# Pseudo-Néron model and Restriction of Sections

A Dissertation Presented

by

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to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

**Doctor of Philosophy**

in

**Mathematics**

Stony Brook University

**August 2020**

**Stony Brook University**

The Graduate School

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Abstract of the Dissertation

**Pseudo-Néron model and Restriction of Sections**

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**2020**

We introduce the notion of pseudo-Néron model and give new examples of varieties admitting pseudo-Néron models other than Abelian varieties. As an application of pseudo-Néron models, given a scheme admitting a finite morphism to an Abelian scheme over a positive-dimensional base, we prove that for a very general genus-0, degree-d curve in the base with d sufficiently large, every section of the scheme over the curve is contained in a unique section over the entire base.

## Acknowledgments

Firstly, I would like to express my sincere gratitude to my advisor Prof. Jason Michael Starr for the continuous support of my Ph.D study and research, for his patience, motivation, and immense knowledge. His guidance helped me in all the time of research and writing of this thesis. I could not have imagined having a better advisor and mentor for my Ph.D study. Doubtless, without his encouragement and help, I will never be able to understand and study algebraic geometry. He is a man who knows almost everything in algebraic geometry.

Besides my advisor, I would like to thank Prof. Xiuxiong Chen and his wife for their help during my undergraduate and graduate years. Their support and encouragement lightened my life in every tough time these years. I will never forget the great time in our annual Thanksgiving party.

# Contents

# 1 Introduction

The main results of this article are Theorem 1.3 and Theorem 1.11 in subsection 1.1. We compare our results with other theorems in literatures in subsection 1.2. The geometric idea to prove Theorem 1.3 is given in subsection 1.3.

## 1.1 Main results

The starting point of this work is a theorem in [10], where Tom Graber and Jason Michael Starr prove the theorem of restriction of sections for families of Abelian varieties (Theorem 1.2). To state their theorems, we cite the following definition from [10].

**Definition 1.1.** ([10], p.312) Let $k$ be an algebraically closed field. Fix a generically finite, generically unramified morphism $u_0 : S \to \mathbb{P}^n_k$. We define

- an $u_0$-*line* is a curve in $S$ of the form $S \times_{\mathbb{P}^n_k} L$ for a line $L \subset \mathbb{P}^n_k$;
- an $u_0$-*conic* is a curve in $S$ of the form $S \times_{\mathbb{P}^n_k} C$ for a plane conic $C \subset \mathbb{P}^n_k$;
- an $u_0$-*line-pair* is a curve in $S$ of the form $S \times_{\mathbb{P}^n_k} L$, where $L = L_1 \cup L_2$ for a pair of incident lines in $\mathbb{P}^n_k$;
- an $u_0$-*smooth-curve* is an irreducible smooth curve in $S$ of the form $S \times_{\mathbb{P}^n_k} C_0$ for a smooth curve $C_0 \subset \mathbb{P}^n_k$;
- an $u_0$-*curve-pair* of degree-$(d+2)$ is a connected curve in $S$ of the form $S \times_{\mathbb{P}^n_k} C$, where $C = C_0 \cup C_1$ is a pair of curves in $\mathbb{P}^n_k$ intersecting transversally at a single closed point such that $C_0$ is a genus zero, smooth curve of degree $d$, and $C_1$ is a smooth conic;
- an $u_0$-*planar surface* is a surface in $S$ of the form $S \times_{\mathbb{P}^n_k} \Sigma$ for a 2-plane $\Sigma \subset \mathbb{P}^n_k$.

Note that, by Bertini's theorem, for sufficiently general line, conic, and plane, the corresponding $u_0$-line, $u_0$-conic, and $u_0$-planar surface will be smooth. By abuse of notations, we will just say line, conic, line-pair, curve-pair, planar surface, and smooth curve in $S$ instead of $u_0$-line, $u_0$-conic, $u_0$-line-pair, $u_0$-curve-pair, $u_0$-planar surface, and $u_0$-smooth-curve.

Let $k$ be an uncountable algebraically closed field. We say a subset of a scheme is *general*, resp. *very general*, if the subset contains an open dense subset, resp. the intersection of a countable collection of open dense subsets. We say that a property of points in a scheme holds *at a general point*, resp. *at a very general point*, if the set where the property holds is a general subset, resp. a very general subset.

Now, we state the main theorem in [10] as following.

**Theorem 1.2.** *([10], Theorem 1.3, p.312) Let $k$ be an uncountable algebraically closed field. Let $S$ be an integral, normal, quasi-projective $k$-scheme of dimension $b \geq 2$. Let $A$ be an Abelian scheme over $S$. For a very general line-pair $C$ in $S$, the restriction map of sections*

$$\text{Sections}(A/S) \to \text{Sections}(A_C/C)$$

*is a bijection. The theorem also holds with $C$ a very general planar surface in $S$. If $\operatorname{char} k = 0$, this also holds with $C$ a very general conic in $S$.*

In this article, we prove that there exists a broader class of varieties for which Theorem 1.2 holds for higher order curve-pairs and smooth curves as the following theorem.

**Theorem 1.3.** *Let $k$ an uncountable algebraically closed field of characteristic zero. Let $S$ be an integral, normal, quasi-projective $k$-scheme of dimension $b \geq 2$. Let $X$ be a smooth $S$-scheme admitting a finite morphism $f : X \to A$ to an Abelian scheme $A$ over $S$. Let $e$ be the fiber dimension of $\operatorname{Iso}(A)$ where $\operatorname{Iso}(A)$ is the isotrivial factor of $A$ (see Definition-Lemma 3.16). Let $d$ be a positive even integer.*

*Then, for $d > 2e - 2$, every section of $X_C$ over a very general genus-0 and degree-$(d+2)$ curve-pair or a very general genus-0, degree-$(d+2)$ smooth curve $C$ is the restriction of a unique global section of $X$ over $S$.*

*Remark* 1.4. Let $X \to S$ be a finite type morphism of locally Noetherian schemes where $S$ is integral. Denote by $K$ the function field of $S$. Then a rational point of the generic fiber $X_K$ is the same as a rational section of $X \to S$. When $X \to S$ is an Abelian scheme, every rational section is a section on the whole $S$, which also holds when $X$ admits a finite morphism to an Abelian scheme over $S$. Thus, the Theorems above claim that we can detect rational points of $X_K$ over the function field $K$ by restricting $X \to S$ to a very general curve in $S$.

In [10], the authors first prove the theorem of restriction of sections for line-pairs. To prove that the result also holds for smooth conics, Néron models is applied to deform line-pairs to smooth conics. We also need this technique in our Theorem 1.3, so we give the definition of Néron models as following.

By a Dedekind scheme, we always mean an irreducible, Noetherian and normal scheme of dimension one. Let $S$ be a Dedekind scheme with function field $K$. Let $X_K$ be a smooth and separated $K$-scheme of finite type. We say that $X$ is an *$S$-model* of $X_K$ if $X$ is an $S$-scheme with generic fiber isomorphic to $X_K$. A Néron model of $X_K$ is an $S$-model satisfying a universal property of extending morphisms. This extends the smooth variety $X_K$ to a family of smooth varieties over $S$. The precise definition is the following.

**Definition 1.5.** ([3], Def.1.2/1, p.12) Let $X_K$ be a smooth and separated $K$-scheme of finite type. A *Néron model* of $X_K$ is an $S$-model $X$ which is *smooth*, separated, and of finite type, and which satisfies the following universal property, called the *Néron mapping property*:

For each smooth $S$-scheme $Y$ and each $K$-morphism $u_K : Y_K \to X_K$ there is a unique $S$-morphism $u : Y \to X$ extending $u_K$.

From the uniqueness of the morphism extension, it is easy to see that a Néron model is unique as soon as it exists. If $X_K$ is an Abelian variety over $K$, then the existence of Néron model is proved in the survey book [3]. However, the Néron model of an Abelian variety is not necessary an Abelian scheme over the Dedekind scheme $S$ (cf. [3], Theorem 1.4/3, p.19).

**Theorem 1.6.** *([3], Theorem 1.4/3, p.19) Let $X_K$ be an abelian variety over $K$. Then $X_K$ admits a Néron model $X$ over $S$.*

The main application of Néron models in the proof of Theorem 1.2 is Lemma 4.13 in [10], p.323. However, going over the proof, it is easy to see that only the existence of extensions of morphisms is needed, and this is also the case for many other applications of Néron models. This leads us to weaken the definition of Néron mapping property, and consider a weak version of Néron model.

**Definition 1.7.** Let $X$ be a flat scheme of finite type over $S$. We say $X$ has the *weak extension property* if for every smooth morphism $Z \to S$ and every $K$-morphism $u_K : Z_K \to X_K$, there exists an $S$-morphism $u : Z \to X$ extending $u_K$.

**Definition 1.8.** Let $X_K$ be a smooth, finite type and $K$-variety. Suppose that $X$ is a flat and finite type scheme over $S$ with generic fiber $X_K$. We say that $X$ is a *pseudo-Néron model of its generic fiber* if $X$ satisfies the weak extension property.

*Remark* 1.9. In Definition 1.7, the extension $u$ of $u_K$ is not unique; however, if $X$ is separated, then the extension is unique. In Definition 1.8, we do not require that $X$ is normal or regular since after an étale base change $T \to S$, $X_T$ is not necessarily normal or regular. And, we stress that, unlike Néron models, a pseudo-Néron model is always not unique since it can be not smooth over $S$.

By [3] Proposition 1.2/8, we know that every Abelian scheme over $S$ satisfies the weak extension property. Moreover, from Theorem 1.6, every Abelian variety has a Néron model, and hence a pseudo-Néron model.

If a smooth variety $X_K$ admits a finite morphism to an Abelian variety $A_K$, then $X_K$ admits a pseudo-Néron model as we will prove in Theorem 2.5. Besides the application of pseudo-Néron model to prove Theorem 1.3, it is natural to ask:

**Question 1.10.** *Is there any other class of varieties, besides Abelian schemes, Abelian varieties and finite cover of Abelian varieties, satisfying the weak extension property or admitting pseudo-Néron models?*

In the first part of this article (Section 2), we give a positive answer to this question. It turns out that the existence of pseudo-Néron models is closely related to the non-existence of rational curves on the variety (Corollary 2.8). Our new example of pseudo-Néron models is the following, which will be proved as Corollary 2.16.

**Theorem 1.11.** *(New Examples) Let $k$ be an uncountable algebraically closed field. Let $S$ be a Dedekind scheme of finite type over $k$ with field of functions $K$ (e.g. $S$ is a smooth curve over $k$). Let $d$ be an integer prime to* $\operatorname{char} k$*. Suppose that $H \subset \mathbb{P}^n_k$ is a very general smooth hypersurface of degree $d \geq 2n-1$. Let $X_K$ be a smooth $K$-variety admitting a finite morphism to $H \times_k K \subset \mathbb{P}^n_K$. Then $X_K$ has a pseudo-Néron model over $S$. In particular, every smooth $K$-subvariety of $H \times_k K$ has a pseudo-Néron model.*

In this second part of this article (Section 3), we will use pseudo-Néron models to restate the Lemma 4.13 in [10] in a more general set up and prove the main result Theorem 1.3.

## 1.2 Review of theorems about sections

Now, we give a brief review of theorems about sections in literatures and compare our main result, Theorem 1.3, with these results.

A complex variety $V$ is said to be *rationally connected* if two general points of $V$ can be joined by a rational curve ([17], Definition 3.2, p.199). In [11], it is proved that a one-parameter family of rationally connected complex varieties has a section.

**Theorem 1.12.** *([11], Theorem 1.1, p.57) Let $f : X \to B$ be a proper morphism of complex varieties with $B$ a smooth curve. If the general fiber of $f$ is rationally connected, then $f$ has a section.*

**Definition 1.13.** ([12], Def.1.2, p.672) Let $\pi : X \to B$ be an arbitrary morphism of complex varieties. By a *pseudosection* of $\pi$ we will mean a subvariety $Z \subset X$ such that the restriction $\pi|_Z : Z \to B$ is dominant with rationally connected general fiber.

In [12], the authors prove the coverse of Theorem 1.12 as following.

**Theorem 1.14.** *([12], Theorem 1.3, p.672, [10], Theorem 1.1, p.311) Let $\pi : X \to B$ be a proper morphism of complex varieties. If $\pi$ admits a section when restricted to a very general sufficiently positive curve in $B$, then there exists a pseudosection of $\pi$.*

However, the theorem asserts only the existence of a pseudosection of $\pi$; it does not claim any direct connection between the sections of $X_C \to C$ over very general curves $C$ and the pseudosection. So the following question is asked in [12] and [10].

**Question 1.15.** *([10], Conjecture 1.2, p.311, [12], Question 7.1, p.689) If $\pi : X \to B$ is a morphism of complex varieties, then for a very general, sufficiently positive curve $C \subset B$, does every section of the restricted family $X_C = \pi^{-1}(C) \to C$ take values in a pseudosection?*

On the other hand, in Theorem 1.14, the genus and degree of the very general curve depend on the relative dimension of $\pi$ (see the statement of theorem in

[12], Theorem 1.3, p.672). The genus and degree can grow enormously fast with respect to the relative dimension of $\pi : X \to B$. So it is natural to ask the following questions.

**Question 1.16.** *([12], Section 7.3, p.689) Can we eliminate the dependence of the family of curves on the relative dimension of $\pi$ in Theorem 1.14?*

The answer of this question is "no". The detailed proof can be found in [27]. A sketch of the argument could also be found in [12], Section 7.3. Then, a further question is the following.

**Question 1.17.** *If the dependence in Theorem 1.14 can not be eliminated, how fast do the genus and degree of the family of curves grow?*

One extreme special case of Question 1.15 and Question 1.16 is that $X$ is an Abelian scheme over $B$. In this case, since the fibers contain no rational curves, every pseudosection is a rational section, and every rational section is everywhere defined. Then, Theorem 1.2 gives positive answers to both Question 1.15 and Question 1.16 when $X$ is an Abelian scheme over $B$.

When $X$ is a smooth scheme admitting a finite morphism to an Abelian scheme $A$ over $B$, Theorem 1.3 gives a positive answer to Question 1.15 and Question 1.17. The genus of curves is zero as in Theorem 1.2. And, the degree of the curves grows at a *linear* rate with respect to the relative dimension of the isotrivial factor of the Abelian scheme.

## 1.3 Idea of the proof

The idea to prove Theorem 1.3 is quite geometric. For a very general point $b \in S$ and a very general genus-0, degree-$d$, smooth curve $m$ containing $b$, there will be a subset $\mathcal{B}_{d,b}$ in $A_b$ (we actually take $\mathcal{B}_{d,b}$ in $\mathrm{Iso}(A)_b$, see subsection 3.6) such that Theorem 1.3 does not hold for sections over $m$ that maps $b$ to points in $\mathcal{B}_{d,b}$. We call this subset the *bad set*, see subsection 3.3.3 and subsection 3.6 for precise definitions. However, if we attach a very general conic $\ell$ to $m$ at a very general point on $m$, the set $\mathcal{B}_{d+2,b}$ for the curve $m \cup \ell$ has dimension strictly less than the dimension of $\mathcal{B}_{d,b}$. Therefore, if we increase the degree of the curve by attaching more conic curves, the bad set will be empty, and so Theorem 1.3 holds for every section over the very general curve.

# 2 Pseudo-Néron Model

## 2.1 Basic properties

In this section, we will assume that $S$ is a *Nagata* Dedekind scheme and $K$ is its function field. Recall that every scheme of finite type over a field is Nagata.

**Lemma 2.1.** *Suppose that $S = \operatorname{Spec} R$ is an affine Nagata Dedekind scheme. Let $Y_K$ be a smooth variety and $Y$ be a normal pseudo-Néron model over $S$. Let $X_K$ be a smooth $K$-variety with a finite $K$-morphism $f : X_K \to Y_K$. Then there exists a flat normal $S$-scheme $X$ admiting a finite morphism $g : X \to Y$ which extends $f$.*

*Proof.* First consider the affine case. Let $\operatorname{Spec} A_K$ be an affine open subset of $Y_K$ and $\operatorname{Spec} B_K = f^{-1}(\operatorname{Spec} A_K)$. Suppose that $\operatorname{Spec} A$ is an open affine in $Y$ with generic fiber $\operatorname{Spec} A_K$, where $K = \operatorname{Frac}(R)$ and $A_K = A \otimes_R K$. We claim that there exists a finite $A$-algebra $C$ such that $B_K = C \otimes_R K$. $\operatorname{Frac}(B_K)$ is a finite field extension of $\operatorname{Frac}(A_K)$ since $B_K$ is finite over $A_K$. Let $B'$ be the integral closure of $A$ in $\operatorname{Frac}(B_K)$. We have the ring $A$ is Nagata ([20], Prop.8.2.29(b), p.340, and Def.8.2.30, p.341), and hence $B'$ is finite over $A$ ([20], Def.8.2.27, p.340). Now, take $C = B_K \cap B'$. Then since $A$ is Noetherian we have $C$ is finite over $A$, and by construction $C$ is the integral closure of $A$ in $B_K$. It is easy to check that $B_K = C \otimes_R K$.

Now, take an affine open covering of $Y_K$, and hence an affine open covering of $X_K$. By the construction in the affine case, $C$ is uniquely determined by $B_K$, $A_K$ and $A$. Thus, we can glue the $\operatorname{Spec} C$ as above to form the normal scheme $X$ with a canonical finite morphism $g : X \to Y$, which is of finite type, separated and flat over $S$. □

**Lemma 2.2.** *Let $Y$ be a separated, flat $S$-scheme of finite type satisfying the weak extension property. Suppose that $X$ is an integral $S$-scheme with a finite $S$-morphism $f : X \to Y$. Then $X$ satisfies the weak extension property.*

*Remark* 2.3. The Dedekind scheme $S$ does not have to be Nagata in this lemma.

