Lagrangian submanifolds near Lagrangian spheres

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Abstract of the Dissertation

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We study local and global Hamiltonian dynamical behaviors of some Lagrangian submanifolds near a Lagrangian sphere S in a symplectic manifold X. When $\dim S = 2$, we show that there is a one-parameter family of Lagrangian tori near S, which are nondisplaceable in X. When $\dim S = 3$, we obtain a new estimate of the displacement energy of S, by estimating the displacement energy of a one-parameter family of Lagrangian tori near S.

In the 2-dimensional case, the proof relies on a computation of the bulk-deformed Floer cohomology of the one-parameter family of Lagrangian tori near S. In the 3-dimensional case, due to the absence of a local bulk cycle, we establish a version of deformed Floer cohomology by using bulk chains with boundary as S. We also make some computations and observations of the classical Floer cohomology by using the symplectic sum formula and Welschinger invariants.

Table of Contents

Contents

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1 Introduction

Let X be a closed symplectic manifold and L be a closed Lagrangian submanifold. A classical problem in symplectic topology cares about the dynamical behavior of L under Hamiltonian isotopies. In particular L is called *nondisplaceable* if it cannot be separated from itself by any Hamiltonian diffeomorphism. That is,

$$L \cap \phi(L) \neq \emptyset, \quad \forall \phi \in Ham(X, \omega).$$

Otherwise L is called *displaceable*. For a displaceable Lagrangian submanifold, there is a notion of displacement energy to characterize how much effort one will need to displace it away. Let H_t be a time-dependent Hamiltonian function on X for $t \in [0,1]$ and ϕ_t be the corresponding Hamiltonian isotopy. The Hofer length of H_t is defined as

$$||H_t||_X = \int_0^1 (\max_X H_t - \min_X H_t) dt$$

and the displacement energy of L is defined as

$$\mathcal{E}_L = \inf\{||H_t||_X \mid L \cap \phi_1(L) = \emptyset\}.$$

If L is nondisplaceable then $\mathcal{E}_L$ is defined to be infinity.

By the work of Gromov [28] and Chekanov [8, 9, 32], the displacement energy is closely related to the least energy of a holomorphic disk with boundary on L. Later this relation has been extended to the torsion part [17, 22] of the Lagrangian Floer cohomology of L, which gives us finer estimates on the displacement energy. In this thesis we mostly focus on the case when $L = S^n$ is a Lagrangian sphere or L being a local Lagrangian submanifold near a Lagrangian sphere. Before stating the main theorems, we recall some known results.

1. When $\dim S^n = n$ is even, then S^n is always nondisplaceable since it has self-intersection number negative two, by the Weinstein neighborhood theorem.
2. When the ambient space is Calabi-Yau and $n \geq 3$, then S^n is always nondisplaceable, proved in [17] and [25].
3. When S^n is homologically non-trivial in X, then S^n is always nondisplaceable, proved in [17].
4. There are examples of displaceable Lagrangian spheres S^3, given by [3, 1] in an open monotone symplectic manifold and [35] in a closed symplectic manifold.

Motivated by the above results, we will study following cases. When $\dim S^n = n = 2$, we show that there is a one-parameter family of Lagrangian tori near S, which is nondisplaceable in X. When the ambient space is Calabi-Yau, we show that there are several one-parameter families of Lagrangian submanifolds near S, which are nondisplaceable in X. When $\dim S^n = n = 3$ and S^n is homologically trivial, we obtain a new estimate of the displacement energy of S^n, by estimating the displacement energy of a one-parameter family of Lagrangian tori near S^n.

1.1 Main results

We start with the local geometry near a Lagrangian n-sphere, where $n = 2, 3$. Let S^n be an n-sphere and (T^*S^n, ω_0) be the total space of its cotangent bundle equipped with the standard symplectic form. It is known that there is a one-parameter family of Lagrangian tori $\{L^n_\lambda\}_{\lambda\in(0,+\infty)}$ in (T^*S^n, ω_0) such that

1. L^n_λ is monotone with monotonicity constant being λ and has minimal Maslov number two;
2. L^n_λ has nonzero Floer cohomology with certain weak bounding cochains, hence it is nondisplaceable in T^*S^n;
3. for any neighborhood of the zero section S^n, L^n_λ is contained in this neighborhood if λ is small enough.

Those Lagrangian tori are toric fibers of T^*S^n, viewed as a singular toric fibration. We will review the explicit construction in Section 4 and Section 5 following [21], [11], [13], where they computed the Gromov-Witten disk potential of L^n_λ.

Then let S be a Lagrangian n-sphere in a symplectic $2n$-manifold X and U be a Weinstein neighborhood of S, which is symplectomorphic to some disk cotangent bundle $(D_rT^*S^n, \omega_0)$. A subfamily of L^n_λ sits in $U = D_rT^*S^n$ and one can ask whether L^n_λ is nondisplaceable globally in X. Note that if L^n_λ is nondisplaceable for all small λ then it implies that the Lagrangian sphere S is also nondisplaceable. We will use this approach to obtain some estimates of the displacement energy of S by estimating the displacement energy of L^n_λ near it.

Now fix $L = L^n_\lambda \in U \subset X$ we want to study pseudoholomorphic disks bounding L within the following condition, see Assumption 3.2 in [4]. How to possibly relax this technical condition will be discussed in Section 5.4, following the work of Charest-Woodward [7].

Condition 1.1. There exists a compatible almost complex structure J such that

1. all non-constant J-holomorphic disks on L have positive Maslov indices;
2. all J-holomorphic disks on L with Maslov index two are regular;
3. all non-constant J-holomorphic spheres have positive first Chern numbers.

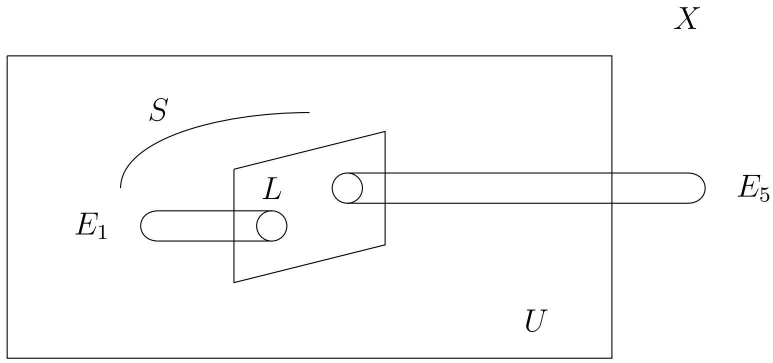

Figure 1: Disk contributions from inside and outside.

For $n = 2$, the above condition is satisfied by a generic almost complex structure. For $n = 3$, a large class of examples satisfying Condition 1.1 is the toric fiber of a symplectic Fano toric manifold. Specific to our case, let X_0 be a nodal toric Fano threefold and let X be the smoothing of X_0. Each node gives us a Lagrangian S^3 and the local tori near the spheres become toric fibers. There is a full classification [27] of 100 nodal toric Fano threefolds, 18 out of which are smooth. In theory one can compute explicitly all the disk potential functions of the toric fibers therein to find the torsion thresholds, by using the combinatorial data from their polytopes. But we do not try to do it here.

Assuming Condition 1.1, the one-pointed open Gromov-Witten invariant n_β is defined (with respect to this particular J), for any disk class $\beta \in \pi_2(X, L)$ with Maslov index two. We consider the sequence

$$\{\beta_k \mid n_\beta \neq 0, E(\beta_k) \leq E(\beta_{k+1})\}_{k=1}^{\infty}$$

of disk classes with Maslov index two, enumerated by their symplectic energy, see Figure 1. From the local study we know that L^n_λ bounds four J-holomorphic disks with Maslov index two inside U, with same energy $E_{1,\lambda}$. (This is true for both cases $n = 2, 3$.) Those are the first four elements in the above sequence if L^n_λ is near S. Let $E_{5,\lambda} = E(\beta_5)$ be the least energy of outside disk contribution. Note that when L^n_λ is close to S then $E_{5,\lambda} >> E_{1,\lambda}$. Our main theorems can be formulated as follows.

Theorem 1.2. *Let X be a closed symplectic 4-manifold which contains a Lagrangian 2-sphere S. Consider the Lagrangian embedding*

$$L^2_\lambda \hookrightarrow U = D_r T^* S^2 \subset X$$

for $\lambda \in (0, \lambda_0)$. Then L^2_λ is nondisplaceable in X.

This theorem relies on and generalizes the work of Fukaya-Oh-Ohta-Ono [21], where they proved the existence of a one-parameter family of nondisplaceable

Lagrangian tori when the ambient space is $S^2 \times S^2$ and the Lagrangian sphere is the anti-diagonal. To deal with possible outside disk contributions from a general ambient space, we use the local computation in [21] and an implicit function theorem in non-Archimedean geometry to control the high energy terms. Another direction of generalization is that we can take the local model as a plumbing of cotangent bundles of 2-spheres. In [39], the author proved the existence of several one-parameter families of nondisplaceable Lagrangian tori near any linear chains of Lagrangian 2-spheres.

The above theorem does not extend directly to higher dimensional cases. On one hand, the techniques therein depends on that a Lagrangian 2-sphere is also a codimension two cycle in a symplectic 4-manifold, hence can be used as a bulk deformation. On the other hand, we already know that there are displaceable Lagrangian 3-spheres [3], [35]. But we are able to give some new estimates of the displacement energy of a Lagrangian 3-sphere, by using a similar philosophy.

Theorem 1.3. *Let X be a closed symplectic 6-manifold which contains a homologically trivial Lagrangian 3-sphere S. Consider the Lagrangian embedding*

$$L^3_\lambda \hookrightarrow U = D_r T^* S^3 \subset X$$

for $\lambda \in (0, \lambda_0)$. If L^3_λ satisfies Condition 1.1 and can be displaced by a Hamiltonian isotopy ϕ_t generated by G_t then

$$||G_t||_X \geq E_{5,\lambda}$$

and

$$||G_t||_X + 2||G_t||_S \geq 2(E_{5,\lambda} - E_{1,\lambda}).$$

Here $||\cdot||_X$ is the Hofer norm and $||\cdot||_S$ is a relative Hofer norm defined by

$$||G_t||_S = \int_0^1 (\max_S H_t - \min_S H_t) dt.$$

By definition we know that $||G_t||_X \geq ||G_t||_S$. But for the above two inequalities we can not say which one is stronger, unless we know the behavior of G_t on S. For example, the displaceable Lagrangian sphere S in [35] can be displaced by a group action. In particular the Hamiltonian function is constant on S, hence $||G_t||_S = 0$ and the second inequality is much stronger than the first one and almost optimal, see Section 7.3.

In the theorem we assume that S is homologically trivial. (Note that a homologically non-trivial Lagrangian sphere always has non-zero Floer cohomology, hence nondisplaceable.) This condition is needed such that S bounds a 4-chain in X and some cylinder counting can be defined. Also it is needed to perform the conifold transition on S in the sense of Smith-Thomas-Yau [37], to compute certain open Gromov-Witten invariants. The smoothings of nodal toric Fano threefolds still satisfy this condition. As a corollary we obtain an estimate of the displacement energy of our Lagrangian sphere S.

Corollary 1.4. *With the same notation in Theorem 1.3, if S can be displaced by a Hamiltonian isotopy ϕ_t generated by G_t then*

$$||G_t||_X \geq \lim_{\lambda\to 0} E_{5,\lambda}, \quad ||G_t||_X + 2||G_t||_S \geq \lim_{\lambda\to 0} 2(E_{5,\lambda} - E_{1,\lambda}) = \lim_{\lambda\to 0} 2E_{5,\lambda}.$$

As λ tends to zero, the parameter $E_{1,\lambda}$ tends to zero and $E_{5,\lambda} - E_{1,\lambda}$ increases to $E_{5,\lambda=0}$. The energy $E_{5,\lambda=0}$ is roughly the least energy of a holomorphic disk with boundary on S. Hence the Hofer norm of the Hamiltonian which displaces S is roughly twice the least energy of a holomorphic disk, with a modification term given by the relative Hofer norm. In practice, the least energy of a holomorphic disk can be bounded from below by the size of the Weinstein neighborhood U. The larger the size of U is, the better this energy estimate will be.

The proof of Theorem 1.3 is to establish a new version of Lagrangian Floer theory to use chains as bulk deformations. More precisely, we not only counts holomorphic strips with Lagrangian boundary conditions, but also counts holomorphic strips with one interior hole, where the interior hole is mapped to another reference Lagrangian submanifold. In our case the reference Lagrangian is a chosen Lagrangian 3-sphere, and we expect that similar constructions should work for other Lagrangian 3-manifolds.

Counting holomorphic cylinders between two non-intersecting Lagrangian submanifolds provides us a map between some quantum invariants of these two Lagrangian submanifolds. For an incomplete list, see [6] and [29] for some geometric applications. In our current setting, this Floer theory is motivated by various works around the *conifold transition*, a surgery that replaces a Lagrangian 3-sphere by a holomorphic $\mathbb{C}P^1$. How geometric invariants change under this transition is an important question in the fields of symplectic topology and enumerative geometry. In particular, some closed Gromov-Witten invariants with point-wise constraints are not preserved under this transition, unless one also takes the open Gromov-Witten invariants on S^3 into account. From this point of view, to compare the Lagrangian Floer theory of a Lagrangian away from the holomorphic $\mathbb{C}P^1$ in the resolved side, it is natural to consider the contributions of bordered curves with disconnected boundaries on both of the sphere S^3 and the Lagrangian. So here we realize this idea in a simple version, where both holomorphic strips and holomorphic strips with one interior hole attached on S^3 are counted. Similar philosophy already started to play an important role in the mirror symmetry ground and we use it to explore applications in symplectic topology.

However, the above philosophy usually expects that the data of all genera should be considered, otherwise what one obtained is not an invariant. Therefore our baby theory only works modulo some energy. Recently, the open Gromov-Witten theory in T^*S^3 with all genera has been successfully related to knot-theoretic invariants by Ekholm-Shende [15]. It would be interesting to try to apply the techniques therein to define a full genus Floer theory, starting with the monotone Lagrangian torus in T^*S^3. Hopefully there will be a correspondence between open Gromov-Witten invariants with coefficients in skein modules and bulk-deformed open Gromov-Witten invariants. Then one may move further to toric compactifications or other general cases, to see how Lagrangian Floer theory and even Fukaya category change under the conifold transition.

Besides introducing this new version of Floer theory, we also carry out some computations of the classical Floer cohomology, which are not deformed by bounding cochains or bulk-deformations. Let (X, ω) be a closed symplectic 6-manifold such that

$$[c_1(TX)] = c \cdot [\omega], \quad c \in \mathbb{R} \tag{1.1}$$

on the image of the Hurewicz map $\pi_2(X; \mathbb{Z}) \to H_2(X; \mathbb{Z})$. We say X is *monotone* if $c > 0$, it is *Calabi-Yau* if $c = 0$ and it is *negatively monotone* if $c < 0$. Note that $\pi_1(S^3) = \pi_2(S^3) = 0$ implies that $\pi_2(X, S) \cong \pi_2(X)$. If (1.1) is satisfied then the two homomorphisms c_1 and ω on the relative homotopy group are also proportional to each other with the same constant c. In particular if X is monotone then S is automatically a monotone Lagrangian submanifold in the usual sense.

First, by a degeneration method [25, 26] from the symplectic cut and sum construction, we can determine the displaceability of S and L_λ when X is Calabi-Yau and negatively monotone. Note that Theorem 1.3 uses cylinder counting to cancel the outside disk contributions to some extent, here we find that the outside disk contributions can be forgotten by perturbing the almost complex structures.

Theorem 1.5. *Let (X, ω) be a Calabi-Yau or negatively monotone symplectic 6-manifold which contains a Lagrangian 3-sphere S. Consider the Lagrangian embedding*

$$L^3_\lambda \hookrightarrow U = D_r T^* S^3 \subset X$$

then there exists a dense subset $\mathcal{J}^{reg}$ of the set of admissible compatible almost complex structures such that for $J \in \mathcal{J}^{reg}$ all J-holomorphic disks with boundary on L^3_λ are contained in U. In particular, L^3_λ is nondisplaceable in X for all λ in a small open interval $(0, \lambda_0)$.

The nondisplaceability of a Lagrangian sphere in a Calabi-Yau manifold was proved in Theorem L [17]. And M.F.Tehrani [25] gave an alternative proof by the symplectic sum and cut method. Here we are using his approach to analyze the Lagrangian submanifolds near the sphere. The degeneration formula actually works for all dimensions $2n \geq 6$. Combined with the Oakley-Usher's families [34] of monotone nondisplaceable Lagrangian submanifolds in T^*S^n we upgrade the above theorem to all dimensions.

Theorem 1.6. *For any integer $n \geq 3$, let (X^{2n}, S^n, ω) be a Calabi-Yau or negatively monotone symplectic manifold with a Lagrangian sphere. Then there are continuum families of Lagrangian submanifolds*

$$L^{k,m}_\lambda \cong (S^1 \times S^k \times S^m)/\mathbb{Z}_2, \quad k, m \in \mathbb{Z}_+, k \leq m, k + m = n - 1, \lambda \in (0, \lambda_0) \subset \mathbb{R}$$

near the Lagrangian sphere S and are nondisplaceable in X.

For readers who are interested in the Lagrangian skeleta of a Calabi-Yau manifold, this theorem helps to show that if a symplectic manifold is the divisor complement of a Calabi-Yau manifold and contains a Lagrangian sphere, then its skeleta must intersect all those $L_\lambda^{k,m}$ near this sphere, see the work [40] by Tonkonog-Varolgunes. In particular, this matches the known fact that T^*S^n, as an affine variety, is never a divisor complement of a Calabi-Yau manifold when $n \geq 3$.

All the above results use a local-to-global method, starting from the local computation to control the global picture. Similar ideas also played an important role in [7], [41] and [43] for other local models, where many geometric applications are obtained.

Next we discuss the case when X is monotone. Since S^3 is simply-connected, orientable and spin, the classical Floer cohomology $HF(S^3; \Lambda(F))$ is well-defined for any finite field F or $F = \mathbb{Z}$, see [23]. Here $\Lambda(F)$ is the Novikov field with F as the ground ring. The underlying complex is the Morse cohomology $H^*(S^3; F) \otimes \Lambda(F)$ with Novikov field coefficients, and the only essential maps to compute the Floer cohomology $HF(S^3; \Lambda(F))$ are

$$\mathfrak{m}_{1;\beta} : H^3(S^3; F) \otimes \Lambda(F) \to H^0(S^3; F) \otimes \Lambda(F)$$

where β is a disk class with Maslov index four. For example, when X has minimal Chern number $N \geq 3$, these maps are zero and we have that $HF(S^3; \Lambda(F)) = H^*(S^3; F) \otimes \Lambda(F)$. When X has minimal Chern number $N = 2$, these maps are two-pointed open Gromov-Witten invariants of class β. When X has minimal Chern number $N = 1$, these maps count (broken) disks connected by Morse flow lines. However, the usual two-pointed open Gromov-Witten invariant of class β is not well-defined due to splittings of disks with Maslov index two.

On the other hand, Welschinger [42] defined F-valued open counts of disks for a Lagrangian submanifold L when $H_1(L; F) \to H_1(X; F)$ is injective. Given a disk class β and sufficient boundary constraints, his invariant n_β^W counts multi-disks weighted by linking numbers. We compare his invariants and the Floer differential and find they are equal to each other.

Theorem 1.7. *Let S be a Lagrangian 3-sphere in a monotone symplectic 6-manifold X. Given a disk class $\beta \in \pi_2(X, S)$ with Maslov index four, we have an equality*

$$\langle \mathfrak{m}_{1;\beta}(PD[pt]), [pt] \rangle = n_{2,\beta}^W \cdot T^{\omega(\beta)}$$

where the pairing on the left is the cohomology-homology pairing and $n_{2,\beta}^W$ is the two-pointed Welschinger invariant of class β.

Therefore we can define a following invariant

$$n_2^W := \sum_{\mu(\beta)=4} n_{2,\beta}^W \in F$$

to determine the Floer cohomology $HF(S;\Lambda(F))$. That is, $HF(S;\Lambda(F)) = \{0\}$ if and only if $n_2^W \neq 0$. One could think this invariant is an analogue of the critical point equation of the disk potential function of a monotone Lagrangian torus. Recall that geometrically the critical point equation (however, at 0) of the potential function is

$$0 = \sum_{\mu(\beta)=2} n_\beta \cdot [\partial\beta] \in H_1(T^n;\mathbb{Z})$$

where n_β is the one-pointed open Gromov-Witten invariant. When our Lagrangian is a sphere, the above theorem says that there is also an equation determining the Floer cohomology, in terms of enumerative invariants. In this setting the equation happens on the level of $H_3(S^3;F)$, which is one-dimensional. Hence we do not need to weight the enumerative invariants by any homology class.

So we hope the Welschinger invariants help to compute the Floer cohomology in certain settings, like the one-pointed open Gromov-Witten invariants in the case of toric fibers. Moreover, we expect to define similar enumerative equations for Lagrangian submanifolds of general topological type. For example, for a Lagrangian $S^3 \times T^n$ we may need both two equations above to determine its Floer cohomology. For a Lagrangian submanifold of which the cohomology groups are generated by elements in certain degrees, we may need a system of equations, one in each degree, to determine its Floer cohomology.

The outline of this article is as follows. In Section 2 we give the background on potential functions with bulk deformations. In Section 3 we review the symplectic sum and cut method and prove Theorem 1.6 and Theorem 1.7. In Section 4 we study the case where S is a Lagrangian 2-sphere and prove Theorem 1.2. Then we move on to the 3-dimensional case. In Section 5 and 6 we construct three types of Floer theories with cylinder corrections and show some geometric properties of these theories. The first model is a disk model with cylinder corrections, which gives us a deformed potential function to do concrete computations. The second and third models are complexes generated by Hamiltonian chords and intersection points respectively, which will be used to study the intersection behavior of our Lagrangians under Hamiltonian perturbations. Once we showed the equivalences between the three models, we can apply them, in Section 7, to obtain estimates of displacement energy and prove Theorem 1.3.

2 Preliminaries

We give a brief summary to the theory of deformed Floer cohomology and potential functions, referring to Section 2 and Appendix 1 in [21] for more details.

First we specify the ring and field that will be used. The Novikov ring Λ_0 and its field Λ of fractions are defined by

$$\Lambda_0 = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{\geq 0}, \lambda_i < \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = +\infty\}$$

and

$$\Lambda = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}, \lambda_i < \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = +\infty\}$$

where T is a formal variable. The maximal ideal of Λ_0 is defined by

$$\Lambda_+ = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} \mid a_i \in \mathbb{C}, \lambda_i \in \mathbb{R}_{>0}, \lambda_i < \lambda_{i+1}, \lim_{i\to\infty} \lambda_i = +\infty\}.$$

We remark that the field Λ is algebraically closed since the ground field is $\mathbb{C}$, see Appendix A in [19]. All the nonzero elements in $\Lambda_0 - \Lambda_+$ are units in Λ_0. Next we define a valuation v on Λ by

$$v(\sum_{i=0}^{\infty} a_i T^{\lambda_i}) = \inf\{\lambda_i \mid a_i \neq 0\}, \quad v(0) = +\infty.$$

This valuation gives us a non-Archimedean norm

$$|a = \sum_{i=0}^{\infty} a_i T^{\lambda_i}| = e^{-v(a)}.$$

2.1 Moduli space of holomorphic disks

Let X be a closed symplectic manifold and L be a Lagrangian submanifold. For our purpose, we assume that L is either a 2-torus or a 3-torus.

Definition 2.1. A bordered semi-stable genus zero curve Σ with $(k+1)$ boundary marked points and l interior marked points is a union of disk components $D^i, i = 1, \cdots, r$ and sphere components $S^j, j = 1, \cdots, s$ satisfying the following conditions.

1. Boundary marked points $z_0, \cdots, z_k \in \cup_{i=1}^r \partial D^i$ and interior marked points $z_1^+, \cdots, z_l^+ \in (\cup_{i=1}^r Int D^i) \cup (\cup_{j=1}^s Int S^j)$.

2. Σ is connected and the dual graph is a tree.

3. If $i \neq i'$ then $D^i \cap D^{i'} = \partial D^i \cap \partial D^{i'}$ contains at most one point, which is called a boundary node.

4. If $j \neq j'$ then $S^j \cap S^{j'}$ contains at most one point, which is called an interior node.

5. $D^i \cap S^j$ contains at most one point, which is called an interior node connecting a disk component and a sphere component.

6. The intersection of any three distinct components, either of disk type or sphere type, is empty.

7. Elements of three types of special points, boundary marked points, interior marked points and nodes, are all mutually distinct.

Let $(\Sigma; z_0, \cdots, z_k; z_1^+, \cdots, z_l^+)$ be a bordered semi-stable genus zero curve with $(k+1)$ boundary marked points and l interior marked points. Let J be a compatible almost complex structure on X and $w : (\Sigma, \partial\Sigma) \to (X, L)$ be a J-holomorphic map. The automorphism group of w is the set of biholomorphic maps $\psi : \Sigma \to \Sigma$ such that $\psi(z_i) = z_i, \psi(z_i^+) = z_i^+$ and $w \circ \psi = w$.

Definition 2.2. A bordered stable genus zero J-holomorphic map is a pair

$$((\Sigma; z_0, \cdots, z_k; z_1^+, \cdots, z_l^+), w)$$

as above, such that its automorphism is finite.

For a class $\beta \in \pi_2(X, L)$, we write $\mathcal{M}_{k+1,l}(X, L; J; \beta)$ as the set of the isomorphism classes of bordered stable genus zero J-holomorphic maps representing class β with $(k+1)$ boundary marked points and l interior marked points. A distinct component $\mathcal{M}^{main}_{k+1,l}(X, L; J; \beta)$ of $\mathcal{M}_{k+1,l}(X, L; J; \beta)$ contains elements with boundary marked points $(z_0, \cdots, z_k)$ located counter-clockwisely on $\partial\Sigma$.

By using boundary marked points and interior marked points, we can study the evaluation maps to L and X. The following foundational results are proved by Fukaya-Oh-Ohta-Ono in Chapter 7 of [17]. We remark that when our Lagrangian torus satisfies the Condition 1.1, there are simplified proofs.

Theorem 2.3. *There are Kuranishi structures on $\mathcal{M}^{main}_{k+1,l}(X, L; J; \beta)$ such that the evaluation maps*

$$ev_i : \mathcal{M}^{main}_{k+1,l}(X, L; J; \beta) \to L, i = 0, \cdots, k$$

and

$$ev_j^+ : \mathcal{M}^{main}_{k+1,l}(X, L; J; \beta) \to X, j = 0, \cdots, l$$

are weakly submersive in the sense of Kuranishi structures.

