Invariants of transverse and annular links in combinatorial link Floer homology

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Abstract of the Dissertation

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In this dissertation, we explore the Ozsváth-Szabó-Thurston transverse invariant and various concordance invariants that could be defined using combinatorial link Floer homology. We prove that non-vanishing of the transverse invariant for a link is equivalent to non-vanishing of the invariant for certain transverse cables of that link. As an application, to these results we generate many infinite families of examples of Legendrian and transversely non-simple topological link types. Then, we give a refinement of the transverse invariant. Finally, we define an annular concordance invariant and study its properties. When specialized to braids, this invariant gives bounds on band rank. We also study the relationship of this invariant with transverse and braid monodromy properties.

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Chapter 1

Introduction and summary of results

Symplectic structures on even dimensional manifolds and related contact structures on odd dimensional manifolds originated from physics. In the presence of contact structures, an important version of knot theory studies Legendrian links which are tangent to the contact planes. Another important class, transverse links are transverse to the contact planes. Legendrian and transverse links play an important role in the study of contact structures in 3-manifolds. They also provide interesting applications to smooth topology such as finding obstructions to slicing a knot. In this thesis, we will explore Legendrian and transverse links and the closely related problem of finding bounds on link cobordisms using the tools of combinatorial link Floer homology.

Contact structure on a 3-manifold admits a surgery diagram represented by Legendrian links. So, it is interesting to study isotopy of Legendrian links through Legendrian links and isotopy of transverse links through transverse links. The classical invariants of Legendrian isotopy are Thurston-Bennequin and rotation numbers. If any topological link type has Legendrian representatives that are not distinguished by the classical invariants, it is called Legendrian non-simple. Legendrian contact homology introduced by Eliashberg and Chekanov [6] is a very powerful tool in this domain that is used in detecting Legendrian non-simplicity. Similarly, the self-linking number is a basic topological invariant for transverse knots. If any two transverse representatives in a topological link type with the same self-linking number are isotopic, then the topological link type is called transversely simple. Otherwise, it is called transversely non-simple. At first, it was not clear if there are any transversely non-simple link types. The first examples of transversely non-simple knots were found by Birman and Menasco [3] using braid foliation techniques and independently by Etnyre and Honda [12] who showed that $(2,3)$ cable of the $(2,3)$ torus knot was transversely non-simple using convex surface theory. Etnyre and Honda also showed that cables of transversely simple links are simple under some conditions.

An effective invariant of transverse knots, θ was defined in combinatorial link Floer homology by Ozsváth, Szabó and Thurston [30]. They also defined the related invariants λ^+ and λ^- for Legendrian links. The invariant θ has been used several times to give examples of transversely non-simple link types. Its especially powerful compared to the classical techniques that focused on specific examples. Usually such examples involve finding

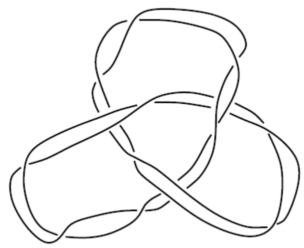


Figure 1.1: A cable of trefoil knot

representatives $\mathcal{T}_1$ and $\mathcal{T}_2$ of a link type $\hat{\theta}(\mathcal{T}_1) = 0$ and $\hat{\theta}(\mathcal{T}_2) \neq 0$.

Cabling is a well-known operation on knots and links [See Figure 1.1]. Cables are interesting objects to study in this context because they provide a natural avenue to look for examples of transversely non-simple link types, and also they provide an interesting insight into some concordance invariants. The work of Etnyre and Honda hints that it might be possible to show that cables of transversely non-simple links are also non-simple under certain conditions. In chapter 3, we study transverse and Legendrian non-simplicity under cabling using combinatorial link Floer homology.

The transverse invariant [30] is defined combinatorially using chain complexes associated to grid diagrams. We can obtain grid diagrams for cables $L_{p,q}$ (for $p \geq 2$) by subdividing the grid of L. Then we consider a special version of combinatorial knot Floer complex $\mathscr{C}_{L_{p,q}}$ associated to the grid diagram of $L_{p,q}$, and a chain complex $p\mathscr{C}$ (obtained by a change of variable from the combinatorial knot Floer complex) associated to the grid diagram of L. Then, we observe that there is a natural inclusion i of this complex to the collapsed grid complex of the cable $L_{p,q}$. The key theorem states that i induces inclusion in homology.

Theorem 1.0.1. *The map i induces an inclusion map on homology i.e.,*

$$H_*(\mathscr{C}_{L_{p,q}}) = H_*(i(p\mathscr{C})) \oplus_{\mathbb{F}_2[V]} \mathcal{R}$$

for some $\mathbb{F}_2[U]$ module $\mathcal{R}$.

The above theorem also implies that non-vanishing of $\hat{\theta}$ of a transverse link L is equivalent to non-vanishing of $\hat{\theta}$ for some transverse representatives of the cable $L_{p,q}$.

Theorem 1.0.2. $\hat{\theta}(L_{p,q}) = 0$ *if and only if* $\hat{\theta}(L) = 0$.

As an application of this theorem, we can obtain new non-simple knots by cabling previously known examples such as $m(10)_{132}$ and any of the infinite families in [1, 20, 39].

In a similar vein, one can also obtain Legendrian cables $K_{p,-pr\pm 1}$ for $r \leq tb(K) + n(K)$ ($n(K)$ is the minimum grid number of L) for knot K. We prove the following result about Legendrian invariants $\lambda^+(K)$ and $\lambda^-(K)$ using Theorem 3.2.1.

Theorem 1.0.3. $\lambda^+(K_{p,-pr\pm1}) = \lambda^-(K_{p,-pr\pm1})$ *if and only if* $\lambda^+(K) = \lambda^-(K)$.

As a corollary, we can show that certain infinite family of cables of $m(5_2)$ is Legendrian non-simple.

In the second part of our thesis, we focus on braids and annular links and certain chain complexes that can be assigned to them. Transverse links are equivalent to braid conjugacy classes up to positive (de)stabilizations. So, we can get more insights into transverse links by studying braid invariants that are well behaved under positive (de)stabilizations.

In chapter 4, we give a refinement of transverse GRID invariant θ. Baldwin, Vela-Vick and Vértesi [2] showed the equivalence of LOSS and GRID transverse invariants for transverse knots in $\mathbb{S}^3$ with the standard contact structure. We consider a deformed filtered complex that is isomorphic to the complex considered by Baldwin, Vela-Vick, and Vértesi. An invariant of braid conjugacy class η can be defined in this complex. By studying stabilization moves in the complex, we show that η is very similar to the analogous invariant Kappa in Khovanov homology [18].

Theorem 1.0.4. $\eta(\beta)$ *is a braid conjugacy class invariant. Also* $\eta(\beta) \leq \eta(\beta_{+stab}) \leq \eta(\beta)+\frac{1}{2}$.

Since it only increases under positive stabilization, one can get a transverse invariant by taking minimum over all braid representatives.

Theorem 1.0.5. *Let* β *be a* N *braid and* $\mathcal{T}$ *be the transverse link represented by* β. *If* $\hat{\theta}(\mathcal{T}) \neq 0$ *then* $\eta(\beta) = \infty$ *and* $-\frac{N}{2} \leq \eta(\beta) \leq \frac{N}{2}$ *otherwise.*

The above theorem shows that η is a refinement of the transverse invariant θ.

In chapter 5, we study annular links and cobordisms. Questions in concordance are particularly interesting in low dimensional topology. If we loosen the conditions and only require that the cylinder be locally flat, rather than smooth, we obtain the topological concordance group. There are examples of slice knots that are not topologically concordant to the unknot. They can be used to give examples of exotic structure in $\mathbb{R}^4$. On a different note the work of Lee Rudolph [35] tells us that braided banded cobordisms are related to ribbon immersed surfaces. This is a potentially effective way of finding ribbon obstruction to slice knot. This was our original motivation that led to a thorough study of annular links using combinatorial link Floer homology.

Inspired by similar work in Khovanov homology by Grigsby-Licata-Wehrli [13], we can define invariant $\mathscr{A}_L$ (and an invariant $\mathcal{A}_L$ which is related by mirroring) of annular links using two filtrations from combinatorial link Floer homology. $\mathscr{A}_L : [0, 2] \to \mathbb{R}$ is a piece-wise linear continuous function constructed using filtered grid complexes of an annular link L. It is an annular link invariant and gives genus bound for strong annular cobordisms (smooth).

Theorem 1.0.6. *If* Σ *is a strong annular cobordism of genus* g *between two annular links* L_1 *and* L_2 *then* $|\mathscr{A}_{L_1}(t) - \mathscr{A}_{L_2}(t)| \leq g(1 - \frac{t}{2})$.

Braided cobordisms are the most interesting examples of annular cobordism. We specialize the invariant $\mathscr{A}$ for braids and braided cobordisms in which case it gives a lower bound on band rank.

Theorem 1.0.7. *Let β be an n-braid with l components and Id_n be the identity n-braid. Then $\mathscr{A}_\beta(t) - \mathscr{A}_{Id_n}(t) \leq \frac{rk_n(\beta)+l-n}{2}(1-\frac{t}{2})$.*

Work of Lee Rudolph shows that every ribbon immersed cobordism is isotopic a braided banded cobordism. So, invariants like $\mathscr{A}_L(t)$ can be used to obtain ribbon obstruction if their behavior is well understood under stabilization.

We define a deformed complex $t\mathbf{C}$. $\mathscr{A}_L$ can also be interpreted as a max grading of a non-torsion element in a deformed knot Floer complex $t\mathbf{C}$.

Theorem 1.0.8. $\mathcal{A}_L(t) = -\mathscr{F}_t([\alpha])$.

A special case of $t\mathbf{C}$ at $t=0$ was considered in chapter 4.

We study the annular invariant under crossing change and braid stabilizations in this complex. We give a slice-Bennequin like lower bound for band rank using these properties of this complex.

Theorem 1.0.9. *Let L be an annular link with l components and $\mathcal{L}$ be the Legendrization of L. Then we have the following inequality,*

$$\mathscr{A}_L(t) \geq \frac{lk(U,L)t}{4} + (1-\frac{t}{2})\frac{tb(\mathcal{L}) + |rot(\mathcal{L})| + l + lk(U,L)}{2}$$

holds for all $t \in [0,2]$.

As an application, we get the following lower bound on the band rank.

Theorem 1.0.10. *If β is a n-braid with l componenets and $\mathcal{L}$ its Legendrization then,*

$$\frac{rk_n(\beta)+l-n}{2} \geq \frac{tb(\mathcal{L}) + |rot(\mathcal{L})| + l}{2}.$$

Finally, we explore some relations with transverse invariants and braid monodromy properties. In particular, we define subsets $\mathscr{M}_t$ of braids such that for $0 \leq t_1 \leq \cdots \leq t_n < 2$,

$$QP \subseteq \mathscr{M}_{t_1} \subseteq \cdots \subseteq \mathscr{M}_{t_n} \subseteq RV.$$

Where QP and RV denotes the monoids of Quasi-positive and right-veering braids respectively.

Theorem 1.0.11. *Membership in $\mathscr{M}_t$ is a transverse invariant and furthermore, $\mathscr{M}_t$ is a monoid.*

In chapter 6, we prove an inequality about τ of cables using grid homological framework developed in chapter 3. We also give a different interpretation of τ as a filtration level of a distinguished class. We propose that this relation could lead to a viable approach for computing invariants like τ and the annular invariant.

Chapter 2

Preliminaries

2.1 Legendrian and transverse links in contact structures

A contact structure on a 3-manifold M is a is a maximally nonintegrable plane field ξ. We say the contact structure is co-orientable if there is a 1-form α such that $\xi = ker\ \alpha$ and $\alpha \wedge d\alpha > 0$. We will consider the standard tight contact structure $(\mathbb{R}^3, \xi_{st})$, with $\xi_{st} = ker(dz - ydx)$. An oriented link $L \subset \mathbb{R}^3$ is called **Legendrian** if it is everywhere tangent to ξ_{st}. An oriented link $L \subset \mathbb{R}^3$ is called **transverse** if it is everywhere transverse to ξ_{st} and $dz - ydx > 0$ along the orientation. Any smooth link can be perturbed by a C^0 isotopy to be Legendrian or transverse. We say that two Legendrian (resp. transverse) links are **Legendrian isotopic** (resp. transversely isotopic) if they are isotopic through Legendrian links (resp. transverse links). We refer the reader to [9] for a thorough exposition of Legendrian and transverse links.

It's convenient to depict a Legendrian link is through its front projection or projection in the $x - z$ plane. A generic front projection has three features: it has no vertical tangencies; it is immersed except at cusp singularities; and at all crossings, the strand of larger slope passes underneath the strand of the smaller slope. Any front projection with these features corresponds to a Legendrian link, with the y coordinate given by the formula $y = \frac{dz}{dx}$.

The two main classical invariants of a Legendrian link L are are the **Thurston − Bennequin invariant** $tb(L)$ and the **rotation number** $r(L)$. They can be easily defined in terms of the front projection $D(L)$ of L. Let $wr(D(L))$ denote the writhe of the projection. Then,

$$tb(L) = wr(D(L)) - \frac{1}{2}\#\{\text{cusps in } D(L)\}$$

and,

$$r(L) = \frac{1}{2}\#(\{\text{downward-oriented cusps}\} - \{\text{upward-oriented cusps}\}).$$

The transverse push-off $\mathcal{T}(L)$ of an oriented Legendrian link L is the transverse link type which can be represented by transverse link arbitrarily close to L. Also, any transverse link can be represented as a transverse push-off of some Legendrian link. The main classical invariant of a transverse link L is the **self-linking number** $sl(L)$. For a transverse push-off, $\mathcal{T}(L)$, $sl(\mathcal{T}(L)) = tb(L) - r(L)$.

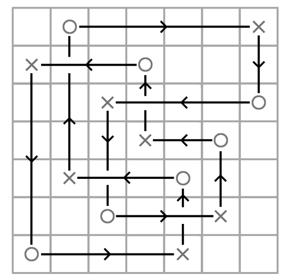


Figure 2.1: Grid diagram of figure eight

2.2 Grid Diagrams

Grid diagrams will be the central tool in our discussion. The combinatorial link Floer complexes are defined using grid diagrams. In this section, we will define grid diagrams, grid moves and how they are related to Legendrian and transverse links.

2.2.1 Toroidal grid diagram and grid moves

Definition 2.2.1. *A planar grid diagram P with grid number n is an $n \times n$ grid with squares marked with X's and O's in a way that no square contains both X and O, and each row and each column contains exactly one X and one O.*

$\mathbb{X}$ will denote the set of squares marked with an X, and $\mathbb{O}$ the ones containing an O. Every planar grid diagram P determines a diagram of an oriented link L in the following way: In each row connect the O-marking to the X-marking, and in each column connect the X-marking to the O-marking with an oriented line segment, such that the vertical segments always pass over the horizontal ones. We call P a planar grid diagram for L. Conversely, every oriented link L can be represented by some planar grid diagram. If L is a link with l components $L_1, L_2, .., L_l$, $\mathbb{X}_i$ (resp. $\mathbb{O}_i$) denotes X marked squares (resp O marked squares)in L_i . Then,we can write $\mathbb{X} = \mathbb{X}_1 \cup \mathbb{X}_2 \cup .. \cup \mathbb{X}_l$ and $\mathbb{O} = \mathbb{O}_1 \cup \mathbb{O}_2 \cup .. \cup \mathbb{O}_l$.

To work in Heegaard Floer homology setting, we find it convenient to transfer the diagram to torus $\mathbb{T}$.

Definition 2.2.2. *A toroidal grid diagram can be obtained by identifying the opposite sides of a planar grid diagram P: its top boundary segment with its bottom one and its left boundary segment with its right one. The resulting diagram D in torus $\mathbb{T}$ is called a toroidal grid diagram, or simply a grid diagram of the link L.*

There are certain moves of grid diagrams that are the equivalent to Reidemeister moves for knot diagrams. They are commutations and stabilizations.

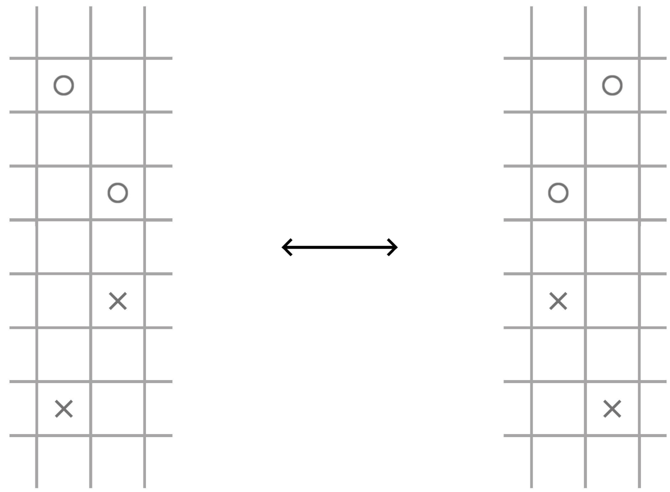


Figure 2.2: Commutation move

Definition 2.2.3. *In a grid diagram D, every column determines a closed interval of real numbers that connect the height of its O-marking with the height of its X-marking. Consider a pair of consecutive columns in D, and suppose that the two intervals associated with them are disjoint, or one contains the interior of the other. We say that the diagram D' differs from D by a column commutation, if it can be obtained by interchanging two consecutive columns of D that satisfy the above condition. A row commutation is defined analogously, using rows in place of columns. Column or row commutations collectively are called commutation moves.*

Definition 2.2.4. *Let D be an $n \times n$ grid diagram. We say that the $(n+1) \times (n+1)$ grid diagram D' differs from D by a stabilization (or that D' is the stabilization of D), if it can be obtained from D in the following way: Choose a marked square in D, and erase the marking in it, in the other marked square in its row and in the other marked square in its column. Then split the row and the column of the chosen marking in D into two, that is, add a new horizontal and a new vertical line to get an $(n+1) \times (n+1)$ grid.*

There are four ways to insert markings in the two new rows and columns to have a grid diagram. When the original square is marked with an X, these are called X: NE, X: NW, X: SE and X: SW [See Fig 2.3]. It turns out that it suffices to consider only these stabilizations for getting all Reidemeister moves.

2.2.2 Grid diagrams of Legendrian and transverse links

The grid diagram D can be viewed as the front projection of a Legendrian link via the following construction. First, we smooth northwest and southeast corners and turn southwest

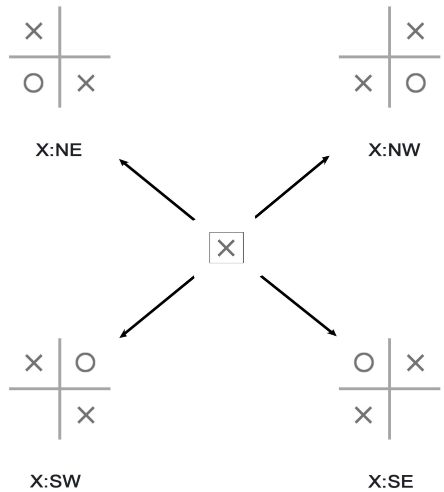


Figure 2.3: Stabilization Moves

and northeast corners into cusps. Then, to avoid vertical tangencies, we tilt the diagram 45° clockwise. Lastly, we reverse all the crossing to ensure the correct crossing convention for a Legendrian front projection [See Figure 2.4]. It is easy to see that if D is a grid diagram of a link L, the Legendrian knot associated to D, denoted by L_D is the Legendrian representative of $m(L)$.

Proposition 2.2.1. *[36] Any Legendrian link type can be represented by some toroidal grid diagram. Two toroidal grid diagrams represent the same Legendrian link type if and only if they can be connected by a sequence of commutation and (de)stabilization of types X: NW and X: SE on the torus.*

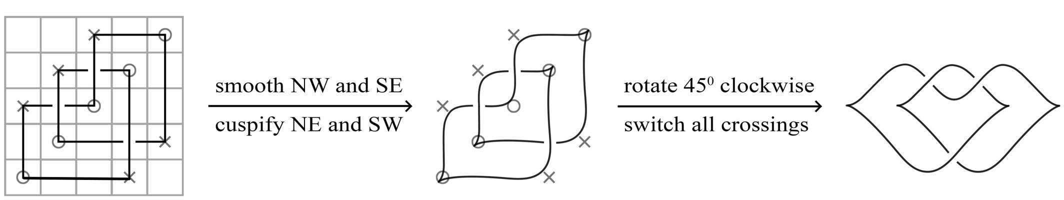


Figure 2.4: Getting the front projection of a Legendrian link from grid diagram

There are also formulas of classical Legendrian and transverse invariants in terms of corners in the grid corresponding to the front projection which will be useful to us. Let x_{NW} (and similarly x_{SW},x_{SE},x_{NE}) denote the number of northwest (similarly southwest, southeast, northeast) X markings, and define o_{NW}, o_{SW}, o_{SE}, o_{SW} similarly. Then,

$$r(L_D) = \frac{1}{2}(x_{NE} + o_{SW} - x_{SW} - o_{NE}) \tag{2.1}$$

and,

$$tb(L_D) = wr(D) - \frac{1}{2}(x_{NE} + o_{SW} + x_{SW} + o_{NE}). \tag{2.2}$$

.

We can also obtain correspondence from transverse links to grid diagrams by thinking of the transverse link as a push-off of a Legendrian link L and then taking the grid diagram D_L corresponding to L.

Proposition 2.2.2. *[36] Any transverse link type can be represented by some toroidal grid diagram. Two toroidal grid diagrams represent the same transverse link type if and only if they can be connected by a sequence of commutation and (de)stabilization of types X: NW, X: SE and X: SW on the torus.*

2.3 Algebraic preliminaries

Now recall some homological algebra that will play an important role in dealing with our main tools in grid homology. Let us assume $\mathcal{R}' = \mathbb{F}_2[U]$.

Definition 2.3.1. *An $\mathcal{R}'$-module M is a* graded $\mathcal{R}'$-module *if it admits a splitting $M = \bigoplus_d M_d$ over $\mathbb{F}_2$, such that $U^\alpha \cdot M_d \subset M_{d-\alpha}$ for each $d \in \mathbb{R}$ and $\alpha \in \mathbb{R}_{\geq 0}$.*

A graded $\mathcal{R}'$-module homomorphism *is a homomorphism $f\colon M \to M'$ between two graded $\mathcal{R}'$-modules that preserves the grading, i.e. that sends M_d to M'_d for every $d \in \mathbb{R}$. An $\mathcal{R}'$-module homomorphism is called homogeneous of degree s, if it maps M_d to M'_{d+s} for all d, $s \in \mathbb{R}$.*

Definition 2.3.2. *A graded chain complex over $\mathcal{R}'$ is a pair (C, ∂), where C is a graded $\mathcal{R}'$-module equipped with $\mathcal{R}'$-module homomorphism $\partial\colon C \to C$. The map ∂ is homogeneous of degree -1 and satisfies $\partial \circ \partial = 0$.*

Definition 2.3.3. *Let (C, ∂) and (C', ∂') be two graded chain complexes over $\mathcal{R}'$. A chain map $f\colon (C, \partial) \to (C', \partial')$ is an $\mathcal{R}'$-module homomorphism with the property $\partial' \circ f = f \circ \partial$. If the chain map f is also a graded homomorphism, then f is a graded chain map.*

Definition 2.3.4. *An isomorphism of graded chain complexes is a graded chain map $f\colon (C, \partial) \to (C', \partial')$ for which there exists another graded chain map $g\colon (C', \partial') \to (C, \partial)$ satisfying the properties $f \circ g = Id_{C'}$ and $g \circ f = Id_C$. Two graded chain complexes are called isomorphic, if there is an isomorphism connecting them.*

We will also consider many cases when $\mathcal{R}'$ will be a $\mathbb{F}_2[U_1, \cdots, U_k]$ module. The above definitions are similarly adapted in those cases.