*Proof. Step 1*: Assume that $S = \operatorname{Spec} R$ is an affine Dedekind scheme. Let $Z$ be an irreducible smooth $S$-scheme with generic fiber $Z_K$ and a $K$-morphism $u_K : Z_K \to X_K$. We note that if $\operatorname{Spec} A$ is an affine open in $X$ then $A \otimes_R K$ is also an integral domain, so the generic fiber $X_K$ is also an integral scheme with the same function field $K(X_K) = K(X)$ as $X$.

First we assume that $X$ and $Y$ are affine. Since $Y$ satisfies the weak extension property, $f_K \circ u_K$ extends to an $S$-morphism $g : Z \to Y$. Denote $\zeta$ by one of the generic points of codimension one irreducible subsets of $Z$. Then, since $Z$ is normal, $\mathcal{O}_{Z,\zeta}$ is a discrete valuation ring. And we have the following commutative diagram

$$\begin{array}{ccc} \operatorname{Spec} K(Z) & \longrightarrow & X \\ \downarrow & \overset{u_\zeta}{\nearrow} & \downarrow f \\ \operatorname{Spec} \mathcal{O}_{Z,\zeta} & \longrightarrow & Y \end{array}$$

where $\operatorname{Spec} K(Z) \to X$ is induced by the map $K(X_K) \to K(Z_K)$. Moreover, by the properness of the morphism $f$, there exists a unique morphism $u_\zeta : \operatorname{Spec} \mathcal{O}_{Z,\zeta} \to X$ making the diagram commute. Since $Z$ is locally of finite type over $S$, the morphism $u_\zeta$ can be extended to a neighborhood $V$ of $\zeta$ in $Z$. We denote this morphism by $u_V : V \to X$. Checking every open affine $\operatorname{Spec} C$ in $V$, since $Z$ is an integral scheme, we have that the generic fiber of the morphism from $\operatorname{Spec} C$ to $X$ is the same as the restriction of $u_K$ because they are giving the same morphism when viewed as restriction of $K(X) \to K(Z)$. Moreover, suppose that there are two codimension one points $\zeta_1$ and $\zeta_2$, and they are giving two extensions $V_1 \to X$ and $V_2 \to X$ respectively. Since $f$ is separated, it is easy to see that these two morphisms agree on every open affine in the overlap $V_1 \cap V_2$ since they have the same generic fiber and hence give the same map in $K(X) \to K(Z)$. Therefore, the morphism $u_K$ can be extended to a rational map defined over every condimension one point on $Z$. Hence, by [3] Lemma 4.4/2, since $X$ is affine, this rational map is actually defined everywhere.

Now, when $X$ and $Y$ are not affine, consider an open affine covering of $Y$, which induces an affine covering of $X$, and extend $u_K$ for every such open affine of $X$. By the same reason as above, since any two extensions give the same morphism on generic fibers, these extensions on affines of $X$ can be glued, and hence $X$ satisfies the weak extension property.

*Step 2*: $S$ is a Dedekind scheme, not necessarily affine. Take an affine covering $\{U_1, \cdots, U_n\}$ of $S$. Let $X_i$, $Y_i$, $Z_i$ and $(u_K)_i$ be the base changes of $X$, $Y$, $Z$ and $u_K$ from $S$ to $U_i$ respectively. From step 1, we know that every $(u_K)_i$ can be extended to the whole $Z_i$. Now, cover each $U_i \cap U_j$ by open affines and check over each these affines. Again, since these extensions give the same morphism on generic fibers of their overlap, the extensions can be glued and give an extension of $u_K$ to the whole $Z$. So $X$ satisfies the weak extension property. □

*Remark* 2.4. Let $\mathcal{C}$ denote the category of normal $S$-schemes with finite morphisms. Then the above lemma asserts that normal $S$-schemes with the weak extension property form a fully faithful subcategory of $\mathcal{C}$.

**Theorem 2.5.** *Let $S$ be a Nagata Dedekind scheme with generic point* $\operatorname{Spec} K$. *Let $X_K$ be a smooth scheme admitting a finite $K$-morphism to a smooth, separated variety $Y_K$ of finite type which has a normal pseudo-Néron model $Y$ over $S$. Then $X_K$ has a normal pseudo-Néron model $X$ over $S$.*

*Proof.* Let $\{U_1, \cdots, U_n\}$ be a finite open affine covering of $S$ and $Y^i$ be the inverse images of each $U_i$ such that they form an open covering of $Y$. Then $Y^i_K$ and $X^i_K = f^{-1}(Y^i_K)$ form an open covering of $Y_K$ and $X_K$ respectively. For each $i$, $Y^i$ is flat, separated and of finite type over the affine Nagata Dedekind scheme $U_i$. Take $X^i$ to be the $U_i$-model of $X^i_K$ as constructed in Lemma 2.1 and $g_i : X^i \to Y^i$ to be the corresponding $U_i$-morphism extending $f|_{Y^i_K}$.

Cover each $Y^i$ by affine opens. Suppose that $\operatorname{Spec} A^i$ and $\operatorname{Spec} A^j$ are two such affine opens in $Y^i$ and $Y^j$ respectively with $i \neq j$. Since $\operatorname{Spec} A^i$ and $\operatorname{Spec} A^j$ are affine schemes over affine bases, $U_i$ and $U_j$, also their generic fibers are affine. Let the inverse image of $\operatorname{Spec} A^i_K$ (resp. $\operatorname{Spec} A^j_K$) in $X^i_K$ (resp. $X^j_K$) be $\operatorname{Spec} B^i_K$ (resp. $\operatorname{Spec} B^j_K$). Then, as in the construction in Lemma 2.1, we can construct an affine open $\operatorname{Spec} C^i$ (resp. $\operatorname{Spec} C^j$) as the integral closure of $A^i$ in $B^i_K$ (resp. $A^j$ in $B^j_K$). Now, by Nike's trick ([29], Prop.5.3.1, p.157), cover $\operatorname{Spec} A_i \cap \operatorname{Spec} A_j$ by principal open affines. Then they have affine generic fibers since $\operatorname{Spec} A^i$ and $\operatorname{Spec} A^j$ have affine bases. Because the process of taking integral closure is unique up to a unique isomorphism and compatible with localization, the affine opens $\operatorname{Spec} C^i$ and $\operatorname{Spec} C^j$ with morphisms $g_i$ and $g_j$ can be glued. By the uniqueness of taking integral closure, we can make the same gluing for other pairs of affine opens in the fixed affine covering of $Y^i$ and $Y^j$. Thus, we obtain a gluing of $X^i$ and $X^j$. Similarly, $X^1, \cdots, X^n$ glue to be an $S$-scheme $X$ admitting a finite $S$-morphism to the normal scheme $Y$.

Take the $S$-model $X$ of $X_K$ as above. By applying Lemma 2.2 to the scheme $X$, we have this normal $S$-scheme satisfies the weak extension property. □

This theorem gives us a strategy. Suppose that $S$ is a Nagata Dedekind scheme. Then every time we have a class of varieties admitting normal pseudo-Néron models, by considering smooth varieties with finite morphisms to the varieties in this class, we will get a new class of varieties admitting normal pseudo-Néron models. As a first result, we know that all varieties admitting finite morphisms to Abelian varieties have normal pseudo-Néron models. In particular, every smooth subvariety of an Abelian variety has a normal pseudo-Néron model.

## 2.2 Application of rational curves

In [19], Qing Liu and Jilong Tong proved theorems about Néron models of smooth proper curves of positive genus, see [19], Theorem 1.1, p.7019, for details. In our situation, their result ([19], Prop.4.13, p.7031) in the higher dimensional case can be used to construct new examples of varieties admitting pseudo-Néron models. We start with the basic notion of rational curves as following.

**Definition 2.6.** ([19], p.7031) Let $V$ be a variety over an algebraically closed field $k$. We say that $V$ *contains a rational curve* if there is a locally closed subscheme of $V$ which is isomorphic to an open dense subscheme of $\mathbb{P}^1_k$.

If $V$ is proper over $k$, then every morphism from an open dense of $\mathbb{P}^1_k$ can be extended to the whole $\mathbb{P}^1_k$ ([20], Cor. 4.1.17, p.119). Moreover, by Lüroth's theorem, our definition is the same as the existence of a nonconstant morphism from $\mathbb{P}^1_k$ to $V$ ([17], Definition 2.6, p.105).

**Proposition 2.7.** *([19], Prop.4.13, p.7031) Let $S$ be a Dedekind scheme with field of functions $K$. Let $X_K$ be a smooth proper variety over $K$. Suppose $X_K$ has a proper regular $S$-model $X$ such that no geometric fiber $X_{\overline{s}}$, $s \in S$, contains a rational curve. Then the smooth locus $X_{sm}$ of $X$ is the Néron model of $X_K$.*

Note that the regularity of $X$ in the above theorem is only used to apply [19] Cor.3.12. Thus, in the case of pseudo-Néron models, the same proof gives the following corollary.

**Corollary 2.8.** *Let $S$ be a Dedekind scheme with field of functions $K$. Let $X_K$ be a smooth proper variety over $K$. Suppose $X_K$ has a proper and flat $S$-model $X$ such that no geometric fiber $X_{\overline{s}}$, $s \in S$, contains a rational curve. Then $X$ satisfies the weak extension property, i.e., $X$ is a pseudo-Néron model of $X_K$.*

The following lemma is well-known.

**Lemma 2.9.** *Let $k$ be an algebraically closed field. Let $f : X \to Y$ be an étale surjective $k$-morphism of proper $k$-varieties. If $X$ does not contain any rational curve, then $Y$ does not contain any rational curve.*

Lemma 2.9 gives an immediate application of Corollary 2.8 as following.

**Corollary 2.10.** *Let $X$ be a proper pseudo-Néron $S$-model of $X_K$ such that no geomertric fiber contains rational curves, as in Corollary 2.8. Suppose that $Y$ is a proper $S$-scheme with smooth generic fiber $Y_K$ and there exists an étale surjective morphism $f : X \to Y$. Then $Y$ is a pseudo-Néron model of $Y_K$.*

There are many varieties which do not contain any rational curve. One of the typical examples is very general hypersurfaces of large degree. We cite the following result.

**Theorem 2.11.** *([25], Theorem 1.2) Let $k$ be an uncountable algebraically closed field. For $d \geq 2n - 1$, a very general hypersurface $X \subset \mathbb{P}^n_k$ of degree $d$ contains no rational curves, and moreover, the locus of hypersurfaces that contain rational curves will have codimension at least $d - 2n + 2$.*

**Lemma 2.12.** *Let $R$ be a Nagata DVR with fraction field $K$ and residue field $k$. Suppose that $X$ is a proper scheme over $R$ with nonempty fibers. If $X_{\overline{k}}$ contains no rational curves, then $X_{\overline{K}}$ also contains no rational curves.*

*Proof.* Suppose that there is a nonconstant $K$-morphism $f_{\overline{K}} : \mathbb{P}^1_{\overline{K}} \to X_{\overline{K}}$. By limit arguments, there exists a discrete valuation ring $T$ with fraction field $L$, finite over $K$, and residue field $l$ such that $R \subset T \subset \overline{K}$, $T$ dominates $R$, and $f_{\overline{K}}$ is the base change of a nonconstant $L$-morphism $f_L : \mathbb{P}^1_L \to X_L$. Consider the generic point of the special fiber $\mathbb{P}^1_l$ which is of codimension one in $\mathbb{P}^1_T$. Using the valuative criterion of properness, $f_L$ extends uniquely to a $T$-morphism $f_T : V \to X_T$ where $V$ is an open dense of $\mathbb{P}^1_T$ containing the generic point of $\mathbb{P}^1_l$ ([20], Prop.4.1.16, p.119), and hence an open dense of $\mathbb{P}^1_l$.

Let $\Gamma$ be the normalization of the schematic closure for the graph of $f_T$. Then the projection $\mathbb{P}^1_T \times_T X_T \to \mathbb{P}^1_T$ induces a birational morphism $\pi : \Gamma \to \mathbb{P}^1_T$.

Since $R$ is Nagata, $T$ is finite over $R$, so $T$ is also Nagata ([20], Def.8.2.27 and Prop.8.2.29, p.340). Thus, all the schemes are Nagata and the normalization morphism is finite. And, hence, $\pi$ is a proper birational morphism. Let $E$ be the exceptional locus of $\pi$. By Abhyankar's lemma ([18], Theorem 4.26, p.112), $E$ is ruled over its image. Let $E'$ be its image. Then $E$ is birationally equivalent over $E'$ to $W \times_{E'} \mathbb{P}^1_{E'}$. Note that, since $f_T$ is defined over $V$, $E'$ is finitely many closed points in the closed fiber $\mathbb{P}^1_l$ and $E$ is of codimension one. Then $W$ is dimension zero over $E'$, hence finitely many closed points. Thus, $W \times_{E'} \mathbb{P}^1_{E'}$ is a finite disjoint copy of $\mathbb{P}^1_{k_j}$ with each $k_j$ a finite extension of $l$. Base change to the algebraic closure $\bar{l}$ of $l$, then each $\mathbb{P}^1_{k_j}$ splits to finitely many disjoint copies of $\mathbb{P}^1_{\bar{l}}$. Now, take one of these copies, say, $C$. If $C$ is mapped to a single point of $X_{\bar{l}}$, then the image of $C$ in $(\mathbb{P}^1_T \times_T X_T) \times_T \bar{l}$ is a single point, contradicting that $\Gamma_{\bar{l}} \to (\mathbb{P}^1_T \times_T X_T) \times_T \bar{l}$ is a finite morphism. Therefore, $C$ is a rational curve in $X_{\bar{l}} = X_{\bar{k}}$. And this contradiction shows that $X_K$ does not contain any rational curve. ☐

**Definition 2.13.** Let $R$ be a DVR with fraction field $K$ and $E = R^\times \cup \{0\}$. Let $H$ be a hypersurface in $\mathbb{P}^n_K$. We say that $(H, f)$ is a *unitary hypersurface* if the defining equation $f$ of $H$ has coefficients in $E$.

**Theorem 2.14.** *Let $R$ be a Nagata DVR with fraction field $K$. Suppose that the residue field $k$ is uncountable and algebraically closed, and $d$ is an integer prime to* $\operatorname{char} k$. *Then, there exists unitary hypersurfaces of degree $d \geq 2n - 1$ in $\mathbb{P}^n_K$ admitting a Néron model.*

*Proof.* Suppose that $X_K = V_+(f)_K \subset \mathbb{P}^n_K$ is a unitary hypersurface defined by an irreducible homogeneous polynomial $f$ of degree $d$ in $n+1$ variables. Since all the nonzero coefficients of $f$ are in the group of units $E$ of $K$, there is no term in $f$ vanishing in the residue field of $R$, so the specialization $X_k = V_+(f)_k$ is a hypersurface of degree $d$ in $\mathbb{P}^n_k$. Conversely, every hypersurface of degree $d$ in $\mathbb{P}^n_k$ arises as a specialization of some unitary hypersurface of degree $d$ in $\mathbb{P}^n_K$.

Set $N = \binom{n+d}{d} - 1$. Let $E$ be the space of unitary hypersurfaces in $\mathbb{P}^n_K$. The argument above gives a surjective map of parameter spaces $F : E \to \mathbb{P}^N_k$ by sending $(X_K, f)$ to its specialization $X_k$. Let $U$ be an intersection of countably many open dense subsets of $\mathbb{P}^N_k$ such that every member in $U$ is smooth without rational curves. Take $X_K \in F^{-1}(U)$ a $K$-point. By Lemma 2.12, there is no rational curve on $X_K$. Let $X = V_+(f) \subset \mathbb{P}^n_R$ be the $R$-model of $X_K$.

Since $f$ is irreducible, $X$ is an integral hypersurface. Thus, $X$ is flat over $\operatorname{Spec} R$ and every nonempty fiber is irreducible of dimension $n-1$([20], Cor.4.3.10, p.137). Let $\mathrm{Fitt}_{n-1}(\Omega^1_{X/R})$ be the $(n-1)$-th Fitting ideal of $\Omega^1_{X/R}$, which is a coherent ideal sheaf of $\mathcal{O}_X$. Then, $\mathrm{Fitt}_{n-1}(\Omega^1_{X_k/k})$ is equal to $(\mathrm{Fitt}_{n-1}(\Omega^1_{X/R})) \cdot \mathcal{O}_{X_k}$ ([6], Cor.20.5, p.498). Since $X_k$ is smooth, $\Omega^1_{X_k/k}$ is locally free of rank $n-1$. Thus, $\mathrm{Fitt}_{n-1}(\Omega^1_{X_k/k})$ is equal to $\mathcal{O}_{X_k}$ ([6], Prop.20.6, p.498). And hence, $\mathrm{Fitt}_{n-1}(\Omega^1_{X/R})$ is equal to $\mathcal{O}_X$. Then, $\Omega^1_{X/R}$ can be locally generated by $n-1$ elements ([6], Prop.20.6, p.498). So $\Omega^1_{X_K/K}$ can be locally generated by $n-1$

elements, and hence locally free of rank $n-1$ ([20], Lemma 6.2.1, p.220, and [3], Prop.2.2/15, p.43). Thus $X_K$ is smooth. At this stage, every fiber of $X$ is smooth and $X$ is flat over $R$, then $X$ is a smooth $R$-scheme ([3], Prop.2.4/8, p.53). Therefore, by Lemma 2.12 and Proposition 2.7, $X$ is the Néron model of $X_K$. □

This theorem gives a direct corollary in the geometric setting as following.

**Corollary 2.15.** *Let $k$ be an uncountable algebraically closed field. Let $S$ be a Dedekind scheme of finite type over $k$ with field of functions $K$ (for example, $S$ is a smooth curve over $k$). Let $d$ be an integer prime to* $\operatorname{char} k$. *Then, a very general smooth hypersurface of degree $d \geq 2n-1$ defined over $k$ in $\mathbb{P}^n_K$ has a Néron model. In particular, the Néron model of such a hypersurface is the constant family over $S$.*

Note that we say a $K$-scheme $X$ is defined over $k$ if there exists a $k$-scheme $Y$ such that $X$ is isomorphic to $Y \times_{\operatorname{Spec} k} \operatorname{Spec} K$ (see [17], Definition 1.15, p.19).