Then for given smooth singular simplices $(f_i : P_i \to L)$ of L and $(g_j : Q_j \to X)$ of X, we can define the fiber product in the sense of Kuranishi structure

$$\begin{aligned} &\mathcal{M}^{main}_{k+1,l}(X, L; J; \beta; \overrightarrow{P}, \overrightarrow{Q}) := \\ &\mathcal{M}^{main}_{k+1,l}(X, L; J; \beta)_{(ev_1,\cdots,ev_k,ev_1^+,\cdots,ev_l^+)} \times_{(f_1,\cdots,f_k,g_1,\cdots,g_l)} (\prod_{i=1}^{k} P_i) \times (\prod_{j=1}^{l} Q_j) \end{aligned} \quad (2.1)$$

Note that we have an extra boundary marked point z_0 and we can study the evaluation map

$$ev_0 : \mathcal{M}^{main}_{k+1,l}(X, L; J; \beta; \overrightarrow{P}, \overrightarrow{Q}) \to L.$$

Theorem 2.4. *([17], Chapter 7 and Appendix A) There exists a multi-valued perturbation of the Kuranishi structure and a suitable triangulation of its zero locus, such that*

$$ev_0 : \mathcal{M}^{main}_{k+1,l}(X, L; J; \beta; \overrightarrow{P}, \overrightarrow{Q}) \to L.$$

is a singular chain, which is called a virtual fundamental chain.

2.2 A-infinity operations and bulk deformations

Now we can construct a filtered A_∞-algebra structure on $H^*(L; \Lambda_0)$ where

$$\mathfrak{m}_k : H^*(L; \Lambda_0)^{\otimes k} \to H^*(L; \Lambda_0)$$

are the A_∞-operations, see section 3 in [19]. The operators $\mathfrak{m}_k$ are defined as

$$\mathfrak{m}_k(x_1, \cdots, x_k) = \sum_{\beta \in \pi_2(X,L)} T^{\omega(\beta)} \cdot \mathfrak{m}_{k;\beta}(x_1, \cdots, x_k)$$

where geometrically $\mathfrak{m}_{k;\beta}(x_1, \cdots, x_k)$ is a chain given by the moduli space of holomorphic disks, representing the class β, with boundary marked points attached on given cocycles $(x_1, \cdots, x_k)$ in L.

Definition 2.5. For smooth singular chains $x_1, \cdots, x_k$ of L and a disk class $\beta \in \pi_2(X, L)$, we define

1. $\mathfrak{m}_{0,\beta}(1) := (ev_0 : \mathcal{M}^{main}_{1,0}(X, L; J; \beta) \to L)$ for $\beta \neq 0$;
2. $\mathfrak{m}_{0,\beta}(1) := 0$ for $\beta = 0$;
3. $\mathfrak{m}_{1,\beta}(x) := (ev_0 : \mathcal{M}^{main}_{2,0}(X, L; J; \beta; x) \to L)$ for $\beta \neq 0$;
4. $\mathfrak{m}_{1,\beta}(x) := (-1)^n \partial x$ for $\beta = 0$, where ∂ is the boundary operator of a singular chain and $n = \dim L$;
5. $\mathfrak{m}_{k,\beta}(x_1, \cdots, x_k) := (ev_0 : \mathcal{M}^{main}_{k+1,0}(X, L; J; \beta; x_1, \cdots, x_k) \to L)$ for any β and $k \geq 2$.

We remark that the operators $\mathfrak{m}_k$ are first defined at the chain level then can be passed to their "canonical model" at the cohomology level. Here we directly use the canonical model at the cohomology level, see [18]. Also here we abuse the notations between singular chains and cochains via the following conventional

Poincaré duality. For a singular chain x in L, the Poincaré dual $PD(x)$, regarded as a current satisfies that

$$\int_x \alpha \mid_x = \int_L PD(x) \wedge \alpha \tag{2.2}$$

for any differential form $\alpha \in \Omega^{\dim x}(L)$.

An element $b \in H^1(L; \Lambda_+)$ is called a weak bounding cochain if it satisfies the A_∞-Maurer-Cartan equation

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \cdots, b) \equiv 0 \mod PD([L]). \tag{2.3}$$

Here $PD([L]) \in H^0(L; \mathbb{Z})$ is the Poincaré dual of the fundamental class and it is the unit of the filtered A_∞-algebra. We denote by $\mathcal{M}_{weak}(L)$ the set of weak bounding cochains of L. If $\mathcal{M}_{weak}(L)$ is not empty then we say L is weakly unobstructed. The filtered A_∞-algebra structure we considered here is the *canonical model* in the Language of [18, 19].

The coefficients of weak bounding cochains can be extended from Λ_+ to Λ_0. For $b \in H^1(L; \Lambda_0)$ we can write $b = b_0 + b_+$ where $b_0 \in H^1(L; \mathbb{C})$ and $b_+ \in H^1(L; \Lambda_+)$. Then we define

$$\mathfrak{m}_{k,\beta}(b, \cdots, b) := e^{\langle \partial\beta, b_0 \rangle} \mathfrak{m}_{k,\beta}(b_+, \cdots, b_+) \tag{2.4}$$

where the pairing $\langle \partial\beta, b_0 \rangle = \int_{\partial\beta} b_0$. Note that if $b_0 = b_0' + 2\pi\sqrt{-1}\mathbb{Z}$ then $e^{\langle \partial\beta, b_0 \rangle} = e^{\langle \partial\beta, b_0' \rangle}$. So the weak bounding cochains with Λ_0 coefficients are actually defined modulo this equivalence. More precisely, they should be regarded as elements in

$$H^1(L; \Lambda_0)\big/H^1(L; 2\pi\sqrt{-1}\mathbb{Z}) := H^1(L; \mathbb{C})\big/H^1(L; 2\pi\sqrt{-1}\mathbb{Z}) \oplus H^1(L; \Lambda_+).$$

Now for a weak bounding cochain b we can deform the A_∞-operations in the following way. Define

$$\mathfrak{m}_k^b(x_1, \cdots, x_k) := \sum_{l=0}^{\infty} \sum_{l_0+\cdots+l_k=l} \mathfrak{m}_{k+l_0+\cdots+l_k}(b^{\otimes l_0}, x_1, b^{\otimes l_1}, x_2, \cdots, x_k, b^{\otimes l_k}).$$

That is, we insert b in all possible ways. Then $\{\mathfrak{m}_k^b\}$ is a new sequence of A_∞-operations on $H^*(L; \Lambda_0)$ which satisfies that

$$\mathfrak{m}_1^b \circ \mathfrak{m}_1^b = 0, \tag{2.5}$$

see Proposition 3.6.10 in [17]. So we can define the deformed Floer cohomology $HF(L, b; \Lambda_0)$ as the cohomology of $\mathfrak{m}_1^b$ whenever b is a weak bounding cochain.

We define a *potential function*

$$\mathfrak{PO} : \mathcal{M}_{weak}(L) \to \Lambda_+$$

by setting

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \cdots, b) = \mathfrak{PO}(b) \cdot PD([L]).$$

The new A_∞-operations $\{\mathfrak{m}_k^b\}$ can be regarded as a deformation of $\{\mathfrak{m}_k\}$ by a weak Maurer-Cartan element b, which is from the cohomology of L itself. Similarly we can deform the A_∞-operations by the cohomology of the ambient symplectic manifold X. Such a deformation is called a *bulk deformation*.

Let $E_l H^*(X; \Lambda_+)$ be the subspace of $H^*(X; \Lambda_0)^{\otimes l}$ which is invariant under the action of the lth symmetric group. Then in [17] a sequence of operators $\{\mathfrak{q}_{l,k;\beta}\}_{l\geq 0; k\geq 0}$ is constructed

$$\mathfrak{q}_{l,k;\beta} : E_l H^*(X; \Lambda_+) \otimes H^*(L; \Lambda_0)^{\otimes k} \to H^*(L; \Lambda_0).$$

Geometrically those operators are chains given by the moduli space of holomorphic disks with boundary marked points attached on given cocycles in L and interior marked points attached on given cocycles in X. And we define the operator $\mathfrak{q}_{l,k} := \sum_\beta T^{\omega(\beta)} \cdot \mathfrak{q}_{l,k;\beta}$.

Definition 2.6. For singular chains $(f_i : P_i \to L)$ of L and $(g_j : Q_j \to X)$ of X and a disk class $\beta \in \pi_2(X, L)$, we define

$$\mathfrak{q}_{l,k;\beta}(Q_1, \cdots, Q_l; P_1, \cdots, P_k) := (ev_0 : \mathcal{M}^{main}_{k+1,l}(X, L; J; \beta; \overrightarrow{P}, \overrightarrow{Q}) \to L)$$

as a singular chain of L.

Again, here we are using the operators constructed on the canonical model and the cohomology-homology duality we mentioned before. When $l = 0$ we have that

$$\mathfrak{q}_{0,k}(1; x_1, \cdots, x_k) = \mathfrak{m}_k(x_1, \cdots, x_k)$$

where $1 \in H^*(X; \Lambda_0)$ is the unit.

Now for any $\mathfrak{b} \in H^*(X; \Lambda_+)$ and $x_1, \cdots, x_k \in H^*(L; \Lambda_0)$ we define

$$\mathfrak{m}_k^{\mathfrak{b}}(x_1, \cdots, x_k) = \sum_{l=0}^{\infty} \mathfrak{q}_{l,k}(\mathfrak{b}^{\otimes l}; x_1, \cdots, x_k).$$

Then $\{\mathfrak{m}_k^{\mathfrak{b}}\}$ also defines a filtered A_∞-algebra structure on $H^*(L; \Lambda_0)$. For a fixed $\mathfrak{b}$, an element $b \in H^1(L; \Lambda_+)$ is called a weak bounding cochain (with respect to $\mathfrak{b}$) if it satisfies the A_∞-Maurer-Cartan equation given by the deformed operators $\{\mathfrak{m}_k^{\mathfrak{b}}\}$. And we write $\mathcal{M}_{weak}(L; \mathfrak{b})$ as the set of weak bounding cochains of L with respect to $\mathfrak{b}$.

To do concrete computations there are two *divisor axioms* for the operators $\mathfrak{m}_k$ and $\mathfrak{q}_{l,k}$. For $\mathfrak{b} \in H^2(X; \Lambda_+), b \in H^1(L; \Lambda_+)$ and $\mu(\beta) = 2$ we have that

$$\begin{aligned} \mathfrak{m}_{k;\beta}(b^{\otimes k}) &= \frac{(b(\partial\beta))^k}{k!} \cdot \mathfrak{m}_{0;\beta}(1); \\ \mathfrak{q}_{l,k;\beta}(\mathfrak{b}^{\otimes l}; x_1, \cdots, x_k) &= \frac{(\mathfrak{b} \cdot \beta)^l}{l!} \cdot \mathfrak{q}_{0,k;\beta}(1; x_1, \cdots, x_k). \end{aligned} \tag{2.6}$$

These are first studied in [16] and we refer to Section 7 in [20] for a proof. Like the extension of b in (2.4), the coefficient of $\mathfrak{b}$ can be also extended from Λ_+ to Λ_0, see Section 11 [20].

Next we put those two deformations together, one from the Lagrangian itself and the other from the ambient space. Define an operator

$$d_{\mathfrak{b}}^b = \sum_{k_0,k_1} \mathfrak{m}_{k_0+k_1+1}^{\mathfrak{b}}(b^{\otimes k_0}, x, b^{\otimes k_1}) : H^*(L;\Lambda_0) \to H^*(L;\Lambda_0). \tag{2.7}$$

When $b \in \mathcal{M}_{weak}(L;\mathfrak{b})$ we have that

$$d_{\mathfrak{b}}^b \circ d_{\mathfrak{b}}^b = 0 \tag{2.8}$$

and the resulting cohomology

$$HF(L,\mathfrak{b},b;\Lambda_0)$$

is called the deformed Floer cohomology of L by the bulk deformation $\mathfrak{b}$. If we expand the summation of $d_{\mathfrak{b}}^b$ we will find that the new differential $d_{\mathfrak{b}}^b$ contains the differential $\mathfrak{m}_1^b$.

$$\begin{aligned} d_{\mathfrak{b}}^b &= \sum_{k_0,k_1} \mathfrak{m}_{k_0+k_1+1}^{\mathfrak{b}}(b^{\otimes k_0}, x, b^{\otimes k_1}) \\ &= \sum_{l,k_0,k_1} \mathfrak{q}_{l,k_0+k_1+1}(\mathfrak{b}^{\otimes l}; b^{\otimes k_0}, x, b^{\otimes k_1}) \\ &= \mathfrak{m}_1^b(x) + \sum_{l\geq 1,k_0,k_1} \mathfrak{q}_{l,k_0+k_1+1}(\mathfrak{b}^{\otimes l}; b^{\otimes k_0}, x, b^{\otimes k_1}). \end{aligned} \tag{2.9}$$

Hence the differential $d_{\mathfrak{b}}^b$ is a sum of the "zeroth order" term $\mathfrak{m}_1^b$ and "higher order" deformations which count holomorphic disks with interior marked points attached on given cocycles in X.

Similarly we define a *bulk-deformed potential function*

$$\mathfrak{PO}^{\mathfrak{b}} : \mathcal{M}_{weak}(L;\mathfrak{b}) \to \Lambda_+$$

by setting

$$\sum_{k=0}^{\infty} \mathfrak{m}_k^{\mathfrak{b}}(b,\cdots,b) = \mathfrak{PO}^{\mathfrak{b}}(b)\cdot PD([L]).$$

From the above discussion we have that $\mathfrak{PO}^{\mathfrak{b}=0}(b) = \mathfrak{PO}(b)$. And for a Lagrangian torus satisfying the Condition 1.1 there is a concrete expression of the bulk-deformed potential function, given by the divisor axiom (2.6).

Let $\beta \in \pi_2(X,L)$ be a relative homotopy class of Maslov index two. Let $\mathcal{M}_{1,0}^{main}(X,L;J;\beta)$ be the moduli space of J-holomorphic disks in Definition 2.5. When J is satisfies the Condition 1.1, the chain

$$\mathfrak{m}_{0,\beta}(1) := (ev_0 : \mathcal{M}_{1,0}^{main}(X,L;J;\beta) \to L)$$

becomes a cycle since there is no J-holomorphic disks with smaller Maslov index. The *one-pointed open Gromov-Witten invariant* is defined as the mapping degree

$$n_\beta = \deg(ev_0 : \mathcal{M}_{1,0}^{main}(X, L; J; \beta) \to L)$$

which is a rational number. If the Lagrangian L is monotone with minimal Maslov number two then n_β is an invariant of the choice of J. If L is not monotone then n_β depends on the choice of a generic J.

Then the bulk-deformed potential function has the following form

$$\mathfrak{PO}_{\mathfrak{b}}^L(b) = \sum_{\mu(\beta)=2} n_\beta e^{\mathfrak{b}\cap\beta} e^{b\cap\partial\beta} T^{\omega(\beta)} \tag{2.10}$$

where

$$\mathfrak{b} \in H^2(X; \Lambda_0), \quad b \in \mathcal{M}_{weak}\left(L, \mathfrak{m}^{\mathfrak{b}}\right).$$

And when L is a Lagrangian torus, this potential function determines the bulk-deformed Floer cohomology.

Theorem 2.7. *(Theorem 2.3, [21]) Let L be a Lagrangian torus in a symplectic manifold X. Suppose that*

$$H^1\left(L; \Lambda_0\right) / H^1\left(L; 2\pi\sqrt{-1}\mathbb{Z}\right) \subset \mathcal{M}_{weak}\left(L, \mathfrak{m}^{\mathfrak{b}}\right)$$

and $b \in H^1\left(L; \Lambda_0\right)$ is a critical point of the potential function $\mathfrak{PO}_{\mathfrak{b}}^L$. Then we have that

$$HF\left(L, \mathfrak{b}, b; \Lambda_0\right) \cong H\left(L; \Lambda_0\right).$$

In particular L is nondisplaceable in X.

Another structural result, Theorem 6.1.20 in [17], tells us a decomposition formula for the deformed Floer cohomology

$$HF(L, \mathfrak{b}, b; \Lambda_0) \cong (\Lambda_0)^a \oplus \left(\bigoplus_{i=1}^{l} \frac{\Lambda_0}{T^{\lambda_i}\Lambda_0}\right) \tag{2.11}$$

where $a \in \mathbb{Z}_{\geq 0}$ is called the Betti number and $\lambda_i \in \mathbb{R}_+$ are called the torsion exponents of the deformed Floer cohomology. It is proved that only the free part of the deformed Floer cohomology is an invariant under Hamiltonian diffeomorphisms, see Theorem J in [17]. Hence it suffices to show that $a > 0$ if we want to prove some L is nondisplaceable. When $a = 0$, the torsion exponents λ_i are closely related to the displacement energy of L, which we will discuss in detail in Section 7.

3 Computations of classical Floer cohomology

In this section we carry out some computations of classical Floer cohomology, which are free of bounding cochains and bulk-deformations.

3.1 Symplectic cut and sum construction

We first summarize the symplectic cut and sum construction to analyze holomorphic disks on our Lagrangian sphere and local tori. The whole construction is fully described in section 2 of [25] and section 3 of [26].

Let

$$Q_n = \{[z_0, \cdots, z_{n+1}] \in \mathbb{C}P^{n+1} \mid z_0^2 = \sum_{j=1}^{n+1} z_j^2\}$$

be the complex quadric hypersurface and

$$D_n = \{[z_0, \cdots, z_{n+1}] \in Q_n \mid z_0 = 0\} \cong Q_{n-1}$$

be the divisor at infinity. Then the real part $Q_{n,\mathbb{R}} = Q_n \cap \mathbb{R}P^{n+1}$ is a Lagrangian n-sphere in (Q_n, ω_{FS}) and $Q_n - D_n$ is a Weinstein neighborhood of $Q_{n,\mathbb{R}}$. Another perspective is that there is a Hamiltonian S^1 action on T^*S^n such that the sphere bundles of it are regular level sets. If we collapse the circles on a fixed sphere bundle then $(D_rT^*S^n, \partial D_rT^*S^n)$ goes to (Q_n, D_n) with a scaled Fubini-Study symplectic form.

Proposition 3.1. *Let (X, S, ω) be a symplectic $2n$-manifold containing a Lagrangian n-sphere S. There exists a symplectic fibration $\pi : (\mathcal{X}, \omega_{\mathcal{X}}) \to \Delta$ with a Lagrangian sub-fibration $\mathcal{S}$. Here Δ is a small disk in $\mathbb{C}$ containing the origin. Let X_z be the fiber at $z \in \Delta$ then we have*

1. *$X_0 = X_- \cup_D X_+$ where both $X_\pm$ are closed smooth symplectic manifolds and $D = X_- \cap X_+$ is a common symplectic hypersurface;*
2. *when $z \neq 0$ the pair $(X_z, \omega_{\mathcal{X}}|_{X_z}, \mathcal{S}_z)$ is symplectically isotopic to the pair (X, ω, S);*
3. *when $z = 0$ then $\mathcal{S}_0$ is in X_- and the pair $(X_-, \omega_{\mathcal{X}}|_{X_-}, \mathcal{S}_0)$ is symplectomorphic to $(Q_n, \omega_{FS}, D_n, Q_{n,\mathbb{R}})$.*

Next we specify the almost complex structures we will use on this fibration. An almost complex structure J on the fibration $\pi : (\mathcal{X}, \omega_{\mathcal{X}}) \to \Delta$ is said to be *admissible* if

1. it is compatible with $\omega_{\mathcal{X}}$ and preserves $\ker d\pi$;
2. it restricts to an almost complex structure on the singular locus D of X_0 and satisfies that

   $$N_J(u, v) \in T_xD \quad \forall u \in T_xD, v \in T_xX_0, x \in D$$

   where N_J is the Nijenhuis tensor of J.

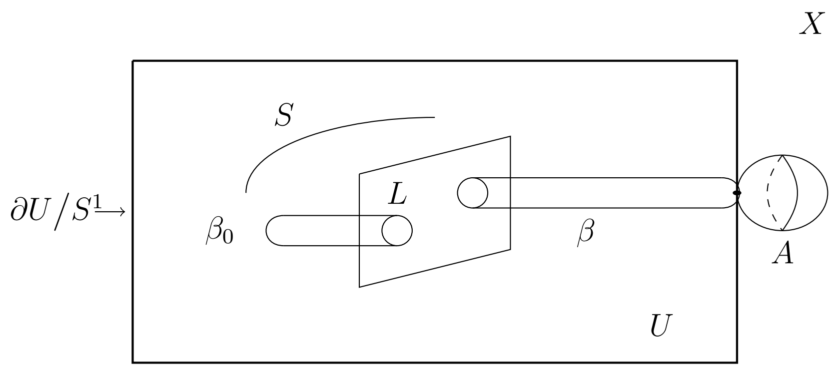

Figure 2: Degeneration of a holomorphic disk.

We denote the set of all admissible almost complex structures on $\mathcal{X}$ by $\mathcal{J}_{\mathcal{X}}$ and the subset of l-differentiable elements by $\mathcal{J}^l_{\mathcal{X}}$. Both spaces $\mathcal{J}_{\mathcal{X}}$ and $\mathcal{J}^l_{\mathcal{X}}$ are non-empty and path-connected.

With respect to an admissible almost complex structure we can compare the first Chern numbers and Maslov indices between (X, ω) and $(X_\pm, \omega_{\mathcal{X}}|_{X_\pm})$. Let $\beta \in H_2(X_-, S; \mathbb{Z})$ and $A \in H_2(X_+; \mathbb{Z})$ such that $\beta \cdot_{X_-} D = A \cdot_{X_+} D$ then we can deform the connected sum of β and A to be a homology class $\beta + A \in H_2(X, S; \mathbb{Z})$ in the smooth fiber. Note that $\partial\beta = 0$ so the pairings $\langle c_1(TX_-), \beta\rangle$ and $\langle c_1(TX), \beta + A\rangle$ are well-defined and we have the following relation.

Proposition 3.2. *With the above notation,*

$$\begin{aligned}\langle c_1(TX), \beta + A\rangle &= \langle c_1(TX_-), \beta\rangle + \langle c_1(TX_+), A\rangle - 2A \cdot_{X_+} D \\ &= \langle c_1(TX_+), A\rangle + (n-2)A \cdot_{X_+} D\end{aligned} \tag{3.1}$$

These two propositions are summaries of Proposition 2.1 in [25] and Proposition 3.1 in [26]. Next we use the symplectic cut and sum construction to show the weakly unobstructedness of a Lagrangian sphere.

3.2 Weakly unobstructedness of Lagrangian spheres

It is proved that any Lagrangian sphere is weakly unobstructed in [17] Corollary 3.8.18. We give an alternative proof by analyzing holomorphic disks with boundary on it. Along this proof we prove Theorem 1.6.

Lemma 3.3. *For a closed smooth relative spin Lagrangian submanifold L, if L does not bound any non-constant J-holomorphic disk with non-positive Maslov index, then L is weakly unobstructed with respect to J. In particular, we have that*

$$H^1(L; \Lambda_+) \subset \mathcal{M}_{weak}(L; \Lambda_+), \quad H^1(L; \Lambda_0)/H^1(L; 2\pi\sqrt{-1}\mathbb{Z}) \subset \mathcal{M}_{weak}(L; \Lambda_0).$$

Proof. We assume that L is relative spin hence orientable. So the Maslov index of a disk class is an even integer. Let $\{\mathfrak{m}_k\}$ be the A_∞ operations defined on $H^*(L;\Lambda_0)$.

First we consider the case with Λ_+ coefficients. By definition (2.3) we need to show that for any $b \in H^1(L;\Lambda_+)$ we have

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b,\cdots,b) \in H^0(L;\Lambda_+).$$

Note that $\mathfrak{m}_k(b,\cdots,b) = \sum_\beta \mathfrak{m}_{k,\beta}(b,\cdots,b)$, and the degree of $\mathfrak{m}_{k,\beta}(b,\cdots,b)$ is $n+\mu(\beta)-2$. Therefore when $\mu(\beta) \geq 4$ the $\mathfrak{m}_{k,\beta}(b,\cdots,b)$ is zero since our Lagrangian is n-dimensional. Moreover if L does not bound any J-holomorphic disk with $\mu(\beta) \leq 0$ the only contribution of $\mathfrak{m}_{k,\beta}$ to $\mathfrak{m}_k$ are from Maslov index two disks. Therefore all $\mathfrak{m}_k(b,\cdots,b)$ are cycles and have the same dimension n. This shows that for any $b \in H^1(L;\Lambda_+)$ the cycle $\mathfrak{m}_k(b,\cdots,b)$ is proportional to $[L]$, which solves the A_∞-Maurer-Cartan equation.

The case with Λ_0 coefficients can be proved similarly once the definition (2.4) is noticed. □

When $n \leq 2$ the moduli space of holomorphic disks bounding a Lagrangian S^n with a non-positive Maslov index has strictly negative dimension. Hence by perturbing the almost complex structure the Lagrangian S^n is weakly unobstructed. Next we assume that $n \geq 3$.

Theorem 3.4. *Let (X,S,ω) be a symplectic $2n$-manifold with a Lagrangian n-sphere S. Then there exists a dense subset $\mathcal{J}^{reg}$ of admissible compatible almost complex structures such that S does not bound any non-constant J-holomorphic disk with non-positive Maslov index for $J \in \mathcal{J}^{reg}$.*

Proof. The proof is also based on a dimension-counting argument. Let $(\mathcal{X},\omega_{\mathcal{X}})$ be the fibration constructed in Proposition 3.1. For an admissible almost complex structure J we study the limit of holomorphic disks from smooth fibers to the central fiber $X_0 = X_- \cup_D X_+$.

Let $\mathcal{M}^{reg}(X_+,A,J)$ be the moduli space of J-holomorphic curves of class $A \in H_2(X_+;\mathbb{Z})$, which are somewhere injective. Then classic result shows that there is a dense subset $\mathcal{J}_{\mathcal{X}}^{reg,A} \subset \mathcal{J}_{\mathcal{X}}$ such that $\mathcal{M}^{reg}(X_+,A,J)$ is a smooth manifold of dimension

$$\dim_{\mathbb{R}} \mathcal{M}^{reg}(X_+,A,J) = 2n-6+2\langle c_1(TX_+),A\rangle$$

for $J \in \mathcal{J}_{\mathcal{X}}^{reg,A}$. In particular if $0 > 2n-6+2\langle c_1(TX_+),A\rangle$ then $\mathcal{M}^{reg}(X_+,A,J)$ is empty. Note that $\mathcal{J}_{\mathcal{X}}^{reg,A}$ is an intersection of countably many open dense subsets of $\mathcal{J}_{\mathcal{X}}$, we can intersect again to get $\mathcal{J}_{\mathcal{X}}^{reg} := \cap_A \mathcal{J}_{\mathcal{X}}^{reg,A}$.