Considering graded chain complexes, the interpretation of homology is the following:

Definition 2.3.5. *Suppose that (C, ∂) is a graded chain complex. Split C into homogeneous submodules as $C = \bigoplus_d C_d$. For each $d \in \mathbb{R}$ consider the homology module $H_d = C_d \cap \operatorname{Ker} \partial / C_d \cap \operatorname{Im} \partial$. The homology of (C, ∂) is the $\mathcal{R}'$-module $H(C) = \bigoplus_d H_d$.*

A graded chain map $f\colon C \to C'$ between two graded chain complexes over $\mathcal{R}'$ induces a well-defined graded map on homology, $H(f)\colon H(C) \to H(C')$. If the induced homomorphism is an isomorphism, then f is called a quasi-isomorphism.

Definition 2.3.6. *Let $f, g\colon (C, \partial) \to (C', \partial')$ be two graded chain maps between graded chain complexes over $\mathcal{R}'$. An $\mathcal{R}'$-module homomorphism $h\colon C \to C'$ is called a chain homotopy from g to f if it is homogeneous of degree 1, and satisfies the equality*

$$f - g = \partial' \circ h + h \circ \partial.$$

We say that f and g are chain homotopic if there exists a chain homotopy between them.

It is easy to show that chain homotopic maps induce the same map on homology.

Definition 2.3.7. *A chain map $f\colon C \to C'$ is a chain homotopy equivalence if there exists a chain map $\phi\colon C' \to C$ with the property that $f \circ \phi$ and $\phi \circ f$ are both chain homotopic to the respective identity maps. In this case, ϕ is called a chain homotopy inverse to f. C and C' are chain homotopy equivalent complexes if there is a chain homotopy equivalence connecting them.*

Now we define the notion of mapping cone that will be useful in some proofs of Chapter 3.

Definition 2.3.8. *Let (C, ∂) and (C', ∂') be two chain complexes over $\mathcal{R}$. The mapping* cone *of a chain map $f\colon C \to C'$ is the chain complex, $\mathrm{Cone}(f) := (C \oplus C', \partial_{\mathrm{Cone}})$, where the differential ∂_{Cone} for an element $(c, c') \in C \oplus C'$ is defined as*

$$\partial_{\mathrm{Cone}}(c, c') = (-\partial(c), \partial(c') + f(c)).$$

Proposition 2.3.1. *Consider the following short exact sequence of chain complexes.*

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0.$$

Then, there exists the long exact sequence of homologies:

$$\ldots \longrightarrow H_{n+1}(C) \longrightarrow H_n(A) \longrightarrow H_n(B) \longrightarrow H_n(C) \longrightarrow \ldots.$$

Corollary 2.3.1. *For C, C' chain complexes, a chain map $f\colon C \to C'$ is a quasi-isomorphism if and only if $H(\mathrm{Cone}(f)) = 0$.*

Proof. There exists a short exact sequence of chain complexes:

$$0 \longrightarrow C' \xrightarrow{\varphi} \mathrm{Cone}(f) \xrightarrow{\psi} C \longrightarrow 0.$$

Consider the associated long exact sequence

$$\ldots \xrightarrow{H_{n+1}(\varphi)} H_{n+1}(\mathrm{Cone}(f)) \xrightarrow{H_n(\psi)} H_n(C) \xrightarrow{H_n(f)} H_n(C') \xrightarrow{H_n(\varphi)} H_n(\mathrm{Cone}(f)) \longrightarrow \ldots$$

If $H(\text{Cone}(f)) = 0$, it is easy to see that $H(f)$ is both a monomorphism and an epimorphism. Therefore, $H(f)$ is an isomorphism.

Now suppose that f is a quasi-isomorphism. Then, $H(f)$ is a monomorphism, thus, because of the exactness, $\text{Ker}(H(f)) = \text{Im}(H(\psi)) = 0$. This also means that $H(\varphi)$ is an epimorphism, since $\text{Ker}(H(\psi)) = \text{Im}(H(\varphi)) = H(\text{Cone}(f))$. But we also know that $H(f)$ is an epimorphism, therefore $\text{Im}(H(f)) = \text{Ker}(H(\varphi)) = H(C')$. Since $H(\varphi)$ maps $H(C')$ to 0, and it is an epimorphism at the same time, we get that $H(\text{Cone}(f)) = 0$.

Proposition 2.3.2. *Suppose that C is a free, graded chain complex over $\mathcal{R}$ that is bounded above. Then $H(C) \neq 0$ if and only if $H(\frac{C}{U^\alpha \cdot C}) \neq 0$ for a fixed $\alpha \in \mathbb{R}_{>0}$.*

Proof. *We assumed that C is free, thus there exists a short exact sequence*

$$0 \longrightarrow C \xrightarrow{\cdot U^\alpha} C \longrightarrow \frac{C}{U^\alpha \cdot C} \longrightarrow 0$$

Considering the associated long exact sequence, it is easy to see that if $H(C) = 0$, then $H(\frac{C}{U^\alpha \cdot C}) = 0$.

Now suppose that $H(C) \neq 0$. Since C is bounded above, $H(C)$ has a homogeneous, non-zero element x with maximal grading. Then x cannot be of the form $y \cdot U^\alpha$ for any $y \in H(C)$, else the grading of x was not maximal. Therefore x must inject to $H(\frac{C}{U^\alpha \cdot C})$, and this way we got a non-zero element of $H(\frac{C}{U^\alpha \cdot C})$. $\square$

Now let us define the notion of filtrations that will be used throughout the text.

Definition 2.3.9. *Let I be a partially ordered set. An I-filtration on a chain complex $\mathcal{C}$ is the choice of subcomplex $\mathcal{F}_i \subseteq \mathcal{C}$ for each $i \in I$, such that $\mathcal{F}_i \subseteq \mathcal{F}_{i'}$ if $i \leq i'$.*

We will be focused on $\mathbb{Z}$, $\frac{1}{2}\mathbb{Z}$ and $\mathbb{R}$ filtrations in our discussion. For the next two definitions, chain complexes $\mathcal{C}$ and $\mathcal{C}'$ are given I-filtrations (I is $\mathbb{Z}$, $\frac{1}{2}\mathbb{Z}$ or $\mathbb{R}$) $\mathcal{F}$ and $\mathcal{F}'$ respectively.

Definition 2.3.10. *A map $f : \mathcal{C} \to \mathcal{C}'$ is said to be filtered of degree $k \in I$ if $f(\mathcal{F}_i) \subseteq \mathcal{F}'_{i+k}$ for all $i \in I$.*

Definition 2.3.11. *Two filtered chain complexes $(\mathcal{C}, \partial)$ and $(\mathcal{C}^{'}, \partial^{'})$ are said to be chain homotopy equivalent if there exists filtered chain maps $f : \mathcal{C} \to \mathcal{C}^{'}$ and $g : \mathcal{C}^{'} \to \mathcal{C}$ and filtered chain homotopies $H : \mathcal{C} \to \mathcal{C}$ and $H^{'} : \mathcal{C}^{'} \to \mathcal{C}^{'}$ satisfying, $gf - Id_{\mathcal{C}} = H\partial + \partial H$ and $fg - Id_{\mathcal{C}'} = H^{'}\partial^{'} + \partial^{'} H$. The maps f and g are called filtered chain homotopy equivalences.*

Sometimes, we will be interested in chain complexes with shifted gradings. The following notation is useful for keeping track of those things. Let $(\mathcal{C}, \partial)$ be a bi-graded chain complex and $\mathcal{C} \bigoplus_{i,j} \mathcal{C}_{i,j}$ where $\mathcal{C}_{i,j}$ is the sub-module generated by elements in bi-grading (i, j). Then, we define a bi-graded chain complex $\mathcal{C}[a, b] = \mathcal{C}'$ so that $\mathcal{C}'_{i,j} = \mathcal{C}_{i-a,j-b}$ for $a, b \in \mathbb{Z}$. Similarly, if we have a filtered graded complex complex $(\mathcal{C}, \partial)$ with filtration $\mathcal{F}$. $\mathcal{C}' = \mathcal{C}[a, b]$ will denote the complex that satisfies $\mathcal{F}^j(\mathcal{C}'_i) = \mathcal{F}^{j-b}(\mathcal{C}_{i-a})$.

2.4 Grid homology

In this section, we will introduce various flavors of grid homology and state the key results that will be used later.

2.4.1 Grid states and gradings

Definition 2.4.1. *A grid state x for a grid diagram D with grid number n consists of n points in the torus such that each horizontal and each vertical circle contains exactly one element of x. The set of grid states for D is denoted by $S(D)$.*

Equivalently, we can regard the generators as n-tuples of intersection points between the horizontal and vertical circles, such that no intersection point appears on more than one horizontal or vertical circle.

Given $x, y \in S(D)$, let $Rect(x, y)$ denote the space of embedded rectangles with the following properties. First of all, $Rect(x, y)$ is empty unless x, y coincide at exactly $n-2$ points. An element r of $Rect(x, y)$ is an embedded disk in $\mathbb{T}$, whose boundary consists of four arcs, each contained in horizontal or vertical circles; under the orientation induced on the boundary of r, the horizontal arcs are oriented from a point in x to a point in y. The set of empty rectangles $r \in Rect(x, y)$ with $x \cap Int(r) = \phi$ is denoted by $Rect^o(x, y)$. More generally, a path from x to y is a 1-cycle γ on $\mathbb{T}$ contained in the union of horizontal and vertical circles such that the boundary of the intersection of γ with the union of the horizontal curves is $y - x$, and a domain Δ from x to y is a two-chain in $\mathbb{T}$ whose boundary $\partial\Delta$ is a path from x to y. The set of domains from x to y is denoted $\Pi(x, y)$.

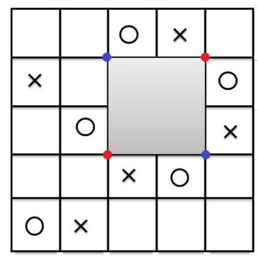


Figure 2.5: Rectangles in a grid diagram of a link

Now we will introduce the gradings that will be used in various flavors of grid homology. **Maslov grading** function $M : S(D) \to \mathbb{Z}$ is defined as

$$M(x) = \mathcal{J}(x - \mathbb{O}, x - \mathbb{O}) + 1.$$

Here for sets P, Q of finitely many points in the grid,

$$\mathcal{J}(P, Q) := \sum_{a \in P} \#\{(a, b) \in (P, Q) | b \text{ has both coordinates strictly greater than the ones of } a\}.$$

For each component L_i , the **Alexander grading** function $A_i : S(D) \to \frac{1}{2}\mathbb{Z}$ is defined as

$$A_i(x) = \mathcal{J}(x - \frac{1}{2}(\mathbb{X} + \mathbb{O}), \mathbb{X}_i - \mathbb{O}_i) - \frac{n_i - 1}{2}.$$

If there is an empty rectangle r between x and y and the Alexander and Masolov gradings also satisfy,

$$M(y) - M(x) = -1 + 2\#(r \cap \mathbb{O})$$

and,

$$A_i(y) - A_i(x) = \#(r \cap \mathbb{O}_i) - \#(r \cap \mathbb{X}_i).$$

Total Alexander grading $A(x)$ is given by the sum of all component grading functions. It satisfies,

$$A(x) = \frac{1}{2}(\mathcal{J}(x - \mathbb{O}, x - \mathbb{O}) - \mathcal{J}(x - \mathbb{X}, x - \mathbb{X}) - \frac{n-1}{2}. \tag{2.1}$$

There is also a winding number formula for computing Alexander gradings of link components. Let $w_{L_i}(q)$ denote the winding number of component L_i around a point q and $(P_i)_{i=1,\cdots,8n}$ be the corners of X and O markings. Then,

$$A_i(x) = -\sum_{p \in x} w_{L_i}(p) + \frac{1}{8}\sum_{j=1}^{8n} w_{L_i}(P_i) - \frac{1}{2}. \tag{2.2}$$

2.4.2 Fully blocked filtered grid complex $\widetilde{\mathcal{GC}}$

Given a toroidal grid diagram D of a link L , we associate to it a chain complex $(\widetilde{\mathcal{GC}}(D), \widetilde{\partial})$ as follows. $\widetilde{\mathcal{GC}}(D)$ is a free $\mathbb{F}_2$ module generated by grid states $S(D)$.

Given $x \in S(G)$, the differential map $\widetilde{\partial} : \widetilde{\mathcal{GC}}(D) \to \widetilde{\mathcal{GC}}(D)$, is defined in the following way ,

$$\widetilde{\partial} x := \sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{O} = \phi} y \quad \forall x \in S(D)$$

It can be shown that $\widetilde{\partial} \circ \widetilde{\partial} = 0$. Also, the Maslov grading when extended as a bi-linear form gives the homological grading of this complex. Moreover, Alexander grading functions $A_i(x)$ gives filtrations when extended as bi-linear form. Therefore, $(\widetilde{\mathcal{GC}}(D), \widetilde{\partial})$ can be thought of as a filtered chain complex. Its homology,denoted by $\widetilde{\mathcal{GH}}(D)$ isomorphic to $\mathbb{F}_2^{2^{n-1}}$. The associated graded complex is denoted by $(\widetilde{GC}(D), \widetilde{\partial_{\mathbb{X}}})$.

2.4.3 Simply blocked filtered grid complex $\widehat{\mathcal{GC}}$

Let D be the grid diagram of an oriented link L with l components. $\mathbb{O}$ is the set of O-markings in D. $s\mathbb{O} \subset \mathbb{O}$ is a subset that contains precisely one O marking from each component of L. Elements of $s\mathbb{O}$ are called special and represented as 'ϕ' in D. The simply blocked filtered chain complex $(\widehat{\mathcal{GC}}(D), \widehat{\partial})$ is defined as -

$$\widehat{\mathcal{GC}}(D) = \text{Free } \mathbb{F}_2[V_1, V_2, ..., V_n] \text{ module over grid states } S(D).$$

The differential ∂ is defined in the following way-

$$\widehat{\partial}x := \sum_{y\in S(D)} \sum_{r\in Rect^o(x,y), r\cap s\mathbb{O}=\phi} V_1^{O_1(r)}...V_m^{O_m(r)}y \quad \forall x \in S(D)$$

The homological grading of a generator $x \in S(D)$ in this complex is again given by Maslov grading. Multiplication by V_i lowers the Maslov grading by 2. $\widehat{\partial}$ lowers Maslov grading by 1.

$(\widehat{\mathcal{GC}}(D), \widehat{\partial})$ also comes with additional filtration for each link component. For a $x \in S(D)$ the i'th link component the Alexander filtration which are given by A_is on generators. Then, A_i's are extended to the whole module so that multiplication by V_k lowers A_i by 1 if $X_k \in \mathbb{X}_i$ and it remains unchanged otherwise.

It can be shown that multiplication by each V_i is filtered chain homotopic to 0. So can be thought of as a $\mathbb{F}_2$ module. Furthermore its homology, denoted by $\widehat{\mathcal{GH}}(L)$, isomorphic to $\mathbb{F}_2^{2^{l-1}}$. Ozsvath and Szabó showed that the filtered chain homotopy type of $(\widehat{\mathcal{GC}}(D), \widehat{\partial})$ is an invariant of L.

$\tau(L)$ is particularly interesting invariant that one can extract from the filtered chain homotopy type.

Definition 2.4.2. *Let G be grid diagram of link L, $\tau(L)$ is defined to be the minimal value of i such that the inclusion $H_0(\mathcal{F}_i(\widehat{\mathcal{GC}}(G)) \to H_0(\widehat{\mathcal{GC}}(G))$ is non zero.*

It turns out that $\tau(L)$ is a smooth concordance invariant. In fact, for a knot, τ gives lower bound on 4-ball genus. It is also possible to define a τ-set by considering other Maslov gradings.

Sometimes we will be interested in the associated graded complex , $(\widehat{GC}(D), \widehat{\partial}_{\mathbb{X}})$.Its homology groups are written as $\widetilde{GH}_i(L, j)$ where i indicates Maslov grading and j indicates Alexander grading. We also have, $\widetilde{GH}(D) \cong \widehat{GH}(L) \otimes W^{\otimes n-l}$. Notice that even though $\widetilde{GH}(D)$ depends on grid number, $\widehat{GH}(L)$ is a link invariant independent of the diagram D. $\widehat{GH}(L)$ is also referred to as combinatorial link Floer homology or simply link Floer homology.

2.4.4 Unblocked filtered grid complex $\mathcal{GC}^-$

The unblocked grid chain complex $(\mathcal{GC}^-(D), \partial^-)$ defined as -

$$\mathcal{GC}^-(D) = \text{Free } \mathbb{F}_2[V_1, V_2, ..., V_n] \text{ module over grid states } S(D).$$

The differential ∂^- is defined in the following way-

$$\partial^- x := \sum_{y\in S(D)} \sum_{r\in Rect^o(x,y)} V_1^{O_1(r)}...V_m^{O_m(r)}y \quad \forall x \in S(D)$$

The unblocked grid homology package also comes with the same extra filtrations (one for each link component). The filtered quasi-isomorphism type of the unblocked grid complex is an invariant of the link L. We will denote its associated graded complex by $(GC^-(D), \partial_{\mathbb{X}}^-)$. The maps given by multiplication by V_i and V_j are chain homotopic in this complex if O_i and O_j are in the same link component. Therefore, $GH^-(D)$ can be thought of as a $\mathbb{F}_2[V_1, V_2, ..., V_l]$ module if the link L represented by D has l components. If L is a knot, it can be shown that $GH^-(D) = F[U] \oplus Tor$ where Tor is the torsion part. Further, the maximal Alexander grading of a non-torsion class is equal to $-\tau(L)$.

2.4.5 Invariants of Legendrian and transverse links

The element $x^+ \in S(D)$, which consists of the intersection points at the upper right corners of the squares containing the markings X in D, is a cycle in $(GC^-(D), \partial_{\mathbb{X}}^-)$. The element $x^- \in S(D)$ consisting of the intersection points at the south west corners of X markings is also a cycle. If L is the Legendrian link corresponding to the grid diagram D, then we know that D represents the topological link type of $m(L)$. There is an interesting formula for the Alexander and Maslov gradings of the distinguished class in terms of the classical invariants of Legendrian link components. First, let us introduce some notations. Suppose $L_1, \cdots, L_l$ are the components of L. Define $tb_i(L)$ to be the linking number of L_i with the Legendrian push-off L'. It can be checked that $tb(L) = tb_1(L) + \cdots + tb_l(L)$. tb_i can be computed in terms of a generic front projection $D(L) = \cup_1^l D(L_i)$ where $D(L_i)$ is associated with L_i.

$$tb_i(L) = wr(D(L_i)) + lk(D(L_i), D(L)\backslash D(L_i)) - \frac{1}{2}\#\{\text{cusps in } D(L_i)\}.$$

Now define rotation numbers

$$r(L_i) := \frac{1}{2}\#(\{\text{downward-oriented cusps in} D(L_i)\} - \{\text{upward-oriented cusps in} D(L_i)\}).$$

Again, it can be checked that $r(L) = r_1(L) + \cdots + r_l(L)$.

Proposition 2.4.1. *Let x^+ and x^- be the distinguished cycles in the grids of $m(L)$ then,*

$$M(x^+) = tb(L) - r(L) + 1 \quad , \quad M(x^-) = tb(L) + r(L) + 1,$$

$$A_i(x^+) = \frac{tb_i(L) - r_i(L) + 1}{2} \quad , \quad A_i(x^-) = \frac{tb_i(L) + r_i(L) + 1}{2},$$

$$A(x^+) = \frac{tb(L) - r(L) + l}{2} \quad \textit{and} \quad A(x^-) = \frac{tb(L) + r(L) + l}{2}.$$

The above formulas will be used later in computations.

The homology classes $[x^+], [x^-] \in GH^-(m(L))$, denoted by $\lambda^+(D)$ and $\lambda^-(D)$ respectively, are called the Legendrian grid invariant of D. For the transverse push-off $\mathcal{T}$ of an oriented Legendrian link L, the transverse grid invariant $\theta^-(D)$ is defined to be $\lambda^+(L) \in GH^-(m(L))$. The following proposition states that the homological class is a well-defined invariant of Legendrian and transverse link types.

Proposition 2.4.2. *[30] Let D and D' be two grid diagrams corresponding to Legendrian link L (similarly transverse link $\mathcal{T}$), then there is an isomorphism*

$$\phi : GH^-(D) \to GH^-(D')$$

such that $\phi(\lambda^+(D)) = \lambda^+(D')$ and $\phi(\lambda^-(D)) = \lambda^-(D')$ (similarly $\phi(\theta^-(D)) = \theta^-(D')$).

Therefore, we choose to write $\lambda^+(D)$ as $\lambda^+(L)$ and $\lambda^-(D)$ as $\lambda^-(L)$ when D corresponds to Legendrian link type L. Similarly, we will write $\theta^-(D)$ as $\theta^-(\mathcal{T})$ when D corresponds to transverse link type $\mathcal{T}$. It is often more useful to consider the projection of $\theta(\mathcal{T})$ into $\widehat{GH}$, which we will call $\hat{\theta}(\mathcal{T})$. Projection of $\hat{\theta}(\mathcal{T})$ into $\widetilde{GH}(D)$ will be denoted as $\tilde{\theta}(\mathcal{T})$. It can be showed that $\hat{\theta}(\mathcal{T}) = 0$ if and only if $\tilde{\theta}(\mathcal{T}) = 0$.

2.4.6 Collapsed grid complexes

Let L be a link with l link components with grid diagram G. There are several collapsed complexes that we can construct from the $\mathbb{F}_2[V_1, V_2, ..., V_n]$ module $\mathcal{GC}^-(G)$ by setting some of the V_i's equal to each other. The collapsed filtered grid complex is defined as $c\mathcal{GC}^-(G) := \frac{\mathcal{GC}^-}{V_{i_1}=V_{i_2}=...=V_{i_l}}$, where O_{i_k} is a O marking belonging in the k'th link component. The associated graded version will be denoted by $cGC^-(G)$. Its homology, $cGH^-(G)$ can be thought of as a $\mathbb{F}_2[V]$ module. In fact, it can be shown that $cGH^-(G) \cong (\mathbb{F}_2[V])^{2^{l-1}} \oplus Tor$ (Here Tor is the torsion part). As usual, $cGH_i^-(G)$ denotes the homology at i'th Maslov grading.

We will also be interested in complexes where we collapse V_i and V_j with markings O_i and O_j belonging in the same component.

Proposition 2.4.3. *Let G be a grid of link L. Suppose markings O_i and O_j belong to some link component L_N. Then, there are filtered quasi-isomorphisms $\frac{\mathcal{GC}^-(G)}{V_i - V_j} \to \mathcal{GC}^-(G) \otimes W_{L_N}$ and $\frac{\widehat{\mathcal{GC}(G)}}{V_i - V_j} \to \widehat{\mathcal{GC}}(G) \otimes W_{L_N}$, where W_{L_N} is a 2 dimensional graded vector space with one generator having (Maslov grading $= 0$, A_{L_N} grading $= 0$, Alexander gradings for other link components $= 0$) and the other generator having (Maslov grading $= -1$, A_{L_N} grading $= -1$, Alexander gradings for other link components $= 0$)*

Proof. Lets consider the short exact sequence

$$0 \longrightarrow \mathcal{GC}^-(G) \xrightarrow{V_i - V_j} \mathcal{GC}^-(G) \longrightarrow \frac{\mathcal{GC}^-(G)}{V_i - V_j} \longrightarrow 0$$

We know that the map given by multiplication by $V_i - V_j$ is chain homotopic to 0. Also , multiplication by $V_i - V_j$ lowers Maslov grading by 2, A_{L_N} grading by 1 and total Alexander grading by 1. Therefore, mapping cone is filter-quasi-isomorphic to $\mathcal{GC}^-(G) \oplus \mathcal{GC}^-(G)[-1, -1, -1]$. So we have a quasi-isomorphism $\frac{\mathcal{GC}^-(G)}{V_i - V_j} \to \mathcal{GC}^-(G) \oplus \mathcal{GC}^-(G)[-1, -1, -1]$ and the conclusion follows. Similarly by setting on of the $U_k = 0$ we can obtain the analogous result for the hat version. $\square$

We can iterate the process in the above proposition as many times as we wish, but we need to be careful when the complex has two markings in two different link components collapsed. It won't make sense to talk about link filtration for those particular components, and we need to talk about combined Alexander gradings.