*Proof.* $S$ is a Nagata Dedekind scheme ([20], Prop.8.2.29, p.340). First assume that $S = \operatorname{Spec} R$ affine. Let $f$ be a homogeneous polynomial of degree $d \geq 2n-1$ defined over $k$ in $\mathbb{P}^n_k$ such that there is no rational curve on the smooth hypersurface $V_+(f)$. Define $X_K = V_+(f)_K$ in $\mathbb{P}^n_K$. Denote $V_+(f)_R \subset \mathbb{P}^n_R$ by $X$, an $R$-model of $X_K$. Then, exactly the same argument as in Theorem 2.14 shows that $X$ is the Néron model of $X_K$.

Now, take a finite affine covering $\{\operatorname{Spec} R_i\}_{i\in I}$ of $S$. Then there exists a Néron model $X^i$ of $X_K$ over $\operatorname{Spec} R_i$ for every $i \in I$. By the uniqueness of Néron model and that Néron model is local on the base ([3], Prop.1.2/3, p.13), $\{X^i\}_{i\in I}$ glues to be a Néron model of $X_K$ over $S$. □

Combining Theorem 2.1 and Corollary 2.15, we get the following corollary.

**Corollary 2.16.** *Keep the notations of Corollary 2.15. Let $X_K$ be a smooth $K$-variety admitting a finite morphism to a very general smooth hypersurface of degree $d \geq 2n-1$ defined over $k$ in $\mathbb{P}^n_K$. Then $X_K$ has a normal pseudo-Néron model over $S$. In particular, every smooth subvariety of a very general hypersurface of degree $d \geq 2n-1$ in $\mathbb{P}^n_K$, where the hypersurface is defined over $k$, has a normal pseudo-Néron model.*

From this Corollary, we see that there are many smooth varieties admitting normal pseudo-Néron model over a smooth curve defined over an uncountable algebraically closed field. In the situation of our corollary, we cannot control the regularity of other fibers except $X_K$. It is a normal model of $X_K$, but in general not a Néron model in the sense of Definition 1.5. Moreover, the variety $X_K$ is not necessarily defined over $k$, unlike the constant case in Corollary 2.15.

## 2.3 Base change properties

The next lemma shows that pseudo-Néron models commute with étale extension of the base scheme, which is the analogue of [3] Prop.1.2/2 (c) for Néron models.

**Lemma 2.17.** *Let $S$ be a Dedekind scheme with function field $K$ and $X_K$ be a smooth $K$-variety with pseudo-Néron model $X$ over $S$. Suppose that $S'$ is another Dedekind scheme with $S' \to S$ étale and the function field of $S'$ is $K'$. Let $X_{S'} = X \times_S S'$ and $X_{K'} = X_K \times_K K'$ be its generic fiber. Then $X_{S'}$ is a pseudo-Néron model of $X_{K'}$.*

*Proof.* Let $Z$ be smooth of finite type over $S'$. Take a $K'$-morphism $Z_{K'} \to X_{K'}$. Then, $Z$ is smooth over $S$ and $Z_{K'}$, as the $K$-generic fiber, is smooth over $K$. By the weak extension property of $X$, there exists an $S$-morphism from $Z \to X$ extending $Z_{K'} \to X_K$. Hence the universal property of fiber products gives $Z \to X'$ as an extension of $Z_{K'} \to X_{K'}$. □

The following lemma is an analogue of [3] Prop.1.2/4. However, since a pseudo-Néron model is not unique, we can not have the converse direction as in [3] Prop.1.2/4.

**Lemma 2.18.** *Let $S$ be a Dedekind scheme with function field $K$, $X$ finite type over $S$ and it is a pseudo-Néron model of its generic fiber. Then, for each closed point $s \in S$, the $\mathcal{O}_{S,s}$-scheme $X_s = X \times_S \mathcal{O}_{S,s}$ is a pseudo-Néron model of its generic fiber.*

*Proof.* Let $Y_s$ be an smooth $\mathcal{O}_{S,s}$-scheme with a $K$-morphism $u_K : Y_{s,K} \to X_{s,K}$. By limit arguments ([3], Lemma 1.2/5), there exists a connected open neighborhood $S'$ of $s$, and a smooth $S'$-scheme $Y'$ such that $Y' \times_{S'} \operatorname{Spec} \mathcal{O}_{S,s} = Y_s$. Lemma 2.17 gives that $X_{S'} = X \times_S S'$ is a pseudo-Néron model of $X_K$ over $S'$. Then, by the weak extension property of $X_{S'}$, $u_K$ extends to an $S'$-morphism $u' : Y' \to X_{S'}$. Therefore, the base change $u = u' \times_{S'} \operatorname{Spec} \mathcal{O}_{S,s}$ is a required extension of $u_K$. □

**Definition 2.19.** Let $S$ be a Dedekind scheme and let $X$ be an $S$-scheme satisfying the weak extension property. We say that $X$ *satisfies the weak extension property universally* if for any $S'$ a Dedekind scheme and for any $S' \to S$ of finite type, the base change $X \times_S S'$ also satisfies the weak extension property.

**Definition 2.20.** Let $S$ be a Dedekind scheme with fraction field $K$. Let $X_K$ be a smooth, separated $K$-scheme of finite type, and let $X$ be a pseudo-Néron model of $X_K$. We say that $X$ is a *universal pseudo-Néron model* of $X_K$ if $X$ satisfies the weak extension property universally.

**Lemma 2.21.** *Keep the notations and hypothesis as in Corollary 2.8. Then, $X$ is a universal pseudo-Néron model of $X_K$.*

*Proof.* Let $g : S' \to S$ be a morphism of Dedekind schemes of finite type and $X' = X \times_S S'$. Take $s \in S$ and $t \in S'$ closed points such that $s = g(t)$. Then the residue field $\kappa(t)$ is finite over $\kappa(s)$, thus $X'_{\overline{t}} = X_{\overline{s}}$, and hence $X'_{\overline{t}}$ does not contain rational curves. And by Lemma 2.12, there is no rational curve on the geometric generic fiber of $X'$. Then $X'$ satisfies the weak extension property by Corollary 2.8. □

**Lemma 2.22.** *Let $S$ be a Dedekind scheme with function field $K$. Let $X_K$ be a smooth and separated variety of finite type over $K$. If $X_K$ has a proper $S$-model satisfying the weak extension property universally, then $X_K$ contains no rational curve.*

*Proof.* Suppose that $X$ is a proper $S$-model of $X_K$, i.e., $X$ is flat, separated and finite type over $S$ satisfying the weak extension property and has generic fiber $X_K$. By Lemma 2.18, we can replace $S$ by $\mathcal{O}_{S,s}$ for any closed point of $S$, and assume that $S = \operatorname{Spec} R$ for some discrete valuation ring R. If $X_K$ contains a rational curve, then there exists a nonconstant $\overline{K}$-morphism $f_{\overline{K}} : \mathbb{P}^1_{\overline{K}} \to X_{\overline{K}}$. By a limit argument as in Lemma 2.12, there exists a DVR in $\overline{K}$ dominating $R$ with fraction field $L$ and residue field $l$ such that $f_{\overline{K}}$ is a base change of a nonconstant morphism $f_L : \mathbb{P}^1_L \to X_L$. Since $X$ is a universal pseudo-Néron model, $X_T$ also satisfies the weak extension property.

Let $C$ be the normalization of the schematic closure of $f_L$. Then, $C$ is a proper normal curve over the field $L$ since $X_L$ is Nagata, and hence $i : C \to X_L$ is a finite morphism. Moreover, $f_L$ has a unique factorization via $C$, i.e., $f_L = i \circ g_L$ where $g_L : \mathbb{P}^1_L \to C$ is a morphism of nonsingular proper curves. Then $g_L$ is finite ([20], Lemma 7.3.10 and Cor.4.4.7). Therefore, $f_L$ is a finite morphism.

Since $X_T$ satisfies the weak extension property, $f_L$ extends to a $T$-morphism $f_T : \mathbb{P}^1_T \to X_T$. Let $f_l$ be the closed fiber of $f_T$. Then, the morphism $f_l$ is nonconstant. The same argument as for $f_L$ shows that $f_l$ is finite. Therefore, $f_T$ is finite ([20], Cor.4.4.7). Then, from Lemma 2.2, $\mathbb{P}^1_T$ satisfies the weak extension property. Now, consider an $L$-isomorphism $\sigma_L : \mathbb{P}^1_L \to \mathbb{P}^1_L$. However, not all these isomorphisms can be extended to be a $T$-morphism $\mathbb{P}^1_T \to \mathbb{P}^1_T$ ([3], Example 5, p.75), contradicting that $\mathbb{P}^1_T$ satisfies the weak extension property. Thus, $X_K$ contains no rational curve. □

*Remark* 2.23. Let $X_K$ be a smooth, separated $K$-scheme of finite type, and let $X$ be a proper $S$-model of $X_K$. The above lemma and theorem give us the following picture:

(i) no rational curve in any geometric fiber,

(ii) universal pseudo-Néron model,

(iii) no rational curve in the generic geometric fiber.

Then, (i)⇒(ii)⇒(iii).

# 3 Theorem of Restriction of Sections

## 3.1 Higher dimensional pseudo-Néron model

We will need the notion of higher dimensional pseudo-Néron model which generalizes definition 4.10 in [10].

**Definition 3.1.** ([10] Definition 4.10) Let S be an integral, regular, separated, Noetherian scheme of dimension $b \geq 1$. A *flat*, finite type, separated morphism $X \to S$ has the *weak extension property* if for every triple $(Z \to S, U, s_U)$ of

(i) a smooth morphism $Z \to S$,

(ii) a dense, open subset $U \subset S$,

(iii) and an $S$-morphism $s_U : Z \times_S U \to X_U$,

there exists a pair $(V, s_V)$ of

(i) an open subset $V \subset S$ containing U and all codimension 1 points of S,

(ii) and an S-morphism $s_V : Z \times_S V \to X$ whose restriction to $Z \times_S U$ is equal to $s_V$.

**Definition 3.2.** Let $S$ be an integral, regular, separated, Noetherian scheme of dimension $b \geq 1$. Let $K$ be the fraction field of $S$, and $X_K$ be a smooth, separated $K$-scheme of finite type. A flat, finite type, separated $S$-scheme $X$ is called a *pseudo-Néron model* of $X_K$ if $X_K$ is isomorphic to its generic fiber and $X$ satisfies the weak extension property as in Definition 3.1.

By a limit argument, it is easy to see that Definition 3.1 (resp. Definition 3.2) implies Definition 1.7 (resp. Definition 1.8) when $S$ is a Dedekind scheme, and they agree when $S = \operatorname{Spec} R$, where $R$ is a DVR. Now, we prove the corresponding results for Lemma 2.1, Lemma 2.2 and Theorem 2.5.

**Lemma 3.3.** *Suppose that $S$ is an integral, regular, separated, Noetherian Nagata scheme of dimension $b \geq 1$ with fraction field $K$. Let $Y_K$ be a smooth $K$-variety and $Y$ be its normal pseudo-Néron model over $S$. Let $X_K$ be a smooth $K$-variety with a finite $K$-morphism $f : X_K \to Y_K$. Then there exists a flat normal $S$-scheme $X$ admiting a finite morphism $g : X \to Y$ which extends $f$.*

*Proof.* Since $S$ is Noetherian, we can cover $S$ by finitely many affine opens. As we assume that $S$ is Nagata, the same proof of Lemma 2.1 and Theorem 2.5 gives the extension as claimed. □

**Lemma 3.4.** *Keep the same hypothesis of $S$ as in Lemma 3.3. Let $Y$ be a separated, flat $S$-scheme of finite type satisfying the weak extension property. Suppose that $X$ is an integral $S$-scheme with a finite $S$-morphism $f : X \to Y$. Then $X$ satisfies the weak extension property.*

*Proof.* Let $U$ be a dense open of $S$, and let $Z$ be a smooth $S$-scheme with a $U$-morphism $t_U : Z_U \to X_U$. Composing this morphism with $f_U$ gives a $U$-morphism $s_U : Z_U \to Y_U$. Since $Y$ satisfies the weak extension property, there exists an open dense $V$ in $S$ containing all the codimension one points, and an extension $s : Z_V \to Y_V$ of $s_U$. Up to replacing $S$ by $V$, we can assume that $V$ is the whole $S$. Cover $S$ and $Y$ by open affines as in Lemma 2.2, then the same proof as in Lemma 2.2 completes the proof. □

Therefore, combining the above two lemmata and the proof of Theorem 2.5, we get the following theorem.

**Theorem 3.5.** *Keep the same hypothesis of $S$ as in Lemma 3.3. Let $X_K$ be a smooth scheme admitting finite $K$-morphism to a smooth, separated variety $Y_K$ of finite type which has a normal pseudo-Néron model $Y$ over $S$. Then $X_K$ has a normal pseudo-Néron model $X$ over $S$.*

*The theorem also holds if $X_K$ and $Y_K$ are replaced by some $X_U$ and $Y_U$ defined over a dense open $U$ of $S$.*

Moreover, Corollary 4.12 in [10] and Theorem 3.5 give us the following corollary.

**Corollary 3.6.** *Let $W$ be an integral, regular, separated, Nagata Noetherian scheme of dimension $b \geq 1$. Let $S$ be a dense open subset of $W$, and let $X$ be a scheme admitting a finite morphism to an Abelian scheme over $S$. There exists an open subset $\widetilde{S}$ of $W$ containing $S$ and all codimension one points, and there exists a normal pseudo-Néron model $\widetilde{X}$ over $\widetilde{S}$ whose restriction over $S$ equals $X$.*

## 3.2 Bertini's theorems for higher order curves

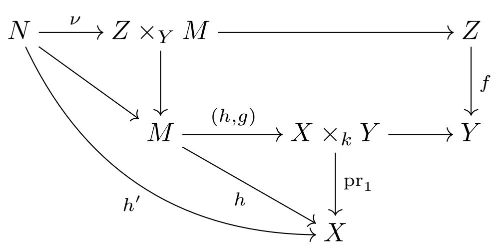
Recall that a scheme $X$ is called *algebraically simply connected* if for every connected scheme $Y$, and every surjective finite étale morphism $f : Y \to X$, the morphism $f$ is an isomorphism ([5], p.97). In this section, by a scheme over a field $k$ or a $k$-scheme, we mean a scheme that is of finite type over $k$.

**Theorem 3.7.** *([21], Prop.3.1) Let $k$ be a field of characteristic zero. Let $X$ be a smooth, algebraically simply connected variety over $k$. Let $N$ be a normal, connected and quasi-projective $k$-scheme. Let $h : N \to X$ be a projective $k$-morphism. If the closed subscheme $N_h$ of $N$ where $h$ is not smooth has codimension at least 2, then the geometric generic fiber of $h$ is connected.*

We include the proof here for completeness.

*Proof.* Let $u : \widetilde{N} \to X$ be the finite part of the Stein factorization of $h$. Since $N_h$ has codimension at least two, $\widetilde{N}_h$ also has codimension at least two. Since $X$ is smooth, $u$ is étale by the Purity Theorem ([14], X, section 3). Since $X$ is algebraically simply connected, $u$ is an isomorphism. Therefore, $h$ has connected fibers. □

**Proposition 3.8.** *Let $k$ be a field. Let $X$ be a smooth, irreducible $k$-scheme that is algebraically simply connected. Let $Y$ be an irreducible quasi-projective $k$-scheme. Let $M$ be a normal, irreducible, quasi-projective $k$-scheme. Let $(h, g) : M \to X \times_k Y$ be a $k$-morphism such that $h$ is projective and surjective, and $g$ is dominant with irreducible geometric generic fiber. Let $Z$ be an irreducible $k$-scheme. Let $f : Z \to Y$ be a finite, surjective $k$-morphism. Denote by $\nu : N \to Z \times_Y M$ the normalization of the fiber product $Z \times_Y M$. Denote by $h' : N \to X$ the composition of $h$ and projection from $N$ to $M$. If the closed subscheme of $N$ where $h'$ is not smooth has codimension at least 2, then the geometric generic fiber of $h'$ is connected.*




*Proof.* Since the geometric generic fiber of $g$ is connected and $Z$ is irreducible, also the normalization $N$ is irreducible. Then the statement reduces to Theorem 3.7 because $h'$ is projective. □

**Definition 3.9.** Let $Y$ be a regular locally Noetherian scheme. Let $f : Z \to Y$ be a finite surjective morphism that is generically étale. The closed subscheme $R$ inside $Z$ where $f$ is not étale is called *the ramification locus* of $f$. The closed subscheme $B = f(R)$ of $Y$ is called *the branch locus* of $f$.

Note that, by Zariski's purity theorem ([20], Exercise 8.2.15(c), p.347), the branch locus $B$ of $f$ is either empty or pure of codimension one if $f$ generically separable. In particular, this holds when $\operatorname{char} k = 0$ and $Y$ is a smooth $k$-scheme.

**Theorem 3.10.** *Let $k$ be an algebraically closed field of characteristic zero. Let $Z$ be a normal $k$-scheme. Let $f : Z \to \mathbb{P}^n_k$ be a finite surjective morphism that is generically étale. Then, for a general smooth curve $C \subset \mathbb{P}^n_k$, the inverse image $f^{-1}(C)$ is a smooth curve.*

*Proof.* This follows from Kleiman-Bertini's Theorem ([15], Theorem III.10.8). □

**Theorem 3.11.** *Let $k$ be an algebraically closed field of characteristic zero. Let $f : Z \to \mathbb{P}^n_k$ be a finite surjective morphism from a normal irreducible variety. Then, for a general genus-0, degree-$d$ smooth curve $C$ in $\mathbb{P}^n_k$, the inverse image $f^{-1}(C)$ is connected.*

*Proof.* Keep the notations in Proposition 3.8. Let $Y$ be projective space $\mathbb{P}^n_k$. Let $X$ be the non-stacky locus inside the stack of genus-0, degree-$d$ stable maps to $\mathbb{P}^n_k$. In other words, $X$ is the maximal open subscheme of this stack. The open subscheme $X$ is algebraically simply connected. Let $M$ be the universal family of curves over $X$, and let $g$ be the universal morphism. Then, the generic geometric fiber of $g$ is connected.