Next let $z_i \in \Delta$ be a sequence converging to 0 and J_i be the restriction of J on X_{z_i}. Consider a sequence of J_i-holomorphic disks of class β in X_{z_i}. Then by Gromov compactness we get a nodal disk in $X_0 = X_- \cup_D X_+$ of class $\beta = \beta' + A$

where $\beta' \in H_2(X_-, S; \mathbb{Z})$ and $A \in H_2(X_+; \mathbb{Z})$. Geometrically this nodal disk is obtained by collapsing the circle where the symplectic cut happens. Assume that we do symplectic cut on the sphere bundle of radius r of the cotangent bundle of S. Then for small ϵ the image of our disk intersects with the sphere bundle of radius $r+\epsilon$. Otherwise this disk is totally contained in $X_- - D$ which contradicts that S is exact in $X_- - D$. Moreover this shows that A is not contained in D since the disk intersects with the sphere bundle of radius $r+\epsilon$, which is in $X_+ - D$. Note that J is admissible therefore the image of the A-part of our holomorphic curve intersects with D in a finite set with positive multiplicities. That is, $A \cdot_{X_+} D > 0$. By choosing $J \in \mathcal{J}_{\mathcal{X}}^{reg}$ we can assume that $0 \leq 2n - 6 + 2\langle c_1(TX_+), A\rangle$. Therefore Proposition 3.2 tells us that

$$\begin{aligned} \mu(\beta) = 2\langle c_1(TX), \beta\rangle &= 2\langle c_1(TX_-), \beta'\rangle + 2\langle c_1(TX_+), A\rangle - 4A \cdot_{X_+} D \\ &= 2\langle c_1(TX_+), A\rangle + 2(n-2)A \cdot_{X_+} D \\ &\geq 6 - 2n + 2(n-2)A \cdot_{X_+} D \\ &\geq 6 - 2n + 2(n-2) \geq 2. \end{aligned} \tag{3.2}$$

This inequality shows that if $\mathcal{M}^{reg}(X_{z_i}, S, \beta, J_i)$ is non-empty then $\mu(\beta) \geq 2$ for large i. Since $(X_{z_i}, \omega_{\mathcal{X}}|_{X_{z_i}}, \mathcal{S}_{z_i})$ is isomorphic to (X, ω, S) the above inequality holds for a neighborhood of J_i in $\mathcal{J}(X, \omega)$. Hence it is proved that there is a dense subset $\mathcal{J}^{reg}$ such that S is weakly unobstructed with respect to $J \in \mathcal{J}^{reg}$. □

The above proof also gives a lower bound of energy of holomorphic disks which bound the Lagrangian sphere. We will use this energy bound when we study the moduli space of cylinders.

Corollary 3.5. *With the notation above, there exists $E_S > 0$ such that all J-holomorphic disks with boundary on S have energy greater than E_S for $J \in \mathcal{J}^{reg}$. The lower bound of E_S depends on the maximal Weinstein neighborhood of S, not on the choice of J.*

Proof. Note that the image of any non-constant J-holomorphic disk cannot be contained in any of the Weinstein neighborhood U. Then after the degeneration the disk breaks into disk parts in the quadric X_- and sphere parts in X_+. Since the Lagrangian sphere is monotone in the quadric X_- the energy of the disk parts is larger than some constant E_S, depending on the size of U. □

Remark 3.6. When (X, S, ω) and J are fixed we always get a lower bound of energy of holomorphic disks, the analytic lower bound. The above bound E_S is obtained from some topological lower bound which is expected to be much larger than the analytic one.

Another corollary of this degeneration formula is that when the symplectic manifold is Calabi-Yau or negatively monotone, any Lagrangian submanifolds in this Weinstein neighborhood U does not bound J-holomorphic disks which are not totally contained in U. Note that Oakley-Usher constructed many families of monotone nondisplaceable Lagrangian submanifolds in T^*S^n. By the degeneration technique we get continuum families of nondisplaceable Lagrangian submanifolds.

Theorem 3.7. *(Oakley-Usher [34]) There exist continuum families of monotone Lagrangian submanifolds*

$$L^{k,m}_{\lambda} \cong (S^1 \times S^k \times S^m)/\mathbb{Z}_2, \quad k, m \in \mathbb{Z}_+, k \le m, k + m = n - 1, \lambda \in (0, +\infty) \subset \mathbb{R}$$

with non-zero Floer cohomology in T^*S^n.

Corollary 3.8. *For any integer $n \ge 3$, let (X^{2n}, S^n, ω) be a Calabi-Yau or negatively monotone symplectic manifold with a Lagrangian sphere. Then there are continuum families of Lagrangian submanifolds*

$$L^{k,m}_{\lambda} \cong (S^1 \times S^k \times S^m)/\mathbb{Z}_2, \quad k, m \in \mathbb{Z}_+, k \le m, k + m = n - 1, \lambda \in (0, \lambda_0] \subset \mathbb{R}$$

near the Lagrangian sphere S and are nondisplaceable in X.

Proof. The Lagrangian submanifolds $L^{k,m}_{\lambda}$ are those in the previous theorem, originally sit in T^*S^n. For a small interval $(0, \lambda_0]$ we assume that all $L^{k,m}_{\lambda}$ are contained in a Weinstein neighborhood U of S. Next we apply the degeneration method to show that they do not bound any holomorphic disk which are not contained in U, with respect to some almost complex structure.

Similar to the proof of Theorem 3.4, let $(\mathcal{X}, \omega_{\mathcal{X}})$ be the fibration constructed in Proposition 3.1. For an admissible almost complex structure J we study the limit of holomorphic disks from smooth fibers to the central fiber $X_0 = X_- \cup_D X_+$. If L bounds a holomorphic disk which is not contained in U then its limit in the singular fiber X_0 is a nodal disk u. The X_+-part of u represents a class $A \in H_2(X_+; \mathbb{Z})$. Since the almost complex structure is admissible, the intersection number $s = A \cdot_{X_+} D$ is positive and finite. Now we choose a class $B \in H_2(X_-; \mathbb{Z})$ such that $B \cdot_{X_-} D = s$. Then we can deform $A + B$ into a homology class in the smooth fiber $H_2(X; \mathbb{Z})$, which we still write as $A + B$. By the Chern number formula Proposition 3.2 we have that

$$\begin{aligned} \langle c_1(TX), B + A\rangle &= \langle c_1(TX_-), B\rangle + \langle c_1(TX_+), A\rangle - 2A \cdot_{X_+} D \\ &= \langle c_1(TX_+), A\rangle + (n-2)A \cdot_{X_+} D. \end{aligned} \tag{3.3}$$

Note that X_- is the quadric hypersurface, which is a monotone symplectic manifold. So $B \cdot_{X_-} D = s > 0$ implies that $\langle \omega_{X_-}, B\rangle > 0$. Moreover $\langle \omega_z, B + A\rangle > 0$ in the smooth fiber. When $n \ge 3$ and X is Calabi-Yau or negatively monotone we obtain that $\langle c_1(TX), B + A\rangle \le 0$ and hence $\langle c_1(TX_+), A\rangle < 0$. Then by perturbing the almost complex structure on X_+ there is no holomorphic curve representing the class A (or its underlying simple curve).

In conclusion, by picking a suitable almost complex structure, our Lagrangian submanifolds $L^{k,m}_{\lambda}$ only bound holomorphic disks inside U. So their Floer cohomology groups are the same as those in T^*S^n, which are non-zero. □

In [34] it was also proved that if we compactify the cotangent bundle to be the quadric then $L_\lambda^{0,m}$ is displaceable in Q_{m+2} for $m \geq 2$. This matches the discussion above that when the ambient space is monotone there will be holomorphic disks coming from outside, which may break the Floer cohomology. The major task of following sections will be studying possible deformations of Floer cohomology to deal with those outside contributions.

3.3 Welschinger invariants and the pearl complex

Now we compare the open Gromov-Witten invariants defined by Welschinger [42] and the Floer differential in the pearl complex.

Let X be a monotone symplectic 6-manifold and S be a Lagrangian 3-sphere in X. Since S is simply-connected and spin, we can define Welschinger invariants with value in any finite field F or $F = \mathbb{Z}$. Fix an orientation and (the unique) spin structure on S. Given $\beta \in \pi_2(X, S)$ and a generic compatible almost complex structure J, we write $\mathcal{M}_r^\beta(X, S; J)$ as the space of simple J-holomorphic disks with boundary on L, representing the class β with r boundary marked points, modulo equivalence. It is an oriented manifold with boundaries and corners of dimension $\mu(\beta) + r$, with an evaluation map to S^r. We also write $\mathcal{M}_r^{\beta_1,\cdots,\beta_k}(X, S; J)$ as the moduli space of simple reducible J-holomorphic disks with k components representing $\beta_1, \cdots, \beta_k$ and have r boundary marked points. Then $\mathcal{M}_r^{\beta_1,\cdots,\beta_k}(X, S; J)$ is an oriented manifold with boundaries and corners of dimension $\mu(\beta_1+\cdots+\beta_k)+r$. Let $\mathcal{M}_{r,int}^{\beta_1,\cdots,\beta_k}(X, S; J)$ be the dense open subset of which the elements are multi-disks with pairwise disjoint boundary components.

In our setting, we only need the case where there are at most two components, to define a linking weight on the moduli space $\mathcal{M}_{r,int}^{\beta_1,\cdots,\beta_k}(X, S; J)$ in a simpler way. It is a locally constant function

$$lk_k : \mathcal{M}_{r,int}^{\beta_1,\cdots,\beta_k}(X, S; J) \to F.$$

When $k = 1$, there is only one component of $\mathcal{M}_{r,int}^{\beta_1}(X, S; J)$, we define $lk_1 = 1$ be the constant function. When $k = 2$, for an element $u \in \mathcal{M}_{r,int}^{\beta_1,\beta_2}(X, S; J)$ we define $lk_2(u) = lk(\partial u)$, the linking number of two boundary components of u. Here we view the two boundary components of u as two disjoint knots in S. Now let β be a disk class of Maslov index four. We set

$$[\mathcal{M}_{\beta,2}(X, S; J)] := \sum_{k=1}^{2} \frac{1}{k!} \sum_{\beta_1+\cdots+\beta_k=\beta} lk_k[\mathcal{M}_{2,int}^{\beta_1,\cdots,\beta_k}(X, S; J)]$$

to define the two-pointed Welschinger invariants.

Theorem 3.9. *(Welschinger, [42]) The chain*

$$ev_*[\mathcal{M}_{\beta,2}(X, S; J)] := \sum_{k=1}^{2} \frac{1}{k!} \sum_{\beta_1+\cdots+\beta_k=\beta} lk_k ev_*[\mathcal{M}_{2,int}^{\beta_1,\cdots,\beta_k}(X, S; J)]$$

is a cycle whose homology class in $H_6(S \times S; F)$ does not depend on the generic choice of J.

The two-pointed Welschinger invariant of class β is defined as

$$n^W_{2,\beta} := \langle ev_*[\mathcal{M}_{\beta,2}(X, S; J)], PD[pt] \cup PD[pt]\rangle \in F,$$

which is independent of a generic choice of J.

Next we review the pearl complex to compute the Floer cohomology, and compare its differential with the two-pointed Welschinger invariant. We refer to [5, 18] for more details about the pearl complex and [23] for the extension to the case of finite characteristics.

Let $f : S \to \mathbb{R}$ be a perfect Morse function on our Lagrangian sphere. That is, f has exactly one critical point of index zero and one critical point of index three. Then the pearl complex contains the following set of data

$$(H^*(S) := H^*(S; F) \otimes \Lambda(F), f, J, \mathfrak{m}_1 := \sum_\beta \mathfrak{m}_{1;\beta})$$

with $H^*(S)$, the Morse cohomology of S, being the underlying complex, a generic compatible almost complex structure J and the differential $\mathfrak{m}_1$. The differential $\mathfrak{m}_1$ counts rigid configurations called "pearl trajectories", which we will explain now. Note that by degree reasons, the only possible non-trivial maps are

$$\mathfrak{m}_{1;\beta} : H^3(S; F) \otimes \Lambda(F) \to H^0(S; F) \otimes \Lambda(F)$$

where β is a disk class with Maslov index four. Then pick a generator $PD[pt] \in H^3(S; R)$ and a generator $PD[S] \in H^0(S; R)$, the map $\mathfrak{m}_{1;\beta}$ is a signed count of following pearl trajectories. Let p, q be the critical points of f corresponding to $PD[pt]$ and $PD[S]$, we consider the space of all possible sequences $(u_1, \cdots, u_k)$ when $1 \le k \le 2$ such that:

1. u_l is a non-constant J-holomorphic disk with boundary on S;
2. If $k = 2$, then $u_1(-1)$ and $u_2(1)$ are connected by a gradient flow line;
3. $u_1(1)$ and p are connected by a gradient flow line;
4. $u_k(-1)$ and q are connected by a gradient flow line;
5. $\sum_{1\le l\le k}[u_l] = \beta$.

Then the space of such pearl trajectories is a compact zero-dimensional manifold, modulo equivalence. We define $\mathfrak{m}_{1;\beta}$ as this signed count, weighted by the symplectic area of the sum of pearls.

Now we are ready to prove Theorem 1.7. The proof is not hard but rather an observation, based on the following lemma.

Lemma 3.10. *Let $f : S \to \mathbb{R}$ be a perfect Morse function on a 3-sphere. Let K_1, K_2 be two disjoint knots in S. Then the linking number $lk(K_1, K_2)$ equals the signed count of Morse flow lines starting from one point on K_1, ending at one point on K_2.*

Proof. Let q be the minimal point of f, consider the preimage of K_2 under the Morse flow ρ. That is, define

$$C := \bigcup_{x \in K_2} \{y \in X \mid \exists t \in \mathbb{R}, \quad \rho^t(y) = x\} \cup \{q\}$$

which is an oriented two-chain in S with boundary as K_2. Then there is a one-to-one correspondence between intersection points of K_1 and C and Morse flow lines starting from one point on K_1, ending at one point on K_2. Moreover, this intersection number between K_1 and C equals the linking number $lk(K_1, K_2)$. Hence we complete the proof.

Note that we assume that f is perfect, for any $x \in K_2$ there is a unique smooth flow line connecting x and q. For a general Morse function, there may be broken flow lines going back to other critical points. We suggest [2] for general discussions. □

Theorem 3.11. *Let S be a Lagrangian 3-sphere in a monotone symplectic 6-manifold X. Given a disk class $\beta \in \pi_2(X, S)$ with Maslov index four, we have an equality*

$$\langle \mathfrak{m}_{1;\beta}(PD[pt]), [pt] \rangle = n^W_{2,\beta} \cdot T^{\omega(\beta)}$$

where the pairing on the left is the cohomology-homology pairing and $n^W_{2,\beta}$ is the two-pointed Welschinger invariant of class β.

Proof. We still fix a generic perfect Morse function f on S to define the pearl complex and $\mathfrak{m}_{1;\beta}$. Let q (p respectively) be the minimal (maximal respectively) point of f. Given two generic points x, y on S and a disk class β with Maslov index four, let $\mathcal{M}_{\beta,2}(X, S; J)$ be the moduli space of multi-disks in Theorem 3.9 and let $\mathcal{M}_{\beta,2}(X, S; (x, y); J)$ be the moduli space of elements such that two marked points go to x and y respectively. Then the two-pointed Welschinger invariant $n^W_{2,\beta}$ is the number of elements in $\mathcal{M}_{\beta,2}(X, S; (x, y); J)$.

Now we construct a one-to-one map between the moduli space of pearl trajectories connecting q and p and the moduli space $\mathcal{M}_{\beta,2}(X, S; (x, y); J)$. Pick an element u in $\mathcal{M}_{\beta,2}(X, S; (x, y); J)$. First, if the underlying disk of u is a single disk u_1. After reparametrization we assume that $u_1(1) = x$ and $u_1(-1) = y$. Since our Morse function is perfect, there is a unique flow line connecting q and x (y and p respectively). So this configuration is counted once in the space of pearl trajectories. On the other hand, a single disk has self-linking number one by definition hence it contributes once to $n^W_{2,\beta}$. Next, if the underlying disk of u is a multi-disk u_1, u_2. (It has at most two components since S is monotone.) Note that if two marked points are both on one component, then we have a Maslov index two disk with two-pointed constraints, which does not happen generically. So we assume that $u_1(1) = x$ and $u_2(-1) = y$. Similarly there is a unique flow line connecting q and x (y and p respectively). Then this configuration is weighted by the number of Morse flow lines from the boundary of u_1 to the boundary of u_2, which is the same as the linking number by Lemma 3.10. Hence the multi-disks are counted by the same number in both moduli spaces.

We remark that to compare the two counts in the equation we need to furthermore compare the orientation data on both sides. That is, compare the orientation conventions in [42] and in [18]. We do not plan to do it here but leave the theorem as proved up to sign. This does not effect our applications since we only care about the vanishing/non-vanishing property of the Floer cohomology of a Lagrangian sphere. □

Therefore we can define an invariant

$$n_2^W := \sum_{\mu(\beta)=4} n_{2,\beta}^W \in F$$

to determine the Floer cohomology $HF(S;\Lambda(F))$. That is, $HF(S;\Lambda(F)) = \{0\}$ if and only if $n_2^W \neq 0$. In particular, when $n_2^W = p \in F = \mathbb{Z}$ this gives a Lagrangian sphere which is a non-trivial object in the Fukaya category with characteristic p but a trivial object in the integral Fukaya category.

A similar story may be also true for a monotone Lagrangian n-sphere for any odd integer n, where its Floer cohomology is determined by counting pearl trajectories with Maslov index $n+1$. Note that Solomon-Tukachinsky [38] and Chen [10] have generalized Welschinger invariants in higher dimensions. We expect their invariants also have some meanings in Floer theory.

4 Lagrangian tori near a Lagrangian 2-sphere

In this section we focus on the case where X is a symplectic 4-manifold containing a Lagrangian 2-sphere S. Our goal is to prove the following theorem.

Theorem 4.1. *Let X be a closed symplectic 4-manifold which contains a Lagrangian 2-sphere S. There is a one-parameter family of Lagrangian embeddings*

$$T^2 \approx L_\lambda^2 \hookrightarrow U = D_r T^* S^2 \subset X$$

for $\lambda \in (0, \lambda_0)$, and each L_λ^2 is nondisplaceable in X.

4.1 Constructions of local Lagrangian 2-tori

Let (T^*S^2, ω_0) be the cotangent bundle of the unit 2-sphere in $\mathbb{R}^3$, equipped with the standard symplectic form. It can be viewed as the smoothing of a node.

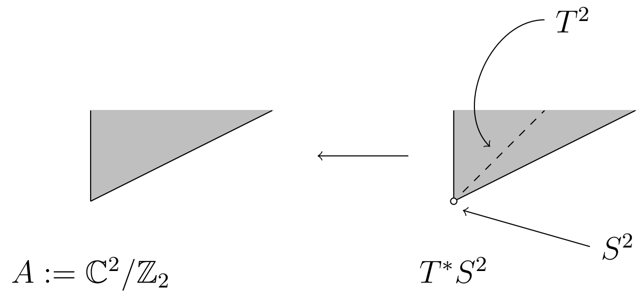

Figure 3: Smoothing a node.

Consider the $\mathbb{Z}_2$-quotient of $\mathbb{C}^2$ by identifying (z_1, z_2) with $(-z_1, -z_2)$. The origin is the fixed point of this action and the resulting quotient $A := \mathbb{C}^2/\mathbb{Z}_2$ is an orbifold with a unique node. This orbifold is a toric orbifold with a moment polytope

$$P = \{u_1 \geq 0\} \cap \{2u_2 - u_1 \geq 0\}$$

where u_1, u_2 are coordinates of $\mathbb{R}^2$. The orbifold A can be also viewed as an affine variety, isomorphic to

$$\{z_1^2 + z_2^2 + z_3^2 = 0\} \subset \mathbb{C}^3.$$

The link of this singularity is $S^3/\mathbb{Z}_2 = \mathbb{R}P^3$ with a contact structure induced from the standard one on the 3-sphere. Let $D_rT^*S^2$ be the disk cotangent bundle with radius $r > 0$ (with respect to the round metric). Then the boundary $\partial D_rT^*S^2$ is diffeomorphic to $S^3/\mathbb{Z}_2 = \mathbb{R}P^3$ and it also has a contact structure, which is isomorphic to the standard one on $\mathbb{R}P^3$. Let $B_\epsilon(0)$ be a ball in $\mathbb{C}^3$ centered at the origin, with radius $\epsilon > 0$. Then we can glue $A - A \cap B_\epsilon(0)$ with $D_rT^*S^2$ for suitable $r = r(\epsilon)$, by using the isomorphic characteristic foliations on their common boundary. The resulting glued manifold is a smooth symplectic manifold, which is symplectomorphic to (T^*S^2, ω_0). Note that $H^2(\mathbb{R}P^3; \mathbb{Q}) = 0$, the glued symplectic form does not depend on the choice of ϵ up to a symplectic diffeomorphism, by the Moser's argument.

For a fixed gluing parameter ϵ, we can make that the toric structure is unchanged outside $A - A \cap B_{2\epsilon}(0)$. Hence the toric fibers outside the gluing region are still Lagrangian tori in the glued symplectic manifold, which is isomorphic to (T^*S^2, ω_0). Since we can choose ϵ as small as we want, actually we have a singular toric fibration with total space as (T^*S^2, ω_0). The base of this fibration is still the polytope P, but over the unique vertex the preimage is a Lagrangian 2-sphere, which corresponds to the zero section of (T^*S^2, ω_0). Alternatively, we can directly construct a singular toric fibration on (T^*S^2, ω_0), since there is a Hamiltonian T^2-action outside the zero section.

Therefore we have a fibration structure

$$\pi : (T^*S^2, \omega_0) \to P$$

where the preimage of an interior point is a Lagrangian torus, see Figure 3. Now we write $L^2_\lambda := \pi^{-1}(\lambda, \lambda)$ for $\lambda \in (0, +\infty)$, which are the Lagrangian tori of our interest.

4.2 Bulk-deformed potential functions of local tori

Now we recall the local computation of the bulk-deformed potential functions of L^2_λ, given by Fukaya-Oh-Ohta-Ono [21].

Theorem 4.2. *The Lagrangian torus L^2_λ has following properties.*

1. *For each $\lambda \in (0, +\infty)$, L^2_λ is a monotone Lagrangian torus with minimal Maslov number two.*
2. *There are four disk classes $\beta_1, \beta_2, \beta_3, \beta_4 \in \pi_2(T^*S^2, L^2_\lambda)$ such that the one-pointed open Gromov-Witten invariant $n_{\beta_i} = 1$. For other disk class $\beta \in \pi_2(T^*S^2, L^2_\lambda)$, the one-pointed open Gromov-Witten invariant $n_\beta = 0$.*
3. *For a chosen basis $e_1, e_2 \in H_2(L^2_\lambda; \mathbb{Z})$, we have that $\partial\beta_1 = e_1, \partial\beta_2 = \partial\beta_3 = e_2, \partial\beta_4 = -e_1 + 2e_2$.*
4. *Let $[S]$ be the homology class of the zero section with a suitable orientation. Then we have the intersection numbers $\beta_1 \cdot [S] = \beta_4 \cdot [S] = 0, \beta_2 \cdot [S] = 1, \beta_3 \cdot [S] = -1$.*

We remark that although in [21] they focus on the case where the ambient manifold is $S^2 \times S^2$, they provided all the essential data for the above theorem, see Theorem 4.1 [21] for a proof. Another proof of the above theorem can be deduced from the work [30] of Lekili-Maydanskiy. They first construct the tori L^2_λ by using a Lefschetz fibration on (T^*S^2, ω_0), then compute all the non-zero one-pointed open Gromov-Witten invariants.

Now we use the local computation to finish the proof of Theorem 4.1.

Proof. Pick a neighborhood U of the Lagrangian sphere S which is symplectomorphic to the disk bundle of the cotangent bundle of a 2-sphere. Then we have a one-parameter family L^2_λ inside U with λ parameterized by an open interval. We study the potential function with bulk deformation of the torus L^2_λ to show it is nondisplaceable in X.

The potential function of L^2_λ counts Maslov index two holomorphic disks. We pick an almost complex structure which agrees the complex structure we use to compute the local one-pointed open Gromov-Witten invariants on U and extend it to a generic one on X, such that the potential function is well-defined. Note that just in U the tori L^2_λ are monotone, the one-pointed open Gromov-Witten invariants are independent of a choice of a regular J. Globally in X, since the Lagrangian submanifold is 2-dimensional, it is weakly unobstructed for a generic almost complex structure J, see Lemma 3.3.

Then for the tori which are close to the Lagrangian sphere S, the holomorphic disks of lowest energy are explicitly known due to the local computation. More precisely, there are four disk classes with non-zero open Gromov-Witten invariants. They are $\beta_1, \beta_2, \beta_3, \beta_4$ shown in Theorem 4.2. All these classes have open Gromov-Witten invariant $n_{\beta_i} = 1$. So the full disk potential function (without bulk deformation) is

$$\mathfrak{PO}(b) = (y_1 + 2y_2 + y_1^{-1}y_2^2)T^\lambda + P(y_1, y_2)T^\mu, \quad \lambda < \mu$$

where the low energy terms are given by Theorem 4.2 and (2.10). Here $b = x_1e_1^* + x_2e_2^* \in H^1(L_\lambda^2; \Lambda_0)$ where e_1^*, e_2^* is a dual basis of e_1, e_2 in Theorem 4.2. And we change the coordinates by setting $y_i = e^{x_i}$.

Now let $\mathfrak{b} = vPD([S]) \in H^2(X; \Lambda_0)$ be a bulk deformation. Note that the intersection numbers are

$$\beta_i \cdot [S] = 0, i = 1, 4; \quad \beta_2 \cdot [S] = 1; \quad \beta_2 \cdot [S] = -1$$

by Theorem 4.2. Then by (2.10) the whole potential function is

$$\mathfrak{PO}^\mathfrak{b}(b) = (y_1 + (e^v + e^{-v})y_2 + y_1^{-1}y_2^2)T^\lambda + P(y_1, y_2, v)T^\mu, \quad \lambda < \mu$$

where $P(y_1, y_2; v, w)$ comes from the contribution of high energy disks, possibly effected by our bulk deformation.

Then the critical point equations, taking derivatives with respect to y_1, y_2, are

$$\begin{cases} 1 - y_1^{-2}y_2^2 + P_1(y_1, y_2, v)T^{\mu-\lambda} = 0 \\ (e^v + e^{-v}) + 2y_1^{-1}y_2 + P_2(y_1, y_2, v)T^{\mu-\lambda} = 0 \end{cases}$$

By a change of coordinate $\tilde{v} = e^v$ we have that

$$\begin{cases} 1 - y_1^{-2}y_2^2 + P_1(y_1, y_2, \tilde{v})T^{\mu-\lambda} = 0 \\ (\tilde{v} + \tilde{v}^{-1}) + 2y_1^{-1}y_2 + P_2(y_1, y_2, \tilde{v})T^{\mu-\lambda} = 0 \end{cases} \tag{4.1}$$

Now we view the above system of equations as two equations with three variables $y_1, y_2, \tilde{v}$. If there are suitable $y_1, y_2, \tilde{v}$ such that (4.1) is satisfied then we can show the Floer cohomology is non-zero for those deformation parameters $y_1, y_2, \tilde{v}$. By introducing the bulk parameter $\tilde{v}$ we have one more dimension for freedom. Note that if we fix any bulk-deformation v then we get non-isolated solutions y_1, y_2, modulo $T^{\mu-\lambda}$. So we change the point of view. We fix a bounding cochain $y_1 = 1$, to find suitable y_2 and $\tilde{v}$.