In our discussion, we will mostly consider a collapsed complex that we will call $\mathscr{C}(D)$ for a grid diagram D.

Definition 2.4.3. *Let D be a grid diagram of a link L. Define $\mathscr{C}(D) = F[U]$ module over grid states $S(D)$ and*

$$\partial x := \sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = \phi} U^{O(r)} y \quad \forall x \in S(D).$$

Proposition 2.4.4. *There is a quasi-isomorphism $\mathscr{C}(D)) \cong GC^-(D) \otimes W^{\otimes n-l}$, where W is a 2 dimensional graded vector space with one generator having (Maslov grading $= 0$, Alexander grading $= 0$) and the other generator having (Maslov grading $= -1$, Alexander grading $= -1$).*

Proof. Proposition 2.4.3 can be specialized to associated graded complex. Then the claim easily follows by iteration.

□

Proposition 2.4.5. *Let D be the grid diagram of link L. The class $[x^+] \in H_*(\mathscr{C}(D))$ is in the U-image if and only if $\hat{\theta}(L) = 0$.*

Proof. Consider the short exact sequence

$$0 \longrightarrow \mathscr{C}(D) \xrightarrow{U} \mathscr{C}(D) \longrightarrow \frac{\mathscr{C}(D)}{U} \longrightarrow 0$$

Therefore, from the induced long exact sequence, we can infer that if the class $[x^+]$ is in U-image then the projection of $[x^+]$ in $\frac{\mathscr{C}(D)}{U}$ is 0. Now notice that $\frac{\mathscr{C}(D)}{U} \cong \widetilde{GH}(L)$ and, the projection of $[x^+]$ there is $\tilde{\theta}(L)$. So we get $\tilde{\theta}(L) = 0$. This implies $\hat{\theta}(L) = 0$. Conversely, $\hat{\theta}(L) = 0$ implies $\tilde{\theta}(L) = 0$. Hence, the short exact sequence implies that $[x^+] \in H_*(\mathscr{C}(D))$ is in the U-image.

□

Proposition 2.4.6. *Let D be the grid diagram of a knot K. Let $x \in H_*(\mathscr{C}_D)$ be the non-torsion element with maximal Alexander grading. Then $\tau(K) = -A(x)$.*

Proof. We know from 2.4.4, $H_*(\mathscr{C}_D) \cong GH^-(K) \otimes W^{\otimes n-1} \cong (F[U] \oplus Tor) \otimes W^{\otimes n-1}$. Therefore, the free part is isomorphic to $F[U]_{(-2\tau(K), -\tau(K))} \otimes W^{\otimes n-1}$. Hence, the conclusion follows. □

2.4.7 Grid complexes of mirror links

Given a grid diagram D (with grid number n) of a link L, let D^* be the diagram obtained by reflecting D through a horizontal axis. Then D^* represents the link $m(L)$. Let M and M^*

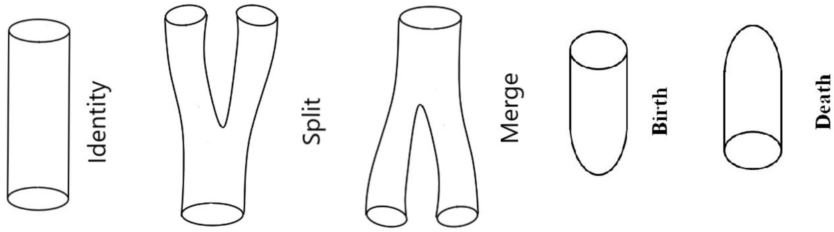


Figure 2.6: Standard link Cobordisms

(resp. A and A^*) denote Maslov (resp Alexander) grading in grids D and D^*. Let $x \to x^*$ be the natural bijection of the grid states induced by reflection. Then we observe that there is also a bijection between $Rect^o(x, y)$ and $Rect^o(y^*, x^*)$. So there is an isomorphism of chain complexes $\widetilde{\mathcal{GC}}(D^*) \cong Hom(\widetilde{\mathcal{GC}}(D), \mathbb{F}_2)$. Now, we can verify that $M(x) + M^*(x^*) = 1 - n$ and $A(x) + A^*(x^*) = l - n$ for a grid state x. Also, using the standard convention the dual complex $\widetilde{GC}^* \cong Hom(\widetilde{\mathcal{GC}}(D), \mathbb{F}_2)$ has grading and filtration level obtained by taking negative of those in $\widetilde{GC}$. Then we get an isomorphism of $\widetilde{\mathcal{GC}}(D^*) \cong \widetilde{\mathcal{GC}}^*(D)[1-n, l-n]$. Now, we can pass to the hat version using Proposition 2.4.3, There, we have $\widehat{\mathcal{GC}}(D^*) \otimes W^{\otimes n-l} \cong \widehat{\mathcal{GC}}^*(D) \otimes W^{*\otimes n-l}[1-l, 0]$. So, we get an isomorphism $\widehat{\mathcal{GC}}(D^*) \cong \widehat{\mathcal{GC}}^*(D)[1-l, 0]$. In fact, we can use the individual Alexander filtrations $\mathcal{F}_i$, $i = 1, \cdots, l$ to put this isomorphism more generally as

$$\widehat{\mathcal{GC}}(D^*) \cong \widehat{\mathcal{GC}}^*(D))[1-l, 0, \cdots, 0].$$

This fact will be used later.

2.5 Cobordisms of links and braids

Definition 2.5.1. *A cobordism, between links L_1 and L_2 in the $\mathbb{S}^3$, is the image of a smooth embedding $f : \Sigma \to \mathbb{S}^3 \times [0, 1]$, where Σ is a compact, oriented surface (not necessarily connected) of genus g such that $\partial\Sigma = L_1 \times \{0\} \sqcup -L_2 \times 1$ and every connected component of Σ has boundary in both L_1 and L_2.*

Any link cobordism can be decomposed into five standard cobordisms. They are identity, birth, death, merge and split cobordisms [See Figure 2.6].

Two knots K_1 and K_2 in the $\mathbb{S}^3$ are called **concordant** if there exists a smooth, properly embedded cylinder in $\mathbb{S}^3 \times [0, 1]$ such that one end of the cylinder is $K_1 \times \{0\}$ and the other is $K_2 \times \{1\}$. This gives us an equivalence relation on the set of knots. A knot K is called **slice** if K is concordant to the unknot. The set ${}^{Knots}/_{concordance\ equivalence}$ forms the concordance group $\mathcal{C}$, where the operation is induced by connected sum. The class of slice knots is the

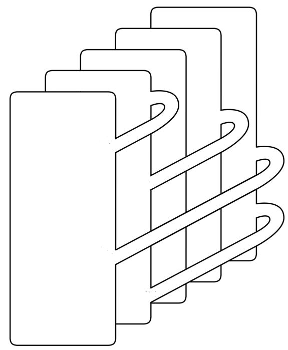


Figure 2.7: A Quasi-positive banded Surface

identity element, and the inverse of $[K]$ is $[-K]$, where $-K$ denotes the reverse of the mirror image of K.

Definition 2.5.2. *A braided cobordism $W \subset S^3 \times [0,1]$ is a smoothly and properly embedded surface, on which the projection $pr2 : S^3 \times [0,1] \to [0,1]$ restricts as a Morse function, with each regular level set $W \cap (S^3 \times t)$ a closed braid in $S^3 \times t$.*

Hughes [19] proved that every link cobordism between braid closures is isotopic rel boundary to a braided cobordism.

Given a braid σ, a band presentation is given by

$$\sigma = \prod_{j=1}^{c} \omega_j \sigma_{i_j}^{\pm 1} \omega_j^{-1}$$

The band rank $rk_n(\sigma) =: min_c\{$ There is band presentation of length $c\}$.

A braid can be written as a product of c bands; then its closure bounds a banded surface (See Fig 2.7) of Euler characteristics $n - c$.

Then it bounds a surface in S^3 with ribbon type singularities(**ribbon immersed surface**). The following theorem by Lee Rudolph [35] gives an intriguing relationship between banded braided surfaces and ribbon immersed surface.

Theorem 2.5.1. *[Rudolph] S is a ribbon-immersed orientable surface in S^3; then it is isotopic to a banded, braided surface of Euler characteristic $n - c$ (from the closure of an n-braid).*

Using this relationship, one can aim to find ribbon obstruction by obstructing certain band ranks of braided cobordisms.

Chapter 3

Cables and transverse invariant

3.1 Grid diagrams for Cables

The (p, q)- cable of a link L, denoted $L_{p,q}$, is the satellite link with pattern the (p, q)-torus knot $T_{p,q}$ (where p indicates the longitudinal winding and q indicates the meridional winding) and companion L. So, we can think of $L_{p,q}$ as the topological type of a link supported on the boundary of a tubular neighborhood of L with slope $\frac{p}{q}$ with respect to the standard framing of the torus, where the longitude is determined by the Seifert framing for L.

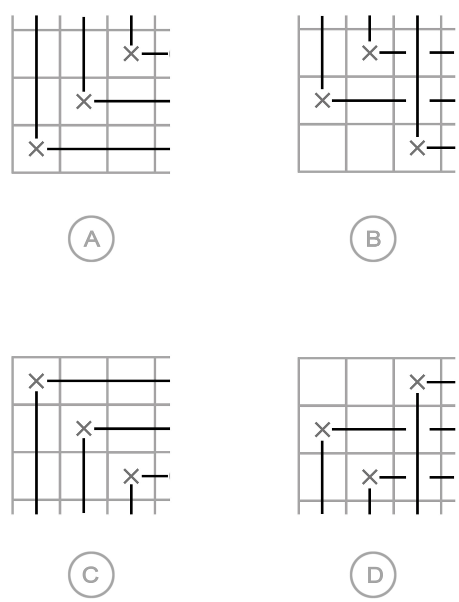


Figure 3.1: Blocks A and C have markings on diagonal. B and D have $p-1$ markings on a diagonal and the last one in a corner

Given a grid D of a link L, we can construct grids of p-cables from the grid by transforming a single square to a $p \times p$ block, so that, empty squares are transformed to empty blocks and marked squares are transformed into four types (A, B, C and D as shown in the Figure 3.1) of blocks. Now to ensure that we get a cable, we need to restrict allowed block types for corners. For top right and bottom left corner corners we use blocks A or B. For top left and bottom right corners we use block C or D. For X marked squares, we will only use blocks of type A. However, then we may not get a cable for $p > 2$. We will rectify this situation in 3.1.

Using the prescription, we can obtain the grid of the cable of topological link type $L_{p,q}$. The Legendrian and transverse link types represented by the grid may depend on the choice of the blocks. So when we refer to $L_{p,q}$ as Legendrian/transverse links we refer to any of those link types. Now we will determine for what values of cabling parameter q, we can obtain Legendrian/transverse representatives of $L_{p,q}$. We first start with the case $p = 2$ and then we will extend the results for $p > 2$.

Remark. *It is possible to make arbitrary choices of blocks to get the grid of certain satellites. However, it is not so clear what kind of satellite we can obtain from this construction.*

Generating 2-cables

For 2-cables, we will only have blocks of type A and C. We will try to use stabilizations to obtain a wide range of twisting coefficients.

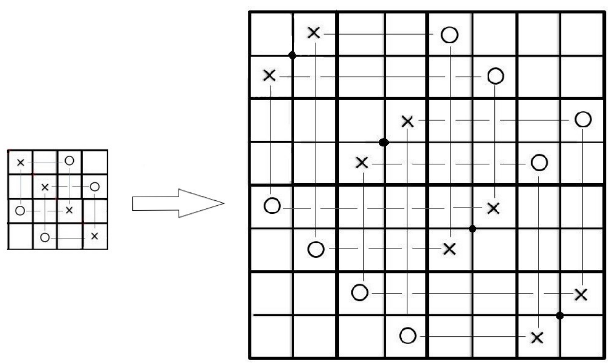


Figure 3.2: Generating grid of a 2 cable

Proposition 3.1.1. *The grid D_2 can be constructed from grid D, link L, by following the above procedure to represent a cable link $L_{2,q}$ if $2wr(D) + o_{SW} + o_{NE} - x_{SE} - x_{NW} \geq q \geq 2wr(D) - o_{SE} - o_{NW} - x_{SE} - x_{NW}$.*

Proof. To determine the cabling coefficient, we need to keep track of contribution for each type of corners and block type [See Figure 3.3] as well as signs of crossings. After putting it all together for all possible block choices, we get the inequality. □

Proposition 3.1.2. *If D represents a Legendrian link L, then we can make certain stabilizations in D followed by appropriate choices of blocks to construct a grid D_2' representing a Legendrian link $L_{2,-q}$ as long as $q \leq n(L) + 2tb(L)$, where $n(L)$ is the minimum grid number of L.*

Proof. First, we realize that mirroring changes the sign of q in the represented Legendrian link.

Since $o_{SE} + x_{SE} \geq 1$ for any grid and X:SE and O:SE stabilizations don't change the Legendrian link type; we can carry out the procedure of replacing a square by blocks after

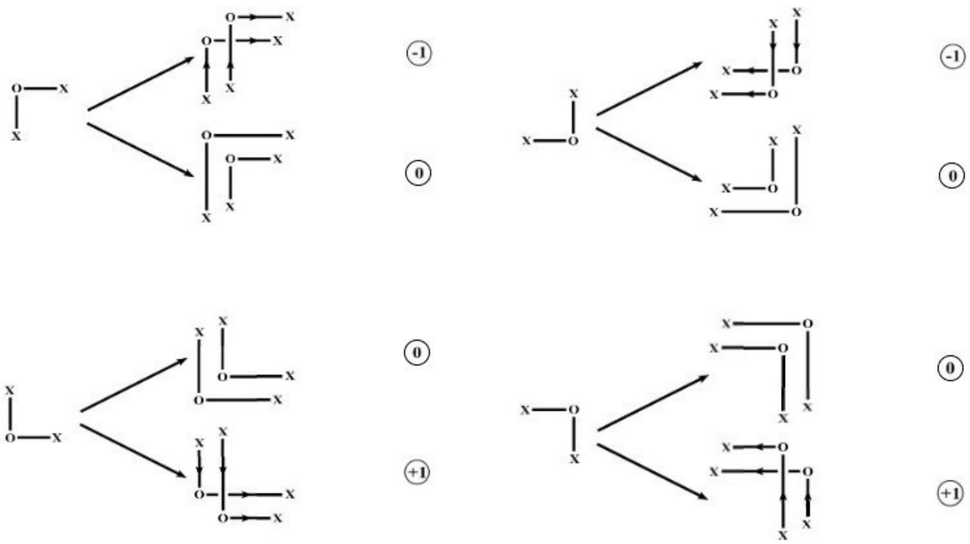


Figure 3.3: Contributions to cabling coefficient for different block types and corners

performing repeated stabilizations on those corners (X:SE stabilization on X:SE corners and O:SE stabilization on O:SE corners) to decrease q by any arbitrary number.

For the upper bound, we could write the upper bound from the previous proposition in terms of tb and n. □

Proposition 3.1.3. *If D represents a transverse link $\mathcal{T}$, then we can make certain stabilizations in D followed by appropriate choices of blocks to construct a grid D'_2 representing a transverse link $\mathcal{T}_{2,q}$ for any $q \in \mathbb{Z}$.*

Proof. First assume that $o_{SW} > 0$. Now we can do a O:SW stabilization on O:SW corners without changing the transverse link type and then repeat the subdivision procedure to increase the value of q by any arbitrary number. Also if $o_{SW} = 0$, we perform a torus translation to ensure $o_{SW} > 0$. This doesn't affect Proposition 3.1.2 because $o_{SE} + x_{SE} \geq 1$ for any grid. So we are able to get an arbitrary integer value for q. □

Generating p -cables for $p > 2$

When $p > 2$, using block A for NW and SE X corners induces a half full twist in the satellite. To get an integer value of the twisting parameter, we need to perform a stabilization on those X corners before replacing the X marked squares by block A [See Figure 3.4]. Again for O markings; we are allowed to use any blocks. There are obvious extensions of the results in the previous section. First, we state the extension of Proposition 3.1.1 -

Proposition 3.1.4. *The grid D_p can be constructed from grid D, representing link L, by following the above procedure to represent a cable link $L_{p,q}$ if $p(wr(D) - x_{SE} - x_{NW}) + o_{SW} + o_{NE} \geq q \geq p(wr(D) - x_{SE} - x_{NW}) - o_{SE} - o_{NW}$.*

For Legendrian links, we are only able to obtain a limited class of cables.

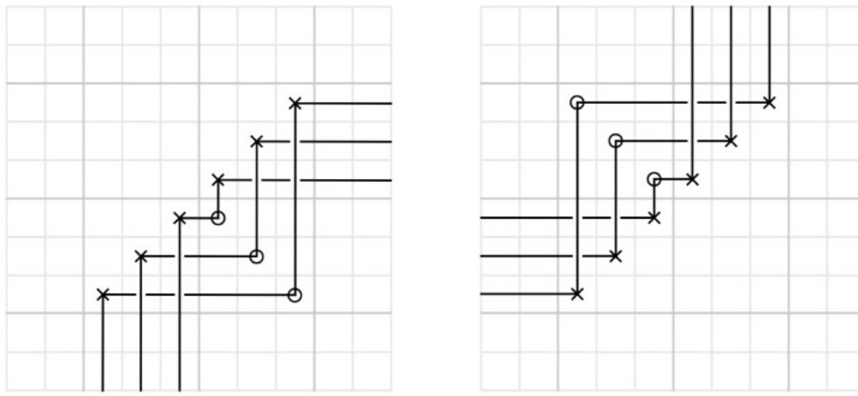


Figure 3.4: Applying X:SE stabilization for X:SE corners and X:NW stabilization for X:NW corners

Proposition 3.1.5. *If D represents a Legendrian link L, then we can make certain stabilizations in D followed by appropriate choices of blocks to construct a grid D'_p representing a Legendrian link $L_{p,-pk\pm 1}$ as long as $k \leq n(L) + tb(L)$, where $n(L)$ is the minimum grid number of L.*

Proof. In Proposition 3.1.4, $wr(D) - x_{SE} - x_{NW} \leq tb(L) + n$. So we can replace the upper bound by tb and n. Also we can make a torus translation to make sure the grid has at least one SE or NW O corner (Alternatively SW or NE O corner). □

But for transverse links, we can extend for full generality

Proposition 3.1.6. *If D represents a transverse link $\mathcal{T}$, then we can make certain stabilizations in D followed by appropriate choices of blocks to construct a grid D'_p representing a transverse link $\mathcal{T}_{p,q}$ for any $q \in \mathbb{Z}$.*

Proof. We can obtain cables with arbitrarily small q from the last proposition. Then we can use the argument from Proposition 3.1.3 again to get arbitrarily large values for q. □

The Legendrian($L_{p,q}$) and transverse ($\mathcal{T}_{n,q}$) link types obtained using this construction may depend on the choices of blocks. However, we notice that choices of blocks don't affect the number of corners of each type. It only affects writhe, which is detected in the cabling coefficient q. Therefore, by Equation 2.1 and 2.2, classical invariants (i.e., tb,r for Legendrian and sl for transverse) of the constructed cable Legendrian/transverse links are determined by the classical invariants of the original link and q.

3.2 Main theorems

Now we are ready to use tools of grid homology to study the constructed cables in Section 3.1. We first define a change of variable in the fully collapsed complex that will be useful for relating the link complex with its cable complex.

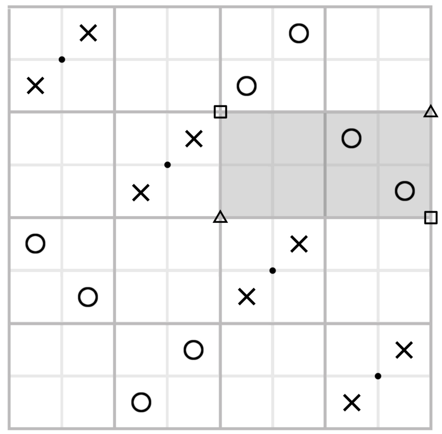


Figure 3.5: States of subcomplex $\mathcal{K}$

Definition 3.2.1. *Let D be a grid diagram of $m(L)$ for some link L and $p \in \mathbb{N}$. Define $p\mathscr{C}$ as $\mathbb{F}_2[V]$ module over grid states $S(D)$ and*

$$\partial_{p\mathscr{C}} x := \sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = \phi} U^{pO(r)} y \text{ for a } x \in S(D)$$

Algebraically $p\mathscr{C}$ is obtained from $\mathscr{C}$ by a change of variable. Therefore it inherits a new Alexander grading A satisfying, $\mathsf{A}(x) = pA(x)$ for $x \in S(D)$ and $\mathsf{A}(U) = -1$. The Maslov grading M from $\mathscr{C}$, can be adapted as M in $p\mathscr{C}$ so that $\mathsf{M}(x) = pM(x)$ for $x \in S(D)$ and $\mathsf{M}(U) = -2$. $\partial_{p\mathscr{C}}$ preserves the A and decreases M by p.

Now let us consider the grid D_p of the p-cable $m(L)_{p,-q} = m(L_{p,q})$ constructed from D using the prescription given in 3.1. $L_{p,q}$ and L will refer to transverse link types corresponding to those grids. Define $i : p\mathscr{C} \to \mathscr{C}(D_p)$ to be the $\mathbb{F}_2[V]$ module map that takes a generator state x in parent grid D to a state in the cable grid D_p obtained by taking union of North-East corners of X in the middle of each block and x. Let $\mathcal{K}$ be the sub-module of $\mathscr{C}(D_p)$ generated by all the states that contain North-East corner of X in the middle of each block [See Figure 3.5].

Proposition 3.2.1. *The map i is a injective chain map and $\mathcal{K}$ is a subcomplex of $\mathscr{C}(D_p)$ isomorphic to $p\mathscr{C}$.*

Proof. It follows from the definition that $i(p\mathscr{C}) = \mathcal{K}$. Now to prove that i is a chain map, we need to verify that $i\partial_{p\mathscr{C}} = \partial_{\mathscr{C}(D_p)} i$. Lets take states $\square, \triangle \in p\mathscr{C}$ such that $\partial_{p\mathscr{C}}(\triangle) = U^{pk} \square + ..$ [as depicted in Fig 3.5]. This implies $i(\partial_{p\mathscr{C}}(\triangle)) = U^{pk}\ i(\square) + \ldots$. Also, we have $\partial_{\mathscr{C}(D_p)}(i(\triangle)) = U^{pk}\ i(\square) + \ldots$ because the shaded rectangle contains p times manys Os in the cable grid. Since there is no rectangle coming out the special points of $\mathcal{K}$, any rectangle coming out of $i(\triangle)$ must join it with another state of the form $i(\square)$ for some $\square$. Hence, the map i satisfies $i\partial_{p\mathscr{C}} = \partial_{\mathscr{C}(D_p)} i$. Also since i is an injective chain map, it follows that $\mathcal{K}$ of is a subcomplex of $\mathscr{C}(D_p)$ isomorphic to $p\mathscr{C}$.