To complete the proof, we need to prove that the singular locus of $h'$ inside $N$ has codimension at least 2. The codimension one subset of $X$ parameterizes degree-$d$, genus-0 curves in $\mathbb{P}_k^n$ that are not transversal to the branch locus. Thus, away from codimension one points in $X$, the fibers are everywhere smooth (Theorem 3.10). Moreover, for a genus-0, degree-$d$ curve that is not transversal to the branch locus, the singularities of the fiber of $h'$ occur only over the intersection points of the curve with the branch locus, and this is codimension one in the fiber. Thus, the total codimension of singular locus of $h'$ in $N$ is at least two. By Proposition 3.8, for a general genus-0, degree-$d$ curve $C$ in $\mathbb{P}_k^n$, the inverse image $f^{-1}(C)$ is connected. □

**Corollary 3.12.** *Let $k$ be an algebraically closed field of characteristic zero. Let $f : Z \to \mathbb{P}_k^n$ be a finite surjective morphism from a normal irreducible variety. Then, for a general genus-0, degree-$d$ smooth curve $C$ in $\mathbb{P}_k^n$, the inverse image $f^{-1}(C)$ is smooth and irreducible.*

**Theorem 3.13.** *Let $k$ be an algebraically closed field of characteristic zero. Let $Z$ be a normal $k$-scheme that is not necessarily connected. Let $f : Z \to S$ be a finite surjective morphism to a smooth, connected, quasi-projective $k$-scheme $S$ where $S$ admits a finite, generically étale morphism to an open dense subset of $\mathbb{P}_k^n$, $u_0 : S \to \mathbb{P}_k^n$. Then, for a general genus-0, degree-$d$ smooth curve $C$ in $S$ (cf. Definition 1.1), the restriction map of sections*

$$\mathrm{Sections}(Z/S) \to \mathrm{Sections}(Z_C/C)$$

*is bijective.*

*Proof.* Shrinking $S$ if necessary, we can assume that $f$ is étale. Also, we can assume that $Z$ is connected. By taking a normal projective compactification of $Z$, Corollary 3.12 shows that for a general genus-0, degree-$d$ curve $C$ the inverse image $f^{-1}(C)$ is an irreducible and smooth curve. If $\deg(f)$ is strictly greater than one, then there is no section for $f$ ([28], Prop.5.3.1, p.165). Since $f$ is flat, $\deg(f|_{f^{-1}(C)})$ equals $\deg(f)$ ([20], Exercise 5.1.25(a), p.176). Thus, there is no section for $f|_{f^{-1}(C)}$ neither. If $\deg(f)$ is one, then the restriction map of sections is trivially bijective. □

## 3.3 Notations and Set up

In the rest of this paper, we will assume that $S$ is a smooth, quasi-projective $k$-variety of dimension$\geq 2$, where $k$ is an uncountable algebraically closed field of characteristic zero. And we fix a generically finite dominant morphism $u_0 : S \to \mathbb{P}_k^n$ so that we can talk about lines and line-pairs, or curves and curve-pairs in $S$ (see Definition 1.1). Note that, without changing any of the results, we can shrink $S$ to a dense open subset and assume further that $u_0$ is a finite, étale morphism onto a dense open subset of $\mathbb{P}_k^n$.

### 3.3.1 Isotrivial quotient and spaces of sections

Let $k$ be an algebraically closed field of characteristic zero. Suppose that $S$ is a smooth variety over $k$ with a finite étale morphism onto a dense, Zariski open subset of $\mathbb{P}^n_k$, say, $u_0 : S \to \mathbb{P}^n_k$. We claim that there exists a smooth projective compactification $W$ of $S$ extending $u_0$. Since $S$ is quasi-projective, let $W$ be the reduced projective completion of $S$. Up to replacing $W$ by its normalization, we can assume $W$ is normal. Then, $W$ is singular only at a codimension two closed subset([20], Prop.4.2.24). Moreover, since the characteristic of $k$ is zero, by Hironaka's resolution of singularities, there exists smooth $W'$ and $W' \to W$, which is birational, projective and an isomorphism on the smooth locus of $W$. So, replacing $W$ by $W'$, we can assume further that $W$ is smooth and projective over $k$, and $S$ is a dense open subset in $W$. We include the following diagram to clarify the situation.

$$\begin{array}{ccc} S & \overset{\text{open dense}}{\hookrightarrow} & W \\ {\scriptstyle \text{finite étale}}\downarrow & & \downarrow \\ \text{Image(S)} & \underset{\text{open dense}}{\hookrightarrow} & \mathbb{P}^n_k \end{array}$$

Now, the image of $W$ contains an open dense subset of $\mathbb{P}^n_k$ since $S$ is finite and étale over a dense open subset of $\mathbb{P}^n_k$. Moreover, since $W$ is projective over $\operatorname{Spec} k$, $W \to \mathbb{P}^n_k$ is projective, and hence the image of $W$ is the whole $\mathbb{P}^n_k$. So the space of conics in $W$ and the space of curve-pairs in $W$ are the same as the space of conics and curve-pairs in $\mathbb{P}^n_k$.

**Notation 3.14.** The smooth projective variety $W$ constructed as above is called *a projective compactification* of $S$.

Recall that an Abelian scheme over $S$ is defined as a proper and smooth $S$-group scheme with connected fibers. Theorem 1.2 gives the result for restriction of sections over line-pairs for a family of Abelian varieties. We hope to generalize Theorem 1.2 to schemes $X$ admitting a finite morphism to some Abelian scheme $A$ over $S$. Unfortunately, in this situation, the trick of taking boundaries fails to apply on $X$ (cf. [10], Lemma 4.3, Lemma 4.4 and Lemma 4.5). So the isotrivial factor of $A$ gives moduli of sections (see Remark 3.47), and hence we have to consider curves of higher degree instead of line-pairs. The process of proof will involve the application of pseudo-Néron models.

We first fix some notations to clarify the situation. Let $A$ be an Abelian scheme over $S$, and $f : X \to A$ a finite $S$-morphism. There exists an open dense subset $V \subset S$ and a finite étale Galois cover $p : V' \to V$ such that the pullback of $A$ to $V'$ is isogenous to a product of a strongly nonisotrivial family of Abelian varieties and a trivial family (see the proof of Theorem 4.7 in [10]). We can assume that $V = S$. And denote $V'$ by $S'$.

**Notation 3.15.** Let $A_0$ be an Abelian variety over $k$ such that $(A_0, v_0)$ is a Chow $S'/k$-trace of $S' \times_S A$ where $v_0 : S' \times_k A_0 \to S' \times_S A$ is a morphism of Abelian schemes over $S'$ ([10], Theorem 3.2 (i), p.315). Then, there exists a strongly nonisotrivial Abelian scheme $Q$ over $S'$ with $v_Q : Q \to S' \times_S A$ a morphism of Abelian schemes, and $\rho_{iso} := v_0 \times v_Q : (S' \times_k A_0) \times_{S'} Q \to S' \times_S A$ is an isogeny of Abelian schemes over $S'$ ([10], Corollary 3.7, p.317). Recall that an isogeny of Abelian schemes is a surjective $S$-group morphism with finite fibers, and such an isogeny must be finite.

Since $A_0 \times_S S'$ is projective over $S'$, the Weil restriction $\mathfrak{R}_{S'/S}(A_0 \times_k S')$ exists ([3], Theorem 7.6/4, p.194). Moreover, since $S$ is a normal scheme, it is geometrically unibranch. Thus, $A$ is projective over $S$ ([26], Théorème XI 1.4). Therefore, the Weil restriction $\mathfrak{R}_{S'/S}(A \times_S S')$ also exists. We can check that the functorial morphism

$$\mathfrak{R}_{S'/S}(v_0) : \mathfrak{R}_{S'/S}(A_0 \times_k S') \to \mathfrak{R}_{S'/S}(A \times_S S')$$

is a closed immersion (see the disscusion after Theorem 6.2 in [4] p.72).

*Proof.* Let $\{S_i\}$ be a finite set of étale neighborhoods of $S$ such that $\coprod_i S_i \to S$ is faithfully flat and the base change of $S' \to S$ by $S_i$ is an open immersion ([3], Prop.2.3/8, p.49). Denote $\coprod_i S_i$ by $T$. Since $T \to S$ is faithfully flat and locally of finite presentation, it suffices to prove that $\mathfrak{R}_{S'/S}(v_0) \times \mathrm{Id}_T$ is a closed immersion ([7], Prop.1.15, p.9). However, since Weil restrictions commute with base change, we are reduced to the case where $S'$ is the disjoint union $\coprod_i S_i$. Therefore, we have the isomorphisms

$$\mathfrak{R}_{S'/S}(A_0 \times_k S') = \prod_i \mathfrak{R}_{S_i/S}(A_0 \times_k S' \times_S S_i) = \prod_i A_0 \times_k S' \times_S S_i,$$

and

$$\mathfrak{R}_{S'/S}(A \times_S S') = \prod_i \mathfrak{R}_{S_i/S}(A \times_S S' \times_S S_i) = \prod_i A \times_S S' \times_S S_i$$

(see the proof of Prop.7.6/5 in [3], p.196). Since $k$ is a field of characteristic zero, the morphism $v_0$ is a closed immersion ([4], p.20 and p.21). Thus, $\mathfrak{R}_{S'/S}(v_0)$ is a closed immersion. □

**Definition-Lemma 3.16.** Let $A \to \mathfrak{R}_{S'/S}(A \times_S S')$ be the functorial morphism of $S$-schemes, which is a closed immersion since $A$ is separated over $S$ ([3], p.197). The *isotrivial factor* $\mathrm{Iso}(A)$ of the Abelian scheme $A$ over $S$ is the fiber product of $A$ and $\mathfrak{R}_{S'/S}(A_0 \times_k S')$ over $\mathfrak{R}_{S'/S}(A \times_S S')$. In other words, the following diagram is Cartesian

$$\begin{array}{ccc} \mathrm{Iso}(A) & \longrightarrow & \mathfrak{R}_{S'/S}(A_0 \times_k S') \\ \downarrow & & \downarrow {\scriptstyle \mathfrak{R}_{S'/S}(v_0)} \\ A & \longrightarrow & \mathfrak{R}_{S'/S}(A \times_S S'). \end{array}$$

It is easy to check that $\mathrm{Iso}(A)$ is a closed Abelian subgroup scheme of $A$ over $S$, and the fiber dimension of $\mathrm{Iso}(A) \to S$ is just $\dim A_0$.

*Proof.* For any $S$-scheme $W$, the Weil restriction $\mathfrak{R}_{S'/S}(A \times_S S')$ represents the functor

$$W \mapsto \mathrm{Hom}_{S'}(W \times_S S', A \times_S S').$$

Thus, $\mathfrak{R}_{S'/S}(A \times_S S')$ is a group scheme over $S$ because $A \times_S S'$ is a group scheme. Moreover, since $A \to \mathfrak{R}_{S'/S}(A \times_S S')$ is induced by the identity on $A \times_S S'$, the functorial morphism $A \to \mathfrak{R}_{S'/S}(A \times_S S')$ is a homomorphism of group schemes. Similarly, $\mathfrak{R}_{S'/S}(A_0 \times_k S')$ is a group scheme over $S$ and $\mathfrak{R}_{S'/S}(A_0 \times_k S') \to \mathfrak{R}_{S'/S}(A \times_S S')$ is a homomorphism of group schemes. Therefore, $\mathrm{Iso}(A)$ is a group scheme over $S$.

Let $T$ be the disjoint union of schemes as in Lemma **??**. Then, $\mathrm{Iso}(A) \times_S T$ is $\prod_i A_0 \times_k S' \times_S S_i$. Thus, $\mathrm{Iso}(A)$ is smooth over $S$ by the standard descent results ([7], Prop.1.15, p.9). Moreover, since $\mathrm{Iso}(A)$ is a closed subscheme of the projective $S$-scheme $A$, $\mathrm{Iso}(A)$ is projective over $S$.

Let $b_i$ be a point in $S_i$ and $b$ be the image of $b_i$ in $S$. Then, $\kappa(b_i)$ is a finite separable extension of $\kappa(b)$. Since surjectivity is stable under base change and $S' \times_S S_i \to S_i$ is an open immersion, the morphism $S' \times_S S_i \to S_i$ is an isomorphism. Let $b''$ be a point in $S' \times_S S_i$ whose image in $S_i$ is $b_i$. Then, $\kappa(b'')$ is the same as $\kappa(b_i)$. Denote the image of $b''$ in $S'$ by $b'$.

$$\begin{array}{ccc} S' & \longleftarrow & S' \times_S S_i \\ {\scriptstyle \text{finite étale}}\downarrow & & \downarrow{\scriptstyle \text{open immersion}} \\ S & \xleftarrow[\text{étale}]{} & S_i \end{array}$$

Then, $\mathfrak{R}_{S'/S}(A \times_S S') \times_S S_i$ equals $\mathfrak{R}_{S' \times_S S_i/S_i}(A \times_S S' \times_S S_i)$ which is $A \times_S S' \times_S S_i$ since $S' \times_S S_i \to S_i$ is an isomorphism. Therefore, the geometric fiber $\mathfrak{R}_{S'/S}(A \times_S S')_{\overline{b}}$ is equal to

$$(A \times_S S' \times_S S_i) \times_{S_i} \mathrm{Spec}\, \overline{\kappa(b_i)}$$

which is the same as

$$A \times_S (S' \times_S S') \times_{S'} \mathrm{Spec}\, \overline{\kappa(b'')}.$$

Let $G$ be the Galois group of $S' \to S$ ([3], Example B, p.139). Then, $S' \times_S S'$ is isomorphic to the disjoint union of $S'$, $G \times S'$. So the geometric fiber is

$$A \times_S (G \times S') \times_{S'} \mathrm{Spec}\, \overline{\kappa(b'')},$$

i.e., a disjoint union of $|G|$ copies of the geometric fiber $A_{\overline{b}}$. The same argument gives that the geometric fiber $\mathfrak{R}_{S'/S}(A_0 \times_k S')_{\overline{b}}$ is a disjoint union of $|G|$ copies of the Abelian variety $A_0 \times_k \mathrm{Spec}\, \overline{\kappa(b)}$.

Because $A \to \mathfrak{R}_{S'/S}(A \times_S S')$ is a closed immersion, this morphism includes the geometric fiber $A_{\overline{b}}$ as one copy of the disjoint union of $|G|$ copies of $A_{\overline{b}}$. Therefore, the geometric fiber of $\mathrm{Iso}(A)$ over $b$ is the Abelian variety $A_0 \times_k \mathrm{Spec}\, \overline{\kappa(b)}$, which is irreducible. As a consequence, $\mathrm{Iso}(A)$ is a smooth, projective group scheme over $S$ with connected geometric fibers. So $\mathrm{Iso}(A)$ is an Abelian $S$-scheme. □

*Remark* 3.17. By Poincaré's complete reducibility theorem, there exists a morphism of $S$-schemes $\pi : A \to \mathrm{Iso}(A)$ such that the composition

$$\mathrm{Iso}(A) \longrightarrow A \xrightarrow{\ \pi\ } \mathrm{Iso}(A)$$

is an isogeny on the generic fiber of $\mathrm{Iso}(A)$. We call such a morphism $\pi$ an *isotrivial quotient* of the Abelian scheme $A$. For the rest of this article, we fix an isotrivial quotient $\pi : A \to \mathrm{Iso}(A)$. Denote by $\rho : \mathrm{Iso}(A) \to S$ the structure morphism of $\mathrm{Iso}(A)$. If $b$ is a point in $S$, we will denote the fiber of $\mathrm{Iso}(A)$ over $b$ by $\mathrm{Iso}(A)_b$.

*Proof.* Over the function field $K$ of $S$, $\mathrm{Iso}(A)_K$ is an Abelian subvariety of $A_K$. By Poincaré's complete reducibility theorem ([2], Theorem 8.9.3, p.267), there is an Abelian subvariety $B$ of $A_K$ such that the restriction of multiplication gives an isogeny

$$m : \mathrm{Iso}(A)_K \times_K B \to A_K.$$

Since $K$ is a perfect field, there is a dual isogeny

$$\widehat{m} : A_K \to \mathrm{Iso}(A)_K \times_K B$$

such that $\widehat{m} \circ m : \mathrm{Iso}(A)_K \times_K B \to \mathrm{Iso}(A)_K \times_K B$ is the multiplication by $\deg(m)$. Let $\mathrm{pr}_1$ be the first projection from $\mathrm{Iso}(A)_K \times_K B$ to $\mathrm{Iso}(A)_K$. Let $\iota : \mathrm{Iso}(A)_K \to A_K$ be the closed immersion. Then, $(\mathrm{pr}_1 \circ \widehat{m}) \circ \iota$ is the multiplication by $\deg(m)$ on $\mathrm{Iso}(A)_K$, which is an isogeny.

Since the Abelian scheme $\mathrm{Iso}(A)$ is a Néron model of $\mathrm{Iso}(A)_K$ ([3], Prop.1.2/8, p.15), the morphism $\mathrm{pr}_1 \circ \widehat{m} : A_K \to \mathrm{Iso}(A)_K$ extends to an open dense subset of $S$ containing all codimension one points of $S$ (Corollary 3.6). Therefore, we have a rational map $\pi : A \dashrightarrow \mathrm{Iso}(A)$. Since $S$ is regular, the rational map $\pi$ is defined everywhere ([3], Cor.8.4/6, p.234). $\square$

**Notation 3.18.** Let $\mathrm{Sections}_b^p(A/S)$ (resp. $\mathrm{Sections}_b^p(X/S)$) be the set of sections of $A$ (resp. $X$) over $S$ such that every section in the set maps $b \in S$ to $p \in \mathrm{Iso}(A)$ via $\pi : A \to \mathrm{Iso}(A)$ (resp. $\pi \circ f$).