Fix $y_1 = 1$ then (4.1) becomes

$$\begin{cases} 1 - y_2^2 + P_1(1, y_2, \tilde{v})T^{\mu-\lambda} = 0 \\ (\tilde{v} + \tilde{v}^{-1}) + 2y_2 + P_2(1, y_2, \tilde{v})T^{\mu-\lambda} = 0 \end{cases} \tag{4.2}$$

which has two equations and two variables $y_2, \tilde{v}$.

Modulo $T^{\mu-\lambda}$ the low energy term equations are

$$\begin{cases} 1 - y_2^2 = 0 \\ \tilde{v} + \tilde{v}^{-1} + 2y_2 = 0 \end{cases}$$

which has isolated solutions at $y_2 = 1, \tilde{v} = -1$ and $y_2 = -1, \tilde{v} = 1$.

We pick a solution $y_2 = 1, \tilde{v} = -1$ modulo $T^{\mu-\lambda}$ and apply the following Lemma 4.3 to (4.2).

1. When the symplectic form on X and λ are both rational, all the energy parameters lie in a finitely generated semigroup in $\mathbb{R}_{\geq 0}$. Hence Lemma 4.3 tells us that in Λ_0^2 near $(1, -1)$, in the sense of non-Archimedean topology, there exists a solution to (4.2). So this solution is in $(\Lambda_0 - \Lambda_+)^2$ which shows that the deformed Floer cohomology is nontrivial.

2. In general the above equations have solutions if the tails P_1, P_2 have finite length, also by Lemma 4.3. Therefore we can truncate those equations to get solutions modulo large power. This shows that the deformed Floer cohomology is nonzero modulo any large energy hence our tori are nondisplaceable. But whether the full Floer cohomology is nontrivial or not is not known.

By the above discussion we complete the proof. The key point is by prescribing one bounding cochain, we look for a suitable bulk deformation and the other bounding cochain, which has nicer solutions modulo higher energy terms. To clarify possible confusion, we remark that when we speak of the critical point equation, we mean for y_1, y_2 variables since we differentiate with respect to them. But once we obtain these equations and look for solutions, it becomes a pure algebraic problem and we can think all $y_1, y_2, \tilde{v}$ are variables. This is because bulk parameters and bounding cochains are "independent" variables and differentiating with respect to bounding cochains does not affect the bulk parameter. $\square$

Lemma 4.3. *Let $F = (f_1(x, y), f_2(x, y))$ be a map from $\mathbb{C}^2$ to $\mathbb{C}^2$ where $f_i(x, y)$ are polynomials in x, y with complex coefficients. Suppose that F has an isolated zero in $\mathbb{C}^2$ then the system of equations*

$$\begin{cases} f_1(x, y) + g_1(x, y; T) = 0 \\ f_2(x, y) + g_2(x, y; T) = 0 \end{cases}$$

have solutions in Λ_0^2 near the zero of F in the non-Archimedean topology. Here $g_i(x, y; T)$ are of the following form

$$g_i(x, y; T) = \sum_{k=1}^{\infty} c_{i,k} x^{l_{i,k}} y^{j_{i,k}} T^{\lambda_k}$$

where

$$c_{i,k} \in \mathbb{C}, \quad l_{i,k}, j_{i,k} \in \mathbb{Z}, \quad 0 < \lambda_k \leq \lambda_{k+1}, \quad \lim_{k\to\infty} \lambda_k = +\infty$$

and λ_k lies in a finitely generated additive semigroup Σ in $\mathbb{R}_{\geq 0}$.

Proof. For complex functions the multiplicity of an isolated zero is stable under small perturbations. This lemma is an analogue in the setting of Novikov ring with the non-Archimedean topology. The proof is essentially given in the weakly nondegenerate case of Theorem 10.4 in [19], where they assume that Σ is generated by a single element such that Σ can be rearranged to be $\mathbb{Z}_{\geq 0}$. When Σ is finitely generated the proof is similar. We deal with the solution space of (10.8) in [19] as a polynomial of several variables instead of a polynomial of one variable. And the target of the projection (10.9) in [19] will be modified to a punctured polydisk. Then other parts of the proof follow similarly. We refer to Lemma 9.18, 10.12, 10.13, 10.14 and 10.15 in [19] for more details. □

In the above proof, we use T^*S^2 as a local model and use the local computation to control abstract outside disk contributions. By using the similar idea of toric degenerations, we can upgrade this theorem to the case where the local model is a linear plumbing of cotangent bundles of 2-sphere, see Theorem 1.1 in [39].

5 A deformed Floer complex

In this section we first review the construction of a family of monotone Lagrangian 3-tori $\{L^3_\lambda\}_{\lambda\in(0,+\infty)}$ in T^*S^3, then study the moduli space of holomorphic cylinders with one end on L^3_λ and the other on the zero section S of T^*S^3. This moduli space of holomorphic cylinders can be used to deform the Floer cohomology of the local torus L^3_λ, when it is near a Lagrangian 3-sphere in a general symplectic 6-manifold. That is, we will construct a Floer complex by counting holomorphic disks and cylinders. A second Floer complex, counting holomorphic strips and strips with one interior hole, will be constructed in Section 5.

5.1 Constructions of local Lagrangian 3-tori

Let T^*S^3 be the cotangent bundle of S^3 with the standard symplectic structure. It admits a Hamiltonian T^3-action outside the zero section. Moreover it admits a Gelfand-Tsetlin system which gives us a singular torus fibration

$$\pi : T^*S^3 \to P \subset \mathbb{R}^3.$$

Here the base P is a convex polytope in $\mathbb{R}^3$, cut out by 4 affine functions

$$x \geq 0; \quad -y \geq 0; \quad x - z \geq 0; \quad z - y \geq 0$$

where (x, y, z) are coordinates in $\mathbb{R}^3$. This polytope P has four faces P_i corresponding to the above four affine functions. A regular fiber over an interior point is a smooth Lagrangian torus and the fiber over the vertex at $(0, 0, 0)$ is a Lagrangian 3-sphere, the zero section. We refer to [11] and [35] for the details of the construction. Similar to the toric case in [12] and [19], the open Gromov-Witten theory of regular fibers of a Gelfand-Tsetlin system was studied in [31], which we state below.

Theorem 5.1. *(Section 9, [31]) Let L be a regular fiber of a Gelfand-Tsetlin system on a symplectic manifold X then we have that*

1. *Each L does not bound any non-constant holomorphic disks with non-positive Maslov index;*
2. *There is a one-to-one correspondence between the holomorphic disks with Maslov index two bounded by L and the faces of the Gelfand-Tsetlin polytope;*
3. *Every class $\beta \in H_2(X, L)$ is Fredholm regular and the one-pointed open Gromov-Witten invariant $n_\beta = 1$.*

Therefore in our case each regular fiber bounds four holomorphic disks with Maslov index two, which span the relative homology $H_2(X, L)$. Moreover when a fiber is over the point $(\lambda, \lambda, 0)$ these four classes have the same symplectic energy. Hence the fiber $L^3_\lambda := \pi^{-1}(\lambda, \lambda, 0)$ is a monotone Lagrangian torus with minimal Maslov number two. This is the one-parameter family of monotone Lagrangian tori in T^*S^3 which are the main objects of following sections. Now for notational simplicity, we write L_λ for L^3_λ. We remark that since L_λ is monotone the number of holomorphic disks in a given class is independent of many auxiliary choices. So $n_\beta = 1$ is not only true for the toric complex structure but also for other regular compatible almost complex structures on T^*S^3.

Another description of this one-parameter family of monotone Lagrangian tori comes from a Lefschetz fibration, see [13] where they also computed all the one-pointed Gromov-Witten invariants. We consider the smoothing

$$Y = \{xy - zw = \epsilon\} \subset \mathbb{C}^4$$

which is symplectomorphic to T^*S^3. It can be embedded into

$$\hat{Y} = \{xy = u - a, zw = u - b\} \subset \mathbb{C}^5$$

where a, b are positive real numbers and $\epsilon = b - a > 0$. The projection $\hat{Y} \to \mathbb{C}$ to the u-variable gives us a double conic fibration with singular fibers over $u = a$ and $u = b$. There is a fiberwise 2-torus action

$$(\theta_1, \theta_2) \cdot (x, y, z, w) = (e^{i\theta_1}x, e^{-i\theta_1}y, e^{i\theta_2}z, e^{-i\theta_2}w) \quad \forall(\theta_1, \theta_2) \in T^2.$$

We call an above torus orbit an equator in the fiber. Then pick a circle in the base $C_r = \{|z| = r, r > b > a\} \subset \mathbb{C}$. The 3-tori formed by crossing an equator with a base circle are of our interest. In particular these tori are monotone with minimal Maslov number two. Note that if we pick a segment connecting a and b and cross the segment with equators which degenerate at endpoints then we get a Lagrangian 3-sphere, Hamiltonian isotopic to the zero section. To compare this one-parameter family of Lagrangian tori with the Oakley-Usher construction [34] we mentioned in Section 3, this family L_λ corresponds to $L_\lambda^{1,1}$.

From above approaches we get all the information to count Maslov two disks with boundary on L_λ so that we can write down the disk potential function explicitly. For one choice of coordinates it is

$$\mathfrak{PO}(b) = x + y^{-1} + xz^{-1} + y^{-1}z, \quad b \in H^1(L_\lambda; \Lambda_0). \tag{5.1}$$

We omit the energy parameter here since L_λ is monotone. It is easy to check that this potential function has a one-dimensional critical loci, which indicates that with respect to some weak bounding cochain the Floer cohomology of L_λ is nonzero hence L_λ is nondisplaceable in T^*S^3.

If we consider a Lagrangian 3-sphere S in a symplectic 6-manifold X then L_λ sits inside a neighborhood of S for small λ. Due to the global symplectic geometry of X our local torus L_λ may bound more higher energy holomorphic disks with Maslov index two. Therefore the potential function may have more higher energy terms and the torus may fail to be nondisplaceable in X. Indeed if the Lagrangian 3-sphere S is displaceable in X, then L_λ is displaceable for small λ.

5.2 Conifold transition

Before constructing the moduli spaces of holomorphic cylinders we first describe some topological aspects of the conifold transition, mostly following [37]. By a 3-fold ordinary double point, or a *node*, we mean a complex singularity analytically equivalent to

$$\{xy - zw = 0\} \subset \mathbb{C}^4.$$

There are two ways to desingularize the node. One is by considering its deformation, or the *smoothing*

$$\{xy - zw = \epsilon\} \subset \mathbb{C}^4$$

which is a complex symplectic smooth hypersurface equipped with the induced symplectic structure on $\mathbb{C}^4$. It is symplectomorphic to the total space of the cotangent bundle of a 3-sphere, no matter ϵ is, while its complex structure depends on ϵ. The other desingularisation is the *small resolution*. We first blow up the singular point, getting a smooth complex manifold with an exceptional divisor $\mathbb{C}P^1 \times \mathbb{C}P^1$, then blow down either $\mathbb{C}P^1$. We have two choices of $\mathbb{C}P^1$ to blow down and the resulting manifolds are related by a flop. The complex structure on either one is canonical while the symplectic structure depends on the size of $\mathbb{C}P^1$.

As a complex manifold, the small resolution is the total space of the holomorphic vector bundle $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{C}P^1$. We say a *conifold transition* by passing from one desingularisation to the other.

Beyond this local picture, the conifold transition was generalized in [37] as a surgery of symplectic 6-manifolds, replacing a Lagrangian 3-sphere by a holomorphic $\mathbb{C}P^1$ with a correct normal bundle. In order to patch the local parameters together, some topological conditions on the symplectic manifold are needed.

Theorem 5.2. *(Theorem 2.9, [37]) Fix a symplectic 6-manifold X with a collection of n disjoint embedded Lagrangian 3-spheres S_i. There is a "good" relation*

$$\sum_i a_i[S_i] = 0 \in H_3(X;\mathbb{Z}), \quad a_i \neq 0 \quad \forall i$$

if and only if there is a symplectic structure on one of the 2^n choices of conifold transitions of X in the Lagrangian S_i, such that the resulting $\mathbb{C}P^1$s are symplectic.

One interesting question is that how symplectic invariants change under conifold transitions. The closed string case, like quantum cohomology, has been more studied by algebraic geometry and by symplectic sum constructions. The open string case like Floer theory is less touched, in particular for a global symplectic manifold, and we will explore some points in this note.

5.3 An example about the quadric hypersurface

Now we discuss a motivating example about the quadric hypersurface. Let

$$Q_3 = \{[z_0, \cdots, z_4] \in \mathbb{C}P^4 \mid z_0^2 = \sum_{j=1}^4 z_j^2\}$$

be the quadric hypersurface in $\mathbb{C}P^4$. It is a monotone symplectic manifold with the induced symplectic structure. And the real part $Q_{3,\mathbb{R}} = Q_3 \cap \mathbb{R}P^4$ is a Lagrangian 3-sphere. We can also obtain Q_3 by performing symplectic cutting on the boundary of a suitable disk bundle of T^*S^3. Then the zero section corresponds to the real part $Q_{3,\mathbb{R}}$ and the boundary of the disk bundle, after quotienting the Hamiltonian S^1-action, becomes the divisor at infinity which is isomorphic to $\mathbb{C}P^1 \times \mathbb{C}P^1$. In this point of view the quadric hypersurface is the "simplest" compactification of T^*S^3 by adding one divisor at infinity.

Note that the symplectic cutting behaves well with respect the moment map

$$\pi : T^*S^3 \to P \subset \mathbb{R}^3$$

we get a singular toric fibration

$$\pi : Q_3 \to P_Q \subset \mathbb{R}^3$$

of Q_3. The new polytope P_Q will be cut out by five affine functions

$$x \geq 0; \quad -y \geq 0; \quad x - z \geq 0; \quad z - y \geq 0; \quad y - x + 1 \geq 0.$$

So compared with the polytope of T^*S^3 there is one more face $y - x + 1 = 0$, which corresponds to the divisor at infinity. Here we fix the constant 1 just for simplicity. The symplectic manifold of the polytope P_Q is only isomorphic to the actual hypersurface Q_3 up to a conformal parameter.

By using the toric degeneration method in [31] the disk potential function of regular fibers can be explicitly computed. For example, over the point $(\frac{1}{3}, -\frac{1}{3}, 0)$ there is a monotone Lagrangian 3-torus L. Its disk potential function is

$$\mathfrak{PO}(b) = x + y^{-1} + xz^{-1} + y^{-1}z + x^{-1}y, \quad b \in H^1(L; \Lambda_0). \tag{5.2}$$

Compared with the case in T^*S^3, there is one more term in the potential function due to the new divisor at infinity. Directly we can check that the new potential function has three critical points, which shows that L carries three different local systems as three different objects in the monotone Fukaya category of Q_3.

Moreover, by the work of Smith [36] the Lagrangian sphere $Q_{3,\mathbb{R}}$ split-generates the monotone Fukaya category with eigenvalue zero. (It also follows from Evans-Lekili [14] since $Q_{3,\mathbb{R}}$ is a Lagrangian $SU(2)$-orbit.) Note that the sum of Betti numbers of Q_3 is four. Therefore the sphere and the monotone torus with three bounding cochains split-generate the whole monotone Fukaya category.

Since the Lagrangian sphere $Q_{3,\mathbb{R}}$ is homologically trivial we can perform conifold transition on it. The resulting manifold $\widetilde{Q_3}$ happens to be toric and one can check that the critical loci of the potential function are six toric fibers with bounding cochains, which match the sum of Betti numbers of $\widetilde{Q_3}$. *Therefore three torus branes are merged and transformed into a sphere brane under the (reversed) conifold transition!* This is a 6-dimensional analogue of 4-dimensional phenomenon in [21], where the "baby conifold transition" of the second quadric hypersurface $Q_2 = \mathbb{C}P^1 \times \mathbb{C}P^1$ was studied.

Hence motivated by [21] all the Lagrangian tori over the line in the polytope connecting the sphere brane and the monotone torus brane are expected to be nondisplaceable. The proof of the 4-dimensional case in [21] considers the bulk-deformed potential functions of these tori, which have critical points for particular bulk deformations. However the same technique fails in our 6-dimensional situation. One reason is that the topology of Q_3 is "too simple" for us. To compute the bulk-deformed potential function explicitly one often uses divisors as bulk deformations. The only 4-cycle of Q_3 is the divisor at infinity. After direct computations we find it does not help us to produce critical points of potential functions of our Lagrangians. This motivates us to use other faces of the polytope as bulk deformations. However, the preimages of other four faces attaching the Lagrangian sphere are four 4-chains, not 4-cycles since they bound the 3-sphere. And we cannot naively use chains as bulk deformation since the squares of some boundary operators are not zero.

If we want to use those 4-chains to perturb the Floer cohomology of our toric fiber, the key problem is to cancel the "boundary effect" of these chains. To achieve this goal we introduce the moduli space of holomorphic cylinders.

Another direction which avoids using these 4-chains is to look at other nodal toric Fano 3-folds. In particular when the second Betti number is large. Then there are more 4-cycles to do bulk deformation and one is more likely to prove the local tori are nondisplaceable since there are more parameters. As we mentioned in the introduction, there is a full classification [27] of 100 nodal toric Fano threefolds where one can do computations explicitly.

5.4 Weakly unobstructedness of local tori

In the last subsection we compactify $D_rT^*S^3$ to be an almost toric manifold such that our local tori become toric fibers. A direct consequence is that the local tori are weakly unobstructed by the structure theorem of holomorphic disks in a Gelfand-Tsetlin fibration. And Condition 1.1 is satisfied. However, for a general symplectic 6-manifold X containing a Lagrangian sphere S, to show that local tori near S are weakly unobstructed is not easy. For example, in the general toric case without assuming the Fano condition, the weakly unobstructedness [19] is proved by using the T^n-action on moduli space of disks. Back to our case, we can first relax (3) in Condition 1.1 to allow J-holomorphic spheres with zero first Chern numbers, as indicated in Remark 3.6 [4]. Next we may use the following theorem from Charest-Woodward, see Chapter 7 and 8 in [7].

Theorem 5.3. *Let (X^{2n}, E, ω) be a rational symplectic manifold with an exceptional divisor E. That is, $E \simeq \mathbb{C}P^{n-1}$ with normal bundle isomorphic to $\mathcal{O}(-1)$. Let L be a local toric fiber near E. Then there exists suitable perturbation data such that the Fukaya algebra of L is weakly unobstructed. Moreover, we have that*

$$H^1(L; \Lambda_0) \subset \mathcal{M}_{weak}(L)$$

hence for any $b \in H^1(L; \Lambda_0)$ the Floer cohomology $HF(L, b)$ is well-defined.

In [7] the weakly unobstructedness is also shown for (local) toric fibers near a reverse flip and for the Clifford torus in a Darboux chart. Their method seems very likely to be applied to our case for any rational symplectic manifold, since our local tori live in a Fano almost toric piece Q_3 after degeneration. That is, when X is rational we hope to prove that the local tori are always weakly unobstructed without assuming Condition 1.1.

But currently we still assume that our local torus satisfies Condition 1.1 for some J.

5.5 Holomorphic disks and cylinders

Let X be a symplectic 6-manifold and S be a Lagrangian 3-sphere in X. We fix a Weinstein neighborhood U of S such that there is a singular toric fibration on U, as we described in the previous subsection. Topologically U is isomorphic to $S^3 \times B^3$ where B^3 is a 3-ball. The preimages of four faces in the moment polytope are four 4-chains $K_i = \pi^{-1}(P_i), i = 1, 2, 3, 4$. Each of them is homeomorphic to $S^3 \times [0, 1]$ with two boundary components. Up to orientation $\partial_0(K_i)$ is the zero section S and $\partial_1(K_i)$ is the generator of $H_3(\partial U; \mathbb{Z})$. First we study some topological condition on S to perform the conifold transition. Let V be a small closed neighborhood containing $X - U$ such that $U \cap V$ is homeomorphic to $S^3 \times S^2 \times [1 - \epsilon, 1]$.

Lemma 5.4. *The Lagrangian sphere S is homologically trivial in X if and only if the inclusion*

$$i : H_3(U \cap V; \mathbb{Z}) \to H_3(V; \mathbb{Z})$$

is trivial.

Proof. Note that $U \cap V$ is homeomorphic to $S^3 \times S^2 \times [1 - \epsilon, 1]$. Consider the Mayer-Vietoris sequence

$$\cdots \to H_3(U \cap V) \to H_3(U) \oplus H_3(V) \to H_3(X) \to H_2(U \cap V) \to \cdots .$$

The inclusion

$$j : H_3(U \cap V) \to H_3(U)$$

is an isomorphism. Hence if the inclusion i is trivial then the composition of two maps

$$H_3(U \cap V) \to H_3(U) \oplus H_3(V) \to H_3(X)$$

is the inclusion $H_3(U) = H_3(U \cap V) \to H_3(X)$, which is zero by the exactness. So S is homologically trivial since $H_3(U)$ is generated by our sphere S.

On the other hand if S is homologically trivial then S bounds a 4-chain K in X. We consider another 3-sphere $S' = S \times \{p\} \times \{1\} \in S^3 \times S^2 \times [1 - \epsilon, 1] = U \cap V$. Then S' bounds a 4-chain K' in X, constructed by a concatenation of K and $S \times \{p\} \times [0, 1]$. Next by a relative Mayer-Vietoris sequence we have that

$$\begin{aligned} &\to H_4(U \cap V) \to H_4(U) \oplus H_4(V, S') \to H_4(X, S') \to \\ &\to H_3(U \cap V) \to H_3(U) \oplus H_3(V, S') \to . \end{aligned}$$

Note that $H_4(U) = H_4(U \cap V) = \{0\}$ and that the last map $H_3(U \cap V) \to H_3(U) \oplus H_3(V, S')$ is injective. So we get $H_4(V, S') \to H_4(X, S')$ is an isomorphism. Therefore the 4-chain K' is homologous to some 4-chain contained in V with boundary S', which shows that S' is homologically trivial in V. That is, the inclusion i is trivial since S' generates $H_3(U \cap V)$. □

Now let (X, S, U) be a triple of a symplectic 6-manifold and a homologically trivial Lagrangian sphere with a Weinstein neighborhood. Then S bounds a 4-chain K hence satisfies the "trivial good condition" in Theorem 4.2. Also by the above lemma all the four K_i's can be completed into four 4-chains in X. In other words the boundary $\partial_1(K_i)$ can be capped in $X - U$. From now on we only consider those "completed" chains and still write them as K_i. So K_i's are 4-chains in X such that $\partial(K_i) = \pm S$ and $K_i \cap U$ is the preimage of P_i. We remark that there may be different choices of K but those differences happen in $H_4(X;\mathbb{Z})$, which can be made away from U. For example any chain $K + A$ for $A \in H_4(X)$ is another choice of a chain with boundary S. Now we fix a "completion" for each K_i and regard them as 4-chains in X. When we consider a local torus $L \subset U$ those K_i's are different elements with one relation in $H_4(X - L;\mathbb{Z})$, or strictly speaking $H_4(X - L, S;\mathbb{Z})$. In order to enumerate the disk classes we need to compute the relative homology group $H_2(X, L;\mathbb{Z})$. (Strictly speaking, the disk classes are in the image of the Hurewicz map $\pi_2(X;\mathbb{Z}) \to H_2(X;\mathbb{Z})$.)

Lemma 5.5. *The relative homology group satisfies that*

$$H_2(X, L;\mathbb{Z}) \cong H_1(L) \oplus H_2(X).$$

Proof. The relative homology exact sequence gives that

$$H_2(L) \to H_2(X) \to H_2(X, L) \to H_1(L) \to H_1(X).$$

Note that L is homologically trivial in U hence also homologically trivial in X, the two inclusions $H_i(L) \to H_i(X), i = 1, 2$ are trivial maps. Then we have that

$$0 \to H_2(X) \to H_2(X, L) \to H_1(L) \to 0.$$

Since $H_1(L) \cong \mathbb{Z}^3$ is free the above short exact sequence splits. □

Roughly speaking when we count holomorphic disks representing a class $\beta \in H_2(X, L)$, the part of $H_1(L) \cong H_2(U, L)$ can be regarded as local contributions and the $H_2(X)$ part will be the contributions from outside.

Now let L be a local torus sufficiently near S and J be a compatible almost complex structure on X satisfying Condition 1.1. We want to study J-holomorphic disks with boundary on L of Maslov index two. When the energy of a holomorphic disk is small, its image lies in the neighborhood U of S, by a modification of Corollary 3.5 or the monotonicity lemma for holomorphic disks [28]. Due to the local classification there are four classes of them, which we call local disk classes $\beta_1, \cdots, \beta_4$. And $\beta_i \cdot K_j = \delta_{ij}$ for homological intersection numbers. Here we regard K_i as a 4-chain in X with just one boundary component S. Note that different "completions" of K_i happen in $H_4(X;\mathbb{Z})$ which can be made away from U. So the intersection number of K_i with β_j does not depend on those choices. But there may be other disk classes which intersect K_i since they can intersect K_i outside U. Their images cannot be totally contained in U so we do not focus on them at the moment. We emphasize that when we write class β_i we mean one of the four local classes.

Let $\mathcal{M}_{l,k}(\beta_i)$ be the set of J-holomorphic maps

$$u : (D, \partial D) \to (X, L)$$

with l interior marked points and k boundary marked points modulo automorphism, representing the class β_i. We first study the case when $l = k = 1$. Let $\mathcal{M}_{1,1}(\beta_i, K_i)$ be the moduli space of holomorphic disks of class β_i with marked points and the interior marked point is mapped to K_i. Suppose that L is close to S then the class β_i has minimal energy among all holomorphic disks, due to the local classification. And this moduli space is compact because of the absence of sphere and disk bubbles. However since K_i is a chain with boundary S, the moduli space might have a codimension one boundary when the interior marked point goes to the boundary of K_i. That is,

$$\partial\mathcal{M}_{1,1}(\beta_i, K_i) = \mathcal{M}_{1,1}(\beta_i, S)$$

where $\mathcal{M}_{1,1}(\beta_i, S)$ is the moduli space of holomorphic disks of class β_i with one interior marked point and the image of this marked point lies on S. Next we will show that when the almost complex structure is nice, the moduli space $\mathcal{M}_{1,1}(\beta_i, S)$ is empty hence $\mathcal{M}_{1,1}(\beta_i, K_i)$ is closed.

Proposition 5.6. *With respect to some almost complex structure J we have that $\mathcal{M}_{1,1}(\beta, S)$ is empty. Here β is any disk class with Maslov index two.*

Proof. The proof uses the degeneration technique in Section 3. First we fix a Weinstein neighborhood U of S and L is in U. Then we chose a smaller Weinstein neighborhood $U' \subset U$ of S such that L is not in U'. That is, U' is symplectomorphic to $D_{r'}T^*S^3$ with the canonical symplectic form with a smaller r'. Now we study the naive moduli space of holomorphic disks without any boundary marked points.

