Extensions of the Mass Angular Momentum Inequality in Mathematical Relativity

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Abstract of the Dissertation

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Inequalities between mass, angular momentum, and charge are motivated by the cosmic censorship conjecture in mathematical relativity. In this dissertation we provide several generalizations which expand the class of data sets in which such inequalities are known to hold.

First, we expand the mass angular momentum inequality to manifolds with minimal surface boundary. In particular we establish a precise mass lower bound for an asymptotically flat Riemannian 3-manifold with nonnegative scalar curvature and minimal surface boundary, in terms of angular momentum and charge. This result does not require the restrictive assumptions of simple connectivity and completeness, which are undesirable from both a mathematical and physical perspective.

Second, we lay out an approach to strengthening the mass angular momentum inequality to the so-called Penrose inequality with angular momentum and charge. Specifically a lower bound for the ADM mass is established in terms of angular momentum, charge, and horizon area in the context of maximal, axisymmetric initial data for the Einstein-Maxwell equations which satisfy the weak energy condition. If, on the horizon, the given data agree to a certain extent with the associated model Kerr-Newman data, then the inequality reduces to the Penrose inequality with angular momentum and charge. In addition, a rigidity statement is also proven whereby equality is achieved if and only if the data set arises from the canonical slice of a Kerr-Newman spacetime.

Finally, we extend a result of Chruściel concerning the existence of Brill coordinates. These coordinates are generally assumed to exist in proofs of the mass angular momentum inequality; thus we can remove this assumption in many cases. We consider simply connected, axisymmetric initial data sets with finitely many asymptotically flat or asymptotically cylindrical ends. Finally we show the extent to which Brill and similar coordinates are unique. A better understanding of these coordinate systems should aid future efforts at proving the Penrose inequality with angular momentum and charge.

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Chapter 1

Introduction and Results

Consider a simply connected, asymptotically flat initial data set (M, g, k, E, B) for the Einstein-Maxwell equations. Here M is a Riemannian 3-manifold with metric g, k is a symmetric 2-tensor representing the second fundamental form of the embedding into spacetime, and (E, B) represents the electromagnetic field. The non-electromagnetic matter energy and momentum densities are given by

$$16\pi\mu_{em} = R + (\mathrm{Tr}_g\, k)^2 - |k|_g^2 - 2(|E|_g^2 + |B|_g^2), \qquad 8\pi J_{em} = \mathrm{div}_g(k - (\mathrm{Tr}_g\, k)g) + 2E \times B, \quad (1.0.1)$$

where R is the scalar curvature and $E \times B$ represents the cross product. It will be assumed that the weak energy condition $\mu_{em} \geq 0$ holds, the data are maximal $\mathrm{Tr}_g\, k = 0$, and that there is no charged matter,

$$\mathrm{div}_g\, E = \mathrm{div}_g\, B = 0. \quad (1.0.2)$$

In addition, the data are taken to be axisymmetric in that the isometry group of (M, g) admits a subgroup isomorphic to $U(1)$, such that all other quantities defining the data are invariant under this $U(1)$ action. The Killing field generator will be denoted by η. In particular, the following Lie derivatives vanish

$$\mathfrak{L}_\eta g = \mathfrak{L}_\eta k = \mathfrak{L}_\eta E = \mathfrak{L}_\eta B = 0. \quad (1.0.3)$$

Heuristic arguments of Penrose [54] may also be used to obtain a conjectured lower bound for the ADM mass m of a spacetime in terms of total angular momentum $\mathcal{J}$ and charge Q, namely

$$m^2 \geq \frac{Q^2 + \sqrt{Q^4 + 4\mathcal{J}^2}}{2}, \quad (1.0.4)$$

Furthermore, similar heuristic arguments of Penrose [54] suggest the stronger inequality

$$m \geq \sqrt{\frac{\mathcal{A}}{16\pi} + \frac{Q^2}{2} + \frac{\pi(Q^4 + 4\mathcal{J}^2)}{\mathcal{A}}} \qquad \text{whenever} \qquad \mathcal{A} \geq 4\pi\sqrt{Q^4 + 4\mathcal{J}^2}, \quad (1.0.5)$$

where $\mathcal{A}$ is the event horizon cross-sectional area. The total angular momentum and charges take the form

$$\mathcal{J} = \frac{1}{8\pi}\int_{S_\infty}(k_{ij} - (\mathrm{Tr}_g\, k)g_{ij})\nu^i\eta^j, \qquad Q_e = \frac{1}{4\pi}\int_{S_\infty} E_i\nu^i, \qquad Q_b = \frac{1}{4\pi}\int_{S_\infty} B_i\nu^i, \quad (1.0.6)$$

with $Q^2 = Q_e^2 + Q_b^2$. Finally the ADM mass is defined by

$$m = \frac{1}{16\pi} \int_{S_\infty} (g_{ij,i} - g_{ii,j})\nu^j. \tag{1.0.7}$$

In these formulas S_∞ represents the limit as $r \to \infty$ for coordinate spheres S_r in the asymptotic end, and ν is the unit outer normal. The above definitions are valid for any asympotitaly flat ends in a manifold. An end is asymptotically flat if it is diffeomorphic to $\mathbb{R}^3 \setminus B(R)$ and has Cartesian coordinates in which the above data satisfy

$$g_{ij} = \delta_{ij} + o_\ell(r^{-\frac{1}{2}}), \quad \partial g_{ij} \in L^2(M_{\text{end}}), \quad k_{ij} = O_{\ell-1}(r^{-\lambda}), \quad \mu_{em}, J^i_{em}, J_{em}(\eta) \in L^1(M_{\text{end}}), \tag{1.0.8}$$

$$E^i = O_{\ell-1}(r^{-\lambda}), \qquad B^i = O_{\ell-1}(r^{-\lambda}), \qquad \lambda > \frac{3}{2}, \tag{1.0.9}$$

for some $\ell \geq 5$.[1]

Inequality (1.0.5) was proposed as a check on the final state conjecture and weak cosmic censorship, in that a counterexample would essentially disprove at least one of these grand conjectures. Details concerning the physical motivation for the most general form of the Penrose inequality are provided in [29]. Furthermore an independent motivation for inequality (1.0.5), based on Bekenstein's entropy bound [3], has been given in [42].

The purpose of this dissertation is to expand the scope in which Equations (1.0.4) and (1.0.5) are known to hold. In Chapter 2 we investigate Equation (1.0.4) in the case where M has a minimal surface boundary. Through the combined work of several authors [13, 15, 17, 22, 23, 48, 57], inequality (1.0.4) has been established when the manifold (M, g) is simply connected, complete, and contains another end which is either asymptotically flat or asymptotically cylindrical. The proof follows a two step procedure, the first of which is to obtain an initial lower bound for the mass in terms a renormalized harmonic map energy. The second consists of minimizing this energy, and showing that the unique global minimizer is the singular harmonic map associated with extreme Kerr-Newman data.

Simple connectivity is used to introduce a specialized coordinate system called Brill coordinates. Brill coordinates are global coordinates (ρ, z, ϕ) on $\mathbb{R}^3$ less a finite set of points in which the metric takes the form

$$g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\phi + A_\rho d\rho + A_z dz)^2. \tag{1.0.10}$$

All functions are independent of ϕ and these coordinates are generally thought of as cylindrical coordinates, with standard coordinate ranges $\rho \in [0, \infty)$, $\phi \in [0, 2\pi)$ and $z \in (-\infty, \infty)$. The existence of Brill coordinates allows for a simple bulk integral expression of the mass. The scalar curvature in such a coordinate system is given by [7]

$$2e^{-2U+2\alpha}R = 8\Delta U - 4\Delta_{\rho,z}\alpha - 4|\nabla U|^2 - \rho^2 e^{-2\alpha}\left(\partial_z A_\rho - \partial_\rho A_z\right)^2, \tag{1.0.11}$$

where Δ is the Laplacian with respect to the flat metric on $\mathbb{R}^3$ and $\Delta_{\rho,z} = \partial_\rho^2 + \partial_z^2$. This form of the scalar curvature will be used repeatedly.

[1] Here $f = o_k(r^{-\ell})$ means that if $s \leq k$ and $\partial_{i_1}, \partial_{i_2}, ..., \partial_{i_s}$ are coordinate vector fields, then $\partial_{i_1}\partial_{i_2}\partial_{i_s} f = o(r^{-\ell-s})$. We will use slightly different versions of this notations throughout the text, and will include the relevant footnote when the meaning is not obvious

Completeness and the asymptotics of the other end also play an important role in that they prevent the appearance of boundary terms in the formula for the mass. Furthermore, simple connectivity is used in the second step to ensure the existence of a twist potential to efficiently encode angular momentum and construct the harmonic map energy. Thus, these hypotheses play fundamental roles in the proof and whether they can be removed has been unclear.

The arguments motivating inequality (1.0.4) require no such hypotheses, and for this reason it has been conjectured that these assumptions are unnecessary. They are also unnatural as the positive mass theorem itself does not require such restrictions. In particular, it is important to allow the initial data to have minimal surface boundary, as these may be interpreted as cross-sections of the event horizon. Furthermore, significant generalizations of the positive mass theorem, including the positive mass theorem with charge [34] and the Penrose inequality (with charge) [49], require a minimal surface boundary to be meaningful. Additionally, from a physical perspective it is undesirable to necessitate the presence of a secondary asymptotic end, as this typically represents the interior of a black hole. Indeed, from the point of view of an outside observer, it is not possible to know the structure of spacetime contained within the event horizon. As for simple connectivity, although topological censorship [31] implies that this is an appropriate assumption for initial data within the domain of outer communication, it says nothing about the fundamental group of the interior black hole region. In fact, it suggests that all nontrivial topology is contained within the black hole, and therefore the combined assumptions of simple connectivity, completeness, and the existence of a secondary asymptotically flat end are not physically justified.

We establish (1.0.4) in generality without the unwanted hypotheses discussed above, for a single black hole. We also obtain a mass lower bound in the multi-black hole case consistent with the lower bound proved under the more restrictive hypotheses in [17, 48]. The main theorem of Chapter 2 is the following:

Theorem 1.0.1. *Let (M, g, k, E, B) be an axisymmetric, maximal initial data set for the Einstein-Maxwell equations with one asymptotically flat end, minimal surface boundary, and satisfying $\mu_{em} \geq 0$ in addition to $J_{em}(\eta) = \operatorname{div}_g E = \operatorname{div}_g B = 0$. If either*

(i) the outermost minimal surface has a single component, or

(ii) the boundary ∂M has one component and M is simply connected,

then

$$m^2 > \frac{Q^2 + \sqrt{Q^4 + 4\mathcal{J}^2}}{2}. \tag{1.0.12}$$

The first point to note is that there is a strict inequality in (1.0.12). This is to be expected since from the heuristic physical arguments leading to (1.0.4), equality should only be achieved if the initial data agree with the canonical slice of an extreme Kerr-Newman spacetime. However the extreme Kerr-Newman data do not possess a compact minimal surface, but rather have a cylindrical end, and therefore do not satisfy the hypotheses of Theorem 1.0.1. The minimal surface boundary, which could consist of many components, together with the asymptotically flat end guarantee the existence of an outermost minimal surface [30], and the assumption that it has one component is analogous to the case of the Penrose inequality treated by Huisken and Ilmanen [40]. In order to treat (1.0.12) in the presence of a multicomponent outermost minimal surface it is most likely that new ideas will be needed, as was the case for the multiple black hole version of the Penrose inequality established by Bray [4]. It is interesting to note that unlike the Penrose inequality, (1.0.12) continues to hold if the boundary ∂M is merely a single component minimal surface but not necessarily outer minimizing. In fact under the assumption treated in (*ii*), one can drop the hypothesis on the outermost minimal surface and replace it with simple connectivity of M to obtain the same conclusion.

The proof of Theorem 1.0.1 is based on a doubling procedure in so-called pseudospherical coordinates, where the data are reflected across the outermost minimal surface. The doubled pseudospherical coordinates yield precisely a Brill coordinate system discussed above. This doubling requires axisymmetry of the outermost minimal surface, a fact that was proven by Bryden. Since it does not appear in the literature, Bryden's proof is included here, see Proposition 2.2.1. This result is of independent interest as it may be applied elsewhere, for example to extensions of the Penrose inequality that include contributions from angular momentum.

We are able to extend Theorem 1.0.1 to allow for certain types of multiple black holes by including a mixture of boundary components and extra asymptotically flat as well as asymptotically cylindrical ends (see [48]). However, as in the case of a complete, simply connected initial data set, the presence of multiple black holes does not immediately yield an explicit expression for the mass lower bound [17, 48]. Rather, the lower bound is given in terms of the reduced harmonic energy of a Weinstein stationary solution [61] to the Einstein-Maxwell equations having the same angular momentum and charge for each black hole. This harmonic energy is denoted by $\mathcal{F}$, and is a function of the angular momenta and charge. It is conjectured that the resulting inequality coincides with the expression (1.0.12) in which $\mathcal{J}$ and Q are the sums of the angular momenta and charge from the different horizon components.

Theorem 1.0.2. *Let (M, g, k, E, B) be an axisymmetric, maximal, asymptotically flat initial data set for the Einstein-Maxwell equations having a minimal surface boundary and a finite number of additional ends each of which is asymptotically flat or asymptotically cylindrical. Assume further that $\mu_{em} \geq 0$ in addition to $J_{em}(\eta) = \mathrm{div}_g E = \mathrm{div}_g B = 0$. If either*

(i) at most one component of the outermost minimal surface encloses components of the boundary ∂M or nonsimply connected domains, or

(ii) the boundary ∂M has one component and M is simply connected,

then

$$m \geq \mathcal{F}(\mathcal{J}_1, \ldots, \mathcal{J}_N, Q_e^1, \ldots, Q_e^N, Q_b^1, \ldots, Q_b^N) \tag{1.0.13}$$

where N is the combined number of additional ends and components of ∂M and $\mathcal{J}_i$, Q_e^i, Q_b^i represent the angular momentum and charge associated with each of these ends and boundary components.

In Chapter 3 we lay out an approach to prove Equation (1.0.5) which is successful in a special case. In order to prove Penrose type inequalities it is customary to replace $\mathcal{A}$ in the maximal case with the area of the outermost minimal surface. Therefore, the manifold (M, g) will be taken to have a boundary consisting of a single component minimal surface. Note that simple connectivity then implies that the boundary must be topologically a 2-sphere, regardless of whether this surface is stable. Moreover, the auxiliary inequality of (1.0.5) is not needed in the single black hole case, since when the minimal surface is stable the area-angular momentum-charge inequality is known to be automatically satisfied [24, 25, 21].

The Penrose inequality without angular momentum and charge was established in the time-symmetric case through the groundbreaking work of Bray [4] and Huisken-Ilmanen [40]. As shown in [62], the addition of charge to this inequality requires the additional assumption of the area-charge inequality in the multiple black hole case. This version of the Penrose inequality was then established in [49, 51] by generalizing Bray's conformal flow. However including horizon area together with angular momentum is quite difficult, and there appear to be only two results in the literature to date in this direction [1, 2], and the approach taken in those articles is based on inverse mean curvature flow. In contrast, the present paper focuses on the techniques used to establish the mass-angular momentum inequalities, namely minimizing renormalized harmonic energies. We refer the reader to the excellent survey [52] for a more detailed account concerning the status of the Penrose inequality.

The results presented here rely on the existence of Weyl coordinates, cylindrical type coordinates associated with the study of stationary axisymmetric black holes. The coordinates play an important role by helping to reduce the Einstein equations to the study of a harmonic map. Details describing this coordinate system for the present situation are discussed. It has been shown in [19] that such a coordinate system can be derived from pseudospherical coordinates, and exist for a general class of asymptotically flat initial data sets. In these coordinates the metric is again in the form of Equation (1.0.10). The minimal surface horizon is identified with the interval $(-m_0, m_0)$ on the z-axis. The constant $m_0 > 0$ is uniquely determined by the geometry of the initial data, and $2m_0$ will be referred to as the horizon rod length. The functions U and α exhibit singular behavior at the horizon and may be modeled by the corresponding functions U_0, α_0 arising from the Schwarzschild solution having mass m_0. We may then write $U = U_0 + \overline{U}$ and $\alpha = \alpha_0 + \overline{\alpha}$, where the remainders $\overline{U}$ and $\overline{\alpha}$ are now uniformly bounded and possess bounded first derivatives even at the horizon. These 'renormalized' functions measure the deviation from the Schwarzschild solution. An important combination of these two which appears in the horizon area formula is $\overline{\beta} := \overline{\alpha} - 2\overline{U}$. Our main result may then be stated as follows.

Theorem 1.0.3. *Let (M, g, k, E, B) be a simply connected, axisymmetric, maximal, asymptotically flat initial data set for the Einstein-Maxwell equations with minimal surface boundary, having non-negative energy density $\mu_{em} \geq 0$, no charged matter, and satisfying the compatibility condition for the existence of a twist potential $J_{em}(\eta) = 0$. Let A_k and $\overline{\beta}_k$ denote the horizon area and Weyl coordinate function for the unique Kerr-Newman black hole sharing the same angular momentum, charge, and horizon rod length as the initial data set. Then*

$$m \geq \sqrt{\frac{A_k}{16\pi} + \frac{Q^2}{2} + \frac{\pi(Q^4 + 4\mathcal{J}^2)}{A_k}} + \frac{1}{4}\int_{-m_0}^{m_0} (\overline{\beta}(0, z) - \overline{\beta}_k(0, z))dz, \tag{1.0.14}$$

and equality occurs if and only if the initial data agree with that of the corresponding Kerr-Newman spacetime.

The hypotheses of this theorem are in agreement with those expected for the conjectured Penrose inequality with angular momentum and charge, except for one missing statement. Namely, in the above result the minimal surface boundary is not required to be outerminimizing, meaning it is not required to have the property that every surface which encloses it has area greater than or equal to $A = |\partial M|$. Theorem 1.0.3 holds under more general circumstances than those for which the Penrose inequality can be valid, and so the resulting inequality (1.0.14) must differ from (1.0.5). Indeed, the most apparent difference arises from the presence of the horizon rod integral involving the functions $\overline{\beta}$ and $\overline{\beta}_k$, which does not appear in the Penrose inequality. This integral measures the discrepancy between the initial data and the model Kerr-Newman solution at the horizon. It is unknown at this time whether this horizon integral is nonnegative under the current hypotheses. One may speculate that nonnegativity is not necessarily guaranteed unless the boundary is outerminimizing. Another difference between (1.0.14) and the conjectured inequality is the presence of the Kerr-Newman horizon area A_k instead of A, although the algebraic structure of this part of the inequality is the same. Despite these differences, one may achieve the desired Penrose inequality under additional assumptions. In particular, if we assume that the initial data is appropriately similar to the model Kerr-Newman solution at the horizon then the conjectured inequality follows.

Corollary 1.0.4. *Under the hypotheses of Theorem 1.0.3, assume further that $A \geq A_k$ and $\overline{\beta}$ is constant on the horizon rod, then*

$$m \geq \sqrt{\frac{A_k}{16\pi} + \frac{Q^2}{2} + \frac{\pi(Q^4 + 4\mathcal{J}^2)}{A_k}}, \tag{1.0.15}$$

and equality occurs if and only if the initial data agree with that of the corresponding Kerr-Newman spacetime. In particular, if $A = A_k$ then the Penrose inequality with angular momentum and charge holds.

This type of result may be considered a generalization of that of Gibbons and Holzegel in [35], who established the Penrose inequality without contributions from angular momentum and charge by utilizing the advantages of Weyl coordinates. In that paper they also had a more stringent condition than that of Corollary 1.0.4, concerning the agreement between the initial data and associated Schwarzschild solution on the horizon. Another related result is that of Chrusciel and Nguyen [19] who utilize pseudospherical coordinates, and obtain a mass lower bound in terms of the horizon rod length.

Finally in Chapter 4 we expand the scope of manifolds for which the previously mentioned Brill coordinates are known to exist. In particular we show that they exist for a large class of manifolds with both asymptotically flat and asymptotically cylindrical ends.

In [13] Chruściel establishes the existence of a Brill coordinate system for axisymmetric, simply connected initial data sets (M^3, g) with one or many asymptotically flat ends. We amend Chruściel's arguments to show the existence of a Brill coordinate system when asymptotically cylindrical ends are allowed. We show the following.

Theorem 1.0.5. *Let (M, g) be of asymptotic order k, for $k \geq 7$, with n ends characterized by (h_i, f_i, ℓ_i). Then there exists a global coordinate system (ρ, z, ϕ) for M in which g takes the form of Equation (1.0.10). The coordinates are defined for $(\rho, z) \in (\mathbb{R}^+ \times \mathbb{R}) \setminus \{(0, a_i)\}_{i=2}^N$, and $\phi \in [0, 2\pi)$. If we define $r = \sqrt{\rho^2 + z^2}$, $r_i = \sqrt{\rho^2 - (z - a_i)^2}$, and $\theta_i = \arctan(\frac{\rho}{z - a_i})$, then the metric components satisfy*

$$A_z = o_{k-3}(r^{-\ell_1 - 1});\ A_\rho = \rho o_{k-3}(r^{-\ell_1 - 2});\ U = o_{k-3}(r^{-\ell_1});\ \alpha = o_{k-4}(r^{-\ell_1})\ \text{as}\ r \to \infty. \tag{1.0.16}$$

For each a_i representing an asymptotically flat end, the metric components satisfy,

$$U = 2\log r_i + C_i + o_{k-4}(r_i^{\ell_i}); \ \alpha = o_{k-4}(r_i^{\ell_i}), \ \ \textit{as} \ \ r_i \to 0, \tag{1.0.17}$$

and for each a_i representing an asymptotically cylindrical end, the components satisfy,

$$U = \log r_i - \log h_i(\theta_i) - \log \frac{f_i(\theta_i)}{\sin\theta_i} + o_{k-4}(r_i^{\ell_i}); \ \alpha = \log \frac{f_i(\theta_i)}{\sin\theta_i} + o_{k-4}(r_i^{\ell_i}), \ \ \textit{as} \ \ r_i \to 0.^2 \tag{1.0.18}$$

A precise definition of an asymptotically cylindrical end is giving in Chapter 4. We conclude with some immediate applications of the above existence theorem.

[2] In [13] the case $\ell = \frac{1}{2}$ is explicitly treated, as this is the most important case for the ADM mass. However as stated in [13], only minor modifications show that the arguments are valid for any $\ell \in (0, 1)$

Chapter 2

The Mass Angular Momentum Charge Inequality for Manifolds with Boundary

2.1 The Doubling Procedure in Pseudospherical Coordinates

Consider the setting of case (*ii*) in Theorem 1.0.1 where (M, g) is axisymmetric, asymptotically flat, and simply connected with a single component minimal surface boundary. It follows from [19, Theorem 2.2] that M is diffeomorphic to $\mathbb{R}^3 \setminus B_{m_1/2}(0)$, and there exists a global system of cylindrical-type coordinates (ρ, z, ϕ) on this domain such that the metric takes the form of Equation (1.0.10). The isothermal part of Equation (1.0.10) is the metric on the orbit space $M/U(1)$, and the remaining part arises from the dual 1-form $\rho^2 e^{-2U}(d\phi + A_\rho d\rho + A_z dz)$ to the Killing field. With the standard transformation $\rho = r\sin\theta$, $z = r\cos\theta$ producing spherical-type coordinates (r, θ, ϕ) with ranges $m_1/2 \leq r < \infty$, $0 \leq \theta \leq \pi$, and $0 \leq \phi < 2\pi$, the fall-off of the metric coefficients in the asymptotically flat end is given by

$$U = o_{\ell-3}(r^{-\frac{1}{2}}), \quad \alpha = o_{\ell-4}(r^{-\frac{1}{2}}), \quad A_\rho = \rho o_{\ell-3}(r^{-\frac{5}{2}}), \quad A_z = o_{\ell-3}(r^{-\frac{3}{2}}). \tag{2.1.1}$$

Furthermore $\alpha = 0$ on the axis $\rho = 0$, and all coefficients are independent of ϕ. Note also that the value $m_1 > 0$ is uniquely determined, and the existence of pseudospherical coordinates does not require the boundary ∂M to be minimal. The mean curvature of a coordinate sphere S_r is

$$H = \frac{2/r + \partial_r(\alpha - 2U)}{\sqrt{e^{-2U+2\alpha} + \rho^2 e^{-2U} A_r^2}} \tag{2.1.2}$$

where $A_r = \sin\theta A_\rho + \cos\theta A_z$, so the assumption of a minimal boundary $\partial M = S_{m_1/2}$ is equivalent to

$$\partial_r\left(U - \frac{1}{2}\alpha\right) = \frac{2}{m_1}. \tag{2.1.3}$$

A particularly advantageous feature of the metric structure (1.0.10) is the simple expression obtained for the scalar curvature [23]

$$2e^{-2U+2\alpha}R = 8\Delta U - 4\Delta_{\rho,z}\alpha - 4|\nabla U|^2 - \rho^2 e^{-2\alpha}\left(A_{\rho,z} - A_{z,\rho}\right)^2, \tag{2.1.4}$$

where Δ is the Laplacian on $\mathbb{R}^3$ with respect to the flat metric $\delta = d\rho^2+dz^2+\rho^2 d\phi^2$ and $\Delta_{\rho,z} = \partial_\rho^2+\partial_z^2$. Moreover the constraint equation (1.0.1), and the assumptions of a maximal slice $\operatorname{Tr} k = 0$ and nonnegative energy density $\mu_{em} \geq 0$ imply that

$$\begin{aligned} R =&16\pi\mu_{em} + |k|^2 + 2\left(|E|^2 + |B|^2\right) \\ \geq& 2\frac{e^{6U-2\alpha}}{\rho^4}|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 + 2\frac{e^{4U-2\alpha}}{\rho^2}\left(|\nabla\chi|^2 + |\nabla\psi|^2\right), \end{aligned} \tag{2.1.5}$$

where v, χ, and ψ are potential functions for angular momentum, electric charge, and magnetic charge respectively. More precisely, the divergence free property of the electric and magnetic fields combined with Cartan's magic formula shows that the 1-forms $\iota_\eta \star E$ and $\iota_\eta \star B$ are closed, where ι and $\star$ denote interior product and the Hodge star operation. Hence simple connectivity yields global potentials satisfying

$$d\chi = \iota_\eta \star E, \qquad d\psi = \iota_\eta \star B. \tag{2.1.6}$$

Furthermore, as shown in [48] the 1-form $2\star(k(\eta)\wedge\eta) - \chi d\psi + \psi d\chi$ is closed exactly when $J_{em}(\eta) = 0$. Therefore under the hypotheses of Theorem 1.0.1 there exists a global twist potential satisfying

$$dv = 2\star(k(\eta)\wedge\eta) - \chi d\psi + \psi d\chi. \tag{2.1.7}$$

The inequality in (2.1.5) then follows in a straightforward way from (2.1.6) and (2.1.7). Moreover, if $\omega = dv + \chi d\psi - \psi d\chi$ then asymptotics [48] for the potentials are expressed by

$$|\omega| = \rho^2 O(r^{-\lambda}), \quad |\nabla\chi| + |\nabla\psi| = \rho O(r^{-\lambda}) \quad \text{as} \quad r \to \infty, \tag{2.1.8}$$

$$|\omega| = O(\rho^2), \quad |\nabla\chi| + |\nabla\psi| = O(\rho) \quad \text{as} \quad \rho \to 0 \quad \text{in} \quad \mathbb{R}^3 \setminus B_{m_1/2}(0). \tag{2.1.9}$$

In addition, since $|\eta| = 0$ on the z-axis all the potential functions are constant there, and the difference of these constants associated with the two connected components $I_+ = \{\rho = 0, z > m_1/2\}$ and $I_- = \{\rho = 0, z < -m_1/2\}$ of the axis yield the angular momentum and charges

$$\mathcal{J} = \frac{1}{4}\left(v|_{I_-} - v|_{I_+}\right), \qquad Q_e = \frac{1}{2}\left(\chi|_{I_-} - \chi|_{I_+}\right), \qquad Q_b = \frac{1}{2}\left(\psi|_{I_-} - \psi|_{I_+}\right). \tag{2.1.10}$$

Typically a mass lower bound in terms of a harmonic map energy is obtained by integrating (2.1.4) over M and applying a version of (2.1.5). This works well when (M, g) is complete, however here the presence of a boundary leads to boundary terms which are not desirable when minimizing the harmonic map energy. Therefore we seek to double the manifold across its boundary, and show that the same strategy may be carried out on the doubled manifold with two ends. The primary difficulty arises from the lack of regularity across the doubling surface. Nevertheless we show that the minimal surface hypothesis is sufficient for the argument to go through.

Consider the conformal map $f : B_{m_1/2} \setminus \{0\} \to \mathbb{R}^3 \setminus B_{m_1/2}$ given by spherical inversion

$$f(\tilde{r}, \tilde{\theta}, \tilde{\phi}) = \left(\left(\frac{m_1}{2}\right)^2 \frac{1}{\tilde{r}}, \tilde{\theta}, \tilde{\phi}\right), \tag{2.1.11}$$

which is expressed in cylindrical coordinates as

$$\rho = \left(\frac{m_1}{2}\right)^2 \frac{\tilde{\rho}}{\tilde{r}^2}, \qquad z = \left(\frac{m_1}{2}\right)^2 \frac{\tilde{z}}{\tilde{r}^2}, \qquad \phi = \tilde{\phi}. \tag{2.1.12}$$

Pulling back the metric to $B_{m_1/2} \setminus \{0\}$ yields

$$\tilde{g} := f^* g = e^{-2\tilde{U}+2\tilde{\alpha}}(d\tilde{\rho}^2 + d\tilde{z}^2) + \tilde{\rho}^2 e^{-2\tilde{U}}(d\tilde{\phi} + \tilde{A}_\rho d\tilde{\rho} + \tilde{A}_z d\tilde{z})^2, \tag{2.1.13}$$

where

$$\tilde{U} = 2\log\tilde{r} + 2\log(2/m_1) + U \circ f, \qquad \tilde{\alpha} = \alpha \circ f, \tag{2.1.14}$$

$$\tilde{A}_\rho = \tilde{r}^{-4}\left(\frac{2}{m_1}\right)^2 \left[(\tilde{z}^2 - \tilde{\rho}^2)A_\rho - 2\tilde{\rho}A_z\right], \qquad \tilde{A}_z = \tilde{r}^{-4}\left(\frac{2}{m_1}\right)^2 \left[(\tilde{\rho}^2 - \tilde{z}^2)A_z - 2\tilde{z}A_\rho\right]. \tag{2.1.15}$$

This leads to a metric and potentials globally defined on the complement of the origin

$$\bar{g} = \begin{cases} g & \text{on } \mathbb{R}^3 \setminus B_{m_1/2}, \\ \tilde{g} & \text{on } B_{m_1/2} \setminus \{0\}. \end{cases} \tag{2.1.16}$$

Similarly, the potentials may also be extended to the ball by setting $\tilde{v} = v \circ f$, $\tilde{\chi} = \chi \circ f$, $\tilde{\psi} = \psi \circ f$ in $B_{m_1/2} \setminus \{0\}$, and the corresponding functions defined on $\mathbb{R}^3 \setminus \{0\}$ will be denoted $\bar{v}$, $\bar{\chi}$, and $\bar{\psi}$. These functions and the metric $\bar{g}$ are $C^{0,1}$ and smooth away from the reflection sphere $S_{m_1/2}$.