**Notation 3.19.** Take a smooth, projective compactification $W$ of $S$. Let $\widetilde{A} \to W_0$ be the Néron model of $A \to S$ where $W_0$ is an open dense subset of $W$ which contains all the codimension one points of $W$. Let $\overline{A} \to W$ be a projective morphism whose restriction over $W_0$ equals $\widetilde{A}$.

Every section of $A$ over $S$ gives a unique rational section of $\overline{A} \to W$, whose maximal domain of definition contains all the codimension one points of $W$ since $A$ has Néron model. Conversely, let $\tau$ be a rational section of $\overline{A} \to W$. The maximal domain of definition of $\tau$ contains $S$ since every geometric fiber of $A \to S$ does not contain any rational curve ([9], Proposition 6.2, p.1234). Thus, the set of rational section of $\overline{A} \to W$ bijectively correspondes to the set of sections of $A \to S$.

There is a Chow variety parameterizing cycles in $\overline{A}$. The Chow variety has countably many irreducible components. There is an open subset of this Chow variety parameterizing cycles $Z \subset \overline{A}$ such that $Z \to W$ is birational. Then, this variety is the parameter space for rational sections of $\overline{A} \to W$. By the correspondence of rational sections of $\overline{A} \to W$ and sections of $A \to S$ as above, also this is the parameter space for sections of $A \to S$. Denote by this parameter space $\mathbf{Sec}(A/S)$. Every irreducible component of $\mathbf{Sec}(A/S)$ is quasi-projective over $k$.

The fiber product $\mathbf{Sec}(A/S) \times_k S$ parameterizes the pairs $([\sigma], b)$ where $\sigma \in \mathrm{Sections}(A/S)$ and $b \in S$. Let $\mathrm{pr}_{\mathrm{ev}} : \mathbf{Sec}(A/S) \times_k S \to S$ be the projection to $S$, i.e., mapping $([\sigma], b)$ to $b$. Denote by $\mathbf{Sec}_b(A/S)$ the product $\mathbf{Sec}(A/S) \times \{b\}$. Let $\Sigma : \mathbf{Sec}(A/S) \times_k S \to A$ be the universal section over $\mathbf{Sec}(A/S)$. Then the morphism $\Sigma_b := \pi \circ (\Sigma|_{\mathbf{Sec}_b(A/S)}) : \mathbf{Sec}_b(A/S) \to \mathrm{Iso}(A)$ maps each $[\sigma]$, a section $\sigma$ of $A$ over $S$, to $\sigma(b) \in \mathrm{Iso}(A)$. Therefore, $\mathbf{Sec}_b(A/S)$ is also a space parameterizing pairs $([\sigma], p)$, a closed point $p \in \mathrm{Iso}(A)$ and a section of $A$ over $S$, $\sigma \in \mathrm{Sections}(A/S)$, such that $\sigma(b) = p$. Denote by $\mathbf{Sec}^p_b(A/S)$ the fiber of $\Sigma_b$ over a point $p \in \mathrm{Iso}(A)_b$. Similarly, we can define $\mathbf{Sec}_b(X/S)$ and $\mathbf{Sec}^p_b(X/S)$.

### 3.3.2 Curves and curve-pairs

Take a smooth, projective compactification $W$ of $S$. Then, there is a projective, surjective morphism $W \to \mathbb{P}^n_k$ extending $u_0 : S \to \mathbb{P}^n_k$. Let $X$ be a scheme over $S$ such that it admits a pseudo-Néron model $\widetilde{X}$ over an open dense $\widetilde{S}$, containing $S$, of codimension at least two in $W$, e.g., $X$ admits a finite morphism to an Abelian scheme $A$ over $S$.

**Definition 3.20.** An *1-pointed, genus 0, degree* $d+2$*, k-curve-pair* $C = (s, [m], t, [\ell])$ in $W$ is a projective, connected, reduced, nodal curve of arithmetic genus 0 and degree $d + 2$ (see Definition 1.1) such that

- $C = m \cup \ell$ where $m$ is a smooth curve of genus 0, degree $d$, $\ell$ is a smooth conic, and $m$ intersects with $\ell$ transversally at a closed point $t$,
- the marked point $s \in m$ is a nonsingular closed point of $C$.

**Notation 3.21.** Let $\overline{\mathcal{M}}^{\leq 2}_{0,1}(W, d + 2)$ be the stack:

$$T \mapsto \begin{Bmatrix} \text{flat, projective families } \mathcal{A}_T \to T \text{ of geneus zero, degree} \\ d+2, \text{ 1-pointed curves in } W, \text{ at worst curve-pairs} \end{Bmatrix}$$

where $T$ is a $k$-scheme. Moreover, if the marked point on every curve is a fixed point $b \in W$, we denote the stack by $\overline{\mathcal{M}}^{\leq 2}_{0,1}(W, d + 2; b)$.

*Remark* 3.22. Our notations are motivated by [1] and [8], but the curves we consider have at most two irreducible components. Also, if $d > 2$, the marked point is always on the irreducible component that is not a smooth conic.

**Notation 3.23.** Denote by $\partial\overline{\mathcal{M}}^{\leq 2}_{0,1}(W, d + 2)$ (resp. $\partial\overline{\mathcal{M}}^{\leq 2}_{0,1}(W, d + 2; b)$) be the substack of $\overline{\mathcal{M}}^{\leq 2}_{0,1}(W, d + 2)$ (resp. $\overline{\mathcal{M}}^{\leq 2}_{0,1}(W, d + 2; b)$) of 1-pointed *curve-pairs.*

Keep the notations in Definition 3.20 and Notation 3.21, the marked point $s \in C$ defines an evaluation morphism

$$\rho_{\mathrm{ev}} : \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2) \to S, (s, [m], t, [\ell]) \mapsto s$$

whose fiber over $b \in S$ is just $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$.

**Notation 3.24.** Denote by $\mathcal{M}_{0,1}(W, d)$ the stack over $k$:

$$T \mapsto \begin{Bmatrix} \text{flat, projective families } \mathcal{B}_T \text{ over } T \text{ of pairs } (s, [m]) \text{ genus zero,} \\ \text{degree } d \text{ smooth curve } m \subset W \text{ with a marked point } s \in m \end{Bmatrix}$$

where $T$ is a $k$-scheme. If the marked point is a fixed closed point $b \in W$, we denote the stack by $\mathcal{M}_{0,1}(W, d; b)$. Similarly, denote by $\mathcal{M}_{0,0}(W, 2)$ the stack of smooth conic curves; note that there is no marked points on conics.

There are forgetful morphisms from $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2)$:

$$\delta_1 : \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2) \to \mathcal{M}_{0,1}(W, d), (s, [m], t, [\ell]) \mapsto (s, [m]),$$
$$\delta_2 : \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2) \to \mathcal{M}_{0,0}(W, 2), (s, [m], t, [\ell]) \mapsto [\ell].$$

**Notation 3.25.** As subspaces of $\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2)$ and $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2)$, denote the open locus of genus-0, degree-$(d+2)$ curves or curve-pairs contained in $\widetilde{S}$ (resp. $S$) by $\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2)$ and $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2)$ (resp. $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)$ and $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)$). Similarly, we can define the subspaces of $\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$ and $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$ for curve or curve-pairs contained in $\widetilde{S}$ and $S$.

*Remark* 3.26. Taking the maximal open subschemes of the stacks in Notation 3.25, we can assume that they are all schemes over $k$. Then, the scheme $\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2)$ is an integral and smooth $k$-scheme and the locus $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2)$ is an integral, smooth divisor in $\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2)$.

**Notation 3.27.** Suppose that $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)^{\mathrm{RS}} \subset \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)$ is the very general subset parameterizing genus zero, degree $d+2$ curve-pairs $C$ in $S$ such that the restriction of sections

$$\mathrm{Sections}(A/S) \to \mathrm{Sections}((A \times_S C)/C) \qquad (\dagger)$$

is bijective. Denote by $\mathcal{M}_{0,1}(S, d)^{\mathrm{RS}} \subset \mathcal{M}_{0,1}(S, d)$ the very general subset parameterizing genus-0, degree-$d$ smooth curves such that (†) is bijective. We will prove the existence of $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)^{\mathrm{RS}}$ and $\mathcal{M}_{0,1}(S, d)^{\mathrm{RS}}$ for $d \geq 2$ an even integer in Corollary 3.42. Similarly, for a fixed very general closed point $b \in S$, we can define $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)^{\mathrm{RS}}$ and $\mathcal{M}_{0,1}(S, d; b)^{\mathrm{RS}}$.

### 3.3.3 Bad sets in parameter spaces

Now we define the bad set for sections and curve-pairs. The fiber product $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2) \times_{\rho_{\mathrm{ev}}, S, \mathrm{pr}_{\mathrm{ev}}} (\mathbf{Sec}(A/S) \times_k S)$, see the notation at the end of Notation 3.19 and the notation after Notation 3.23, parameterizes the pairs of sections and curve-pairs $((b, [m], t, [\ell]), ([\sigma], p))$ where $\sigma$ is a section of $A$ over $S$ mapping $b$ to $p$ and the marked point on the curve $m$ is a point $b \in S$.

For a pair of closed points $b \in S$, $p \in \mathrm{Iso}(A)_b$, and let $\sigma$ be a section of $A$ over $S$ that maps $b$ to $p$. Denote by $C = m \cup \ell$ a curve-pair in $S$ where $m$ is a smooth curve, $\ell$ is a conic. Consider the following two properties,

(i) $\mathrm{Sections}_b^p(A/S) \to \mathrm{Sections}_b^p((A \times_S C/C)$ is bijective;

(ii) $\mathrm{Sections}_b^p((X \times_{A,\sigma} S)/S) \to \mathrm{Sections}_b^p((X \times_{A,\sigma} S \times_S \ell)/\ell)$ is bijective,

where the maps of the sets of sections are restrictions and the fiber product $X \times_{A,\sigma} S$ comes from the section $\sigma$ from $S$ to $A$ mapping $b$ to $p$.

**Definition 3.28.** The subset

$$\mathcal{B}\mathrm{ad}(d+2) \subset \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2) \times_{\rho_{\mathrm{ev}}, S, \mathrm{pr}_{\mathrm{ev}}} (\mathbf{Sec}(A/S) \times_k S)$$

such that either (i) is false or (ii) is false is called *the bad set of sections and curve-pairs*. Denote by $\mathcal{B}\mathrm{ad}(d+2; b)$ the fiber of $\mathcal{B}\mathrm{ad}(d+2)$ over $b \in S$, and we call $\mathcal{B}\mathrm{ad}(d+2; b)$ *the bad set of sections and curve-pairs marked by b.*

*Remark* 3.29. By Theorem 3.13, there is a very general subset $\mathcal{U} \subset \mathcal{M}_{0,0}(S, 2)$ such that for each $[\ell] \in \mathcal{U}$ and each $\sigma \in \mathrm{Sections}(A/S)$ the property (ii) holds. Then, $\mathcal{B}\mathrm{ad}(d+2)$ is contained in the complement of the very general subset

$$(\delta_2^{-1}(\mathcal{U}) \cap \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)^{\mathrm{RS}}) \times_{\rho_{\mathrm{ev}}, S, \mathrm{pr}_{\mathrm{ev}}} (\mathbf{Sec}(A/S) \times_k S).$$

*Remark* 3.30. The subset $\mathcal{B}\mathrm{ad}(d+2; b)$ is contained in $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b) \times_k \mathbf{Sec}_b(A/S)$. Denote by $\phi_1$ and $\phi_2$ the projections from $\mathcal{B}\mathrm{ad}(d+2; b)$ to $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$ and $\mathbf{Sec}_b(A/S)$. The composition of $\phi_2$ and $\Sigma_b$ (see Notation 3.19) gives

$$\phi_3 : \mathcal{B}\mathrm{ad}(d+2; b) \to \mathrm{Iso}(A)_b, \text{ by } \{(b, [m], t, [\ell]), ([\sigma], p)\} \mapsto p.$$

**Definition 3.31.** Denote by $\mathcal{B}\mathrm{ad}(d+2; b, p)$ the fiber of $\phi_3$ over $p \in \mathrm{Iso}(A)_b$. If

$$\{(b, [m], t, [\ell]), ([\sigma], p)\} \in \mathcal{B}\mathrm{ad}(d+2; b, p),$$

$p$ is called a *bad point* for the curve-pair $(b, [m], t, [\ell])$, and $\sigma$ is called a *bad section* for $(b, [m], t, [\ell])$. Otherwise, $(b, [m], t, [\ell])$ is called *good for the section* $\sigma \in \mathrm{Sections}_b^p(A/S)$. If $(b, [m], t, [\ell])$ is good for every section in $\mathrm{Sections}_b^p(A/S)$, the point $p$ is called a *good point* for $(b, [m], t, [\ell])$.

**Definition 3.32.** An irreducible smooth curve $C$ with a marked point $b \in S$ is called *good for a section* $\sigma$ *in* $\mathrm{Sections}_b^p(A/S)$ if the following two properties hold

(i) $\text{Sections}_b^p(A/S) \to \text{Sections}_b^p((A \times_S C)/C)$ is bijective,

(ii) $\text{Sections}_b^p((X \times_{A,\sigma} S)/S) \to \text{Sections}_b^p((X \times_{A,\sigma} S \times_S C)/C)$ is bijective,

where the maps of the set of sections are restrictions and the fiber product $X \times_{A,\sigma} S$ comes from the section $\sigma$ from $S$ to $A$. And $p$ is a *good point* for $(b, [C])$ if $(b, [C])$ is good for every section in $\text{Sections}_b^p(A/S)$.

The same construction as for curve-pairs shows that there is a subset $\mathcal{B}\text{ad}(d)$ in the fiber product of $\mathcal{M}_{0,1}(S, d)$ and $\mathbf{Sec}(A/S) \times_k S$ over $S$ via evaluation maps such that $\mathcal{B}\text{ad}(d)$ parameterizes smooth curves making either (i) false or (ii) false. Also, $\mathcal{B}\text{ad}(d)$ is contained in a countable union of closed subsets.

*Remark* 3.33. Precisely, we should use different notations for bad sets of curve-pairs and curves since they are in different parameter spaces. Since in our proof of Theorem 1.3 every family of curves is clear from context, we apply the same notation $\mathcal{B}\text{ad}$ and just indicate the degrees of the curves.

#### 3.3.4 Universal curves and universal sections

**Notation 3.34.** Let $\mathcal{C}_{\overline{\mathcal{M}}(W)} \subset \overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b) \times_k W$ be the universal family of curves over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$. Denote the open subset of universal family of curves in $\widetilde{S}$ (resp. $S$) by $\mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})}$ (resp. $\mathcal{C}_{\overline{\mathcal{M}}(S)}$).

**Notation 3.35.** Fix closed points $b \in S$ and $p \in \text{Iso}(A)_b$. Denote by $H_b^p$ be the scheme parameterizing sections $\gamma$ of $X_C \to C$ where $C$ is a curve in $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$ *and* $\gamma$ *maps* $b$ *to* $p$ *via* $\pi \circ f$.

Equivalently, we have the following diagram, where $\Phi$ is the composition of the structure morphism of $H_b^p$ over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$ and the open immersion $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b) \to \overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$ and all squares are Cartesian.

$$\begin{array}{ccccc}
H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} \times_{\widetilde{S}} \widetilde{X} & \longrightarrow & \widetilde{X} \times_{\widetilde{S}} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} & \longrightarrow & \widetilde{X} \\
\downarrow & & \downarrow & & \downarrow \\
H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} & \longrightarrow & \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} & \longrightarrow & \widetilde{S} \\
\downarrow & & \downarrow & & \\
H_b^p & \xrightarrow{\Phi} & \overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b) & &
\end{array}$$

We note that $H_b^p$ is locally of finite type over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$, and hence over $k$, but may have infinitely many irreducible components. Every irreducible component is quasi-projective over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$.

**Notation 3.36.** There is a universal section of $H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S},d+2;b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} \times_{\widetilde{S}} \widetilde{X} \to H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S},d+2;b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})}$. We compose it with the top row of the digram above, and denote the morphism by

$$\varrho : H_b^p \times_{\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S},d+2;b)} \mathcal{C}_{\overline{\mathcal{M}}(\widetilde{S})} \to \widetilde{X},$$

which factors through the inclusion $X \to \widetilde{X}$.

## 3.4 Restrictions of Sections for Abelian Schemes

### 3.4.1 Main pseudo-Néron model theorem

In this subsection, we prove the key theorem that applies pseudo-Néron models to the problem of restriction of sections. The proof is similar to Lemma 4.13 of [10], but we prove a relative version of this result, i.e., for fixed $b \in S$ and $p \in \mathrm{Iso}(A)_b$.

**Lemma 3.37.** *Let $X \to S$ be a morphism locally of finite type of regular Noetherian schemes. Let $Z$ be a codimension one regular closed subscheme of $X$, and suppose that $Z \to S$ is smooth. Then, there exists an open subset $U$ of $X$ that contains $Z$ such that $U \to S$ is smooth.*

*Proof.* See Appendix B. □

Recall that $W$ is a smooth, projective compactification of $S$.