The boundary $\partial U'$ is a contact hypersurface in $U \subset X$. We perform the neck stretching operation in symplectic field theory along $\partial U'$. Equivalently we degenerate the almost complex structures through a sequence J_k. Let $\{u_k\}$ be a sequence of J_k-holomorphic disks representing a disk class β. The limit u_∞ is a broken holomorphic building. Then we collapse the Reeb orbits in $\partial U'$. The resulting curve is a nodal curve with one boundary component on L.

Next we do dimension counting to show that there is no component in the top level. After quotienting the S^1-action on $\partial U'$ the top level U' becomes the quadric hypersurface Q_3. The bottom level $X - U'$ becomes the (big) resolved side of the conifold transition. One can think that we collapse the neighborhood U' to a node then resolve it. In particular, our local torus becomes a (local) toric fiber in the bottom level hence it still satisfies Condition 1.1. Suppose that for the nodal curve the component in the quadric is of class A and the component not in the quadric is of class β'. If $A \neq 0$ we write s as the intersection number of A and the divisor $Q_2 \subset Q_3$. Then we have the Maslov index formula

$$\mu(\beta) = 2c_1(TQ_3)(A) + \mu(\beta') - 4s = 2s + \mu(\beta') \geq 2s + 2.$$

The second equality uses that $c_1(TQ_3) = 3[Q_2]$ and the last inequality uses Condition 1.1 so that $\mu(\beta') \geq 2s \geq 2$. Therefore if the class β has Maslov index two then the nodal curve can not have a component in the top level Q_3. That is, for a Maslov index two class β, the images of all holomorphic disks representing β with respect to some J do not intersect U'. □

Remark 5.7. The above proposition tells us that when the complex structure is good and there is no Hamiltonian perturbation of L, there is actually *no* holomorphic disks touching S representing certain classes. One essential reason is that $D_rT^*S^3$ is "positive enough" to force holomorphic curves lie outside a neighborhood of S. This fact is also proved in Section 7 of [15] by a SFT stretching argument to identify open Gromov-Witten invariants under conifold transitions.

Hence with respect to the above almost complex structure J, the moduli space $\mathcal{M}_{1,1}(\beta_i, K_i)$ is closed and carries a fundamental cycle with dimension three. Define

$$n_i := \deg(ev_0 : \mathcal{M}_{1,1}(\beta_i, K_i) \to L).$$

Then with the help of the conifold transition we can relate these numbers n_i on the smooth side with corresponding numbers n'_i on the resolved side.

Corollary 5.8. *The corresponding one-pointed open Gromov-Witten invariants with the same class are equal. That is,*

$$n_i = n'_i = 1, \quad \forall i = 1, 2, 3, 4.$$

Proof. From the above proposition, for any regular J satisfying Condition 1.1 and being admissible with the degeneration, there is a small neighborhood U' such that the images of J-holomorphic disks representing β_i do not intersect U'. In particular it works for the toric complex structure, away from the sphere S. We first collapse the sphere to a point such that locally our Weinstein neighborhood U becomes a toric orbifold. Then we blow up the orbifold point in a toric way. Since all our holomorphic disks are away from S, we can assume this blow up does not affect the moduli spaces $\mathcal{M}_{1,1}(\beta_i, K_i)$. Of course there will be a new disk class corresponding the exceptional divisor but the old moduli spaces are the same.

Then the number $n'_i := \deg(ev_0 : \mathcal{M}_{1,1}(\beta_i, K_i) \to L)$, which is defined in the resolved toric side, is known to be one. Since the moduli space and evaluation map are the same as those in the smooth side, we obtain that $n_i = n'_i = 1$. □

Note that the dimension counting argument in Proposition 5.6 only works for disk classes with Maslov index two. (Under Condition 1.1, holomorphic disks with Maslov index two are minimal.) And to define the Floer cohomology we also need to consider holomorphic disks with Maslov index four. Let β be a disk class with Maslov index four, the moduli space $\mathcal{M}_{1,1}(\beta, K_i)$ may have a codimension one boundary component when the interior marked point going to $\partial K_i = S$. And we can not exclude it as a priori. Next we use the moduli space of cylinders to cancel this possible boundary component.

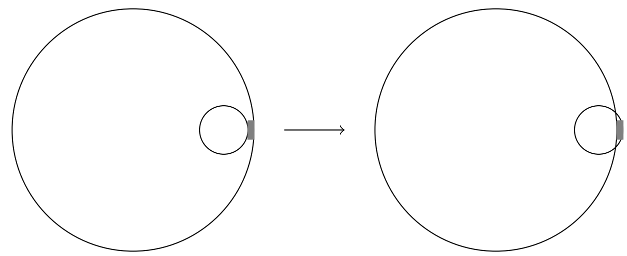

Figure 4: Degeneration when circle ends meet.

We will construct another moduli space $\mathcal{M}^{cy}_{1,1}(\beta, S)$. The elements in the moduli space $\mathcal{M}^{cy}_{1,1}(\beta, S)$ are holomorphic cylinders with two Lagrangian boundary conditions, one on L and one on S.

We write the domain as

$$A_{\epsilon,p} = \{z \in \mathbb{C} \mid |z| \leq 1, |z - p| \geq \epsilon, \epsilon < 1 - |p|\}$$

where $0 < \epsilon < 1$ is a conformal parameter and p is a point in the (open) unit disk. Topologically the domain is an annulus with two disjoint boundaries C_ϵ and C_1. With respect to an almost complex structure J in Proposition 5.6, we consider the J-holomorphic maps

$$\{u : A_{\epsilon,p} \to X \mid u(C_1) \in L, u(C_\epsilon) \in S\}.$$

And the moduli space $\widetilde{\mathcal{M}}^{cy}_{1,k}(\beta, S)$ contains all such maps u representing a homotopy class β with one marked points on the boundary C_1, modulo automorphisms. Note that S is simply-connected, the set of all such class β can be identified with the relative homology group $H_2(X, L)$. The moduli space $\widetilde{\mathcal{M}}^{cy}_{1,k}(\beta_i, S)$ are not compact since there will be domain degenerations. Next we compactify this moduli space.

Theorem 5.9. *There is a compactification* $\mathcal{M}^{cy}_{1,1}(\beta, S)$ *of* $\widetilde{\mathcal{M}}^{cy}_{1,1}(\beta, S)$*, such that it has a unique codimension one boundary component*

$$\partial^{cy}\mathcal{M}^{cy}_{1,1}(\beta, S) = -\mathcal{M}_{1,1}(\beta, S)$$

with respect to suitably chosen orientations.

Proof. The construction of the compactification is by adding all possible degenerations. And the verification of the compactness will be proved by a gluing method.

First we consider the case when p is fixed but ϵ goes to zero. Then in the limit we add a holomorphic disk with one interior point attaching on S. Conversely we need to do the gluing to resolve this interior point. The gluing analysis here is similar to the gluing when one study open Gromov-Witten theory and the boundary class of the given disk class is trivial. We describe the construction here following Proposition 3.8.27 and Subsection 7.4.1 in [17].

For a holomorphic disk with an interior point mapping to S, the idea to "blow up" this interior point to get a holomorphic cylinder is first glue a constant disk to this point then convert this boundary gluing to a interior gluing. Let $D(1)$ be the unit disk in $\mathbb{C}$. Consider a holomorphic map

$$u : D(1) \to X, \quad u(\partial D(1)) \in L$$

with two marked points. One marked point $z_0 = (1, 0)$ on the boundary and one interior marked point $w_0 = (0, 0)$ with $u(w_0) \in S$. Let $D(\sigma)$ be a small disk with one boundary marked point z_1 and Σ be a nodal surface such that

$$\Sigma = D(1) \sqcup D(\sigma) \big/ (0, 0) \sim (0, 0).$$

Then we consider a holomorphic map w_u, which is induced from u.

$$w_u(z) = \begin{cases} u(z) & z \in D(1), \\ u(w_0) & z \in D(\sigma). \end{cases}$$

That is, the restriction of the map w_u on $D(\sigma)$ is the constant map. Next, several standard steps give us the gluing conclusion.

1. First we smooth the singular point of Σ as an interior singular point to get the pregluing map, without being holomorphic.

2. Then we apply the implicit function theorem to get a genuine holomorphic cylinder with two boundary marked points z_0 and z_1. Here z_0 is on the positive boundary and z_1 is on the negative boundary.

3. We forget the marked point z_1 by a forgetful map. The image of the forgetful map is parameterized by the small disk $D(\sigma)$.

4. In the end we check that the implicit function theorem and the forgetful map is S^1-equivariant with respect to the standard rotation action on $D(\sigma)$. And we modulo this action to obtain a neighborhood of u as $u \times D(\sigma) \big/ S^1 = u \times [0, \sigma)$.

This cylinder-to-disk degeneration gives us a codimension one boundary component $\partial^{cy} \mathcal{M}^{cy}_{1,1}(\beta, S)$, which matches the moduli space $\mathcal{M}_{1,1}(\beta, S)$ up to an orientation.

The second case is that p is fixed and ϵ goes to $1 - |p|$. That is, two boundary C_1 and C_ϵ meet. In the limit a small region between these two circle boundaries converges to a holomorphic strip, see Figure 4. Since this strip splits from a finite energy map, itself also has finite energy. Hence the two ends of the strip converge to intersection points of S and L, which is empty. In conclusion such a degeneration does not happen.

The third case is that when ϵ goes to zero and p goes to C_1.

1. When $\lim \frac{\epsilon}{1-|p|} = c > 0$. Then by a conformal change the domain becomes a disk with an annulus bubble, with the modulus of the annulus bubble determined by c. Note that our class β has Maslov index four. By Condition 1.1 the only possible case is a disk and an annulus with both Maslov index two. However, an annulus with Maslov index two can be excluded in the same way in Proposition 5.6. Hence there is no such a degeneration.

2. When $\lim \frac{\epsilon}{1-|p|} = 0$. It is similar case as above, the annulus bubble become actually a disk bubble. So we have two disks with both Maslov index two, one has an interior point attached on S. This degeneration can be excluded by Proposition 5.6.

3. When $\lim \frac{\epsilon}{1-|p|} = +\infty$. Then two circle boundaries meet much faster than ϵ goes to zero. This degeneration will end up with a holomorphic strip as in the second case. So we exclude it in the same way.

Other cases include disk and sphere bubbles. The only possible disk bubble has Maslov index two, which gives an annulus with Maslov index two. So we exclude it as above. The sphere bubble will be a codimension two (or higher) phenomenon hence we do not discuss it here.

In conclusion we add all possible degenerations to compactify the moduli space. And there is a unique codimension one boundary component which comes from the circle boundary C_ϵ shrinking to a point. □

Then we can glue the two moduli spaces along the common boundary to obtain a new moduli space.

Corollary 5.10. *For a disk class β with Maslov index four there are fundamental chains on $\mathcal{M}_{1,1}(\beta, K_i)$ and $\mathcal{M}^{cy}_{1,1}(\beta, S)$ such that we can glue them along their boundaries to obtain a moduli space*

$$\mathcal{M}_{1,1}(\beta, K_i + S) = \mathcal{M}_{1,1}(\beta, K_i) \sqcup \mathcal{M}_{1,1}(\beta, S) / \partial\mathcal{M}_{1,1}(\beta, K_i) \sim -\partial\mathcal{M}^{cy}_{1,1}(\beta, S).$$

Now we consider the case when there are more boundary marked points to insert more data as inputs. Given $x_1, \cdots, x_k$ being singular chains in L and a general class $\beta \in H_2(X, L)$ with Maslov index two or four. Then holomorphic disks representing β may split. Let $\mathcal{M}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i)$ be the moduli space of holomorphic disks with one interior marked point attached on K_i, with $k+1$ boundary marked points and the last k points attached on $x_1, \cdots, x_k$ respectively. Let $\widetilde{\mathcal{M}}^{cy}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i)$ be the moduli space of holomorphic disks with one interior hole attached on S, with $k+1$ boundary marked points and the last k

points attached on $x_1, \cdots, x_k$ respectively. First we can compactify it on one end, where $\epsilon = 0$, like what we did in Theorem 5.9. Then we deal with other types of possible degenerations by using the general theory in [17]. We write the compactified moduli space as $\mathcal{M}^{cy}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i)$. Therefore we obtained two compact moduli spaces such that they have a common boundary component. Next we glue these two moduli spaces along this particular common boundary, where the interior hole collapses to an interior point.

Theorem 5.11. *With above notations, there are fundamental chains on two moduli spaces $\mathcal{M}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i)$ and $\mathcal{M}^{cy}_{1,k+1}(\beta; (x_1, \cdots, x_k); S)$ such that we can glue them along one of their common boundaries to obtain a moduli space*

$$\begin{aligned} &\mathcal{M}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i + S) \\ =&\mathcal{M}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i) \sqcup \mathcal{M}_{1,k+1}(\beta; (x_1, \cdots, x_k); S) / \sim \end{aligned} \tag{5.3}$$

where the equivalence $\sim$ is

$$\partial \mathcal{M}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i) \sim -\partial \mathcal{M}^{cy}_{1,k+1}(\beta; (x_1, \cdots, x_k); S).$$

By using the first boundary marked point we get a singular chain

$$ev : \mathcal{M}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i + S) \to L.$$

The expected dimension of this virtual fundamental chain is

$$\dim \mathcal{M}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i + S) = \mu(\beta) + k + 1 - \sum_{j=1}^{k} (3 - d_j)$$

where d_j is the dimension of the singular chain x_j.

Proof. We first use the gluing method in Theorem 5.9 to deal with the domain degeneration of holomorphic cylinders. Then the general theory in [17] helps us to compactify the moduli space with respect to disk/sphere bubbles, as well as to insert singular chains by the boundary marked points. In the end we should obtain two compact moduli spaces, each has several codimension one boundary components. Then we glue these two along a common boundary component which comes from the degeneration of holomorphic cylinders. □

These chains

$$ev : \mathcal{M}_{1,k+1}(\beta; (x_1, \cdots, x_k); K_i + S) \to L$$

will play the role of $\mathfrak{q}_{l,k;\beta}$ when the interior marked point is attached on a chain K_i, not a cycle. But we are only able to define it for $l = 0, 1$. Since if there are more than one marked points then we need to study the moduli space of holomorphic maps with more interior holes, which we do not know how to do it so far.

In practice we only use the cases when $k = 0, 1$. For $\mathfrak{b} = PD(K_i)$ we define

$$\mathfrak{q}^{cy,\mathfrak{b}}_{1,1;\beta} : H^*(L) \to H^*(L)$$

by

$$x \mapsto (ev : \mathcal{M}_{1,2}(\beta; x; K_i + S) \to L) \tag{5.4}$$

and extend it linearly over Λ_+. That is, for $\mathfrak{b} = w \cdot PD(K)$ with $w \in \Lambda_+$, we define

$$\mathfrak{q}^{cy,\mathfrak{b}}_{1,1;\beta}(x) := w \cdot \mathfrak{q}^{cy,PD(K)}_{1,1;\beta}(x). \tag{5.5}$$

Similarly we define $\mathfrak{q}^{cy,\mathfrak{b}}_{1,0;\beta}$ as the chain

$$w \cdot (ev : \mathcal{M}_{1,1}(\beta; K_i + S) \to L) \tag{5.6}$$

with coefficient w. Note that our Lagrangian torus is three-dimensional, the operators $\mathfrak{q}^{cy,\mathfrak{b}}_{1,1;\beta}$ are non-zero only when $\mu(\beta) = 2$ or 4 and the chains $\mathfrak{q}^{cy,\mathfrak{b}}_{1,0;\beta}$ are non-zero only when $\mu(\beta) = 2$.

We remark here we abuse the notations between singular chains and cochains via the following conventional Poincaré duality. For a singular chain x in L, the Poincaré dual $PD(x)$, regarded as a current satisfies that

$$\int_x \alpha \mid_x = \int_L PD(x) \wedge \alpha \tag{5.7}$$

for any differential form $\alpha \in \Omega^{\dim x}(L)$. Then we define the operator $\mathfrak{q}^{cy,\mathfrak{b}}_{1,k}$ to be

$$\mathfrak{q}^{cy,\mathfrak{b}}_{1,k} = \sum_\beta \mathfrak{q}^{cy,\mathfrak{b}}_{1,k;\beta} \cdot T^{\omega(\beta)} \tag{5.8}$$

for $k = 0, 1$. By the Gromov compactness theorem the right hand side converges in the non-Archimedean topology. Note that the those operators are initially defined on the tensor product of singular chains. By a homotopy transfer lemma we should be able to consider their "canonical model" where the domain is the cohomology group. The argument is similar to the case of the genuine operators $\mathfrak{q}_{l,k}$ in [17]. So we omit the proof and directly use $\mathfrak{q}^{cy,\mathfrak{b}}_{1,k}$ as in the canonical model.

Next for $\mathfrak{b} = wPD(K_i)$ we define a $\mathfrak{b}$-deformed potential function

$$\mathfrak{PO}^{cy,\mathfrak{b}} : H^1(L; \Lambda_+) \to \Lambda_+.$$

For a group homomorphism

$$\rho : \pi_1(L) = H_1(L; \mathbb{Z}) \to \Lambda_0 - \Lambda_+$$

it can be regarded as an element in $H^1(L; \Lambda_+)$. Then we define

$$\mathfrak{PO}^{cy,\mathfrak{b}}(\rho) = \sum_\beta e^{\rho(\partial\beta)} T^{\omega(\beta)} (\mathfrak{m}_{0;\beta}(1) + \mathfrak{q}^{cy,\mathfrak{b}}_{1,0;\beta}(1)) \tag{5.9}$$

where $\mathfrak{m}_{0;\beta}$ is the (undeformed) A_∞-structure on $H^*(L)$, see Section 2. Here

$$\mathfrak{m}_{0;\beta}(1) = PD([L])(\mathfrak{m}_{0;\beta})$$

and

$$\mathfrak{q}^{cy,\mathfrak{b}}_{1,0;\beta}(1) = PD([L])(\mathfrak{q}^{cy,\mathfrak{b}}_{1,0;\beta})$$

are pairings between cochains and chains, which give us two numbers.

In order to compute this potential function explicitly we need to the numbers $\mathfrak{m}_{0;\beta}(1)$ and $\mathfrak{q}^{cy,\mathfrak{b}}_{1,0;\beta}(1)$. By the degree computation it is enough to only consider β with Maslov index two. Hence Corollary 5.8 tells us the mapping degrees are all one when $\beta = \beta_i$ is a basic disk class. For example when $\mathfrak{b} = wPD(K_1)$, with respect to a chosen basis of $H^1(L;\mathbb{Z})$ (the same basis as in (5.1)), the potential function is

$$\mathfrak{PO}^{cy,\mathfrak{b}}(\rho) = ((1+w)x + y^{-1} + xz^{-1} + y^{-1}z)T^{\omega(\beta)} + H(w,x,y,z,T) \qquad (5.10)$$

where $H(w,x,y,z,T)$ are higher energy terms. Note that for the usual bulk-deformation, the effect of $\mathfrak{b} = wPD(K)$ is e^w, for a cycle K. Here our operators only gives the "zeroth-order" and "first-order" approximation $1+w$.

As we mentioned before, by this cylinder counting we try to use the chain K_i as a bulk deformation. Now we define the $\mathfrak{b}$-deformed Floer complex, analogous to (2.7) and (2.9).

Definition 5.12. For $\mathfrak{b} = w \cdot PD(K_i)$ with $w \in \Lambda_+$ and $\rho \in H^1(L;\Lambda_+)$, we define the operator

$$\partial^\rho_{cy,\mathfrak{b}} : H^*(L;\Lambda_+) \to H^*(L;\Lambda_+)$$

by

$$\partial^\rho_{cy,\mathfrak{b}}(x) = \sum_\beta e^{\rho(\partial\beta)}\mathfrak{q}^{cy,\mathfrak{b}}_{1,1;\beta}(x) \cdot T^{\omega(\beta)}.$$

The deformed complex is defined by

$$(H^*(L;\Lambda_+), d^\rho_{cy,\mathfrak{b}} = \delta^\rho + \partial^\rho_{cy,\mathfrak{b}}).$$

Here δ^ρ is similarly defined as

$$\delta^\rho := \mathfrak{m}^\rho_1(x) = \sum_\beta e^{\rho(\partial\beta)}\mathfrak{m}_{1,\beta}(x) \cdot T^{\omega(\beta)}.$$

Remark 5.13. In this section we define the operators $\mathfrak{m}^\rho_1$ and $\partial^\rho_{cy,\mathfrak{b}}$ by using local systems

$$\rho : \pi_1(L) = H_1(L;\mathbb{Z}) \to \Lambda_0 - \Lambda_+$$

which is different in the usual definition of bulk-deformed potential functions, where weak bounding cochains are used. But under Condition 1.1 there is no disk bubbles with non-positive Maslov indices, these two approaches are the same.

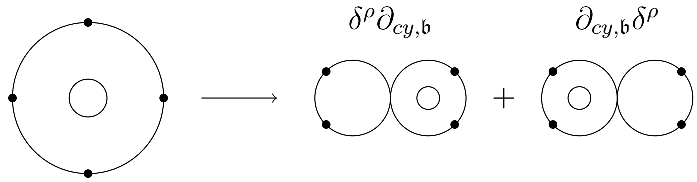

Figure 5: Splitting of disks with one interior hole.

This is proved in Section 4.1 in [24] for the genuine bulk deformation case with all operators $\mathfrak{q}_{l,k;\beta}$. And here we only need to adapt the proof therein for our operators $\mathfrak{q}^{cy,\mathfrak{b}}_{1,k;\beta}$. More precisely, the proof boils down to prove the divisor axiom for the operator $\mathfrak{q}_{1,k;\beta}$, which is given by the integration-along-fiber technique on the moduli spaces of disks, see Section 4.1 in [24] and Lemma 7.1 in [20] for the proof, or Section 3 in [16] for more original statements.

Compared with the operator $\mathfrak{q}_{l,k}$ in Section 2, our deformed operator $d^\rho_{\mathfrak{b}}$ is just a sum of the "zeroth-order" and the "first-order" terms in (2.7). Hence it only gives a cohomology theory modulo some energy.

Proposition 5.14. *The operator $d^\rho_{cy,\mathfrak{b}}$ satisfies that*

$$(d^\rho_{cy,\mathfrak{b}})^2 \equiv 0 \mod T^{2v(\mathfrak{b})}.$$

Hence we have a cohomology modulo $T^{2v(\mathfrak{b})}$ which we write as $HF_{cy}(L;(\mathfrak{b},\rho))$.

Proof. The definition $d^\rho_{cy,\mathfrak{b}} = \delta^\rho + \partial^\rho_{cy,\mathfrak{b}}$ tells that

$$(d^\rho_{cy,\mathfrak{b}})^2 = (\delta^\rho)^2 + \delta^\rho\partial^\rho_{cy,\mathfrak{b}} + \partial^\rho_{cy,\mathfrak{b}}\delta^\rho + (\partial^\rho_{cy,\mathfrak{b}})^2.$$

The first term $(\delta^\rho)^2$ vanishes since δ^ρ itself is a differential, due to the Condition 1.1. The last term $(\partial^\rho_{cy,\mathfrak{b}})^2$ vanishes modulo $T^{2v(\mathfrak{b})}$ by definition, see (5.5).

Next we consider the sum of the second and the third terms $\delta^\rho\partial^\rho_{cy,\mathfrak{b}} + \partial^\rho_{cy,\mathfrak{b}}\delta^\rho$. It vanishes by splitting of holomorphic disks in all possible ways. That is, we study the one-dimensional moduli spaces and look at their boundaries. The sphere bubble is a codimension two phenomenon hence generically we omit it. In the definition of $\partial^\rho_{cy,\mathfrak{b}}$ we glue the moduli space of cylinders with the moduli space of disks with one interior marked points. So two such codimension one boundaries canceled with each other. The only codimension one boundaries are from disk breaking, which result in the sum $\delta^\rho\partial^\rho_{cy,\mathfrak{b}} + \partial^\rho_{cy,\mathfrak{b}}\delta^\rho$, see Figure 5 for a picture. Since they are boundaries of a compact one-manifold, their sum (counted with signs) is zero. □

Remark 5.15. The operator $d^\rho_{cy,\mathfrak{b}}$ is not the bulk-deformed differential defined in Section 2, but an approximation since we only consider the case with one interior marked points. This is the reason why the genuine bulk-deformed differential is a differential but ours is only a differential modulo some energy.

As we mentioned before, if we want to define a genuine differential then we need to consider counting holomorphic disks with arbitrarily many interior holes to cancel the boundary effect that K is not a cycle. However the full version of higher genus Floer theory will be difficult and out of the scope of this note. So we just leave it as a possible direction for the future.

Therefore we obtain a cohomology theory for a fixed bulk chain $\mathfrak{b} = w \cdot PD(K_i)$. Its underlying complex is the singular cohomology of L and its differential counts a combination of holomorphic disks and cylinders. An advantage of this cohomology is that we can do explicit computation by the help of the $\mathfrak{b}$-deformed potential function. For example, the existence of a critical point of the potential function gives us a non-vanishing result of the cohomology.

Proposition 5.16. *If the potential function $\mathfrak{PO}^{cy,\mathfrak{b}}(\rho)$ for L has a critical point for some $(\mathfrak{b}, \rho)$ modulo T^E, $E < 2v(\mathfrak{b})$. Then the deformed Floer cohomology satisfies that*

$$HF_{cy}(L; (\mathfrak{b}, \rho)) \cong H^*(L; \frac{\Lambda_0}{T^E \Lambda_0}) \cong (\frac{\Lambda_0}{T^E \Lambda_0})^{\oplus 8}.$$

Proof. By a direct computation below we can find that if there is a critical point for some $(\mathfrak{b}, \rho)$ modulo T^E then the deformed boundary operator $d^\rho_{cy,\mathfrak{b}} \equiv 0$ modulo T^E. So the cohomology is isomorphic to the underlying complex.