The form of the metric (2.1.13) guarantees that the scalar curvature of $\bar{g}$ satisfies the equation (2.1.4) on all of $\mathbb{R}^3 \setminus \{0\}$. Moreover it also satisfies the lower bound in (2.1.5). To see this observe that

$$|\nabla\chi|^2 \circ f = (\partial_\rho \chi)^2 \circ f + (\partial_z \chi)^2 \circ f = \left(\frac{2}{m_1}\right)^4 \tilde{r}^4 \left[(\partial_{\tilde{\rho}}\tilde{\chi})^2 + (\partial_{\tilde{z}}\tilde{\chi})^2\right] = \left(\frac{2}{m_1}\right)^4 \tilde{r}^4 |\tilde{\nabla}\tilde{\chi}|^2, \tag{2.1.17}$$

and similarly

$$|\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 \circ f = \left(\frac{2}{m_1}\right)^4 \tilde{r}^4 |\tilde{\nabla}\tilde{v} + \tilde{\chi}\tilde{\nabla}\tilde{\psi} - \tilde{\psi}\tilde{\nabla}\tilde{\chi}|^2. \tag{2.1.18}$$

Combining this with (2.1.5), (2.1.12), and (2.1.14) shows that in $B_{m_1/2} \setminus \{0\}$

$$\begin{aligned} \tilde{R} =& R \circ f \\ \geq& 2\frac{e^{6U\circ f - 2\alpha\circ f}}{(\rho\circ f)^4} |\nabla v + \chi\nabla\psi - \psi\nabla\chi|^2 \circ f + 2\frac{e^{4U\circ f - 2\alpha\circ f}}{(\rho\circ f)^2}\left(|\nabla\chi|^2 \circ f + |\nabla\psi|^2 \circ f\right) \\ =& 2\frac{e^{6\tilde{U} - 2\tilde{\alpha}}}{\tilde{\rho}^4} |\tilde{\nabla}\tilde{v} + \tilde{\chi}\tilde{\nabla}\tilde{\psi} - \tilde{\psi}\tilde{\nabla}\tilde{\chi}|^2 + 2\frac{e^{4\tilde{U}-2\tilde{\alpha}}}{\tilde{\rho}^2}\left(|\tilde{\nabla}\tilde{\chi}|^2 + |\tilde{\nabla}\tilde{\psi}|^2\right). \end{aligned} \tag{2.1.19}$$

It follows that the scalar curvature of the doubled metric $\bar{g}$ satisfies the desired lower bound on $\mathbb{R}^3 \setminus \{0\}$ away from the sphere $S_{m_1/2}$. Although the metric is not sufficiently regular across this sphere to have a pointwise defined scalar curvature on this surface, the fact that it is a minimal surface with respect to both inner and outer domains guarantees that $\bar{R}$ satisfies the inequality distributionally. Furthermore the minimal surface property allows for the fundamental mass lower bound in terms of scalar curvature, despite the lack of metric regularity.

In order to establish the mass lower bound it is necessary to note that the doubled manifold $(\bar{M}, \bar{g})$, where $\bar{M} = \mathbb{R}^3 \setminus \{0\}$, possesses two asymptotically flat ends. Indeed, at the additional end near the origin the metric coefficients and potentials satisfy the asymptotics

$$\bar{U} = 2\log r + C + o_1(r^{\frac{1}{2}}), \quad \bar{\alpha} = o_1(r^{\frac{1}{2}}), \quad \bar{A}_\rho = \rho o_1(r^{-\frac{5}{2}}), \quad \bar{A}_z = o_1(r^{-\frac{3}{2}}), \tag{2.1.20}$$

$$|\bar{\omega}| = \rho^2 O(r^{\lambda-6}), \quad |\nabla\bar{\chi}| + |\nabla\bar{\psi}| = \rho O(r^{\lambda-4}) \quad \text{as} \quad r \to 0, \tag{2.1.21}$$

for some constant C. Here and in what follows, unless stated otherwise, the tilde notation will be removed from coordinates within the domain $B_{m_1/2} \setminus \{0\}$.

Lemma 2.1.1. *The doubled manifold $(\bar{M}, \bar{g})$ possesses two asymptotically flat ends, and the mass is given by*

$$m = \frac{1}{32\pi}\int_{\mathbb{R}^3}\left(2e^{-2\bar{U}+2\bar{\alpha}}\,\bar{R} + 4|\nabla\bar{U}|^2 + \rho^2 e^{-2\bar{\alpha}}(\bar{A}_{\rho,z} - \bar{A}_{z,\rho})^2\right)dx, \tag{2.1.22}$$

where dx is the Euclidean volume element.

Proof. Although the metric $\bar{g}$ is only Lipschitz across $S_{m_1/2}$, the fact that this sphere is a minimal surface guarantees that a particular combination of coefficients has improved regularity, namely $\bar{U} - \frac{1}{2}\bar{\alpha} \in C^{1,1}$. Moreover, this is all that is needed to establish (2.1.22).

First observe that in light of (2.1.3)

$$\lim_{r\to\frac{m_1}{2}^+} \partial_r\left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) = \frac{2}{m_1}. \tag{2.1.23}$$

It suffices then to show that the limit from inside $B_{m_1/2}$ yields the same value. For emphasis we will use the tilde notation to perform this computation. By (2.1.14)

$$\partial_{\tilde{r}}\tilde{U} = \frac{2}{\tilde{r}} + \partial_r U\frac{\partial r}{\partial\tilde{r}} = \frac{2}{\tilde{r}} - \left(\frac{m_1}{2}\right)^2\frac{1}{\tilde{r}^2}\partial_r U. \tag{2.1.24}$$

Therefore (2.1.3) implies

$$\partial_{\tilde{r}}\tilde{U}|_{\tilde{r}=\frac{m_1}{2}} = \frac{4}{m_1} - \partial_r U|_{r=\frac{m_1}{2}} = \frac{2}{m_1} - \frac{1}{2}\partial_r\alpha|_{r=\frac{m_1}{2}}. \tag{2.1.25}$$

On the other hand

$$\partial_{\tilde{r}}\tilde{\alpha} = -\left(\frac{m_1}{2}\right)^2\frac{1}{\tilde{r}^2}\partial_r\alpha, \tag{2.1.26}$$

and therefore the desired conclusion follows

$$\lim_{r\to\frac{m_1}{2}^-} \partial_r\left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) = \frac{2}{m_1}. \tag{2.1.27}$$

We will now show that (2.1.22) holds. According to [13]

$$m = \lim_{r\to\infty}\frac{1}{4\pi}\left(\int_{S_r}\partial_r\left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) + \frac{1}{2}\int_{W_r}\frac{\bar{\alpha}}{\rho}\right), \tag{2.1.28}$$

where $W_r = \{\rho = r, -r < z < r\}$ is the wall of the cylinder or radius r. Next observe that (2.1.23) and (2.1.27) yield

$$\begin{aligned}\int_{S_r} \partial_r \left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) &= \int_{B_r \setminus B_{m_1/2}} \Delta \left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) dx + \lim_{r \to \frac{m_1}{2}^+} \int_{S_r} \partial_r \left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) \\ &= \int_{B_r \setminus B_{m_1/2}} \Delta \left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) dx + \lim_{r \to \frac{m_1}{2}^-} \int_{S_r} \partial_r \left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) \\ &= \int_{B_r \setminus B_{m_1/2}} \Delta \left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) dx + \int_{B_{m_1/2}} \Delta \left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) dx,\end{aligned} \tag{2.1.29}$$

since the asymptotics (2.1.20) show that

$$\lim_{r \to 0} \int_{S_r} \partial_r \left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) = 0. \tag{2.1.30}$$

Moreover, since $\bar{\alpha}$ is continuous across $S_{m_1/2}$, vanishes away from the origin on the z-axis, and satisfies (2.1.20)

$$\lim_{r \to \infty} \int_{W_r} \frac{\bar{\alpha}}{\rho} = \int_{\mathbb{R}^3} \frac{1}{\rho} \partial_\rho \bar{\alpha} dx. \tag{2.1.31}$$

Finally, since (2.1.4) holds globally on $\bar{M}$ we have

$$\Delta \left(\bar{U} - \frac{1}{2}\bar{\alpha}\right) + \frac{1}{2\rho}\partial_\rho \bar{\alpha} = \Delta \bar{U} - \frac{1}{2}\Delta_{\rho,z}\bar{\alpha} = \frac{1}{4}e^{-2\bar{U}+2\bar{\alpha}}\bar{R} + \frac{1}{2}|\nabla \bar{U}|^2 + \frac{1}{8}\rho^2 e^{-2\bar{\alpha}} \left(\bar{A}_{\rho,z} - \bar{A}_{z,\rho}\right)^2. \tag{2.1.32}$$

The desired mass formula (2.1.22) now follows by combining (2.1.28), (2.1.29), (2.1.31), and (2.1.32). □

Lemma 2.1.1 relates the mass to an energy functional with the help of (2.1.5) and (2.1.19). Namely together they imply

$$m \geq \mathcal{I}(\Psi), \tag{2.1.33}$$

where $\Psi = (\bar{U}, \bar{v}, \bar{\chi}, \bar{\psi})$ and

$$\mathcal{I}(\Psi) = \frac{1}{8\pi} \int_{\mathbb{R}^3} \left(|\nabla \bar{U}|^2 + \frac{e^{4\bar{U}}}{\rho^4} |\nabla \bar{v} + \bar{\chi}\nabla\bar{\psi} - \bar{\psi}\nabla\bar{\chi}|^2 + \frac{e^{2\bar{U}}}{\rho^2} \left(|\nabla\bar{\chi}|^2 + |\nabla\bar{\psi}|^2\right) \right) dx. \tag{2.1.34}$$

The functional $\mathcal{I}$ may be interpreted as the reduced harmonic energy [48] for maps $\Psi : \mathbb{R}^3 \setminus \{0\} \to \mathbb{H}^2_{\mathbb{C}}$ into the complex hyperbolic plane. Note that the asymptotics (2.1.1), (2.1.8), (2.1.20), and (2.1.21) guarantee that $\mathcal{I}(\Psi)$ is finite precisely when $\lambda > \frac{3}{2}$.

Proof of Theorem 1.0.1 (ii). Since the map Ψ is smooth away from the sphere $S_{m_1/2}$, Lipschitz across this surface, and satisfies the asymptotics (2.1.1), (2.1.8), (2.1.9), (2.1.20), (2.1.21), the gap bound of Schoen and Zhou [57] applies to yield

$$\mathcal{I}(\Psi) - \mathcal{I}(\Psi_0) \geq C \left(\int_{\mathbb{R}^3} \mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi, \Psi_0) dx \right)^{\frac{1}{3}}, \tag{2.1.35}$$

where Ψ_0 is the renormalized harmonic map associated with the extreme Kerr-Newman black hole possessing the same angular momentum and charge as Ψ, and $\mathrm{dist}_{\mathbb{H}_{\mathbb{C}}}$ denotes distance in the complex hyperbolic plane. In particular, together with (2.1.33) we obtain

$$m \geq \mathcal{I}(\Psi_0). \tag{2.1.36}$$

The desired inequality (1.0.12) now follows since

$$\mathcal{I}(\Psi_0)^2 = \frac{Q^2 + \sqrt{Q^4 + 4\mathcal{J}^2}}{2}. \tag{2.1.37}$$

Consider now the case in which equality holds in (1.0.12). This implies, with the help of (2.1.33), (2.1.35), and (2.1.37), that $\Psi = \Psi_0$ and in particular $U = U_0$. However this is a contradiction since the asymptotics (2.1.20) show that $U = 2\log r + O(1)$ as $r \to 0$, whereas the corresponding asymptotics for the extreme Kerr-Newman map are given by $U_0 = \log r + O(1)$. This difference arises from the fact that Ψ arises from an asymptotically flat geometry near the origin, while the extreme Kerr-Newman initial data possess instead an asymptotically cylindrical end in this location. □

2.2 The Outermost Minimal Surface in Axisymmetry

Let (M, g) be as in case (i) in Theorem 1.0.1. Since this manifold is asymptotically flat and possesses a minimal surface boundary, there exists a unique outermost minimal surface [30, 40] which is a compact embedded smooth hypersurface Σ. The term outermost refers to the fact that there are no other minimal surfaces homologous to Σ which lie outside it with respect to the asymptotic end. The set $M \setminus \Sigma$ consists of one unbounded component, and perhaps several bounded components the union of which will be denoted by Ω, so that $\partial\Omega = \Sigma$. Each component of the outermost minimal surface must be a topological 2-sphere [33], since Σ is outer area minimizing in that it has the least area among all surfaces which enclose it. Furthermore, according to [40, Lemma 4.1] $M \setminus \Sigma$ is diffeomorphic to the complement of a finite number of open 3-balls in $\mathbb{R}^3$ with disjoint closure. Here we show that the property of axisymmetry for the ambient manifold descends to Σ.

Proposition 2.2.1. *If (M, g) is axisymmetric then the outermost minimal surface is also axisymmetric.*

Proof. Suppose that the outermost minimal surface Σ is not axisymmetric. Let φ_t denote the flow of the axisymmetric Killing field η, so that $\partial_t \varphi_t = \eta \circ \varphi_t$. Lack of axisymmetry implies that η is not tangential to Σ at all points. Therefore there exists a nonzero $t_0 \sim 0$ such that a domain within $\varphi_{t_0}(\Sigma)$ lies outside of Σ. Note that φ_t is a flow by isometries so $\varphi_{t_0}(\Sigma)$ is a minimal surface, and it is still an embedded 2-sphere.

Let $\mathcal{S}$ denote the compact set which is the union of all compact immersed minimal surfaces within M, and define the trapped region $\mathcal{T}$ to be the union of $\mathcal{S}$ with all the bounded components of $M \setminus \mathcal{S}$. The trapped region is compact and its topological boundary is comprised of embedded smooth minimal 2-spheres [40, Lemma 4.1]. In fact the outermost minimal surface arises as the boundary $\partial\mathcal{T}$. In light of this, and the fact that a portion of $\varphi_{t_0}(\Sigma)$ lies outside Σ, it follows that $\Sigma \neq \partial\mathcal{T}$ contradicting the uniqueness of the outermost minimal surface. □

Proof of Theorem 1.0.1 (i). Assume that the outermost minimal surface Σ has a single component. From the discussion above we then have that $M_0 = M \setminus \Omega$ is diffeomorphic to the complement of a 3-ball in $\mathbb{R}^3$. Proposition 2.2.1 guarantees that ∂M_0 is axisymmetric, and hence M_0 is axisymmetric. It follows that (M_0, g, k, E, B) satisfies the hypotheses of Theorem 1.0.1 (ii), and has the same mass, angular momentum, and charge as the original data. Part (ii) may now be applied to obtain (1.0.12). □

2.3 Multiple Black Holes

Consider the setting of case (ii) in Theorem 1.0.2 where (M, g) is axisymmetric, asymptotically flat, simply connected, with a single component minimal surface boundary, and a finite number n of additional ends each of which is asymptotically flat or asymptotically cylindrical; see [48] for a definition of asymptotically cylindrical ends. The additional ends may be interpreted physically as individual black holes. A version of pseudospherical coordinates exists for this situation, where each additional end is represented by a puncture on the z-axis and again the boundary component is identified with a coordinate sphere.

Proposition 2.3.1. *Under the hypotheses of Theorem 1.0.2 (ii), M is diffeomorphic to $\left(\mathbb{R}^3 \setminus B_{m_1/2}(0)\right)\setminus \cup_{i=1}^n \{p_i\}$ and there exists a global system of cylindrical-type coordinates (ρ, z, ϕ) such that g takes the form* (1.0.10) *with $\alpha = 0$ whenever $\rho = 0$. Each puncture p_i represents an additional end in which the metric coefficients have the asymptotics* (2.1.20) *in the asymptotically flat case, or*

$$U = \log r_i + O_1(1), \qquad \alpha = o_1(r_i^{\frac{1}{2}}), \qquad A_\rho = \rho o_1(r_i^{-\frac{5}{2}}), \qquad A_z = o_1(r_i^{-\frac{3}{2}}), \tag{2.3.1}$$

in the asymptotically cylindrical case. Here r_i denotes the Euclidean distance to the puncture p_i.

Proof. The proof is nearly identical to that of [19, Theorem 2.2], and thus we only give an outline. Since M is simply connected the single boundary component ∂M must topological be a 2-sphere by [38, Lemma 4.9]. The boundary may then be filled in with a 3-ball, and the metric extended to this domain to obtain a complete, axisymmetric, simply connected Riemannian manifold $(\hat{M}, \hat{g})$ with $n+1$ asymptotic ends. According to [13, 46] M is diffeomorphic to $\mathbb{R}^3 \setminus \cup_{i=1}^n \{p_i\}$ with the punctures p_i lying on the $\hat{z}$-axis of a global system of Brill coordinates $(\hat{\rho}, \hat{z}, \phi)$ in which $\hat{g}$ has the structure (1.0.10). The orbit space $M/U(1)$ is identified with the $\hat{\rho}\hat{z}$-half plane, and may be doubled across the axis so that $(\hat{\rho}, \hat{z})$ parameterize $\mathbb{R}^2$ minus the axis punctures. Within this plane the projection of ∂M is given by a smooth closed curve γ which intersects the $\hat{z}$-axis at two points, and bounds a disc. Using the Riemann mapping theorem, a conformal transformation of the plane may now be applied which maps γ to a circle centered at the origin of radius $m_1/2$. The new coordinates obtained from this map are the desired pseudospherical coordinates (ρ, z, ϕ). Although the punctures may move under this transformation, they will remain on the axis since the mapping is axisymmetric. Lastly, the conformal property of the map preserves the metric structure (1.0.10). □

Simple connectivity of M yields potentials v, χ, and ψ satisfying (2.1.6), (2.1.7) as well as the asymptotics (2.1.8), (2.1.21) in the asymptotically flat ends. In asymptotically cylindrical ends [48]

$$|\omega| = \rho^2 O(r_i^{\lambda-5}), \qquad |\nabla\chi| + |\nabla\psi| = \rho O(r_i^{\lambda-3}) \quad \text{as} \quad r_i \to 0. \tag{2.3.2}$$

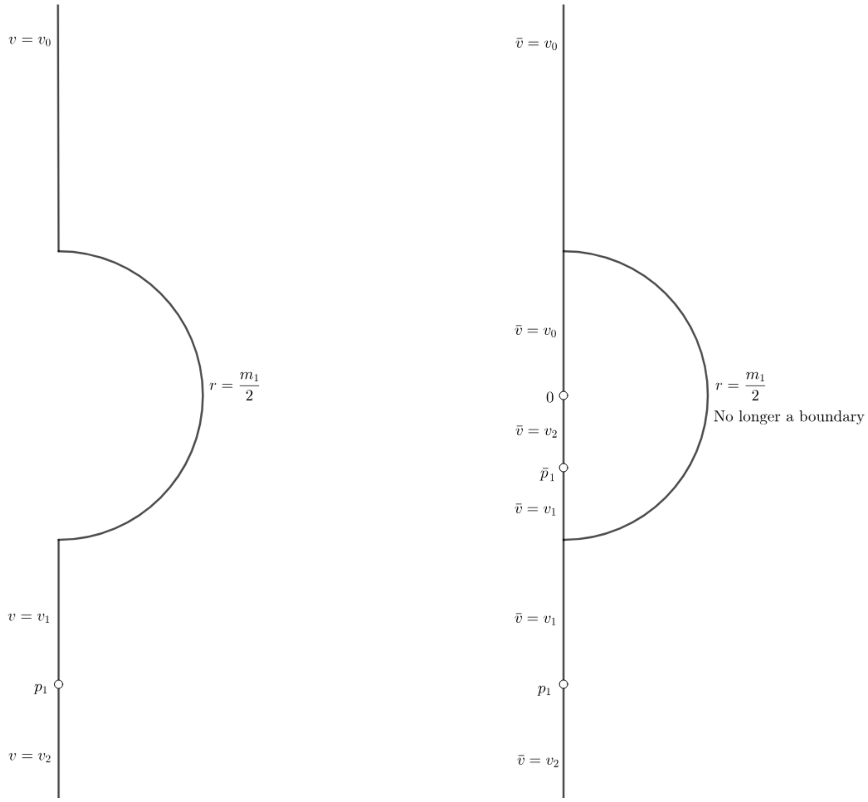

Figure 2.1: Doubling with additional ends

The punctures $\{p_i\}_{i=1}^n$ and ball $B_{m_1/2}$ break up the z-axis into a sequence of connected component intervals $\{I_j\}_{j=1}^{n+2}$ on which each of the potentials is constant; this is sometimes referred to as a 'rod structure'. As in (2.1.10), the difference of two such constants associated to the intervals surrounding a puncture yields the angular momentum or charge associated to the black hole represented by the puncture. Following Section 2.1 we may double the manifold and potentials across the sphere $S_{m_1/2}$ to obtain a manifold $(\bar{M}, \bar{g})$ and functions $\bar{v}$, $\bar{\chi}$, $\bar{\psi}$. The only difference that occurs concerns the number of asymptotic ends. Previously the new manifold had two asymptotically flat ends, however now $\bar{M}$ has $2n+2$ asymptotic ends. In Figure 2.1 a diagram of the doubling rod structure in the orbit space is shown, where a single additional end occurs below the circle of radius $m_1/2$ at point p_1. After doubling, this end is reflected inside the circle to point $\bar{p}_1$ which represents another end of the same asymptotic type. As before the origin also becomes an asymptotically flat end, associated with the designated asymptotically flat end at infinity. Notice, as is shown in the diagram, that the potential constants on the axis also reflect inside in such a way that they are smooth across the sphere $S_{m_1/2}$ and so that the angular momentum and charge of each end inside the $B_{m_1/2}$ has the same value with opposite sign as that associated with the corresponding puncture outside the ball. Therefore the total angular momentum and total charge of $\bar{M}$, computed by adding all the individual contributions of each end, agrees with the total angular momentum and total charge of M.

The presence of additional asymptotically flat and cylindrical ends does not affect the proof of

Lemma 2.1.1, as well as scalar curvature lower bounds (2.1.5) and (2.1.19). It follows that as before

$$m \geq \mathcal{I}(\Psi). \tag{2.3.3}$$

Proof of Theorem 1.0.2. Consider case (*ii*). It remains to show that the renormalized energy may be minimized by a harmonic map

$$\mathcal{I}(\Psi) \geq \mathcal{I}(\Psi_1). \tag{2.3.4}$$

Here Ψ_1 is the unique renormalized harmonic map from $\mathbb{R}^3 \setminus \{z\text{-axis}\} \to \mathbb{H}^2_{\mathbb{C}}$, having the same potential constants as Ψ. Such solutions have been constructed in [48], and the corresponding gap bound (2.1.35) was established there as well. Thus, by setting $\mathcal{I}(\Psi_1) = \mathcal{F}$ we obtain the desired result.

Consider now case (*i*). As in the proof of Theorem 1.0.1 (*i*) let M_0 denote the region exterior to the outermost minimal surface Σ, with respect to the designated asymptotically flat end. Then M_0 is diffeomorphic to the complement of a finite number of open 3-balls and a finite number of points in $\mathbb{R}^3$, where the point punctures represent asymptotically cylindrical ends and the boundary of the 3-balls are the components of Σ. By assumption, at most one component Σ_1 of the outermost minimal surface encloses components of the boundary ∂M or nonsimply connected domains. If $\Sigma_1 = \emptyset$ then M is simply connected and has no boundary, and therefore this theorem follows from [48]. If $\Sigma_1 \neq \emptyset$ let M_1 denote the region of M outside of Σ_1. The hypotheses then imply that M_1 has a single component minimal surface boundary, is simply connected, and has a finite number of additional asymptotically flat and cylindrical ends. By Proposition 2.2.1 $\partial M_1 = \Sigma_1$ is axisymmetric, so that M_1 itself is axisymmetric. The initial data (M_1, g, k, E, B) now satisfy the hypotheses of Theorem 1.0.2 (*ii*), and (1.0.13) follows. □

Chapter 3

An Approach to the Penrose Inequality with Charge and Angular Momentum

3.1 The Mass Formula and Reduced Harmonic Energy

An initial data set (M, g, k, E, B) as in Theorem 1.0.3 admits a global set of Weyl coordinates [19] (ρ, z, ϕ) in which the metric takes the form (1.0.10) and the scalar curvature is of the form (1.0.11). Since there is a single black hole, or rather one minimal surface boundary component, the z-axis is broken up into three intervals or 'rods' $(-\infty, -m_0)$, $(-m_0, m_0)$, (m_0, ∞) in which the two semi-infinite rods are the axis and the finite rod represents the horizon boundary ∂M. The value $m_0 > 0$ is uniquely determined by the geometry of the initial data. Let U_0 and α_0 denote the metric coefficients in Weyl coordinates for the Schwarzschild solution having this same rod structure; note that m_0 is then the mass of this Schwarzschild spacetime. If $r_+ = \sqrt{\rho^2 + (z - m_0)^2}$ and $r_- = \sqrt{\rho^2 + (z + m_0)^2}$ denote the Euclidean distances to the poles $p_+ = (0, m_0)$ and $p_- = (0, -m_0)$ in the ρz-plane, then

$$U_0 = \frac{1}{2} \log \frac{r_- + r_+ - 2m_0}{r_+ + r_- + 2m_0}, \qquad \alpha_0 = \frac{1}{2} \log \frac{(r_- + r_+)^2 - 4m_0^2}{4r_- r_+}. \tag{3.1.1}$$

These functions blow-up on the horizon but are finite along the axis. In particular

$$U_0 = -\frac{m_0}{r} + O\left(\frac{1}{r^2}\right), \qquad \alpha_0 = O\left(\frac{1}{r^2}\right) \quad \text{as } r := \sqrt{\rho^2 + z^2} \to \infty, \tag{3.1.2}$$

$$U_0 = \frac{1}{2} \log\left(\frac{z - m_0}{z + m_0}\right) + O(\rho^2), \qquad \alpha_0 = O(\rho^2) \quad \text{as } \rho \to 0 \text{ and } |z| \geq m_0 + \epsilon, \tag{3.1.3}$$

$$U_0 = \log \rho + O(1), \qquad \alpha_0 = \log \rho + O(1) \quad \text{as } \rho \to 0 \text{ and } |z| \leq m_0 - \epsilon, \tag{3.1.4}$$

where $\epsilon > 0$. These Schwarzschild coefficients play the role of singular part for the metric coefficients of (1.0.10). That is, we may write $U = U_0 + \overline{U}$ and $\alpha = \alpha_0 + \overline{\alpha}$ where $\overline{U}$ and $\overline{\alpha}$ remain bounded. In fact, this decomposition has the following regularity properties which are proved in Section 3.5 and rely on the minimal surface condition at the boundary.

Lemma 3.1.1. *Under the assumptions of Theorem 1.0.3 the renormalized functions* $\overline{U}$ *and* $\overline{\alpha}$ *are smooth away from the horizon rod, and have continuous first derivatives everywhere except possibly at the poles* $p_{\pm}$ *where they are bounded. At infinity* $\overline{U} = O_1(r^{-1/2-\epsilon})$ *and* $\overline{\alpha} = O_1(r^{-1/2-\epsilon})$ *for some* $\epsilon > 0$.