□

Proposition 3.2.2. *The map i sends the distinguished cycles $[x^+]$ and $[x^-]$ in $p\mathscr{C}$ to the distinguished cycles $[x^+]$ and $[x^-]$ respectively in $\mathscr{C}(D_p)$. It shifts the Alexander grading by $\frac{(p-1)(q-1)}{2}$ and Maslov grading by $(p-1)(q-1)$.*

Proof. It is obvious from the construction that i sends the distinguished states x^+ and x^- in $p\mathscr{C}$ to the distinguished states x^+ and x^- respectively in $\mathscr{C}(D_p)$. Also it is easy to see that i respects relative Alexander and Maslov grading. Hence, we just need to compute the Alexander and Maslov grading difference of the distinguished state in the respective complexes. Using 12.7.5 in [36], it is equal to

$$A(i(x^+)) - \mathsf{A}(x^+) = \frac{sl(L_{p,q})+1}{2} - \frac{p(sl(L)+1)}{2} = \frac{sl(L_{p,q})-p\cdot sl(L)-(p-1)}{2}.$$

To compute this quantity lets assume L has braid representative with index N and that $L_{p,q}$ has r twists with respect to blackboard framing. Then, $q = p \cdot wr(L) + r$ and $L_{p,q}$ has a braid representative with index Np and $wr(L_{p,q}) = p^2 \cdot wr(L) + r(p-1)$. We also know that for braid β of index n, $sl(\beta) = wr(\beta) - n$. Hence, it is equal to -

$$\frac{(wr(L_{p,q})-Np)-p(wr(L)-N)-(p-1)}{2} = \frac{(p-1)(p\cdot wr(L)+r-1)}{2} = \frac{(p-1)(q-1)}{2}.$$

Similarly, $M(i(x^+)) - \mathsf{M}(x^+) = (sl(L_{p,q})+1) - p(sl(L)+1) = (p-1)(q-1)$. □

Proposition 3.2.3. *If $\hat{\theta}(L) = 0$ then $\hat{\theta}(L_{p,q}) = 0$*

Proof. $\hat{\theta}(L) = 0$ implies that $[x^+]$ is in the U-image in the homology of complex $p\mathscr{C}$ i.e., $[x^+] = Uy$ for some y. Then, $i_*([x^+]) = i_*(Uy) = Ui_*(y)$ is also in U-image in the homology of complex $\mathscr{C}(D_p)$. So, it follows that $\hat{\theta}(L_{p,q}) = 0$.

□

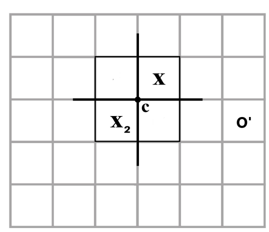


Figure 3.6: Special point c and markings around it

Theorem 3.2.1. *The map i induces an inclusion map on homology i.e.,*

$$H_*(\mathscr{C}_{L_{p,q}}) = H_*(i(p\mathscr{C})) \oplus_{\mathbb{F}_2[V]} \mathcal{R}$$

for some $\mathbb{F}_2[U]$ module $\mathcal{R}$.

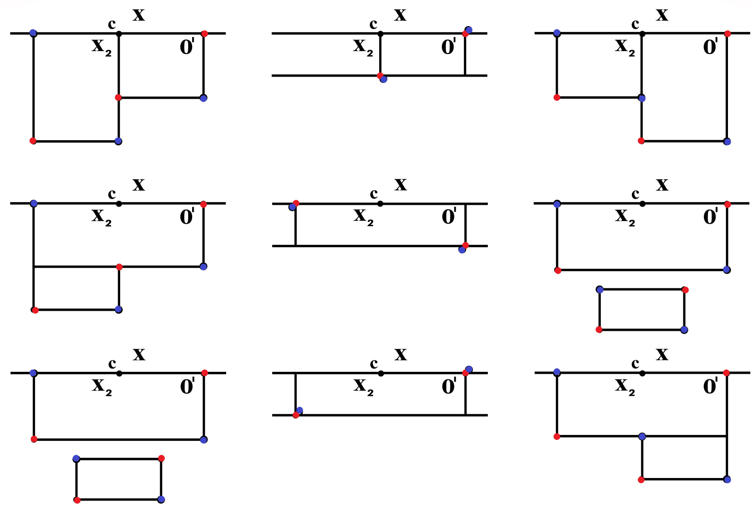


Figure 3.7: Juxtapositions of rectangles from the red state to blue state in Equation (2). First row, second row and third row represents $H_{X_2} \circ H_{O'}$, $H_{X_2,O'} \circ {\partial_N}^N$ and ${\partial_N}^N \circ H_{X_2,O'}$ respectively.

Proof. Lets consider one of the $n \times n$ block in the grid [See Figure 3.6]. There are two X markings inside the block around the special point c. The north-east square is marked with X; the south-west square is marked with X_2, and they intersect at c. Let O' be the O marking in the row containing X_2. We will write, $\mathscr{C}(D_p) = \mathcal{S} \oplus \mathcal{N}$, where $\mathcal{S}$ is a sub-module generated by all states that contain a special point c and $\mathcal{N}$ is a sub-module generated by all states that don't contain c. Since there are no rectangles coming out of the special point c, $\mathcal{S}$ is a subcomplex as before. Therefore, the differential of the complex can be written as, $\partial = \begin{bmatrix} {\partial_S}^S & {\partial_N}^S \\ 0 & {\partial_N}^N \end{bmatrix}$.

So $\mathscr{C}(D_p)$ can be seen as $Cone({\partial_N}^S)$. If ${\partial_N}^S$ induces 0 map on homology then it will follow that $H_*(\mathscr{C}(D_p), \partial) = H_*(\mathcal{S}, {\partial_S}^S) \oplus H_*(\mathcal{N}, {\partial_N}^N)$. This verification is an adaptation of the stabilization invariance proof in [36].

Define chain map $H_{X_2} : \mathcal{S} \longrightarrow \mathcal{N}$ as,

$$H_{X_2}(x) = \sum_{y \in S(D_p)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = X_2} U^{O(r)} y$$

${\partial_N}^S \circ H_{X_2}$ counts contributions from juxtapositions of rectangles where the first one goes out of the special point c and the second one goes into c. The only scenario that allows this is when we have the thin vertical or horizontal annulus containing X_2. Hence, we have ${\partial_N}^S \circ H_{X_2} = U + U = 0$. So if we can show that H_{X_2} is an injective map on homology, it will follow that ${\partial_N}^S$ induces 0 map on homology.

To see that, we define chain maps $H_{O'} : \mathcal{N} \longrightarrow \mathcal{S}$,

$$H_{O'}(x) = \sum_{y \in S(D_p)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = \phi, r \ni O'} U^{O(r)-1} y$$

and, $H_{X_2,O'} : \mathcal{N} \longrightarrow \mathcal{N}$,

$$H_{X_2,O'}(x) = \sum_{y \in S(D_p)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = X_2, r \ni O'} U^{O(r)-1} y$$

Then we have the following identity

$$H_{X_2} \circ H_{O'} + H_{X_2,O'} \circ {\partial_N}^N + {\partial_N}^N \circ H_{X_2,O'} = Id \tag{3.1}$$

To verify the identity, we see that the left-hand side counts some juxtapositions of rectangles from $\mathcal{N}$ to $\mathcal{N}$ which only contains the markings X_2 and O'. For such domains, these contributions cancel [See Figure 3.7] unless the domain is a horizontal annulus (containing O' and X_2) in which case it gives the right-hand side.

This verifies that H_{X_2} is an injective map on homology.

Now by iterating this procedure (starting with performing the same operation on $\mathcal{S}$), we can make the subcomplex include all such special points and recover the subcomplex as mentioned earlier $\mathcal{K}$. That will imply the decomposition and the fact that i induces inclusion map on homology.

□

Remark. *The statement of Theorem 3.2.1 is ungraded but after taking the degrees into account, Proposition 3.2.2 implies that*

$$H_{pM+(p-1)(q-1)}(\mathscr{C}_{L_{p,q}}, pA + \frac{(p-1)(q-1)}{2}) \text{ has a } H_M(\mathscr{C}_L, A) \text{ summand.}$$

As a corollary, we obtain out key theorem -

Theorem 3.2.2. $\hat{\theta}(L_{p,q}) = 0$ *if and only if* $\hat{\theta}(L) = 0$.

Proof. We already know one direction from Proposition 3.2.3. Now by Theorem 3.2.1, we know that i induces inclusion (as $\mathbb{F}_2[V]$ module) on homology. So $\hat{\theta}(L_{p,q}) = 0$ implies $[x^+]$ is in U-image in $H_*(\mathscr{C}(D_p))$ and hence in $H_*(p\mathscr{C})$. It follows that $\hat{\theta}(L) = 0$. □

Now, for a Legendrian knot K, recall that we can construct a Legendrian cable $K_{p,-pk\pm 1}$ for each $k \leq tb(K) + n(K)$. Instead of looking at $\lambda^+(K)$ or $\lambda^-(K)$ individually, it is more useful to consider the sum $\lambda^+(K) + \lambda^-(K)$ that will be denoted by $\eta(K)$.

Theorem 3.2.3. $\lambda^+(K_{p,-pr\pm1}) = \lambda^-(K_{p,-pr\pm1})$ *if and only if* $\lambda^+(K) = \lambda^-(K)$.

Proof. We want to show that $\eta(K) = 0$ if and only if $\eta(K_{p,-pk\pm1}) = 0$. First, we need to show that $\eta(K) = 0$ if and only if its projection $\eta'(K)$ to the fully collapsed complex is 0. Actually, this is true for any homology class. Suppose D is a grid diagram for K. Let us consider the short exact sequence in 2.4.4 again.

$$0 \longrightarrow GC^-(D) \xrightarrow{V_i - V_j} GC^-(D) \longrightarrow \frac{GC^-(D)}{V_i - V_j} \longrightarrow 0$$

From the induced long exact sequence, we can infer that projection of any homology class $\alpha = [\xi]$ is 0 then ξ is in the image of $V_i - V_j$ which implies $\alpha = [\xi] = 0$ since $V_i - V_j$ is null-homotopic. Conversely if $\alpha = 0$ then obviously its projection is 0. Iteration of this argument proves our claim.

Now the conclusion follows from Theorem 3.2.1 since $i(\eta'(K)) = \eta'(K_{p,-pk\pm1})$. □

3.3 Examples of Legendrian and transversely non-simple links

Now, let K and K' be two transverse links with same topological type and self-linking number such that, $\hat{\theta}(K) = 0$ and $\hat{\theta}(K') \neq 0$. By our construction, $K_{p,q}$ and ${K'}_{p,q}$ also represent transverse links with same topological type and self-linking number, but they are not isotopic as $\hat{\theta}$ vanishes for only one of them. So we can combine our result with the already known examples to generate various infinite families of transversely non-simple link type. The following proposition gives an example -

Proposition 3.3.1. *The topological link type* $m(10_{132})_{p,q}$ *is transversely non-simple for* $p \geq 2, q \in \mathbb{Z}$.

Proof. $m(10_{132})$ has transverse representatives $\mathcal{T}_1$ and $\mathcal{T}_2$ with same self-linking number such that $\hat{\theta}(\mathcal{T}_1) = 0$ and $\hat{\theta}(\mathcal{T}_2) \neq 0$. The conclusion follows. □

In the same vein, further examples can be obtained for cables of $m(10_{161})$,$m(7_2)$ etc.

We know that vanishing of η distinguishes Chekanov [6] pair in knot type $m(5_2)$. The following proposition shows that some of its cables are also Legendrian non-simple.

Proposition 3.3.2. *The topological link type* $m(5_2)_{p,-pk\pm1}$ *is Legendrian non-simple for* $p \geq 2, k \leq 8$.

Proof. There are $\mathcal{K}$ and $\mathcal{K}'$ in $m(5_2)$ with $tb = 1$ and $r = 0$ such that $\eta(\mathcal{K}) = 0$ and $\eta(\mathcal{K}') \neq 0$. Also they have grid diagrams with grid number 7. Hence, $\mathcal{K}_{p,-pk\pm1}$ and ${\mathcal{K}'}_{p,-pk\pm1}$ can be constructed for $k \leq 8$ in $m(5_2)_{p,-pk\pm1}$ so that they have the same tb and r. So by applying Theorem 3.2.3, we have $\eta(\mathcal{K}_{p,-pk\pm1}) = 0$ and $\eta({\mathcal{K}'}_{p,-pk\pm1}) \neq 0$. Therefore, $m(5_2)_{p,-pk\pm1}$ is a Legendrian non-simple link type for each $p \geq 2$ and $k \leq 8$. □

Chapter 4

Refinement of $\hat{\theta}$ invariant

4.1 Introduction

In this chapter, we will focus on grid complexes of braids. We will consider grid diagram of a braid $U \cup m(\beta)$ that will represent the braid β. This will allow us to define an invariant for braid conjugacy classes. Now if we look at the rotationally symmetric contact structure $\xi_{rot} = ker(dz - ydx + xdy)$, any closed braid around z-axis can be made transverse to contact planes. Conversely, any transverse link can also be represented as a closed braid. Now under this correspondence, the Transverse Markov Theorem [28] tells us

$$\text{Transverse links} \cong {}^{Braids}\!\big/_{\text{positive (de)stabilizations}}.$$

So if we know how the invariant changes under the operation of positive stabilization, we can hope to find a transverse invariant from the braid conjugacy class invariant.

Baldwin, Vela-Vick and Vértesi [2] showed the equivalence of LOSS and GRID transverse invariants for transverse knots in $\mathbb{S}^3$ with the standard contact structure. Their work involved passing to the knot Floer complex $CFK^{-,2}(-U \cup \beta)$ which comes with an extra filtration $\mathcal{F}^{-U}$. A key proposition in their work showed that the distinguished class $[x_4]$ generated $H_{top}(\mathcal{F}^{-U}_{bot})$. In view of that, it is natural to ask if the filtered quasi-isomorphism itself contains any information about the transverse knot represented by β. From a different perspective, we consider a filtered grid complex $\mathcal{C}_{U\cup\beta}$ which turns out to be isomorphic to $CFK^{-,2}(-U \cup \beta)$. We can independently show that filtered quasi-isomorphism type of that complex is an invariant of the braid conjugacy class. Moreover, by studying the crossing change moves we study how it changes under positive stabilization. We can then extract a numerical invariant $\eta(\beta)$ from the complex that is similar to the braid conjugacy class invariant κ defined by Hubbard and Saltz [18] in Khovanov homology.

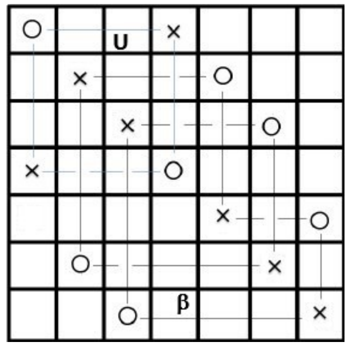


Figure 4.1: Diagram of $U \cup m(\beta)$

4.2 Braid grid complex

4.2.1 Definition

Let β be a N-braid. We can consider the the grid diagram D of $U \cup m(\beta)$ [See Fig. 4.1] where the unknot U (oriented clockwise) acts as a braid axis. Also we assume that the unknot is linked negatively with the braid. Let $\mathbb{X} = \{X_1, X_2, X_3, \cdots, X_n\}$ and $\mathbb{O} = \{O_1, O_2, O_3, \cdots, O_n\}$ be the sets of X markings and O markings respectively where X_1, X_2, O_1 and O_2 represent the markings of the unknot U. We will use the notation $O_\beta(r)$ to denote the number of O markings belonging to the β component inside a rectangle r.

Definition 4.2.1. *Define the chain complex $(\mathcal{C}_{U\cup\beta}(D), \partial)$ as a $\mathbb{F}_2[V_3, V_4, \cdots, V_n]$-module over grid states $S(D)$*

$$\partial x := \sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = \phi} V_3^{O_3(r)} V_4^{O_4(r)} \cdots V_n^{O_n(r)} y \quad \forall x \in S(D)$$

From now on we will refer to a $\mathbb{F}_2[V_3, V_4, \cdots, V_n]$-module as $\mathcal{R}$-module.

Proposition 4.2.1. $\partial \circ \partial = 0$

Proof. We need to observe that O_β is additive for a juxtaposition of rectangles and then the rest of the proof is identical to $\partial_{\mathbb{X}}^- \circ \partial_{\mathbb{X}}^- = 0$ proof in ordinary grid link homology. □

Let A_β be the sum of Alexander gradings for components of β. It is easy to see that the complex $\mathcal{C}_{U\cup\beta}$ is A_β graded. We also observe that it is $-A_U$ gives a filtration which we will call $\mathcal{F}_U$.

Proposition 4.2.2. *$\mathcal{C}_{U\cup\beta}$ is A_β graded and $\mathcal{F}_U$ filtered.*

Proof. Let y be a state appearing in the expansion of differential of x. Then,

$$A_\beta(V_3^{O_3(r)} \cdots V_n^{O_n(r)} y) - A_\beta(x) = A_\beta(y) - A_\beta(x) - O_\beta(r) = 0.$$

And,

$$\mathcal{F}_U(V_3^{O_3(r)} \cdots V_n^{O_n(r)} y) - \mathcal{F}_U(x) = \mathcal{F}_U(y) - \mathcal{F}_U(x) \leq 0$$

□

4.2.2 Invariance

The goal of this section is to prove the following theorem

Theorem 4.2.1. *$\mathcal{F}_U$ filtered quasi-isomorphism type of $(\mathcal{C}_{U\cup\beta}(D), \partial)$ is a braid conjugacy class invariant.*

Firstly, we realize that grid diagram of any braid in the conjugacy class of β can be obtained using the following moves

$$D\Big/\text{commutation moves} + \text{(de)stabilization moves on the components of } \beta.$$

This is true since any braid word move can be made with grid moves on the β component and for conjugation, we need to use commutation moves with U along with grid moves on β component. So to prove this invariance, we would like to explore the maps induced by commutation and stabilization moves in grid diagram. We will show that those maps can be obtained by taking quotients of the maps in the usual grid link complex. The reader is referred to [36] for more details.

Commutation moves

Consider two grid diagrams D and D' of $U \cup \beta$ that differ by a commutation move. We can represent these two diagrams in the same picture so that the X and O markings are fixed, and two of the vertical circles are curved [See Figure 4.2]. Denote the horizontal circles of D by $\{\alpha_1, ...\alpha_n\}$ and its vertical circles by $\{\beta_1, ...\beta_n\}$. Then the set of horizontal circles of D' is also $\{\alpha_1, ...\alpha_n\}$ and its vertical circles are given by $\{\beta_1, ..., \beta_{i-1}, \gamma_i, \beta_{i+1}, ..., \beta_n\}$. The vertical circles β_i and γ_i intersect at two points. The complement of $\beta_i \cup \gamma_i$ in the grid, consists of two bigons. Consider the one, of which the western boundary is a part of β_i, and the eastern boundary is a part of γ_i. We label the two intersection points by s and s' so that s is the southern vertex and s' is the northern vertex of that bigon.

We will need to define the pentagons in this picture to define the map corresponding to the commutation move.

Definition 4.2.2. *Consider two grid states $x \in S(D)$ and $y' \in S(D')$. A pentagon Π from x to y' is an embedded disk in the torus that satisfies the following conditions:*

- *The boundary of Π is the union of five arcs lying in α_j, β_j or γ_i for i and for some j.*
- *Four corners of Π are in $x \cup y'$.*
- *If we consider any corner point of Π, it is the intersection of two curves of $\{\alpha_j, \beta_j, \gamma_i\}_{j=1}^n$. These two curves divide a small disk on the torus into four quadrants, and Π intersects exactly one of them.*
- *The horizontal segments in the boundary of Π point from the components of x to the components of y'.*

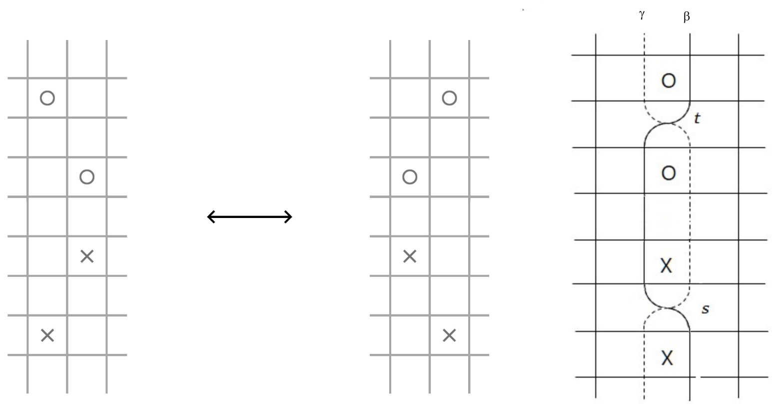


Figure 4.2: Diagrams in commutation move

We use the notation $\mathrm{Pent}(x, y')$ for the set of pentagons going from x to y'. Observe that the set $\mathrm{Pent}(x, y')$ consists of at most one element, and it is empty unless x and y' share exactly $n-2$ elements. From the above properties of a pentagon follows that its fifth corner point is the distinguished point s. Pentagons from y' to x are defined similarly. However, in that case, the fifth vertex is given by the distinguished point t. A pentagon $\Pi \in Pent(x, y')$ an empty pentagon if it doesn't contain any point of x in the interior. The set of empty pentagons from x to y' is denoted by $Pent^{\circ}(x, y')$.

Define the $\mathcal{R}$-module map $P\colon \mathcal{C}_{U\cup\beta}(D) \to \mathcal{C}_{U\cup\beta}(D')$ by the formula:

$$P(x) = \sum_{y' \in S(D')} \sum_{\Pi \in Pent^{\circ}(x,y'), \Pi \cap \mathbb{X} = \phi} V_3^{O_3(\Pi)} \cdots V_n^{O_n(\Pi)} \cdot y'$$

Proposition 4.2.3. *P is A_β graded and $\mathcal{F}_U$ filtered.*

Proof. Suppose Π is an empty pentagon from x to y' in the expansion of $P(x)$. Then,

$$A_U(x) - A_U(y') = O_U(\Pi) \text{ and } A_\beta(x) - A_\beta(y') = O_\beta(\Pi).$$

The conclusion follows by taking the sum with proper weights.

□

Proposition 4.2.4. *The map P is a chain map.*

Proof. Consider the $\mathbb{F}_2[V_1, V_2, \cdots, V_n]$-module map $\mathbf{P} : GC^-(D) \to GC^-(D)$ given by

$$\mathbf{P}(x) = \sum_{y' \in S(D')} \sum_{\Pi \in Pent^{\circ}(x,y'), \Pi \cap \mathbb{X} = \phi} V_1^{O_1(\Pi)} V_2^{O_2(\Pi)} V_3^{O_3(\Pi)} \cdots V_n^{O_n(\Pi)} \cdot y'$$

We know $\mathbf{P}$ is a chain map [See Lemma 5.1.4 in [36]]. Then, P is induced map on the quotient $\frac{GC^-(D)}{(V_1-1)(V_2-1)}$. Therefore, P is a chain map since $\mathbf{P}$ is a chain map. □

Proposition 4.2.5. *The map P is a quasi-isomorphism.*

Proof. Let $Hex(x, y)$ for the set of hexagons going from x to y.

Definition 4.2.3. *We call a hexagon $h \in \mathrm{Hex}(x, y)$ an empty hexagon if*

$$x \cap Int(h) = y \cap Int(h) = \emptyset.$$

The set of empty hexagons from x to y is denoted by $Hex^\circ(x, y)$.

