac{-4}{5760}p_2$$

$$\hat{A}_3 = 0$$

$$\hat{A}_4 = \tfrac{1}{464486400}(208p_2^2 - 192p_4)$$

$$\hat{A}_5 = 0$$

$$\hat{A}_6 = \tfrac{1}{2678117105664000}(-769728p_2^3 + 719872p_3^2 + 1476352p_2p_4 - 707584p_6).$$

Here $\hat{A}_i$ denotes the degree $4i$ term in the $\hat{A}$ polynomial.

From here we obtain the following full set of congruences. Along with $\hat{A}_6$, each of the following expressions must be an integer:

$$\sigma_1 \cdot \hat{A} = \tfrac{-97}{638668800}p_2^3 + \tfrac{37}{159667200}p_2p_4 - \tfrac{1}{39916800}p_6$$

$$\sigma_2 \cdot \hat{A} = \tfrac{1}{29030400}p_2^3 - \tfrac{29}{806400}p_2p_4 + \tfrac{31}{604800}p_6$$

$$\sigma_3 \cdot \hat{A} = \tfrac{1}{10080}p_2p_4 - \tfrac{4}{945}p_6$$

$$\sigma_4 \cdot \hat{A} = \tfrac{1}{1440}p_2p_4 + \tfrac{19}{240}p_6$$

$$\sigma_5 \cdot \hat{A} = \tfrac{-1}{2}p_6$$

$$\sigma_6 \cdot \hat{A} = p_6$$

$$\sigma_1^2 \cdot \hat{A} = \tfrac{-19}{362880}p_2^3 + \tfrac{1}{15120}p_2p_4$$

$$\sigma_1^3 \cdot \hat{A} = \tfrac{-1}{216}p_2^3$$

$$\sigma_1\sigma_2 \cdot \hat{A} = \tfrac{-1}{60480}p_2^3 - \tfrac{11}{2520}p_2p_4$$

$$\sigma_1^2\sigma_2 \cdot \hat{A} = \tfrac{1}{36}p_2^3$$

$$\sigma_1\sigma_2^2 \cdot \hat{A} = \tfrac{-1}{6}p_2^3$$

$$\sigma_1\sigma_3 \cdot \hat{A} = \tfrac{1}{18}p_2p_4$$

$$\sigma_1\sigma_4 \cdot \hat{A} = \tfrac{-1}{6}p_2p_4$$

$$\sigma_2^3 \cdot \hat{A} = p_2^3$$

$$\sigma_2\sigma_3 \cdot \hat{A} = \tfrac{-1}{3}p_2p_4$$

$$\sigma_2\sigma_4 \cdot \hat{A} = p_2p_4$$

Some of the above congruences evidently imply some of the others. Before we proceed to consider the system of congruences, we now recall that we require $\hat{A}(M) = 1$ and $\hat{A}(M, T_{\mathbb{C}}) = 0$. Calculating the top degree of the latter to be

$$\begin{aligned}\hat{A}(M, T_{\mathbb{C}})_6 &= (\hat{A}(M) \cdot ch(TM \otimes \mathbb{C}))_{24} \\ &= \tfrac{-8389}{52835328000}p_2^3 + \tfrac{9707}{39626496000}p_2p_4 - \tfrac{311}{9906624000}p_6,\end{aligned}$$

we obtain

$$p_6 = \tfrac{-25167}{4976}p_2^3 + \tfrac{9707}{1244}p_2p_4.$$

The first requirement $\hat{A} = 1$ gives us

$$p_2p_4 = \frac{873600000p_2^3 - 311 \cdot 2678117105664000}{1257984000}.$$

Now we have the following two equations:

$$\begin{aligned}p_2p_4 &= \tfrac{25}{36}p_2^3 - 662086656 \\ p_6 &= \tfrac{13}{36}p_2^3 - 5166298368\end{aligned}$$

and the following congruences:

$$\frac{25}{648}p_2^3 - 36782592 \in \mathbb{Z}$$

$$\frac{1}{216}p_2^3 \in \mathbb{Z}$$

$$\frac{-79}{25920}p_2^3 + \frac{14450304}{5} \in \mathbb{Z}$$

$$\frac{-1}{155520}p_2^3 - \frac{218944}{5} \in \mathbb{Z}$$

$$\frac{1507}{51840}p_2^3 - \frac{2047292016}{5} \in \mathbb{Z}$$

$$\frac{-227}{155520}p_2^3 + \frac{109011232}{5} \in \mathbb{Z}$$

$$\frac{-1}{155520}p_2^3 - \frac{1204984}{5} \in \mathbb{Z}$$

Since we also require p_2^3, p_2p_4, p_6 to be integers, we have the following full system of congruences:

$$648 \mid p_2^3 \qquad 7535p_2^3 \equiv 51840 \bmod 259200$$

$$395p_2^3 \equiv 103680 \bmod 129600 \qquad -1135p_2^3 \equiv 466560 \bmod 777600$$

$$-5p_2^3 \equiv 622080 \bmod 777600 \qquad -5p_2^3 \equiv 622080 \bmod 777600$$

Denoting by x the integer $\frac{p_2^3}{648}$, the remaining congruences (after the first above) are equivalent to:

$$x \equiv -192 \bmod 240$$

$$x \equiv 48 \bmod 80$$

$$x \equiv 48 \bmod 240$$

$$x \equiv 8 \bmod 40.$$

A solution of this system with a particularly nice property is $x = 4608$, i.e. $p_2^3 = 2985984$, since then $p_2^3 = 144^3$.