Let us now use this decomposition of the metric coefficients to compute the ADM mass. Recall from [19] that if S_∞ represents the limit as $r \to \infty$ for coordinate spheres S_r then the mass is given by

$$m = \frac{1}{8\pi} \int_{S_\infty} \left[\partial_r(2U - \alpha) + \frac{\alpha}{r} \right] d\sigma. \tag{3.1.5}$$

The boundary terms at infinity in this formula arise from integrating the scalar curvature formula (1.0.11). Observe that

$$\begin{aligned} \int_{\mathbb{R}^3} \Delta_{\rho,z}\alpha dx &= \int_{\mathbb{R}^2_+} 2\pi\rho\Delta_{\rho,z}\alpha d\rho dz \\ &= \lim_{\varepsilon\to 0} \int_{\rho=\varepsilon} 2\pi(\alpha - \rho\partial_\rho\alpha)dz + \lim_{r\to\infty} \int_{\partial D_r^+} 2\pi(\rho\partial_r\alpha - \alpha\sin\theta)ds \\ &= \lim_{\varepsilon\to 0} \int_{\rho=\varepsilon} 2\pi(\alpha - \rho\partial_\rho\alpha)dz + \int_{S_\infty} \left(\partial_r\alpha - \frac{\alpha}{r}\right) d\sigma. \end{aligned} \tag{3.1.6}$$

Here D_r^+ is the half disk of radius r, and $\rho = r\sin\theta$ and $z = r\cos\theta$. Furthermore

$$\int_{\mathbb{R}^3} \Delta U dx = \int_{S_\infty} \partial_r U d\sigma - \lim_{\varepsilon\to 0} \int_{\rho=\varepsilon} 2\pi\rho\partial_\rho U dz, \tag{3.1.7}$$

and since $U_0 = O_1(r^{-1})$ as $r \to \infty$ with U_0 harmonic

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla U|^2 dx &= \int_{\mathbb{R}^3} |\nabla(U_0 + \overline{U})|^2 dx \\ &= \int_{\mathbb{R}^3} \left(|\nabla\overline{U}|^2 + \nabla(U_0 + 2\overline{U}) \cdot \nabla U_0 \right) dx \\ &= \int_{\mathbb{R}^3} |\nabla\overline{U}|^2 dx - \lim_{\varepsilon\to 0} \int_{\rho=\varepsilon} (U_0 + 2\overline{U})\partial_\rho U_0 d\sigma + \int_{S_\infty} (U_0 + 2\overline{U})\partial_r U_0 d\sigma \\ &= \int_{\mathbb{R}^3} |\nabla\overline{U}|^2 dx - \lim_{\varepsilon\to 0} \int_{\rho=\varepsilon} 2\pi\rho(U_0 + 2\overline{U})\partial_\rho U_0 dz. \end{aligned} \tag{3.1.8}$$

Therefore by integrating the scalar curvature formula, and putting all these computations together, we find that

$$\begin{aligned} 8\pi m = &\int_{\mathbb{R}^3} \left[|\nabla\overline{U}|^2 + \frac{1}{2}e^{-2U+2\alpha}R + \frac{1}{4}\rho^2 e^{-2\alpha}(\partial_z A_\rho - \partial_\rho A_z)^2 \right] dx \\ &+ \lim_{\varepsilon\to 0} \int_{\rho=\varepsilon} \left[4\pi\rho\partial_\rho U + 2\pi(\alpha - \rho\partial_\rho\alpha) - 2\pi\rho(U_0 + 2\overline{U})\partial_\rho U_0 \right] dz. \end{aligned} \tag{3.1.9}$$

Consider now the boundary integrals in (3.1.9). Computations show that

$$\lim_{\varepsilon\to 0} \int_{\rho=\varepsilon} \rho\partial_\rho U dz = \lim_{\varepsilon\to 0} \int_{\rho=\varepsilon} \varepsilon\partial_\rho U_0(\varepsilon, z) dz = 2m_0, \tag{3.1.10}$$

and

$$\lim_{\varepsilon\to 0}\int_{\rho=\varepsilon}\left[\alpha-\rho\partial_\rho\alpha-\rho(U_0+2\overline{U})\partial_\rho U_0\right]dz=\lim_{\varepsilon\to 0}\int_{\substack{\rho=\varepsilon\\ |z|<m_0}}\left(\alpha_0+\overline{\alpha}-\rho\partial_\rho\alpha_0-U_0-2\overline{U}\right)dz$$
$$=\int_{-m_0}^{m_0}(\overline{\alpha}-2\overline{U})(0,z)dz. \tag{3.1.11}$$

Furthermore, simple connectedness and the divergence free condition for the electric and magnetic fields gives rise to electromagnetic potentials [48, Section 2]

$$d\psi=F(\eta,\cdot),\qquad d\chi=\star F(\eta,\cdot), \tag{3.1.12}$$

where F is the field strength tensor and $\star$ denotes the Hodge star operation. Similarly the compatibility condition $J_{em}(\eta)=0$ guarantees the existence of a charged twist potential

$$dv=k(\eta)\times\eta-\chi d\psi+\psi d\chi. \tag{3.1.13}$$

Since the initial data are maximal, nonnegativity of the energy density $\mu_{em}\geq 0$ implies the following lower bound [48, Section 2] for scalar curvature

$$R\geq|k|_g^2+2(|E|_g^2+|B|_g^2)\geq 2\frac{e^{6U-2\alpha}}{\rho^4}|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2+2\frac{e^{4U-2\alpha}}{\rho^2}(|\nabla\chi|^2+|\nabla\psi|^2). \tag{3.1.14}$$

Putting all this together yields the mass lower bound

$$m\geq\frac{1}{8\pi}\int_{\mathbb{R}^3}\left(|\nabla\overline{U}|^2+\frac{e^{4U}}{\rho^4}|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2+\frac{e^{2U}}{\rho^2}(|\nabla\chi|^2+|\nabla\psi|^2)\right)dx$$
$$+\frac{1}{4}\int_{-m_0}^{m_0}(\overline{\alpha}(0,z)-2\overline{U}(0,z))dz+m_0. \tag{3.1.15}$$

Related formulas were obtained in [13, 19] and [35] in different settings.

The volume integral on the right-hand side of (3.1.15) is directly related to the harmonic energy of maps between $\mathbb{R}^3\setminus\Gamma\to\mathbb{H}^2_{\mathbb{C}}$, where $\Gamma=\{\rho=0,|z|>m_0\}$ is the axis. More precisely, let $\tilde{\Psi}=(u,v,\chi,\psi):\mathbb{R}^3\setminus\Gamma\to\mathbb{H}^2_{\mathbb{C}}$ and consider the harmonic energy of this map on a bounded domain $\Omega\subset\mathbb{R}^3\setminus\Gamma$:

$$E_\Omega(\tilde{\Psi})=\int_\Omega|\nabla u|^2+e^{4u}|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2+e^{2u}\left(|\nabla\chi|^2+|\nabla\psi|^2\right)dx. \tag{3.1.16}$$

Set $u=U-\log\rho$, then the reduced energy $\mathcal{I}_\Omega$ of the renormalized map $\Psi=(\overline{U},v,\chi,\psi)$ is related to the harmonic energy of $\tilde{\Psi}$ by

$$\mathcal{I}_\Omega(\Psi)=E_\Omega(\tilde{\Psi})+\int_{\partial\Omega}(2\overline{U}+U_0-\log\rho)\partial_\nu(\log\rho-U_0)d\sigma, \tag{3.1.17}$$

where ν denotes the unit outer normal to the boundary $\partial\Omega$ and

$$\mathcal{I}_\Omega(\Psi)=\int_\Omega|\nabla\overline{U}|^2+\frac{e^{4U}}{\rho^4}|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2+\frac{e^{2U}}{\rho^2}\left(|\nabla\chi|^2+|\nabla\psi|^2\right)dx. \tag{3.1.18}$$

Observe that the volume integral of (3.1.15) is exactly the reduced energy on $\mathbb{R}^3$, which will be denoted by $\mathcal{I}(\Psi)$. The relation (3.1.17) is established through an integration by parts, using the fact that $\log\rho$ and U_0 are harmonic on $\mathbb{R}^3\setminus\Gamma$. Namely

$$\begin{aligned}\mathcal{I}_\Omega(\Psi) =& \int_\Omega \left(|\nabla(u-U_0+\log\rho)|^2 + e^{4u}|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2 + e^{2u}(|\nabla\chi|^2+|\nabla\psi|^2)\right)dx\\ =& \int_\Omega |\nabla u|^2 + \nabla(2u-U_0+\log\rho)\cdot\nabla(\log\rho-U_0)dx\\ &+\int_\Omega e^{4u}|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2+e^{2u}(|\nabla\chi|^2+|\nabla\psi|^2)dx\\ =& \int_\Omega \left(|\nabla u|^2 + e^{4u}|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2+e^{2u}(|\nabla\chi|^2+|\nabla\psi|^2)\right)dx\\ &+\int_{\partial\Omega}(2u-U_0+\log\rho)\partial_\nu(\log\rho-U_0)d\sigma\\ =&E_\Omega(\tilde{\Psi})+\int_{\partial\Omega}(2\overline{U}+U_0-\log\rho)\partial_\nu(\log\rho-U_0)d\sigma.\end{aligned} \tag{3.1.19}$$

The functional $\mathcal{I}$ may be considered a regularization of E since the infinite term $\int|\nabla(\log\rho-U_0)|^2$ has been removed, and since the two functionals differ only by a boundary term they must have the same critical points.

Let $\tilde{\Psi}_k=(u_k,v_k,\chi_k,\psi_k)$ denote the harmonic map associated with the Kerr-Newman solution, and let Ψ_k be the corresponding renormalized map where $u_k=U_k-\log\rho=\overline{U}_k+U_0-\log\rho$. It follows that Ψ_k is a critical point of $\mathcal{I}$. As will be shown in Section 3.3, Ψ_k realizes the global minimum for $\mathcal{I}$.

Theorem 3.1.2. *Suppose that $\Psi=(\overline{U},v,\chi,\psi)$ is smooth and satisfies the asymptotics* (3.2.4)-(3.2.14). *If $v|_\Gamma=v_k|_\Gamma$, $\chi|_\Gamma=\chi_k|_\Gamma$, and $\psi|_\Gamma=\psi_k|_\Gamma$ then there exists a constant $C>0$ such that*

$$\mathcal{I}(\Psi)-\mathcal{I}(\Psi_k)\geq C\left(\int_{\mathbb{R}^3}\mathrm{dist}^6_{\mathbb{H}^2_{\mathbb{C}}}(\Psi,\Psi_k)dx\right)^{\frac{1}{3}}. \tag{3.1.20}$$

3.2 Asymptotics in Weyl Coordinates

In order to minimize the functional $\mathcal{I}(\Psi)$ it is necessary to choose the appropriate asymptotics for the map Ψ. The asymptotics will be guided by the principle of having a finite reduced energy, however the convexity minimization argument of the next section will in general require stronger asymptotics than that which is optimal for integrability. It will be useful to first record the asymptotics of the Schwarzschild metric coefficients near the poles, namely a computation shows that

$$e^{U_0}=O(r_+^{1/2})\quad\text{as } r_+\to 0 \text{ and } z\geq m_0,\qquad e^{U_0}=O(\rho r_+^{-1/2})\quad\text{as } r_+\to 0\text{ and } z\leq m_0, \tag{3.2.1}$$

$$e^{U_0}=O(\rho r_-^{-1/2})\quad\text{as } r_-\to 0\text{ and } z\geq -m_0,\qquad e^{U_0}=O(r_-^{1/2})\quad\text{as } r_-\to 0\text{ and } z\leq -m_0, \tag{3.2.2}$$

$$e^{U_0-\alpha_0}=O(r_\pm^{1/2})\qquad\text{as } r_\pm\to 0. \tag{3.2.3}$$

According to Lemma 3.1.1 we have

$$\overline{U}\in C^{0,1}(\mathbb{R}^3),\qquad \overline{U}=O_1(r^{-1/2-\epsilon})\quad\text{as } r\to\infty, \tag{3.2.4}$$

which is enough to guarantee that the first term of $\mathcal{I}(\Psi)$ is finite. Consider now the potential terms and set $\omega = dv + \chi d\psi - \psi d\chi$. In order to achieve integrability at infinity and near the axes away from the poles we will require, for $\lambda > \frac{3}{2}$, the following asymptotics

$$|\omega| = \rho^2 O(r^{-\lambda}), \quad |\nabla\chi| + |\nabla\psi| = \rho O(r^{-\lambda}) \qquad \text{as } r \to \infty, \tag{3.2.5}$$

$$|\omega| = O(\rho^2), \quad |\nabla\chi| + |\nabla\psi| = O(\rho) \qquad \text{as } \rho \to 0 \text{ and } |z| > m_0, \tag{3.2.6}$$

$$|\chi|, |\psi| = \text{const} + \rho^2 O(r^{-\lambda}) \qquad \text{as } r \to \infty, \tag{3.2.7}$$

$$|\chi|, |\psi| = \text{const} + O(\rho^2) \qquad \text{as } \rho \to 0 \text{ and } |z| > m_0, \tag{3.2.8}$$

from which it follows that

$$|\nabla v| = \rho O(r^{-\lambda+1}) \quad \text{as } r \to \infty, \qquad |\nabla v| = O(\rho) \quad \text{as } \rho \to 0 \text{ and } |z| > m_0. \tag{3.2.9}$$

It remains to prescribe asymptotics near the poles and in a neighborhood of the horizon rod. By (3.2.1), $e^{4U} = O(r_+^2)$ or $e^{4U} = O(\rho^4 r_+^{-2})$ near p_+ if $z \geq m_0$ or $z \leq m_0$ respectively. It follows that the second term in $\mathcal{I}(\Psi)$ is integrable near p_+ if

$$|\omega| = \rho^2 O(r_+^{-3/2}) \quad \text{for } z \geq m_0, \qquad |\omega| = O(r_+^{1/2}) \quad \text{for } z \leq m_0. \tag{3.2.10}$$

Similarly, near p_- we will impose

$$|\omega| = O(r_-^{1/2}) \quad \text{for } z \geq -m_0, \qquad |\omega| = \rho^2 O(r_-^{-3/2}) \quad \text{for } z \leq -m_0. \tag{3.2.11}$$

Analogous considerations lead to the condition near p_+

$$|\nabla\chi| + |\nabla\psi| = \rho O(r_+^{-1}) \quad \text{for } z \geq m_0, \qquad |\nabla\chi| + |\nabla\psi| = O(1) \quad \text{for } z \leq m_0, \tag{3.2.12}$$

and near p_-

$$|\nabla\chi| + |\nabla\psi| = O(1) \quad \text{for } z \geq -m_0, \qquad |\nabla\chi| + |\nabla\psi| = \rho O(r_-^{-1}) \quad \text{for } z \leq -m_0. \tag{3.2.13}$$

Next observe that since $e^U = O(\rho)$ near the interior of the horizon rod, if

$$|\omega| = |\nabla\chi| = |\nabla\psi| = O(1) \qquad \text{as } \rho \to 0 \text{ and } |z| < m_0, \tag{3.2.14}$$

then the last two terms of the reduced energy are integrable in this region.

Lastly we record additional asymptotics that follow from above and will be needed in the following section. Assuming that the value of the potentials on the axes agree with those of the potentials for the Kerr-Newman map Ψ_k, we may integrate on lines perpendicular to the axes and near $p_\pm$ to obtain

$$|v - v_k| + |\chi - \chi_k| + |\psi - \psi_k| = O(\rho^2 r_\pm^{-1}) \qquad \text{as } r_\pm \to 0 \text{ and } |z| \geq m_0. \tag{3.2.15}$$

For $|z| \leq m_0$, integrating on horizontal lines will not yield such an estimate since the two sets of potentials do not necessarily agree on the horizon rod. Thus, we integrate along radial lines emanating from the poles $p_\pm$ to find

$$|v - v_k| + |\chi - \chi_k| + |\psi - \psi_k| = O(r_\pm) \qquad \text{as } r_\pm \to 0 \text{ and } |z| \leq m_0. \tag{3.2.16}$$

3.3 Minimizing the Functional

In this section it will be shown that the renormalized Kerr-Newman harmonic map Ψ_k is the global minimizer of the functional $\mathcal{I}$, among competitors Ψ satisfying the asymptotics of Section 3.2. This is based on the convexity of harmonic energy E for nonpositively curved target spaces under geodesic deformations. Such a strategy has been used successfully in connection with mass-angular momentum-charge inequalities in [17, 48, 57], where the minimizer arises from extreme black holes. Here we will extend this method to the setting of nondegenerate black holes. The difficulty arises from the fact that the convexity property does not pass directly from E to $\mathcal{I}$ since the energy is applied to singular maps. To get around this problem a cut-and-paste procedure is employed in which the regularized map Ψ is approximated by maps $\Psi_{\delta,\varepsilon}$ which agree with Ψ_k on certain domains. More precisely, let $\delta, \varepsilon > 0$ be small parameters and set $\Omega_{\delta,\varepsilon} = \{\delta < r_\pm;\ r < 2/\delta;\ \rho > \varepsilon\}$ and $\mathcal{A}_{\delta,\varepsilon} = B_{2/\delta} \setminus \Omega_{\delta,\varepsilon}$, where $B_{2/\delta}$ is the coordinate ball of radius $2/\delta$. Then $\Psi_{\delta,\varepsilon} = (\overline{U}_{\delta,\varepsilon}, v_{\delta,\varepsilon}, \chi_{\delta,\varepsilon}, \psi_{\delta,\varepsilon})$ will be constructed so that

$$\operatorname{supp}(\overline{U}_{\delta,\varepsilon} - \overline{U}_k) \subset B_{2/\delta}, \qquad \operatorname{supp}(v_{\delta,\varepsilon} - v_k, \chi_{\delta,\varepsilon} - \chi_k, \psi_{\delta,\varepsilon} - \psi_k) \subset \Omega_{\delta,\varepsilon}. \tag{3.3.1}$$

If $\tilde{\Psi}^t_{\delta,\varepsilon}$, $t \in [0,1]$ is a geodesic in $\mathbb{H}^2_{\mathbb{C}}$ connecting $\tilde{\Psi}^1_{\delta,\varepsilon} = \tilde{\Psi}_{\delta,\varepsilon}$ and $\tilde{\Psi}^0_{\delta,\varepsilon} = \tilde{\Psi}_k$, then $\tilde{\Psi}^t_{\delta,\varepsilon} \equiv \Psi_k$ outside $B_{2/\delta}$ and $v^t_{\delta,\varepsilon} = v_k, \chi^t_{\delta,\varepsilon} = \chi_k$, and $\psi^t_{\delta,\varepsilon} = \psi_k$ on a neighborhood of $\mathcal{A}_{\delta,\varepsilon}$. We then have that $\overline{U}^t_{\delta,\varepsilon} = \overline{U}_k + t(\overline{U}_{\delta,\varepsilon} - \overline{U}_k)$ on this domain. The fact that this expression is linear in t, together with convexity of the harmonic energy produces

$$\frac{d^2}{dt^2}\mathcal{I}(\Psi^t_{\delta,\varepsilon}) \geq 2\int_{\mathbb{R}^3} |\nabla \operatorname{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon}, \Psi_k)|^2 dx. \tag{3.3.2}$$

Furthermore, since Ψ_k is a critical point it follows that

$$\frac{d}{dt}\mathcal{I}(\Psi^t_{\delta,\varepsilon})|_{t=0} = 0. \tag{3.3.3}$$

The gap bound of Theorem 3.1.2 is then obtained by integrating (3.3.2), applying a Sobolev inequality, and taking the limit as $\delta, \varepsilon \to 0$. Each of these steps will now be justified. Repeated use of the asymptotics in Section 3.2 will be made, sometimes implicitly without reference to a particular equation.

The following cut-off functions are needed to construct the approximations $\Psi_{\delta,\varepsilon}$. Namely

$$\varphi_\delta = \begin{cases} 0 & \text{if } r_\pm \leq \delta, \\ |\nabla\varphi_\delta| \leq \frac{2}{\delta} & \text{if } \delta < r_\pm < 2\delta, \\ 1 & \text{if } r_\pm \geq 2\delta, \end{cases} \tag{3.3.4}$$

$$\varphi^1_\delta = \begin{cases} 1 & \text{if } r \leq \frac{1}{\delta}, \\ |\nabla\varphi^1_\delta| \leq 2\delta & \text{if } \frac{1}{\delta} < r < \frac{2}{\delta}, \\ 0 & \text{if } r \geq \frac{2}{\delta}, \end{cases} \tag{3.3.5}$$

$$\phi_\varepsilon = \begin{cases} 0 & \text{if } \rho \leq \varepsilon, \\ \frac{\log(\rho/\varepsilon)}{\log(\sqrt{\varepsilon}/\varepsilon)} & \text{if } \varepsilon < \rho < \sqrt{\varepsilon}, \\ 1 & \text{if } \rho \geq \sqrt{\varepsilon}. \end{cases} \tag{3.3.6}$$

The first step deals with neighborhoods of the poles $p_\pm$. Let $\mathcal{F}_\delta(\Psi) = (\overline{U}, v_\delta, \chi_\delta, \psi_\delta)$ where

$$(v_\delta, \chi_\delta, \psi_\delta) = (v_k, \chi_k, \psi_k) + \varphi_\delta(v - v_k, \chi - \chi_k, \psi - \psi_k), \tag{3.3.7}$$

so that the potentials of $\mathcal{F}_\delta(\Psi)$ and Ψ_k agree on $B_\delta(p_+) \cup B_\delta(p_-)$.

Lemma 3.3.1. *Suppose that* $\Psi \equiv \Psi_k$ *outside* $B_{2/\delta}$*, then* $\lim_{\delta\to 0} \mathcal{I}(\mathcal{F}_\delta(\Psi)) = \mathcal{I}(\Psi)$.

Proof. Write

$$\mathcal{I}(\mathcal{F}_\delta(\Psi)) = \sum_\pm \left[\mathcal{I}_{r_\pm<\delta}(\mathcal{F}_\delta(\Psi)) + \mathcal{I}_{\delta<r_\pm<2\delta}(\mathcal{F}_\delta(\Psi))\right] + \mathcal{I}_{r_\pm>2\delta}(\mathcal{F}_\delta(\Psi)), \tag{3.3.8}$$

where $r_\pm > 2\delta$ denotes the complement of $B_{2\delta}(p_+) \cup B_{2\delta}(p_-)$. Then according to the dominated convergence theorem (DCT)

$$\mathcal{I}_{r_\pm \geq 2\delta}(\mathcal{F}_\delta(\Psi)) = \mathcal{I}_{r_\pm\geq 2\delta}(\Psi) \to \mathcal{I}(\Psi). \tag{3.3.9}$$

Furthermore since the potentials of $\mathcal{F}_\delta(\Psi)$ and Ψ_k agree on $r_\pm < \delta$, and $e^U \leq ce^{U_k}$ as $|\overline{U}|$ and $|\overline{U}_k|$ are bounded near $p_\pm$, the second and third integrands of $\mathcal{I}_{r_\pm<\delta}(\mathcal{F}_\delta(\Psi))$ converge to zero in light of the finite reduced energy of Ψ_k. The first integrand involving $|\nabla\overline{U}|$ also tends to zero since this function remains bounded.

Now consider

$$\mathcal{I}_{\delta<r_\pm<2\delta}(\mathcal{F}_\delta(\Psi)) = \underbrace{\int_{\delta<r_\pm<2\delta} |\nabla\overline{U}|^2}_{I_1} + \underbrace{\int_{\delta<r_\pm<2\delta} \frac{e^{4U}}{\rho^4}|\omega_\delta|^2}_{I_2} + \underbrace{\int_{\delta<r_\pm<2\delta} \frac{e^{2U}}{\rho^2}(|\nabla\chi_\delta|^2 + |\nabla\psi_\delta|^2)}_{I_3}, \tag{3.3.10}$$

and note that $I_1 \to 0$ by the DCT. Next compute

$$\begin{aligned} \omega_\delta =& \varphi_\delta\omega + (1-\varphi_\delta)\omega_k + (v - v_k)\nabla\varphi_\delta + (\chi_k\psi - \psi_k\chi)\nabla\varphi_\delta \\ &+ \varphi_\delta(1-\varphi_\delta)[(\psi-\psi_k)\nabla(\chi-\chi_k) - (\chi-\chi_k)\nabla(\psi-\psi_k)], \end{aligned} \tag{3.3.11}$$

and use properties of the cut-off function to find

$$\begin{aligned} I_2 \leq& C\int_{\delta<r_\pm<2\delta} \left(\frac{e^{4U}}{\rho^4}|\omega|^2 + \frac{e^{4U_k}}{\rho^4}|\omega_k|^2 + \frac{e^{4U}}{r_\pm^2\rho^4}|v-v_k|^2 + \frac{e^{4U}}{r_\pm^2\rho^4}|\chi_k\psi - \psi_k\chi|^2\right) \\ &+ C\int_{\delta<r_\pm<2\delta} \frac{e^{4U}}{\rho^4}\left(|\psi-\psi_k|^2|\nabla(\chi-\chi_k)|^2 + |\chi-\chi_k|^2|\nabla(\psi-\psi_k)|^2\right). \end{aligned} \tag{3.3.12}$$

The first and second terms converge to zero by the DCT and finite reduced energies of Ψ and Ψ_k. The third term may be estimated with the help of (3.2.15) and (3.2.16), namely

$$\int_{\delta<r_\pm<2\delta} \frac{e^{4U}}{r_\pm^2\rho^4}|v-v_k|^2 \leq \int_{\delta<r_\pm<2\delta} Cr^{-2} \to 0, \tag{3.3.13}$$

and similar considerations apply for the fourth term. For the fifth term employ (3.2.12), (3.2.13), (3.2.15), and (3.2.16) to find

$$\int_{\delta<r_\pm<2\delta} \frac{e^{4U}}{\rho^4}|\psi-\psi_k|^2|\nabla(\chi-\chi_k)|^2 \leq \int_{\delta<r_\pm<2\delta} C \to 0, \tag{3.3.14}$$

and similarly for the sixth term. This shows that $I_2 \to 0$. Lastly, analogous reasoning yields $I_3 \to 0$. □

Consider now the asymptotically flat end and set

$$\mathcal{F}^1_\delta(\Psi) = \Psi_k + \varphi^1_\delta(\Psi - \Psi_k), \tag{3.3.15}$$

so that $\mathcal{F}^1_\delta(\Psi) = \Psi_k$ on $\mathbb{R}^3 \setminus B_{2/\delta}$. Then as is shown in [48, Lemma 4.2]

$$\lim_{\delta\to 0} \mathcal{I}(\mathcal{F}^1_\delta(\Psi)) = \mathcal{I}(\Psi). \tag{3.3.16}$$

Next we treat the cylindrical regions around the axis and horizon rod, and will make use of the domains

$$\mathcal{C}_{\delta,\varepsilon} = \{\rho < \varepsilon;\ \delta < r_\pm;\ r < 2/\delta\}, \tag{3.3.17}$$

$$\mathcal{W}^1_{\delta,\varepsilon} = \{\varepsilon < \rho < \sqrt{\varepsilon};\ \delta < r_\pm;\ r \le 2/\delta;\ |z| > m\}, \tag{3.3.18}$$

$$\mathcal{W}^2_{\delta,\varepsilon} = \{\varepsilon < \rho < \sqrt{\varepsilon};\ \delta < r_\pm;\ |z| < m\}. \tag{3.3.19}$$

Let $\mathcal{G}_\varepsilon(\Psi) = (\overline{U}, v_\varepsilon, \chi_\varepsilon, \psi_\varepsilon)$ where

$$(v_\varepsilon, \chi_\varepsilon, \psi_\varepsilon) = (v_k, \chi_k, \psi_k) + \phi_\varepsilon(v - v_k, \chi - \chi_k, \psi - \psi_k), \tag{3.3.20}$$

so that the potentials of $\mathcal{G}_\varepsilon(\Psi)$ and Ψ_k agree on $\rho < \varepsilon$.

Lemma 3.3.2. *Fix $\delta > 0$. Assume that the potentials of Ψ and Ψ_k agree on $B_\delta(p_+) \cup B_\delta(p_-)$, and $\Psi \equiv \Psi_k$ outside $B_{2/\delta}$, then $\lim_{\varepsilon\to 0} \mathcal{I}(\mathcal{G}_\varepsilon(\Psi)) = \mathcal{I}(\Psi)$.*

Proof. Write

$$\mathcal{I}(\mathcal{G}_\varepsilon(\Psi)) = \mathcal{I}_{\mathcal{C}_{\delta,\varepsilon}}(\mathcal{G}_\varepsilon(\Psi)) + \mathcal{I}_{\mathcal{W}^1_{\delta,\varepsilon}}(\mathcal{G}_\varepsilon(\Psi)) + \mathcal{I}_{\mathcal{W}^2_{\delta,\varepsilon}}(\mathcal{G}_\varepsilon(\Psi)) + \mathcal{I}_{\mathbb{R}^3\setminus(\mathcal{C}_{\delta,\varepsilon}\cup\mathcal{W}^1_{\delta,\varepsilon}\cup\mathcal{W}^2_{\delta,\varepsilon})}(\mathcal{G}_\varepsilon(\Psi)). \tag{3.3.21}$$

Since the potentials of Ψ and Ψ_k agree on $B_\delta(p_\pm)$, the DCT and finite reduced energy imply that

$$\mathcal{I}_{\mathbb{R}^3\setminus(\mathcal{C}_{\delta,\varepsilon}\cup\mathcal{W}^1_{\delta,\varepsilon}\cup\mathcal{W}^2_{\delta,\varepsilon})}(\mathcal{G}_\varepsilon(\Psi)) \to \mathcal{I}(\Psi). \tag{3.3.22}$$

Furthermore since the potentials of $\mathcal{G}_\varepsilon(\Psi)$ and Ψ_k agree on $\mathcal{C}_{\delta,\varepsilon}$, and $e^U \le ce^{U_k}$ on this region, the second and third integrands of $\mathcal{I}_{\mathcal{C}_{\delta,\varepsilon}}(\mathcal{G}_\varepsilon(\Psi))$ converge to zero in light of the finite reduced energy of Ψ_k. The first integrand involving $|\nabla\overline{U}|$ also tends to zero since this function remains bounded.