Define the $\mathcal{R}$-module homomorphism $H \colon \mathcal{C}_{U\cup\beta}(D) \to \mathcal{C}_{U\cup\beta}(D)$ for each $x \in S(D)$ by the formula:

$$H(x) = \sum_{y \in S(D)} \left(\sum_{h \in \mathrm{Hex}^\circ(x,y)} V_3^{O_3(h)} \cdots V_n^{O_n(h)} \right) \cdot y.$$

Now lets consider the $\mathbb{F}_2[V_1, V_2, \cdots, V_n]$-module map $\mathbf{H} : GC^-(D) \to GC^-(D)$ given by

$$\mathbf{H}(x) = \sum_{y \in S(D)} \sum_{h \in Hex^\circ(x,y'), h \cap \mathbb{X} = \phi} V_1^{O_1(h)} V_2^{O_2(h)} V_3^{O_3(h)} \cdots V_n^{O_n(h)} \cdot y.$$

Then, H is induced map on the quotient $\frac{GC^-(D)}{(V_1-1)(V_2-1)}$. The map $\mathbf{H}$ satisfies [See Lemma 5.1.6 in [36]]

$$\partial^- \circ \mathbf{H} + \mathbf{H} \circ \partial = Id - \mathbf{P}' \circ \mathbf{P}.$$

It follows that the induced quotient map is chain homotopy between $P \circ P'$ and identity. Therefore, the conclusion follows.

□

Stabilization moves

Let D be a grid diagram. By performing a stabilization of type $X \colon SW$, we get the diagram D' . Number the markings in the way that O_i is the newly-introduced O-marking, O_{i+1} is in the consecutive row below O_i, X_i and X_{i+1} lie in the same row as O_i and O_{i+1}, respectively, i.e. $\begin{array}{c|c} X_i & O_i \\ \hline & X_{i+1} \end{array}$.

Denote c the intersection point of the new horizontal and vertical circles in D'. Considering this point, we can partition the grid states of the stabilized diagram D' into two parts, depending on whether or not they contain the intersection point c. Define the sets $I(D')$ and $N(D')$ so that $x \in I(D')$ if c is included in x, and $x \in N(D')$ if c is not included in x. Now $S(D') = I(D') \cup N(D')$ gives a decomposition of $\mathcal{C}_{U\cup\beta}(D') \cong I \oplus N$, where I and N denote the $\mathcal{R}$-modules spanned by the grid states of $I(D')$ and $N(D')$ respectively.

There is a one-to-one correspondence between grid states of $I(D')$ and grid states of $S(D)$: Let

$$e : I(D') \to S(D), \quad x \cup \{c\} \mapsto x.$$

So the map e is well defined for the generators of GC^-. Then we can extend the definition of e to $\mathcal{C}_{U\cup\beta}(D')$ by first linearly extending to to GC^- and then passing to the quotient.

Proposition 4.2.6. *The map e is a filtered quasi-isomorphism.*

Proof. The same argument works in this case. The map e can be seen as a quotient of a filtered quasi-isomorphism that is defined in GC^- [See Section 5.2 in [36]]. Then, taking quotients of the homotopy equivalences show that e is a filtered quasi-isomorphism.

□

4.2.3 Relation with $CFK^{-,2}$ complex

We review the construction used in [2] and show how their chain complex is related to ours. A multi-pointed Heegaard diagram for an oriented link $L \subset Y$ is given by a ordered tuple $H = (\Sigma, \alpha, \beta, z_L, w_L \cup w_f)$, where w_f is the set of free base points. A grid diagram can be naturally viewed as a multi-pointed Heegard diagram embedded in torus. Lets denote the the grid of $\beta \cup U$ as the multi-pointed Heegard diagram, $\mathcal{H}_1 = (T^2, \alpha, \beta, z_\beta \cup z_U, w_\beta \cup w_U)$. Here $'z'$s denote the X markings and $'w'$s denote the O markings. If we drop the points z_U, then we get the variant $\mathcal{H}_4 = (T^2, \alpha, \beta, z_\beta, w_\beta \cup w_U)$ where the set w_U work as free basepoints. This variant is considered in [2] to introduce the reformulation of GRID invariant. $CFK^{-,2}(\mathcal{H}_4)$ denotes the knot Floer complex associated with $\mathcal{H}_4$.

Proposition 4.2.7. *$\mathcal{C}_{U\cup\beta}$ is filtered quasi-isomorphic to $CFK^{-,2}(\mathcal{H}_4)$.*

Proof. It is easy to see that these complexes are isomorphic after reversal of roles of X and O markings in the unknot component, which also explains the orientation convention of the unknot in these two complexes.

□

In their paper [2] , they also show that $H_{top}({\mathcal{F}_U}^{bot}(\mathcal{C}_{U\cup\beta}))$ is generated by the distinguished cycle $[x_4]$ which in our complex is the state consting of north-east corners of X-markings (Here *bot* is minimum value of $\mathcal{F}_U$). We will use this relationship to refine the transverse invariant.

4.2.4 Properties of $\mathcal{C}_{U\cup\beta}$

Proposition 4.2.8. *Multiplication by V_i is chain homotopic to multiplication by V_j if O_i and O_j belong to the same link component in β.*

Proof. Let X_k be the X-marking that is in same row as O_m and in the same column as O_n. Define $H_{X_k} : \mathcal{C}_{U\cup\beta} \to \mathcal{C}_{U\cup\beta}$,

$$H_{X_k}(x) := \sum_{y\in S(D)} \sum_{r\in Rect^o(x,y), r\cap\mathbb{X}=X_k} V_3^{O_3(r)} V_4^{O_4(r)} \cdots V_n^{O_n(r)} y \quad \forall x \in S(D)$$

Then, in the compositions of rectangles appearing in $\partial H_{X_k} + H_{X_k}\partial$, contributions from all but two annuli containing X_k cancel. So we get,

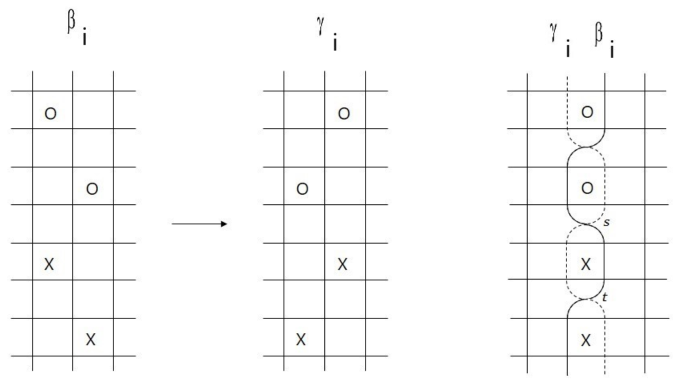


Figure 4.3: Crossing change move

$$\partial H_{X_k} + H_{X_k}\partial = V_m - V_n.$$

It follows that V_m and V_n are chain homotopic. Iterating this argument shows that V_i and V_j are chain homotopic if O_i and O_j belong to the same link component.

□

In view of the last result, we can think of $\mathcal{C}_{U\cup\beta}$ as a $\mathbb{F}_2[V_{i_1}, \cdots, V_{i_l}]$-module. We can also consider the complex $c\mathcal{C}_{U\cup\beta} \cong \frac{\mathcal{C}_{U\cup\beta}}{V_{i_1}=\cdots=V_{i_l}}$. It can be easily seen that $\mathcal{F}_U$ filtered quasi-isomorphism type of $c\mathcal{C}_{U\cup\beta}$ is also a braid conjugacy class invariant and its homology can be thought of as a $\mathbb{F}_2[V]$-module.

4.3 Refinement of $\hat{\theta}$ invariant

4.3.1 Crossing change move

Suppose L^+ is obtained from L^- by changing a negative crossing to positive crossing. The crossing change maps between their GC^- version of grid complexes was defined in Chapter 6 of [36]. Now, we will study the effect of crossing change map in the complex. First, we discuss the effect of changing a positive crossing to negative in the braid β. The maps associated with crossing change move will also appear for positive and negative stabilizations.

Suppose the diagram D^- representing $U \cup \beta^-$is obtained by cross commutation of the two columns from the diagram D^+ representing $U \cup \beta^+$ (See Figure 4.3.3). The vertical circle β_i in D^+ is replaced by the dotted vertical circle γ_i in D^- and, they intersect at two points s and t. We define the $\mathcal{R}$-module maps $c_- : \mathcal{C}_{U\cup\beta^+}(D^+) \to \mathcal{C}_{U\cup\beta^-}(D^-)$ and

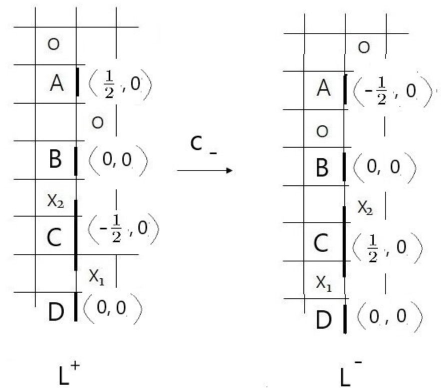


Figure 4.4: Local change in (A_β, A_U) grading

$c_+ : \mathcal{C}_{U\cup\beta^+}(D^-) \to \mathcal{C}_{U\cup\beta^-}(D^+)$ for a grid state $x \in S(D^+)$ and $y' \in S(D^-)$ respectively in the following way

$$c_-(x) = \sum_{y\in S(D^-)} \sum_{p\in Pent_s{}^o(x,y),\ \mathbb{X}\cap p=\phi} V_3^{O_p(r)} V_4^{O_4(p)} \cdots V_n^{O_n(p)} y$$

and,

$$c_+(y') = \sum_{x'\in S(D^+)} \sum_{p\in Pent_t{}^o(y',x'),\ \mathbb{X}\cap p=\phi} V_3^{O_3(p)} V_4^{O_p(r)} \cdots V_n^{O_n(p)} x'.$$

Proposition 4.3.1. *The map c_- is A_β graded and $\mathcal{F}_U$ filtered. The map c_+ is A_β graded of degree 1 and $\mathcal{F}_U$ filtered.*

Proof. To compute degrees of c_- and c_+ as in Lemma 6.2.1 of [36], we can compare the pentagons appearing in those maps with rectangles. First notice, that each state s in D^+ corresponds to a state $\phi(s)$ in D^- by mapping the point in β_i vertical circle to the γ_i circle. Now, the 4 X and O markings divide the vertical circles β_i and γ_i into four intervals A, B, C and D. We consider states that contains a point in on of the the four intervals A, B, C and D [See Figure 4.4] between the four special markings. We can make local computations in each case (based on relative position of markings) to compute the difference in gradings between a state and its corresponding state. Now if y is a term appearing in $c_-(x)$, the there is a pentagon from x to y. To each pentagon from a state in D^+ to a state in D^-, we can associate a rectangle [See Figure 4.5] in D^+ from x to $\phi(y)$. This allows us to compute grading change under c_- by the following formula

$$A(x) - A(y) = (A(x) - A(\phi^-(y)) + (A(\phi^-(y) - A(y)).$$

The first term $A(x)-A(\phi(y))$ just counts extra contribution in the associated rectangle as the differential preserves degree and filtration, and the second term $A(\phi(y)-A(y)$ computes the local change. In this case, it is easy to see the $\mathcal{F}_U$ filtration is unaffected. Similarly, we

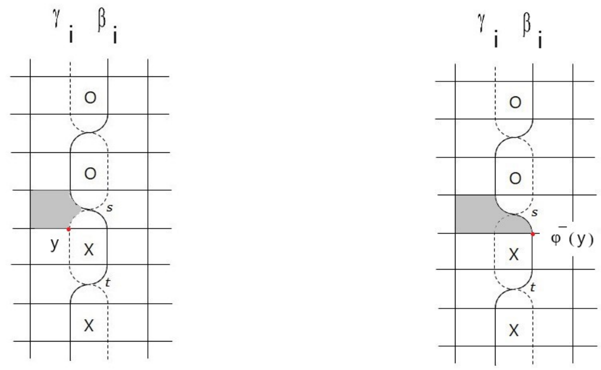


Figure 4.5: A left pentagon and its associated rectangle

can compute grading change under c_+. In our case, there is no change in $\mathcal{F}_U$ filtration level for either c_- or c_+ as there no local change in A_U grading or any extra X marking belonging to the unknot component in the associated rectangles. So those maps are $\mathcal{F}_U$ filtered. Then, the computation of A_β grading change under c_- and c_+ turns out to be identical to Lemma 6.2.1 of [36].

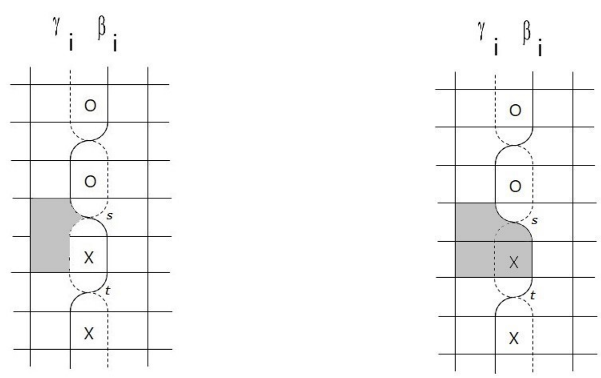


Figure 4.6: Associated rectangle for a left pentagon in interval C

Let y be the term, appearing in $c_-(x)$ and assume there is a left pentagon from x to y.
Case 1: y is type B Since there are no extra markings in the associated rectangles [See

Figure 4.5] we have,

$$A_\beta(y) - A_\beta(x) = A_\beta(\phi^-(y)) - A_\beta(x) + A_\beta(y) - A_\beta(\phi^-(y)) = 0$$

and

$$\mathcal{F}_U(y) - \mathcal{F}_U(x) = \mathcal{F}_U(\phi^-(y)) - \mathcal{F}_U(x) + \mathcal{F}_U(y) - \mathcal{F}_U(\phi^-(y)) \leq \mathcal{F}_U(\phi^-(y)) - \mathcal{F}_U(y) = 0.$$

Case 2: y is type C In this case, we have an extra X marking belonging to the braid β [See Figure 4.6] in the associated rectangle. So,

$$A_\beta(y) - A_\beta(x) = A_\beta(\phi^-(y)) - A_\beta(x) + A_\beta(y) - A_\beta(\phi^-(y)) = 1 - 1 = 0$$

and

$$\mathcal{F}_U(y) - \mathcal{F}_U(x) = \mathcal{F}_U(\phi^-(y)) - \mathcal{F}_U(x) + \mathcal{F}_U(y) - \mathcal{F}_U(\phi^-(y)) \leq \mathcal{F}_U(\phi^-(y)) - \mathcal{F}_U(y) = 0.$$

For right pentagons, we consider initial corners in B and C and the computation works similarly. We can also compute grading and filtered degree of c_+ using the same technique. □

Proposition 4.3.2. *The map c_- and c_+ are chain maps.*

Proof. Again, consider the $\mathbb{F}_2[V_1, V_2, \cdots, V_n]$-module map $\mathbf{c}_- : GC^-(D^+) \to GC^-(D^-)$ given by

$$\mathbf{c}_-(x) = \sum_{y' \in S(D^-)} \sum_{p \in Pent_s{}^\circ(x,y'), p \cap \mathbb{X} = \phi} V_1^{O_1(p)} V_2^{O_2(p)} V_3^{O_3(p)} \cdots V_n^{O_n(p)} \cdot y'$$

Then, c_- is induced map on the quotient $\frac{GC^-(D)}{(V_1-1)(V_2-1)}$. Therefore, c_- is a chain map since $\mathbf{c}_-$ is a chain map. A similar argument shows that c_+ is a chain map. □

The chain maps c_- and c_+ induce the desired maps C_- and C_+ on the homologies.

Proposition 4.3.3. *$c_+ \circ c_-$ and $c_- \circ c_+$ are chain homotopic to multiplication by V_i.*

Proof. For $x_-, y_- \in S(D^-)$, let $\mathrm{Hex}^\circ_{s,t}(x_-, y_-)$ denote the set of empty hexagons with two consecutive corners at s and at t in the order consistent with the orientation of the hexagon. The set $\mathrm{Hex}^\circ_{s,t}$ for $x_+, y_+ \in S(D^+)$ is defined analogously.

Let $H_- : \mathcal{C}_{U\cup\beta}(D^-) \to \mathcal{C}_{U\cup\beta}(D^-)$ be the $\mathcal{R}$-module map whose value on any $x_- \in S(D^-)$ is

$$H_-(x_-) = \sum_{y_- \in S(D^-)} \sum_{h \in \mathrm{Hex}^\circ_{s,t}(x_-,y_-)} V_3^{O_3(h)} \cdots V_n^{O_n(h)} \cdot y_-.$$

The analogous map $H_+ : \mathcal{C}_{U\cup\beta}(D^+) \to \mathcal{C}_{U\cup\beta}(D^+)$ is defined in the same way using $\mathrm{Hex}^\circ_{s,t}(x_+, y_+)$.

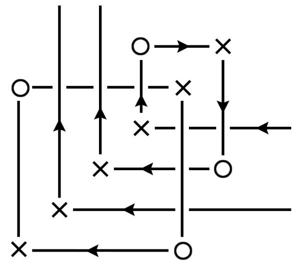


Figure 4.7: Positive stabilization

Following the lines of the proof of Proposition 6.1.1. in [36], we can easily verify that H_+ is a chain homotopy between $c_+ \circ c_-$ and the multiplication by V_i, and that H_- is a chain homotopy between $c_- \circ c_+$ and V_i, i.e.:

$$\partial \circ H_+ + H_+ \circ \partial = c_+ \circ c_- + V_i,$$
$$\partial \circ H_- + H_- \circ \partial = c_- \circ c_+ + V_i.$$

□

4.3.2 Positive stabilization

Suppose we have a positive stabilization diagram D^+ (See Figure 4.7) and D^- is obtained by cross commutation of the two columns in the left one belonging to the unknot and the other one belonging to the braid β as shown in Figure 4.7. It is easy to see that then D^- represents $U \cup \beta$. We define the $\mathcal{R}$-module maps $PS_+ : \mathcal{C}_{U\cup\beta_{+stab}}(D^-) \to \mathcal{C}_{U\cup\beta}(D^+)$ and $PS_- : \mathcal{C}_{U\cup\beta}(D^+) \to \mathcal{C}_{U\cup\beta_{+stab}}(D^-)$ for a grid state $x \in S(D^+)$ and $y' \in S(D^-)$ respectively in the following way:

$$PS_-(x) = \sum_{y\in S(D^-)} \sum_{p\in Pent_s{}^o(x,y),\ \mathbb{X}\cap p=\phi} V_3^{O_p(r)} V_4^{O_4(p)} \cdots V_n^{O_n(p)} y$$

$$PS_+(y') = \sum_{x'\in S(D^+)} \sum_{p\in Pent_s{}^o(y',x'),\ \mathbb{X}\cap p=\phi} V_3^{O_3(p)} V_4^{O_p(r)} \cdots V_n^{O_n(p)} x'$$

Proposition 4.3.4. *The maps PS_- is $\mathcal{F}_U$ filtered and PS_+ is $\mathcal{F}_U$ filtered of degree $\frac{1}{2}$.*

Proof. We inspect the intervals A, B, C and D again [See Figure 4.8]. Local change in A_U can be computed easily using the winding number formula [Equation 2.2] for A_U. Then, we can use the fact that $\mathcal{F}_U = -A_U$ for grid states to compute filtration change. If y is a term appearing in $PS_-(x)$ and there is an empty left pentagon p (pentagon to the the left of vertical circle β_i or γ_i) from x to y. Notice that, that the terminal generator y is either of

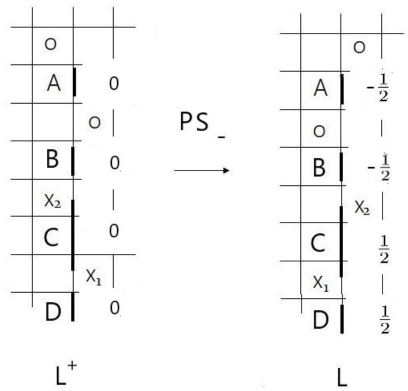


Figure 4.8: Change in local A_U grading

type B or C [See Lemma 6.2.1 of [36]]. So we can use associated left rectangles for the left pentagons to compute grading change as in Proposition 4.3.1.

Case 1: y is type B Since there are no extra markings in the associated rectangles [See Figure 4.5] we have,

$$\mathcal{F}_U(y) - \mathcal{F}_U(x) = \mathcal{F}_U(\phi^-(y)) - \mathcal{F}_U(x) + \mathcal{F}_U(y) - \mathcal{F}_U(\phi^-(y)) \leq \mathcal{F}_U(\phi^-(y)) - \mathcal{F}_U(y) = -\frac{1}{2}.$$

Case 2: y is type C In this case, we have an extra X marking belonging to the unknot [See Figure 4.6] in the associated rectangle. So,

$$\mathcal{F}_U(y) - \mathcal{F}_U(x) = \mathcal{F}_U(\phi^-(y)) - \mathcal{F}_U(x) + \mathcal{F}_U(\phi^-(y)) - \mathcal{F}_U(y) \leq -1 + \mathcal{F}_U(\phi^-(y)) - \mathcal{F}_U(y) = -1 + \frac{1}{2} = -\frac{1}{2}.$$

Similarly for a right pentagon (pentagon to the the right of vertical circle β_i or γ_i), we compare it with a right rectangle. Here the initial corner is either of type B or type C. In each case, we get filtration change = 0. Therefore, PS_- is $\mathcal{F}_U$ filtered of degree 0.

Now, let y' be a term appearing in $PS_+(x')$.

Case 1: y' is type B From Figure 4.9, we observe that the associated rectangle has one less O marking belonging to the unknot. So we have,

$$\mathcal{F}_U(y') - \mathcal{F}_U(x') = \mathcal{F}_U(\phi(y')) - \mathcal{F}_U(x') + \mathcal{F}_U(y') - \mathcal{F}_U(\phi(y')) \leq 1 + \mathcal{F}_U(y') - \mathcal{F}_U(\phi(y')) = \frac{1}{2}.$$

Case 2: y' is type C There are no additional markings. So we have,

$$\mathcal{F}_U(y') - \mathcal{F}_U(x') = \mathcal{F}_U(\phi(y')) - \mathcal{F}_U(x') + \mathcal{F}_U(y') - \mathcal{F}_U(\phi(y')) \leq \mathcal{F}_U(y') - \mathcal{F}_U(\phi(y')) = \frac{1}{2}.$$

Case 3: y' is type D Again there are no additional markings. Therefore,

$$\mathcal{F}_U(y') - \mathcal{F}_U(x') = \mathcal{F}_U(\phi(y')) - \mathcal{F}_U(x') + \mathcal{F}_U(y') - \mathcal{F}_U(\phi(y')) \leq \mathcal{F}_U(y') - \mathcal{F}_U(\phi(y')) = \frac{1}{2}.$$

Case 4: y' **is type A** Again the associated rectangle has one less O marking belonging to the unknot. So we have,

$$\mathcal{F}_U(y') - \mathcal{F}_U(x') = \mathcal{F}_U(\phi(y')) - \mathcal{F}_U(x') + \mathcal{F}_U(y') - \mathcal{F}_U(\phi(y')) \leq 1 + \mathcal{F}_U(y') - \mathcal{F}_U(\phi(y')) = \frac{1}{2}.$$

Hence, the map PS_+ is $\mathcal{F}_U$ filtered of degree $\frac{1}{2}$.