We may now start building a 24–dimensional Poincaré duality rational homotopy type with prescribed Pontryagin classes to achieve the above requirements. Note that any manifold satisfying all the above conditions (along with having those Pontryagin numbers which involve odd Pontryagin classes equal to 0) with $p_2^3 = 144^3$, has the following Pontryagin numbers:

$$p_2^3 = 2985984$$
$$p_2p_4 = -660013056$$
$$p_6 = -5165220096.$$

Calculating the signature of such a manifold from

$$L_6 = \tfrac{1}{638512875}(2828954p_6 - 159287p_2p_4 + 8718p_2^3)$$

we obtain $\sigma = -22720000$.

Now consider the algebra over $\mathbb{Q}$ generated by α in degree 8 with $\alpha^4 = 0$, and 22720000 variables z_i in degree 12 such that $z_i \cdot \alpha = 0$, $z_i \cdot z_j = 0$ for $i \neq j$, $z_i^3 = 0$, and $z_i^2 + \alpha^3 = 0$. Realize this algebra by a rational homotopy type, and take the fundamental class to be the dual of α^3. Prescribe the Pontryagin classes as $p_1 = 0$, $p_2 = 144\alpha$, $p_3 = 0$, $p_4 = -4583424\alpha^2$, $p_5 = 0$, $p_6 = -5165220096\alpha^3$. Then the spin version of the realization theorem tells us there is a simply connected smooth closed spin 24–manifold realizing this rational cohomology ring and Pontryagin numbers.

This problem has already been solved, in a much stronger fashion (with a 7–connected manifold; in particular $p_1 = 0$ integrally) by Hopkins and Mahowald [Hop02, remark after Corollary 6.26].

3.6 On the realization of symplectic algebras and rational homotopy types by closed symplectic manifolds

We survey how a conjecture of Thurston on the existence of a symplectic structure on a cohomologically symplectic almost complex manifold has been answered in dimension 4, but remains open in higher dimensions. We then answer a question of Oprea–Tralle on the realizability of symplectic algebras by symplectic manifolds in the negative in dimensions divisible by 4, along with a question of Lupton–Oprea in all even dimensions. Our answer to the first question implies a negative answer in all even dimensions ≥ 6 to another question of Oprea–Tralle on the possibility of algebraic conditions on the rational homotopy minimal model of a smooth manifold implying the existence of a symplectic structure on the manifold. The content of this section, with some modification, appeared in [Mil21].

At the end of his famous two-page paper providing an example of a symplectic non-Kähler compact 4–manifold, Thurston [Th76] posed the following conjecture:

Conjecture 3.6.1. *([Th76]) Every closed $2k$–manifold which has an almost complex structure τ and a real cohomology class α such that $\alpha^k \neq 0$ has a symplectic structure realizing τ and α.*

Due to foundational results of Taubes and Witten in Seiberg–Witten theory, one can find counterexamples to this conjecture in dimension 4 (the argument to follow has been well-known, see e.g. [Gom01, Example p.49]). Indeed, the oriented connected sum $\#_{i=1}^{2\ell+1}\mathbb{CP}^2$ for any $\ell \geq 1$ contains elements in H^2 not squaring to zero and admits an almost complex structure compatible with the orientation, but does not admit a compatible symplectic structure. By a classical result of Wu, one knows that a closed oriented four–manifold M admits an almost complex structure if and only if there is a class $c \in H^2(M;\mathbb{Z})$ such that its reduction mod 2 is the second Stiefel–Whitney class w_2 and $\int_M c^2 = 2\chi + 3\sigma$, where χ is the Euler characteristic and σ is the signature. Since $w_2(\#_{i=1}^{2\ell+1}\mathbb{CP}^2) = (1,1,\ldots,1) \in \mathbb{Z}_2^{\oplus 2\ell+1} \cong H^2(\#_{i=1}^{2\ell+1}\mathbb{CP}^2;\mathbb{Z}_2)$ and $2\chi + 3\sigma = 10\ell + 9$, we see that $c = (3,1,3,1,\ldots,1,3) \in H^2(\#_{i=1}^{2\ell+1}\mathbb{CP}^2;\mathbb{Z})$ satisfies these conditions. Now, if $\#_{i=1}^{2\ell+1}\mathbb{CP}^2$ were to admit a symplectic structure realizing this almost complex structure, then by a theorem of Taubes, since $b_2^+ > 1$ (here b_2^+ is the dimension of the positive-definite subspace of the intersection form), it would have a non-vanishing Seiberg–Witten invariant. However, due to Witten, if a manifold is a connected sum of manifolds each with $b_2^+ \geq 1$, then the Seiberg–Witten map is identically zero (see [Ko95, Corollary 4.1(2)]).