The domain $\mathcal{W}^1_{\delta,\varepsilon}$ concerns a neighborhood of the axis of rotation, and therefore $\mathcal{I}_{\mathcal{W}^1_{\delta,\varepsilon}}(\mathcal{G}_\varepsilon(\Psi)) \to 0$ according to Lemma 4.4 of [48]. Now consider

$$\mathcal{I}_{\mathcal{W}^2_{\delta,\varepsilon}}(\mathcal{G}_\varepsilon(\Psi)) = \underbrace{\int_{\mathcal{W}^2_{\delta,\varepsilon}} |\nabla\overline{U}|^2}_{I_1} + \underbrace{\int_{\mathcal{W}^2_{\delta,\varepsilon}} \frac{e^{4U}}{\rho^4}|\omega_\varepsilon|^2}_{I_2} + \underbrace{\int_{\mathcal{W}^2_{\delta,\varepsilon}} \frac{e^{2U}}{\rho^2}(|\nabla\chi_\varepsilon|^2 + |\nabla\psi_\varepsilon|^2)}_{I_3}, \tag{3.3.23}$$

and notice that $I_1 \to 0$ since $|\nabla\overline{U}|$ remains bounded. Next observe that

$$\begin{aligned}\omega_\varepsilon =& \phi_\varepsilon\omega + (1-\phi_\varepsilon)\omega_k + (v - v_k)\nabla\phi_\varepsilon + (\chi_k\psi - \psi_k\chi)\nabla\phi_\varepsilon \\ &+ \phi_\varepsilon(1-\phi_\varepsilon)[(\psi-\psi_k)\nabla(\chi-\chi_k) - (\chi-\chi_k)\nabla(\psi-\psi_k)].\end{aligned} \tag{3.3.24}$$

The asymptotics of the cut-off function then yield

$$\begin{aligned}I_2 \leq & C\int_{\mathcal{W}^2_{\delta,\varepsilon}}\left(\frac{e^{4U}}{\rho^4}|\omega|^2+\frac{e^{4U_k}}{\rho^4}|\omega_k|^2+(\log\varepsilon)^{-2}\rho^{-2}|v-v_k|^2+(\log\varepsilon)^{-2}\rho^{-2}|\chi_k\psi-\psi_k\chi|^2\right)\\ &+C\int_{\mathcal{W}^2_{\delta,\varepsilon}}\left(|\psi-\psi_k|^2|\nabla(\chi-\chi_k)|^2+|\chi-\chi_k|^2|\nabla(\psi-\psi_k)|^2\right).\end{aligned} \tag{3.3.25}$$

The first two terms converge to zero by the finite reduced energies. Furthermore according to (3.2.14), $|v-v_k|=O(1)$ and thus

$$\int_{\mathcal{W}^2_{\delta,\varepsilon}}(\log\varepsilon)^{-2}|v-v_k|^2\leq C\int_{\mathcal{W}^2_{\delta,\varepsilon}}(\log\varepsilon)^{-2}\rho^{-2}=O\left((\log\varepsilon)^{-1}\right)\to 0. \tag{3.3.26}$$

Analogous considerations may be used to treat the fourth term. Lastly, since $|\psi-\psi_k|$ and $|\nabla(\chi-\chi_k)|$ remain bounded the fifth term tends to zero, and similarly for the sixth. □

We are now in a position to construct the appropriate approximation to Ψ via the cut and paste operations by composition

$$\Psi_{\delta,\varepsilon}=\mathcal{G}_\varepsilon\left(\mathcal{F}_\delta\left(\mathcal{F}^1_\delta(\Psi)\right)\right). \tag{3.3.27}$$

Then according to (3.3.16) and Lemmas 3.3.1 and 3.3.2,

$$\lim_{\delta\to 0}\lim_{\varepsilon\to 0}\mathcal{I}(\Psi_{\delta,\varepsilon})=\mathcal{I}(\Psi). \tag{3.3.28}$$

Proof of Theorem 3.1.2. As in the introduction to this section let $\tilde{\Psi}^t_{\delta,\varepsilon}$ be the geodesic deformation connecting $\tilde{\Psi}_k$ to $\tilde{\Psi}_{\delta,\varepsilon}$. Due to the properties of the approximation the first component of the geodesic is $\overline{U}^t_{\delta,\varepsilon}=\overline{U}_k+t(\overline{U}_{\delta,\varepsilon}-\overline{U}_k)$ on $\mathcal{A}_{\delta,\varepsilon}$, and in particular $\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon},\Psi_k)=|\overline{U}_{\delta,\varepsilon}-\overline{U}_k|$ on this domain. These two observations, together with the asymptotics near the poles $p_\pm$ show that one may differentiate under the integral sign to directly compute the second variation and find

$$\frac{d^2}{dt^2}\mathcal{I}_{\mathcal{A}_{\delta,\varepsilon}}(\Psi^t_{\delta,\varepsilon})\geq\int_{\mathcal{A}_{\delta,\varepsilon}}2|\nabla(\overline{U}_{\delta,\varepsilon}-\overline{U}_k)|^2=\int_{\mathcal{A}_{\delta,\varepsilon}}2|\nabla\,\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon},\Psi_k)|^2. \tag{3.3.29}$$

On the domain $\Omega_{\delta,\varepsilon}$, the relation (3.1.17) between reduced and harmonic energies may be used. Due to the linearity of $\overline{U}^t_{\delta,\varepsilon}$ in t, the boundary term of (3.1.17) vanishes when computing the second variation so that

$$\frac{d^2}{dt^2}\mathcal{I}_{\Omega_{\delta,\varepsilon}}(\Psi^t_{\delta,\varepsilon})=\frac{d^2}{dt^2}E_{\Omega_{\delta,\varepsilon}}(\tilde{\Psi}^t_{\delta,\varepsilon})\geq\int_{\Omega_{\delta,\varepsilon}}2|\nabla\,\mathrm{dist}_{\mathbb{H}^2_{\mathbb{C}}}(\Psi_{\delta,\varepsilon},\Psi_k)|^2, \tag{3.3.30}$$

where the inequality is obtained from the convexity of harmonic energy [57]. Since $\Omega_{\delta,\varepsilon}$ and $\mathcal{A}_{\delta,\varepsilon}$ are complementary in $B_{2/\delta}$, and the geodesic deformation is constant outside of this large ball, it follows that (3.3.2) holds.

Next, let $\bar{\delta}<\delta$ and $\bar{\varepsilon}<\varepsilon$, and observe that since Ψ_k is a critical point

$$\frac{d}{dt}\mathcal{I}_{\Omega_{\bar{\delta},\bar{\varepsilon}}}(\Psi^t_{\delta,\varepsilon})|_{t=0}=-\sum_{\pm}\int_{\partial B_{\bar{\delta}}(p_\pm)}2(\overline{U}_{\delta,\varepsilon}-\overline{U}_k)\partial_\nu\overline{U}_k-\int_{\partial\mathcal{C}_{\bar{\delta},\bar{\varepsilon}}}2(\overline{U}_{\delta,\varepsilon}-\overline{U}_k)\partial_\nu\overline{U}_k, \tag{3.3.31}$$

where ν is the unit normal pointing towards infinity. In addition, using the constancy of the potentials and linearity of $\overline{U}^t_{\delta,\varepsilon}$ on $\mathcal{A}_{\bar{\delta},\bar{\varepsilon}}$ we find that

$$\begin{aligned}\frac{d}{dt}\mathcal{I}_{\mathcal{A}_{\bar{\delta},\bar{\varepsilon}}}(\Psi^t_{\delta,\varepsilon})|_{t=0} = &\int_{\mathcal{A}_{\bar{\delta},\bar{\varepsilon}}} 2\nabla\overline{U}_k\cdot\nabla(\overline{U}_{\delta,\varepsilon}-\overline{U}_k)+4(\overline{U}_{\delta,\varepsilon}-\overline{U}_k)\frac{e^{4U_k}}{\rho^4}|\omega_k|^2\\ &+\int_{\mathcal{A}_{\bar{\delta},\bar{\varepsilon}}} 2(\overline{U}_{\delta,\varepsilon}-\overline{U}_k)\frac{e^{2U_k}}{\rho^2}\left(|\nabla\chi_k|^2+|\nabla\psi_k|^2\right).\end{aligned} \tag{3.3.32}$$

Since $|\overline{U}|+|\nabla\overline{U}|$ is uniformly bounded, (3.3.31) tends to zero as $\bar{\varepsilon}\to 0$ followed by $\bar{\delta}\to 0$, and the same holds for (3.3.32) since it may be estimated by the reduced energy of Ψ_k on $\mathcal{A}_{\bar{\delta},\bar{\varepsilon}}$.

We may now integrate (3.3.2) two times and use a Sobolev inequality to obtain the inequality (3.1.20) of Theorem 3.1.2 with Ψ replaced by $\Psi_{\delta,\varepsilon}$. In light of (3.3.28), the desired result follows by taking the limits as $\varepsilon\to 0$ and then $\delta\to 0$. □

3.4 Proof of the Main Results

We first show that under the assumptions of Theorem 1.0.3 the potentials and quantities arising from Weyl coordinates satisfy the asymptotics stated in Section 3.2. Lemma 3.1.1 guarantees that $\overline{U}$ behaves in a manner consistent with (3.2.4). Next, as is shown in [48]

$$\frac{e^{6U-2\alpha}}{\rho^4}|\nabla v+\chi\nabla\psi-\psi\nabla\chi|^2\leq|k|^2_g. \tag{3.4.1}$$

Consider a domain near the poles $p_\pm$ with $|z|\geq m_0$, then using (3.2.1)-(3.2.3) we find that

$$|\nabla v+\chi\nabla\psi-\psi\nabla\chi| = O(\rho^2 e^{-2U}e^{-U+\alpha}) = O(\rho^2 r_\pm^{-3/2}), \tag{3.4.2}$$

since $|k|_g$ remains bounded. Similarly if $|z|\leq m_0$

$$|\nabla v+\chi\nabla\psi-\psi\nabla\chi| = O(\rho^2 e^{-2U}e^{-U+\alpha}) = O(r_\pm^{1/2}), \tag{3.4.3}$$

which confirms (3.2.10) and (3.2.11). Near the horizon rod away from the poles, that is $|z|<m_0$, the asymptotics (3.1.4) imply

$$|\nabla v+\chi\nabla\psi-\psi\nabla\chi| = O(\rho^2 e^{-2U}e^{-U+\alpha}) = O(1), \tag{3.4.4}$$

confirming part of (3.2.14).

For the electromagnetic potentials recall that from [48],

$$\frac{e^{4U-2\alpha}}{\rho^2}\left(|\nabla\chi|^2+|\nabla\psi|^2\right)\leq|E|^2_g+|B|^2_g. \tag{3.4.5}$$

Again the right-hand side is bounded near the poles, so for $|z|\geq m_0$ we have

$$|\nabla\chi|+|\nabla\psi| = O(\rho e^{-U}e^{-U+\alpha}) = O(\rho r_\pm^{-1}), \tag{3.4.6}$$

and for $|z| \leq m_0$

$$|\nabla\chi| + |\nabla\psi| = O(\rho e^{-U} e^{-U+\alpha}) = O(1). \tag{3.4.7}$$

This shows that (3.2.12) and (3.2.13) hold. Analogously, near the horizon rod with $|z| < m_0$

$$|\nabla\chi| + |\nabla\psi| = O(1), \tag{3.4.8}$$

which fulfills (3.2.14). Furthermore the asymptotics in a neighborhood of the axis, (3.2.6) and (3.2.8), may be obtained in similar fashion. Lastly, (3.2.5) and (3.2.7) follow from asymptotic flatness.

We are now in a position to establish Theorem 1.0.3. As shown above, the map Ψ arising from the initial data satisfies the hypotheses of Theorem 3.1.2. Therefore, together with (3.1.15) the following lower bound for the mass is achieved

$$m \geq \frac{1}{8\pi}\mathcal{I}(\Psi_k) + \frac{1}{4}\int_{-m_0}^{m_0} (\overline{\alpha}(0,z) - 2\overline{U}(0,z))dz + m_0. \tag{3.4.9}$$

Let m_k and A_k denote the mass and horizon area of the Kerr-Newman solution associated with the map Ψ_k. Then since the Kerr-Newman solution is known to saturate the Penrose inequality

$$\begin{aligned} m_k =& \sqrt{\frac{A_k}{16\pi} + \frac{Q^2}{2} + \frac{\pi(Q^4 + 4\mathcal{J}^2)}{A_k}} \\ =& \frac{1}{8\pi}\mathcal{I}(\Psi_k) + \frac{1}{4}\int_{-m_0}^{m_0} (\overline{\alpha}_k(0,z) - 2\overline{U}_k(0,z))dz + m_0. \end{aligned} \tag{3.4.10}$$

It follows that

$$m \geq \sqrt{\frac{A_k}{16\pi} + \frac{Q^2}{2} + \frac{\pi(Q^4 + 4\mathcal{J}^2)}{A_k}} + \frac{1}{4}\int_{-m_0}^{m_0} (\overline{\beta}(0,z) - \overline{\beta}_k(0,z))dz, \tag{3.4.11}$$

which is the desired inequality. In the case that this inequality is saturated we must have $\Psi = \Psi_k$ by Theorem 3.1.2. Several other quantities arising from the derivation of (3.1.15) vanish, from which it may be shown that the initial data (M, g, k, E, B) agrees with that of the canonical slice of the Kerr-Newman spacetime; details are given in [48, Section 2].

We will now establish Corollary 1.0.4. If $\overline{\beta}$ is constant on the horizon rod then

$$e^{\frac{1}{2m_0}\int_{-m_0}^{m_0}\overline{\beta}(0,z)dz} = \frac{1}{2m_0}\int_{-m_0}^{m_0} e^{\overline{\beta}(0,z)}dz = \frac{A}{16\pi m_0^2}. \tag{3.4.12}$$

The same equality holds for β, A replaced by β_k, A_k since β_k is also constant on the horizon. Therefore if we assume that $A \geq A_k$, then

$$\int_{-m_0}^{m_0} \overline{\beta}(0,z)dz \geq \int_{-m_0}^{m_0} \overline{\beta}_k(0,z)dz, \tag{3.4.13}$$

which together with (3.4.11) yields the desired inequality. The case of equality here is treated as above. □

3.5 Weyl Coordinates

Here we prove Lemma 3.1.1. In [19] the existence of Weyl coordinates was established by first constructing so called pseudospherical coordinates (ρ_s, z_s, ϕ), in which the initial data boundary ∂M is represented by a semi-circle of radius $\frac{m_0}{2}$ about the origin in the $\rho_s z_s$-plane. This contrasts with Weyl coordinates in which the boundary takes the form of an interval on the z-axis in the orbit space. Pseudospherical coordinates are valid on the planar region $\mathbb{C}_+ \setminus D_{m_0/2} = \{\rho_s + iz_s \mid \rho_s > 0, r_s > m_0/2\}$, where $r_s^2 = \rho_s^2 + z_s^2$. In these coordinates the metric takes the standard 'Brill' form

$$g = e^{-2U_s+2\alpha_s}(d\rho_s^2 + dz_s^2) + \rho_s^2 e^{-2U_s}(d\phi + A_{\rho_s} d\rho_s + A_{z_s} dz_s)^2. \tag{3.5.1}$$

This structure for the metric is preserved under any coordinate change of the plane which yields a conformal transformation, and Weyl coordinates are a particular example of this. The metric coefficients are axisymmetric, smooth up to the boundary in $\mathbb{C}_+ \setminus D_{m_0/2}$ with $\alpha_s = 0$ on the z_s-axis, and satisfy the fall-off

$$U_s = O_1(r_s^{-1/2-\epsilon}), \quad \alpha_s = O_1(r_s^{-1/2-\epsilon}), \quad A_{\rho_s} = O_1(r_s^{-3/2-\epsilon}), \quad A_{z_s} = O_1(r_s^{-3/2-\epsilon}). \tag{3.5.2}$$

Weyl coordinates (ρ, z, ϕ) are constructed from pseudospherical coordinates as follows. Define complex coordinates $\zeta_s = \rho_s + iz_s$ and $\zeta = \rho + iz$ and consider the holomorphic diffeomorphism $f : \mathbb{C}_+ \setminus D_{m_0/2} \to \mathbb{C}_+$ given by

$$\zeta = f(\zeta_s) = \zeta_s - \frac{m_0^2}{4\zeta_s} \qquad \Rightarrow \qquad \rho = \frac{\rho_s(r_s^2 - \frac{m_0^2}{4})}{r_s^2}, \quad z = \frac{z_s(r_s^2 + \frac{m_0^2}{4})}{r_s^2}. \tag{3.5.3}$$

Observe that

$$\frac{\partial \zeta}{\partial \zeta_s} = 1 + \frac{m_0^2}{4\zeta_s^2}, \tag{3.5.4}$$

which is smooth up to the boundary of $\mathbb{C}_+ \setminus D_{m_0/2}$ and is nonzero except at the points $\zeta_s = \pm\frac{m_0}{2}i$. Thus by the inverse function theorem, the inverse transformation is holomorphic and has bounded derivatives away from the poles $\zeta = \pm m_0 i$ of the horizon. Near these points we have

$$\left|\frac{\partial \zeta}{\partial \zeta_s}\right| \geq C^{-1}|\zeta_s \mp \frac{m_0}{2}i| \qquad \Rightarrow \qquad \left|\frac{\partial \zeta_s}{\partial \zeta}\right| \leq \frac{C}{|\zeta_s \mp \frac{m_0}{2}i|}. \tag{3.5.5}$$

In particular, all first derivatives of the real and imaginary parts admit the bound

$$\left|\frac{\partial \rho_s}{\partial \rho}\right| + \left|\frac{\partial \rho_s}{\partial z}\right| + \left|\frac{\partial z_s}{\partial \rho}\right| + \left|\frac{\partial z_s}{\partial z}\right| \leq \frac{C}{|\zeta_s \mp \frac{m_0}{2}i|} \tag{3.5.6}$$

near the poles.

The relationship between U, α of Weyl coordinates and U_s, α_s of pseudospherical coordinates is given by [19]

$$U(\rho, z) = U_s(\rho_s, z_s) - \log\frac{\rho_s}{\rho}, \qquad \alpha(\rho, z) = \alpha_s(\rho_s, z_s) + \log\frac{|\zeta_s|^2 - \frac{m_0^2}{4}}{|\zeta_s^2 + \frac{m_0^2}{4}|}. \tag{3.5.7}$$

Note that the second term on the right-hand side of both expressions depends only on the coordinate transformation. For the Schwarzschild solution

$$U_{s,0}(\rho_s, z_s) = -2\log\frac{2r_s + m_0}{2r_s}, \qquad \alpha_{s,0}(\rho_s, z_s) = 0, \tag{3.5.8}$$

and the expressions for the Schwarzschild data U_0 and α_0 in Weyl coordinates may then be obtained from the above formulas. We may then write $U = U_0 + \overline{U}$ and $\alpha = \alpha_0 + \overline{\alpha}$ where

$$\overline{U}(\rho, z) := U(\rho, z) - U_0(\rho, z) = U_s(\rho_s, z_s) - U_{s,0}(\rho_s, z_s), \tag{3.5.9}$$

and

$$\overline{\alpha}(\rho, z) := \alpha(\rho, z) - \alpha_0(\rho, z) = \alpha_s(\rho_s, z_s). \tag{3.5.10}$$

It immediately follows that $\overline{U}$ and $\overline{\alpha}$ are uniformly bounded and satisfy the desired decay at infinity. Furthermore since U_s, $U_{s,0}$, and α_s are smooth, the regularity properties of $\overline{U}$ and $\overline{\alpha}$ depend on the coordinate transformation f^{-1}, and the only possible issues arise at the poles.

Consider the partial derivative

$$\frac{\partial \overline{U}}{\partial \rho} = \left(\frac{\partial U_s}{\partial \rho_s} - \frac{\partial U_{s,0}}{\partial \rho_s}\right)\frac{\partial \rho_s}{\partial \rho} + \left(\frac{\partial U_s}{\partial z_s} - \frac{\partial U_{s,0}}{\partial z_s}\right)\frac{\partial z_s}{\partial \rho}. \tag{3.5.11}$$

Since the horizon is a minimal surface

$$\frac{\partial}{\partial r_s}(U_s - \frac{1}{2}\alpha_s) = \frac{2}{m_0} = \frac{\partial U_{s,0}}{\partial r_s} \qquad \text{when } r_s = \frac{m_0}{2}. \tag{3.5.12}$$

In particular this holds at $(\rho_s, z_s) = (0, \pm m_0/2)$. Moreover, since $\alpha_s = 0$ on the axis and ∂_{r_s} coincides with $\pm\partial_{z_s}$ there, we have

$$\left(\frac{\partial U_s}{\partial z_s} - \frac{\partial U_{s,0}}{\partial z_s}\right)\left(0, \pm\frac{m_0}{2}\right) = 0. \tag{3.5.13}$$

Next, use the fact that all functions are axisymmetric to find

$$\frac{\partial U_s}{\partial \rho_s}\left(0, \pm\frac{m_0}{2}\right) = \frac{\partial U_{s,0}}{\partial \rho_s}\left(0, \pm\frac{m_0}{2}\right) = 0. \tag{3.5.14}$$

Therefore the first derivatives of $U_s - U_{s,0}$ vanish at the poles. This, combined with the smoothness of this function up to the boundary, shows that even though $\partial_\rho \rho_s$ and $\partial_\rho z_s$ may blow-up at these points in a manner controlled by (3.5.6), the full expression (3.5.11) remains bounded. Similar considerations may be used to treat the $\partial_z \overline{U}$ and the derivatives of $\overline{\alpha}$.

Chapter 4

Brill Coordinates for Initial Data with Cylindrical Ends

4.1 Definition and Properties of Cylindrical Ends

The results of the previous chapters are proven for axisymmetric initial data sets on $\mathbb{R}^3$ with a global Brill like coordinate system (ρ, z, ϕ). Thus it important to know when an axisymmetric initial data set has such a representation. In [13] Chruściel establishes the existence of a Brill coordinate system for axisymmetric, simply connected initial data sets (M^3, g) with one or many asymptotically flat ends. We use nearly the same definition of asymptotically flat as above, however we allow slightly more general falloff.

$$g_{ij} - \delta_{ij} = o_k(r^{-\ell}) \tag{4.1.1}$$

as $r \to \infty$ for some $\ell > 0$.

In this chapter we do not need the additional structure provided by k, B, E or the constraint equations. We will refer to a coordinate system (x^i) satisfying Equation (4.1.1) as an AF coordinate system, and M_{ext} as a submanifold of M in which equation (4.1.1) holds. An asymptotically cylindrical end is defied analogously, and will be abbreviated AC.

Definition 4.1.1. *An open submanifold M_{ext} of a Riemannian 3 manifold (M, g) is said to be an asymptotically cylindrical end of order k if there exists an $\ell \in (0,1)$ and a diffeomorphism between M and $\{r \in \mathbb{R}; r > R\} \times S^2$ for some $R > 0$, such the metric g satisfies*

$$||g - \overline{g}||_{\overline{g}} = O_k(r^{-\ell}),^{1} \tag{4.1.2}$$

where

$$\overline{g} = h^2(\frac{1}{r^2}dr^2 + g_s), \tag{4.1.3}$$

h is a function on S^2, and g_s is a metric on S^2. Further we assume that $\overline{g}$ admits a killing vector $\overline{\eta}$ which is tangent to S^2 and has periodic orbits.

The hypothesis $\mathcal{L}_{\overline{\eta}}\overline{g} = 0$, with $\overline{\eta}$ tangent to S^2 implies $\mathcal{L}_{\overline{\eta}}\frac{1}{r^2}dr^2$=0, and hence

$$0 = \mathcal{L}_{\overline{\eta}}(\frac{h^2}{r^2}dr^2) + \mathcal{L}_{\overline{\eta}}(s^2h^2g_s) = \overline{\eta}(h^2)\frac{1}{r^2}dr^2 + s^2\overline{\eta}(h^2)g_s + s^2h^2\mathcal{L}_{\overline{\eta}}g_s \tag{4.1.4}$$

If we apply the 2-form $\overline{\eta}(h^2)\frac{1}{r^2}dr^2 + s^2\overline{\eta}(h^2)g_s + s^2h^2\mathcal{L}_{\overline{\eta}}g_s$ to (∂_r, ∂_r), the second and third term give no contribution and thus $\overline{\eta}(h) = 0$. Thus we also have $\mathcal{L}_{\overline{\eta}}g_s = 0$, so $\overline{\eta}$ is a killing vector for g_s. Since η has periodic orbits, we can find global coordinates (θ, ϕ) on S^2 such that $\eta = \partial_\phi$ and $g_s = a^2(d\theta^2 + f(\theta)^2d\phi^2)$ for some positive constant a and some nonnegative function f. Thus

$$\overline{g} = h(\theta)^2(\frac{1}{r^2}dr^2 + a^2(d\theta^2 + f(\theta)^2d\phi^2)). \tag{4.1.5}$$

Making the reparametrization $\tilde{r}^a = r$ we have $\frac{1}{r^2}dr^2 = \frac{a^2}{\tilde{r}^2}d\tilde{r}^2$. We can factor out the a and write,

$$\overline{g} = a^2h(\theta)^2(\frac{1}{\tilde{r}^2}d\tilde{r}^2 + d\theta^2 + f(\theta)^2d\phi^2), \tag{4.1.6}$$

at which point we can absorb the constant a into h. Note that under this transformation the falloff becomes $||g - \overline{g}|||_{\overline{g}} = o_k(\tilde{r}^{-\ell a})$. Using this information we formulate the following definition.

Definition 4.1.2. *An AC end of order k is said to be characterized by (h, f, ℓ) if there exists a coordinate system (τ, θ, ϕ) for M_{ext} such that*

$$\overline{g} = h(\theta)^2(\frac{1}{r^2}dr^2 + d\theta^2 + f(\theta)^2d\phi^2). \tag{4.1.7}$$

and

$$||g - \overline{g}||_{\overline{g}} = o_k(r^{-\ell}) \tag{4.1.8}$$

By the above discussion, every AC end is characterized by (h, f, ℓ) for some functions h and f and some $\ell \in (0, 1)$. We will use this form for $\overline{g}$ throughout the paper. In the above definition h is any smooth positive function on S^2 which is independent of ϕ, and f is a nonnegative function for which $d\theta^2 + f^2d\phi^2$ is a smooth metric on S^2. In particular we must have $h'(0) = h'(\pi) = 0$, $f(0) = f(\pi) = 0$, $f'(0) = 1$ and $f'(\pi) = -1$.

Our model space appears to differ from that which appears in much of the literature, however if we make the coordinate change $\tau = \log r$ we may write

$$\overline{g} = h(\theta)^2(d\tau^2 + d\theta^2 + f(\theta)^2d\phi^2), \tag{4.1.9}$$

so that g fits into the class of conformally cylindrical metrics considered in [18]. Further, in our definition the asymptotics hold at r and τ go to infinity. This is done so that our definitions and theorems will more closely resemble the corresponding results for AF ends. In the literature one often defines asymptotics as τ approaches $-\infty$. The two approaches are clearly equivalent. The cylindrical metrics in this paper are modeled after the neck of the $t = 0$ slice of an extreme Kerr black hole. For extreme Kerr we have $h(\theta)^2 = m^2(1 + \cos^2(\theta))$ and $f(\theta)^2 = \frac{4\sin^2(\theta)}{(1+\cos^2(\theta))^2}$, see [18].

[1] In this case we do not have explicit coordinate system. However if we let (θ, ϕ) be the standard coordinates on S^2 and (x, y, z) the Cartesian coordinates corresponding to the spherical coordinates (r, θ, ϕ), then $f = O_k(r^{-\ell})$ takes on the same meaning as in the AF case, with coordinate vector fields ∂_x, ∂_y, and ∂_z.

As we will be using the above definitions as hypotheses repeatedly throughout the paper, we introduce the following definition for the sake of brevity

Definition 4.1.3. *A Riemannian 3 manifold* (M, g) *is said to be of asymptotic order* k *characterized by* (h_i, f_i, ℓ_i)*, if it is the union of a compact set* K *and a finite number of ends* M_i*,* $i \in \{1, ..., n\}$*, where* M_1 *is an asymptotically flat end of order* k *and* M_i *is either an asymptotically flat end of order* k *or an asymptotically cylindrical end of order* k *characterized by* (h_i, f_i, ℓ_i)*. Further* (M, g) *is assumed to be simply connected and axisymmetric.*

In practice one generally works with initial data sets that have at least one asymptotically flat end. The arguments of this paper can be used to create Brill coordinates for initial data sets which only have asymptotically cylindrical ends, however the statement of our main theorem is simpler if we place an asymptotically flat end at infinity. This is the setting of Theorem 1.0.5.

The Theorem is proven as follows. As in [13], simple connectedness and axisymmetry imply that the quotient $M/U(1)$ is diffeomorphic to a half plane with a finite number of points along the axis removed. We may double $M/U(1)$ across its boundary to obtain a manifold $\overline{N}$ which is diffeomorphic to $\mathbb{R}^2$ minus a finite number of points, where each removed point represents either an asymptotically flat or asymptotically cylindrical end. In Section 2 we describe the natural induced metric, q, on $\overline{N}$ and show that in each cylindrical end q is conformal to an asymptotically flat metric in a natural way. In Section 3 we use the AF and AC character of the metric q to construct isothermal coordinates (ρ, z) on $\overline{N}$ such that q takes the form $q = e^{2u}(d\rho^2 + dz^2)$, and reflection across the z axis is an isometry of q. We then pull back the coordinates to M and the relationship between q and g gives us the representation in equation (1.0.10).

In the remaining sections of the paper, we complete the proof of Theorem 1.0.5 by analyzing the falloff of the metric components, and then use this falloff to establish the positivity of ADM mass for any AF end of M. In the final section we prove an estimate on η in AC ends which will be used extensively in Section 2.

4.2 Analysis of Quotient Metric

For any axisymmetric manifold (M, g), there is a natural induced metric q on the quotient manifold $M/U(1)$. Given any local cross section N for $M/U(1)$ and $X, Y \in T_pN$, q is given by

$$q(X, Y) = g(X, Y) - \frac{g(\eta, X)g(\eta, Y)}{g(\eta, \eta)}. \tag{4.2.1}$$

For $p \in N$ we can also express g at p in terms of q and the one form $\eta^\flat$. That is, for all $X, Y \in T_pM$

$$g(X, Y) = q(P_\eta X, P_\eta Y) + \frac{g(\eta, X)g(\eta, Y)}{g(\eta, \eta)}, \tag{4.2.2}$$

where $P_\eta : T_pM \to T_pN$ is the projection along η. Further if x^A, $A = 1, 2$ are local coordinates for N and hence $M/U(1)$, we can propagate them off N by requiring $\mathcal{L}_\eta x^A = 0$ and define the third coordinate ϕ by, $\phi = 0$ on N and $\mathcal{L}_\eta \phi = 1$. In these coordinates $\eta = \partial_\phi$ and $P_\eta(X^A\partial_A + X^\phi\partial_\phi) = X^A\partial_A$ so Equation (4.2.2) takes the form

$$g = q_{AB}dx^A dx^B + g(\eta,\eta)(d\phi + \theta_A dx^A)^2, \tag{4.2.3}$$

Recall that our definition of axisymmetric presupposes that $\mathcal{A}$ is nonempty, and it is well known that each component of $\mathcal{A}$ is a geodesic for M, thus $\mathcal{A}$ is often referred to as the axis of M. Furthermore, the quotient map $\pi : M \to M/U(1)$ maps $\mathcal{A}$ diffeomorphically to the boundary of $M/U(1)$. As stated in the introduction, we let $\overline{N}$ denote the doubling of $M/U(1)$ across its boundary. In [1], Chruściel shows the following:

Theorem 4.2.1. *If (M, g) is of asymptotic order k, then the metric q and the one form θ are k times differentiable on $\overline{N}$. In particular q is smooth up to the boundary of $M/U(1)$.*

This fact is nontrivial because of the factor of $\frac{1}{g(\eta,\eta)}$ appearing in Equation (4.2.1). The proof relies on and expression for the metric g in geodesic coordinates about $\mathcal{A}$ and not on the asymptotic behavior of g, so Chruściel's proof goes through unmodified in our case.

Before preceding we will introduce some notation. Throughout this section we will use capital letters A, B, C... to denote x or z and lower case letters a, b, c... to denote x or y, while standard indices i, j, k... will still mean any of x, y, and z.