□

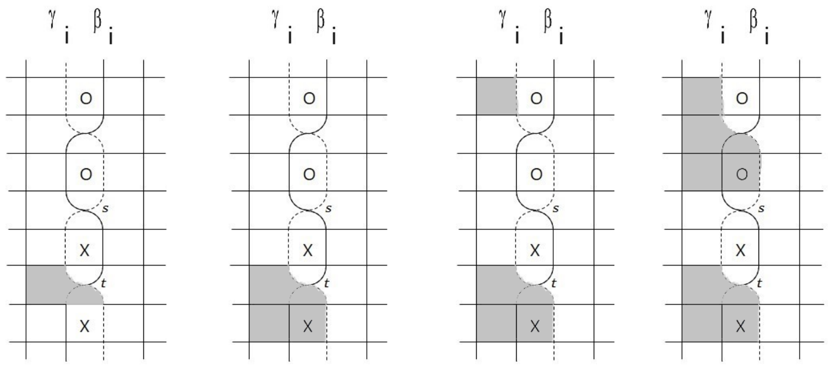


Figure 4.9: Associated rectangle for interval A, B, C and D in the map PS_+

Proposition 4.3.5. *The maps PS_- and PS_+ are chain maps.*

Proof. This same argument shows that $\partial \circ PS_- + PS_- \circ \partial = 0$ and $\partial \circ PS_+ + PS_+ \circ \partial = 0$. □

Proposition 4.3.6. *PS_- and PS_+ are quasi-isomorphisms.*

Proof. Let us revisit the homotopy equivalence maps from the previous section again. In this case, the same argument shows

$$\partial \circ H_+ + H_+ \circ \partial = PS_+ \circ PS_- + 1,$$
$$\partial \circ H_- + H_- \circ \partial = PS_- \circ PS_+ + 1.$$

The conclusion follows. □

4.3.3 Refinement of θ

Let x_4 be the distinguished cycle consisting of northeast corners of X-markings in D as described before. Let V be the subset of homology generated by V_i's. Any class belonging in V is said to be in V-image.

Definition 4.3.1. *Define $\eta(\beta) = \min \{ \ k \mid [x_4]$ is in the V-image in $H_*({\mathcal{F}_U}^k(\mathcal{C}_{U\cup\beta})\}$.*

Theorem 4.3.1. *$\eta(\beta)$ is a braid conjugacy class invariant. Also $\eta(\beta) \leq \eta(\beta_{+stab}) \leq \eta(\beta)+\frac{1}{2}$.*

Proof. From [2], we know that $H_{top}({\mathcal{F}_U}^{bot}(\mathcal{C}_{U\cup\beta}))$ is generated by $[x_4]$. So we can reinterpret the definition as $\eta(\beta) = \min \{\ k \mid$ the natural inclusion map $i : H_{top}({\mathcal{F}_U}^{bot}(\mathcal{C}_{U\cup\beta})) \to \frac{H_*({\mathcal{F}_U}^{k}(\mathcal{C}_{U\cup\beta}))}{V}$ is trivial $\}$. It follows that $\eta(\beta)$ is a braid conjugacy class invariant. Since PS_- and PS_+ are filtered quasi-isomorphisms, the second claim follows from the filtered degrees of the maps PS_- and PS_+.

□

Using the above theorem, we know that $\eta(\beta)$ is non decreasing under positive stabilization. So it is possible to define a transverse invariant $\bar{\eta}(\beta)$ by taking minimum over all braid representatives.

Theorem 4.3.2. *Let β be a N-braid and $\mathcal{T}$ be the transverse link represented by β. If $\hat{\theta}(\mathcal{T}) \neq 0$ then $\eta(\beta) = \infty$ and $-\frac{N}{2} \leq \eta(\beta) \leq \frac{N}{2}$ otherwise.*

Proof. From [2], we know that there is a inclusion map $I : HFK^-(m(\beta)) \to HFK^{-,2}(\mathcal{H}_4)$ sending $\theta(\mathcal{T})$ to the distinguished class $[x_4] \in HFK^{-,2}(\mathcal{H}_4)$. Therefore, if $\hat{\theta}(\mathcal{T}) \neq 0$ then $[x_4]$ can't be in the V image in any filtration level in $HFK^{-,2} \cong H(\mathcal{C}_{U\cup\beta})$ and vice-versa. Hence, it follows that $\eta(\beta) = \infty$ if and only if $\hat{\theta}(\mathcal{T}) \neq 0$. Also since $[x_4]$ generates the top Maslov grading in the bottom filtration in $HFK^{-,2}$. From the winding number formula [Equation 2.2], we have $bot = A_U(x^+) = -N + \frac{1}{8}4(N+1) - \frac{1}{2} = -\frac{N}{2}$. It follows that $\eta(\beta) \geq bot = -\frac{N}{2}$. Also since $\mathcal{F}_U^{\frac{N}{2}}(\mathcal{C}_{U\cup\beta}) = \mathcal{C}_{U\cup\beta}$, it is obvious that $\frac{N}{2} \geq \eta(\beta)$.

□

It also follows that the transverse invariant $\bar{\eta}(\beta)$ is atleast as strong as $\hat{\theta}$.

Proposition 4.3.7. *If β^+ is obtained from β^- by changing a negative crossing to positive crossing then, $\eta(\beta^+) \geq \eta(\beta^-)$.*

Proof. It can be shown that the crossing change map C_- sends the cycle $[x_4]$ in $H(\mathcal{C}_{U\cup\beta^+})$ to $[x_4]$ in $H(\mathcal{C}_{U\cup\beta^-})$. Since C_- is filtered of degree 0, the conclusion follows. □

4.3.4 Negative stabilization

Now lets consider a negative stabilization diagram as in Figure 4.10. Then we can define analogous maps $NS_- : \mathcal{C}_{U\cup\beta_{-stab}}(D^+) \to \mathcal{C}_{U\cup\beta}(D^-)$ and $NS_+ : \mathcal{C}_{U\cup\beta}(D^-) \to \mathcal{C}_{U\cup\beta_{-stab}}(D^+)$ be the analogous $\mathcal{R}$-module maps. Again the same argument shows that these are chain maps. However, it is easily checked that they are not filtered quasi-isomorphisms through the next proposition.

Proposition 4.3.8. *$NS_+ \circ NS_-$ is chain homotopic to V_i.*

Proof. This also follows from the discussion in Proposition 4.3.3.

□

Hence, unlike positive stabilizations these maps are not quasi-isomorphisms. In fact, in the next proposition, we show that negative stabilizations have a special value.

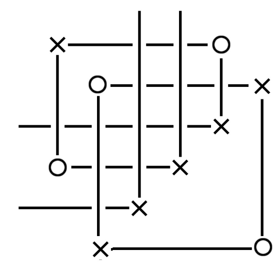


Figure 4.10: Negative stabilization

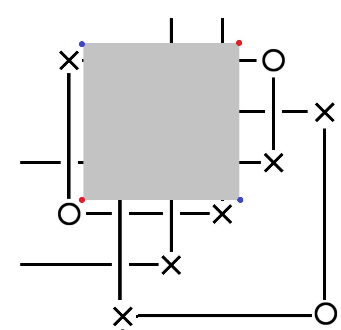


Figure 4.11: Rectangle to the distinguished state in a negative stabilization

Proposition 4.3.9. $\eta(\beta_{-stab}) = \frac{-N+1}{2}$.

Proof. Let r be red colored state in Figure 4.11 where x_4 is depicted by the blue colored state. It is easy to see that $r \in \mathcal{F}^{\frac{-N+1}{2}}(\mathcal{C}_{U\cup\beta})$. By considering the rectangles from r, we find that either they must contain a O marking in the β component or they connect to the distinguished state x_4. So, $\partial r = x_4 + V_i(..)$. It follows that $[x_4]$ is V-image in $\mathcal{F}^{\frac{-N+1}{2}}(\mathcal{C}_{U\cup\beta})$ and the conclusion follows. $\square$

Remark. *These properties of $\eta(\beta)$ are very similar to the Kapppa invariant [18] in Khovanov homology which served as a motivation for defining η. It is not clear at this point how one can compute the transverse $\bar{\eta}(\beta)$. However for Kapppa invariant, computation suggests that the minimum of Kappa over all braid representatives is probably an effective transverse invariant.*

Chapter 5

Annular Invariant

In this chapter, we will define an invariant of annular links. The invariant is a piece-wise linear function from $[0, 2] \to \mathbb{R}$. The construction is analogous construction by Grigsby, Wehrli and Licata [13] in Khovanov homology. It gives a lower bound on annular cobordisms. It also has an alternative description in terms of max grading of a non-torsion element of a deformed complex $t\mathbf{C}$. We recover the braid complex from the last chapter in the special case $t = 0$. We study the invariant under crossing change and stabilizations using the deformed complex. The knowledge of maps associated with crossing change and stabilizations allows us to define braid monoids with properties similar to those defined by Grigsby, Wehrli and Licata [13].

5.1 Annular concordance invariant

5.1.1 Definition

Let L be an oriented link in $\mathbb{R}^3$. An annular link is $L' = L \cup U$ where U is an unknot [See Fig 5.1]. We will assume that U is oriented clockwise in x-y plane. So $-U$ will indicate the anticlockwise orientation.

We will consider the grid chain complex $(\widehat{\mathcal{GC}}(D), \widehat{\partial})$, where D is a toroidal grid diagram

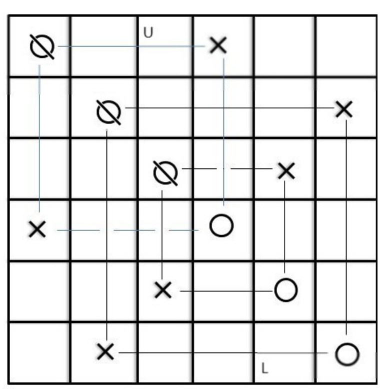


Figure 5.1: Grid of $U \cup L$

of $U \cup L$; U being the unknot (oriented clockwise) and L an oriented link. Suppose D has grid number n and l is the number of components of L. We can write the set of non-special O- markings as $\mathbb{O} \setminus s\mathbb{O} = \{O_1, O_2, .., O_{n-l}\}$.

We will call Alexander filtration for the unknot U as A_U and $(A_1, ..., A_l)$ = Alexander filtrations for l components of L. We denote the sum $A = A_1 + .. + A_l$ as simply A_L and $A = A_U + A_L$ is the total Alexander grading.

Definition 5.1.1. *Define* $\mathcal{F}_t(x)$ *to be* $\frac{t}{2}A_U(x) + (1 - \frac{t}{2})A_L(x)$ *for each* $0 \le t \le 2$ *and* $x \in \widehat{\mathcal{GC}}(D)$.

Since $\mathcal{F}_t(x)(\hat{\partial}(x)) \le \mathcal{F}_t(x)(x)$, $\mathcal{F}_t$ gives filtration levels defined as $\mathcal{F}_t^s(\widehat{\mathcal{GC}}) = \{a \in \widehat{\mathcal{GC}}(D) | \mathcal{F}_t(a) \le s\}$ for every $s \in \mathbb{R}$. These filtration levels $.. \subseteq \mathcal{F}_t^s \subseteq \mathcal{F}_t^{s+1} \subseteq ..$ naturally induce filtration levels in the homology.

Since $U \cup L$ has $l+1$ components from [36], we know that $\widehat{\mathcal{GH}}(U \cup L) = H_*(\widehat{\mathcal{GC}}, \widehat{\partial}) \cong (\mathbb{F}_2)^{2^l}$ and $\widehat{\mathcal{GH}}_0(U \cup L) \cong \mathbb{F}_2$ ($\widehat{\mathcal{GH}}_0$ is the homology at maslov grading 0)

Definition 5.1.2. *Define* $\mathscr{A}_L^U(t) := min\{s | H_0(\mathcal{F}_t^s(\widehat{\mathcal{GC}}), \widehat{\partial}) \xrightarrow{i} \widehat{\mathcal{GH}}_0$ *is nontrivial* $\}$.

Since $\mathcal{F}_t^s(\widehat{\mathcal{GC}}) = \widehat{\mathcal{GC}}$ for sufficiently large s and $\mathcal{F}_t^s(\widehat{\mathcal{GC}})$ is empty for sufficiently small s, it follows that $\mathscr{A}_L^U(t)$ is a finite real number.

Analogously, since the homology at Maslov grading $-l$ has rank equal to 1, we can define

Definition 5.1.3. *Define* $\mathsf{A}_L^U(t) := min\{s | H_{-l}(\mathcal{F}_t^s(\widehat{\mathcal{GC}}), \widehat{\partial}) \xrightarrow{i} \widehat{\mathcal{GH}}_{-l}$ *is nontrivial* $\}$.

We will drop U for making the notation look less cumbersome (with a slight abuse of notation) and write $\mathscr{A}_L^U(t)$ and $\mathsf{A}_L^U(t)$ as $\mathscr{A}_L(t)$ and $\mathsf{A}_L(t)$.

Since filtered quasi-isomorphism type of the complex is an invariant of the link $U \cup L$, it follows that $\mathscr{A}_L(t)$ and $\mathsf{A}_L(t)$ are annular link invariants.

5.1.2 Adding special O-markings

Now we will show that if we add any number of special O markings the invariant $\mathscr{A}_L(t)$ (defined in the same way in that diagram) remains the same.

In particular, we may define the invariant in the $\widetilde{\mathcal{GC}}$ version which is useful for computations. First, we notice that $\widetilde{\mathcal{GC}} \cong \widetilde{GH} \otimes W^{\otimes n-l-1}$. So $\widetilde{\mathcal{GH}}_0$ and $\widetilde{\mathcal{GH}}_{1-n}$ have rank 1.

Definition 5.1.4. $\widetilde{\mathscr{A}_L(t)} := min\{s | H_0(\mathcal{F}_t^s(\widetilde{\mathcal{GC}}), \widetilde{\partial}) \xrightarrow{i} \widetilde{\mathcal{GH}}_0$ *is nontrivial* $\}$.

We have the following equivalence,

Proposition 5.1.1. $\mathscr{A}_L(t) = \widetilde{\mathscr{A}_L(t)}$.

Proof. Let,

$$W = 2 \text{ dimensional graded vector space generated by } \{v_L^+, v_L^-\}$$
$$[\text{with } (M, A_U, A_L)(v_L^+) = (0,0,0) \text{ and } (M, A_U, A_L)(v_L^-) = (-1,0,-1)].$$

and

$$W_U = \text{ 2 dimensional graded vector space generated by } \{v_U^+, v_U^-\}$$
$$[\text{with } (M, A_U, A_L)(v_U^+) = (0,0,0) \text{ and } (M, A_U, A_L)(v_U^-) = (-1,-1,0)].$$

From Proposition 2.4.3, we know that $(\widetilde{\mathcal{GC}}, \widetilde{\partial})$ is $(A_L{,}A_U)$ filtered quasi-isomorphic to $(\widehat{\mathcal{GC}} \otimes W^{\otimes n-l-2} \otimes W_U, \widehat{\partial})$. So it follows that $\mathscr{A}_L(t) = \widetilde{\mathscr{A}_L(t)}$. ☐

Similarly, we can define

Definition 5.1.5. $\widetilde{\mathsf{A}_L}(t) := min\{s|H_{1-n}(\mathcal{F}_t^s(\widetilde{\mathcal{GC}}), \widetilde{\partial}) \xrightarrow{i} \widetilde{\mathcal{GH}}_{1-n}$ *is nontrivial* $\}$ *[where* n *is the grid number of* $U \cup L$*].*

Then, we have the following relation

Proposition 5.1.2. $\widetilde{\mathsf{A}}_L(t) = \mathsf{A}_L(t) - \frac{t}{2} - (n-l-2)(1-\frac{t}{2})$.

Proof. Again, since homology at Maslov grading $1-n$ has rank 1, we have $(\widetilde{\mathcal{GC}}, \widetilde{\partial})$ is $(A_L{,}A_U)$ filtered quasi-isomorphic to $(\widehat{\mathcal{GC}} \otimes W^{\otimes n-l} \otimes W_U, \widehat{\partial})$. Now, $\mathcal{F}^t(\widetilde{\mathcal{GH}}(U \cup L)_{1-n} = \mathcal{F}^t(\widetilde{\mathcal{GH}}(U \cup L)_l \otimes v_U^- \otimes v_L^{-\otimes(n-l-2)})$. Therefore, it follows from the definition that $\widetilde{\mathsf{A}}_L(t) = \mathsf{A}_L(t) - \frac{t}{2} - (n-l-2)(1-\frac{t}{2})$ ☐

5.1.3 Properties of the invariant

Proposition 5.1.3. $\mathscr{A}_{-L}^{-U}(t) = \mathscr{A}_L^U(t)$ *and* $\mathcal{A}_L^U(t) = \mathcal{A}_{-L}^{-U}(t)$.

Proof. As in [3], we can consider a map Φ from the diagram of $U \cup L$ to the reflection of the diagram along a diagonal which is a diagram of $-(U \cup L)$. We have $A_U(\Phi(x)) = A_U(x)$ and $A_L(\Phi(x)) = A_L(x)$. Then Φ is a filtered isomorphism. Therefore, it follows that if both orientations are reversed, the invariants don't change. ☐

Proposition 5.1.4. $\mathscr{A}_{L_1 \sqcup L_2}(t) = \mathscr{A}_{L_1}(t) + \mathscr{A}_{L_2}(t)$.

Proof. We observe that $(L_1 \sqcup L_2) \cup U$ can be viewed as $(L_1 \cup U_1)\#(L_2 \cup U_2)$ where the connected sum is obtained as $(U_1 \# U_2) \sqcup (L_1 \sqcup L_2)$. Let A_U be equal to $A_{U_1} + A_{U_2}$. From the Kunneth formula (See Theorem 11.1 in [29]), we know that $\widehat{\mathcal{GC}}((L_1 \cup U_1)\#(L_2 \cup U_2))$ is (A_U, A_{L_1}, A_{L_2})-filtered isomorphic to $\widehat{\mathcal{GC}}(L_1 \cup U_1) \otimes \widehat{\mathcal{GC}}((L_2 \cup U_2)$. Therefore, they are also $\mathscr{F}_t$ filtered isomorphic and the conclusion immediately follows. ☐

Proposition 5.1.5. $\mathscr{A}_{-L}(t) = -\mathscr{A}_L(t)$.

Proof. There is an annular cobordism from $L \sqcup -L$ to null. So $\mathscr{A}_{L\sqcup -L}(t) = 0$. Then we can apply Proposition5.1.4. $\square$

Proposition 5.1.6. *$\mathscr{A}_L(t) = -\mathcal{A}_{m(L)}(t)$.*

Proof. Let D be the grid diagram for $L \cup U$, and let D^* be the diagram obtained by reflecting D through a horizontal axis. Then D^* represents $m(L \cup U)$.
We know that $\widehat{GC}(D^*)$ is filtered isomorphic to $\widehat{GC}^*(D)[l-1, 0, \cdots, 0]$ [See Section 2.4.7].

This implies $H_*(\mathcal{F}_t^s(\widehat{GC}(D^*))) = H_{*-(1-l)}(\mathcal{F}^{*s}_t(\widehat{GC}^*(D)))$. Therefore, the conclusion follows.

$\square$

We can extract two invariant functions from the invariant $\mathscr{A}_L(t)$. For any $t_0 \in [0, 2)$, the slope function

$$m_{t_0}(L) := \lim_{t \to t_0^+} \frac{A_L(t) - A_L(t_0)}{t - t_0}.$$

We will also assume $m_2(L) = 0$. We also define the y-value function $y_{t_0}(L) := \mathscr{A}_L(t_0) - t_0 m_{t_0}(L)$.

Proposition 5.1.7. *(i)$\mathscr{A}_L(t)$ is a continuous piece-wise linear function.*
(ii) At a non-singular point t_0, the slope m_{t_0} is equal to $\frac{A_U(x_0)-A_L(x_0)}{2}$ for some generator $x_0 \in \widehat{\mathcal{GC}}(U \cup L)$.
(iii)If t_0 is a singular point, then the absolute value of change in slope $|\Delta m_{t_o}|$ is equal to $|\frac{A_L(x_2)-A_L(x_1)}{t_0}|$ for some generators $x_1, x_2 \in \widehat{\mathcal{GC}}(U \cup L)$.

Proof. There are only finitely many elements $x_i \in \widehat{\mathcal{GC}}(U \cup L)$ that are generators of homology at Maslov degree 0. Lets consider all linear functions $G_{x_i}(t) = \frac{t}{2}A_U(x_i) + (1 - \frac{t}{2})A_L(x_i)$. Then, $\mathscr{A}_L(t) = \min_i G_{x_i}(t)$. It follows that $\mathscr{A}_L(t)$ is a continuous piece-wise linear.

At each non-singular point t_0, there must be some generator x_0 such that $\mathscr{A}_L(t) = G_{x_0}(t)$ $\forall t \in (t_0 - \delta, t_0 + \delta)$ for some $\delta > 0$. Hence, the slope of $G_{x_0}(t)$ is equal to m_{t_0}.

At a singular point t_0 assume there is a generator generator x_1 that assumes the value of the invariant for t's slightly less than t_0 and x_2 assumes the value for t's slightly greater than t_0. Then at t_0, we must have $\frac{t_0}{2}A_U(x_1) + (1 - \frac{t_0}{2})A_L(x_1) = \frac{t_0}{2}A_U(x_2) + (1 - \frac{t_0}{2})A_L(x_2)$. So, $t_0 \Delta m_{t_0} = A_L(x_2) - A_L(x_1)$ and the conclusion follows. $\square$

Let τ be the Cavallo's invariant [4] for links.

Proposition 5.1.8. *$\mathscr{A}_L(\frac{1}{2}) = \frac{\tau(U \cup L)}{2}$.*

Proof. We have, $\mathcal{F}_{\frac{1}{2}} = \frac{1}{2}A_U + \frac{1}{2}A_L = \frac{1}{2}A$. Therefore, $\mathscr{A}_L^U(\frac{1}{2}) := min\{s | H_0(\mathcal{F}^s_{\frac{1}{2}}(\widehat{\mathcal{GC}}), \widehat{\partial}) \xrightarrow{i} \widehat{\mathcal{GH}}_0$ is nontrivial $\} = \frac{1}{2}\tau(U \cup L)$. $\square$

5.1.4 Some computations

We will give two sample computations for torus braids and trivial braid. We will derive a more general formula for quasi-positive braids in Proposition 5.3.5.

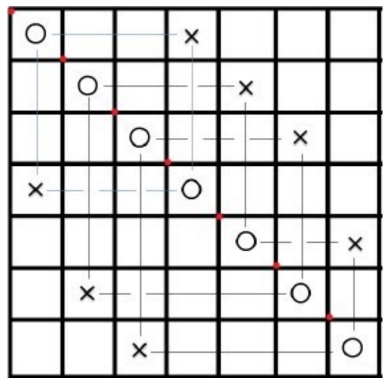


Figure 5.2: Torus braid

Proposition 5.1.9. *$\mathscr{A}_{T_{p,q}}(t) = \frac{pq-q+l}{2} + \frac{t}{4}(p+q-pq-l)$ where $T_{p,q}$ is the torus braid with l components and $0 \leq p \leq q$.*

Proof. In case of annular torus links $T_{p,q}$ with $p \leq q$, the grid diagram(of $T_{p,q} \cup U$) [See Fig 5.2] contains a unique generator x_{NWO} [37] of the filtered complex at Maslov grading 0. We can compute the A_U and A_L gradings of the generator x_{NWO} representing the cycle to determine the invariant. We have,

$$A_U(x_{NWO}) = \frac{p}{2} \text{ [Using Equation 2.2]}$$

and

$$A_{T_{p,q}}(x_{NWO}) = \frac{pq-q+l}{2} \text{ [Using Equation 2.1] .}$$

Therefore,

$$\mathscr{A}_{T_{p,q}}(t) = \frac{pq-q+l}{2} + \frac{t}{4}(p+q-pq-l)$$

□

Proposition 5.1.10. *$\mathscr{A}_{I_n}(t) = \frac{n}{2}$ where I_n is the trivial braid with n strands.*

Proof. As in the case of annular torus links, the grid diagram contains a unique cycle in Maslov grading 0 represented by the generator x_{NWO}. Also,

$$A_U(x_{NWO}) = \frac{n}{2} \text{ [Using Equation 2.2]}$$

and

$$A_{I_n}(x_{NWO}) = \frac{n}{2} \text{ [Using Equation 2.1] .}$$

Therefore,

$$\mathscr{A}_{I_n}(t) = \frac{n}{2}$$

□

5.1.5 Annular concordance

Definition 5.1.6. *An annular cobordism Σ between two annular links $L_1 \in S^3 \times \{0\}$ and $L_2 \in S^3 \times \{1\}$ is an embedded surface in $S^3 \times [0,1]$ which is disjoint from z-axis in each $S^3 \times \{i\}$ for $i \in [0,1]$ and satisfying $\partial\Sigma = L_1 \sqcup -L_2$.*

Any annular cobordism can be represented by a sequence of an identity, annular split, annular merge, annular birth, and annular death cobordisms. An annular cobordism Σ between two links L_1 and L_2 is called **strong** if the connected components of Σ are knot cobordisms between a two components of L_1 and L_2. Any strong annular cobordism can be perturbed so that it is a composition of torus cobordisms, annular birth followed by merge cobordisms and split followed by annular death cobordisms.