In dimensions ≥ 6, these arguments from Seiberg–Witten theory do not directly apply, and Conjecture 3.6.1 remains open. From the realization theorem for almost complex manifolds, one can easily see that any symplectic algebra in dimension 6 is realized by a closed almost complex manifold, as c_1 and c_2 can be chosen as for $\mathbb{CP}^3$, with c_3 (which is independent of c_1^3 and c_1c_2) chosen to evaluate to the Euler characteristic.

We will address the following variations of this conjecture:

Conjecture 3.6.2. *([OT06, §6.5 Conjecture 3] [HT08], [Tr00]) For every symplectic algebra H over $\mathbb{R}$, there is a closed symplectic manifold M such that $H^*(M;\mathbb{R}) \cong H$.*

Recall, a *Poincaré duality algebra* (over the field $\Bbbk = \mathbb{Q}$ or $\mathbb{R}$) of dimension n is a finite-dimensional graded-commutative algebra H over $\Bbbk$ such that $H^n \cong \Bbbk$ and the pairing $H^* \otimes H^{n-*} \to \Bbbk$ given by $\alpha \otimes \beta \mapsto \mu(\alpha\beta)$ is non-degenerate for some (and hence any) choice of non-zero element $\mu \in (H^n)^*$. By a *symplectic algebra* we mean a Poincaré duality algebra of dimension $2k$ for which there exists an element $\alpha \in H^2$ such that $\alpha^k \neq 0$. Hence, for simplicity, the adjectives "Poincaré duality" and "symplectic" will indicate properties of an algebra, not additional structure; $H^*(M;\mathbb{R}) \cong H$ in the above conjecture will mean isomorphism of algebras. In dimensions $n = 4k$, a choice of orientation class μ lets one consider the signature of the induced pairing on H^{2k}. The pairing with respect to $a\mu$ will have the same signature for $a > 0$, and the opposite signature for $a < 0$; thus the signature of a $4k$–dimensional Poincaré duality algebra is well-defined up to sign.

Question 3.6.3. *([LO04, Remark 2.11]) Does a manifold that has rational cohomology algebra a symplectic algebra admit a symplectic structure?*

In line with our previous definition, by a manifold we mean a connected orientable closed smooth manifold without a choice of orientation; hence admitting a symplectic (or almost complex) structure means admitting a symplectic form (or almost complex structure) inducing one of the two possible orientations on the manifold. Manifolds with symplectic rational cohomology algebras are also known as *cohomologically symplectic* (or *c–symplectic*) [Tr00].

Question 3.6.4. *([OT06, §6.5 Problem 4], [Tr00]) Are there algebraic conditions on the minimal model $(\mathcal{M}_M, d)$ of a compact manifold M implying the existence of a symplectic structure on M?*

To answer Conjecture 3.6.2 in dimensions that are multiplies of four, we will use a restriction on the topology of closed almost complex manifolds due to Hirzebruch. For Question 3.6.3, we will employ simply connected rational homology spheres not admitting spin^c structures in dimensions greater than five. This will immediately imply a negative answer to Question 3.6.4, in dimensions six and greater, when restricted to simply connected manifolds. In the non-simply connected (or more generally, non-nilpotent) case, one must first decide on what is meant by a minimal model in the sense of rational homotopy. However, we observe that any such notion which is invariant under weak homotopy equivalence of rationalizations in the sense of Bousfield–Kan cannot detect the existence of a symplectic form on a given manifold.

3.6.1 Some symplectic algebras not realized by closed symplectic manifolds

We provide counterexamples to Conjecture 3.6.2 in dimensions of the form $4k$. Consider for example

$$H = H^*\left((S^2)^{2k} \# \#_{i=1}^{j} (S^1 \times S^{4k-1}); \mathbb{R}\right)$$

for odd j. Taking α to be the sum of the images of generators of $H^2(S^2; \mathbb{R})$ under the inclusion

$$H^2(S^2; \mathbb{R}) \hookrightarrow H^2((S^2)^{2k}; \mathbb{R}) \hookrightarrow H,$$

we see that $\alpha^{2k} \neq 0$, and so H is a symplectic algebra. Note that the signature σ of the realizing oriented manifold

$$(S^2)^{2k} \# \#_{i=1}^{j} (S^1 \times S^{4k-1})$$

is 0, and so the signature of any oriented manifold M with $H^*(M;\mathbb{R}) \cong H$ is 0, as the signature of a Poincaré duality algebra (with respect to any orientation class) is invariant up to sign under algebra isomorphisms of Poincaré duality algebras. On the other hand, the Euler characteristic satisfies $\chi = 2^{2k} - 2j \equiv 2 \bmod 4$ as j is odd. By [Hirz87, p.777], a closed almost complex $4k$–manifold with the induced orientation satisfies the congruence $\chi \equiv (-1)^k \sigma \bmod 4$, so we conclude that H cannot be realized by an almost complex manifold; in particular it cannot be realized by a symplectic manifold. We emphasize that this conclusion depends only on the algebra H, and so we have the following:

Theorem 3.6.5. *There are symplectic algebras H over $\mathbb{R}$ in every dimension $4k$, $k \geq 1$, such that there is no closed symplectic manifold M with $H^*(M;\mathbb{R}) \cong H$.*