In addition to knowing that the metric q is smooth on $\overline{N}$, we must also know the asymptotic behavior of q. In the AF case the following is shown in [13]:

Theorem 4.2.2. *Let M_{ext} be any AF end on a manifold (M, g) of asymptotic order $k \geq 3$. Then there exists an $R \geq 0$ and AF coordinates (x, y, z) on M_{ext} such that the plane $\{y = 0\} \cap \{r \geq R\}$ is transverse to η except at $x = z = 0$ where η vanishes. Furthermore the coordinates (x, z) form asymptotically flat coordinates for $M_{ext}/U(1)$, i.e.*

$$q_{AB} - \delta_{AB} = o_{k-3}(r^{-\ell}). \tag{4.2.4}$$

We will prove that analogous statement for AC ends.

Theorem 4.2.3. *Let M_{ext} be any AC end on a manifold (M, g) of order $k \geq 7$ characterized by (h, f, ℓ). Then there exists an $R \geq 0$ and coordinates (x, y, z) on M_{ext} such that the plane $\{y = 0\} \cap \{r \geq R\}$ is transverse to η except at $x = 0$ where η vanishes. Furthermore in the coordinates (x, z), the metric q satisfies*

$$q_{AB} - \frac{h(\theta)^2}{r^2}\delta_{AB} = o_{k-3}(r^{-\ell-2}) \text{ where } \theta = \arctan(\frac{x}{z}). \tag{4.2.5}$$

To prove this theorem, we start by constructing nice Cartesian coordinates (x, y, z) for M_{ext}. Our construction is highly dependent on the following estimate for the killing vector η.

Proposition 4.2.4. *Let η^i be a killing vector for an axisymmetric three manifold (M, g). Let M_{ext} be an AC end of M of orderk ≥ 7 characterized by (h, f, ℓ). Then there exists Cartesian coordinates (x, y, z) for M_{ext} such that*

$$g = h(\theta)^2(\frac{1}{r^2}\delta + \frac{1}{\rho^4}(f(\theta)^2 - \sin^2\theta)(xdy - ydx)^2) + O_k(r^{-\ell-2})dx^i dx^j, \tag{4.2.6}$$

$$\eta^i \partial_i = x\partial_y - y\partial_x + o_k(r^{1-\ell})\partial_i \quad (4.2.7)$$

and

$$\eta_i dx^i = \frac{h(\theta)^2 f(\theta)^2}{\rho^2}(xdy - ydx + o_k(r^{1-\ell})dx^i). \quad (4.2.8)$$

The proof of this estimate is straightforward, but lengthy, and is given in Section 4.7. It is worth noting that since $f(0) = f(\pi) = 0$ and $f'(0) = -f'(\pi) = 1$ we have $c\sin(\theta) \leq f(\theta) \leq C\sin(\theta)$ for some positive constants c, C. Since $\frac{\sin(\theta)}{\rho} = \frac{1}{r}$, this implies $\frac{f(\theta)}{\rho} = O(r^{-1})$. Thus in the above Cartesian coordinates, all metric components of g are $O(r^{-2})$. We will use ω^i_j to denote the antisymmetric 3 by 3 matrix whose only nonzero entries are $\omega^1_2 = -\omega^2_1 = -1$. Using this we may write $\eta^i = \omega^i_j x^j + O(r^{1-\ell})$.

Using Proposition 4.2.4, we can prove the following:

Proposition 4.2.5. *Let M_{ext} be any AC end on a manifold (M, g) of asymptotic order $k \geq 7$ characterized by (h, f, ℓ), then there exists a Cartesian coordinate system (x, y, z) on M_{ext} such that, g satisfies Equation* (4.2.6)*,*

$$\eta^x = -y + o_k(r^{1-\ell}),\ \eta^y = x + o_k(r^{1-\ell}),\ \text{and } \eta^z = 0. \quad (4.2.9)$$

Further η_i satisfies Equation (4.2.8)*, and if ψ_s represents the flow of η, we have*

$$x \circ \psi_\pi = -x,\ y \circ \psi_\pi = -y,\ \text{and } z \circ \psi_s = z. \quad (4.2.10)$$

Proof. Let $(\hat{x}, \hat{y}, \hat{z})$ be an arbitrary set of coordinates for M_{ext} given by Proposition 4.2.4. Let $\hat{\eta}^i$ be the associated killing vector and ψ_s the flow of $\hat{\eta}^i$. As in [13], our estimate for $\hat{\eta}^i$ implies that ψ_s is given by

$$\psi_s(\hat{x}^i) = (\cos(s)\hat{x} - \sin(s)\hat{y} + f^{\hat{x}}(s, \hat{x}^i), \sin(s)\hat{x} + \cos(s)\hat{y} + f^{\hat{y}}(s, \hat{x}^i), \hat{z} + f^{\hat{z}}(s, \hat{x}^i)), \quad (4.2.11)$$

where the $f^{\hat{x}^i}$ are error terms, each of order $o_{k+1}(\hat{r}^{1-\ell})$. Using this we can construct the desired coordinates on M_{ext}. Define

$$x = \frac{\hat{x} - \hat{x} \circ \psi_\pi}{2},\ y = \frac{\hat{y} - \hat{y} \circ \psi_\pi}{2},\ z = \frac{1}{2\pi}\int_0^{2\pi} \hat{z} \circ \psi_s ds. \quad (4.2.12)$$

Simple calculations using the form of ψ_s lead to

$$x^i = \hat{x}^i + o_{k+1}(\hat{r}^{1-\ell}) \quad (4.2.13)$$

and hence

$$\frac{\partial x^i}{\partial \hat{x}^j} = \delta^i_j + o_k(\hat{r}^{-\ell}), \quad (4.2.14)$$

where (x^i) is used to represent the coordinates (x, y, z). Define $r = \sqrt{x^2 + y^2 + z^2}$, and note that Equation (4.2.13) implies

$$r = \hat{r} + o_{k+1}(\hat{r}^{1-\ell}) = \hat{r} + o_{k+1}(r^{1-\ell}). \tag{4.2.15}$$

By the implicit function theorem, for R large enough, the x^i form a coordinate system on M_{ext}. Further Equation (4.2.12) implies $x \circ \psi_\pi = -x$, $y \circ \psi_\pi = -y$, and $z \circ \psi_s = z$ for all $s \in [0, 2\pi)$. It remains to show that g satisfies Equation (4.2.6) and that η^i is of the desired form. Equation (4.2.14) and the fact that the metric components are $O(r^{-2})$ imply that

$$\hat{g}_{ij} = g_{ij} + o_k(r^{\frac{-5}{2}}), \tag{4.2.16}$$

and hence g satisfies Equation (4.2.6).

Finally Equations (4.2.13) and (4.2.14) imply

$$\hat{\eta}^i = \frac{\partial \hat{x}^i}{\partial x^j}\eta^j = (\delta^i_j + o_k(r^{-\ell}))(\omega^j_l \hat{x}^l + o_k(r^{1-\ell})) = \omega^i_l \hat{x}^l + o_k(r^{1-\ell}) = \omega^i_l x^l + o_k(r^{1-\ell}) \tag{4.2.17}$$

and hence $\hat{\eta}^i = \eta^i + o_k(r^{1-\ell})$. Thus since $\hat{\eta}^x = \hat{y} + o_k(\hat{r}^{1-\ell})$, and $\hat{\eta}^y = -\hat{x} + o_k(\hat{r}^{1-\ell})$, we conclude $\eta^x = y + o_k(r^{1-\ell})$, and $\eta^y = -x + o_k(r^{1-\ell})$. Since $z \circ \psi_s = z$ we must have $\eta^z = 0$, so η^i satisfies the desired condition. The estimate for η_i follows from the estimate for η^i as in the proof of Proposition 4.2.4 given in Section 4.7. □

Next we prove several additional properties of the coordinate system (x, y, z) which will allow us to prove Theorem 4.2.3.

Lemma 4.2.6. *Under the coordinate system (x, y, z) the metric g satisfies*

$$\frac{\partial g_{ab}}{\partial x^c}(0,0,z) = 0,\ \frac{\partial g_{zz}}{\partial x^a}(0,0,z) = 0 \text{ and } g_{az}(0,0,z) = 0, \tag{4.2.18}$$

where $a, b, c \in \{x, y\}$.

Proof. The fact that $\psi_\pi(x, y, z) = (-x, -y, z)$ implies $d\psi_\pi(\partial_x) = -\partial_x$, $d\psi_\pi(\partial_y) = -\partial_y$, and $d\psi_\pi(\partial_z) = \partial_z$. Thus since ψ_π is an isometry

$$g(\partial_a, \partial_b)_{(0,y,z)} = g(d\psi_\pi(\partial_a), d\psi_\pi(\partial_b))_{\psi_\pi(0,y,z)} = g(-\partial_a, -\partial_b)_{(0,-y,z)} = g_{ab}(0,-y,z). \tag{4.2.19}$$

Thus $g_{ab}(0, y, z)$ is an even function of y so all odd order y derivatives vanish at $x = y = 0$, i.e. $\frac{\partial^{2l+1} g_{ab}}{\partial y^{2l+1}}(0,0,z) = 0$ for all $l \in \mathbb{N}$. Similarly $g_{ab}(x, 0, z) = g_{ab}(-x, 0, z)$ so we have $\frac{\partial^{2l+1} g_{ab}}{\partial x^{2l+1}}(0,0,z) = 0$ for all $l \in \mathbb{N}$. When $l = 1$ we have the first desired equation $\frac{\partial g_{ab}}{\partial x^c}(0,0,z) = 0$.

Similarly we can calculate

$$g(\partial_z, \partial_z)_{(0,y,z)} = g(d\psi_\pi(\partial_z), d\psi_\pi(\partial_z))_{\psi_\pi(0,y,z)} = g(\partial_z, \partial_z)_{(0,-y,z)} = g_{zz}(0,-y,z). \tag{4.2.20}$$

Thus $\frac{\partial^{2l+1} g_{zz}}{\partial y^{2l+1}}(0,0,z) = 0$ and in the same way $\frac{\partial^{2l+1} g_{zz}}{\partial x^{2l+1}}(0,0,z) = 0$ for all $l \in \mathbb{N}$. When $l = 1$ we have the second desired equation $\frac{\partial g_{zz}}{\partial x^a}(0,0,z) = 0$.

Finally we have

$$g(\partial_a, \partial_z)_{(0,y,z)} = g(d\psi_\pi(\partial_a), d\psi_\pi(\partial_z))_{\psi_\pi(0,y,z)} = g(-\partial_a, \partial_z)_{(0,-y,z)} = -g_{az}(0,-y,z), \tag{4.2.21}$$

so $g_{az}(0,-y,z)$ is an odd function in y and hence $g_{az}(0,-y,z) = 0$ and $\frac{\partial^{2l} g_{az}}{\partial y^{2l}}(0,0,z) = 0$ for all $l \in \mathbb{N}$. □

We now prove a few properties of η^i which will allow us to estimate η^i near $\mathcal{A}$.

Lemma 4.2.7. *Under the coordinate system (x, y, z), the killing vector η^i satisfies*

$$\eta^i(0,0,z) = 0,\ \nabla_i \eta^z(0,0,z) = \nabla_z \eta^i(0,0,z) = 0, \tag{4.2.22}$$

and

$$\nabla_i \nabla_j \eta^k(0,0,z) = \nabla_i \nabla_j \eta_k(0,0,z) == \partial_a \partial_b \eta^c(0,0,z) == \partial_a \partial_b \eta_c(0,0,z) = 0 \tag{4.2.23}$$

where $a, b, c \in \{x, y\}$. In addition we have $\{(0,0,z)\} = \mathcal{A} \cap M_{ext}$.

Proof. Since $\psi_\pi(x,y,z) = (-x,-y,z)$, the fixed point set of ψ_π is precisely $\{(0,0,z)\}$. Since all nontrivial orbits of ψ have period 2π, we conclude $(0,0,z)$ is a fixed point of ψ_s for all s. Thus $M_{ext} \cap \mathcal{A} = \{(0,0,z)\}$, and $\eta^i(0,0,z) = 0$.

Now since $\eta^z(x,y,z) = 0$ for all (x,y,z) we have $\partial_i \eta^z = 0$. Thus $\nabla_i \eta^z(0,0,z) = \partial_i \eta^z(0,0,z) + \Gamma^z_{ij}\eta^j(0,0,z) = 0$. Since η^i is a killing vector $\nabla_i \eta^z = 0$ implies $\nabla_z \eta^i = 0$. Finally, the killing equations imply that $\nabla_i \nabla_j \eta^k = R^i_{lij}\eta^l$, so since $\eta^i(0,0,z) = 0$ we have $\nabla_i \nabla_j \eta^k(0,0,z) = 0$. Lowering an index we have $\nabla_i \nabla_j \eta_k(0,0,z) = 0$. Finally we expand the second covariant derivative as

$$0 = \nabla_a \nabla_b \eta^c(0,0,z) = (\partial_a \nabla_b \eta^c + \Gamma^c_{aj} \nabla_b \eta^j - \Gamma^j_{ab} \nabla_j \eta^c)(0,0,z). \tag{4.2.24}$$

Expanding the Christoffel symbols in terms of the metric and applying Equations (4.2.18) and (4.2.22) we obtain

$$0 = \partial_a \partial_b \eta^c(0,0,z). \tag{4.2.25}$$

and similarly for the lowered index version. □

Define $\lambda^i_j(z) = \frac{\partial \eta^i}{\partial x^j}(0,0,z)$ and $\lambda_{ij}(z) = g_{ik}(0,0,z)\lambda^k_j$. Since $\eta^z = 0$ and $\eta^i(0,0,z) = 0$ we have $\lambda^z_i = \lambda^i_z = 0$. Further note that Equation (4.2.7) implies $\lambda^a_b(z) = \omega^a_b + o_{k-1}(r^{-\ell})$. Finally the vanishing of η on the axis implies $\lambda_{xx} = \lambda yy = 0$. We can now establish the necessary estimates for η near the axis.

Lemma 4.2.8. *In the region $M_{ext} \cap \{(x,y,z) :\ z^2 \geq \rho^2\}$ the killing vector satisfies*

$$\eta^a = \lambda^a_b x^b + o_{k-3}(r^{-\ell-2})\rho^3 \tag{4.2.26}$$

and

$$\eta_a = \lambda_{ab} x^b + o_{k-3}(r^{-4})\rho^3 \tag{4.2.27}$$

In addition we may write

$$\eta_i = \frac{f(\theta)^2 h(\theta)^2}{\rho^2}(\lambda^i_b x^b + o_{k-3}(r^{-\ell})\rho) \tag{4.2.28}$$

Proof. To estimate the value of η^a at the point (x,y,z) we consider the segment γ connecting $(0,0,z)$ to (x,y,z). Let $\partial_\rho = \frac{x\partial_x + y\partial_y}{(x^2+y^2)^{1-\ell}} = \frac{x\partial_x+y\partial_y}{\rho}$ denote the tangent vector along this segment. From Equation (4.2.7) we have $\frac{\partial^3\eta^a}{\partial\rho^3} = o_{k-3}(r^{-\ell-2})$. Integrating once along γ we have $\frac{\partial^2\eta^a}{\partial\rho^2}(x,y,z) = o_{k-3}(r^{-\ell-2})\rho + \frac{\partial^2\eta^a}{\partial\rho^2}(0,0,z) = o_{k-3}(r^{-\ell-2})\rho$ where $\frac{\partial^2\eta^a}{\partial\rho^2}(0,0,z)$ vanishes by Lemma 4.2.7. Integrating two more times along γ we have

$$\frac{\partial\eta^a}{\partial\rho}(x,y,z) = o_{k-3}(r^{-\ell-2})\rho^2 + \frac{\partial\eta^a}{\partial\rho}(0,0,z), \tag{4.2.29}$$

and

$$\eta^a(x,y,z) = o_{k-3}(r^{-\ell-2})\rho^3 + \rho\frac{\partial\eta^a}{\partial\rho}(0,0,z) + \eta^a(0,0,z) = o_{k-3}(r^{-\ell-2})\rho^3 + \rho\frac{\partial\eta^a}{\partial\rho}(0,0,z), \tag{4.2.30}$$

where $\eta^a(0,0,z) = 0$ by Lemma 4.2.7. Finally we note that by definition $\rho\frac{\partial\eta^a}{\partial\rho}(0,0,z) = \lambda^a_b x^b$ and hence $\eta^a = \lambda^a_b x^b + o_{k-3}(r^{-\ell-2})\rho^3$ as desired.

We can follow the same procedure for estimating η_a, however we start from the fact that $\eta_a = O_k(r^{-1})$ and hence $\frac{\partial^3\eta^a}{\partial\rho^3} = O_{k-3}(r^{-4})$. Upon integrating three times we obtain $\eta_a = \lambda_{ab}x^b + o_{k-3}(r^{-4})\rho^3$ □

We know $\eta^z = 0$ in all of M_{ext}, however we do require a precise estimate of η_z near the axis.

Lemma 4.2.9. *In the region $M_{ext} \cap \{(x,y,z) : z^2 \geq \rho^2\}$ the killing vector satisfies*

$$\eta_z = O_{k-1}(r^{-3})\rho^2 + O_{k-3}(r^{-4})\rho^3 \tag{4.2.31}$$

Proof. Following the procedure of Lemma 4.2.8 we begin from the starting point $\eta_z = O_k(r^{-1})$. In this case Equation (4.2.22) implies the first order terms vanish, so

$$\eta_z = \rho^2\frac{\partial^2\eta_z}{\partial\rho} + O_{k-3}(r^{-4})\rho^3 \tag{4.2.32}$$

Expanding the second covariant derivative yields $0 = \nabla_a\nabla_b\eta_z = \partial_a\partial_b\eta_z - 2\Gamma^c_{az}\lambda_{bc}$. Here the Christoffel symbols are in general $O_{k-1}(r^{-1})$, and hence $\partial^2_\rho\eta_z = O_{k-1}(r^{-3})$. Equation (4.2.31) follows. □

We can now show that $\{(x,0,z) : x \geq 0\}$ can be used as a global cross section for $M_{ext}/U(1)$ and hence that we can use (x,z) as global coordinates for $M_{ext}/U(1)$.

Lemma 4.2.10. *Given the coordinates (x,y,z), there exists an $R > 0$ such that η^i is transverse to the submanifold $\overline{N}_{ext} := \{(x,0,z) : x^2 + z^2 \geq R^2\}$ except at $x = 0$.*

Proof. We will require different estimates for η^i in the region $x^2 \geq z^2$ and $x^2 \leq z^2$. First, for $x^2 \geq z^2$ we recall that $\eta^i = \omega^i_j x^j + o_k(r^{1-\ell})$ where $\omega^y_x = 1$, so if $y = 0$ we have $\eta^y = x + o_k(r^{1-\ell})$. Since $x^2 \geq z^2$, we have $x^2 \geq \frac{1}{2}r^2$, so $x > o_k(r^{1-\ell})$ for large r. So there exists a R such that $\eta^y > 0$ in the region $\{(x,0,z) : x^2 + z^2 \geq R, x^2 \geq z^2\}$, and thus η^i is transverse in this region.

Now suppose $z^2 \geq x^2$, and thus z is comparable with r. By Lemma 4.2.8 we have $\eta^y(x,0,z) = \lambda^y_x x + o_{k-3}(r^{-\ell-2})x^3$. Since $\lambda^y_x = 1 + o_{k-1}(r^{-\ell})$ we have the estimate

$$\eta^y(x,0,z) = (1 + o_{k-1}(r^{-\ell}))x + o_{k-3}(r^{-\ell-2})x^3 = (1 + o_{k-1}(r^{-\ell}))x. \tag{4.2.33}$$

Thus by taking R large enough we can ensure $\eta^y(x,0,z) \neq 0$ whenever $x^2 + z^2 \geq R$. We conclude there exists an R such that η is transverse in the region $\{(x,0,z) : x^2 + z^2 \geq R,\ x \neq 0\}$. □

The above Lemma implies that $N_{ext} := \overline{N}_{ext} \cap \{x \geq 0\}$ is a global cross section for $M_{ext}/U(1)$ and that we can use (x,z) as global coordinates for $M_{ext}/U(1)$. We can now prove the main result of the section.

Proof of Theorem 4.2.3. All that remains is to show q satisfies Equation (4.2.5) in the coordinates (x,z). To show this we use our previously obtained estimates on the killing vector η, and Equation (4.2.1), or equivalently $q_{AB} = g_{AB} + \frac{\eta_A \eta_B}{g(\eta,\eta)}$.

From Equation (4.2.6) and the fact that $y = 0$ on N_{ext} we obtain $g_{AB} = \frac{h(\theta)^2}{r^2}\delta_{AB} + o_{k-3}(r^{-\ell-2})$. When restricted to N_{ext}, r for the coordinates (x,y,z) agrees with r for the coordinates (x,z), thus we need only show that $\frac{\eta_A \eta_B}{g(\eta,\eta)} = o_{k-3}(r^{-\ell-2})$ throughout N_{ext}. This is done by considering two regions, $x \geq z$ and $x \leq z$, where will will show the stronger statement $\frac{\eta_A \eta_B}{g(\eta,\eta)} = o_{k-3}(r^{-2\ell-2})$

First suppose $x \geq z$ and recall that the estimates for η imply $g(\eta,\eta) = \frac{h(\theta)^2 f(\theta)^2}{\rho^2}(\rho^2 + o_k(r^{2-\ell}))$, thus in the plane $y = 0$, $g(\eta,\eta) = \frac{h(\theta)^2 f(\theta)^2}{x^2}(x^2 + o_k(r^{2-\ell}))$. Now since $\eta_A = \frac{h(\theta)^2 f(\theta)^2}{\rho^2}(\omega^A_i x^i + o_k(r^{1-\ell}))$, and $\frac{h(\theta)^2 f(\theta)^2}{\rho^2} = O_k(r^{-2})$, in the plane $y = 0$ we have $\eta_x = o_k(r^{-\ell-1})$ and $\eta_z = o_k(r^{-\ell-1})$. Since $\frac{\rho^2}{h(\theta)^2 f(\theta)^2} = O_k(r^2)$, we conclude

$$\frac{\eta_A \eta_B}{g(\eta,\eta)} = \frac{\rho^2}{h(\theta)^2 f(\theta)^2}\frac{o_k(r^{-\ell-2})}{(x^2 + o_k(r^{3/2}))} = \frac{o_k(r^{-2\ell})}{x^2 + o_k(r^{3/2})}. \tag{4.2.34}$$

The fact that $x \geq \frac{1}{2}r$ implies $x^2 + o_k(r^{2-\ell}) = O_k(x^2) = O_k(r^2)$ so $\frac{\eta_A \eta_B}{g(\eta,\eta)} = O(r^{-2\ell-2})$. Our control over the derivatives of the numerator and denominator imply $\frac{\eta_A \eta_B}{g(\eta,\eta)} = o_k(r^{-2\ell-2})$ as desired.

Now suppose $x \leq z$. On N_{ext} the estimates in Lemma 4.2.8 are of the form $\eta^a = \lambda^a_x x + o_{k-3}(r^{-\ell-2})x^3$ and $\eta_i = \frac{h(\theta)^2 f(\theta)^2}{\rho^2}(\lambda^i_x x + o_{k-3}(r^{-\ell})x)$. For the numerator of $\frac{\eta_A \eta_B}{g(\eta,\eta)}$ we recall that $\lambda^x_x = o_{k-1}(r^{-\ell})$ and $\lambda^z_x = 0$ so $\eta_x = o_{k-3}(r^{-\ell-2})x$ and $\eta_z = o_{k-3}(r^{-\ell-2})x$. Thus in any case $\eta_A \eta_B = o_{k-3}(r^{-2\ell-4})x^2$. For the denominator we have

$$\frac{\rho^2}{h(\theta)^2 f(\theta)^2} g(\eta,\eta) = \frac{\rho^2}{h(\theta)^2 f(\theta)^2}\eta^a \eta_a = (\lambda^a_x x + o_{k-3}(r^{-\ell-2})x^3)(\lambda^a_x x + o_{k-3}(r^{-\ell})x). \tag{4.2.35}$$

When $a = y$ we have leading term $(\lambda^y_x)^2 x^2 = (1 + o_{k-1}(r^{-\ell}))x^2$. All other terms for $a = y$ and $a = x$ are $o_{k-3}(r^{-\ell})x^2$ and thus we have $g(\eta,\eta) = \frac{h(\theta)^2 f(\theta)^2}{\rho^2}(1 + o_{k-1}(r^{-\ell}))x^2$. Combining this with our estimate for the numerator we have

$$\frac{\eta_A \eta_B}{g(\eta,\eta)} = \frac{\rho^2}{h(\theta)^2 f(\theta)^2}\frac{o_{k-3}(r^{-2\ell-4})x^2}{(1 + o_{k-1}(r^{-\ell}))x^2} = O_k(r^2)\frac{o_{k-3}(r^{-2\ell-4})}{1 + o_{k-1}(r^{-\ell})} = o_{k-3}(r^{-2\ell-2}) \tag{4.2.36}$$

as desired.

□

4.3 Isothermal Coordinates

In [13] the following theorem is proved concerning the existence of isothermal coordinates

Theorem 4.3.1. *Let (N, q) be a 2 manifold with n AF ends N_i, each of order $k-3$ which is diffeomorphic to $\mathbb{R}^2/\{a_i\}_{i=2}^N$ such that each puncture represents an AF end. Further for $1 \leq i \leq n$ let x_i^A be asymptotically flat coordinates in the i^{th} end, and let $\tilde{r}_i$ denote the corresponding radial function. Then there exists a unique function u such that $e^{-2u}q$ is flat throughout N, $u = o_{k-4}(r_1^{-\ell})$ as $r_1 \to \infty$, and*

$$u = \log C_i + 2\log \tilde{r}_i + o_{k-4}(\tilde{r}_i^{-\ell}) \tag{4.3.1}$$

as $\tilde{r}_i \to \infty$ for $i \geq 2$ and some constants C_i. Furthermore this conformal factor compactifies each of the asymptotically flat ends except N_1 so that the metric $e^{-2u}q$ extends to a complete flat metric on $\mathbb{R}^2$.

We will use this result to show the existence and uniqueness of the desired conformal factor when cylindrical ends are allowed. For conciseness we first introduce the following definition:

Definition 4.3.2. *An open submanifold N_{ext} of a Riemannian 2 manifold (N, q) is said to be an asymptotically cylindrical end of order k if there exists a diffeomorphism between N and $\mathbb{R} \setminus B(R)$ for some $R > 0$, such that in local coordinates (x, z) on N_{ext} obtained from $\mathbb{R} \setminus B(R)$ the metric q satisfies*

$$q_{AB} - \frac{h(\theta)^2}{r^2}\delta_{AB} = o_{k-3}(r^{-\ell-2}), \tag{4.3.2}$$

where h is a positive smooth function of $\theta = \arctan(\frac{x}{z})$.

The coordinates (x, z) will be called AC coordinates and N_{ext} will be referred to as an AC end of N.

Theorem 4.3.3. *Let (N, q) be a 2 manifold with n AF or AC ends N_i, each of order $k-3$. Further suppose N is diffeomorphic to $\mathbb{R}^2/\{a_i\}_{i=2}^N$ such that each puncture represents an AF or AC end, and N_1 is an AF end at infinity. Further for $1 \leq i \leq n$ let x_i^A be AF or AC coordinates in the i^{th} end, and let $\tilde{r}_i$ denote the corresponding radial function. For each N_i which is AC, let h_i denote the conformal factor appearing in Equation (4.3.2). Then there exists a unique function u such that $e^{-2u}q$ is flat throughout N, $u = o_{k-4}(r_1^{-\ell})$ as $r_1 \to \infty$, and for $i \geq 2$ and some constants C_i,*

$$u = \log C_i + 2\log \tilde{r}_i + o_{k-4}(\tilde{r}_i^{-\ell}) \tag{4.3.3}$$

if N_i is AF, while

$$u = \log C_i + \log h_i + \log \tilde{r}_i + o_{k-4}(\tilde{r}_i^{-\ell}) \tag{4.3.4}$$

if N_i is AC. Furthermore this conformal factor compactifies each of the AF and AC ends except N_1 so that the metric $e^{-2u}q$ extends to a complete flat metric on $\mathbb{R}^2$.

Proof. We first show existence. Let σ be a smooth positive function which is equal to $\frac{r_i^2}{h_i^2}$ in each N_i which is AC and equal to 1 in each N_i which is AF. Now consider the metric σq. Since $\sigma = 1$ on each AF end, the coordinates x_i^A which are AF for q, are also AF for σq. Now if N_i is an AC end for q, and x_i^A the corresponding AC coordinates, then $q_{AB} - \frac{h_i^2}{\tilde{r}_i^2}\delta_{AB} = o_{k-3}(\tilde{r}_i^{-\ell-2})$ in N_i, so by construction $\sigma q_{AB} - \delta_{AB} = o_{k-3}(r^{-\ell})$. Thus x_i^A are AF coordinates for N_i in the metric σq. Hence the Riemannian manifold $(N, \sigma q)$ satisfies the hypotheses of Theorem 4.3.1. Let $\hat{u}$ denote the conformal factor such that $\sigma e^{-2\hat{u}} q$ satisfies the conclusions of Theorem 4.3.1. Thus the conformal factor $u := \hat{u} - \ell \log \sigma$ is such that $e^{-2u} q = \sigma e^{-2\hat{u}} q$ is flat and every end except N_1 is compactified. Thus it remains to check that u has the desired falloff. Since $\sigma = 1$ in each flat end, $u = \hat{u}$ in each such end, and by Theorem 4.3.1. Thus $u = o_{k-4}(r_1^{-\ell})$ as $r_1 \to \infty$, and for $i \geq 2$,

$$u = \log C_i + 2 \log \tilde{r}_i + o_{k-4}(\tilde{r}_i^{-\ell}) \tag{4.3.5}$$

if N_i is AF. If N_i is an *AC* end for q, then in N_i we have

$$u = \hat{u} - \ell \log \sigma = \hat{u} + \log h_i - \log \tilde{r}_i = \log C_i + \log h_i + \log \tilde{r}_i + o_{k-4}(\tilde{r}_i^{-\ell}) \tag{4.3.6}$$

as desired.