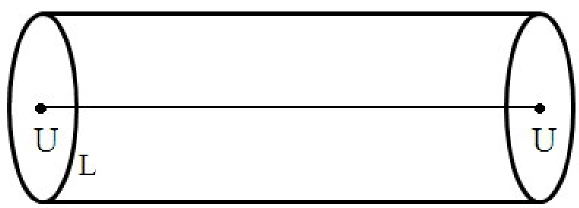


Figure 5.3: Identity cobordism

By taking slices of a cobordism, any cobordism can also be seen as a movie in the link diagram. An identity cobordism is a cobordism between a link the movie is represented by the three Reidemeister moves performed. For birth and death, the movie corresponds introduction or deletion of an unknotted circle. For merge or split cobordisms the change is represented by perturbing the link diagram like Figure 5.6. Similarly, we can easily figure out annular version of the movie [See [13] for more details].

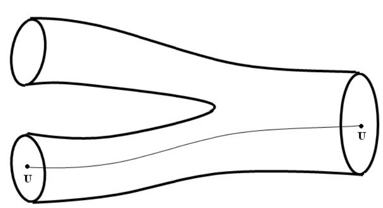


Figure 5.4: Annular merge cobordism

We know that each of these moves induces filtered maps of some degree. We check that these maps are both A_U and A_L filtered. We also compute the $\mathcal{F}_t$ grading shift that will give bounds on cobordism genus. Our construction follows the prescription given in [37].

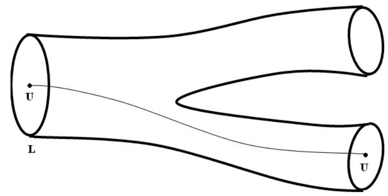


Figure 5.5: Annular Split cobordism

1. **Identity:** These maps are filtered quasi-isomorphisms, hence $\mathcal{F}_t$ filtered of degree 0.

2. **Split:** If the grid diagram D_2 of L_2 is obtained from D_1 of L_1 by a split move, then it can represented as the effect of swapping the positions of two Xs(in one of the components of L_1 in $2*2$ block. Notice that we need an extra special marking in D_2 for $\widehat{\mathcal{GC}}$ version. So, we consider the $\widetilde{\mathcal{GC}}$ version as done in [37].

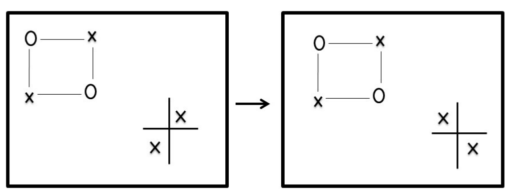


Figure 5.6: Grid move corresponding to Merge and Split cobordisms

Consider $Id : \widetilde{\mathcal{GC}}(D_1) \to \widetilde{\mathcal{GC}}(D_2)$. It induces an isomorphism in homology as it is unaffected by O swaps. So we just need to compute A_U and A_L degree shifts.

Now, we have the formula [From Equation 2.2] for alexander filtration for unknot in terms of winding numbers- $A_U(\mathbf{x}) = \sum_{x\in\mathbf{x}} w_U(x) + \frac{1}{8}\sum_{j=1}^{8n} w_U(p_j) - 1$, where p_j are corners of Xs and Os. Clearly swapping two Xs in one of the components of L doesn't affect any of the winding numbers. Therefore, A_U degree shift is 0.
Now $A_{L_1}(x)-A_{L_2}(x) = \frac{1}{2}(\mathcal{J}(x-\mathbb{X}_{L_1}, x-\mathbb{X}_{L_1})-\mathcal{J}(x-\mathbb{X}_{L_2}, x-\mathbb{X}_{L_2}))+\frac{1}{2} = \frac{1}{2}(\mathcal{J}(x,\mathbb{X}_{L_2})-\mathcal{J}(x,\mathbb{X}_{L_1}) + \mathcal{J}(\mathbb{X}_{L_2},x) - \mathcal{J}(\mathbb{X}_{L_1},x) + \mathcal{J}(\mathbb{X}_{L_1},\mathbb{X}_{L_1}) - \mathcal{J}(\mathbb{X}_{L_2},\mathbb{X}_{L_2})) + \frac{1}{2} = 1$[Since the quantity $\mathcal{J}(\mathbb{X}_{L_1},\mathbb{X}_{L_1}) - \mathcal{J}(\mathbb{X}_{L_2},\mathbb{X}_{L_2}) = 1$ from the diagram]. Here $\mathbb{X}_{L_1}$ are the X's in the grid diagram of L_1 and $\mathbb{X}_{L_2}$ are X's in grid diagram of L_2.

Passing to the $\widehat{\mathcal{GC}}$ version, we get a quasi-isomorphism $\Phi_{merge} : \widehat{\mathcal{GC}}(L_1) \to \widehat{\mathcal{GC}}(L_2) \otimes W$ of $\mathcal{F}_t$ filtered of degree $1 - \frac{t}{2}$.

3. **Merge:** We use the same construction for the merge move. A_U degree shift is 0 by the same argument, but for the A_L grading, since number of link component is decreasing

by 1 here, $A_{L_1}(x) - A_{L_2}(x) = \frac{1}{2}(\mathcal{J}(x - \mathbb{X}_{L_1}, x - \mathbb{X}_{L_1}) - \mathcal{J}(x - \mathbb{X}_{L_2}, x - \mathbb{X}_{L_2})) - \frac{1}{2} = 0$.

Again by passing to the $\widehat{\mathcal{GC}}$ version, we get a quasi-isomorphism $\Phi_{split} : \widehat{\mathcal{GC}}(L_1) \otimes W \to \widehat{\mathcal{GC}}(L_2)$ of $\mathcal{F}_t$ filtered of degree 0.

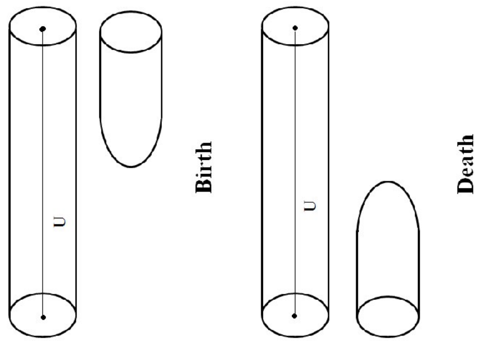


Figure 5.7: Annular Birth and death cobordisms

4. **Birth:** If L_2 (with grid D_2 is obtained from L_1 (with grid D_1) by birth, we know from [37] that there is a quasi-isomorphism from $\widehat{\mathcal{GC}}(L_1)$ to $\widehat{\mathcal{GC}}(L_2)$ given by

   $s(x) = \sum_{y \in S(D_1)} \sum_{H \in s\mathscr{L}(i(x),y,x), H \cap s\mathbb{O} = \phi} V_1^{n_1(H)} ... V_m^{n_m(H)} y$ for any $x \in S(D_1)$. Here $s\mathscr{L}(i(x), y, x)$ are snail like domains centered at c joining $i(x)$ to y and $n_i(H)$ is the number of times its passes through O_i.

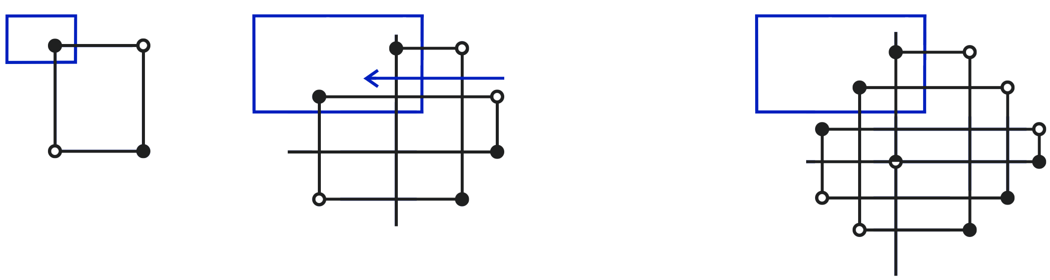


Figure 5.8: Snail like domains in the birth move

Now let us assume O_1 be the non-special O marking in U. Now, for any $H \in s\mathscr{L}(i(x), y, x)$ we have $A_U(x) - A_U(y) = -\sum($ unknot winding numbers for x points $)+ \sum($ unknot winding numbers for y points $) = n_1(H)$.Therefore $A_U(x) = A_U(V_1^{n_1(H)} ... V_m^{n_m(H)} y)$. So s is A_U filtered.

We already know that s is A filtered from [2]. Therefore, s is $\mathcal{F}_t$ filtered of degree 0.

5. **Death:** If we compose annular birth cobordism with a merge then we get a cobordism **BM** which induces a quasi-isomorphism $\Phi_{BM} : \widehat{\mathcal{GC}}(L_1) \to \widehat{\mathcal{GC}}(L_2)$ which is $\mathcal{F}_t$ filtered of degree 0. It can also be seen as $\Phi^*_{BM} : \widehat{\mathcal{GC}}(L_2^*) \to \widehat{\mathcal{GC}}(L_1^*)$. It follows that Φ^*_{BM} is also $\mathcal{F}_t$ filtered of degree 0. Now we observe that the cobordism from L_2^* to L_1^* is a split move followed by annular death,i.e. $\Phi^*_{BM} = \Phi_{Split} \circ \Phi_{Death}$. Therefore, annular death induces the quasi-isomorphism Φ_{Death} which is $\mathcal{F}_t$ filtered of degree $-1 + \frac{t}{2}$.

We will call annular merge cobordism followed by an annular split cobordism(or annular split cobordism followed by an annular merge cobordism) a **torus cobordism**. Clearly torus cobordism induces a quasi-isomorphism $\Phi_T : \widehat{\mathcal{GC}}(L_1) \to \widehat{\mathcal{GC}}(L_2)$ of filtered degree $1 - \frac{t}{2}$.

Theorem 5.1.1. *If Σ is a strong annular cobordism of genus g between two annular links L_1 and L_2 then* $\left|\mathscr{A}_{L_1}(t) - \mathscr{A}_{L_2}(t)\right| \leq g(1 - \frac{t}{2})$.

Proof. Since any strong cobordism of genus g can be written as composition of g torus cobordisms, some annular birth followed by merge and some split followed by annular death cobordisms. Both annular birth followed by merge and some split followed by annular death are filtered of degree 0. So, we get a quasi-isomorphism $\Phi : \widehat{\mathcal{GC}}(L_1) \to \widehat{\mathcal{GC}}(L_2)$ with filtered degree $g(1 - \frac{t}{2})$. So we have a commutative diagram

$$\begin{array}{ccc}
H_0(\mathcal{F}_t^s(\widehat{\mathcal{GC}}(U \cup L_1)) & \xrightarrow{\Phi} & H_0(\mathcal{F}_t^{s+g(1-\frac{t}{2})}(\widehat{\mathcal{GC}}(U \cup L_2)) \\
\downarrow i & & \downarrow i \\
\widehat{\mathcal{GH}}_0(U \cup L_1) & \xrightarrow{Id} & \widehat{\mathcal{GH}}_0(U \cup L_2) \\
\wr\Vert & & \wr\Vert \\
\mathbb{F}_2 & & \mathbb{F}_2
\end{array}.$$

It follows that $\mathscr{A}_{L_2}(t) \leq \mathscr{A}_{L_1}(t) + g(1 - \frac{t}{2})$. Similarly by looking at the cobordism from L_2 to L_1, we can show that $\mathscr{A}_{L_1}(t) \leq \mathscr{A}_{L_2}(t) + g(1 - \frac{t}{2})$.

□

Corollary 5.1.1. *If L_1 and L_2 are annular concordant then $\mathscr{A}_{L_1}(t) = \mathscr{A}_{L_2}(t)$.*

5.2 A t-modified annular chain complex

In this section, we define the modified annular grid complex $t\mathbf{C}$. Then, we will link it to the annular concordance invariant.

5.2.1 Definition

Let D be a grid diagram of annular link $U \cup L$. Let $\mathbb{X} = \{X_1, X_2, X_3, \cdots, X_n\}$ and $\mathbb{O} = \{O_1, O_2, O_3, \cdots, O_n\}$ be the sets of X markings and O markings respectively where X_1, X_2, O_1 and O_2 represent the markings of the unknot U. For $0 \leq t = 2\frac{p}{q} \leq 2$ where p, q are co-prime non-negative integers, we define the following modified link Floer complex.

Definition 5.2.1. *Define* $t\mathbf{C}(D) = \mathbb{F}_2[V_1, V_2, V_3, \cdots, V_{n-1}]$ *module over grid states* $S(D)$ *and for a* $x \in S(D)$*,*

$$\partial_t x := \sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = \phi} V_1^{pO_1(r)} V_2^{pO_2(r)} V_3^{(q-p)O_3(r)} \cdots V_{n-1}^{(q-p)O_{n-1}(r)} V_1^{(q-p)O_n(r)} y$$

We first show that ∂_t is indeed a differential.

Proposition 5.2.1. $\partial_t \circ \partial_t = 0$.

Proof. Let $\mathscr{C}_{U \cup L}$ be the $\mathbb{F}_2[W_1, W_2, \cdots, W_n]$ module over grid states $S(D)$

$$\partial_{\mathbb{X}} x := \sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = \phi} W_1^{p(q-p)O_1(r)} W_2^{p(q-p)O_2(r)} \cdots W_n^{p(q-p)O_n(r)} y$$

($\mathscr{C}_{U \cup L}$,$\partial_{\mathbb{X}}$) is a chain complex obtained $GC^-(U \cup L)$ by change of variables $V_i \to W_i^{p(q-p)}$. Now, consider the quotient complex $\frac{\mathscr{C}_{U \cup L}}{W_1^{q-p} - W_n^p}$. After setting, $V_1 = W_1^{q-p} = W_n^p$, $V_2 = W_2^{q-p}$ and $V_i = W_i^p$ for $i > 2$, we observe that the differential $\partial_{\mathbb{X}}$ becomes ∂_t in the quotient. Hence, $t\mathbf{C} \cong \frac{\mathscr{C}_{U \cup L}}{W_1^{q-p} - W_n^p}$ and ∂_t is just the restriction of $\partial_{\mathbb{X}}$ to the quotient. The conclusion follows easily.

□

$t\mathbf{C}$ is not Maslov graded. We define a function, $\mathscr{F}_t(x) = \frac{pA_U(x) + (q-p)A_L(x)}{q}$ for $x \in S(D)$ which is extended to $t\mathbf{C}$ by setting $\mathscr{F}_t(V_i) = -\frac{1}{q}$. Similarly define, $F_t(x) = \frac{(2p-2q)A_U(x) - \frac{p}{2}A_L(x)}{q}$ for $x \in S(D)$ and $F_t(V_i) = -\frac{1}{q}$

Proposition 5.2.2. $t\mathbf{C}$ *is* $\mathscr{F}_t$ *graded and* F_t *filtered.*

Proof. Let y be a state appearing in the expansion of differential of x. Then,

$$\mathscr{F}_t(V_1^{pO_1(r)} V_2^{pO_2(r)} V_3^{(q-p)O_3(r)} \cdots V_{n-1}^{(q-p)O_{n-1}(r)} V_1^{(q-p)O_n(r)} y) - \mathscr{F}_t(x)$$

$$= \mathscr{F}_t(y') - \mathscr{F}_t(x) - \frac{1}{q}(pO_U(r) + (q-p)O_L(r)) = 0.$$

and

$$F_t(V_1^{pO_1(r)} V_2^{pO_2(r)} V_3^{(q-p)O_3(r)} \cdots V_{n-1}^{(q-p)O_{n-1}(r)} V_1^{(q-p)O_n(r)} y) - F_t(x)$$

$$= F_t(y') - F_t(x) - \frac{1}{q}(pO_U(r) + (q-p)O_L(r)) = \frac{1}{q}((p-2q)O_U(r) + (\frac{p}{2} - q)O_L(r)) \leq 0.$$

□

The homology of $t\mathbf{C}$ will be denoted by $tH_*(\mathbf{C})$.

5.2.2 Invariance of $t\mathbf{C}$

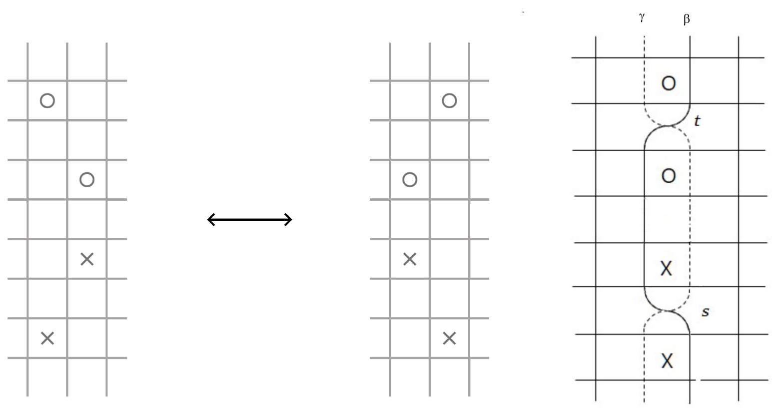
The goal of this section is to prove the following theorem

Theorem 5.2.1. *F_t filtered quasi-isomorphism type of $t\mathbf{C}$ is an annular link invariant.*

The proof is identical to the invariance proof of the last chapter. We sketch the details for completeness. The key observation is that we need to check invariance under commutation moves and stabilization moves of the component L.

Commutation move

Let D and D' be two grid diagrams that differ by a commutation move and we assume the same notation we saw in the last chapter.



Define the $\mathcal{R}$-module map $P\colon t\mathbf{C}(D) \to t\mathbf{C}(D')$ by the formula:

$$P(x) = \sum_{y' \in S(D')} \sum_{\Pi \in Pent^\circ(x,y'), \Pi \cap \mathbb{X} = \phi} V_1^{pO_1(\Pi)} V_2^{pO_2(\Pi)} V_3^{(q-p)O_3(\Pi)} \cdots V_{n-1}^{(q-p)O_{n-1}(\Pi)} V_1^{(q-p)O_n(\Pi)} \cdot y'$$

Proposition 5.2.3. *P is $\mathscr{F}_t$ graded and F_t filtered.*

Proof. Suppose Π is an empty pentagon from x to y' in the expansion of $P(x)$. Then,

$$A_U(x) - A_U(y') = O_U(\Pi) \text{ and } A_L(x) - A_L(y') = O_L(\Pi)$$

The conclusion follows by taking the sum with proper weights.

□

The following proposition can be proved analogously.

Proposition 5.2.4. *The map P is a chain map.*

Proof. Consider the $\mathbb{F}_2[W_1, V_2, \cdots, W_n]$-module map $\mathbf{P} : GC^-(D) \to GC^-(D)$ that gives quasi-isomorphism between two diagrams differing by commutation move [See Lemma 5.1.4 in [36]]. Again, P is the induced map on the quotient complex $\frac{\mathscr{C}_{U \cup L}}{W_1^{q-p} - W_n^p}$. Hence, P is a chain map. $\square$

Now, we define an analogous $\mathcal{R}$-module homomorphism $P' \colon t\mathbf{C}(D') \to t\mathbf{C}(D)$. For a grid state $x \in S(D')$, let

$$P'(x') = \sum_{y \in S(D)} \left(\sum_{p \in \mathrm{Pent}^\circ(x', y)} V_1^{pO_1(\Pi)} V_2^{pO_2(\Pi)} V_3^{(q-p)O_3(\Pi)} \cdots V_{n-1}^{(q-p)O_{n-1}(\Pi)} V_1^{(q-p)O_n(\Pi)} \right) \cdot y.$$

Again. We will show that the two maps P and P' are homotopy inverses of each other.

Define the $\mathcal{R}$-module homomorphism $H \colon t\mathbf{C}(D) \to t\mathbf{C}(D)$ for each $x \in S(D)$ by the formula:

$$H(x) = \sum_{y \in S(D)} \left(\sum_{h \in \mathrm{Hex}^\circ(x, y)} V_1^{pO_1(h} V_2^{pO_2(h)} V_3^{(q-p)O_3(h)} \cdots V_{n-1}^{(q-p)O_{n-1}(h)} V_1^{(q-p)O_n(h)} \right) \cdot y.$$

An analogous map $H' \colon t\mathbf{C}(D') \to t\mathbf{C}(D')$ can be defined by counting empty hexagons from $S(D')$ to itself. Again, H and H' are maps induced in the quotient complex from $\mathbf{H}$ and $\mathbf{H}'$.

Proposition 5.2.5. *The map $H \colon t\mathbf{C}(D) \to t\mathbf{C}(D)$ provides a chain homotopy from the chain map $P' \circ P$ to the identity map on $t\mathbf{C}(D)$.*

Proof. To have that H is a chain homotopy from $P' \circ P$ to the identity map on $t\mathbf{C}(D)$, we need to verify the following identity which are true since $\mathbf{H}$ satisfies it. [See Lemma 5.1.6 in [36]].

$$\partial_t \circ H + H \circ \partial_t = Id - P' \circ P.$$

$\square$

Now by putting it all together, we get -

Theorem 5.2.2. *Let D and D' be two grid diagrams that differ by a commutation move. Then*

$$tH(\mathbf{C})(D) \cong tH(\mathbf{C})(D').$$

Stabilization moves

We assume the same notation from the last chapter. As before, there is a one-to-one correspondence e between grid states of $I(D')$ and grid states of $S(D)$. It is defined as-

$$e : I(D') \to S(D), \quad x \cup \{c\} \mapsto x.$$

Then, we linearly extend e to $t\mathbf{C}$.

Proposition 5.2.6. *The map e is a filtered quasi-isomorphism.*

Proof. The same argument works in this case. e can be seen as a quotient of a filtered quasi-isomorphism that is defined in GC^- [See Section 5.2 in [36]]. Then, taking quotients of the homotopy equivalences show that e is a filtered quasi-isomorphism.

□

5.2.3 Relation with the annular concordance invariant

Definition 5.2.2. *For $0 \leq t = \frac{p}{q} \leq 1$, Define $t\mathscr{C} = F[V]$ module over grid states $S(D)$ and*

$$\partial_t x := \sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = \phi} V^{pO_U(r)+(q-p)O_L(r)} y \text{ for a } x \in S(D)$$

So $t\mathscr{C}$ is obtained from $t\mathbf{C}$ by setting all $V_i's$ equal to each other.