Note that these examples (by taking coefficients in $\mathbb{Q}$ instead of $\mathbb{R}$) provide an answer in the negative to Question 3.6.3 in dimensions that are multiples of four. Alternatively, we can answer this question negatively in all even dimensions ≥ 6 as follows: consider the Wu manifold $W = SU(3)/SO(3)$ of dimension 5; this is a simply connected rational homology sphere which does not admit a spin^c structure. We consider the product $S^1 \times W$ and the result of performing surgery on the S^1 embedded in this product; this procedure is known as *spinning* the manifold W [Suc90]. The result of spinning a simply connected rational homology sphere of dimension n is a simply connected [Br72, Theorem IV.1.5] rational homology sphere of dimension $n+1$ [Suc90, Lemma 2.1], and the resulting manifold admits a spin^c structure if and only if the original manifold does [AM19, Proposition 2.4]. By iterating this procedure we can produce a simply connected rational homology sphere M^n of any dimension $n \geq 5$ not admitting a spin^c structure, therefore not admitting an almost complex (and in particular a symplectic) structure. For even n we can then take the connected sum $M^n \# \mathbb{CP}^{n/2}$ of this rational homology sphere with $\mathbb{CP}^{n/2}$ to obtain a cohomologically symplectic but not symplectic manifold:

Theorem 3.6.6. *There are cohomologically symplectic manifolds in all dimensions* $2k$, $k \geq 2$, *that do not admit a symplectic structure.*

3.6.2 The existence of a symplectic structure cannot be detected from the rational homotopy model

We now address Question 3.6.4. For any simply connected symplectic manifold X of dimension at least six, consider the connected sum $M \# X$ (to form the connected sum we choose any orientation on M and X), where M is a non-spin^c simply connected rational homology sphere as in the previous section. The collapse map $M \# X \to X$ is a rational homotopy equivalence, and so the minimal models of these manifolds are isomorphic while only one of them admits a symplectic structure (with respect to *some* orientation), as $M \# X$ does not admit a spin^c structure. Since X was an arbitrary simply connected symplectic manifold, we conclude that there can be no algebraic condition on minimal models of simply connected manifolds which implies the manifold admits a symplectic structure.

In the non-simply connected case, the classical theory for simply connected spaces of finite type due to Sullivan extends immediately to spaces with nilpotent fundamental group which acts nilpotently on the higher homotopy groups, and the algebraic information encoded in the minimal model directly corresponds to geometric information. Bousfield and Kan extended the procedure of rationalizing spaces to all path-connected spaces in two ways [BoKa71]: the $\mathbb{Q}$–completion and the fiberwise $\mathbb{Q}$–completion, both restricting to the classical rationalization on nilpotent spaces (see [RWZ19] for an overview). A map $X \to Y$ induces a weak homotopy equivalence of $\mathbb{Q}$–completions if it induces an isomorphism on rational homology [BoKa71], and it induces a weak homotopy equivalence of fiberwise $\mathbb{Q}$–completions if it induces an isomorphism on fundamental groups and on rationalized higher homotopy groups (see [RWZ19, Theorem 3]). Substantial progress has been made in algebraically encoding spaces up to these notions of equivalence, extending the classical theory of rational homotopy minimal models; see [GHT00], [BFMT18].

We now observe that for any (not necessarily simply connected) symplectic manifold X, there is another manifold, not admitting a symplectic structure, which is equivalent to X under either of the above notions. Consider again the collapse map $M\#X \to X$, where M is a non-spin^c simply connected rational homology sphere; this map induces an isomorphism on rational homology and hence a weak homotopy equivalence of $\mathbb{Q}$–completions. The map induces an isomorphism of fundamental groups, and to verify it induces an isomorphism on $\pi_{\geq 2} \otimes \mathbb{Q}$, we proceed as follows: pick basepoints and consider the induced map $\widetilde{M\#X} \to \widetilde{X}$ on universal covers. Since M is simply connected, the space $\widetilde{M\#X}$ can be visualized as the universal cover of X, with a small disk $\mathbb{D}_i$ around each preimage $\tilde{b}_i$ of the basepoint of X (chosen to coincide with the center of the disk at which the connected sum with M is performed) replaced by $M\#\mathbb{D}_i$. The map on universal covers $\widetilde{M\#X} \to \widetilde{X}$ is then the collapse map $M\#\mathbb{D}_i \to \mathbb{D}_i$ applied at each of these disks, and the identity elsewhere. Now consider the open cover of $\widetilde{M\#X}$ given by a small neighborhood of $\bigcup_i (M\#\mathbb{D}_i)$ and the complement of $\bigcup_i (M\#\mathbb{D}_i)$, along with the open cover of $\widetilde{X}$ given by a small neighborhood of $\bigcup_i \mathbb{D}_i$ and the complement of $\bigcup_i \mathbb{D}_i$. Applying the naturality of the Mayer–Vietoris sequence in homology to these open covers, by the five lemma we see that the map $\widetilde{M\#X} \to \widetilde{X}$ induces an isomorphism on rational homology. Thus, since these spaces are simply connected, it induces an isomorphism on rational homotopy groups, and from the naturality of the long exact sequence in homotopy for fibrations and the five lemma again we conclude that $M\#X \to X$ induces an isomorphism on $\pi_{\geq 2} \otimes \mathbb{Q}$. Therefore the fiberwise $\mathbb{Q}$–completions of these spaces are also equivalent. In conclusion, we have:

Theorem 3.6.7. *There are no algebraic conditions on the minimal model $(\mathcal{M}_M, d)$ of a manifold M implying the existence of a symplectic structure on M, in dimensions six or greater.*

Here by a minimal model we mean any object (in particular, the classical minimal models in the case of finite-type nilpotent spaces) which is invariant up to isomorphism under weak homotopy equivalence of rationalizations in either sense of Bousfield–Kan.