Let $\rho \in H^1(L; \Lambda_+)$ and $e_i, i = 1, 2, 3$ be a set of generators of $H^1(L; \mathbb{Z})$. Then any $b \in H^1(L; \Lambda_+)$ can be written as $\rho(x) = \sum_{i=1}^3 x_i e_i$. For notational simplicity we assume that $\rho(x) = x_1 e_1$. Then we have that

$$\begin{aligned}
\frac{\partial}{\partial x_1}\mathfrak{PO}^{cy,\mathfrak{b}}(\rho(x)) =& \frac{\partial}{\partial x_1}\sum_\beta e^{\rho(\partial\beta)} T^{\omega(\beta)}(\mathfrak{m}_{0;\beta}(1) + \mathfrak{q}^{cy,\mathfrak{b}}_{1,0;\beta}(1)) \\
=& \frac{\partial}{\partial x_1}\sum_\beta e^{x_1 e_1(\partial\beta)} T^{\omega(\beta)}(\mathfrak{m}_{0;\beta}(1) + \mathfrak{q}^{cy,\mathfrak{b}}_{1,0;\beta}(1)) \\
=& \sum_\beta (e_1(\partial\beta)) \cdot e^{x_1 e_1(\partial\beta)} T^{\omega(\beta)}(\mathfrak{m}_{0;\beta}(1) + \mathfrak{q}^{cy,\mathfrak{b}}_{1,0;\beta}(1)) \\
=& \sum_\beta e^{x_1 e_1(\partial\beta)} T^{\omega(\beta)}(\mathfrak{m}_{1;\beta}(e_1) + \mathfrak{q}^{cy,\mathfrak{b}}_{1,1;\beta}(e_1)) \\
=& \delta^\rho(e_1) + \partial^\rho_{cy,\mathfrak{b}}(e_1) = d^\rho_{cy,\mathfrak{b}}(e_1).
\end{aligned} \tag{5.11}$$

The third last equality again uses the divisor axiom, see (2.6). Therefore if all the partial derivatives of $\mathfrak{PO}^{cy,\mathfrak{b}}$ vanishes then our deformed Floer boundary operators vanishes on $H^1(L; \Lambda_+)$. Since L is a torus of which the cohomology is generated by degree one elements, we can perform an induction to show that the deformed Floer boundary operator vanishes on the whole $H^*(L; \Lambda_+)$. We refer to Section

13 in [19] for the induction process and the extension from $\rho \in H^1(L;\Lambda_+)$ to $\rho \in H^1(L;\Lambda_0)$. □

In the next section we will relate the cohomology $HF_{cy}(L;(\mathfrak{b}, b))$ to another model of cohomology such that the underlying complex is generated by Hamiltonian chords with ends on L and a Hamiltonian perturbation $\phi(L)$. The first cohomology $HF_{cy}(L;(\mathfrak{b}, b))$ is for computational purpose and the later cohomology is more geometrical. Once we established the equivalence between these two theories we get a critical points theory to detect the displacement energy of L.

Remark 5.17. In the definition of $HF_{cy}(L;(\mathfrak{b}, b))$ we use the fact that with respect to some J there is no holomorphic disk touching S with Maslov index two. This condition is not necessary, but just for computational purposes since the potential function is explicitly known by the conifold transition.

In general when there is Hamiltonian perturbation, Maslov two disks may touch the sphere. Then we use the same gluing technique to cancel this possible codimension one boundary. More precisely, we will glue the moduli spaces inductively. We start with minimal holomorphic disks. (Under Condition 1.1, holomorphic disks with Maslov index two are minimal.) The corresponding moduli spaces have codimension one boundary where the disks touch the sphere. Then we use the moduli spaces of holomorphic cylinders of the same class to cancel this boundary. Next we move to the disks with Maslov index four, the corresponding moduli spaces are manifolds with boundaries and corners. We first cap the lowest strata coming from the splitted disks (with Maslov index two) touching the sphere. Then we cap the codimension one boundary coming from disks which do not split but touch the sphere. After capping all the strata where disks touching the sphere, the boundary of the capped moduli spaces only contains disk/sphere splittings. Then we can define boundary operators and show that they give us a cohomology modulo some energy.

Therefore Definition 5.12 and Proposition 5.14 should be understood as a special case of capping moduli spaces, where only Maslov four disks are considered, to do concrete computations.

6 A second deformed Floer complex

Now we will construct another deformed Floer complex and study its change of filtration under Hamiltonian diffeomorphisms. With the same notations in the previous section, we fix a triple (X, S, U) and a local torus L inside U. We still assume that S is homologically trivial and fix the choices of completions of K_i such that they are regarded as 4-chains in X.

6.1 Definition of the complex

Let H_t be a time-dependent Hamiltonian function on X and let ϕ be its time-one Hamiltonian diffeomorphism. We first review the Floer complex generated by the Hamiltonian chords with ends on L, which is called the *dynamical* version of Floer theory in [22].

Consider the path space

$$\Omega(L) = \{l : [0,1] \to X \mid l(0) \in L, l(1) \in L\}.$$

We fix a base path $l_a \in \Omega(L)$ for each component $a \in \pi_0(\Omega(L))$. Let $[l,w]$ be a pair such that $l \in \Omega(L)$ and $w : [0,1]^2 \to X$ satisfying

$$w(s,0) \in L, w(s,1) \in L, w(0,t) = l_a(t), w(1,t) = l(t).$$

Then we define the dynamical action functional, with respect to H_t, to be

$$\mathcal{A}_{H_t,l_a}([l,w]) = \int w^*\omega + \int_0^1 H_t(l(t))dt. \tag{6.1}$$

on the space of pairs $[l,w]$. The critical points of this action functional are Hamiltonian chords. We write the set of critical points as

$$CF(L,H_t) = \{[l,w] \mid l'(t) = X_{H_t}(l(t))\}.$$

For a critical point $[l,w]$ the path l corresponds to a geometric intersection point in $L \cap \phi(L)$ since $\phi(l(0)) = l(1) \in L$. When H_t is generic there are only finitely many of them. We remark that the set of critical points has a decomposition with respect to the different components $a \in \pi_0(\Omega(L))$. We define the action functionals and study their critical points on different components separately.

Now we equip L with local systems. For any group homomorphism

$$\rho : \pi_1(L) \to \Lambda_0 - \Lambda_+$$

we choose a flat Λ_0-bundle $(\mathcal{L}, \nabla_\rho)$ such that its holonomy representation is ρ. Then we define the cochain complex as

$$CF((L,\rho),H_t;\Lambda_0) := \bigoplus_{[l,w]\in CF(L,H_t)} \hom(\mathcal{L}_{l(0)}, \mathcal{L}_{l(1)}) \otimes_{\mathbb{C}} \Lambda_0. \tag{6.2}$$

Here $\mathcal{L}_{l(i)}$ is the fiber of the bundle $\mathcal{L}$ over $l(i)$ and $\hom(\mathcal{L}_{l(0)}, \mathcal{L}_{l(1)})$ is the homomorphism induced by the path l.

Next we consider smooth maps

$$u(\tau,t) : \mathbb{R} \times [0,1] \to X, \quad u(\tau,0) \in L, \quad u(\tau,1) \in L$$

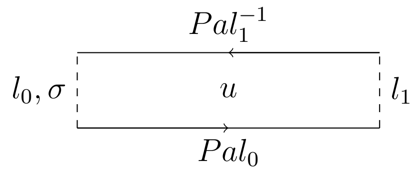

Figure 6: Composition of parallel transport maps.

such that $u(-\infty, t) = l_0(t), u(\infty, t) = l_1(t)$ for some l_0, l_1 to define the parallel transport maps. Let B be the homotopy class of u and $\sigma \in \hom(\mathcal{L}_{l_0(0)}, \mathcal{L}_{l_0(1)})$ then we define

$$Comp_{(B,\sigma)} : \hom(\mathcal{L}_{l_1(0)}, \mathcal{L}_{l_1(1)}) \to \hom(\mathcal{L}_{l_1(0)}, \mathcal{L}_{l_1(1)})$$

by

$$Comp_{(B,\sigma)} = Pal_0 \circ \sigma \circ Pal_1^{-1} \tag{6.3}$$

where Pal_i is the parallel transport along the path $u(\tau, i) \in L$ for $i = 0, 1$, see Figure 6. And the composition map is a homotopy invariant.

Lemma 6.1. *The definition of the composition map only depends on the homotopy class B of u, not on the choice of u.*

Now we can define the Floer coboundary operator with local systems. Let

$$\begin{aligned}&\mathcal{M}([l_0, w_0], [l_1, w_1]) \\ =&\{u(\tau, t) : \mathbb{R} \times [0, 1] \to X \mid \partial_\tau u + J(\partial_t u - X_{H_t}) = 0, \\ &u(\tau, 0) \in L, u(\tau, 1) \in L, u(-\infty, t) = l_0(t), u(\infty, t) = l_1(t)\}\end{aligned}$$

be the moduli space of holomorphic maps connecting $[l_0, w_0]$ and $[l_1, w_1]$. Then for a fixed ρ we define

$$\delta^\rho : CF((L, \rho), H_t; \Lambda_0) \to CF((L, \rho), H_t; \Lambda_0)$$

as

$$\begin{aligned}&\delta^\rho(\sigma \otimes [l_0, w_0]) \\ =&\sum_{[l_1, w_1]} Comp_{(w_0 - w_1, \sigma)} \otimes \sharp\mathcal{M}([l_0, w_0], [l_1, w_1])[l_1, w_1] \cdot T^{\omega([w_1 - w_0])}.\end{aligned} \tag{6.4}$$

Here the sum is over all $[l_1, w_1]$ such that the corresponding moduli space is zero-dimensional. And the number $\sharp\mathcal{M}([l_0, w_0], [l_1, w_1])$ is a signed count.

Proposition 6.2. *Under the Condition 1.1, the coboundary operator is well-defined and satisfies that $(\delta^\rho)^2 = 0$.*

Proof. The proof is similar to the case when a Lagrangian torus is monotone, where the self-Floer cohomology is well-defined, see Theorem 16.4.10 in [33]. Note that Condition 1.1 excludes possible disk bubbles, with non-positive Maslov indices, splitting from the holomorphic strips. For disk bubbles with Maslov index two, they appear in pairs on L and cancel with each other. We do not need to consider disk bubbles with higher Maslov indices since we are looking at a one-dimensional moduli space to show the square of δ^ρ is zero. □

We call the above cohomology given by δ^ρ the Floer cohomology with local systems. Next we want to deform it further by counting strips with an interior marked point/an interior hole.

The aim is to define a new operator

$$\partial_K : CF((L,\rho), H_t; \Lambda_0) \to CF((L,\rho), H_t; \Lambda_0).$$

Here we write K as one of K_i for notational simplicity. First we describe the domain we will use to count holomorphic maps. Consider the domain

$$Strip_{\epsilon,r} = \{(\tau,t) \in \mathbb{R}\times[0,1] \subset \mathbb{C} \mid \tau^2 + (t-r)^2 \geq \epsilon^2\}.$$

Let $C(\epsilon)$ denote the circle boundary $\tau^2 + (t-r)^2 = \epsilon^2$ of $Strip_{\epsilon,r}$. We put the interior hole centered at $(0,r)$ with radius $\epsilon \in (0, \min\{r, 1-r\})$. The radius ϵ determines the complex structure on the domain. And we write $Strip = Strip_{0,r}$ as the usual holomorphic strip in $\mathbb{C}$. We put the τ-coordinate of the center of the circle to be 0 to cancel the translation action.

Now we consider several moduli spaces. For a pair $([l_0,w_0],[l_1,w_1])$ let

$$\widetilde{\mathcal{M}}_1(([l_0,w_0],[l_1,w_1]);K)$$

be the moduli space of holomorphic strips with one interior marked point at $(0,r)$, where the interior point is mapped to K. More precisely, it contains maps $u : Strip \to X$ such that

$$u(\tau,0)\in L, \quad u(\tau,1)\in L, \quad u(-\infty,t)=l_0, \quad u(\infty,t)=l_1$$

and

$$u(0,r)\in K$$

where the map u represents the class $\beta = w_0 - w_1$. And let

$$\widetilde{\mathcal{M}}_1^{cy}(([l_0,w_0],[l_1,w_1]);S)$$

be the moduli space of holomorphic strips with one interior hole, where the hole is mapped to S. It contains maps from domain $Strip_{\epsilon,r}$ for all (ϵ,r). And u satisfies the same Lagrangian boundary condition as above: the line boundaries are mapped to L and two ends converge to given chords l_0, l_1. One extra boundary condition is that the circle boundary is mapped to S.

Figure 7: Counting strips with one interior marked point and one hole.

The elements in both types of moduli spaces satisfy the same perturbed holomorphic equation

$$\partial_\tau u + J(\partial_t u - X_{H_t}) = 0.$$

The differences are that they are from different domains and have different boundary conditions.

Note that $\partial K = S$ and S is simply connected, the homotopy classes in these two types of moduli spaces can be identified. Similar to the discussion in Section 5 we want to compactify these moduli spaces and glue them together along a common boundary for the same class $\beta = w_0 - w_1$.

Proposition 6.3. *For fixed generators $[l_0, w_0]$ and $[l_1, w_1]$, there are compactification*

$$\mathcal{M}_1(([l_0, w_0], [l_1, w_1]); K) \supseteq \widetilde{\mathcal{M}}_1(([l_0, w_0], [l_1, w_1]); K)$$

and compactification

$$\mathcal{M}_1^{cy}(([l_0, w_0], [l_1, w_1]); S) \supseteq \widetilde{\mathcal{M}}_1^{cy}(([l_0, w_0], [l_1, w_1]); S).$$

Each of them has a particular boundary component such that

$$\partial_K \mathcal{M}_1(([l_0, w_0], [l_1, w_1]); K) = -\partial_{cy} \mathcal{M}_1^{cy}(([l_0, w_0], [l_1, w_1]); S)$$

and we can glue them on this component to get a compact moduli space

$$\begin{aligned} &\mathcal{M}_1(([l_0, w_0], [l_1, w_1]); K + S)) = \\ &\mathcal{M}_1(([l_0, w_0], [l_1, w_1]); K) \sqcup \mathcal{M}_1^{cy}(([l_0, w_0], [l_1, w_1]); S)/\sim \end{aligned}$$

where the equivalence relation is

$$\partial_K \mathcal{M}_1(([l_0, w_0], [l_1, w_1]); K) \sim -\partial_{cy} \mathcal{M}_1^{cy}(([l_0, w_0], [l_1, w_1]); S).$$

Proof. To get the compactification we add several types of degenerations: strip breaking, disk/sphere bubbles and domain degeneration. The cases of the strip breaking and disk/sphere bubbles are more standard in Floer theory. So we mainly explain the domain degeneration involving two parameters ϵ and r. The former is the radius of the interior hole and the later is the vertical position of the center of the hole. Suppose that we have a sequence of parameters $\{(\epsilon_i, r_i)\}_{i=1}^{+\infty}$, we will discuss by cases of possible degenerations.

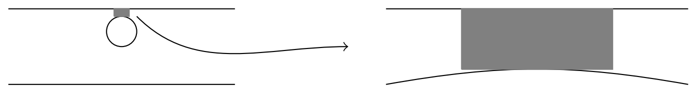

Figure 8: Zoom in on the region where two boundaries meet.

1. If $\inf_i\{\epsilon_i\} > 0$ and $\epsilon_i + r_i \to 1$ or $-\epsilon_i + r_i \to 0$. Geometrically the circle boundary approaches to the strip boundary while the radius of the circle is bounded from below. We will show that this type of degeneration does not happen since our S and L are disjoint. Without losing generality we assume that $\epsilon_i + r_i \to 1$ with $\epsilon_i \equiv \epsilon_0 > 0$ for some constant ϵ_0. Then we can scale a neighborhood of the point $(0, \epsilon_i + r_i)$ such that locally we have a holomorphic strip u_i with one boundary on L and with one curved boundary on S, see Figure 8. To compactify such a degeneration we need to add a genuine holomorphic strip u_∞ in the moduli space, since in the limit the curved boundary becomes a usual boundary. However, note that such a strip u_∞ has finite energy because it splits from a finite energy solution. By exponential decay estimate we know $\lim_{\tau\to\pm\infty} u_\infty(\tau, t)$ converges to the intersections of L and S, which is empty by our assumption. Hence such a degeneration will not appear.

2. If $\epsilon_i \to 0$ and $\{r_i\}$ stays the interior of the strip. In the limit we have a holomorphic strip with one interior marked point. Then we can perform the same gluing argument as we did in Section 5. That is, we glue this end with the moduli space of strips with on interior marked point as we did before, to cancel this end of boundary.

3. If $\epsilon_i \to 0$ and $\{r_i\}$ goes to one strip boundary. Without losing generality we assume that $\lim_i(r_i) = 1$. Then we consider the ratio $\frac{\epsilon_i}{1-r_i}$ and there are different possibilities.

   (a) If $\lim_i \frac{\epsilon_i}{1-r_i} = +\infty$, the case is similar to (1.) and we use the fact $L \cap S = \emptyset$ to exclude this degeneration.

   (b) If $\lim_i \frac{\epsilon_i}{1-r_i} = R$ for some constant $R > 0$, after a conformal change this degeneration is equivalent to an annulus bubble on the boundary. So we put this type of limit of solutions into the compactification.

   (c) If $\lim_i \frac{\epsilon_i}{1-r_i} = R = 0$, then after a conformal change it is a disk bubble, with one interior point attaching to S.

In conclusion, to get the compactification we add broken curves in (2.), (3.b), (3.c) and broken strips. Next we glue the particular boundary component in (2.) with the moduli space of holomorphic strips with one interior marked point, as we did in Theorem 5.9.

We write

$$\partial_K \mathcal{M}_1(([l_0, w_0], [l_1, w_1]); K)$$

as the boundary component containing elements when the interior marked point is mapped to $S = \partial K$. And we write

$$\partial_{cy} \mathcal{M}_1^{cy}(([l_0, w_0], [l_1, w_1]); S)$$

as the boundary component containing elements in (2.). These two boundary components are the same since they contain the same set of elements. Then we glue these two compactified moduli spaces along this common boundary component.

We remark that if the class β is energy minimal then this boundary component is the only boundary part. So after gluing we will get a closed moduli space. □

Now we can define an operator deformed by K. With the fixed ρ we define

$$\partial_K CF((L, \rho), H_t; \Lambda_0) \to CF((L, \rho), H_t; \Lambda_0)$$

as

$$\begin{aligned} &\partial_K(\sigma \otimes [l_0, w_0]) \\ = &\sum_{[l_1, w_1]} Comp_{(w_0 - w_1, \sigma)} \otimes \sharp \mathcal{M}_1(([l_0, w_0], [l_1, w_1]); K + S))[l_1, w_1] \cdot T^{\omega([w_1 - w_0])}. \end{aligned} \tag{6.5}$$

Here the sum is also over all $[l_1, w_1]$ such that the corresponding moduli space is zero-dimensional.

Moreover we can define this operator for $K = w \cdot K_i$ with $w \in \Lambda_+$ by just extending it Λ_+-linearly. That is, we define $\partial_{w \cdot K} := w \cdot \partial_K$. Then we set $d_K^\rho = \delta^\rho + \partial_K$ and study when d_K^ρ gives us a differential.

Proposition 6.4. *The operator d_K^ρ satisfies that*

$$(d_K^\rho)^2 = (\delta^\rho + \partial_K)^2 \equiv 0 \mod T^{2v(K)}.$$

Proof. By definition we have that

$$(d_K^\rho)^2 = (\delta^\rho)^2 + \delta^\rho \partial_K + \partial_K \delta^\rho + (\partial_K)^2.$$

Assuming Condition 1.1 the operator δ^ρ itself is a differential hence $(\delta^\rho)^2 = 0$. The last term $(\partial_K)^2$ vanishes by the filtration reason. We just need to show that $\delta^\rho \partial_K + \partial_K \delta^\rho = 0$. This is obtained by considering one-dimensional moduli spaces of holomorphic cylinder with one interior hole and study the breaking of such strips, see Figure 9. By Proposition 6.3 we have a list of possible degenerations. Now we discuss them by cases.

The first type of degeneration, which is strip breaking, corresponds to the sum $\delta^\rho \partial_K + \partial_K \delta^\rho$.

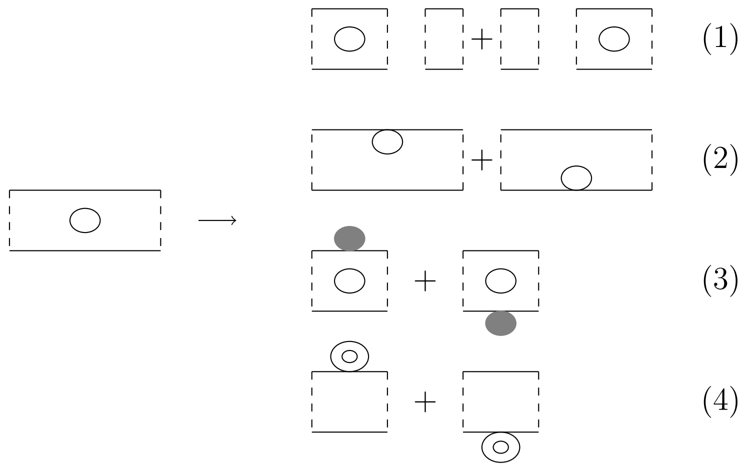

Figure 9: Degenerations of a one-dimensional moduli space.

The second type of degeneration corresponds to disk bubbles with Maslov index two. Since we assume the Condition 1.1 there is no holomorphic disks with non-positive Maslov index. In this case disk bubbles on two components of line boundaries cancel with each other by the invariance of one-point open Gromov-Witten invariants.

The third type of degenerations are annuls bubbles. Note that the moduli spaces of annuli with Lagrangian boundary conditions is one dimension higher than the moduli space of holomorphic disks with the same homotopy class (we use that S is simply connected). So the annuli bubble is at least a codimension two phenomenon by the assumption of Condition 1.1. We ignore it as we ignore the sphere bubbles. Note that by a similar degeneration argument in Proposition 5.6, our Lagrangian torus does not bound annuli with negative indices.

In conclusion the codimension one boundaries of the moduli space are listed in Figure 9. Terms (2) and (4) can not happen by various conditions. Two terms in (3) cancel with each other. So the only contribution is $\delta^\rho \partial_K + \partial_K \delta^\rho$, which corresponds to (1) and should be zero as a signed count. This completes our proof that d_K^ρ is a differential modulo $T^{2v(K)}$. □

Therefore the operator d_K^ρ defines a differential modulo $T^{2v(K)}$ and we can talk about the cohomology modulo this energy. We write this cohomology as

$$HF_{cy}((L,\rho),(L,\rho),H_t;K).$$

In the next subsection we will study how this cohomology behaves with respect to the choice of Hamiltonian H_t. Then we can obtain the desired energy estimate. The key point is that how the energy of a holomorphic strip with one interior hole change under a Hamiltonian diffeomorphism.

Before we deal with a general Hamiltonian diffeomorphism, we look at the case when H_t is C^2-small. Let ϕ be the time-1 flow of H_t. We assume that $L \cap \phi(L)$ is transversal and $S \cap \phi(L) = \emptyset$. Then we can define a similar cohomology theory $HF_{int,cy}((L,\rho),(\phi(L),\rho);K)$ where the underlying complex is generated by intersection points of L and $\phi(L)$. We call it the *intersection model*. The differential is also a sum of two operators, one counts the usual holomorphic strips and the other counts holomorphic strips with one interior hole. Here the pair $(\phi(L),\rho)$ is actually $(\phi(L),(\phi^{-1})^*\rho)$ but for notational simplicity we just write it as $(\phi(L),\rho)$.

Proposition 6.5. *The intersection model gives a cohomology theory*

$$HF_{int,cy}((L,\rho),(\phi(L),\rho);K)$$

with coefficients $\Lambda_0/T^{2v(K)}\Lambda_0$.

Proof. We need to show that the square of the differential is zero. It can be done by the same argument as in Proposition 6.4, using the assumption that both $S \cap \phi(L) = \emptyset$ and $S \cap L = \emptyset$. Since H_t is C^2-small, two Lagrangians L and $\phi(L)$ both satisfy Condition 1.1 with a common J. And the counts of holomorphic disks with Maslov index two are the same. Hence possible disk bubbles on L and $\phi(L)$ cancel with each other. Then the proof in Proposition 6.4 works for this intersection model.

For a general Hamiltonian perturbation, there may happen a wall-crossing phenomenon for holomorphic disks with Maslov index two. So this intersection model is only defined with a small perturbation. □

Remark 6.6. With the assumption that H_t is C^2-small we can prove that these two theories are equivalent as filtered cohomology groups. But we do not need this fact in our following context. The intersection model just plays a transition role between the disk model (coming from the potential function) and the chord model. In practice we will use a chord model of which the generators are chords with one end on L and the other end on $\phi(L)$. And the result about displacement energy will be proved by a limit argument since we can take ϕ arbitrarily small, see Theorem 6.14.

6.2 Change of filtration under Hamiltonian isotopies

Let ϕ be the time-one flow of H_t (not necessarily C^2-small) such that L and $\phi(L)$ intersect transversally. Then the cohomology $HF_{cy}((L,\rho),(L,\rho),H_t;K)$ is well-defined with coefficient $\Lambda_0/T^{2v(K)}\Lambda_0$. We can view the cohomology group as a Λ_0-module. Now we study how the choice of H_t change the cohomology.

This deformed Floer complex is a modification of the Floer complex with bulk deformations and can be regarded as its "first order approximation". Note that the differential is a sum of two operators. The dependence of H_t on the usual differential δ^ρ with local systems is well-studied in [17] and [22]. So we focus on the part which involves the operator ∂_K. Actually we will prove a new energy estimate to construct different chain maps and chain homotopies then the rest argument will follow the same proof in Section 6 and 7 in [22].

First we recall some relevant backgrounds on the *geometric* version of Floer theory and the *dynamical* one.

Let L_0, L_1 be a pair of two closed Lagrangian submanifolds of X. We consider their Hamiltonian deformations L_0', L_1'. That is, there are Hamiltonian isotopies

$$\phi_{H_0} = \{\phi_{H_0}^s\}_{0\le s\le 1}, \quad \phi_{H_1} = \{\phi_{H_1}^s\}_{0\le s\le 1}$$

such that

$$\phi_{H_0}^1(L_0) = L_0', \quad \phi_{H_1}^1(L_1) = L_1'.$$

Set

$$\psi^t = \phi_{H_0}^t \circ (\phi_{H_0}^1)^{-1} \circ \phi_{H_1}^{1-t} \circ (\phi_{H_1}^1)^{-1} \tag{6.6}$$

and

$$\tilde{H}_t = H_{0,t} - H_{1,1-t} \circ \phi_{H_0}^1 \circ (\phi_{H_0}^t)^{-1}. \tag{6.7}$$

Then one can directly check that $\psi^0 = (\phi_{H_0}^1)^{-1}, \psi^1 = (\phi_{H_1}^1)^{-1}$ and

$$\frac{d}{dt}\psi^t(p) = X_{\tilde{H}_t}(\psi^t(p)).$$

Now we fix the pairs L_0, L_1 and L_0', L_1'. The geometric version of the Floer complex $CF^*(L_0', L_1')$ is generated by the intersection points

$$p \in L_0' \cap L_1'$$

where p can be regarded as a constant element in the path space

$$\Omega(L_0', L_1') = \{l : [0,1] \to X \mid l(0) \in L_0', l(1) \in L_1'\}.$$

We fix a base path $l_a' \in \Omega(L_0', L_1')$ for each component $a \in \pi_0(\Omega(L_0', L_1')$. Let $[l, w]$ be a pair such that $l \in \Omega(L_0', L_1')$ and $w : [0,1]^2 \to X$ satisfying

$$w(s,0) \in L_0', w(s,1) \in L_1', w(0,t) = l_a'(t), w(1,t) = l(t).$$

Then we define the geometric action functional

$$\mathcal{A}_{l_a'}([l,w]) = \int w^*\omega \tag{6.8}$$

on the space of pairs $[l, w]$.