**Theorem 3.38.** *Suppose that:*

- *$X$ is smooth projective over $S$,*
- *$X$ has a pseudo-Néron model $\widetilde{X}$ over $W$, and*
- *every geometric fiber $X_{\overline{s}}$ for $s \in S$ does not contain any rational curve.*

*Then, sections of $X \to S$ over genus-0, degree-$(d+2)$ smooth curves specialize to sections over genus-0, degree-$(d+2)$ curve-pairs. More precisely, any irreducible component $H_0$ of $H_b^p$ which dominates $\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$ also dominates $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$. That is, the intersection of the image $\Phi(H_0)$ with $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$ contains a dense open subset of $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(\widetilde{S}, d+2; b)$.*

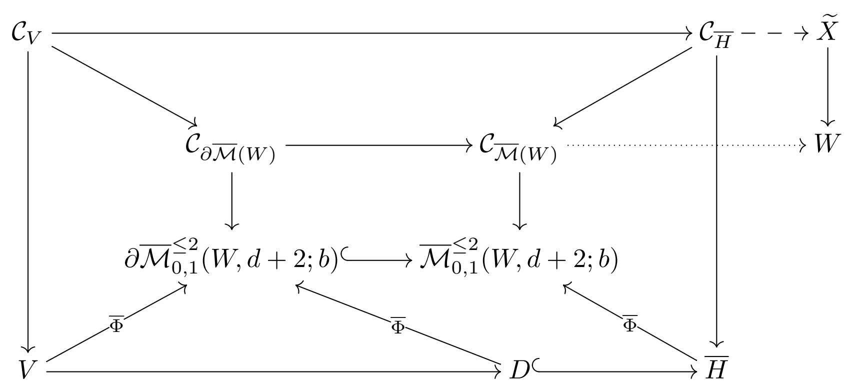
*Proof.* As in [10], we consider the diagram:

$$\begin{array}{ccccc} \mathcal{C}_{\partial\overline{\mathcal{M}}(W)} & \longrightarrow & \mathcal{C}_{\overline{\mathcal{M}}(W)} & \longrightarrow & W \\ \downarrow & & \downarrow & & \\ \partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b) & \longrightarrow & \overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b) & & \end{array}$$

where $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)}$ is the universal family of curve-pairs with nodes of curve-pairs deleted so that $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)} \to \partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2; b)$ is smooth. Also, the morphisms $\mathcal{C}_{\overline{\mathcal{M}}(W)} \to W$ and the composition $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)} \to W$ are smooth.

Since $H_0$ is quasi-projective over $\overline{\mathcal{M}}_{0,1}^{\le 2}(\widetilde{S}, d+2; b)$, we can choose a compactification $\overline{H}$ of $H_0$ such that $\overline{H}$ is normal and $\Phi$ extends to a proper surjection $\overline{\Phi} : \overline{H} \to \overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2; b)$. Thus, $D := \overline{\Phi}^{-1}(\partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2; b))$ is nonempty, and we may assume that $D$ is irreducible. Since $\overline{H}$ is normal, it is regular at a general point of $D$. Set $D_{\text{red}}$ as the reduced structure of $D$. Denote by $\overline{H}^{reg}$ the regular locus of $\overline{H}$. By the generic smoothness theorem, there is a dense open $V \subset D_{\text{red}} \cap \overline{H}^{reg}$ such that $\overline{\Phi} : D_{\text{red}} \to \partial\overline{\mathcal{M}}_{0,1}^{\le 2}(W, d+2; b)$ is smooth on $V$.

Denote by $\mathcal{C}_{\overline{H}}$ the base change $\mathcal{C}_{\overline{\mathcal{M}}(W)} \times_{\overline{\mathcal{M}}_{0,1}^{\le 2}(W,d+2;b)} \overline{H}$. Then, $\mathcal{C}_V \to W$ is smooth, $\mathcal{C}_V \to \mathcal{C}_{\overline{H}}$ is an immersion and $V$ is contained in the regular locus of $\overline{H}$. We summarize the objects in the following diagram (cf. [10], p.324).




By Lemma 3.37, there exists a maximal open subset $U$ of $\mathcal{C}_{\overline{H}}$ containing $\mathcal{C}_V$ such that $U \to W$ is smooth. The universal section $\varrho$ (see Notation 3.36) gives a rational map $U \dashrightarrow X$, which is marked as the dashed arrow from $\mathcal{C}_{\overline{H}}$ to $\widetilde{X}$ in the diagram above. Denote this rational map also by $\varrho$.

Now, let $\widetilde{W} \subset W$ be the image of $U \to W$. By construction, $\widetilde{W}$ contains $S$ and an open dense subset of the image of $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)}$ in $W$. Since $\widetilde{X}$ is a pseudo-Néron model of $X$ over $S$, $U \dashrightarrow \widetilde{X}$ is well-defined outside a codimension two subset of $\widetilde{W}$ by the weak extension property. Every such codimension two subset in $\widetilde{W}$ can be avoided by a general smooth curve or curve-pair in $S$. This gives an open dense subset $U'$ of $U$ such that $U'$ contains an open dense subset of $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)}$ and $\varrho$ is well-defined on $U'$. As a consequence, every curve-pair intersecting an open subset of $\mathcal{C}_{\partial\overline{\mathcal{M}}(W)}$ with nodes deleted can be lifted. Moreover, since there is no rational curve on every geometric fiber of $X \to S$, any morphism from a punctured curve-pair to $X$ can be extended to the node by Abhyankar's lemma ([17], Theorem (1.9.3), p.290). Since $H_b^p$ parameterizes the space of genus zero, degree-$(d+2)$ curves that can be lifted (Notation 3.35), an open subset of $V$ is contained in the image of $H_0$, i.e., $H_0$ also dominates $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(W, d+2; b)$. $\square$

*Remark* 3.39. Technically, that the characteristic of $k$ is zero is required to apply the generic smoothness theorem.

### 3.4.2 Inductive pseudo-Néron deforming step

Now we prove the existence of the very general subset $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)^{\mathrm{RS}}$ of $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2)$, see Notation 3.27. That is, the very general subet parameterizing genus zero, degree-$(d+2)$ curve-pairs $C$ such that the restriction

$$\mathrm{Sections}(A/S) \to \mathrm{Sections}((A \times_S C)/C)$$

is bijective. In [10] (Theorem 1.2), it was proved that this very general subset exists if $C$ is a line-pair. And since we work over a field of characteristic zero, the same is true for a very general smooth conic by using the pseudo-Néron model (Theorem 1.2). The following inductive step gives the relative version of Theorem 1.2 for very general genus-0, degree-$(d+2)$ curves, and curve-pairs with fixed $b \in S$, $p \in \mathrm{Iso}(A)_b$.

**Corollary 3.40.** *Fix $b \in S$ and $p \in \mathrm{Iso}(A)_b$ closed points. Suppose that for a very general genus-0, degree-$(d+2)$ curve-pair $C = m \cup \ell$, where $\ell$ is a smooth conic, every section in $\mathrm{Sections}_b^p(X_C/C)$ is the restriction of a unique section in $\mathrm{Sections}_b^p(X/S)$, see Notation 3.18. Then, for a very general genus-0, degree-$(d+2)$ irreducible smooth curve containing $b$, every section over this curve mapping $b$ to $p$ is the restriction of a unique section of $X$ over $S$.*

*Proof.* Since $k$ has characteristic zero, $f : X \to A$ is generically unramified. Thus, up to shrinking $S$, we assume that $f$ is finite and unramified. Let $\mathcal{Y}$ be the scheme parameterizing sections of $X$ over $S$ mapping $b$ to $p$. Then, because $f$ is unramified, $H_b^p$ has reduced fibers over $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$. Moreover, there is a $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$-morphism $\Upsilon : \overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b) \times_k \mathcal{Y} \to H_b^p$ that maps $\{[C_1]\} \times \{[\sigma]\}$ to $[\sigma|_{C_1}]$ for each $[C_1] \in \overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$ and $[\sigma] \in \mathcal{Y}$. The image of this morphism is a union of irreducible components of $H_b^p$.

Suppose that there exists an irreducible component of $H_b^p$ dominating $\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$ such that a generic section parameterized by this component is not a restriction of a section of $X$ over $S$. By Theorem 3.38, this irreducible component also dominates $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$. But then by hypothesis this component intersects with the image of $\Upsilon$, which contradicts that the fibers of $H_b^p \to \overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2; b)$ are reduced. $\square$

**Lemma 3.41.** *Suppose that every section of $A$ over a very general genus-0, degree-$d$, irreducible, smooth curve is contained in a unique section of $A$ over $S$. Then, for very general points $b'$ and $b''$ in $S$ and for a very general conic $\ell$ containing $b'$ and a very general genus-0, degree-$d$ curve $m$ containing $b''$ with $d \geq 2$ such that $\ell$ and $m$ intersect at a very general point $c$, every section in $\mathrm{Sections}(A_{m\cup\ell}/m \cup \ell)$ is the restriction of a unique section in $\mathrm{Sections}(A/S)$.*

*Proof.* Denote by $M'$ the space of conics in $S$ and $M''$ the space of genus-0, degree-$d$ curves. Denote by $C'$, resp. $C''$, the universal family of curves over $M'$, resp. $M''$ (Definition A.4 (i)). Let $\gamma$ be a section in Sections$(A_{m\cup\ell}/m\cup\ell)$. Then, by Theorem 1.2, $\gamma|_\ell$ is contained in a unique global section $\sigma \in$ Sections$(A/S)$, and the same is true for $\gamma|_m$ by hypothesis. Let $\tau \in$ Sections$(A/S)$ be the unique global section extending $\gamma|_m$. Set $r = 2\dim(S) - 2$. The transversal Grassmannian $G'$ is the Grassmannian parameterizing $r$-dimensional linear subvarieties $N'$ of $M'$, and similarly define $G''$ for $M''$ (Definition A.4 (vii)).

For very general $m$ and $\ell$, they are contained in some very general linear subvariety $N'$ of $M'$ and linear subvariety $N''$ of $M''$ respectively. Denote by $u'_S$ and $u''_S$ the composed morphisms

$$C' \to S \times_{\operatorname{Spec} k} M' \to S,$$

and

$$C'' \to S \times_{\operatorname{Spec} k} M'' \to S$$

(Definition A.4 (iii)). The base change of $\sigma$, resp. $\tau$, to $C'$, resp. $C''$, gives a $u'_S$-multisection $\sigma_1$, resp. $u''_S$-multisection $\tau_1$ of $A \to S$ (Definition A.3). Consider the 2-pointed bi-gon $(C'_{t'} \cup_c C''_{t''}, b', b'')$ parameterized by a point $t' \in N'$, resp. $t'' \in N''$, whose curve $C'_{t'}$ contains $c$ and $b'$, resp. whose curve $C''_{t''}$ contains $c$ and $b''$ (see the statement of Lemma A.10). Denote the image of $\sigma_1$, resp. $\tau_1$ in $C' \times_S A$, resp. $C'' \times_S A$ by $\Omega'$, resp. $\Omega''$ (Definition A.3).

The following composition

$$\Omega' \times_{N'} C'_{N'} \to A \times_S (C'_{N'} \times_{N'} C'_{N'}) \xrightarrow{Id_A \times u^{[2]}_{N',S}} A \times_S (S \times_{\operatorname{Spec} k} S)$$

defines a $(S \times_{\operatorname{Spec} k} S, \mathrm{pr}_1)$-multisection of $A \to S$ (Lemma A.7). Let the image of the restriction of this multisection on the fiber $\mathrm{pr}_1^{-1}(b')$ in $A$ be $\Omega'_{M',N',b'}$, and similarly define $\Omega''_{M'',N'',b''}$ (see the notations in Lemma A.10). The base change of $\sigma$ via the composition

$$C'_{N'} \times_{N'} C'_{N'} \xrightarrow{u^{[2]}_{N',S}} S \times_{\operatorname{Spec} k} S \xrightarrow{\mathrm{pr}_1} S$$

gives a section of $A \times_S (C'_{N'} \times_{N'} C'_{N'})$ over $C'_{N'} \times_{N'} C'_{N'}$, which is the same as the base change of $\sigma$ via

$$C'_{N'} \times_{N'} C'_{N'} \xrightarrow{\mathrm{pr}_1} C'_{N'} \xrightarrow{u'_S} S.$$

Thus, the image of $\sigma$ in $A$ equals the image of $\Omega' \times_{N'} C'_{N'}$ in $A$. The restriction of $\Omega' \times_{N'} C'_{N'}$ on the $\mathrm{pr}_1$-fiber $\mathrm{pr}_1^{-1}(b')$ is the restriction of $\sigma$ on the curves $C'_{t'}$ containing $b'$ and parameterized by $t' \in N'$. Therefore, the image of $\gamma|_\ell$ is contained in $\Omega'_{M',N',b'}$, and the same argument shows that the image of $\gamma|_m$ is contained in $\Omega''_{M'',N'',b''}$. By the Bi-gon Lemma (Lemma A.10), for a very general pair $(N', N'', b', c, b'')$ in $G' \times_{\operatorname{Spec} k} G'' \times_{\operatorname{Spec} k} S \times_{\operatorname{Spec} k} S \times_{\operatorname{Spec} k} S$, $\gamma$ comes from a unique section of $A$ over $S$. □

**Corollary 3.42.** *Let $S$ be a smooth, quasi-projective $k$-scheme of dimension $b \geq 2$. Let $A$ be an Abelian scheme over $S$. For a very general curve-pair $C = m \cup \ell$ in $S$ such that the degree of $m$ is even, $\ell$ is a smooth conic, the restriction map of sections*

$$\text{Sections}(A/S) \to \text{Sections}(A_C/C)$$

*is a bijection. This also holds for $C$ a very general genus-0, irreducible, smooth curve of even degree in $S$.*

*Proof.* Take $X = A$ in Corollary 3.40. In Lemma 3.41, take $m$ as a conic curve. Then by Theorem 1.2 and Lemma 3.41 the result holds for a very general $m \cup \ell$. Next, use Corollary 3.40 to deform $m \cup \ell$ to a very general genus-0, irreducible smooth curve of degree 4. Attach a very general conic to this curve at a very general point and apply Lemma 3.41 and Corollary 3.40 again. Then, the corollary follows by induction. □

## 3.5 Moduli of bad points caused by Iso($A$)

**Lemma 3.43.** *Let $A$ and $B$ be two Abelian varieties over a field $k$. Then, there are at most countably many homomorphism of Abelian varieties from $A$ to $B$.*

*Proof.* See Appendix B. □

**Lemma 3.44.** *Let $C$ be a smooth curve in $S$. Then, for fixed $b \in C$ and $p \in \text{Iso}(A)_b$ closed points, there are at most countably many sections of $X$ (resp. $A$, resp. $\text{Iso}(A)$) over $C$ that map $b$ to $p$. And there are at most countably many sections of $X$ (resp. $A$, resp. $\text{Iso}(A)$) over $S$ that map $b$ to $p$.*

*Proof.* It suffices to prove the statement for the Abelian scheme $A$ since $X \to A$ is finite and $\text{Iso}(A)$ is a closed subscheme of $A$.

First suppose that $A = A_0 \times_k S$ for some abelian variety $A_0$ over $k$. Since the inclusion $C \to S$ is fixed, giving a morphism from $C$ to $A_0 \times_k S$ is the same as giving a morphism from $C$ to $A_0$. Up to a translation we can assume that the image of $p$ is the identity in $A_0$. Then, this is equivalent to specifying a homomorphism $(\text{Jac}(C), 0)$ to $(A_0, 0)$, where $\text{Jac}(C)$ is the Jacobian of $C$. By Lemma 3.43, there are at most countably many such homomorphisms, and hence at most countably many sections from $C$ to $A$ that map $b$ to $p$.

Now, suppose that $A$ is not a trivial family of Abelian varieties. However, by a finite, étale and Galois base change $S' \to S$, we have an isogeny of Abelian $S'$-schemes,

$$\rho_{iso} : (A_0 \times_k S') \times_{S'} Q \to A \times_S S'$$

where $A_0 \times_k S'$ is a trivial family of Abelian varieties over $S'$ and $Q$ is a strongly nonisotrivial Abelian scheme over $S'$.

Consider the product $(A_0 \times_k S') \times_{S'} Q$. Let $b' \in S'$ and $p' \in A_0 \times_k S'$. Every section of $(A_0 \times_k S') \times_{S'} Q$ over $C' = C \times_S S'$ that maps $b'$ to $p'$ comes from a section of $A_0 \times_k S'$ over $C'$ mapping $b'$ to $p'$ and a section of $Q$ over $C'$. There are at most countably many sections of $A_0 \times_k S'$ over $C'$ mapping $b'$ to $p'$. And since $Q$ is strongly nonisotrivial, there are at most countably many section of $Q$ over $S'$ ([10], Lemma 3.6, p.316). Thus, there are at most countably many sections of $(A_0 \times_k S') \times_{S'} Q$ over $C'$ mapping $b'$ to $p'$.

By the standard descent result, $\mathrm{Hom}_S(C, A) \to \mathrm{Hom}_{S'}(C', A \times_S S')$ is injective ([3], Theorem 6.1/6 (a), p.135). And, if a morphism from $C$ to $A$ is an immersion after the base change by $S'$, so is the original morphism ([13], $\mathrm{IV}_2$, Prop.2.7.1). So the problem reduces to counting the sections of $A \times_S S'$ over $C'$ mapping a fixed point $b' \in S'$ to a fixed point $p' \in A \times_S S'$. Let

$$\tau_{iso} : A \times_S S' \to (A_0 \times_k S') \times_{S'} Q$$

be the dual isogeny of $\rho_{iso}$. Let $p''$ be the image of $p'$ under $\tau_{iso}$. Then, since $\tau_{iso}$ is finite, for every section $\sigma''$ in $\mathrm{Sections}_{b'}^{p''}((A_0 \times_k S') \times_{S'} Q/C')$, there are at most finitely many sections of $A \times_S S'$ over $C'$ lifting $\sigma''$ and mapping $b'$ to $p'$. However, since $\mathrm{Sections}_{b'}^{p''}((A_0 \times_k S') \times_{S'} Q/C')$ is at most countable, $\mathrm{Sections}_{b'}^{p'}(A \times_S S'/C')$ is at most countable. Putting all these together, $\mathrm{Sections}_b^p(A/C)$ is at most countable.