We note that the same argument, using non-spin^c simply connected rational homology spheres, shows that the existence of a (stable) almost complex structure cannot be implied by algebraic conditions on the minimal model:

Corollary 3.6.8. *There are no algebraic conditions on the minimal model of a manifold M implying the existence of a complex structure (or more generally a stable almost complex structure) on M, in dimensions six or greater.*

3.6.3 Another variation of Thurston's conjecture

It seems that the following question, another variation of Conjecture 3.6.1, is still unanswered in all dimensions ≥ 4:

Question 3.6.9. *Is there a symplectic algebra which is realized by a closed almost complex manifold but not realized by a closed symplectic manifold?*

Currently there are no known topological obstructions to a closed smooth manifold admitting a symplectic structure beyond those of admitting an almost complex structure and having a symplectic cohomology algebra. A possible direction presents itself as it seems that for all known examples of closed symplectic $2n$–manifolds, the Betti numbers b_i for $i \leq n$ satisfy the non-decreasing property $b_0 \leq b_2 \leq b_4 \leq \cdots$ and $b_1 \leq b_3 \leq \cdots$ [Cho16, Question 1.1]. A proof that this property holds for all closed symplectic manifolds would immediately enable one to provide counterexamples to Conjectures 3.6.1 and 3.6.2, along with the above question.

3.7 An 8–manifold not admitting a spin^h structure

In Section 3.5, we produced a spin manifold by employing a version of the realization theorem adapted to spin manifolds. Using the classical smooth realization theorem of Sullivan, combined with integrality statements coming from index theory, one can produce closed smooth manifolds not admitting certain reductions of its tangent bundle. For example, one can ask whether a manifold of dimension n admits a spin, spin^c, or spin^h structure, i.e. whether the classifying map for the tangent bundle factors through the corresponding classifying space $BSpin(n), BSpin^c(n), BSpin^h(n)$. An orientable manifold admits a spin structure if $w_2(TM) = 0$; it admits a spin^c structure if $w_2(TM) = w_2(E)$ for some $SO(2)$–bundle E; it admits a spin^h structure if $w_2(TM) = w_2(E)$ for some $SO(3)$–bundle E. Every orientable manifold of dimension ≤ 3 admits a spin structure; $\mathbb{CP}^2$ and its products with spheres give examples of manifolds in all dimensions ≥ 4 not admitting spin structure. Further, every orientable manifold of dimension ≤ 4 admits a spin^c structure; while the Wu manifold $SU(3)/SO(3)$ in dimension 5 does not. Since every manifold of dimension ≤ 5 admits a codimension 3 immersion into Euclidean space, one sees that the normal bundle of an orientable manifold in these dimensions provides an E such that $w_2(TM) = w_2(E)$ and hence the manifold admits a spin^h structure.

We produce an 8–manifold not admitting a spin^h structure; to the author's knowledge this is the first recorded example of such a manifold (at the time of writing; this example appears in [AM21]). We apply the integrality statement in [B99, Theorem 5],

$$\int_M 2\cosh\big(\tfrac{\sqrt{p_1(E)}}{2}\big)\hat{A}(TM) \in \mathbb{Z}$$

for an orientable manifold with $SO(3)$–bundle E such that $w_2(TM) = w_2(E)$. This following computation was carried out by the author in 2019, and subsequently significantly expanded on in joint work with M. Albanese [AM21].

Take $H = \mathbb{Q}[\alpha]/(\alpha^3)$ where $\deg(\alpha) = 4$, with fundamental class μ such that $\langle \alpha^2, \mu\rangle = 1$. The signature is 1, and for $p_1 \in H^4$, $p_2 \in H^8$, one calculates the conditions in the rational realization theorem to be that $\langle p_1^2, \mu\rangle$ and $\langle p_2, \mu\rangle$ are integers satisfying

$$\langle 7p_2 - p_1^2, \mu\rangle = 45$$
$$\langle 5p_1^2 - 2p_2, \mu\rangle \equiv 0 \bmod 3.$$

Observe that the second condition follows from the first. Denoting $p_1 = x\alpha$ and $p_2 = y\alpha^2$, these conditions are thus equivalent to x and y being integers satisfying

$$7y - x^2 = 45.$$

For X a spin^h manifold with normal bundle E, recall that $w_2(E) = w_2(TX)$, and since $w_2(TX)^2 \equiv p_1(TX) \bmod 2$, we conclude that there is a class $\gamma \in H^4(X;\mathbb{Z})$ such that $p_1(E) = p_1(TX) + 2\gamma$. The integrality theorem above then gives us

$$\int_X \tfrac{2p_1^2}{384} + \tfrac{\gamma^2}{48} - \tfrac{p_1^2}{96} + \tfrac{14p_1^2 - 8p_2}{5760} \in \mathbb{Z}.$$