For the above Lagrangian submanifolds L_0, L_1 and a time-dependent Hamiltonian $\tilde{H}_t$, the dynamical version of the Floer complex is generated by the solutions of Hamilton's equation

$$\{x \in \Omega(L_0, L_1) \mid \dot{x} = X_{\tilde{H}_t}(x)\}.$$

For a fixed base path l_a and a pair $[x, w]$, the dynamical action functional is defined as

$$\mathcal{A}_{\tilde{H}_t, l_a}([x, w]) = \int w^* \omega + \int_0^1 \tilde{H}_t(l(t))dt. \tag{6.9}$$

Here the base path l_a is given by $l_a(t) = \psi^t(l'_a(t))$.

Now the two versions of Floer complexes can be regarded as filtered complexes with respect to their action functionals. And those two Floer theories are related by the following transformation. For a generator $[l', w']$ of the geometric version Floer theory and a generator $[l, w]$ of the dynamical Floer theory, we have that

$$\mathfrak{g}^+_{H_0, H_1} : [l', w'] \mapsto [l, w]$$

given by

$$l(t) = \psi^t(l'(t)), \quad w(s, t) = \psi^t(w'(s, t)).$$

This map $\mathfrak{g}^+_{H_0, H_1}$ preserves the action up to a constant

$$c(\tilde{H}_t; l_a) := \int_0^1 \tilde{H}_t(l_a(t))dt.$$

That is,

$$\mathcal{A}_{\tilde{H}_t, l_a} \circ \mathfrak{g}^+_{H_0, H_1}([l', w']) = \mathcal{A}_{l'_a}([l', w']) + c(\tilde{H}_t; l_a),$$

see Lemma 4.2 in [22]. Also by the discussion therein we can make this constant to be zero by choosing the base chord l_a properly. So in the following we forget this constant term in our estimates.

Next we introduce the notion of the *perturbed* Cauchy-Riemann equation to study the relation between these two versions of Floer theories. Let $\chi_+(\tau) : \mathbb{R} \to \mathbb{R}$ be a smooth function such that

$$\chi_+(\tau) = \begin{cases} 0 & \tau \leq -2, \\ 1 & \tau \geq -1, \end{cases} \quad \chi'_+(\tau) \geq 0$$

and $\chi_-(\tau) = 1 - \chi_+(\tau)$. Also we will use a family of smooth bump functions $\chi_N(\tau)$ for $N \geq 1$, satisfying

$$\chi_N(\tau) = \begin{cases} 0 & |\tau| \geq N+1, \\ 1 & |\tau| \leq N, \end{cases}$$

and

$$\chi'_N(\tau) \geq 0, \forall \tau \in [-N-1, -N], \quad \chi'_N(\tau) \leq 0, \forall \tau \in [N, N+1].$$

In particular, we assume that on $[-N-1, -N]$ ($[N, N+1]$ respectively) the function χ_N is a translation of χ_+ (χ_- respectively). For $N \leq 1$ we define $\chi_N(\tau) = \chi_1(\tau) \cdot N$ such that $\chi_N(\tau)$ converges to the zero function as N goes to zero.

From now on we assume that our pairs L_0, L_1 and L_0', L_1' intersect with each other transversally. Since we can achieve this by perturbations with arbitrarily small Hamiltonian, this assumption does not affect the conclusions involving estimates of Hofer energy. The perturbed Cauchy-Riemann equation of $u(\tau, t) : \mathbb{R} \times [0,1] \to X$ is the following

$$\begin{cases} \frac{\partial u}{\partial \tau} + J(\frac{\partial u}{\partial t} - \chi(\tau) X_{H_t}(u)) = 0, \\ u(\tau, 0) \in L_0, \quad u(\tau, 1) \in L_1. \end{cases} \tag{6.10}$$

Here $J = J^s = \{J_t^s\}_{0 \le t \le 1}$ is a family of compatible almost complex structures, $\chi(\tau) = \chi_{\pm,N}(\tau)$ is one of the bump functions we defined before. And H_t is defined as in (6.7) but we only move one Lagrangian submanifold here. So most terms in (6.7) are just identity maps. Similarly we can define the perturbed Cauchy-Riemann equation where the domain is $Strip_\epsilon$, a strip with one interior hole.

The energy of a solution u is defined as

$$E_{(J,\chi(\tau),\tilde{H}_t)}(u) = \int |\frac{\partial u}{\partial \tau}|_J^2$$

and we will study the moduli space of finite energy solutions. First we review the energy estimate of solutions when the domain is a strip without holes.

Lemma 6.7. *(Lemma 5.1, [22]) Let u be a finite energy solution of the perturbed Cauchy-Riemann equation with domain Strip. Then we have that*

$$\begin{aligned} E_{(J,\chi(\tau),\tilde{H}_t)}(u) = \int u^*\omega + \int_0^1 \tilde{H}_t(u(+\infty, t))dt \\ - \int_{-\infty}^{\infty} \chi'(\tau) \int_0^1 \tilde{H}_t(u) dt d\tau. \end{aligned} \tag{6.11}$$

When the domain is a strip with one interior hole we can do the similar computation. As expected, the result has one more term involving the integral on the circle boundary. We will compute by cases when $\chi = \chi_+, \chi = \chi_-$ and $\chi = \chi_N$. First we fix the center of the interior hole at $(0, \frac{1}{2})$ and write

$$Strip_\epsilon := Strip_{\epsilon, \frac{1}{2}} = \{(\tau, t) \in \mathbb{R} \times [0,1] \subset \mathbb{C} \mid \tau^2 + (t - \frac{1}{2})^2 \ge \epsilon^2\}$$

to do the computation.

Lemma 6.8. *Let u be a finite energy solution of the perturbed Cauchy-Riemann equation with domain $Strip_\epsilon$. Then we have that*

$$\begin{aligned} E_{(J,\chi(\tau),\tilde{H}_t)}(u) = \int u^*\omega + \int_0^1 \tilde{H}_t(u(+\infty, t))dt \\ - \int_{-\infty}^{\infty} \chi'(\tau) \int_0^1 \tilde{H}_t(u) dt d\tau + \int_{C(\epsilon)} \tilde{H}_t(u) \end{aligned} \tag{6.12}$$

when $\chi(\tau) = \chi_+(\tau)$.

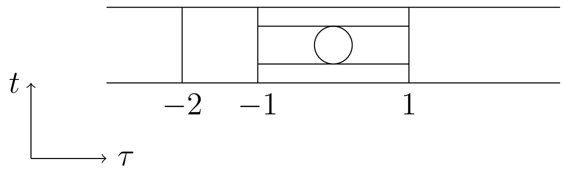

Figure 10: Divide $Strip_\epsilon$ into regions to do integration.

Proof. We prove the lemma by a direct computation.

$$\begin{aligned} E_{(J,\chi(\tau),\tilde{H}_t)}(u) &= \int_{Strip_\epsilon} |\frac{\partial u}{\partial \tau}|_J^2 = \int_{Strip_\epsilon} \omega(\frac{\partial u}{\partial \tau}, J\frac{\partial u}{\partial t}) \\ &= \int_{Strip_\epsilon} \omega(\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial t} - \chi(\tau) X_{\tilde{H}_t}(u)) \\ &= \int_{Strip_\epsilon} \omega(\frac{\partial u}{\partial \tau}, \frac{\partial u}{\partial t}) - \int_{Strip_\epsilon} \omega(\frac{\partial u}{\partial \tau}, \chi(\tau) X_{\tilde{H}_t}(u)) \\ &= \int_{Strip_\epsilon} u^*\omega - \int_{Strip_\epsilon} \chi(\tau)\cdot d\tilde{H}_t(u)(\frac{\partial u}{\partial \tau}) \\ &= \int_{Strip_\epsilon} u^*\omega - \int_{Strip_\epsilon} \chi(\tau)\cdot \frac{\partial}{\partial \tau}\tilde{H}_t(u). \end{aligned} \tag{6.13}$$

Next we consider the last term.

$$\begin{aligned} &\int_{Strip_\epsilon} \chi(\tau)\cdot \frac{\partial}{\partial \tau}\tilde{H}_t(u) \\ =&\int_{Strip_\epsilon, \tau\le -2} \chi(\tau)\cdot \frac{\partial}{\partial \tau}\tilde{H}_t(u) + \int_{Strip_\epsilon, -2\le\tau\le -1} \chi(\tau)\cdot \frac{\partial}{\partial \tau}\tilde{H}_t(u) \\ +&\int_{Strip_\epsilon, -1\le\tau\le 1} \chi(\tau)\cdot \frac{\partial}{\partial \tau}\tilde{H}_t(u) + \int_{Strip_\epsilon, 1\le\tau} \chi(\tau)\cdot \frac{\partial}{\partial \tau}\tilde{H}_t(u) \end{aligned} \tag{6.14}$$

For $\tau \le -2$, the integral is zero since $\chi(\tau)$ is zero. For $-2 \le \tau \le -1$, the integral is

$$\begin{aligned} &\int_{-2}^{-1}\int_0^1 \chi(\tau)\cdot \frac{\partial}{\partial \tau}\tilde{H}_t(u) \\ =&\int_{-2}^{-1} \chi(\tau)\cdot \frac{\partial}{\partial \tau}\int_0^1 \tilde{H}_t(u)dtd\tau \\ =&(\chi(\tau)\cdot \int_0^1 \tilde{H}_t(u)dt)|_{-2}^{-1} - \int_{-2}^{-1}\chi'(\tau)\int_0^1 \tilde{H}_t(u)dtd\tau \\ =&\int_0^1 \tilde{H}_t(u(-1,t))dt - \int_{-2}^{-1}\chi'(\tau)\int_0^1 \tilde{H}_t(u)dtd\tau. \end{aligned} \tag{6.15}$$

Similarly for $1 \le \tau$, the integral is

$$\begin{aligned} &\int_1^{+\infty}\int_0^1 \chi(\tau)\cdot\frac{\partial}{\partial\tau}\tilde{H}_t(u) \\ &= \int_0^1 \tilde{H}_t(u(+\infty,t))dt - \int_0^1 \tilde{H}_t(u(1,t))dt. \end{aligned} \tag{6.16}$$

Now we consider the terms involving the interior hole. For $-1 \le \tau \le 1$ we have that $\chi(\tau) \equiv 1$ and the integral can be split as

$$\begin{aligned} &\int_{Strip_\epsilon, -1\le\tau\le 1} \frac{\partial}{\partial\tau}\tilde{H}_t(u) \\ &= \int_{-1}^1\int_{\frac{1}{2}+\epsilon}^1 \frac{\partial}{\partial\tau}\tilde{H}_t(u)dtd\tau + \int_{-1}^1\int_0^{\frac{1}{2}-\epsilon} \frac{\partial}{\partial\tau}\tilde{H}_t(u)dtd\tau \\ &+ \int_{-1}^{-\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}}\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \frac{\partial}{\partial\tau}\tilde{H}_t(u)dtd\tau + \int_{\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}}^1\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \frac{\partial}{\partial\tau}\tilde{H}_t(u)dtd\tau. \end{aligned} \tag{6.17}$$

Direct computation gives that

$$\begin{aligned} &\int_{-1}^1\int_{\frac{1}{2}+\epsilon}^1 \frac{\partial}{\partial\tau}\tilde{H}_t(u)dtd\tau = \int_{\frac{1}{2}+\epsilon}^1 \tilde{H}_t(u(1,t))dt - \int_{\frac{1}{2}+\epsilon}^1 \tilde{H}_t(u(-1,t))dt \\ &\int_{-1}^1\int_0^{\frac{1}{2}-\epsilon} \frac{\partial}{\partial\tau}\tilde{H}_t(u)dtd\tau = \int_0^{\frac{1}{2}-\epsilon} \tilde{H}_t(u(1,t))dt - \int_0^{\frac{1}{2}-\epsilon} \tilde{H}_t(u(-1,t))dt \end{aligned} \tag{6.18}$$

and

$$\begin{aligned} &\int_{-1}^{-\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}}\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \frac{\partial}{\partial\tau}\tilde{H}_t(u)dtd\tau \\ &= \int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \tilde{H}_t(u(-\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2},t))dt - \int_{-\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \tilde{H}_t(u(-1,t))dt \\ &\int_{\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}}^1\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \frac{\partial}{\partial\tau}\tilde{H}_t(u)dtd\tau \\ &= -\int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \tilde{H}_t(u(\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2},t))dt + \int_{-\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \tilde{H}_t(u(1,t))dt. \end{aligned} \tag{6.19}$$

Put all (6.15)-(6.19) into (6.14) we get the desired estimate. Here we write

$$\begin{aligned} &\int_{C(\epsilon)} \tilde{H}_t(u) \\ &= \int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \tilde{H}_t(u(\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2},t))dt - \int_{\frac{1}{2}-\epsilon}^{\frac{1}{2}+\epsilon} \tilde{H}_t(u(-\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2},t))dt. \end{aligned}$$

In particular we have that

$$-||\tilde{H}_t||_S \leq -2\epsilon \cdot ||\tilde{H}_t||_S \leq \int_{C(\epsilon)} \tilde{H}_t(u) \leq 2\epsilon \cdot ||\tilde{H}_t||_S \leq ||\tilde{H}_t||_S$$

for all $\epsilon \in (0, \frac{1}{2})$. □

By the same computation when $\chi(\tau) = \chi_-$ we have that

Lemma 6.9. *Let u be a finite energy solution of the perturbed Cauchy-Riemann equation with domain $Strip_\epsilon$. Then we have that*

$$\begin{aligned} E_{(J,\chi(\tau),\tilde{H}_t)}(u) = \int u^*\omega - \int_0^1 \tilde{H}_t(u(-\infty, t))dt \\ - \int_{-\infty}^{\infty} \chi'(\tau) \int_0^1 \tilde{H}_t(u)dtd\tau + \int_{C(\epsilon)} \tilde{H}_t(u) \end{aligned} \tag{6.20}$$

when $\chi(\tau) = \chi_-(\tau)$.

And when $\chi(\tau) = \chi_N$ we have that

Lemma 6.10. *Let u be a finite energy solution of the perturbed Cauchy-Riemann equation with domain $Strip_\epsilon$. Then we have that*

$$E_{(J,\chi(\tau),\tilde{H}_t)}(u) = \int u^*\omega - \int_{-\infty}^{\infty} \chi'(\tau) \int_0^1 \tilde{H}_t(u)dtd\tau + \int_{C(\epsilon)} \tilde{H}_t(u) \tag{6.21}$$

when $\chi(\tau) = \chi_N(\tau)$.

The above three lemmas provide necessary energy estimates for us to establish the chain maps and chain homotopies when we change the Hamiltonian functions H_t. More precisely, they give the estimates of maximal energy loss for chain maps. Now we explain how to use them in our situations.

In the formula (6.12) there are four terms. The first two terms correspond to the actions of the input and output generators of the strip. The last two terms correspond to the "energy loss". Note that $\chi_+(\tau) \geq 0$ and $\chi_+(-\infty) = 0, \chi_+(+\infty) = 1$ we have that the maximal energy loss is

$$-\int_0^1 \max_X H_t dt - 2\epsilon||H_t||_S \geq -\int_0^1 \max_X H_t dt - ||H_t||_S \tag{6.22}$$

for any solution u in Lemma 6.8. Similarly the maximal energy loss is

$$\int_0^1 \min_X H_t dt - 2\epsilon||H_t||_S \geq \int_0^1 \min_X H_t dt - ||H_t||_S \tag{6.23}$$

for any solution u in Lemma 6.9. We remark that both Lemma 6.8 and Lemma 6.9 estimate the energy of the solution over the domain $Strip_\epsilon$ where the interior

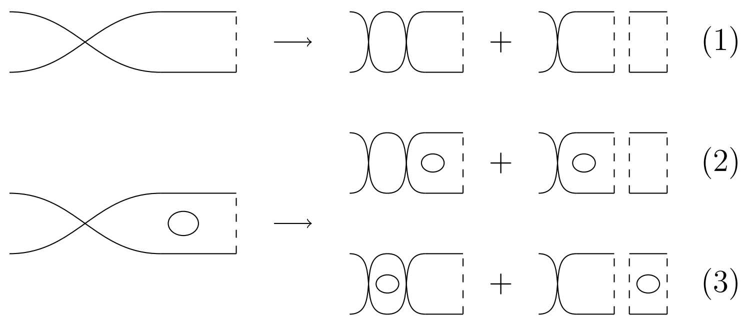

Figure 11: Degenerations of solutions of the perturbed Cauchy-Riemann equation.

hole is centered at $(0, \frac{1}{2})$. If we move the center of the hole to (τ, r) then similar estimate can only be weaker. For example, when the hole is contained outside the support of $\chi(\tau)$ then the fourth term in (6.12) will be zero. When the hole is not contained in the region where $\chi(\tau) = 1$, the fourth term will only be smaller than the case we did in (6.12) because $\chi(\tau) \leq 1$ and $\chi'(\tau) \geq 0$. In conclusion, the above estimates of maximal energy loss work for all the case when we move the center of the interior hole.

Next we construct the chain maps in our settings. We fix a C^2-small perturbation φ such that $L \cap \varphi(L)$ transversally and $\varphi(L) \cap S = \emptyset$. Now for a Hamiltonian G_t, let ϕ be its time-one flow. When $L \cap \phi(\varphi(L))$ is transversal we can also define the cohomology

$$HF_{cy}((L, \rho), (\varphi(L), \rho), G_t; K)$$

where the generators are chords of G_t with ends on L and $\varphi(L)$. Here we remark that when φ is small L and $\varphi(L)$ have the same one-pointed open Gromov-Witten invariants. Hence we can define this cohomology generated by chords with ends on L and $\varphi(L)$, similar to Proposition 6.4. For a general Hamiltonian isotopy there may be wall-crossing phenomenon of the one-pointed invariants which can not be prevented only by Condition 1.1.

Then we use the perturbed Cauchy-Riemann equation to construct chain maps

$$CF_{int,cy}((L, \rho), (\varphi(L), \rho); K) \to CF_{cy}((L, \rho), (\varphi(L), \rho), G_t; K)$$

and

$$CF_{cy}((L, \rho), (\varphi(L), \rho), G_t; K) \to CF_{int,cy}((L, \rho), (\varphi(L), \rho); K).$$

We remark that the two maps are constructed by using the cut-off functions χ_+ and χ_- respectively. Then chain homotopy map is constructed by using the cut-off function χ_N.

Proposition 6.11. *Let (X, S, U, L) be a Lagrangian 3-sphere, a Weinstein neighborhood and a local torus we fixed before. Let (H_t, φ) and (G_t, ϕ) be generic Hamiltonians such that H_t is C^2-small. Then there are two maps*

$$\Phi_+ : CF_{int,cy}((L, \rho), (\varphi(L), \rho); K) \to CF_{cy}((L, \rho), (\varphi(L), \rho), G_t; K)$$

and

$$\Phi_- : CF_{cy}((L,\rho),(\varphi(L),\rho),G_t;K) \to CF_{int,cy}((L,\rho),(\varphi(L),\rho);K)$$

are chain maps.

Proof. The proof is similar to the proof of Theorem 6.2 in [22]. The only difference is that we apply our energy estimate of the change of filtration when the domain has an interior hole. So this difference results in the extra term $||H||_S$.

First for a fixed cut-off function χ_+ we define a chain map

$$\Phi_+ : CF_{int,cy}((L,\rho),(\varphi(L),\rho);K) \to CF_{cy}((L,\rho),(\varphi(L),\rho),G_t;K)$$

by $\Phi_+ = T^{E_+}(\Phi_{+,0} + \Phi_{+,1})$. Here

$$\Phi_{+,0}(p) = \sum_{[l,w]} \sharp\mathcal{M}_0(p,[l,w]) \cdot [l,w]$$

and

$$\Phi_{+,1}(p) = \sum_{[l,w]} \sharp\mathcal{M}_1(p,[l,w]) \cdot [l,w] \cdot T^{v(K)}.$$

The energy weights T^{E_+} is necessary since we want to consider the map over Λ_0. Note that there will be energy loss for the perturbed Cauchy-Riemann equation. And the maximal energy loss is computed in (6.22). So if we set

$$E_+ = \int_0^1 \max_X G_t dt + ||G_t||_S$$

then we get a map which does not decrease the energy.

We explain the moduli spaces as follows. The moduli space $\mathcal{M}_0(p,[l,w])$ contains solutions of the perturbed Cauchy-Riemann equation when the domain is a genuine strip. The moduli space $\mathcal{M}_1(p,[l,w])$ is obtained by gluing two moduli spaces

$$\mathcal{M}_1(p,[l,w]) = \mathcal{M}_{1,pt}(p,[l,w]) \sqcup \mathcal{M}_{1,hole}(p,[l,w])/\sim$$

where $\mathcal{M}_{1,pt}(p,[l,w])$ contains solutions of the perturbed Cauchy-Riemann equation when the domain is a strip with one interior marked point, and the moduli space $\mathcal{M}_{1,hole}(p,[l,w])$ contains solutions when the domain is a strip with one interior hole. And the gluing is understood as we did in defining ∂_K.

Next we show that Φ_+ is a chain map. That is,

$$\Phi_+ d^\rho_{K,int} + d^\rho_K \Phi_+ \equiv 0 \mod T^{2v(K)}.$$

Note that

$$\begin{aligned} &\Phi_+ d^\rho_{K,int} + d^\rho_K \Phi_+ \\ =&T^{E_+}(\Phi_{+,0} + \Phi_{+,1})(\delta^\rho_{int} + \partial_{K,int}) + T^{E_+}(\delta^\rho + \partial_K)(\Phi_{+,0} + \Phi_{+,1}) \end{aligned} \tag{6.24}$$

and there are eight terms in the full expansion. After compensating the energy loss by T^{E_+}, the sum

$$T^{E_+}(\Phi_{+,1}\partial_{K,int} + \partial_K\Phi_{+,1}) \equiv 0 \mod T^{2v(K)}$$

by the energy reason. So we need to check the remaining sum of six terms is zero. The proof is by studying all types of degenerations of one-dimensional moduli spaces. By a similar argument in Proposition 6.4, we assume that there is no contribution from sphere bubble, disk bubble and annulus bubble to the codimension one boundary. Then there are six types of degenerations for the moduli spaces $\mathcal{M}_0(p,[l,w])$ and $\mathcal{M}_1(p,[l,w])$, shown in Figure 11. In particular, the terms in (1) correspond to

$$\Phi_{+,0}\delta^{\rho}_{int} + \delta^{\rho}\Phi_{+,0}$$

which are from the boundary components of $\mathcal{M}_0(p,[l,w])$. Hence the sum, weighted by T^{E_+}, vanishes. Similarly the terms in (2) correspond to

$$\Phi_{+,1}\delta^{\rho}_{int} + \delta^{\rho}\Phi_{+,1}$$

and the terms in (3) correspond to

$$\Phi_{+,0}\partial_{K,int} + \partial_K\Phi_{+,0}.$$

Therefore the sum of these four terms, weighted by $T^{E_+ + v(K)}$, vanishes. In conclusion we have that the sum of these eight terms in (6.24) is zero and Φ_+ is a chain map. In the same way we can construct

$$\Phi_- = T^{E_-}(\Phi_{-,0} + \Phi_{-,1})$$

as a chain map by a chosen cut-off function χ_-. Here

$$E_- = -\int_0^1 \min_X G_t dt + ||G_t||_S.$$

Then $\Phi_\pm$ induce maps in the cohomology level, which we still write as $\Phi_\pm$. □

Next we construct chain homotopy maps such that $\Phi_- \circ \Phi_+$ is chain homotopic to some inclusion map.

Proposition 6.12. *With the same notations in the previous proposition, the composition*

$$\Phi_- \circ \Phi_+ : HF_{int,cy}((L,\rho),(\varphi(L),\rho);K) \to HF_{int,cy}((L,\rho),(\varphi(L),\rho);K)$$

equals the inclusion-induced map

$$\mathfrak{i} := T^E(\mathfrak{i}_0 + \mathfrak{i}_1) : HF_{int,cy}((L,\rho),(\varphi(L),\rho);K) \to HF_{int,cy}((L,\rho),(\varphi(L),\rho);K).$$

Here $E = E_+ + E_- = ||G_t||_X + 2||G_t||_S$.

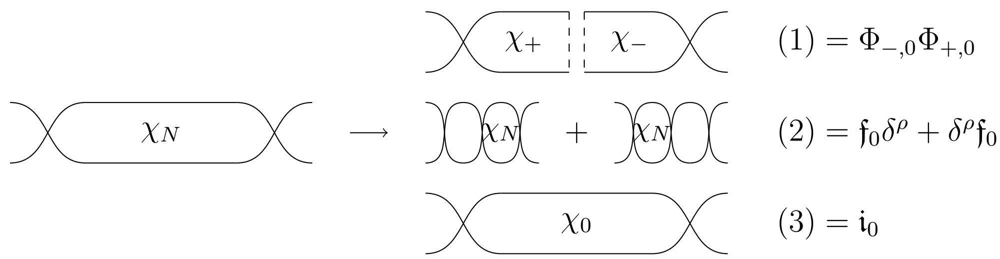

Figure 12: Degenerations in $\mathcal{M}_0^{para}$.

Proof. The chain homotopy maps are constructed by using the perturbed Cauchy-Riemann equation with cut-off function χ_N. Consider the one-parameter moduli spaces

$$\widetilde{\mathcal{M}}_0^{para} = \bigcup_{N\in[0,+\infty)} \{N\} \times \mathcal{M}_0^N(p,q)$$

and

$$\widetilde{\mathcal{M}}_1^{para} = \bigcup_{N\in[0,+\infty)} \{N\} \times \mathcal{M}_1^N(p,q)$$

parameterized by N. Here the moduli space $\mathcal{M}_0^N(p,q)$ contains solutions of the perturbed Cauchy-Riemann equation with cut-off function χ_N where the domain is a genuine strip. The moduli space $\mathcal{M}_1^N(p,q)$ contains solutions of the perturbed Cauchy-Riemann equation with cut-off function χ_N where the domain is a strip with one interior hole. The energy estimate in Lemma 6.10 tells that for a solution u in $\mathcal{M}_0^N(p,q)$ or $\mathcal{M}_1^N(p,q)$, we always have that

$$\begin{aligned} E_{(J,\chi_N(\tau),G_t)}(u) &= \int u^*\omega - \int_{-\infty}^{\infty} \chi_N'(\tau) \int_0^1 G_t(u) dt d\tau + \int_{C(\epsilon)} G_t(u) \\ &\leq \int u^*\omega + ||G_t||_X + ||G_t||_S \end{aligned}$$

which is uniformly bounded from above, independent of N. Then we can compactify $\widetilde{\mathcal{M}}_0^{para}$ and $\widetilde{\mathcal{M}}_1^{para}$ to obtain $\mathcal{M}_0^{para}$ and $\mathcal{M}_1^{para}$, by adding possible broken curves. In particular, we deal with the codimension one boundary from domain degenerations in $\mathcal{M}_1^N(p,q)$ by gluing it with the moduli space where the domain is a strip with one interior marked point, as we did before.