For uniqueness we will apply the uniqueness of Theorem 4.3.1. Explicitly, suppose $e^{-2v} q$ is flat and v satisfies the falloff of Theorem 4.3.3. Let σ be as above and define $\hat{v} = v + \frac{1}{2}\log(\sigma)$. Then we note that $\hat{v}$ satisfies the conclusions of Theorem 4.3.1 when applied to the metric σq. By the uniqueness of Theorem 4.3.1 we have $\hat{v} = \hat{u}$ and thus $u = v$ as desired. □

We can now apply this theorem to the doubled manifold $\overline{N}$.

Theorem 4.3.4. *Let (M, g) be of asymptotic order k with n ends. Let $(\overline{N}, q)$ be the doubling of $M/U(1)$ with the induced quotient metric and isometry ψ. Then there exists a global coordinate system (ρ, z) for $\overline{N}$ under which q has the form $q = e^{2u}(d\rho^2 + dz^2)$, and the isometry ψ is given by reflection across the z axis. Further u satisfies the falloff given in Theorem 4.3.3.*

Proof. Simple connectedness an axisymmetry of M imply that $\overline{N}$ is diffeomorphic to $\mathbb{R}^2/\{a_i\}_{i=2}^n$. Further Theorem 4.2.1 implies that q is smooth, while Theorems 4.2.2 and 4.2.3 imply that $\overline{N}$ is AF at infinity and each puncture of $\overline{N}$ corresponds to either an AF or and AC end of $(\overline{N}, q)$. Thus $(\overline{N}, q)$ satisfies the hypotheses of Theorem 4.3.3 and we can let u be the corresponding conformal factor. Finally, we may consider $e^{-2u} q$ a metric on $\mathbb{R}^2$ because the conformal factor compactifies each puncture.

We claim u is invariant under ψ. Since ψ is a diffeomorphism the pullback $\psi^*(e^{-2u} q)$ is a flat metric on $\mathbb{R}^2$. This pullback is given by $\psi^*(e^{-2u} q) = e^{-2u\circ\psi} \psi^* q = e^{-2u\circ\psi} q$ since ψ is an isometry of q. We wish to apply the uniqueness of Theorem 4.3.3, so we must show $u \circ \psi$ satisfies the desired falloff in each end. Given an AC end $\overline{N}_i$, Theorem 4.2.3 implies that there exists AC coordinates (x_i, z_i) for $\overline{N}_i$ such that $\psi(x_i, z_i) = (-x_i, z_i)$ for large $\tilde{r}_i$. Thus in these coordinates, $\tilde{r}_i$ and $\tilde{\theta}_i$ are invariant under ψ. Hence $u = \log C_i + \log h_i + \log \tilde{r}_i + o_{k-4}(\tilde{r}_i^{-\ell})$ implies $u \circ \psi = \log C_i + \log h_i + \log \tilde{r}_i + o_{k-4}(\tilde{r}_i^{-\ell})$. Similarly Theorem 4.2.2 implies $u \circ \psi$ has the desired falloff in any AF end. By uniqueness we conclude $u \circ \psi = u$.

Now since ψ is an isometry for q, and u is invariant under ψ, ψ is also an isometry of $e^{-2u} q$. Since $e^{-2u} q$ is a flat metric on $\mathbb{R}^2$, there exists global coordinates (v, w) such that $e^{-2u} q = dv^2 + dw^2$.

Further, by a rigid translation we may assume the origin of this coordinate system lies on $\mathcal{A}$, and by a rotation assume that ∂_w coincides with the tangent vector to $\mathcal{A}$ at the origin. Now clearly the curve $\gamma(t) = (0, t)$ is a geodesic for $e^{-2u}q$. Since ψ is an isometry of $e^{-2u}q$ whose fixed point set is $\mathcal{A}$, we also have $\mathcal{A}$ is a geodesic for $e^{-2u}q$. Thus by the uniqueness of geodesics $\mathcal{A} = \{(0, v); z \in \mathbb{R}\}$. Now each geodesic $\gamma_v(t) = (t, w)$ of $e^{-2u}q$ gets mapped by ψ to a geodesic on the other side of $\mathcal{A}$ whose initial tangent vector is perpendicular to $\mathcal{A}$. Thus we must have $\psi(\gamma_w(t)) = (-t, w)$, and hence for all (v, w) we have $\psi(v, w) = (-v, w)$. For convenience we rename the coordinates (ρ, z) and have the desired result. □

We must prove a few more facts about our new coordinate system (ρ, z). In particular if we define $r_i = \sqrt{\rho^2 + (z - a_i)^2}$ for $i \geq 2$ as in Theorem 1.0.5, we would like to know the behavior of u as $r_i \to 0$. The fact that e^{-2u} compactifies the punctures implies that $(\overline{x}_i, \overline{z}_i) := (\frac{x_i}{\tilde{r}_i^2}, \frac{z_i}{\tilde{r}_i^2})$ can be used as coordinates in a neighborhood of the i^{th} puncture. Further $e^{-2u}q$ at the origin of this coordinate system is $\frac{\delta_{AB}}{C_i^2}$. Note that $(\rho, z - a_i)$ are orthogonal coordinates for $e^{-2u}q$ with the same origin as $(\overline{x}_i, \overline{z}_i)$. Further the line $(0, \overline{z}_i)$ corresponds to the line $(0, z - a_i)$. Thus we must have $\rho = \frac{\overline{x}_i}{C_i} + O(\tilde{r}_i^2)$ and $z = \frac{\overline{z}_i}{C_i} + a_i + O(\tilde{r}_i^2)$. In particular $r_i = \frac{1}{C_i \tilde{r}_i} + O_{k-4}(r_i^2)$. This relationship along with the falloff given in Theorem 4.3.4 imply

$$u = -\log C_i - 2\log r_i + o_{k-4}(r_i^\ell) \tag{4.3.7}$$

if N_i is AF, and

$$u = \log h_i - \log r_i + o_{k-4}(r_i^\ell) \tag{4.3.8}$$

if N_i is AC.

In addition the relationship between $(\overline{x}_i, \overline{z}_i)$ and $(\rho, z - a_i)$ implies a relationship between angular coordinate $\tilde{\theta}_i = \arctan(\frac{x_i}{z_i}) = \arctan(\frac{\overline{x}_i}{\overline{z}_i})$ and the angular coordinate $\theta_i := \arctan(\frac{\rho}{z - a_i})$. In particular we have

$$\theta_i = \tilde{\theta}_i + O_{k-4}(r_i). \tag{4.3.9}$$

If $i = 1$, then the fact that $u = o_{k-4}(r_1^{-\ell})$ implies that (ρ, z) are AF coordinates for M_1 in the metric q. Thus if we define $r = \sqrt{\rho^2 + z^2}$ as in Theorem 1.0.5 we have $u = o_{k-4}(r^{-\ell})$ as $r \to \infty$. In the next section we use these fact about the falloff of u complete the construction of Brill coordinates.

4.4 Final Construction

We now have all of the tools to prove Theorem 1.0.5. The construction of Brill coordinates precedes as in [13], and the falloff of the metric components is computed similarly.

Proof of Theorem 1.0.5. For $1 \leq i \leq n$ let (x_i, y_i, z_i) be the AF or AC coordinates constructed in Section 2. Then let N be a global cross section of $M/U(1)$ such that in each end M_i, $N \cap M_i$ agrees with the set $(x_i, 0, z_i) : x_i \geq 0$. We can do this because it was shown in Section 2 that the set $\{(x_i, 0, z_i) : x_i \geq 0\}$ is traverse to η.

By Theorem 4.3.4 we have global coordinates (ρ, z) for $M/U(1)$, and hence (ρ, z) are also coordinates for N. We set $\phi = 0$ on N and propagate the coordinates (p, z, ϕ) off N by requiring $\mathcal{L}_\eta \rho = \mathcal{L}_\eta z = 0$ and $\mathcal{L}_\eta \phi = 1$. Since N is a global cross section, every point p in M is given by (ρ, z, ϕ) for some $\rho \in [0, \infty)$, $z \in (-\infty, \infty)$ and $\phi \in [0, 2\pi)$ where $(\rho, 0, z)$ is the unique point on N in the same orbit as p. Thus (ρ, z, ϕ) forms a global coordinate system for M. Furthermore $\eta = \partial_\phi$ and and $P_\eta(X^A\partial_A + X^\phi\partial_\phi) = X^A\partial_A$. Hence by Equation (4.2.3) the metric in these coordinates is given by

$$g = q_{AB}dx^A dx^B + g(\eta, \eta)(d\phi + A_\rho d\rho + A_z dz)^2 \tag{4.4.1}$$

where $A_\rho = \rho\frac{\eta_\rho}{\rho g(\eta,\eta)}$ and $A_z = \frac{\eta_z}{g(\eta,\eta)}$. Further all of these functions are independent of ϕ. To get this into the form of Equation (1.0.10) we simply define U and α by $e^{-2U} = \frac{g(\eta,\eta)}{\rho^2}$ and $\alpha = u + U$. Then since $q_{AB}dx^A dx^B = e^{2u}(d\rho^2 + dz^2)$ we have

$$g = e^{-2U+2\alpha}(d\rho^2 + dz^2) + \rho^2 e^{-2U}(d\phi + A_\rho d\rho + A_z dz)^2 \tag{4.4.2}$$

As in the statement of the Theorem, we now define $r = \sqrt{\rho^2 + z^2}$ and $\hat{r}_i = \sqrt{\rho^2 + (z - a_i)^2}$ for $i \geq 2$. It remains to calculate the falloff of the metric components. This is done for M_1 and any other AF end in [13]. Thus we will only treat the AC case.

Let M_i be an AC end of M, and let $N_i := M_i \cap N$. In Section 2 we obtained estimates for η in the coordinates (x_i, y_i, z_i). In particular we know that on N_i we have

$$\rho^2 e^{-2U} = g(\eta, \eta) = h_i(\tilde{\theta}_i)^2 f_i(\tilde{\theta}_i)^2(1 + o_{k-3}(\tilde{r}_i^{-\ell})). \tag{4.4.3}$$

Using the relationship between (x_i, z_i) and (ρ, z) derived at the end of Section 3 this becomes

$$e^{-2U} = \frac{h(\tilde{\theta}_i)^2 f(\tilde{\theta}_i)^2}{\rho^2}(1 + o_{k-3}(r_i^\ell)), \tag{4.4.4}$$

and thus

$$U = \log \rho - \log h_i(\tilde{\theta}_i) - \log f_i(\tilde{\theta}_i) + o_{k-3}(r_i^\ell). \tag{4.4.5}$$

Using the relationship between ρ, r_i, and θ_i we may write this as

$$U = \log r_i - \log h_i(\tilde{\theta}_i) - \log \frac{f_i(\tilde{\theta}_i)}{\sin \tilde{\theta}_i} + o_{k-3}(r_i^\ell). \tag{4.4.6}$$

Now using Equation (4.3.8) we can directly estimate $\alpha = u + U$ as

$$\alpha = -\log \frac{f_i(\tilde{\theta}_i)}{\sin \tilde{\theta}_i} + o_{k-4}(r_i^\ell). \tag{4.4.7}$$

Functions in both U and α remain in terms of $\tilde{\theta}_i$, the angular coordinate for (x_i, z_i). To solve this we use Equation (4.3.9) and the fact that all derivatives of f_i and h_i are bounded to obtain

$$f_i(\tilde{\theta}_i) = f_i(\theta_i) + o_{k-4}(r^{1-\ell}), \quad h_i(\tilde{\theta}_i) = h_i(\theta_i) + o_{k-4}(r^\ell). \tag{4.4.8}$$

Thus we in fact have

$$U = \log r_i - \log h_i(\theta_i) - \log \frac{f_i(\theta_i)}{\sin\theta_i} + o_{k-3}(r_i^\ell), \quad \alpha = -\log \frac{f_i(\theta_i)}{\sin\theta_i} + o_{k-4}(r_i^\ell). \tag{4.4.9}$$

Finally in [13] it is shown that A_z and A_ρ are given in the slice $y = 0$ by $A_z = \frac{\eta_z}{g(\eta,\eta)}$ and $A_\rho = \rho\frac{\eta_x}{xg(\eta,\eta)}$.

and the proof of Theorem 1.0.5 is complete. □

We will use in the following section that $\alpha = 0$ in the z which follows from the fact that g is assumed to have no conical singularities. The proof of this fact is given in [13]. As a consistency check we simply note that $\lim_{\theta_i \to 0} \log \frac{f_i(\theta_i)}{\sin\theta_i} = 0$, which is a necessary condition for α to vanish on the axis.

4.5 Application to ADM Mass

The ADM mass of an asymptotically flat end with asymptotically flat coordinates, (x, y, z), is given by

$$m := lim_{R\to\infty} \frac{1}{16\pi} \int_{S_R} (g_{ij,j} - g_{jj,i}) dS_i, \tag{4.5.1}$$

where $dS_i = \partial_i \rfloor (dx \wedge dy \wedge dz)$, and S_R can be taken to be any piecewise differentiable surface homologous to the coordinate sphere of radius R such that $\lim_{R\to\infty} \inf\{r(p) : p \in S_R\} = \infty$.

In [13] it is shown that the positive mass theorem holds for asymptotically flat metrics of the form (1.0.10) given certain falloff the for metric components. Here we modify the arguments for the case when asymptotically cylindrical ends are present. In particular we prove the following:

Theorem 4.5.1. *Let (M, g) be of asymptotic order k, with $k \geq 6$, and $n \geq 2$ ends. Further suppose that the scalar curvature $R^{(3)} \geq 0$. Then the ADM mass m_i of any asymptoticly flat end M_i satisfies $0 < m_i \leq \infty$*

with $m_i < \infty$ if and only if

$$R^{(3)} \in L^1(\mathbb{R}^3),\ \ DU,\ \rho(A_{\rho,z} - A_{z,\rho}) \in L^2(\mathbb{R}^3) \tag{4.5.2}$$

Proof. Assume without loss of generality that we are trying to determine the mass of M_1. Let (ρ, z, ϕ) be the Brill coordinates for M given by Theorem 1.0.5. As in [13] we can convert to standard Cartesian coordinates by setting

$$x = \rho\cos\phi,\ \ y = \rho\sin\phi. \tag{4.5.3}$$

The falloff of the metric components given by Theorem 1.0.5 then imply that (x, y, z) is an asymptotically flat coordinate system for M_1. Our Brill coordinate system has the same behavior of that obtained in [13] except near the punctures, so the integral in Equation (4.5.1) can be computed in an identical way with the exception of our application of Stokes' Theorem.

We define C_R to be the boundary of the solid cylinder

$$\mathcal{C}_R := \{-R \leq z \leq R,\, 0 \leq \rho \leq R\}. \tag{4.5.4}$$

It is then shown in [13] that the mass integral of Equation (4.5.1) reduces to

$$m = \lim_{R\to\infty} \frac{1}{4\pi} \int_{C_R} \partial_i (U - \frac{1}{2}\alpha) dS_i + \frac{1}{8\pi} \int_{W_R} \alpha d\phi dz, \tag{4.5.5}$$

where

$$W_R := \{-R \le z \le R,\, \rho = R\} \tag{4.5.6}$$

represents the wall of the boundary cylinder C_R. We will use $B_{1/R}(a_i)$ to denote the coordinate ball of radius $1/R$ about the i^{th} puncture and $S_{1/R}(a_i)$ to denote the corresponding boundary sphere. Define

$$\hat{\mathcal{C}}_R := \mathcal{C}_R \backslash \cup_{i=2}^n B_{1/R}(0,0,a_i) \tag{4.5.7}$$

and note that, for large R, the boundary of $\hat{\mathcal{C}}_R$ is the union of C_R and $\cup_{i=2}^n S_{1/R}(a_i)$. The unit outer normal of $S_{1/R}(a_i)$ is $-\partial_{r_i}$.Thus Stokes Theorem implies that

$$\int_{C_R} \partial_i (U - \frac{1}{2}\alpha) dS_i = \Sigma_{i=2}^n \int_{S_{1/R}(a_i)} \partial_{r_i} (U - \frac{1}{2}\alpha) dA_i + \int_{\hat{\mathcal{C}}_R} \Delta_\delta (U - \frac{1}{2}\alpha) dx^3 \tag{4.5.8}$$

where Δ_δ denotes the Euclidean Laplacian, and dA_i is the Euclidean area form on $S_{1/R}(a_i)$. The falloff of $U - \ell\alpha$ is given by Equation (1.0.17) or (1.0.18) depending on whether $S_{1/R}(a_i)$ encloses an AF end or an AC end. In either case, since $\partial_{r_i} f_i(\theta_i) = \partial_{r_i} h_i(\theta_i) = 0$, we have $\partial_{r_i}\alpha = o_{k-5}(r_i^{-\ell})$ and $\partial_{r_i} U = O_{k-5}(r_i^{-1})$. Thus for large R, $\sup\{|\partial_{r_i}(U - \frac{1}{2}\alpha)(p)| : p \in S_{1/R}(a_i)\} \le R$. Since the coordinate area of $S_{1/R}(a_i)$ is $\frac{4\pi}{R^2}$ we can conclude $\lim_{R\to\infty} |\int_{S_{1/R}(a_i)} \partial_{r_i}(U - \frac{1}{2}\alpha) dA_i| \le \lim_{R\to\infty} C\frac{4\pi}{R^2} R = 0$.

To estimate the second term in Equation (4.5.5), we recall that α is independent of ϕ and $\alpha = 0$ along the z axis. Thus we can write

$$\alpha(R,z) = \int_0^R \partial_\rho \alpha(\rho, z) d\rho + \alpha(0,z) = \int_0^R \partial_\rho \alpha(\rho, z). \tag{4.5.9}$$

Note that this equality is not necessarily true for the finite set of z values corresponding to punctures. However that finite set not affect the second integral in Equation (4.5.5), and we have

$$\frac{1}{2} \int_{W_R} \alpha d\phi dz = \frac{1}{2} \int_{\mathcal{C}_R} \partial_\rho \alpha d\rho d\phi dz = \frac{1}{2} \int_{\mathcal{C}_R} \frac{\partial_\rho \alpha}{\rho} dx^3. \tag{4.5.10}$$

Combining this with Equation (4.5.8) we obtain

$$m = \lim_{R\to\infty} \frac{1}{4\pi} \int_{\hat{\mathcal{C}}_R} \Delta_\delta (U - \frac{1}{2}\alpha) dx^3 + \frac{1}{8\pi} \int_{\mathcal{C}_R} \frac{1}{\rho} \partial_\rho \alpha dx^3 \tag{4.5.11}$$

Falloff for α implies

$$\lim_{R\to 0} \int_{B_{1/R}(a_i)} \frac{1}{\rho} \partial_\rho \alpha dx^3 = 0, \tag{4.5.12}$$

so we may write the whole limit as an integral over $\hat{\mathcal{C}}_R$, i.e.

$$m = \lim_{R\to\infty} \frac{1}{4\pi} \int_{\hat{\mathcal{C}}_R} \Delta_\delta (U - \frac{1}{2}\alpha) + \frac{1}{2\rho} \partial_\rho \alpha dx^3 \tag{4.5.13}$$

We now apply the formula for scalar curvature, R^3 in Brill Coordinates. We have

$$R^{(3)} = -\frac{4}{e^{-2U+2\alpha}}(-\Delta_\delta(U - \frac{1}{2}\alpha) + \frac{1}{2}|\nabla U|^2 - \frac{1}{2\rho}\partial_\rho \alpha + \frac{\rho^2 e^{-2\alpha}}{8}(\rho\partial_z B - \partial_\rho A)^2) \tag{4.5.14}$$

Applying this formula, all terms in the integrand become nonnegative, and we can apply the dominated convergent theorem to obtain

$$m = \frac{1}{16\pi}\int_{\mathbb{R}^3}[R^{(3)} + \frac{1}{2}\rho^2 e^{-4\alpha+2U}(\rho\partial_z B - \partial_\rho A)^2]e^{-2U+2\alpha} + 2|\nabla U|^2 dx^3. \tag{4.5.15}$$

Observe that this is precisely Equation 3.9 in [13]. Since $R^{(3)} \geq 0$ by hypothesis we conclude $m \geq 0$. We now demonstrate when m is finite. Note that R^3 is bounded in each AC end. Further $e^{-2U+2\alpha} = O(\frac{1}{r_i^2})$ near each AC end. Thus $R^3 e^{-2U+2\alpha}$ is integrable near each puncture representing an AC end. Similarly for AF ends $R^3 = O(r_i^{2+\ell})$ and $e^{-2U+2\alpha} = O(\frac{1}{r_i^4})$, so $R^{(3)}e^{-2U+2\alpha}$ is integrable near each puncture representing an AF end. Since $e^{-2U+2\alpha} \to 1$ as $r \to \infty$ we have $\int_{\mathbb{R}^3} R^{(3)}e^{-2U+2\alpha}dx^3 < \infty$ if and only if $R^{(3)} \in L^1(\mathbb{R}^3)$.

Now since α is bounded $\int_{\mathbb{R}^3}\rho^2 e^{-2\alpha}(\rho\partial_z B - \partial_\rho A)^2 dx^3 < \infty$ if and only if $\rho(\rho\partial_z B - \partial_\rho A) \in L^2(\mathbb{R}^3)$. We conclude $m < \infty$ if and only if $R^{(3)} \in L^1(\mathbb{R}^3)$, DU, $\rho(A_{\rho,z} - A_{z,\rho}) \in L^2(\mathbb{R}^3)$.

Finally suppose $m = 0$, then arguments identical to those in [13] show that g is flat, and thus by simple connectedness isometric to $\mathbb{R}^3$. Since we assumed g has at least 2 ends, this is impossible, so we must have $m > 0$.

□

4.6 Uniqueness of Brill Type Coordinates

We will discuss the uniqueness of Brill coordinates systems, as well as the pseudospherical and Weyl coordinates used above. In any case we show that the coordinates are essentially unique once one chooses a cross section N for $M/U(1)$.

We begin with the case where (M, g) is an axisymmetric, simply connected Riemannian 3-manifold with one asymptotically flat end. By Theorem 2.7 in [13], we know there exists a global Brill coordinates system for (M, g). We prove the following uniqueness theorem:

Theorem 4.6.1. *Let (M, g) be a simply connected axisymmetric Riemannian manifold which is the union of a compact set and an asymptotically flat end. Let (ρ, z, ϕ) and $(\tilde{\rho}, \tilde{z}, \tilde{\phi})$ be two Brill coordinate systems for M satisfying the conclusions of Theorem 2.7 in [13]. Then*

$$\rho = \tilde{\rho},\ z = \tilde{z} + c,\ \phi = \tilde{\phi} + f(\rho, z),\ U = \tilde{U},\ \alpha = \tilde{\alpha},\ \text{and}\ \partial_z A_\rho + \partial_\rho A_z = \partial_z \tilde{A}_\rho + \partial_\rho \tilde{A}_z, \tag{4.6.1}$$

where c is a constant and f is a smooth axisymmetric function satisfying $f = b + o_1(r^{-\ell})$.

Let (ρ, z, ϕ) and $(\tilde{\rho}, \tilde{z}, \tilde{\phi})$ be two Brill coordinate systems for (M, g). The two coordinate systems can be seen as the result of applying Chruściel's construction of Brill coordinates using two different cross sections, N and $\tilde{N}$, of the orbit space $M/U(1)$; however we do not assume the coordinates were obtained in this way. The Theorem follows from a series of lemmas.

Lemma 4.6.2. *Given the two coordinate systems (ρ, z, ϕ) and $(\tilde{\rho}, \tilde{z}, \tilde{\phi})$ for the manifold (M, g) defined above, we have $\rho = \tilde{\rho}$ and $z = \tilde{z} + c$ for some constant c.*

Proof. Consider the orbit space $M/U(1)$. For the coordinate system (ρ, z, ϕ) we have $\partial_\phi = \eta$, so the hypersurface $\phi = 0$ intersects each orbit of η precisely once. Thus (ρ, z) are global coordinates for $M/U(1)$. By the definition of the quotient metric we have $q = e^{-2U+2\alpha}(d\rho^2 + dz^2)$. Since U and α are axisymmetric functions and g is smooth we have $\partial_\rho^{2k+1}(-2U + 2\alpha)(0, \rho) = 0$ for all k. Thus $(M/U(1), q)$ can be doubled across its boundary and by allowing $\rho \in (-\infty, \infty)$ we obtain a smooth metric on $\mathbb{R}^2$ equal to $q = e^{-2U+2\alpha}(d\rho^2 + dz^2)$. By definition $U(-\rho, z) = U(\rho, z)$ and $\alpha(-\rho, z) = \alpha(\rho, z)$.

Since the quotient metric q is independent of the choice of cross section, we may use the same procedure to obtain $q = e^{-2\tilde{U}+2\tilde{\alpha}}(d\tilde{\rho}^2 + d\tilde{z}^2)$, where we now allow $(\tilde{\rho}, \tilde{z}) \in \mathbb{R}^2$.

Thus (ρ, z) and $(\tilde{\rho}, \tilde{z})$ are both isothermal coordinate systems for q. A standard argument shows that the map $(\rho, z) \to (\tilde{\rho}, \tilde{z})$, satisfies the Cauchy Riemann equations and thus is a conformal bijection from $\mathbb{R}^2$ to $\mathbb{R}^2$. A standard argument using Picard's Theorem implies such a map must be complex linear, so if we write $\zeta = \rho + iz$ and $\tilde{\zeta} = \tilde{\rho} + i\tilde{z}$ then $\tilde{\zeta} = f(\zeta) = a\zeta + b$ for some complex numbers a and b. Since η vanishes precisely where ρ and $\tilde{\rho}$ vanish, f fixes the imaginary axis, $\rho = 0$, so $a \in \mathbb{R}$ and $ib \in \mathbb{R}$. By definition $a = \frac{\partial p_1}{\partial \rho}$ and $0 = \frac{\partial p_1}{\partial z}$. Thus $\partial_\rho = a\partial_{\tilde{\rho}}$.. By our falloff hypotheses $q(\partial_\rho, \partial_\rho)$ and $q(\partial_{\tilde{\rho}}, \partial_{\tilde{\rho}})$ both approach 1 as r and $\tilde{r}$ approach infinity respectively. By the form of f, $f = az + b$, $\tilde{r}$ goes to infinity as r goes to infinity. Thus $|a| = 1$. Since f preserves orientation we conclude that $\tilde{\zeta} = f(\zeta) = \zeta + b$ where $ib \in \mathbb{R}$ and thus $\rho = \tilde{\rho}$ and $z = \tilde{z} + b$ as desired.

□

With this relationship between the coordinates we can now compare the metric parameters.

Lemma 4.6.3. *Let (ρ, z, ϕ) and $(\tilde{\rho}, \tilde{z}, \tilde{\phi})$ be two coordinate systems for a manifold (M, g), in which the metric takes the form of Equation 1.0.10. Further suppose $\rho = \tilde{\rho}$ and $z = \tilde{z} + c$. Then $U(\rho, z) = \tilde{U}(\rho, z + c)$ and $\alpha(\rho, z) = \tilde{\alpha}(\tilde{\rho}, \tilde{z} + c)$*

Proof. From the proof of Lemma 4.6.2 we have $q = e^{-2U+2\alpha}(d\rho^2 + dz^2) = e^{-2\tilde{U}+2\tilde{\alpha}}(d\rho^2 + dz^2)$ and the relationship between (ρ, z) and $(\tilde{\rho}, \tilde{z})$ implies $d\rho^2 = d\tilde{\rho}^2$ and $dz^2 = d\tilde{z}^2$. Thus $-2U + 2\alpha = -2\tilde{U} + 2\tilde{\alpha}$ as functions on $M/U(1)$. Since both sides of the equation are constant on the orbits of η we have $-2U + 2\alpha = -2\tilde{U} + 2\tilde{\alpha}$ globally. We see from the form of g that $g(\eta, \eta) = g(\partial_\phi, \partial_\phi) = \rho^2 e^{-2U}$. However we also have $g(\eta, \eta) = g(\partial_{\tilde{\phi}}, \partial_{\tilde{\phi}}) = \tilde{\rho}^2 e^{-2\tilde{U}}$. Since $g(\eta, \eta)$ is a geometric quantity and $\rho = \tilde{\rho}$ we conclude $U(\rho, z) = \tilde{U}(\tilde{\rho}, \tilde{z} + c)$. Now using $-2U + 2\alpha = -2\tilde{U} + 2\tilde{\alpha}$ we obtain $\alpha(\rho, z) = \tilde{\alpha}(\tilde{\rho}, \tilde{z} + c)$ as desired. □

For future use we note that in general we do not have $\partial_\rho = \partial_{\tilde{\rho}}$ because $\frac{\partial \phi}{\partial \tilde{\rho}} \neq 0$. However most of the functions, f, we encounter are axisymmetic, i.e. they satisfy $\partial_\phi f = 0$. In this case we do in fact have $\partial_\rho f = \partial_{\tilde{\rho}} f$, and $\partial_z f = \partial_{\tilde{z}}$. When dealing with axisymmetric functions we will use ∂_ρ and $\partial_{\tilde{\rho}}$ interchangeably.

Lemma 4.6.3 implies that the only components of the metric that can change significantly are ϕ, A_ρ and A_z. We now put the desired constraints on their behavior.

Lemma 4.6.4. *Let (ρ, z, ϕ) and $(\tilde{\rho}, \tilde{z}, \tilde{\phi})$ be two coordinate systems for a manifold (M, g), in which the metric takes the form of Equation 1.0.10. Further suppose $\rho = \tilde{\rho}$, $z = \tilde{z} + c$, $U(\rho, z) = \tilde{U}(\rho, z + c)$*

and $\alpha(\rho, z) = \tilde{\alpha}(\tilde{\rho}, \tilde{z} + c)$. Then we have $\phi = \tilde{\phi} + f(\rho, z)$ and $\partial_z A_\rho - \partial_\rho A_z = \partial_{\tilde{z}} \tilde{A}_\rho - \partial_\rho \tilde{A}_z$. Here f is a smooth axisymmetric function satisfying $f = c + o_1(r^{-\ell})$.