Now, take $x = -168a + 240$ and $y = 4032a^2 - 11520a + 8235$ for any integer a; these integers satisfy $7y - x^2 = 45$ and so there is an 8–manifold X with x^2 and y as its Pontryagin numbers $\int_X p_1^2$ and $\int_X p_2$. If this manifold were to admit a spin^h structure, then there would be a class $\gamma \in H^4(X;\mathbb{Z})$ such that the above integrality statement holds; the free part of this class γ would be some integer multiple c of the generator of $H^4(X;\mathbb{Z})$ whose square integrates to 1 over the manifold. The integrality theorem would then simplify to

$$c^2 - y + 6 \equiv 0 \bmod 48.$$

Since $y - 6 \equiv 21 \bmod 48$, and 21 is not a quadratic residue modulo 48, the congruence $c^2 \equiv y - 6 \bmod 48$ does not have a solution, and thus the obtained manifold cannot admit a spin^h structure.

We observe that the argument above goes through even in the presence of cohomology outside of degrees $0, 4, 8$, so we have the following:

Theorem 3.7.1. *For any simply connected commutative differential graded algebra A over $\mathbb{Q}$ of cohomological dimension 8 satisfying Poincaré duality on cohomology, with $b_4 = 1$, there* *exist infinitely many pairwise non-homeomorphic closed smooth 8–manifolds not admitting spin^h structures with the rational homotopy type of A. (In particular, the cohomology algebra of each manifold is isomorphic to HA.)*

3.8 An almost complex rational $\mathbb{HP}^3$

We calculate that there exists a closed simply connected almost complex 12–manifold with the rational homotopy type of $\mathbb{HP}^3$. A classical result of Massey states that no $\mathbb{HP}^n$ admits an almost complex structure; a rational $\mathbb{HP}^1 = S^4$ does not admit an almost complex structure by a quick signature argument, and a rational $\mathbb{HP}^2$ does not admit an almost complex structure by a calculation with the Stong congruences. In general, the only dimension in which there exists a closed almost complex manifold with sum of Betti numbers 3 is dimension 4, by Zhixu Su and Jiahao Hu's independent resolutions of a case left open in [AM19].

Take the rational algebra $\mathbb{Q}[x_4]/(x^4)$ and take any rational space with this as its cohomology, e.g. the homotopy fiber of the fourth power map $K(\mathbb{Q}, 4) \to K(\mathbb{Q}, 16)$. Choose as fundamental class the linear dual of the cohomology class $[x^3]$. Note that necessarily $c_1 = 0$ rationally, and so we will have to satisfy two sets of congruences in order to produce an almost complex manifold with this rational homotopy type.

Taking into consideration that $c_1 = c_3 = c_5 = 0$ rationally, the first set of SU congruences, coming from the condition $z \cdot Td \in \mathbb{Z}$ for all $z \in \mathbb{Z}[e_1^c, e_2^c, \ldots]$, come down to

$$10c_2^3 - 9c_2c_4 + 2c_6 \in 60480\mathbb{Z}$$

$$c_2c_4 + 2c_6 \in 240\mathbb{Z}$$

$$-c_2^3 + 4c_2c_4 \in 12\mathbb{Z}$$

$$c_2^3 - 16c_2c_4 \in 12\mathbb{Z}$$

$$c_6 \in 4\mathbb{Z}$$

in addition to the Chern numbers being integers. As before, it is understood that the products of Chern classes above have been paired with the fundamental class, for notational convenience.

The second set of SU congruences, coming from the condition $w \cdot \hat{A}(p_i) \in 2\mathbb{Z}$ for all $w \in \mathbb{Z}[e_1^p, e_2^p, \ldots]$, when translated into Chern classes, gives us

$$\tfrac{1}{6048}c_2^3 - \tfrac{1}{6720}c_2c_4 + \tfrac{1}{30240}c_6 \in 2\mathbb{Z}$$
$$-\tfrac{1}{120}c_2c_4 - \tfrac{1}{60}c_6 \in 2\mathbb{Z}$$
$$-\tfrac{1}{3}c_2^3 + \tfrac{4}{3}c_2c_4 \in 2\mathbb{Z}$$
$$-\tfrac{1}{12}c_2^3 + \tfrac{1}{3}c_2c_4 + \tfrac{1}{2}c_6 \in 2\mathbb{Z}$$

and the signature being 0 gives us, from the L–polynomial,

$$5c_2^3 - 36c_2c_4 - 68c_6 = 0.$$

Now, $c_2 = ax$ for some rational number a. Since $\langle a^3x^3, [X]\rangle = a^3$ must be an integer, a must be an integer. Also, $c_4 = bx^2$ for some rational number b; note that it does not follow that b is an integer, but let us require $b \in \mathbb{Z}$ regardless. Then, since $\chi = 4$, we have $c_4 = 4$, and simplifying the above congruences gives us the follow system of Diophantine equations:

$$-a^3 + 4ab \in 24\mathbb{Z}$$
$$ab + 8 \in 1920\mathbb{Z}$$
$$5a^3 - 36ab = 248$$

This system has a solution of $a = -2$, $b = 4$ (in fact, unique among integers), and hence by the above theorem we obtain the desired manifold.