Replacing the Jacobian of a smooth curve by the Albanese variety of $S$ ([24], Theorem 5.7.13, p.141), the result for $\mathrm{Sections}_b^p(X/S)$, resp. $\mathrm{Sections}_b^p(A/S)$, resp. $\mathrm{Sections}_b^p(\mathrm{Iso}(A)/S)$ follows immediately. □

**Lemma 3.45.** *Fix closed points $b \in S$, $p$ in $\mathrm{Iso}(A)_b$. Let $\sigma$ be a section in $\mathrm{Sections}_b^p(A/S)$. Recall Definition 3.31 and Definition 3.32 for good curves for sections.*

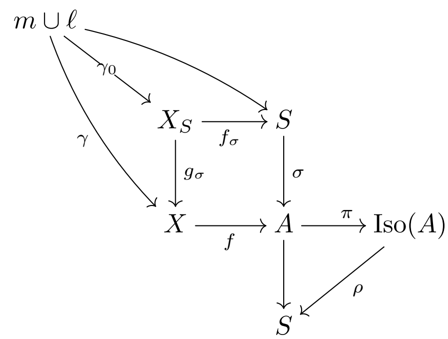
*(1). For a very general smooth conic $\ell$ and a very general genus-0, degree-d curve $m$ containing $b$ with $d \geq 2$ such that $\ell$ and $m$ intersect at a very general point, every section in $\mathrm{Sections}_b^p(X_{m\cup\ell}/m \cup \ell)$ that maps to $\sigma|_{m\cup\ell}$ is the restriction of a unique section in $\mathrm{Sections}_b^p(X/S)$ that maps to $\sigma$ if $m \cup \ell$ is good for $\sigma$.*

*(2). Let $C$ be a very general genus-0, degree-d, irreducible smooth curve marked by $b$ with $d \geq 2$. Then, every section in $\mathrm{Sections}_b^p(X_C/C)$ that maps to $\sigma|_C$ is the restriction of a unique section in $\mathrm{Sections}_b^p(X/S)$ that maps to $\sigma$ if $C$ is good for $\sigma$.*

*(3). Conversely, if $p$ is a bad point for a very general irreducible smooth curve $C$, then there exists a section in $\mathrm{Sections}_b^p(X_C/C)$ that cannot be extended uniquely.*

*Proof.* (1). Let $f_\sigma : X \times_{A,\sigma} S$ be the finite morphism arising from base change of $f$ by $\sigma$. Denote $g_\sigma : X_S \to X$ to be the base change of $\sigma$ by $f$. For any two different sections in $\mathrm{Sections}_b^p(X/S)$, the intersection in $X$ maps in $S$ to a proper closed subset of $S$. Moreover, there are at most countably many sections in $\mathrm{Sections}_b^p(X_m/m)$. For any two distinct such sections, the intersection in $X$ maps in $S$, via the structure morphism of $X$, to a proper closed subset of $S$. The complement of all these closed subsets is a very general subset of $S$. Denote this very general subset by $S_0$.

Take $c \in S_0$ as the intersection point of $m$ and $\ell$. Let $\gamma \in \mathrm{Sections}_b^p(X_{m\cup\ell}/m\cup\ell)$ such that $(f\circ\gamma)|_m$ is contained in the section $\sigma$ in $\mathrm{Sections}_b^p(A/S)$. Then form the following diagram.




Since $X_S$ is the fiber product of $X$ and $S$ via $f$ and $\sigma$, every section of $f_\sigma$ over $S$ (resp. over $m\cup\ell$) arises from a unique section of $X$ over $S$ (resp. over $m\cup\ell$). Thus, $\gamma$ gives a section, $\gamma_0$, of $f_\sigma$ over $m\cup\ell$ such that $g_\sigma\circ\gamma_0=\gamma$. Since $m\cup\ell$ is *good* for $\sigma$, $\gamma_0|_\ell$ is contained in a unique section of $f_\sigma$ over $S$, say, $\tau$. Then, $g_\sigma\circ\tau$ is a section of $X$ over $S$ such that $f\circ g_\sigma\circ\tau=\sigma$. By construction, $(g_\sigma\circ\tau)|_\ell$ equals $g_\sigma\circ(\gamma_0|_\ell)$, which is $\gamma|_\ell$. Let $\gamma_1$ be the restriction of $g_\sigma\circ\tau$ on $m$. Suppose that $\gamma_1\neq\gamma|_m$, then $\gamma_1(c)=g_\sigma\circ\tau(c)=\gamma(c)$ is in the intersection of the images of $\gamma_1$ and $\gamma|_m$. Since $\gamma_1$ is a also section of $X$ over $m$ mapping $b$ to $p$, $c$ is in the complement of $S_0$, contradicting the choice of $c$. Therefore, $\gamma_1$ equals $\gamma|_m$, and $\gamma$ extends to a unique section of $X$ over $S$, which is $g_\sigma\circ\tau$. The statement (2) follows from the same proof as (1).

(3). Suppose that $p$ is a bad point for $C$. For every two distinct sections in $\mathrm{Sections}_b^p(A/S)$, the intersection of their images in $A$ maps to a proper closed subset of $S$. Remove these countably many closed subsets from $S_0$. Denote this very general subset by $S^\circ$. Take $C$ such that $S^\circ\cap C\neq\emptyset$ and a section $\tau\in\mathrm{Sections}_b^p(A/S)$ such that $C$ is bad for $([\tau],p)$. Since $p$ is a bad point, there exists a section of $\gamma$ of $X$ over $C$ mapping $b$ to $p$ such that either $\gamma_0$ cannot be extended, the extension is not unique, or $f\circ\gamma$ cannot be extended. If $f\circ\gamma$ cannot be extended, $\gamma$ does not have an extension. If $\gamma_0$ cannot be extended, then $\gamma$ cannot be extended. If the extension is not unique, these different extensions gives distinct extensions of $\gamma$ as in the proof of the first part. □

**Corollary 3.46.** *Fix a very general point $b\in S$ and a point $p$ in $\mathrm{Iso}(A)_b$. Then, for a very general conic $\ell$ and a very general genus-0, degree-d curve $m$ containing $b$ with $d$ even and $d\geq 2$ such that $\ell$ and $m$ intersect at a very general point, every section in $\mathrm{Sections}_b^p(X_{m\cup\ell}/m\cup\ell)$ is the restriction of a unique section in $\mathrm{Sections}_b^p(X/S)$.*

*Proof.* Consider the very general subset $S^\circ$ as in Lemma 3.45. For a very general $b$, and very general $m\cup\ell$, the restriction of sections

$$\mathrm{Sections}_b^p(A/S)\to\mathrm{Sections}_b^p(A_{m\cup\ell}/m\cup\ell)$$

is bijective by Corollary 3.42.

Let $c \in m \cap S^\circ$ be a closed point. For every $\sigma$ in $\mathrm{Sections}_b^p(A/S)$, there is a general family of conic curves $\mathcal{N}_2(\sigma, c)$ such that every section of $f_\sigma$ over $\ell$ extends to a unique section of $f_\sigma$ by Bertini's theorem (Theorem 3.13). Take a very general conic $\ell$ that is contained in $\mathcal{N}_2(\sigma, c)$ for every $\sigma \in \mathrm{Sections}_b^p(A/S)$. Now, for every section $\gamma \in \mathrm{Sections}_b^p(X_{m\cup\ell}/m\cup\ell)$, $f \circ \gamma$ is contained in a unique section $\sigma \in \mathrm{Sections}_b^p(A/S)$. Therefore, by Lemma 3.45 (1), $\gamma$ is contained in a unique section of $\mathrm{Sections}_b^p(X/S)$. □

*Remark* 3.47. For a fixed $p \in \mathrm{Iso}(A)_b$, Corollary 3.46 claims the existence of good curve-pairs for sections in $\mathrm{Sections}_b^p(A/S)$. However, such a good curve-pair might be bad for other choices $p_0 \in \mathrm{Iso}(A)_b$. And as we vary the point $p_0$, the bad sets $\mathcal{B}\mathrm{ad}(d+2; b, p_0)$ might sweep out the moduli space $\partial\overline{\mathcal{M}}_{0,1}^{\leq 2}(S, d+2) \times_{\rho_{\mathrm{ev}}, S, \mathrm{pr}_{\mathrm{ev}}} (\mathbf{Sec}(A/S) \times_k S)$, see Definition 3.28. To resolve this problem, we have to increase the degree of curve-pairs (Theorem 1.3).

## 3.6 Proof of The Main Theorem

Now, we can give the proof of Theorem 1.3.

*Proof.* Let $C_1$ be a very general conic curve containing a very general point $b_1 \in S$. Let $\mathcal{B}_{2,b_1}$ be the image of $\phi_3(\mathcal{B}\mathrm{ad}(2; b_1))$, i.e. the set of bad points $p_1 \in \mathrm{Iso}(A)_{b_1}$ for $C_1$. By Corollary 3.46, $\mathcal{B}_{2,b_1}$ is a proper subset of $\mathrm{Iso}(A)_{b_1}$. Take $C_2$ a very general conic intersecting with $C_1$ at a very general point $c_2$, and a very general point $b_2$ on $C_2$. Denote by $\Delta_2'(C_1 \cup C_2, b_1)$ the union of the images of $C_1 \cup C_2$ under bad sections $\sigma$ of $A$ over $S$ mapping $b_1$ to some $p \in \mathcal{B}_{2,b_1}$. Let $\Delta_2(C_1 \cup C_2, b_1)$ be the image of $\Delta_2'(C_1 \cup C_2, b_1)$ in $\mathrm{Iso}(A)$ under $\pi$. Then, $\Delta_2(C_1 \cup C_2, b_1)$ is contained in $\rho^{-1}(C_1 \cup C_2)$ and $\Delta_2(C_1 \cup C_2, b_1) \cap \mathrm{Iso}(A)_{b_1}$ equals $\mathcal{B}_{2,b_1}$. Define $\Delta_2(C_1 \cup C_2, b_2)$ in the same way for points in $\mathcal{B}_{2,b_2}$.

By choosing $C_2$, $c_2$ and $b_2$ very generally, $\Delta_2(C_1 \cup C_2, b_2) \cap \mathrm{Iso}(A)_{b_1}$ will intersect $\mathcal{B}_{2,b_1}$ transversally. Moreover, since $C_1 \cup_{c_2} C_2$ is very general, we may assume that

$$\mathrm{Sections}(A/S) \to \mathrm{Sections}(A_{C_1 \cup_{c_2} C_2}/C_1 \cup_{c_2} C_2)$$

are bijective by Corollary 3.42.

Let $p$ be a point in $\mathcal{B}_{2,b_1}$, but not in $\Delta_2(C_1 \cup C_2, b_2)$. Let $\sigma$ be a section in $\mathrm{Sections}_{b_1}^p(A/S)$. If $\sigma(b_2)$ does not belong to $\mathcal{B}_{2,b_2}$, $(b_1, [C_1], c_2, [C_2])$ is good for $\sigma$. If $\sigma(b_2)$ is in $\mathcal{B}_{2,b_2}$, then $\sigma(b_1)$ is in $\Delta_2(C_1 \cup C_2, b_2)$, which contradicts the choice of $p$. Thus, $C_1 \cup C_2$ is good for every section in $\mathrm{Sections}_{b_1}^p(A/S)$, and $p$ is a good point for this marked curve-pair. Now, take a point $p$ in the set $\Delta_2(C_1 \cup C_2, b_2) \cap \mathrm{Iso}(A)_{b_1}$, but not in $\mathcal{B}_{2,b_1}$. Denote by $\gamma$ a section of $X$ over $C_1 \cup C_2$ mapping $b_1$ to $p$. Let $p_2$ be the image of $\gamma(b_2)$ in $\mathrm{Iso}(A)$. Let $\sigma$ be a section of $A$ over $S$ extending $f \circ \gamma$. Since $p$ is not in $\mathcal{B}_{2,b_1}$, $(b_2, [C_2], c_2, [C_1])$ is good for $([\sigma], p_2)$. By Lemma 3.45 (1), $\gamma$ extends to a unique section of $X$ over

$S$ mapping $b_2$ to $p_2$ and $b_1$ to $p$. Denote by $\mathcal{B}'_{4,b_1}$ the set of points in $\mathrm{Iso}(A)_{b_1}$ such that for each $p \in \mathcal{B}'_{4,b_1}$ the restriction of sections

$$\mathrm{Sections}^p_{b_1}(X/S) \to \mathrm{Sections}^p_{b_1}(X_{C_1 \cup_{c_2} C_2}/C_1 \cup_{c_2} C_2)$$

is not a bijection. By the argument above, $\mathcal{B}'_{4,b_1}$ is contained in the intersection of $\Delta_2(C_1 \cup C_2, b_2)$ and $\mathcal{B}_{2,b_1}$. Therefore, $\dim \mathcal{B}'_{4,b_1}$ is strictly less than $\dim \mathcal{B}_{2,b_1}$.

Let $C_{1,2}$ be a very general, genus-0, degree-4, irreducible, smooth curve containing $b_1$. By Corollary 3.40 and Lemma 3.45 (3), the bad set of points $\mathcal{B}_{4,b_1}$ for $(b_1, [C_{1,2}])$ is contained in $\mathcal{B}'_{4,b_1}$. Attach a very general conic $C_3$ to $C_{1,2}$ at a very general point $c_3$. Then inductively, we get a decreasing sequence of dimensions

$$\dim \mathcal{B}_{2,b_1} > \dim \mathcal{B}'_{4,b_1} \geq \dim \mathcal{B}_{4,b_1} > \dim \mathcal{B}'_{6,b_1} \geq \dim \mathcal{B}_{6,b_1} > \cdots .$$

Then, for $d > 2e - 2$ an even number, $\mathcal{B}'_{d+2,b_1}$ for $(b_1, [m], c, [\ell])$, a very general genus-0, degree-$(d+2)$ curve-pair, is empty, and hence every section of $X$ over $m \cup \ell$ is the restriction of a unique section. And, by Corollary 3.40, this is also true for very general irreducible smooth curves of genus-0, degree-$(d+2)$. $\square$

# Appendix A The Bi-gon Lemma

Let $k$ be an algebraically closed field. In the statement of the Bi-gon Lemma, also $k$ will be uncountable. Let $B$ be an irreducible, quasi-projective $k$-scheme of dimension $\geq 2$.

**Definition A.1.** For every $k$-morphism of locally finite type $k$-schemes, $f : R \to S$, for every integer $\delta \geq 0$, the **$\delta$-locus of** $f$, $E_{f,\geq\delta} \subseteq R$, is the union of all irreducible components of fibers of $f$ that have dimension $\geq \delta$. The **$\delta$-image of** $f$, $F_{f,\geq\delta} \subseteq S$, is the image under $f$ of $E_{f,\geq\delta}$.

**Lemma A.2.** *([15], Exercise II.3.22, p.95) For every locally finite type morphism $f$ and every integer $\delta \geq 0$, the subset $E_{f,\geq\delta}$ of $R$ is closed. If also $f$ is quasi-compact, resp. proper, then the subset $F_{f,\geq\delta}$ of $S$ is constructible, resp. closed.*

**Definition A.3.** For every proper, surjective morphism $\rho : Y \to B$, for every pair $(T, w)$ of an integral scheme $T$ and a dominant, finite type morphism,

$$w : T \to B,$$

a $(T, w)$-**multisection of** $\rho$ is a pair $(\Omega, v)$ of an irreducible scheme $\Omega$ and a proper morphism $v = (v_Y, v_T)$,

$$v : \Omega \to Y \times_B T, \quad v_Y : \Omega \to Y, \quad v_T : \Omega \to T,$$

such that $v_T$ is surjective and generically finite. Since $v$ is proper, also the image $(v(\Omega), v(\Omega) \hookrightarrow Y \times_B T)$ is a $(T, w)$-multisection of $\rho$. This is the **image multisection** of $(\Omega, v)$.

For every pair $((\Omega', v'), (\Omega'', v''))$ of $(T, w)$-multisections, denote the fiber product of $v'$ and $v''$ by

$$(\pi' : P_{v',v''} \to \Omega', \quad \pi'' : P_{v',v''} \to \Omega''), \quad v' \circ \pi' = v'' \circ \pi''.$$

The **special subset** $S_{v',v''}$ **of the pair** is the closed image in $T$ of $P_{v',v''}$.

**Definition A.4. (i).** For an integral, quasi-projective $k$-scheme $M$, a **family of smooth, proper, connected curves over** $M$ is a smooth, proper morphism,

$$\overline{u}_M : \overline{C} \to M,$$

whose geometric fibers are connected curves.

**(ii).** For every open immersion

$$\iota : C \to \overline{C}$$

whose image is dense in every fiber of $\overline{u}$, the composite morphism $u_M = \overline{u}_M \circ \iota$ is a **family of smoothly compactifiable curves over** $M$.

**(iii).** A **family of curves to** $B$ is a pair $(M, u)$ of an irreducible, quasi-projective $k$-scheme $M$ and a proper morphism $u = (u_Y, u_M)$,

$$u : C \to B \times_{\text{Spec } k} M, \quad u_B : C \to B, \quad u_M : C \to M,$$

such that $u_M$ is a family of smoothly compactifiable curves over $M$.

**(iv).** The family of curves to $B$ is **connecting**, resp. **minimally connecting**, if the following induced $k$-morphism is dominant, resp. dominant and generically finite,

$$u^{(2)} : C \times_M C \to B \times_{\text{Spec } k} B, \quad \text{pr}_i \circ u^{(2)} = u_B \circ \text{pr}_i, i = 1, 2.$$

By definition, both $u_M \circ \text{pr}_1$ and $u_M \circ \text{pr}_2$ are equal as morphisms from $C \times_M C$ to $M$; denote this common morphism by $\widetilde{u}_M$. Denote by $\widetilde{u}^{(2)}$ the induced morphism

$$(u^{(2)}, \widetilde{u}_M) : C \times_M C \to B \times_{\text{Spec } k} B \times_{\text{Spec } k} M.$$

**(v).** A connecting family of curves to $B$ is a **Bertini family** if for every integral $k$-scheme $\widetilde{B}$ and for every finite, surjective $k$-morphism $\phi : \widetilde{B} \to B$, the induced morphism $\widetilde{B} \times_B C \to M$ has integral geometric generic fiber. Denote by $M_\phi$ the maximal open subscheme of $M$ over which $\widetilde{B} \times_B C$ has integral geometric fibers.

**(vi).** An integral closed subvariety $N$ of $M$ is **transversal** if the following family of curves to $B$ is minimally connecting,

$$(N, u \times \text{Id}_N : C \times_M \to B \times_{\text{Spec } k} M \times_M N).$$

Such a subvariety is $\phi$-**Bertini** if $N$ intersects the open $M_\phi$.