Proof. Consider the function $\phi - \tilde{\phi}$ on M. Since $\eta = \partial_\phi = \partial_{\tilde{\phi}}$ we have $\partial_\phi(\phi - \tilde{\phi}) = \partial_{\tilde{\phi}}(\phi - \tilde{\phi}) = 0$ so the difference is independent of ϕ and we can write $\phi - \tilde{\phi} = f(\rho, z)$. From the form of the metric we see $A_z = \tilde{A}_z + \partial_z f$ and $A_\rho = \tilde{A}_\rho + \partial_\rho f$. Thus we compute

$$\partial_z A_\rho - \partial_\rho A_z = \partial_z \tilde{A}_\rho + \partial_z \partial_\rho f - \partial_\rho \tilde{A}_z - \partial_z \partial_\rho f = \partial_z \tilde{A}_\rho + \partial_\rho \tilde{A}_z. \tag{4.6.2}$$

Thus we have the desired relationship between $A_z, A_\rho, \tilde{A}_z$, and $\tilde{A}_\rho$. To obtain the desired falloff for f we simply note that the falloffs for A_z and $\tilde{A}_z$ imply that $\partial_z f = o(r^{-\ell-1})$ and $\partial_\rho f = \rho o(r^{\ell-2})$. Integrating these falloffs gives $f = c + o_1(r^{-\ell})$. We know f is independent of ϕ and smooth away from the z axis because the coordinate charts are smooth. Since $\partial_\rho f = 0$ on the $z-$axis we conclude f is globally smooth and axisymmetric. □

Theorem 4.6.1 is a direct consequence of applying Lemmas 4.6.2, 4.6.3, and 4.6.4 to the given coordinates. In addition we can prove what essentially amounts to a converse of Theorem 4.6.1.

Theorem 4.6.5. *Let (M, g) be a simply connected axisymmetric Riemannian manifold which is the union of a compact set and an asymptotically flat end. Let (ρ, z, ϕ) be Brill coordinates with metric functions U, α, A_z, A_ρ. Suppose $\tilde{A}_z, \tilde{A}_\rho$ are axisymmetric functions, smooth away from the axis, and satisfying the compatibility condition $\partial_z A_\rho - \partial_\rho A_z = \partial_{\tilde{z}} \tilde{A}_\rho - \partial_\rho \tilde{A}_z$ as well as the falloff conditions $\tilde{A}_z = o(r^{-\ell-1})$ and $\tilde{A}_\rho = \rho o(r^{\ell-2})$. Then there exist coordinates $(\rho, z, \tilde{\phi})$ where g is in Brill form with metric functions $U, \alpha, \tilde{A}_z, \tilde{A}_\rho$.*

Proof. Define $f_\rho(\rho, z) = A_\rho - \tilde{A}_\rho$, and $f_z(\rho, z) = A_z - \tilde{A}_z$. The equation $\partial_z A_\rho - \partial_\rho A_z = \partial_{\tilde{z}} \tilde{A}_\rho - \partial_\rho \tilde{A}_z$ implies $\partial_z f_\rho = \partial_\rho f_z$, and the falloff conditions for $A_z, A_\rho, \tilde{A}_z$, and $\tilde{A}_\rho$ imply that f_ρ and f_z are integrable on $\mathbb{R}^{\not{2}}_{+}$. Now define $f(\rho, z)$ by first by $f(0,0) = 0$, then $f(\rho, 0) = \int_0^\rho f_\rho(0, s) ds$ and finally

$$f(\rho, z) = \int_0^z f_z(\rho, w) dw + f(\rho, 0). \tag{4.6.3}$$

By the compatibility condition

$$\partial_\rho f(\rho, z) = \int_0^z \partial_\rho f_z(\rho, w) dw + \partial_\rho f(\rho, 0) = \int_0^z \partial_w f_\rho(\rho, w) dw + f_\rho(\rho, 0) = f_\rho(\rho, z), \tag{4.6.4}$$

and clearly $\partial_z f(\rho, z) = f_z(\rho, z)$. Thus if we define $\tilde{\phi} = \phi + f(\rho, z)$ and take $\tilde{\phi}$ modulo 2π it is easily seen that the coordinates $(\rho, z, \tilde{\phi})$ form a Brill coordinate system for (M, g) with metric functions $U, \alpha, \tilde{A}_z, \tilde{A}_\rho$. □

Similarly we may consider (M, g) and $(\tilde{M}, \tilde{g})$, two manifolds with Brill coordinates (ρ, z, ϕ) and $(\tilde{\rho}, \tilde{z}, \tilde{\phi})$ such that $U = \tilde{U}$ and $\alpha = \tilde{\alpha}$ as functions on $\mathbb{R}^+_2$ and such that $A_z, A_\rho, \tilde{A}_z$ and $\tilde{A}_\rho$ satisfy the compatibility condition as functions on $\mathbb{R}^+_2$. The above argument implies that the map defined by $\rho = \tilde{\rho}$, $z = \tilde{z}$ and $\phi = \tilde{\phi} + f$ is an isomorphism between (M, g) and $(\tilde{M}, \tilde{g})$.

Now we by make the necessary modifications to the above arguments in the case where there are multiple asymptotically flat or asymptotically cylindrical ends. We prove the following:

Theorem 4.6.6. *Let* (M, g) *be a simply connected axisymmetric Riemannian manifold which is the union of a compact set and a finite number of asymptotically flat or asymptotically cylindrical ends. Let* (ρ, z, ϕ) *and* $(\tilde{\rho}, \tilde{z}, \tilde{\phi})$ *be two Brill coordinate systems for* M *with coordinate ranges* $(\rho, z) \in \mathbb{R}^2_+ \setminus \{a_i\}_{i=2}^n$ *and* $(\tilde{\rho}, \tilde{z}) \in \mathbb{R}^2_+ \setminus \{b_i\}_{i=2}^n$*. Finally we assume that both coordinates systems have the same end at infinity. In particular we assume that there exists a neighborhood* V_i *of each* a_i*, such that the image of* V_i *under the map* $(\rho, z, \phi) \to (\rho', z', \phi')$ *is a bounded set. Then*

$$\rho = \tilde{\rho},\ z = \tilde{z} + c,\ \phi = \tilde{\phi} + f(\rho, z),\ U = \tilde{U},\ \alpha = \tilde{\alpha},\ \text{and}\ \partial_z A_\rho + \partial_\rho A_z = \partial_z \tilde{A}_\rho + \partial_\rho \tilde{A}_z, \tag{4.6.5}$$

where c *is a constant and* f *is a smooth axisymmetric function satisfying* $f = b + o_1(r^{-\ell})$*.*

Proof. The main step of the proof is again to show that $\rho = \tilde{\rho}$ and $z = \tilde{z} + c$. Once this is established the remaining conclusions follow precisely as in the one end case. We again have the quotient metric defined by $q = e^{-2U+2\alpha}(d\rho^2 + dz^2) = e^{-2\tilde{U}+2\tilde{\alpha}}(d\tilde{\rho}^2 + d\tilde{z}^2)$. By doubling we obtain a smooth metric on $\mathbb{R}^2 \setminus \{a_i\}$ and $\mathbb{R}^2 \setminus \{b_i\}$. Since (ρ, z) and $(\tilde{\rho}, \tilde{z})$ are isothermal coordinates the map $(\rho, z) \to (\tilde{\rho}, \tilde{z})$ is conformal and can be thought of as a complex holomorphic bijection $f : \mathbb{C} \setminus \{a_i\} \to \mathbb{C} \setminus \{b_i\}$. We claim f can be extended to a bijection from $\mathbb{C} \to \mathbb{C}$. First we extend he range and consider $f : \mathbb{C} \setminus \{a_i\} \to \mathbb{C}$. Let a_i be one of the removed points. Then by hypothesis we know f is bounded in a neighborhood of a_i. Thus a_i is a removable singularity of f and we may define $f(a_i) = \lim_{z \to a_i} f(z)$. Applying this procedure to all of the removed points we obtain a holomorphic map $f : \mathbb{C} \to \mathbb{C}$.

To show f is a bijection let $z_0 \notin \{a_i\}$ and suppose $f(z_0) = f(a_1)$. Choose ϵ so small that $B_\epsilon(z_0)$ and $B_\epsilon(a_1)$ are disjoint. Since f is conformal it maps the open balls $B_\epsilon(z_0)$ and $B_\epsilon(a_1)$ to two balls around open balls around $f(z_0)$. Thus

$$f(B_\epsilon(z_0)) \cap f(B_\epsilon(a_1) \setminus \{a_1\}) \neq \emptyset. \tag{4.6.6}$$

This contradicts the fact that f is bijective on $\mathbb{C} \setminus \{a_i\}$. Thus $f(a_1) = b_j$ for some j and filling in one hole we obtain a bijection $f : \mathbb{C} \setminus \{a_i\}_{\{i \neq 1\}} \to \mathbb{C} \setminus \{b_i\}_{\{i \neq j\}}$. Repeating this procedure we conclude f maps $\{a_i\}$ bijectively onto $\{b_i\}$. Thus we obtain a bijection $f : \mathbb{C} \to \mathbb{C}$. Since f fixes the imaginary axis the same argument used in the proof for one end implies $f(\zeta) = \zeta + ic$ where $c \in \mathbb{R}$. Thus $\rho = \tilde{\rho}$, and $z = \tilde{z} + c$ for some constant c as desired. □

We can use similar techniques to analyze the uniqueness of the pseudoshperical and Weyl coordinates. We prove the following:

Theorem 4.6.7. *Let* (M, g) *be a simply connected axisymmetric Riemannian manifold with boundary which is the union of a compact set and an asymptotically flat end. Let* (ρ, z, ϕ) *and* $(\tilde{\rho}, \tilde{z}, \tilde{\phi})$ *be two pseudospherical coordinate systems for* M *with coordinate ranges* $(\rho, z) \in \mathbb{R}^2_+ \setminus \{\rho^2 + z^2 \leq m_1^2/4\}$ *and* $(\rho', z') \in \mathbb{R}^2_+ \setminus \{\rho^2 + z^2 \leq m_2^2/4\}$*. Then* $m_1 = m_2$ *and*

$$\rho = \tilde{\rho},\ z = \tilde{z} + c,\ \phi = \tilde{\phi} + f(\rho, z),\ U = \tilde{U},\ \alpha = \tilde{\alpha},\ \text{and}\ \partial_z A_\rho + \partial_\rho A_z = \partial_z \tilde{A}_\rho + \partial_\rho \tilde{A}_z, \tag{4.6.7}$$

where c *is a constant and* f *is a smooth axisymmetric function satisfying* $f = b + o_1(r^{-\ell})$*.*

Proof. The proof that $m_1 = m_2$ is given in [19]. This proof is essentially complete if we can show $\rho = \tilde{\rho}$ and $z = \tilde{z}$. We again consider the doubled quotient metric q on $\mathbb{R}^2 \setminus \{\rho^2 + z^2 \leq m_1^2/4\}$. The map

$$f : \mathbb{R}^2 \setminus \{\rho^2 + z^2 \leq m_1^2/4\} \to \mathbb{R}^2 \setminus \{\tilde{\rho}^2 + \tilde{z}^2 \leq m_1^2/4\}, \tag{4.6.8}$$

is again conformal bijection. By performing an inversion about the circle or radius $m_1/2$ in both the domain and range we obtain a conformal bijection

$$\tilde{f} : \{\rho^2 + z^2 \leq m_1^2/4\} \setminus \{(0,0)\} \to \{\tilde{\rho}^2 + \tilde{z}^2 \leq m_1^2/4\} \setminus \{(0,0)\}. \tag{4.6.9}$$

Since $\tilde{f}$ is bounded near 0, 0 is a removable singularity and we may define $\tilde{f}(0) = \lim_{z\to 0} \tilde{f}(z)$. As in he multiple end case, the bijectivity of $\tilde{f}$ on $\{\rho^2 + z^2 \leq m_1^2/4\} \setminus \{0\}$ implies that the extension satisfies $\tilde{f}(0) = 0$. Thus we obtain a conformal bijection $\tilde{f} : \{\rho^2 + z^2 \leq m_1^2/4\} \to \{\tilde{\rho}^2 + \tilde{z}^2 \leq m_1^2/4\}$. Thus $\tilde{f}$ is a Mobious transformation. Since $\tilde{f}$ fixes the origin it must be a rotation, and since $\tilde{f}$ fixes the imaginary axis and preserves the orientation it must the identity. Using the equation for the circle inversion we conclude

$$\frac{\rho}{r^2} = \frac{\tilde{\rho}}{\tilde{r}^2} \text{ and } \frac{z}{r^2} = \frac{\tilde{z}}{\tilde{r}^2}. \tag{4.6.10}$$

We can square these equations and sum them to obtain $\frac{\rho^2+z^2}{r^4} = \frac{\tilde{\rho}^2+\tilde{z}^2}{\tilde{r}^4}$ and thus $\frac{1}{r^2} = \frac{1}{\tilde{r}^2}$. Hence we must in fact have $\rho = \tilde{\rho}$ and $z = \tilde{z}$ as desired.

The remaining equalities in the proof follow from Lemmas 4.6.3 and 4.6.4 just as in the proof of Theorem 4.6.1. □

We can use the uniqueness for pseudospherical coordinates to immediately obtain uniqueness of Weyl coordinates.

Theorem 4.6.8. *Let (M, g) be a simply connected axisymmetric Riemannian manifold with boundary which is the union of a compact set and an asymptotically flat end. Let (ρ, z, ϕ) and $(\tilde{\rho}, \tilde{z}, \tilde{\phi})$ be two Weyl coordinate systems for M with coordinate ranges $(\rho, z) \in \mathbb{R}^2_+ \setminus \{\rho = 0, |z| < m_1\}$ and $(\rho', z') \in \mathbb{R}^2_+\{\rho = 0, |z| < m_2\}$. Then $m_1 = m_2$ and*

$$\rho = \tilde{\rho},\ z = \tilde{z} + c,\ \phi = \tilde{\phi} + f(\rho, z),\ U = \tilde{U},\ \alpha = \tilde{\alpha},\ \text{and } \partial_z A_\rho + \partial_\rho A_z = \partial_z \tilde{A}_\rho + \partial_\rho \tilde{A}_z, \tag{4.6.11}$$

where c is a constant and f is a smooth axisymmetric function satisfying $f = b + o_1(r^{-\ell})$.

Proof. In [19], Chruściel defines a rotated Joukovsky transformation given by $\zeta = f(\zeta_S) = \zeta_S - \frac{m_1^2}{4\zeta_S}$ where $\zeta = \rho + iz$ represent the Weyl coordinates and $\zeta_S = \rho_S + iz_S$ represents pseudospherical coordinates. The map f is a holomorphic bijection from $\mathbb{R}^2 \setminus \{\rho_S^2 + z_S^2 \leq m_1^2/4\} \to \mathbb{R}^+_2$. We can perform the inverse map , f^{-1}, to both the (ρ, z) coordinates and the $(\tilde{\rho}, \tilde{z})$ coordinates to obtain two sets of pseudospherical coordinates (ρ_S, z_S) and $(\tilde{\rho}_S, \tilde{z}_S)$. By Theorem 4.6.7 the (ρ_S, z_S) and $(\tilde{\rho}_S, \tilde{z}_S)$ coordinates must coincide. We conclude $\rho = \tilde{\rho}$ and $z = \tilde{z}$ and the remainder of the theorem follows from Lemmas 4.6.3 and 4.6.4.

□

4.7 Killing Vector Estimates

Throughout the body of this paper we take $\overline{g}$ to be of the form $\overline{g} = h(\theta)^2(\frac{1}{r^2}dr^2 + d\theta^2 + f(\theta)^2 d\phi^2)$. For this section we will reparametrize θ and set $\tau = \log r$ so that $\overline{g}$ can be written as $\overline{g} := \tilde{h}(\theta)^2 d\tau^2 + a(d\theta^2 + \tilde{f}(\theta)^2 d\phi^2)$ for some functions $\tilde{h}$ and $\tilde{f}$. As $\tilde{h}$ and $\tilde{f}$ satisfy the same hypotheses as h and f respectively, we will abuse notation and also refer to these functions as h and f. In (τ, θ, ϕ) coordinates the falloff of g becomes $||g - \overline{g}||_{\overline{g}} = o_k(e^{-\ell\tau})$ as $\tau \to \infty$, and in particular in (τ, θ, ϕ) coordinates we have

$$g = h(\theta)^2 d\tau^2 + a(d\theta^2 + f(\theta)^2 d\phi^2) + G_{ij}(\tau, \phi, \theta) dx^i dx^j, \tag{4.7.1}$$

where $G_{ij} = o_k(e^{-\ell\tau})$, $G_{i\phi} = f o_k(e^{-\ell\tau}) f$ and $G_{\phi\phi} = o_k(e^{-\ell\tau}) f^2$ where $i, j \in \{\theta, \tau\}$.[2]

The purpose of this section is to prove Proposition 4.2.4. The main step is to show that if $\eta^i\partial_i$ is a killing vector for g with periodic orbits of period 2π, then after perhaps a change of coordinates on S^2 we have $||\eta^i\partial_i - \partial_\phi||_{\overline{g}} = ||\eta_i dx^i - af^2 d\phi||_{\overline{g}} = o_k(e^{-\ell\tau})$. Before estimating the killing vectors we must prove a few properties of g and $\overline{g}$.

Lemma 4.7.1. *Let $h(\theta)$ be a smooth positive function on S^2 which is independent of ϕ. If h satisfies the equation*

$$hh'' - h'h' = b, \tag{4.7.2}$$

for a constant b, then h is a constant function.

Proof. Suppose Equation (4.7.2) holds. We claim b must equal 0. Since h is a smooth function on S^2 we must have $h'(0) = h'(\pi) = 0$. To show $b = 0$ we split into three cases based on the sign of $h''(0)$.

If $h''(0) = 0$ then since $h'(0) = 0$ Equation (4.7.2) directly implies $b = 0$.

Now suppose $h''(0) > 0$. Then since $h > 0$, Equation (4.7.2) at $\theta = 0$ implies $b > 0$. Further since $h''(0) > 0$ we know $h' > 0$ on some interval $(0, \delta)$. If t is the next zero of h' we must have $h''(t) \leq 0$. We know such a zero exists because $h'(\pi) = 0$. Then Equation (4.7.2) at $\theta = t$ implies $b \leq 0$. This is a contradiction so we cannot have $h''(0) > 0$. The $h''(0) < 0$ case is identical, so we conclude $h''(0) = 0$ and furthermore $b = 0$.

Now since $h > 0$, Equation (4.7.2) with $b = 0$ implies $h'' = \frac{h'h'}{h} \geq 0$. Thus h' is a nondecreasing function. Since $h'(0) = h'(\pi) = 0$ we must have $h' = 0$, so h is a constant function. □

Lemma 4.7.2. *Let g_s be a metric on S^2 of the form $g_s := a^2(d\theta^2 + f(\theta)^2 d\phi^2)$ where (θ, ϕ) are the standard global coordinates. Let η_i be a killing one form for g_s. Then either $\eta_i dx^i = a^2 f^2 d\phi$, or $f = \sin^2\theta$, in which case g_s is the standard round metric on the sphere of radius a.*

Proof. Let η_i be a killing one form for g_s. Then η_i satisfies the following equations,

$$\partial_\phi \eta_\phi + ff'\eta_\theta = 0 \tag{4.7.3}$$

$$\partial_\theta \eta_\phi + \partial_\phi \eta_\theta - 2\frac{f'}{f}\eta_\phi = 0 \tag{4.7.4}$$

[2] Throughout this section we will use $u = o_k(e^{-\ell\tau})$ to mean all derivatives of u up to k^{th} order by vector fields of bounded length are $O(e^{-\ell\tau})$. In particular if $k \geq 1$ then $\partial_\tau u = O(e^{-\ell\tau})$, $\partial_\theta u = O(e^{-\ell\tau})$, $\frac{1}{f}\partial_\phi u = O(e^{-\ell\tau})$.

$$\partial_\theta \eta_\theta = 0 \tag{4.7.5}$$

We can solve this system by taking the θ derivative of Equation (4.7.3) and using Equation (4.7.4) to obtain

$$0 = \partial_\theta \partial_\phi \eta_\phi + (ff')'\eta_\theta = \partial_\phi(-\partial_\phi \eta_\theta + 2\frac{f'}{f}\eta_\phi) + (ff')'\eta_\theta = -\partial_\phi^2 \eta_\theta - 2(f')^2\eta_\theta + (ff')'\eta_\theta. \tag{4.7.6}$$

Thus we have the equation $\partial_\phi^2 \eta_\theta + ((f')^2 - ff'')\eta_\theta = 0$ where we know η_θ is independent of θ and $((f')^2 - ff'')$ is independent of ϕ. Taking the θ derivative gives the equation $((f')^2 - ff'')'\eta_\theta = 0$. Since η_θ is independent of θ we must have either $\eta_\theta = 0$ everywhere or $((f')^2 - ff'')' = 0$.

Suppose $((f')^2 - ff'')' = 0$. This gives us the ODE $(f')^2 - ff'' = b$ for some constant b. Since g_s must be a smooth metric for S^2 we have the initial conditions $f(0) = f''(0) = 0$ and $f'(0) = 1$. Thus $b = 1$, and the general solution to this equation is $f(\theta) = \pm C_1 \sin(C_1(x + C_2))$. Our initial conditions imply $C_2 = 0$, and since $f'(0) = 1$ we conclude that $f = \sin(\theta)$. Thus in this case g_s is the standard round metric. □

Now note the metrics g and $\overline{g}$ have a coordinate singularity at $\theta = 0$ and $\theta = \pi$, and that not all Christoffel symbols are bounded as we approach these singularities. For $\epsilon > 0$ and define $A_\epsilon = \{(\tau, \theta, \phi) : \theta \in [\epsilon, \pi - \epsilon]\}$.

Lemma 4.7.3. *In the coordinates (τ, θ, ϕ), the metric $g = h(\theta)^2 d\tau^2 + a^2(d\theta^2 + f(\theta)^2 d\phi^2) + G_{ij}(\tau, \theta, \phi)dx^i dx^j$ has Christoffel symbols and curvature components which satisfy*

$$\Gamma^i_{jk} = O_{k-1}(1) \tag{4.7.7}$$

.

$$R^i_{jkl} = O_{k-2}(1) \tag{4.7.8}$$

on the set A_ϵ.

Proof. First note that Equation (4.7.8) follows from Equation (4.7.7), and the expression of curvature in coordinates $R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^s_{jk}\Gamma^l_{is} - \Gamma^s_{ik}\Gamma^l_{js}$. Now to prove Equation (4.7.7) we just use the standard expression for the Christoffel symbols $\Gamma^k_{ij} = \frac{1}{2}g^{km}(\partial_i g_{jm} + \partial_j g_{im} - \partial_m g_{ij})$. All derivatives of metric components are $O_{k-1}(1)$, and all metric components of the diagonal metric $\overline{g} = h(\theta)^2 d\tau^2 + a(d\theta^2 + f^2 d\phi^2)$ are bounded away from zero on A_ϵ. Thus $\overline{g}^{ij} = O_k(1)$ on A_ϵ. Further the falloff of g gives us $g^{ij} = \overline{g}^{ij} + o_k(e^{-\ell\tau})$ on A_ϵ. Thus we conclude $\Gamma^k_{ij} = O_{k-1}(1)$ on A_ϵ as desired. □

We can now obtain a weak bound for the components of a killing one forms for g. Now will use an analog of a proof by Chruściel to show the following:

Proposition 4.7.4. *Suppose η_i is a killing one form for the metric $g = h(\theta)^2 d\tau^2 + a(d\theta^2 + f(\theta)^2 d\phi^2) + G_{ij}(\tau, \theta, \phi)dx^i dx^j$ where $G_{ij} = o_k(e^{-\ell\tau})$ for some $k \geq 2$. Then for some constant β we have $\eta_i = o_1(e^{\beta\tau})$ on the set A_ϵ.*

Proof. We will make use of the following equations,

$$\partial_\tau \eta_i = \nabla_\tau \eta_i + \Gamma^k_{\tau i}\eta_k \tag{4.7.9}$$

$$\partial_\tau \nabla_i \eta_j = \nabla_\tau \nabla_i \eta_j + \Gamma^l_{\tau i}\nabla_l \eta_j + \Gamma^l_{\tau j}\nabla_i \eta_l \tag{4.7.10}$$

$$\nabla_i \nabla_j \eta_k = R^m_{ijk}\eta_m \tag{4.7.11}$$

The first two equations hold for any vector field and follow from the expression for the covariant derivative in coordinates. The third equation follows from the killing equation $\nabla_i \eta_j + \nabla_j \eta_i = 0$ and the Bianchi identity. Define $\psi := \Sigma_i {\eta_i}^2 + \Sigma_{i,j}(\nabla_i \eta_j)^2$. We claim that $|\partial_\tau \psi| \leq C\psi$ pointwise. To show this we begin by computing

$$\partial_\tau \psi = 2(\Sigma_i \eta_i(\partial_\tau \eta_i) + \Sigma_{i,j}(\nabla_i \eta_j)(\partial_\tau \nabla_i \eta_j)). \tag{4.7.12}$$

To estimate the first term on the right hand side we apply Equation (4.7.9) and Young's inequality to obtain

$$\Sigma_i \eta_i(\partial_\tau \eta_i) = \Sigma_i \eta_i(\nabla_\tau \eta_i + \Gamma^k_{\tau i}\eta_k) = C(\Sigma_i \eta_i^2 + \Sigma_i(\nabla_\tau \eta_i)^2) \leq C\psi \tag{4.7.13}$$

for some constant C, where we have used the fact that the Christoffel symbols are bounded. Bounding the term $\Sigma_{i,j}(\nabla_i \eta_j)(\partial_\tau \nabla_i \eta_j)$ similarly follows from Equations (4.7.10), (4.7.11) and several applications of Young's inequality. Together these bounds imply $|\partial_\tau \psi| \leq C\psi$ as desired.

Now since $|\partial_\tau \psi| \leq C\psi$, we apply Gronwall's inequality to obtain $|\psi(\tau, \theta, \psi)| \leq C_1(1 + e^{\int_1^\tau C_2 d\tau}) \leq C_1(1 + e^{\beta\tau})$ for some $\beta \geq 0$. Since $\eta_i \leq \sqrt{\psi}$, and $\partial_i \eta_j \leq \sqrt{\psi}$ we have $\eta_i = O_1(e^{\tau\beta/2})$. Thus for $\beta' \geq \frac{\beta}{2}$ we have $\eta_i = o_1(e^{\beta'\tau})$ as desired. □

We will need estimates for the higher order derivatives of η_i. To do this we will use the killing equations of g. We will work on the set A_ϵ so we can assume $g^{ij} = \overline{g}^{ij} + o_k(e^{-\ell\tau})$ and the Christoffel symbols of g satisfy $\Gamma^k_{ij} = \overline{\Gamma}^k_{ij} + o_{k-1}(e^{-\ell\tau})$.

We now prove a general regularity result for killing one forms.

Proposition 4.7.5. *Suppose η_i is a killing one form for a 3-manifold g, and let (x^i) be a coordinate system for g under which the Christoffel symbols of g are $O_{k-1}(1)$. Further suppose that $\eta_i = o_1(e^{\beta\tau})$ where $\beta \geq 0$. Then for all $s \leq k$ we have $\eta_i = o_s(e^{\beta\tau})$.*

Proof. Suppose $\eta_i = o_1(e^{\beta\tau})$. Then by the hypothesis on the Christoffel symbols we have $\Gamma^k_{ij}\eta_k = o_1(e^{\beta\tau})$. Thus the killing equations, $\partial_i \eta_j + \partial_j \eta_i = 2\Gamma^k_{ij}\eta_k$, can be written as

$$\partial_i \eta_j + \partial_j \eta_i = o_1(e^{\beta\tau}) \tag{4.7.14}$$

$$\partial_\tau \eta_\tau = o_1(e^{\beta\tau}) \tag{4.7.15}$$

$$\partial_\phi \eta_\tau + \partial_\tau \eta_\phi = o_1(e^{\beta\tau}) \tag{4.7.16}$$

$$\partial_\theta \eta_\tau + \partial_\tau \eta_\theta = o_1(e^{\beta\tau}) \tag{4.7.17}$$

$$\partial_\phi \eta_\phi = o_1(e^{\beta\tau}) \tag{4.7.18}$$

$$\partial_\theta \eta_\phi + \partial_\phi \eta_\theta = o_1(e^{\beta\tau}) \tag{4.7.19}$$

$$\partial_\theta \eta_\theta = o_1(e^{\beta\tau}) \tag{4.7.20}$$

We first establish that $\eta_i = o_2(e^{\beta\tau})$. First setting $i = j$ in Equation (4.7.14) implies $\partial_m\partial_i\eta_i = O(e^{\beta\tau})$. Since we may commute partials it remains to estimate $\partial_i\partial_j\eta_k$ when i, j, and k are all distinct. Taking the θ derivative of Equation (4.7.14) when $i = \phi, j = \tau$, the ϕ derivative when $i = \theta, j = \tau$, and the τ derivative when $i = \theta, j = \phi$ gives

$$\partial_\theta\partial_\phi\eta_\tau + \partial_\theta\partial_\tau\eta_\phi = O(e^{\beta\tau}) \tag{4.7.21}$$

$$\partial_\phi\partial_\theta\eta_\tau + \partial_\phi\partial_\tau\eta_\theta = O(e^{\beta\tau}) \tag{4.7.22}$$

$$\partial_\tau\partial_\theta\eta_\phi + \partial_\tau\partial_\phi\eta_\theta = O(e^{\beta\tau}) \tag{4.7.23}$$

Solving this system and using the fact that partial derivatives commute gives $\partial_i\partial_j\eta_k = O(e^{\beta\tau})$ as desired.

We prove higher order estimates by induction on s. Suppose $\eta_i = o_s(e^{\beta\tau})$ for some $2 \le s \le k-1$. Now the right hand side of Equation (4.7.14) can be replaced by $o_s(e^{\beta\tau})$.

We will use this to estimate the derivatives of the components of order $s+1$. Again setting $i = j$ we have $\partial^\zeta\partial_i\eta_i = O(e^{\beta\tau})$ for any multi-index ζ with $|\zeta| \le s$.

Thus for η_τ it remains to estimate $\partial^\zeta\eta_\tau$ when ζ is a multi-index of the form $(0, p, q)$ where $p + q = s + 1$. Since $s + 1 \ge 3$ either $p \ge 2$ or $q \ge 2$.