Bibliography

[AM19] Albanese, M. and Milivojević, A., 2019. *On the minimal sum of Betti numbers of an almost complex manifold*. Differential Geometry and its Applications, 62, pp.101-108.

[AM21] Albanese, M. and Milivojević, A., 2021. *Spin^h and further generalisations of spin*. Journal of Geometry and Physics, p.104174.

[Wal60] Wallace, A.H., 1960. *Modifications and cobounding manifolds*. Canadian Journal of Mathematics, 12, pp.503-528.

[Ba76] Barge, J., 1976. *Structures différentiables sur les types d'homotopie rationnelle simplement connexes*. Annales scientifiques de l'École Normale Supérieure (Vol. 9, No. 4, pp. 469-501).

[B99] Bär, C., 1999. *Elliptic symbols*. Mathematische Nachrichten, 201(1), pp.7-35.

[BoKa71] Bousfield, A.K. and Kan, D.M., 1971. *Localization and completion in homotopy theory*. Bulletin of the American Mathematical Society, 77(6), pp.1006-1010.

[Br62] Browder, W., 1962, August. *Homotopy type of differentiable manifolds*. In Proceedings of the Aarhus Symposium (pp. 42-46).

[Br72] Browder, W., 1972. *Surgery on Simply Connected Manifolds*. Ergeb. der Math, (2).

[BFMT18] Buijs, U., Félix, Y., Murillo, A. and Tanré, D., 2018. *Homotopy theory of complete Lie algebras and Lie models of simplicial sets*. Journal of Topology, 11(3), pp.799-825.

[Ch46] Chern, S.S., 1946. *Characteristic classes of Hermitian manifolds*. Annals of Mathematics, pp.85-121.

[Cho16] Cho, Y., 2016. *Unimodality of Betti numbers for Hamiltonian circle actions with index-increasing moment maps*. International Journal of Mathematics, 27(05), p.1650043.

[CN20] Crowley, D. and Nordström, J., 2020. *The rational homotopy type of $(n-1)$-connected manifolds of dimension up to $5n-3$*. Journal of Topology, 13(2), pp.539-575.

[DGMS75] Deligne, P., Griffiths, P., Morgan, J. and Sullivan, D., 1975. *Real homotopy theory of Kähler manifolds*. Inventiones mathematicae, 29(3), pp.245-274.

[DKS89] Dwyer, W.G., Kan, D.M. and Smith, J.H., 1989. Homotopy commutative diagrams and their realizations. Journal of Pure and Applied Algebra, 57(1), pp.5-24.

[GeM00] Müller, S. and Geiges, H., 2000. *Almost complex structures on 8-manifolds.* L'Enseignement Mathématique, 46(1/2), pp.95-108.

[GHT00] Gómez-Tato, A., Halperin, S. and Tanré, D., 2000. *Rational homotopy theory for non-simply connected spaces.* Transactions of the American Mathematical Society, 352(4), pp.1493-1525.

[Gom01] Gompf, R.E., 2001. *The topology of symplectic manifolds.* Turkish Journal of Mathematics, 25(1), pp.43-60.

[GM81] Griffiths, P., Morgan, J.W. and Morgan, J.W., 1981. *Rational homotopy theory and differential forms* (Vol. 16). Boston: Birkhäuser.

[HT08] Hajduk, B. and Tralle, A., 2008. *Exotic smooth structures and symplectic forms on closed manifolds.* Geometriae Dedicata, 132(1), pp.31-42.

[Hilb02] Hilbert, D., 1902. *Mathematical problems.* Bulletin of the American Mathematical Society, 8(10), pp.437-479.

[Hirz60] Hirzebruch, F., 1960. *Komplexe mannigfaltigkeiten. In International Congress of Mathematicians* (pp. 119-136). Cambridge University Press.

[Hirz87] Hirzebruch, F., 1987. *Gesammelte Abhandlungen-Collected Papers I: 1951-1962* (Vol. 2). Springer.

[HiBeJu92] Hirzebruch, F., Berger, T., Jung, R. and Berger, T., 1992. *Manifolds and modular forms* (Vol. 20). Braunschweig: Vieweg.

[Hop02] Hopkins, M.J., 2002. *Algebraic topology and modular forms.* arXiv preprint math/0212397.

[Kad08] Kadeishvili, T., 2008. *Cohomology C_∞-algebra and Rational Homotopy Type.* arXiv preprint arXiv:0811.1655.

[Kahn69] Kahn, P.J., 1969. *Obstructions to extending almost X-structures.* Illinois Journal of Mathematics, 13(2), pp.336-357.

[KerMil63] Kervaire, M.A. and Milnor, J.W., 1963. *Groups of homotopy spheres: I.* Annals of Mathematics, pp.504-537.

[Ko95] Kotschick, D., 1995. *The Seiberg-Witten invariants of symplectic four–manifolds.* Séminaire Bourbaki, 38, pp.195-220.

[LO94] Lupton, G. and Oprea, J., 1994. *Symplectic manifolds and formality.* Journal of Pure and Applied Algebra, 91(1-3), pp.193-207.