Under transversality assumptions, both of the moduli spaces $\mathcal{M}_0^{para}(p,q)$ and $\mathcal{M}_1^{para}(p,q)$ have dimension one when $p = q$. Now we study the boundary of the these two moduli spaces. By similar argument before, we assume there is no disk bubble, sphere bubble or annulus bubble. Then the boundary components of $\mathcal{M}_0^{para}(p,p)$ have four types degenerations (listed in Figure 12) and the boundary components of $\mathcal{M}_1^{para}(p,p)$ have seven types of degenerations (listed in Figure 13). We remark that there is another type of degenerations in $\mathcal{M}_1^{para}(p,p)$ which we deal with the same strategy as before, by gluing this boundary component with the boundary of moduli space with one interior marked point. Hence we omit it in Figure 13.

Now we look at the chain homotopy equation

$$\Phi_- \circ \Phi_+ - \mathfrak{i} = d^{\rho}_{K,int}\mathfrak{f} + \mathfrak{f} d^{\rho}_{K,int} \tag{6.25}$$

where

$$\begin{aligned} &\Phi_+ = T^{E_+}(\Phi_{+,0} + \Phi_{+,1});\\ &\Phi_- = T^{E_-}(\Phi_{-,0} + \Phi_{-,1});\\ &\mathfrak{i} = T^{E}(\mathfrak{i}_0 + \mathfrak{i}_1);\\ &\mathfrak{f} = \mathfrak{f}_0 + \mathfrak{f}_1;\\ &d^{\rho}_{K,int} = \delta^{\rho} + \partial_{K,int}. \end{aligned}$$

We explain the operators and corresponding moduli spaces as follows. Operators $\Phi_{+,0}, \Phi_{+,1}, \Phi_{-,0}, \Phi_{-,1}$ are chain maps defined in the previous proposition. The operator $d^{\rho}_{K,int} = \delta^{\rho} + \partial_{K,int}$ is the differential to define the cohomology. Operators $\mathfrak{f}_0, \mathfrak{f}_1$ will be defined as chain homotopy maps between $\Phi_- \circ \Phi_+$ and $\mathfrak{i}$.

All four operators $\mathfrak{f}_0, \mathfrak{f}_1, \mathfrak{i}_0, \mathfrak{i}_1$ are defined from $CF_{int,cy}((L, \rho), (\varphi(L), \rho); K)$ to itself. The operator $\mathfrak{i}_0$ is the identity map, which comes from the "zero end" moduli space $\mathcal{M}^0_0(p, p)$ as a boundary of $\mathcal{M}^{para}_0$. Note that when $p = q$ and $\chi_N = \chi_0 \equiv 0$ the only element in $\mathcal{M}^0_0(p, p)$ is the constant map. Similarly the operator $\mathfrak{i}_1$ is the identity map weighted by $T^{v(K)}$. The operator $\mathfrak{f}_0$ is defined by using the perturbed Cauchy-Riemann equation with bump function χ_N. And the operator $\mathfrak{f}_1$ is defined by using the perturbed Cauchy-Riemann equation with bump function χ_N, when the domain is a strip with an interior marked points mapping to K. We also weight $\mathfrak{f}_1$ by $T^{v(K)}$.

So in the full expansion of the chain homotopy equation there are 14 terms. The following three terms

$$\partial_{K,int}\mathfrak{f}_1, \quad \mathfrak{f}_1\partial_{K,int}, \quad T^E\Phi_{-,1}\Phi_{+,1} \equiv 0 \mod T^{2v(K)}$$

by energy reason. And the remaining 11 terms correspond to the 11 types of degenerations in the moduli spaces $\mathcal{M}^{para}_0(p, p)$ and $\mathcal{M}^{para}_1(p, p)$, which form the boundary components of two compact one-dimensional manifolds. Therefore we proved the chain homotopy property. □

Remark 6.13. The above two propositions are proved assuming some analytic results. First, Condition 1.1 is necessarily used to exclude disk and annulus bubbles. Moreover, the regularity and the gluing theory of the moduli spaces of perturbed Cauchy-Riemann equations is assumed. When the domain is a genuine strip this moduli space is discussed in [22]. And we expect the same analytic argument therein can be applied here when the domain has one interior hole.

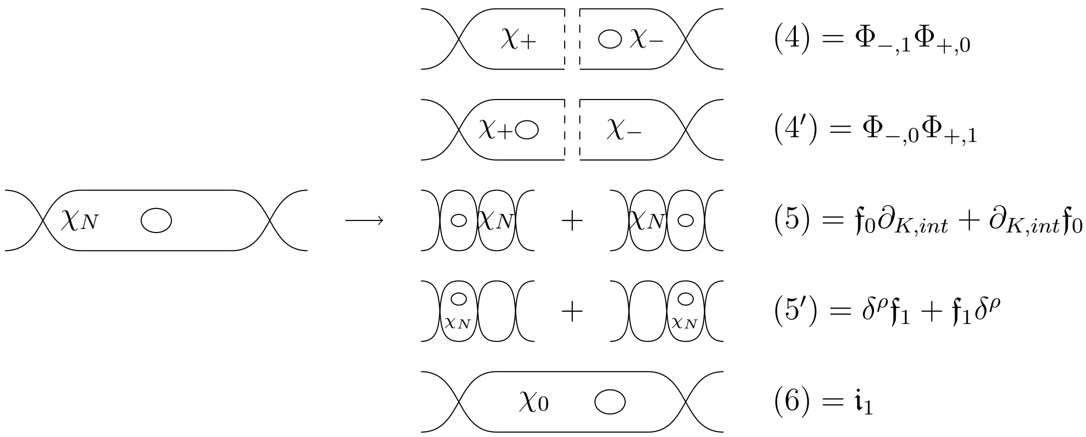

Figure 13: Degenerations in $\mathcal{M}_1^{para}$.

6.3 Relations among three deformed Floer complexes

So far we defined three complexes to describe a new version of deformed Floer cohomology. For the first one, the disk model,

$$HF_{cy}(L;(\mathfrak{b}=K,\rho))$$

the underlying complex is the singular cohomology of L and the differential counts holomorphic disks and holomorphic annuli, twisted by a local system ρ. The second one, the intersection model,

$$HF_{int,cy}((L,\rho),(\varphi(L),\rho);K)$$

and the third one, the chord model,

$$HF_{cy}((L,\rho),(\varphi(L),\rho),G_t;K)$$

are defined by first choosing suitable (H_t,φ) and (G_t,ϕ) then counting holomorphic strips with a possible interior hole. (Note that in the definition we assume that both H_t and G_t are generic and H_t is small.) For the genuine Floer cohomology with bulk deformations, it is known that these three cohomology theories are equivalent over the Novikov field Λ (Proposition 8.24 [20]) and have a good Lipschitz property over the Novikov ring Λ_0 (Theorem 6.2 [22]). Now we will discuss the relations among these three models in our setting.

The disk model, of which the cohomology is determined by the potential function, is used for concrete computation once we know the potential function. The displacement results are given by the change of torsion exponents of the chord model, where large Hamiltonian perturbation is allowed. And to connect these two models we need the intersection model, where only small Hamiltonian perturbation is considered.

Theorem 6.14. *Suppose that the potential function $\mathfrak{PO}^{cy,\mathfrak{b}}(\rho)$ for L has a critical point for some $(\mathfrak{b}, \rho)$ modulo T^E, $E \leq 2v(\mathfrak{b})$. If there is a Hamiltonian G_t with time-1 flow ϕ such that $L \cap \phi(L) = \emptyset$ then it satisfies that $||G_t||_X + 2||G_t||_S \geq E$.*

Proof. First the existence of the critical point shows that

$$HF_{cy}(L; (\mathfrak{b}, \rho)) \cong H^*(L; \frac{\Lambda_0}{T^E \Lambda_0}) \cong (\frac{\Lambda_0}{T^E \Lambda_0})^{\oplus 8} \neq \{0\}$$

by Proposition 5.16.

Next we choose a C^2-small (H_t, φ) such that $L \cap \varphi(L)$ is transversal. Then the cohomology

$$HF_{int,cy}((L, \rho), (\varphi(L), \rho); K)$$

is well-defined for $(\mathfrak{b} = K, \rho)$. We can construct chain maps between the two theories $HF_{cy}(L; (\mathfrak{b}, \rho))$ and $HF_{int,cy}((L, \rho), (\varphi(L), \rho); K)$. In the case of genuine Floer cohomology with bulk deformations, the chain maps are constructed in Section 8 [20]. So we combine the proof therein with the special case when the domain has one interior hole in the previous subsection, to get the chain maps and chain homotopies with new energy estimates. Note that H_t is C^2-small, the Condition 1.1 is preserved and $\varphi_t(L) \cap S$ is always empty. Hence the discussion in previous subsections all works. Then we obtain that

$$HF_{int,cy}((L, \rho), (\varphi(L), \rho); K) \cong \bigoplus_{i=1}^{8} (\frac{\Lambda_0}{T^{E_i} \Lambda_0})$$

where $|E - E_i| < ||H_t||_X + 2||H_t||_S$ for all i. That is, under the small perturbation H_t the torsion exponents are also slightly perturbed, by the amount of some Hofer norms.

Therefore we have transited from the disk model to the intersection model. Next the estimates in previous subsection help us to transit from the intersection model to the chord model, where large Hamiltonian perturbation is allowed. Suppose that there is a Hamiltonian G_t with time-one flow ϕ such that $L \cap \phi(\varphi(L)) = \emptyset$. From the definition we know that

$$HF_{cy}((L, \rho), (\varphi(L), \rho), G_t; K) = \{0\}$$

and $\Phi_+ = \Phi_- = 0$. Proposition 6.12 tells that

$$\Phi_- \circ \Phi_+ : HF_{int,cy}((L, \rho), (\varphi(L), \rho); K) \to HF_{int,cy}((L, \rho), (\varphi(L), \rho); K)$$

equals the inclusion-induced map

$$T^{E_0}(\mathfrak{i}_0 + \mathfrak{i}_1) : HF_{int,cy}((L, \rho), (\varphi(L), \rho); K) \to HF_{int,cy}((L, \rho), (\varphi(L), \rho); K)$$

where $E_0 = ||G_t||_X + 2||G_t||_S$. Therefore we have that

$$0 = T^{E_0}(\mathfrak{i}_0 + \mathfrak{i}_1) : HF_{int,cy}((L, \rho), (\varphi(L), \rho); K) \to HF_{int,cy}((L, \rho), (\varphi(L), \rho); K).$$

So $E_0 > \max_i\{E_i\}$ for all i. Let $||H_t|| \to 0$ we obtain that $||G_t||_X + ||G_t||_S \geq E$.

In conclusion, for any Hamiltonian diffeomorphism ψ which displaces L there is a small amount $\epsilon(\psi) > 0$ such that any pair (H_t, φ) with $||H_t|| < \epsilon(\psi)$ then ψ also displaces $\varphi(L)$ from L. Hence we can use those small (H_t, φ) to do the above energy estimate for ψ, which completes the proof. □

The above theorem is parallel to Theorem 5.11 in [19] for potential functions without bulk deformation and Theorem 7.7 in [22] for potential functions with bulk deformation. We just adapt the proof therein by combining our energy estimates in this section.

7 Estimates of displacement energy

Now we estimate the displacement energy of a local torus. First we fix a Weinstein neighborhood U of S such that U admits a singular toric fibration as we described earlier. And we fix a local torus $L \subset U$ near S. Let J be a compatible almost complex structure on X which agrees with the almost toric complex structure on U and satisfies Condition 1.1. Then the one-point open Gromov-Witten invariant n_β is defined with respect to J, for a disk class $\beta \in \pi_2(X, L)$ with Maslov index two. We consider the sequence

$$\{\beta_k \mid n_\beta \neq 0, E(\beta_k) \leq E(\beta_{k+1})\}_{k=1}^{\infty}$$

of disk classes with Maslov index two, enumerated by their symplectic energy. We know that L bounds four J-holomorphic disks with Maslov index two inside U, with same energy E_1. Those are the first four elements in the above sequence if L is near S. Let $E_5 = E(\beta_5)$ be the least energy of outside disk contribution. Note that since L is close to S we actually have that $E_{5,\lambda} >> E_{1,\lambda}$.

If we do symplectic cutting on ∂U then U becomes the quadric Q_3 with a scaled Fubini-Study form. Since the almost complex structure J agrees with the toric one on U there will be a new J-holomorphic disk with Maslov index two, intersecting the divisor at infinity. We write the energy of this new disk as E_{cut}. Note that $E_5 \geq E_{cut}$ since the image of this disk class goes out from U. And E_{cut} just depends on the size of U and it is independent of J if J is toric on U, while E_5 depends on J. So here E_{cut} plays the role of a "universal lower bound" for all E_5 among all J satisfying Condition 1.1. In the following we just write E as E_5, for notational simplicity.

7.1 First estimate

Let L be a local torus, we will first show its displacement energy is greater than or equal to E. This is directly from the decomposition formula of the Floer cohomology, which do not need the bulk deformation by the chain K.

Let $H^*(L;\Lambda_0)$ be the singular cohomology of L with Novikov coefficients. By the weakly unobstructedness of L there is a differential

$$\delta^\rho : H^*(L;\Lambda_0) \to H^*(L;\Lambda_0)$$

for any $\rho \in H^1(L;\Lambda_0)$. The cohomology given by δ_ρ is the Floer cohomology $HF(L,\rho;\Lambda_0)$ with a local system ρ. Note that under the energy filtration E we only have the four basic disk classes, which "cancel with each other" for some ρ_0. That is, from (5.1), the disk potential function is

$$\mathfrak{PO}(\rho) = (x + y^{-1} + xz^{-1} + y^{-1}z)T^{E_1} \mod T^E, \quad \rho \in H^1(L_\lambda;\Lambda_0). \tag{7.1}$$

So it has a critical point at $\rho_0 = (x = 1, y = 1, z = -1)$. Hence by the decomposition formula (2.11) we have

$$HF(L,\rho_0;\Lambda_0) \cong (\bigoplus_{i=1}^{l} \frac{\Lambda_0}{T^E\Lambda_0}) \mod T^E.$$

Therefore in the decomposition of $HF(L,\rho_0;\Lambda_0)$ the least torsion exponent is great than or equal to E. And Theorem J in [17] gives that $\mathcal{E}_L \geq E$.

7.2 Second estimate

For the second estimate will use the deformed Floer cohomology of a local torus. This new cohomology is an analogue to the Floer cohomology with bulk deformations. But here we use chains instead of cycles to do the deformation.

First we compute the deformed potential function (5.10) using the chain K as a bulk deformation, here K is a fixed completion of K_1.

Theorem 7.1. *Let $\mathfrak{b} = w \cdot PD(K), w \in \Lambda_+$ be a bulk chain then the $\mathfrak{b}$-deformed potential function is*

$$\mathfrak{PO}^{cy,\mathfrak{b}}(\rho) = ((1+w)x + y^{-1} + xz^{-1} + y^{-1}z)T^{E_1} + H(w,x,y,z,T) \mod T^{2v(\mathfrak{b})}$$

where $H(w,x,y,z,T)$ is the higher energy part.

Proof. The key point is that the potential function only depends on holomorphic disks with Maslov index two. And when there is no Hamiltonian perturbation, there is no holomorphic cylinders to count. Hence the $\mathfrak{b}$-deformed potential function looks the same as the (first order approximation of) usual potential function with bulk deformation, modulo some energy.

More precisely, in Proposition 5.6 we have two moduli spaces which are identified between the smoothed side and the resolved side. On the resolved side the moduli space contains holomorphic disks with one interior marked points attached to the cycle $w \cdot PD(\widetilde{K})$. By the computation in toric case [20] we know its contribution to the potential function is $1+w$, since we only consider the zeroth and the first operators. (For the full bulk deformation the contribution will be e^w, see the divisor axioms in (2.6).) And by our assumption on the choice of the completion K, other local disk classes $\beta_2, \beta_3, \beta_4$ do not intersection K. Therefore on the smoothed side the contribution of the chain $w \cdot PD(K)$ is also $1+w$ since two moduli spaces are identified and they give the same one-point open invariants. Then by filling these information in the definition (5.9) we obtain the $\mathfrak{b}$-deformed potential function in the smoothed side. $\square$

Next we can compute the critical points of this deformed potential function. The critical points equation will be

$$\begin{aligned} 0 &= 1 + w + z^{-1} + \frac{\partial H}{\partial x} \mod T^{2v(\mathfrak{b})} \\ 0 &= 1 + z - y^{-2}\frac{\partial H}{\partial y} \mod T^{2v(\mathfrak{b})} \\ 0 &= -xz^{-2} + y^{-1} + \frac{\partial H}{\partial z} \mod T^{2v(\mathfrak{b})}. \end{aligned} \tag{7.2}$$

If this system of equations has solutions in $\Lambda_0 - \Lambda_+$ then by Theorem 6.10 we have an estimate of the displacement energy $\mathcal{E}_L$ of L. We view (7.2) as a system of three equations with four variables (w, x, y, z) hence we have freedom to prescribe the value of one of the variables. So we set $x = 1$ to these equations and view w, y, z as variables. The existence of suitable solution w, y, z is assured by an implicit function theorem in the setting of Novikov ring, see Lemma 4.3.

Next we directly check that $w = 0, y = 1, z = -1$ is a nondegenerate solution of (7.2) modulo higher energy terms. By the Gromov compactness theorem the higher energy part H in the potential function is a Laurent polynomial since we work modulo $T^{2v(K)}$. (In general the potential function could be a Laurent series with energy going to infinity.) Hence our system of equations fits in a 3-dimensional version of Lemma 4.3 and the whole system of critical point equation (7.2) has a suitable solution modulo $T^{2v(\mathfrak{b})}$.

Note that under the energy filtration E_1 we already has a critical point. So we only need to perturb the higher energy terms with filtration larger than or equal to E. Hence the deformation $\mathfrak{b}$ does not have low energy part below $E - E_1$. A more careful study of the choice of $wPD(K)$ shows that it has the following form

$$\mathfrak{b} = wPD(K) = w_1 PD(K) + w_2 PD(K) + \cdots + w_j PD(K)$$

such that

$$\mathfrak{PO}^{cy,\mathfrak{b}}(\rho) = ((1+w)x + y^{-1} + xz^{-1} + y^{-1}z)T^{E_1} + H(w, x, y, z, T) \mod T^{2v(\mathfrak{b})}$$

has a critical point and

$$E - E_1 \leq v(w_1) < v(w_2) < \cdots < v(w_j).$$

Therefore by Theorem 6.14 we know that if ϕ displaces L then its corresponding Hamiltonian functions G_t satisfy that

$$||G_t||_X + 2||G_t||_S \geq 2v(\mathfrak{b}) \geq 2(E - E_1).$$

This completes the proof of Theorem 1.3.

Now we explain the proof of Corollary 1.4. Let G_t be a time-dependent Hamiltonian function and ϕ be its time-one map such that $S \cap \phi(S) = \emptyset$. Then there is a small neighborhood U which is also displaced by ϕ. Note that for a small number λ', all local tori L_λ are contained in U if $\lambda \in (0, \lambda')$ and are displaced by ϕ. Therefore we know that

$$||G_t||_X \geq E_{5,\lambda}, \quad ||G_t||_X + 2||G_t||_S \geq 2(E_{5,\lambda} - E_{1,\lambda})$$

for all $\lambda \in (0, \lambda')$. As λ goes to zero, the energy $E_{1,\lambda}$ decreases and $E_{5,\lambda}$ increases hence we complete the proof of Corollary 1.4.

7.3 Examples of displaceable Lagrangian spheres

Now we briefly review Pabiniak's construction [35] of displaceable Lagrangian 3-spheres and show our theoretical estimate is almost optimal in this case.

Consider the Lie group $SU(3)$. We identify the dual of its Lie algebra $\mathfrak{su}^*(3)$ with the vector space of 3×3 traceless Hermitian matrices. Then the group $SU(3)$ acts on $\mathfrak{su}^*(3)$ by conjugation. Through a regular point $diag(a, b, -a-b)$, the action orbit M is a smooth 6-dimensional symplectic manifold with the Kostant-Kirillov symplectic form.

We fix a regular point $diag(a, b, -a-b)$ with $a > b \geq 0$ and write the orbit as M. The symplectic form on M is monotone if and only if $b = 0$. There is a Gelfand-Tsetlin fibration $\Gamma : M \to \mathbb{R}^3$. For a matrix $A \in M$ let $a_1(A) \geq a_2(A)$ denote the two eigenvalues of the 2×2 top left minor of A, and let $a_3(A) = a_{11}$ be the $(1,1)$ entry of A. Then the system $\Gamma(A) = (a_1(A), a_2(A), a_3(A))$ gives the fibration map. Let (x, y, z) be the coordinates of $\mathbb{R}^3$. The image polytope (see Figure 14) of Γ is given by affine functions

$$a \geq x \geq b;$$

$$b \geq y \geq -a-b;$$

$$x \geq z \geq y.$$

This Gelfand-Tsetlin fibration Γ can be viewed as a smooth torus fibration away from the fiber $\Gamma^{-1}(b, b, b)$ since the three functions (a_1, a_2, a_3) integrate to a 3-torus action. There is a unique non-smooth point (b, b, b) in the polytope, of which the fiber $S = \Gamma^{-1}(b, b, b)$ is a smooth Lagrangian 3-sphere. So this fibration is a compactification of the fibration on T^*S^3 by putting divisors at infinity, see Section 5.1. And the parameter b measures the symplectic form on this compactification.

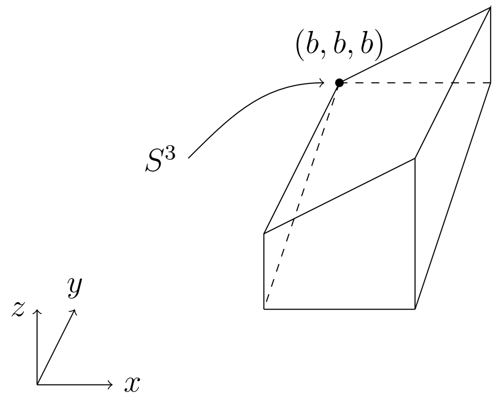

Figure 14: Moment polytope for the fibration Γ.

Moreover we can consider the standard action of the maximal torus of $SU(3)$, which gives us a subaction of the Gelfand-Tsetlin action. This 2-torus action has a moment map $\mu : M \to \mathbb{R}^2$. We have the following commutative diagram

$$\begin{array}{ccc} M & \xrightarrow{\Gamma} & \mathbb{R}^3 \\ & \searrow^{\mu} & \downarrow{\scriptstyle pr} \\ & & \mathbb{R}^2 \end{array}$$

where we view $\mathbb{R}^2 = \{x + y + z = 0\} \subset \mathbb{R}^3$. The projection map is given by

$$pr(x, y, z) = (z, x + y - z, -x - y).$$

Consider the permutation matrix

$$P = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

which is an element of $SU(3)$. Then the conjugation with P is a Hamiltonian action on M. Note that for $A = [a_{ij}] \in M$

$$\mu(PAP^{-1}) = (a_{11}, a_{33}, a_{22}).$$

So we have that

$$\mu(S) = \mu(\Gamma^{-1}(b, b, b)) = pr(b, b, b) = (b, b, -2b)$$

and

$$\mu(PSP^{-1}) = (b, -2b, b).$$

In particular if $b \neq 0$ then the Lagrangian 3-sphere S will be displaced by this group action. We also remark that when $b = 0$ the Lagrangian 3-sphere S is monotone and is proved to be nondisplaceable by Cho-Kim-Oh [11].

In [31] it is calculated that S bounds two holomorphic disks with energy $2\pi(a+2b)$ and $2\pi(a-b)$. Moreover the Floer cohomology $HF(S,S;\Lambda)$ vanishes. Next we assume that $b > 0$ so that $2\pi(a+2b) > 2\pi(a-b)$. By Chekanov's theorem the displacement energy $\mathcal{E}_S$ of S is greater than $2\pi(a-b)$. For the Hamiltonian action by P, its corresponding Hamiltonian function is the inner product with the vector $diag(0,\pi,-\pi)$. That is, for a fiber $\Gamma^{-1}(x,y,z)$ over the point (x,y,z) the Hamiltonian function is constant on the fiber and can be written as

$$H(x,y,z) = (0,\pi,-\pi)\cdot pr(x,y,z) = \pi(2x+2y-z).$$

From the polytope we can check that

$$\max_M H = H(a,b,b) = \pi(2a+b), \quad \min_M H = H(b,-a-b,b) = \pi(-2a-b).$$

Hence we have that

$$\int_0^1 (\max_M H - \min_M H)dt = 2\pi(2a+b).$$

In particular $H\mid_S \equiv H(b,b,b) = 3b$. So for this Hamiltonian we have that

$$||H||_M = 2\pi(2a+b), \quad ||H||_S = 0$$

and

$$||H||_M + 2||H||_S = ||H||_M = 2\pi(2a+b) \geq 2E_5 := \lim_{\lambda\to 0} 2E_{5,\lambda} = 4\pi(a-b).$$

This matches our theoretical prediction in Theorem 1.3. And when $a >> b \geq 0$ we have that $2\pi(2a+b)$ is close to $4\pi(a-b)$, which shows that the estimate is almost optimal in this case.

One can also check the case of the displaceable Lagrangian $S^3 \subset \mathbb{C}^2 \times \mathbb{C}P^1$. Consider the following Lagrangian embedding

$$S^3 \to \mathbb{C}^2 \times \mathbb{C}P^1, \quad x \mapsto (i(x), -h(x))$$

where i is the inclusion of the unit sphere and h is the Hopf map. The symplectic form on $\mathbb{C}^2 \times \mathbb{C}P^1$ is the standard one times the Fubini-Study form. Let H be a Hamiltonian on $\mathbb{C}^2$ which displaces the unit sphere and $G(z_1,z_2) := H(z_1)$ be a Hamiltonian on $\mathbb{C}^2 \times \mathbb{C}P^1$. Then G displaces the Lagrangian sphere and $||G||_{\mathbb{C}^2\times\mathbb{C}P^1} = ||H||_{\mathbb{C}^2}$. Moreover, it is known that $||H||_{\mathbb{C}^2}$ can be chosen to be arbitrarily close to π.

However, the Hamiltonian H takes maximal and minimal values on the unit sphere hence G takes maximal and minimal values on the Lagrangian sphere S. So we have $||G||_{\mathbb{C}^2\times\mathbb{C}P^1} = ||G||_S$. Note that

$$H_2(\mathbb{C}^2 \times \mathbb{C}P^1, S) \cong H_2(\mathbb{C}^2 \times \mathbb{C}P^1) \cong H_2(\mathbb{C}P^1)$$

hence the minimal energy of a holomorphic disk bounding S is $\pi = \int_{\mathbb{C}P^1} \omega_{FS}$. And our estimate gives that $3||G||_{\mathbb{C}^2\times\mathbb{C}P^1} \geq 2\pi$, which is not a contradiction but not very powerful for this example.

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