If $p \ge 2$ then let $\gamma = \zeta - (0, 1, 0)$, and take ∂^γ of Equation (4.7.14) with $i = \phi, j = \tau$ to obtain $\partial^\zeta\eta_\tau = O(e^{\beta\tau}) - \partial^\zeta\partial_\tau\eta_\phi$. Since $p - 1 \ne 0$, our estimate for $\partial^\zeta\partial_\phi\eta_\phi$ implies $\partial^\gamma\partial_\tau\eta_\phi = O(e^{\beta\tau})$ so we have $\partial^\zeta\eta_\tau = O(e^{\beta\tau})$. If $q \ge 2$, an analogous argument using Equation (4.7.14) with $i = \theta, j = \tau$ implies $\partial^\zeta\eta_\tau = O(e^{\beta\tau})$. We conclude $\eta_\tau = o_{s+1}(e^{\beta\tau})$. An analogous argument implies $\eta_\phi = o_{s+1}(e^{\beta\tau})$ and $\eta_\theta = o_{s+1}(e^{\beta\tau})$. By induction we conclude that for any $s \le k$ we have $\eta_i = o_s(e^{\beta\tau})$ as desired. □

We can apply the above Proposition to conclude that $\eta_i = o_k(e^{\beta\tau})$ on A_ϵ. Our next task is to reduce the exponent β. Recall that on the set A_ϵ we have $g^{ij} = \overline{g}^{ij} + o_k(e^{-\ell\tau})$ and the Christoffel symbols of g satisfy $\Gamma^k_{ij} = \overline{\Gamma}^k_{ij} + o_{k-1}(e^{-\ell\tau})$.

Note that the only nonzero Christoffel symbols for $\overline{g}$ are given by $\overline{\Gamma}^\tau_{\tau\theta} = \overline{\Gamma}^\tau_{\theta\tau} = \frac{h'}{h}$, $\overline{\Gamma}^\theta_{\tau\tau} = -hh'$, $\overline{\Gamma}^\phi_{\phi\theta} = \overline{\Gamma}^\phi_{\theta\phi} = \frac{f'}{f}$ and $\overline{\Gamma}^\theta_{\phi\phi} = -ff'$ where primes represent derivatives with respect to θ. Since $\Gamma^k_{ij} = \overline{\Gamma}^k_{ij} + o_{k-1}(e^{-\ell\tau})$ we have the following killing equations

$$\partial_\tau\eta_\tau + hh'\eta_\theta = o_s(e^{(\beta-\ell)\tau}) \tag{4.7.24}$$

$$\partial_\phi\eta_\tau + \partial_\tau\eta_\phi = o_s(e^{(\beta-\ell)\tau}) \tag{4.7.25}$$

$$\partial_\theta\eta_\tau + \partial_\tau\eta_\theta - 2\frac{h'}{h}\eta_\tau = o_s(e^{(\beta-\ell)\tau}) \tag{4.7.26}$$

$$\partial_\phi\eta_\phi + ff'\eta_\theta = o_s(e^{(\beta-\ell)\tau}) \tag{4.7.27}$$

$$\partial_\theta\eta_\phi + \partial_\phi\eta_\theta - 2\frac{f'}{f}\eta_\phi = o_s(e^{(\beta-\ell)\tau}) \tag{4.7.28}$$

$$\partial_\theta \eta_\theta = o_s(e^{(\beta-\ell)\tau}) \tag{4.7.29}$$

In order to get the improved estimate we will show $\eta_i = O(e^{(\beta-\ell)\tau})$ and $\partial_i \eta_j = O(e^{(\beta-\ell)\tau})$ for all i and j. Then we may apply Proposition 4.7.5 to obtain higher order estimates. To begin we prove the following Lemma:

Lemma 4.7.6. *Suppose $s \geq 2$ and $\beta \in \mathbb{R}$. Then Equations* (4.7.24) (4.7.26) *and* (4.7.29) *imply that either h is constant, or $\eta_\theta = o_{s-1}(e^{(\beta-\ell)\tau})$.*

Proof. Taking the θ derivative of Equation (4.7.24) and applying Equation (4.7.26) we have

$$o_{s-1}(e^{(\beta-\ell)\tau}) = \partial_\theta \partial_\tau \eta_\tau + \partial_\theta(hh'\eta_\theta) = \partial_\tau(-\partial_\tau \eta_\theta + 2\frac{h'}{h}\eta_\tau + o_s(e^{(\beta-\ell)\tau})) + (hh')'\eta_\theta + hh'\partial_\theta\eta_\theta \tag{4.7.30}$$

Simplifying the right hand side an absorbing terms of order $o_{s-1}(e^{(\beta-\ell)\tau})$ we have

$$o_{s-1}(e^{(\beta-\ell)\tau}) = -\partial_\tau^2 \eta_\theta + 2\frac{h'}{h}\partial_\tau \eta_\tau + (hh')'\eta_\theta \tag{4.7.31}$$

Now using Equation (4.7.24) to replace $\partial_\tau \eta_\tau$ this becomes

$$o_{s-1}(e^{(\beta-\ell)\tau}) = -\partial_\tau^2 \eta_\theta - 2(h')^2\eta_\theta + (hh')'\eta_\theta = -\partial_\tau^2\eta_\theta + (hh'' - h'h')\eta_\theta \tag{4.7.32}$$

Taking one more θ derivative and again using that $\partial_\theta \eta_\theta = o_s(e^{(\beta-\ell)\tau})$ we can conclude

$$(hh'' - h'h')'\eta_\theta = o_{s-2}(e^{(\beta-\ell)\tau}) \tag{4.7.33}$$

Now since $(hh'' - h'h')'$ is independent of τ we must have either $(hh'' - h'h')' = 0$, or $\eta_\theta = o_{s-2}(e^{(\beta-\ell)\tau})$. If $(hh'' - h'h')' = 0$, then by Lemma 4.7.1 we must have h constant. Thus either h is constant or $\eta_\theta = o_{s-2}(e^{(\beta-\ell)\tau})$ as desired. □

We can now prove the following reduction of β.

Proposition 4.7.7. *Let η_i be a killing one form for the metric $g = h(\theta)^2 d\tau^2 + a(f(\theta)^2 d\phi^2 + d\theta^2) + G_{ij}(\tau, \phi, \theta)dx^i dx^j$, with $G_{ij} = o_k(e^{-\ell\tau})$ for some $k \geq 4$. Further assume that on A_ϵ, $\eta_i = o_s(e^{\beta\tau})$ for some constant $\beta > 1 - \ell$, and $4 \leq s \leq k$. Then we can improve the estimate to $\eta_i = o_s(e^{(\beta-\ell)\tau})$ on A_ϵ.*

Proof. As discussed above we must show only $\eta_i = O(e^{(\beta-\ell)\tau})$ and $\partial_i\eta_j = O(e^{(\beta-\ell)\tau})$ for all i and j. By Lemma 4.7.6 we know h is constant or $\eta_\theta = o_{s-2}(e^{(\beta-\ell)\tau})$. In either case Equation (4.7.24) implies $\partial_\tau \eta_\tau = o_{s-2}(e^{(\beta-\ell)\tau})$. We can integrate this quantity with respect to τ to obtain $\eta_\tau = c(\phi, \theta) + o_{s-2}(e^{(\beta-\ell)\tau})$ for some differentiable function c on S^2 which can be taken to be $c(\phi,\theta) = \eta_\tau(1, \phi, \theta)$. Since c is a smooth function on the sphere, c and all of its derivatives are bounded. Since $\beta > 1 - \ell$, $c = O(1) \leq O(e^{(\beta-\ell)\tau})$, and $\partial_i c = O(1) \leq O(e^{(\beta-\ell)\tau})$. Hence $\eta_\tau = o_{s-2}(e^{(\beta-\ell)\tau})$, so we have the desired estimates for η_τ and all of its derivatives.

Now, using Equation (4.7.25), we can write $\partial_\tau \eta_\phi$ as a difference of functions which are $o_{s-3}(e^{(\beta-\ell)\tau})$, so we have $\partial_\tau\eta_\phi = o_{s-3}(e^{(\beta-\ell)\tau})$. If we integrate this estimate with respect to τ we obtain $\eta_\phi = c(\phi,\theta) + o_{s-3}(e^{(\beta-\ell)\tau})$, and again since $\beta - \ell > 0$ we have $\eta_\phi = o_{s-3}(e^{(\beta-\ell)\tau})$. Since $s \geq 3$ by hypothesis, we have the desired estimate on η_ϕ and all of its derivatives.

Similarly Equation (4.7.26) and our estimate $\eta_\tau = o_{s-2}(e^{(\beta-\ell)\tau})$ imply that $\partial_\tau \eta_\theta = o_{s-3}(e^{(\beta-\ell)\tau})$, and integration with respect to τ gives $\eta_\theta = o_{s-3}(e^{(\beta-\ell)\tau})$. We conclude $\eta_i = o_1(e^{(\beta-\ell)\tau})$ for all i. Applying Proposition 4.7.5 we conclude $\eta_i = o_s(e^{(\beta-\ell)\tau})$ as desired. □

Note that if we apply the above process starting with $\eta_i = o_s(e^{\beta\tau})$ for $\beta \leq 1-\ell$ be obtain η_i and all of its derivatives are uniformly bounded. Furthermore Lemma 4.7.6 applies for any value of β and hence either h is constant, or $\eta_\theta = o_{k-2}(e^{-\ell\tau})$. Thus iterating Proposition 4.7.7, we obtain the following Corollary

Corollary 4.7.8. *Let η_i be a killing one form for the metric $g = h(\theta)^2 d\tau^2 + a(f(\theta)^2 d\phi^2 + d\theta^2) + G(\tau,\phi,\theta)$, with $G_{ij} = o_k(e^{-\ell\tau})$ for some $k \geq 5$. Then on the set A_ϵ, $\eta_i = O_k(1)$ where the subscript k denotes that the derivatives up to order k are also $O(1)$. Further either h is constant or $\eta_\theta = o_{k-2}(e^{-\ell\tau})$ on A_ϵ.*

In order to show that our killing one form converges to a killing one form on the sphere as $\tau \to \infty$, we must obtain a better estimate on the τ derivative of the components.

Proposition 4.7.9. *Let η_i be a killing one form for $g = h(\theta)^2 d\tau^2 + a(f(\theta)^2 d\phi^2 + d\theta^2) + G_{ij}(\tau,\phi,\theta)dx^i dx^j$, with $G_{ij} = o_k(e^{-\ell\tau})$ for some $k \geq 6$. Then on the set A_ϵ we have $\partial_\tau \eta_i = o_{k-6}(e^{-\ell\tau})$.*

Proof. We know $\eta_i = O_k(1)$ by Corollary 4.7.8, so we have the following killing equations

$$\partial_\tau \eta_\tau + hh'\eta_\theta = o_{k-1}(e^{-\ell\tau}) \tag{4.7.34}$$

$$\partial_\phi \eta_\tau + \partial_\tau \eta_\phi = o_{k-1}(e^{-\ell\tau}) \tag{4.7.35}$$

$$\partial_\theta \eta_\tau + \partial_\tau \eta_\theta - 2\frac{h'}{h}\eta_\tau = o_{k-1}(e^{-\ell\tau}) \tag{4.7.36}$$

$$\partial_\phi \eta_\phi + ff'\eta_\theta = o_{k-1}(e^{-\ell\tau}) \tag{4.7.37}$$

$$\partial_\theta \eta_\phi + \partial_\phi \eta_\theta - 2\frac{f'}{f}\eta_\phi = o_{k-1}(e^{-\ell\tau}) \tag{4.7.38}$$

$$\partial_\theta \eta_\theta = o_{k-1}(e^{-\ell\tau}) \tag{4.7.39}$$

Our goal now is to show that this system of equations implies $\partial_\tau \eta_i = o_{k-6}(e^{-\ell\tau})$. Lemma 4.7.6 applies as above and implies that either $\eta_\theta = o_{k-3}(e^{-\ell\tau})$ or $h = 0$, so will will treat these two cases.

Suppose $\eta_\theta = o_{k-3}(e^{-\ell\tau})$. Then immediately the above equations become

$$\partial_\tau \eta_\tau = o_{k-3}(e^{-\ell\tau}) \tag{4.7.40}$$

$$\partial_\phi \eta_\tau + \partial_\tau \eta_\phi = o_{k-1}(e^{-\ell\tau}) \tag{4.7.41}$$

$$\partial_\theta \eta_\tau - 2\frac{h'}{h}\eta_\tau = o_{k-4}(e^{-\ell\tau}) \tag{4.7.42}$$

$$\partial_\phi \eta_\phi = o_{k-1}(e^{-\ell\tau}) \tag{4.7.43}$$

$$\partial_\theta \eta_\phi - 2\frac{f'}{f}\eta_\phi = o_{k-4}(e^{-\ell\tau}) \tag{4.7.44}$$

Note that Equation (4.7.42) is equivalent to

$$h^2\partial_\theta(\frac{\eta_\tau}{h^2}) = o_{k-4}(e^{-\ell\tau}). \tag{4.7.45}$$

Dividing by h^2, integrating with respect to θ, and multiplying by h^2 we have $\eta_\tau = o_{k-4}(e^{-\ell\tau}) + h^2B(\tau,\phi)$. Further we can take B independent of τ because $\partial_\tau\eta_\tau = o_{k-3}(e^{-\ell\tau})$.

Similarly Equation (4.7.44) implies $\eta_\phi = o_{k-4}(e^{\frac{-\tau}{2}}) + f^2C(\tau)$. We now plug these equations for η_τ and η_ϕ into Equation (4.7.41) to obtain

$$h^2\partial_\phi B(\phi) + f^2\partial_\tau C(\tau) = o_{k-5}(e^{-\ell\tau}) \tag{4.7.46}$$

Now the first term on the left hand side is independent of τ, and the second term is independent of ϕ. Thus taking the τ derivative yields

$$\partial_\tau\partial_\tau C_\tau(\tau) = o_{k-6}(e^{-\ell\tau}) \tag{4.7.47}$$

Hence we can write $\partial_\tau C(\tau) = c + o_{k-6}(e^{-\ell\tau})$ for some constant c. In the same way $\partial_\phi B(\phi) = b + o_{k-6}(e^{-\ell\tau})$ for some constant b. Returning to Equation (4.7.46) we obtain

$$h^2b + f^2c = o_{k-6}(e^{-\ell\tau}). \tag{4.7.48}$$

Since h and f are independent of τ this implies

$$h^2b + f^2c = 0. \tag{4.7.49}$$

Now since $\lim_{\theta\to 0} f = 0$ and $\lim_{\theta\to 0} h \neq 0$, by taking ϵ small enough, we must have $b = 0$. Since f is not identically zero we must also have $c = 0$. Hence $\partial_\tau\eta_\phi = o_{k-5}(e^{-\ell\tau}) + f^2\partial_\tau D(\tau) = o_{k-6}(e^{\frac{-\tau}{2}})$. Thus we have the desired estimate on the τ derivative of all components.

Now suppose instead we have $h = 0$. Then we have the following killing equations

$$\partial_\tau\eta_\tau = o_{k-1}(e^{-\ell\tau}) \tag{4.7.50}$$

$$\partial_\phi\eta_\tau + \partial_\tau\eta_\phi = o_{k-1}(e^{-\ell\tau}) \tag{4.7.51}$$

$$\partial_\theta\eta_\tau + \partial_\tau\eta_\theta = o_{k-1}(e^{-\ell\tau}) \tag{4.7.52}$$

$$\partial_\phi\eta_\phi + ff'\eta_\theta = o_{k-1}(e^{-\ell\tau}) \tag{4.7.53}$$

$$\partial_\theta\eta_\phi + \partial_\phi\eta_\theta - 2\frac{f'}{f}\eta_\phi = o_{k-1}(e^{-\ell\tau}) \tag{4.7.54}$$

$$\partial_\theta\eta_\theta = o_{k-1}(e^{-\ell\tau}) \tag{4.7.55}$$

First, taking the θ derivative of 4.7.52 we obtain

$$\partial_\theta^2\eta_\tau + \partial_\theta\partial_\tau\eta_\theta = o_{k-2}(e^{-\ell\tau}) \tag{4.7.56}$$

Commuting ∂_θ and ∂_τ and using (4.7.39) this implies

$$\partial_\theta^2\eta_\tau = o_{k-2}(e^{-\ell\tau}). \tag{4.7.57}$$

Similarly by taking the τ derivative of (4.7.52), commuting partials, and applying (4.7.55) we have

$$\partial_\tau^2 \eta_\theta = o_{k-2}(e^{-\ell\tau}). \tag{4.7.58}$$

Taking the τ derivative of (4.7.51), commuting partials and applying (4.7.50), we have

$$\partial_\tau^2 \eta_\phi = o_{k-2}(e^{-\ell\tau}). \tag{4.7.59}$$

Taking the τ derivative of (4.7.53) we obtain $\partial_\tau\partial_\phi\eta_\phi + ff'\partial_\tau\eta_\theta = 0$. Commuting partial derivatives and using (4.7.41) we can write the first term as $-\partial_\phi\partial_\phi\eta_\tau$, so we have

$$-\partial_\phi\partial_\phi\eta_\tau + ff'\partial_\tau\eta_\theta = o_{k-2}(e^{-\ell\tau}). \tag{4.7.60}$$

Taking two θ derivatives of the equation we note that $-\partial_\theta\partial_\theta\partial_\phi\partial_\phi\eta_\tau = \partial_\phi\partial_\phi\partial_\theta\partial_\theta\eta_\tau = o_{k-4}(e^{-\ell\tau})$ by Equation (4.7.57). Further $\partial_\theta\partial_\tau\eta_\theta = \partial_\tau\partial_\theta\eta_\theta = o_{k-2}(e^{-\ell\tau})$ by Equation (4.7.55). Thus we have

$$(ff')''\partial_\tau\eta_\theta = o_{k-4}(e^{-\ell\tau}) \tag{4.7.61}$$

Now since $\partial_\theta\partial_\tau\eta_\theta = o_{k-2}(e^{-\ell\tau})$ we must either have $\partial_\tau\eta_\theta = o_{k-4}(e^{-\ell\tau})$ or $(ff')'' = 0$ everywhere.

Suppose to the contrary $(ff')'' = 0$. Then $ff' = a\theta + b$ for some $a, b \in \mathbb{R}$. However, since g_s is a smooth metric on S^2 we have $f(0) = 0$ which implies $b = 0$ and $f(\pi) = 0$ which implies $a = 0$. Thus $ff' = 0$. Now f must be positive on $(0, 2\pi)$, so $f' = 0$ on $(0, 2\pi)$. Thus f is constant on $(0, 2\pi)$ and by continuity $f(0) = 0$ implies $f = 0$. This is a contradiction and we conclude

$$\partial_\tau\eta_\theta = o_{k-4}(e^{-\ell\tau}). \tag{4.7.62}$$

By (4.7.52) this immediately gives

$$\partial_\theta\eta_\tau = o_{k-4}(e^{-\ell\tau}) \tag{4.7.63}$$

If we take the θ derivative of (4.7.51) we have

$$\partial_\theta\partial_\tau\eta_\phi = o_{k-5}(e^{-\ell\tau}) \tag{4.7.64}$$

Finally if we take the τ derivative of (4.7.38) we have

$$2\frac{f'}{f}\partial_\tau\eta_\phi = o_{k-5}(e^{-\ell\tau}) \tag{4.7.65}$$

Since $\frac{f'}{f}$ is independent of τ, we must have $\frac{f'}{f} = 0$ everywhere or $\partial_\tau\eta_\phi = o_{k-5}(e^{-\ell\tau})$. We know $f'(0) = 1$ so, again assuming ϵ is small enough, we must have $\partial_\tau\eta_\phi = o_{k-5}(e^{-\ell\tau})$. We thus have the desired estimates on τ in all cases. □

We can now obtain the desired convergence result.

Proposition 4.7.10. *Let η_i be a killing one form for $g = h(\theta)^2 d\tau^2 + a^2(f(\theta)^2 d\phi^2 + d\theta^2) + G_{ij}(\tau, \phi, \theta)dx^i dx^j$, with $G_{ij} = o_k(e^{-\ell\tau})$ for some $k \geq 7$. Further suppose η has periodic orbits with period 2π. Then by changing coordinates on S^2 if necessary $\eta_i dx^i = a^2 f^2 d\phi + o_k(e^{-\ell\tau})$ on A_ϵ.*

Proof. Since $\partial_\tau \eta_i = o_{k-6}(e^{-\ell\tau})$ on A_ϵ, the functions $\eta_i(\tau,\phi,\theta)$ converge uniformly in τ to functions $Y_i(\phi,\theta)$ on $S^2 \cap A_\epsilon$. Further, since $k \geq 7$ we have $\partial_\tau\partial_j\eta_i = O(e^{-\ell\tau})$, so $\partial_j\eta_i$ converges uniformly in τ to a function on $S^2 \cap A_\epsilon$. Since convergence is uniform we can interchange the limit and differentiation and obtain

$$\lim_{\tau\to\infty} \partial_j\eta_i = \partial_j Y_i \tag{4.7.66}$$

for $j \in \{\theta,\phi\}$. Further letting ϵ go to zero we can define Y_i on S^2 except at $\theta = 0$ and $\theta = \pi$, and Equation (4.7.66) still holds. Equation (4.7.66), along with Equations (4.7.37), (4.7.38), and (4.7.39) imply that Y_i is a killing one form for g_s. Since Y_i defined except at isolated points, it can be completed to a global killing field for g_s. By Lemma 4.7.2, either $Y_\theta = 0$ or g_s is a standard round metric. Furthermore Corollary 4.7.8 implies that either $Y_\theta = 0$ or h is constant. Thus the only case when $Y_\theta \neq 0$ occurs when $\overline{g} = d\tau^2 + a(d\theta^2 + \sin^2\theta d\phi^2)$. In this case we can make a coordinate change on S^2 so that $Y_\theta = 0$. Thus we may assume without loss of generality that Y_θ=0. The killing equations for S^2, and the fact that the orbits of η have period 2π imply $Y_i dx^i = a^2 f^2 d\phi$. Our task is now to show $\eta_i dx^i = af^2 d\phi^2 + o_s(e^{-\ell\tau})dx^i$ on A_ϵ for all $s \leq k$. This is done by induction on s and is clearly equivalent to showing $\eta_i - Y_i = o_s(e^{-\ell\tau})$. The case $s = 1$ follows from the fact that $\partial_\tau\eta_i = O(e^{-\ell\tau})$ and $\partial_\tau\partial_j\eta_i = O(e^{-\ell\tau})$. Now suppose that for $2 \leq s \leq k$ we have $\eta_i - Y_i = o_{s-1}(e^{-\ell\tau})$. To show the next order estimate we will show $\partial_k\partial_j(\eta_i - Y_i) = o_{s-2}(e^{-\ell\tau})$. We take the difference of the killing equations for η_i and the killing equations for Y_i to obtain

$$\partial_\tau(\eta_\tau - Y_\tau) + hh'(\eta_\theta - Y_\theta) = o_{s-1}(e^{-\ell\tau}) \tag{4.7.67}$$

$$\partial_\phi(\eta_\tau - Y_\tau) + \partial_\tau(\eta_\phi - Y_\phi) = o_{s-1}(e^{-\ell\tau}) \tag{4.7.68}$$

$$\partial_\theta(\eta_\tau - Y_\tau) + \partial_\tau(\eta_\theta - Y_\theta)) - 2\frac{h'}{h}(\eta_\phi - Y_\phi) = o_{s-1}(e^{-\ell\tau}) \tag{4.7.69}$$

$$\partial_\phi(\eta_\phi - Y_\phi) + ff'(\eta_\theta - Y_\theta) = o_{s-1}(e^{-\ell\tau}) \tag{4.7.70}$$

$$\partial_\theta(\eta_\phi - Y_\phi) + \partial_\phi(\eta_\theta - Y_\theta) - 2\frac{f'}{f}(\eta_\phi - Y_\phi) = o_{s-1}(e^{-\ell\tau}) \tag{4.7.71}$$

$$\partial_\theta(\eta_\theta - Y_\theta) = o_{s-1}(e^{-\ell\tau}). \tag{4.7.72}$$

The induction hypothesis implies that the zeroth order terms on the left hand side can be absorbed into the right hand side. Thus every equation is of the form $\partial_j(\eta_i - Y_i) + \partial_i(\eta_j - Y_j) = o_{s-1}(e^{-\ell\tau})$. As in the proof of Proposition 4.7.5 we can differentiate and solve this system to obtain $\partial_k\partial_j(\eta_i - Y_i) = o_{s-1}(e^{-\ell\tau})$. By induction we conclude $\eta_i dx^i = af^2 d\phi^2 + o_k(e^{-\ell\tau})$ on A_ϵ as desired. □

The above Proposition implies $|\eta^i\partial_i - Y^i\partial_i|_{\overline{g}} = |\eta_i dx^i - Y_i dx^i|_{\overline{g}} = O_k(e^{-\ell\tau})$ on A_ϵ for any ϵ. By changing coordinates on S^2 so there is no singularity at $\theta = 0$ and $\theta = \pi$ and using the fact that $|g - \overline{g}| = O_k(e^{-\ell\tau})$ if follows that $|\eta^i\partial_i - Y^i\partial_i|_{\overline{g}} = |\eta_i dx^i - Y_i dx^i|_{\overline{g}} = O_k(e^{-\ell\tau})$ globally. This can be seen using a technique similar to that of Proposition 4.7.4.

Proposition 4.7.11. *Let η_i be a killing one form for a metric g satisfying $||g - \overline{g}||_{\overline{g}} = O_k(e^{-\ell})$, for some $k \geq 7$. Further suppose η has periodic orbits with period 2π. Then by changing coordinates on S^2 if necessary $||\eta^i - \partial_\phi||_{\overline{g}} = O(e^{-\ell})$.*

Proof. Given a point (θ, ϕ) on S^2, Proposition 4.7.10 implies that $||\eta^i(\theta, \phi) - \partial_\phi(\theta, \phi)||_{\overline{g}} = O(e^{-\ell})$ as $\tau \to \infty$, except possibly at the points $\theta = 0$ and $\theta = \pi$. By compactness, the global estimate $||\eta^i - \partial_\phi||_{\overline{g}} = O(e^{-\ell})$ follows if we can show the pointwise estimate, $||\eta^i(\theta, \phi) - \partial_\phi(\theta, \phi)||_{\overline{g}} = O(e^{-\ell})$, also holds for $\theta = 0$ and $\theta = \pi$. We will show the case $\theta = 0$.

Let (x, y) be geodesic coordinates for g_s about the point $\theta = 0$. We will use (τ, x, y) as our coordinate system about the line $\theta = 0$ in $\mathbb{R} \times S^2$. We will again use Γ^i_{jk} and $\overline{\Gamma}^i_{jk}$ as the Christoffel symbols for g and $\overline{g}$ respectively. In this coordinate system the Christoffel symbols are clearly bounded near $\theta = 0$ and the falloff $||g - \overline{g}||_{\overline{g}} = O_k(e^{-\ell})$ implies $\Gamma^i_{jk} - \overline{\Gamma}^i_{jk} = o_{k-1}(e^-\ell)$. Let $Y_i dx^i$ represent the one form dual to ∂_ϕ in these coordinates. By the pointwise estimate obtained above, we know $\eta_i(x, y) - Y_i(x, y) = O_k(e^{-\ell\tau})$ except possibly at the origin.

Consider a point $(\tau, x, 0)$ for some $x > 0$. We have $\eta_i(x, 0) - Y_i(x, 0) = o_k(e^{-\ell\tau})$. As in the proof of Proposition 4.7.4 define

$\psi := \Sigma_i(\eta_i - Y_i)^2 + \Sigma_{i,j}(\nabla_i(\eta_j - Y_j)^2)$. We claim that $|\partial_x\psi| \leq C\psi + D$ where C is a constant and $D = o_k(e^{-\ell\tau})$. To show this we begin by computing

$$\partial_\tau\psi = 2(\Sigma_i\eta_i(\partial_\tau\eta_i) + \Sigma_{i,j}(\nabla_i\eta_j)(\partial_\tau\nabla_i\eta_j)). \tag{4.7.73}$$

Now a modified version of Gronwalls inequality implies that $\eta_i(0, 0) - Y_i(0, 0) \leq \eta_i(x, 0) * f(x)$ and similarly $\partial_j(\eta_i(0, 0) - Y_i(0, 0)) \leq \eta_i(x, 0) * f(x)$. Since x is fixed we have $\eta_i(0, 0) - Y_i(0, 0) = o_1(e^{\ell\tau})$. The techniques of Proposition 4.7.5 allow us to bootstrap to obtain $\eta_i(0, 0) - Y_i(0, 0) = o_k(e^{\ell\tau})$. Since the coordinate vector fields in (τ, x, y) coordinates have length bounded away from zero we conclude $|\eta_i - Y_i|_g = o_k(e^{-\ell\tau})$ globally.

□

We are now ready to return to the (r, θ, ϕ) coordinates used in the body of the paper. Note that in (r, θ, ϕ) coordinates we still have $Y^i\partial_i = \partial_\phi$, and thus the above estimate translates to $|\eta^i\partial_i - Y^i\partial_i|_{\overline{g}} = o_k(r^{-\ell})$. We can now prove Proposition 4.2.4. We will use (x, y, z) to denote the standard Cartesian coordinates corresponding to the spherical coordinates (r, θ, ϕ).

Proof of Proposition 4.2.4. This is primarily an exercise in changing coordinates. The above arguments imply that we may assume $|\eta - \partial_\phi|_{\overline{g}} = o_k(r^{-\ell})$.

Now using the fact that $\frac{1}{r^2}dr^2 + d\theta^2 + \sin^2\theta d\phi^2 = \frac{1}{r^2}\delta$ and $d\phi^2 = \frac{1}{\rho^4}(xdy - ydx)^2)$ we see that in Cartesian coordinates $\overline{g} = h(\theta)^2(\frac{1}{r^2}\delta + \frac{1}{\rho^4}(f(\theta)^2 - \sin^2\theta)(xdy - ydx)^2)$. Further we see from this representation that for $i \in \{x, y, z\}$, $\frac{c}{r^2} \leq |\partial_i|_{\overline{g}} \leq \frac{C}{r^2}$ for some constants c, C. Thus the falloff $|g - \overline{g}|_{\overline{g}} = o_k(r^{-\ell})$ translates to $|g_{ij} - \overline{g}_{ij}| = o_k(r^{-\ell-2})$. Thus Equation (4.2.6) holds.

Similarly since $\partial_\phi = x\partial_y - y\partial_x$ and $|\eta^i\partial_i - \partial_\phi|_{\overline{g}} = |\eta_i dx^i - h^2f^2d\phi|_{\overline{g}} = o_k(r^{-\ell})$ we have Equation (4.2.7) and Equation (4.2.8). □

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