**(vii).** The **transversal dimension** is $d := 2\dim(B) - 2$. For every integer $e$ with $0 \leq e \leq d$, the **transversal Grassmannian** $G_e$ is the open subscheme of the Grassmannian parameterizing $e$-dimensional linear sections $N$ of $M$.

**Lemma A.5.** *For every connecting family of curves* $(M, u)$*, for every integer* $e$ *with* $0 < e \leq d$*, a general point of* $G_e$ *parameterizes a linear section* $N$ *of* $M$ *that is geometrically integral.*

*Proof.* This follows from a Bertini Connectedness Theorem, [16] Théorème 6.10. □

**Lemma A.6.** *For every connecting, Bertini family* $(M, u)$*, for every integer* $e$ *with* $0 \leq e \leq d$*, a general point of* $G_e$ *parameterizes a linear section* $N$ *of* $M$ *such that the induced morphism* $u_N^{(2)}$ *is generically finite. For every finite, surjective* $k$*-morphism* $\phi : \widetilde{B} \to B$*, every general* $N \in G_d$ *is transversal and* $\phi$*-Bertini.*

*Proof.* Generic finiteness of $u_N^{(2)}$ is proved by induction on $e$. The base case is when $e = 0$. Since the family is connecting, the morphism $u$ is generically finite to its image. Thus, the morphism $\widetilde{u}^{(2)}$ is generically finite to its image. The 1-relative locus $E$ of $\widetilde{u}^{(2)}$ is a proper, closed subset of $C \times_M C$. For the induced morphism,

$$\widetilde{u}_M|_E : E \to M,$$

the 2-relative locus $E_{\geq 2}$ of this morphism is a closed subset of $E$. Since $E$ is a proper closed subset of $C \times_M C$, and since the geometric generic fiber of $\widetilde{u}_M$ is irreducible, the proper closed subset $E_{\geq 2}$ is disjoint from this fiber. Thus, the image $F_{\geq 2}$ of $E_{\geq 2}$ in $M$ is a constructible subset that does not contain the generic point, i.e., it is not Zariski dense. Denote by $M^o$ the open subset of $M$ that is the complement of the closure of the image of $F_{\geq 2}$. For every singleton $N$ of a closed point of $M^o$, the restrictions to $N$ of $\widetilde{u}^{(2)}$ and $u^{(2)}$ are equal; refer to this common restriction by $u_N^{(2)}$. Since $\widetilde{u}^{(2)}$ is generically finite on the fiber over $N$ by construction, also $u_N^{(2)}$ is generically finite. This establishes the base case.

For the induction step, assume that the result is proved for an integer $e$ satisfying $0 \leq e < 2\dim(B) - 2$. Then for a general linear subvariety $N$ of $M$ of dimension $e$, the image of $u_N^{(2)}$ has dimension $e + 2 < 2\dim(B)$. Thus, the image is not Zariski dense. Since $u^{(2)}$ is dominant, a general point of $B \times_{\text{Spec } k} B$ is contained in the image of $u^{(2)}$ over a general point of $M$, say $m$. Let $N'$ be the intersection of $M$ with the span of $N$ and $m$. Then $N'$ is a linear subvariety of $M$ of dimension $e + 1$. By Lemma A.5, for $N$ general and for $m$ general, the linear section $N'$ is geometrically integral. Thus, the image of $u_{N'}^{(2)}$ is a geometrically integral scheme that is strictly larger than the image of $u_N^{(2)}$. Thus, the dimension of the image of $u_{N'}^{(2)}$ is strictly larger than the dimension of the image of $u_N^{(2)}$. Since $u_N^{(2)}$ is generically finite, and since $N'$ has dimension precisely 1 larger than the dimension of $N$, also $u_{N'}^{(2)}$ has dimension precisely 1 larger than the dimension of $u_N^{(2)}$. Thus, also $u_{N'}^{(2)}$ is generically finite to its image.

In particular, for $e$ equal to $d$, since $u_N^{(2)}$ is generically finite and the domain and target both have the same dimension, the image of $u_N^{(2)}$ contains a nonempty Zariski open subset of $B \times_{\text{Spec } k} B$. By hypothesis, $B \times_{\text{Spec } k} B$ is integral. Thus, the image contains a dense Zariski open subset of $B \times_{\text{Spec } k} B$. Therefore $(N, u_N)$ is a minimally connecting family, i.e., $N$ is transversal.

Finally, by hypothesis, the open subscheme $M_\phi$ contains the generic point of $M$, and hence this open subscheme is dense. Therefore, a general $N$ intersects $M_\phi$. ∎

**Lemma A.7.** *For every minimally connecting family of curves to $B$,*

$$(N, (u_{N,B}, u_N) : C_N \to B \times_{\text{Spec } k} N),$$

*for every $(C_N, u_{N,B})$-multisection $(\Omega, v)$ of $\rho$, the scheme $\Omega \times_N C_N$ is irreducible, and the following composition $\widetilde{v}_B$ is a multisection of $\rho$ relative to $B \times_{\text{Spec } k}$*

$B \xrightarrow{pr_1} B$,

$$\Omega \times_N C_N \xrightarrow{v_N \times Id_{C_N}} Y \times_B C_N \times_N C_N \xrightarrow{Id_Y \times u^{[2]}_{N,B}} Y \times_B B \times_{Spec\ k} B.$$

*Proof.* Since the morphism

$$u_N : C_N \to N$$

is flat with integral geometric fibers, the following base change morphism is flat with integral geometric fibers,

$$\mathrm{pr}_\Omega : \Omega \times_N C_N \to \Omega.$$

Since $\Omega$ is irreducible, and since $\mathrm{pr}_\Omega$ is flat with integral geometric generic fiber, also $\Omega \times_N C_N$ is irreducible.

Since $v_C$ is surjective and generically finite, and since $u_N$ is flat, also the following morphism is surjective and generically finite,

$$v_C \times \mathrm{Id}_{C_N} : \Omega \times_N C_N \to C_N \times_N C_N.$$

Since $(N, u_N)$ is minimally connecting, the following morphism is dominant and generically finite,

$$u_N^{[2]} : C_N \times_N C_N \to B \times_{\mathrm{Spec}\ k} B.$$

Thus, the composition is dominant and generically finite. This composition equals the composition of $\widetilde{v}_B$ with the morphism

$$\rho \times \mathrm{Id}_B : Y \times_{\mathrm{Spec}\ k} B \to B \times_{\mathrm{Spec}\ k} B.$$

Thus, the morphism $\widetilde{v}_B$ is a multisection of $\rho$. □

For every pair of connecting families of curves to $B$,

$$(M', u' : C' \to B \times_{\mathrm{Spec}\ k} M'), \quad (M'', u'' : C'' \to B \times_{\mathrm{Spec}\ k} M''),$$

denote by $G$, resp. by $G'$, the open subscheme of the Grassmannian parameterizing $d$-dimensional linear sections $N'$ of $M'$, resp. $N''$ of $M''$, that are transversal; by Lemma A.6, there is a dense open subscheme parameterizing linear sections that are transversal.

**Lemma A.8.** *For every pair of connecting families of curves to $B$ as above that are Bertini families, for every pair*

$$(\Omega', v'), \quad (\Omega'', v'')$$

*of a $(C', u'_B)$-multisection of $\rho$ and a $(C'', u''_B)$-multisection of $\rho$, for a general pair $(N', N'') \in G' \times_{Spec\ k} G''$, the families $(N', u' \times Id_{N'})$ and $(N'', u'' \times Id_{N''})$ are minimal connecting families of curves to $B$. Also, for a general pair $(b', b'') \in B \times_{Spec\ k} B$, the family $(N', u'_{N'})$, resp. $(N'', u''_{N''})$, is a Bertini family for the image in $Y$ of the multisection $\widetilde{v}'_{B,b'}$, resp. $\widetilde{v}''_{B,b''}$ of $\rho$, obtained by restricting to the fiber of $pr_2 : B \times_{Spec\ k} B \to B$ over $b'$, resp. over $b''$.*

*Proof.* By Lemma A.7, each of $(\Omega'_{N'} \times_{N'} C'_{N'}, \widetilde{v}'_B)$ and $(\Omega''_{N''} \times_{N''} C''_{N''}, \widetilde{v}''_B)$ is a $\mathrm{pr}_1$-multisection of $\rho$. Thus, for general $(b', b'') \in B \times_{\mathrm{Spec}\ k} B$, the restriction of the $\mathrm{pr}_1$-multisection $\Omega'_{N'} \times_{N'} C'_{N'}$, resp. $\Omega''_{N''} \times_{N''} C''_{N''}$, to the $\mathrm{pr}_2$-fiber over $b'$, resp. over $b''$, maps dominantly and generically finitely to $B$, i.e., each of the finitely many irreducible components of the the restriction is a $u'$-multisection of $\rho$, resp. a $u''$-multisection of $\rho$. Denote the image in $Y$ of this finite union of multisection by $\widetilde{v}'_{B,b'}$, resp. by $\widetilde{v}''_{B,b''}$. By Lemma A.6, for $N''$ general applied to the finitely many irreducible components of $\widetilde{v}'_{B,b'}$, the family $(N'', u''_{N''})$ is transversal and Bertini relative to $\widetilde{v}'_{B,b'}$. Similarly, for $N'$ general, the family $(N', u'_{N'})$ is transversal and Bertini relative to $\widetilde{v}''_{B,b''}$. □

**Lemma A.9.** *Assume that $k$ is algebraically closed and uncountable. With the same hypotheses as above, for a countable family of $(M', u')$-multisections, $(\Omega'_{i'}, v'_{i'})_{i' \in I'}$, with pairwise distinct images in $Y \times_B C'$, resp. for a countable family of $(M'', u'')$-multisections, $(\Omega''_{i''}, v''_{i''})_{i'' \in I''}$, with pairwise distinct images in $Y \times_B C''$, if $(N', N'') \in G' \times_{Spec\ k} G''$ and $(b', b'') \in B \times_{Spec\ k} B$ are very general, then for every $(i', i'') \in I' \times I''$, the conclusion holds for $(\Omega'_{i'}, v'_{i'})$ and $(\Omega''_{i''}, v''_{i''})$.*

*Proof.* For each $(i', i'')$, by Lemma A.8, there exists a dense open $W_{i',i''}$ of $G' \times_{\mathrm{Spec}\ k} G'' \times_{\mathrm{Spec}\ k} B \times_{\mathrm{Spec}\ k} B$ parameterizing $(N', N'', b', b'')$ such that Lemma A.8 holds. Thus, for every $(N', N'', b', b'')$ in the countable intersection $\cap_{(i',i'')} W_{i',i''}$, the conclusion of the lemma holds for every $(\Omega'_{i'}, v'_{i'})$ and $(\Omega''_{i''}, v''_{i''})$. □

**Lemma A.10.** *(The Bi-gon Lemma) With hypotheses as in the previous lemma, for a very general $(N', N'', b', b, b'')$ in $G' \times_{Spec\ k} G'' \times_{Spec\ k} B \times_{Spec\ k} B \times_{Spec\ k} B$, for a very general 2-pointed bi-gon $(C = C'_{t'} \cup_b C''_{t''}, b', b'')$ parameterized by a point $t' \in N'$, resp. $t'' \in N''$, whose curve $C'_{t'}$ contains $b$ and $b'$, resp. whose curve $C''_{t''}$ contains $b$ and $b''$, the only sections $\sigma$ of $\rho$ over $C$ whose restriction to $C'_{t'}$ is in some $\Omega'_{M',N',b',i'}$ and whose restriction to $C''_{t''}$ is in some $\Omega''_{M'',N'',b'',i''}$ are those that come from global sections $\Omega'_{M',N',b',i'} = \Omega = \Omega''_{M'',N'',b'',i''}$ over $B$.*

*Proof.* Let $W$ denote the countable intersection of $W_{i',i''}$ inside $G' \times_{\mathrm{Spec}\ k} G'' \times_{\mathrm{Spec}\ k} B \times_{\mathrm{Spec}\ k} B$ as in the proof of the previous lemma. Let $(N', N'', b', b'')$ be an element of $W$. Consider the countable collection of image multisections $(\Omega'_{M',N',b',i'})_{i'}$ and $(\Omega''_{M'',N'',b'',i''})$ of $\rho$ as closed subschemes of $Y$. For every pair $(i'_1, i'_2)$ of distinct elements of $I'$, the special subset $S_{i'_1,i'_2}$ associated to $\Omega'_{M',N',b',i'_1}$ and $\Omega'_{M',N',b',i'_2})$ is a proper closed subset of $B$, and similarly for the special subset $S_{i''_1,i''_2}$ associated to every pair $(i''_1, i''_2)$ of distinct elements of $I''$. Finally, for every $i' \in I'$ and every $i'' \in I''$, the special subset $S_{i',i''}$ associated to $\Omega'_{M',N',b',i'}$ and $\Omega''_{M'',N'',b'',i''})$ is a proper closed subset except in those cases where $\Omega'_{M',N',b',i'}$ equals $\Omega''_{M'',N'',b'',i''}$.

Choose $b$ to be a very general point of $B$ that is contained in none of these special subsets that is a proper closed subsets of $B$. The matching condition at $b$ for a section $\sigma$ implies that $\sigma(C)$ is contained in $\Omega'_{M',N',b',i'} = \Omega''_{M'',N'',b'',i''}$ for unique $\Omega'_{M',N',b',i'}$ and $\Omega''_{M'',N'',b'',i''}$ in their respective countable multisections.

By the previous lemma, for every $i' \in I'$, if the restriction of $\Omega_{M',N',b',i'}$ over $C''_{t''}$ has a section, then the multisection $\Omega_{M',N',b',i'}$ is a global section. Similarly, for every $i'' \in I''$, if the restriction of $\Omega_{M'',N'',b'',i''}$ over $C'_{t'}$ has a section, then the multisection $\Omega_{M'',N'',b'',i''}$ is a global section. Thus, for every section $\sigma$ as in the previous paragraph, $\Omega'_{M',N',b',i'} = \Omega''_{M'',N'',b'',i''}$ is a global section. $\square$

# Appendix B Proofs of Two Lemmas

For completeness, we prove two lemmas that are already in the literature in this appendix.

Proof of Lemma 3.37.

*Proof.* This is a local problem, so we can assume that $S = \operatorname{Spec} R$ and $X$ is a closed subscheme of $W = \mathbb{A}^n_R$ defined by $g_1, \cdots, g_r$. Let $Z$ is defined by $g$ in $\mathcal{O}_X$. Let $z \in Z$ and $dw_1, \cdots, dw_n$ be a basis of $(\Omega^1_{W/S})_z$. Then, up to a re-indexing, $g_{t+1}, \cdots, g_{n-t-2}, g$ generate the ideal sheaf defining $Z$ and $dw_1, \cdots, dw_t, dg_{t+1}, \cdots, dg_{n-t-2}, dg$ generate $(\Omega^1_{W/S})_z$ ([3], Prop.2.2/7, p.39). Let $Y$ be the closed subscheme of $\mathbb{A}^n_R$ defined by $g_{t+1}, \cdots, g_{n-t-2}$. Then, we have $X \subset Y$, a closed subscheme. Since both $dw_1, \cdots, dw_n$ and $dw_1, \cdots, dw_t, dg_{t+1}, \cdots, dg_{n-t-2}, dg$ are basis of $(\Omega^1_{W/S})_z$, there exists some $w_{t+1}$ such that $dw_1, \cdots, dw_t, dw_{t+1}, dg_{t+1}, \cdots, dg_{n-t-2}$ form a basis of $(\Omega^1_{W/S})_z$. By the Jacobian criterion, $Y$ is smooth at $z$ over $S$. Thus, locally at $z$, $Y$ is regular ([20], Theorem 4.3.36, p.142). Therefore, $X$ and $Y$ are regular schemes of the same dimension locally at $z$ with $X \subset Y$. We get $X = Y$ locally at $z$, and hence $X$ is smooth at $z$ over $S$. Consider every point $z \in Z$, there will be an open subset $U \subset X$ such that $U \to S$ is smooth. □

Proof of Lemma 3.43.

*Proof.* Since Abelian varieties are projective, there exists a very ample sheaf $\mathcal{L}$ on $A \times_k B$. Then, for every homomorphism $u : (A, 0) \to (B, 0)$, the graph $G_u$ in $A \times_k B$ has a Hilbert polynomial $P(t)$ with respect to $\mathcal{L}$. Let $\operatorname{Hom}^P_k(A, B)$ be the scheme parameterizing homomorphisms from $A$ to $B$ with Hilbert polynomial $P(t)$. Then, $\operatorname{Hom}^P_k(A, B)$ is quasi-projective over $k$. Now, take a homomorphism $u$ from $A$ to $B$. The Zariski tangent space of $\operatorname{Hom}^P_k(A, B)$ at $[u]$ is isomorphism to the $k$-vector space of global sections of

$$\mathcal{E} = \operatorname{Hom}_{\mathcal{O}_A}(u^*\Omega^1_{B/k}, \mathcal{I}_0)$$

where $\mathcal{I}_0$ is the ideal sheaf defining the origin 0 in $A$. Since $B$ is an Abelian variety, $\Omega^1_{B/k}$ is isomorphic to the trivial locally free sheaf $\Omega_0 \otimes_k \mathcal{O}_B$ where $\Omega_0$ is dual space $T^*_{B,0}$ of the Zariski tangent space $T_{B,0}$ of $B$ at the origin ([23], (iii), p.39). Thus, $\mathcal{E}$ is equal to

$$\operatorname{Hom}_k(\Omega_0, k) \otimes_k \mathcal{I}_0.$$

Since $A$ is projective and by the exact sequence of the ideal sheaf $\mathcal{I}_0$ and structure sheaf of the origin $\mathcal{O}_A/\mathcal{I}_0$ ([20], Cor.3.3.21), there is no nonzero global section for $\mathcal{I}_0$, and hence the finite direct sum $\mathcal{E}$ of $\mathcal{I}_0$ does not have nonzero global section. Therefore, $\mathrm{H}^0(A, \mathcal{E}) = 0$ and $[u]$ is an isolated point of the quasi-projective $k$-variety $\operatorname{Hom}^P_k(A, B)$. So there are at most countably many such homomorphisms. □

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