[Mar90] Markl, M., 1990. *The rigidity of Poincaré duality algebras and classification of homotopy types of manifolds.* In Théorie de L'homotopie: Colloque CNRS-NSF-SMF Au CIRM Du 11 Au 15 Juillet 1988 (Vol. 191, p. 221). Société Mathématique de France.

[MP11] May, J.P. and Ponto, K., 2011. More concise algebraic topology: localization, completion, and model categories. University of Chicago Press.

[MichConf40] *Lectures in Topology*, The University of Michigan, Conference of 1940.

[MiHu73] Milnor, J.W. and Husemoller, D., 1973. *Symmetric bilinear forms* (Vol. 73). New York: Springer-Verlag.

[Mil21] Milivojević, A., 2021. *On the realization of symplectic algebras and rational homotopy types by closed symplectic manifolds.* Proceedings of the American Mathematical Society.

[Mill79] Miller, T.J., 1979. *On the formality of* $K-1$ *connected compact manifolds of dimension less than or equal to* $4K-2$. Illinois Journal of Mathematics, 23(2), pp.253-258.

[Nov64] S.P. Novikov, *Homotopy-equivalent smooth manifolds I*, Izv. Akad. Nauk SSSR Ser. Mat. , 28 : 2 (1964) pp. 365–474

[OT06] Tralle, A. and Oprea, J., 2006. *Symplectic manifolds with no Kähler structure.* Springer.

[Po38] L. S. Pontryagin, *A classification of continuous transformations of a complex into a sphere.* Dokl. Akad. Nauk SSSR, 1938.

[Po42] Pontryagin, L.S., 1942. *Characteristic cycles of smooth manifolds.* In Dokl. Akad. Nauk SSSR (Vol. 35, No. 2, pp. 35-39).

[Q68] Quillen, D.G., 1968. *Some remarks on etale homotopy theory and a conjecture of Adams.* Topology, 7(2), pp.111-116.

[Q69] Quillen, D., 1969. *Rational homotopy theory.* Annals of Mathematics, pp.205-295.

[RWZ19] Rivera, M., Wierstra, F. and Zeinalian, M., 2019. *Rational homotopy equivalences and singular chains.* arXiv preprint arXiv:1906.03655.

[Sm62] Smale, S., 1962. *On the structure of manifolds. American Journal of Mathematics*, 84(3), pp.387-399.

[Sp64] Spivak, M., 1964. *On spaces satisfying Poincare duality* (Doctoral dissertation, Princeton University).

[St35] Stiefel, E., 1935. *Richtungsfelder und Fernparallelismus in* n*–dimensionalen Mannigfaltigkeiten* (Doctoral dissertation, ETH Zürich).

[Stong65a] Stong, R.E., 1965. *Relations among characteristic numbers—I.* Topology, 4(3), pp.267-281.

[Stong65b] Stong, R.E., 1966. *Relations among characteristic numbers—II*. Topology, 5(2), pp.133-148.

[Su09] Su, Z., 2009. *Rational homotopy type of manifolds*. PhD thesis, Indiana University.

[Su14] Su, Z., 2014. *Rational analogs of projective planes*. Algebraic & Geometric Topology, 14(1), pp.421-438.

[Suc90] Suciu, A.I., 1990. *Iterated spinning and homology spheres*. Transactions of the American Mathematical Society, 321(1), pp.145-157.

[Sull65] Sullivan, D., 1965. *Triangulating homotopy equivalences* (Doctoral dissertation, Princeton University).

[Sull70] Sullivan, D., 1970. *Geometric Topology. Part I. Localization, periodicity and Galois symmetry*. Notes, MIT.

[Sull74] Sullivan, D., 1974. *Genetics of homotopy theory and the Adams conjecture*. Annals of Mathematics, pp.1-79.

[Sull77] Sullivan, D., 1977. *Infinitesimal computations in topology*. Publications Mathématiques de l'IHÉS, 47, pp.269-331.

[TayWil79] Taylor, L. and Williams, B., 1979. *Local surgery: foundations and applications*. In Algebraic Topology Aarhus 1978 (pp. 673-695). Springer, Berlin, Heidelberg.

[Thom54] Thom, R., 1954. *Quelques propriétés globales des variétés différentiables*. Commentarii Mathematici Helvetici, 28(1), pp.17-86.

[Th76] Thurston, W.P., 1976. *Shorter Notes: Some Simple Examples of Symplectic Manifolds*. Proceedings of the American Mathematical Society, 55(2), pp.467-468.

[Tr00] Tralle, A., 2000. *Homotopy properties of closed symplectic manifolds*. Univ. Iagel. Acta Math, 38, pp.105-128.

[Wh35] Whitney, H., 1935. *Sphere-spaces*. Proceedings of the National Academy of Sciences of the United States of America, 21(7), p.464.

[Wh36] Whitney, H., 1936. *Differentiable manifolds*. Annals of Mathematics, pp.645-680.

[Wh44] Whitney, H., 1944. *The self-intersections of a smooth n–manifold in $2n$–space*. Annals of Math, 45(220-446), p.180.

[Wh57] Whitney, H., 1957. *Geometric integration theory*.

[Wu52] Wu, W., *Sur les classes caractéristiques des structures fibrées sphériques*, Publ. Inst. Math. Univ. Strasbourg, (11).