Proposition 5.2.7. *Multiplication by V_i^q is chain homotopic to multiplication by V_j^q in $t\mathbf{C}$ if O_i and O_j belong to the same link component in L.*

Proof. Let X_k be the X-marking that is in same row as O_m and in the same column as O_n. Define $H_{X_k} : t\mathbf{C} \to t\mathbf{C}$,

$$H_{X_k}(x) := \sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r \cap \mathbb{X} = X_k} V_1^{pO_1(r)} V_2^{pO_2(r)} V_3^{(q-p)O_3(r)} \cdots V_{n-1}^{(q-p)O_{n-1}(r)} V_1^{(q-p)O_n(r)} y \quad \forall x \in S(D)$$

Then, if O_m is one of the markings belonging to L

$$\partial H_{X_k} + H_{X_k} \partial = V_m^q - V_n^q$$

It follows that V_m^q and V_n^q are chain homotopic. Iterating this argument shows that V_i^q and V_j^q are chain homotopic if O_i and O_j belong to the same link component in L.

□

In light of Proposition 5.2.7, we can think of $t\mathbf{C}$ as a $\mathbb{F}_2[V_{i_1}, \cdots, V_{i_l}]$-module. We can also consider the complex $ct\mathbf{C} \cong \frac{t\mathbf{C}}{V_{i_1} = \cdots = V_{i_l}}$. It can be easily seen that F_t filtered quasi-isomorphism type of $ct\mathbf{C}$ is also an annular link invariant and its homology (denoted by $ctH_*(\mathbf{C})$) can be thought of as a $\mathbb{F}_2[V]$-module. Now, we are ready to relate $t\mathscr{C}$ with $ct\mathbf{C}$

Let $\mathbf{W}_L$ be a vector space with two generators, one in 0 $\mathscr{F}_t$-grading and the other in $(1-t)$ $\mathscr{F}_t$-grading. Let $\mathbf{W}_U$ be a vector space with two generators one in 0 $\mathscr{F}_t$-grading and the other in t $\mathscr{F}_t$-grading (Similar to proposition 5.1.1).

Proposition 5.2.8. *$t\mathscr{C}$ is quasi-isomorphic to $ct\mathbf{C} \otimes \mathbf{W}_L^{n-l-2} \otimes \mathbf{W}_U$ where l denotes the number of components in L.*

Proof. Using Proposition 5.2.7 and a short exact sequence similar to one considered earlier, we can derive the relation. □

Proposition 5.2.9. *$tH(\mathscr{C}) \cong (\mathbb{F}_2[V])^{2^{n-1}} \bigoplus Tor$, where n is the grid number of $U \cup L$.*

Proof. Let $\phi : t\mathscr{C} \to \frac{t\mathscr{C}}{V-1}$ be the projection onto quotient.
Now the quotient, $\frac{t\mathscr{C}}{V-1} \cong \widetilde{\mathcal{GC}}(-(U \cup L))$ under the natural identification (we will call the identification map χ).
We also know that $[\xi] \in tH(\mathscr{C})$ is non-torsion element if and only if $H(\phi)([\xi]) \neq 0$. So $ker(H(\phi)) = Tor(t\mathscr{C})$.
Now since $\widetilde{\mathcal{GC}}(-(U \cup L)) \cong \mathbb{F}_2^{2^l}$, it follows that rank of free part of $tH(\mathscr{C})$ is 2^{n-1}. Hence we can conclude that $tH(\mathscr{C}) \cong (\mathbb{F}_2[V])^{2^{n-1}} \bigoplus Tor$.

□

We can also compute the free rank of $ct\mathbf{C}$.

Proposition 5.2.10. *If an annular link $U \cup L$ has l components.*

$$\frac{ctH_*(\mathbf{C}(D))}{Tor(ctH_*(\mathbf{C}(D))} \cong \mathbb{F}_2[V]^{2^{l-1}}.$$

Proof. The conclusion follows from Proposition 5.2.8 and 5.2.9.

□

The distinguished class $[x^+]$ in $ctH_*(\mathbf{C}) \subset tH(\mathscr{C})$ is a non-torsion element since its image under $\phi \circ \chi$ is $[x^{NEX}]$ in $\widetilde{\mathcal{GC}}(-(U \cup L))$, which is non trivial of Maslov grading $1-n$. There is an unique generator of that grading in $\widetilde{\mathcal{GC}}(-(U \cup L))$. So, there must be a class $[\alpha]$ in $ctH_*(\mathbf{C}) \subset tH(\mathscr{C})$ for which $[x^+] = V^k[\alpha] + [\beta]$ where $[\beta]$ is a torsion class and $k \geq 0$ is maximum. Similarly, it can be seen that $[x^-]$ is also a non-torsion element in the same tower. This gives the following relation with the annular concordance invariant $\mathcal{A}_L(t)$.

Theorem 5.2.3. $\mathcal{A}_L(t) = -\mathscr{F}_t([\alpha])$.

Proof. Let $\mathcal{F}_t$ be the filtration given by $\frac{t}{2}A_{-U} + (1-\frac{t}{2})A_{-L}$ on $\widetilde{\mathcal{GC}}(-(U \cup L))$. It follows from the definitions that $\mathcal{F}_t(x) = -\mathscr{F}_t(x) - \frac{t}{2} - (n-l-2)(1-\frac{t}{2})$. Since $H(\phi \circ \chi)([\alpha]) \neq 0$. It follows from the definition of $\widetilde{\mathcal{A}_L}(t)$ and 4.3 that $\widetilde{\mathcal{A}_L^U}(t) = \widetilde{\mathcal{A}_{-L}^{-U}}(t) \leq \mathcal{F}_t((\phi \circ \chi)([\alpha])) = -\mathscr{F}_t([\alpha]) - \frac{t}{2} - (n-l-2)(1-\frac{t}{2})$. Therefore, using 4.2 we obtain $\mathcal{A}_L(t) \leq -\mathscr{F}_t([\alpha])$. Conversely, if we take an element $a \neq 0 \in \widetilde{\mathcal{GC}}_{1-n}(-(U \cup L))$ with $\widetilde{\mathcal{A}_L}(t) = \mathcal{F}_t(a)$. Since $H(\phi \circ \chi)([\alpha]) = [x^{NEX}] = [a]$, it follows that $\mathcal{A}_L(t) = \mathcal{F}_t(a) + \frac{t}{2} + (n-l-2)(1-\frac{t}{2}) \geq \mathcal{F}_t([a]) + \frac{t}{2} + (n-l-2)(1-\frac{t}{2}) = -\mathscr{F}_t([\alpha])$. Hence, the equality follows.

□

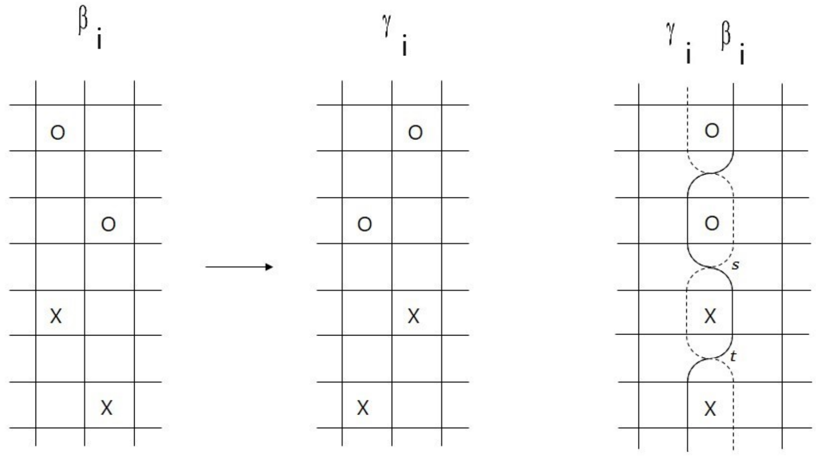


Figure 5.9: Crossing change move in $t\mathbf{C}$

5.2.4 Crossing change

Let L_+ and L_- be two annular links with grids D_+ and D_- that differ only in a crossing in the L component.

Proposition 5.2.11. *There are $\mathcal{R}$-module maps*

$$C_-\colon ctH(\mathbf{C})(L_+) \to ctH(\mathbf{C})(L_-) \quad \textit{and} \quad C_+\colon ctH(\mathbf{C})(L_-) \to ctH(\mathbf{C})(L_+)$$

where C_- is homogeneous and preserves the $\mathscr{F}_t$ grading, and C_+ is homogeneous and shifts $\mathscr{F}_t$ degree by $-(1-\frac{t}{2})$. $C_- \circ C_+$ and $C_+ \circ C_-$ are both the multiplication by V^{q-p}.

Here again, we have the same picture as in the previous chapter. We assume that we have the same notation.

Proof. Fix grid states $x_+ \in S(D_+)$ and $x_- \in S(D_-)$. We again use the notation $Pent^\circ_s(x_+, x_-)$ for the set of empty pentagons from x_+ to x_- containing s as a vertex, and similarly, $Pent^\circ_s(x_-, x_+)$ for the set of empty pentagons from x_- to x_+ with one vertex at t.

Consider the $\mathcal{R}$-module maps $c_-\colon ct\mathbf{C}(D_+) \to ct\mathbf{C}(D_-)$ and $c_+\colon ct\mathbf{C}(D_-) \to ct\mathbf{C}(D_+)$ defined on a grid state $x_+ \in S(D_+)$ and $x_- \in S(D_-)$ respectively in the following way:

$$c_-(x_+) = \sum_{y_- \in S(D_-)} \sum_{\Pi \in \mathrm{Pent}^\circ_s(x_+, y_-)} V_1^{pO_1(\Pi)} V_2^{pO_2(\Pi)} V_3^{(q-p)O_3(\Pi)} \cdots V_{n-1}^{(q-p)O_{n-1}(\Pi)} V_1^{(q-p)O_n(\Pi)} \cdot y_-.$$

$$c_+(x_-) = \sum_{y_+ \in S(D_+)} \sum_{\Pi \in \mathrm{Pent}^\circ_t(x_-, y_+)} V_1^{pO_1(\Pi)} V_2^{pO_2(\Pi)} V_3^{(q-p)O_3(\Pi)} \cdots V_{n-1}^{(q-p)O_{n-1}(\Pi)} V_1^{(q-p)O_n(\Pi)} \cdot y_+.$$

□

Proposition 5.2.12. *The map c_- preserves the $\mathscr{F}_t$-grading and c_+ drops the $\mathscr{F}_t$-grading by $(1-\frac{t}{2})$.*

Proof. The grading changes can be computed like chapter 4 by considering local computations for each interval. Let y be the term, appearing in $c_-(x)$ and assume there is a left pentagon from x to y.

Case 1: y is type B In this case we saw that, $A_L(y) - A_L(x) = A_L(\phi^-(y)) - A_L(x) + A_L(y) - A_L(\phi^-(y)) = 0$ and $\mathcal{A}_U(y) - \mathcal{A}_U(x) = \mathcal{A}_U(\phi^-(y)) - \mathcal{A}_U(x) + \mathcal{A}_U(y) - \mathcal{A}_U(\phi^-(y)) = 0$. So $\mathscr{F}_t(y) - \mathscr{F}_t(x) = 0$.

Case 2: y is type C In this case, $A_L(y) - A_L(x) = A_L(\phi^-(y)) - A_L(x) + A_L(y) - A_L(\phi^-(y)) = 1-1 = 0$ and $\mathcal{A}_U(y) - \mathcal{A}_U(x) = \mathcal{A}_U(\phi^-(y)) - \mathcal{A}_U(x) + \mathcal{A}_U(y) - \mathcal{A}_U(\phi^-(y)) = 0$. So $\mathscr{F}_t(y) - \mathscr{F}_t(x) = 0$.

For right pentagons, the computation is same except we consider initial corners in B and C.

Similarly, we can compute the grading shift for c_+.

□

Proposition 5.2.13. *The maps c_- and c_+ are chain maps.*

Proof. Again the proof is similar to the one given in chapter 4.

□

The above chain maps c_- and c_+ induce the desired maps C_- and C_+ on the homologies. In order to verify Proposition 5.2.11, we have to show that $C_- \circ C_+$ and $C_+ \circ C_-$ are both the multiplication by V^q. For this, we can find chain homotopies between the composites $c_- \circ c_+$ respectively $c_+ \circ c_-$ and multiplication by V^{q-p} by taking quotient similar to the proof in the last chapter.

Therefore, we have that $C_- \circ C_+$ and $C_+ \circ C_-$ are both the multiplication by V^{q-p}.

5.2.5 Stabilizations

Now, we will look at stabilizations. First, we will need to define the notion of stabilization of annular links. We define negative stabilization L^- to be the annular link obtained from of an annular link L by adding a linked negative crossing [See Figure 5.10]. Similarly, we define positive stabilization L^+ by generalizing the picture of positive stabilization in Figure 4.10. Obviously, for braids, these correspond to the standard braid stabilizations. We will abuse notations by referring to both grids of annular links L and L^+ by L and L^+ respectively.

We define the $\mathcal{R}$-module maps $PS_- : ct\mathbf{C}(L^+) \to ct\mathbf{C}(L)$ and $PS_+ : ct\mathbf{C}(L) \to ct\mathbf{C}(L^+)$ for a grid state $x \in S(L^+)$ and $y' \in S(L)$ respectively in the following way:

$$PS_-(x) = \sum_{y \in S(L)} \sum_{\Pi \in \mathrm{Pent}_s^\circ(x_+, y_-)} V_1^{pO_1(\Pi)} V_2^{pO_2(\Pi)} V_3^{(q-p)O_3(\Pi)} \cdots V_{n-1}^{(q-p)O_{n-1}(\Pi)} V_1^{(q-p)O_n(\Pi)} \cdot y$$

$$PS_+(y') = \sum_{x' \in S(L^+)} \sum_{\Pi \in \mathrm{Pent}_t^\circ(x_-, y_+)} V_1^{pO_1(\Pi)} V_2^{pO_2(\Pi)} V_3^{(q-p)O_3(\Pi)} \cdots V_{n-1}^{(q-p)O_{n-1}(\Pi)} V_1^{(q-p)O_n(\Pi)} \cdot x'$$

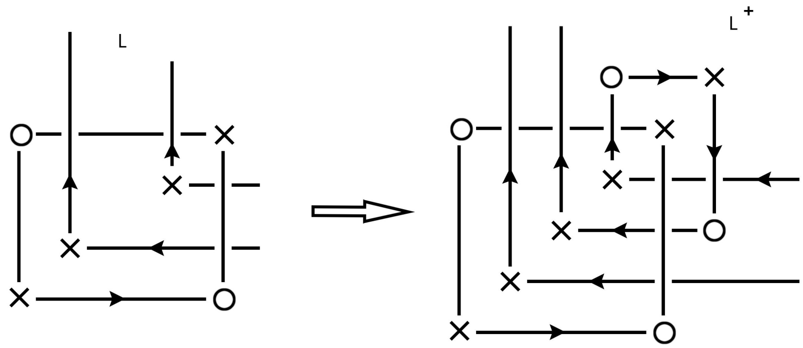


Figure 5.10: Negative stabilization of an annular link

For a negative stabilzation L^- of L, we have the $\mathcal{R}$-module maps $NS_+ : ct\mathbf{C}(L^-) \to ct\mathbf{C}(L)$ and $NS_- : ct\mathbf{C}(L) \to ct\mathbf{C}(L^-)$ for a grid state $x \in S(L^+)$ and $y' \in S(L)$ defined as usual.

Now, the following propositions are derived just like the last chapter.

Proposition 5.2.14. *The maps NS_-, NS_+, PS_- and PS_+ are chain maps.*

Proposition 5.2.15. *The maps $PS_- \circ PS_+$ and $PS_+ \circ PS_-$ are both homotopic to V^{q-p}. The maps $NS_- \circ NS_+$ and $NS_+ \circ NS_-$ are both homotopic to V^p.*

Now we will compute grading shifts of these maps that will be key for understanding behavior of the invariant under positive/negative stabilizations.

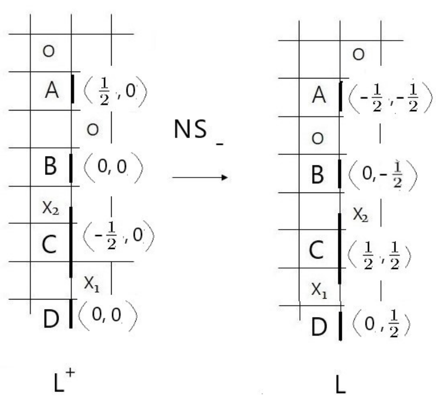


Figure 5.11: Change in local values of (A, A_U) gradings

Proposition 5.2.16. *The maps NS_- shifts $\mathscr{F}_t$ by $\frac{1}{2}(1-t)$ and NS_+ shifts $\mathscr{F}_t$ by $-\frac{1}{2}$. The maps PS_- shifts $\mathscr{F}_t$ by $-\frac{1}{2}$ and PS_+ shifts $\mathscr{F}_t$ by $\frac{t-1}{2}$.*

Proof. We again make local computations for each of the intervals and keep track of local change in both $A = A_L + A_U$ and A_U gradings [See Figure 5.11]. We know how A grading changes locally from Lemmma 6.2.1 of [36] and we use the winding number formula (Proposition 11.2.6. in [36]) for computing the A_U grading change. If y is a term appearing in $NS_-(x)$ and there is an empty left pentagon p (pentagon to the the left of vertical circle β_i or γ_i) from x to y and from Lemma 6.2.1 of [36] that the terminal generator y is either of type B or C. So we can use associated left rectangles for the left pentagons to compute grading change.

Case 1: y is type B Since there are no extra markings in the associated rectangles we have,

$$\mathscr{F}_t(y) - \mathscr{F}_t(x) = \text{ Local change } = \frac{1}{2}(1-\frac{t}{2}) + (-\frac{1}{2})\frac{t}{2} = \frac{1-t}{2}.$$

Case 2: y is type C In this case, we have an extra X marking belonging to the unknot in the associated rectangle. So,

$$\mathscr{F}_t(y) - \mathscr{F}_t(x) = \text{ Local change } -\frac{t}{2} = \frac{1}{2}\cdot\frac{t}{2} + \frac{1}{2}(1-\frac{t}{2}) - \frac{t}{2} = \frac{1-t}{2}.$$

Similarly for a right pentagon (pentagon to the the right of vertical circle β_i or γ_i), we compare it with a right rectangle. Here the initial corner is either of type B or type C. In each case, we get grading change $= \frac{1-t}{2}$.

Now, let y' be a term appearing in $NS_+(x')$.

Case 1: y' is type B From Figure 4.9, we observe that the associated rectangle has one extra X and O markings belonging to L and one less O marking belonging to the unknot. So we have,

$$\mathscr{F}_t(y') - \mathscr{F}_t(x') = \text{ Local change } -\frac{t}{2} = -\frac{1-t}{2} - \frac{t}{2} = -\frac{1}{2}.$$

Case 2: y' is type C There are no additional markings. So we have,

$$\mathscr{F}_t(y') - \mathscr{F}_t(x') = \text{ Local change } = -\frac{1}{2}(1-\frac{t}{2}) - \frac{t}{2}\cdot\frac{1}{2} = -\frac{1}{2}.$$

Case 3: y' is type D There is an additional X marking belonging to L. Therefore,

$$\mathscr{F}_t(y') - \mathscr{F}_t(x') = \text{ Local change } -(1-\frac{t}{2}) = \frac{1}{2}(1-\frac{t}{2}) - \frac{1}{2}\cdot\frac{t}{2} - (1-\frac{t}{2}) = -\frac{1}{2}.$$

Case 4: y' is type A Now, the associated rectangle has one extra X marking belonging to L and one less O marking belonging to the unknot. So we have,

$$\mathscr{F}_t(y') - \mathscr{F}_t(x') = \text{ Local change } -(1-\frac{t}{2}) - \frac{t}{2} = \frac{1}{2}(1-\frac{t}{2}) + \frac{1}{2}\cdot\frac{t}{2} - 1 = -\frac{1}{2}.$$

Hence, the map NS_+ is $\mathscr{F}_t$ graded of degree $-\frac{1}{2}$. Similarly, we compute the PS_- and PS_+ shifts. □

5.2.6 Inequalities involving the annular concordance invariant

Proposition 5.2.17. *If the annular links L_+ and L_- differ in a crossing change, then for $t \in [0,2]$*

$$\mathcal{A}_{L_-}(t) \leq \mathcal{A}_{L_+}(t) \leq \mathcal{A}_{L_-}(t) + (1 - \frac{t}{2})$$

and

$$\mathscr{A}_{L_-}(t) \leq \mathscr{A}_{L_+}(t) \leq \mathscr{A}_{L_-}(t) + (1 - \frac{t}{2}).$$

Proof. Consider a non-torsion element $\xi \in ctH(\mathbf{C})(L_-)$ that has grading $-\mathcal{A}_{L_-}(t)$. Since $C_- \circ C_+$ and $C_+ \circ C_-$ are both the multiplication by V^{q-p}, $C_+(\xi)$ is non-torsion. The $\mathcal{F}_t$ grading of $C_+(\xi)$ is $-\mathcal{A}_{L_-}(t) - (1 - \frac{t}{2})$. Also since it is in the tower of the distinguished class $[x^+]$, by Theorem 5.2.3 $-\mathcal{A}_{L_+}(t) \geq -\mathcal{A}_{L_-}(t) - (1 - \frac{t}{2})$. Similarly, if $\sigma \in ctH(\mathbf{C})(L_+)$ is a non-torsion element with grading $-\mathcal{A}_{L_+}(t)$, then its image $C_-(\eta)$ has grading $-\mathcal{A}_{L_+}(t)$ too. Again since $C_-(\eta)$ is non-torsion and in the tower of the distinguished class, $-\mathcal{A}_{L_+}(t) \leq -\mathcal{A}_{L_-}(t)$. For the second inequality, we use the fact that mirroring takes $\mathcal{A}_t$ to $-\mathscr{A}_t$. $\square$

Proposition 5.2.18. *Let L^+ and L^- denote the positive and negative stabilization of an annular link L. Then for $t \in [0,2]$*

$$\mathcal{A}_L(t) - \frac{1}{2} \leq \mathcal{A}_{L^-}(t) \leq \mathcal{A}_L(t) - \frac{1-t}{2}$$

and

$$\mathcal{A}_L(t) - \frac{1}{2} \leq \mathcal{A}_{L^+}(t) \leq \mathcal{A}_L(t) + \frac{1-t}{2}$$

Also,

$$\mathscr{A}_L(t) - \frac{1-t}{2} \leq \mathscr{A}_{L^-}(t) \leq \mathscr{A}_L(t) + \frac{1}{2}$$

and

$$\mathscr{A}_L(t) + \frac{1-t}{2} \leq \mathscr{A}_{L^+}(t) \leq \mathscr{A}_L(t) + \frac{1}{2}.$$

Proof. The proof for $\mathcal{A}_L$ inequalities is similar to the last proposition. Instead of using C_+ and C_-, we use PS_+, PS_-, NS_+ and NS_- to derive the inequalities. Then, we obtain $\mathscr{A}_L$ inequalities by taking the mirror. $\square$

Since $\mathscr{A}_L(t) = -\mathcal{A}_{m(L)}(t)$, Theorem 5.2.3 implies $\mathscr{A}_L(t) = \mathscr{F}(t)([\alpha])$ where $[\alpha]$ is the maximum non-torsion class in $[x^+]$ tower in $ct\mathbf{C}(m(L))$. We can use this relationship to derive a slice-Bennequin type inequality. Given a link L, a Legendrization of L is a Legendrian link whose topological link type is L. A slice-Bennequin type inequality relates classical invariants of the Legendrization with a concordance invariant of the topological link type.

Theorem 5.2.4. *Let L be an annular link with l components and $\mathcal{L}$ be a Legendrization of L. Then we have the following inequality,*

$$\mathscr{A}_L(t) \geq \frac{lk(U,L)t}{4} + (1-\frac{t}{2})\frac{tb(\mathcal{L}) + |rot(\mathcal{L})| + l + lk(U,L)}{2}$$

holds for all $t \in [0,2]$.

Proof. By Theorem 5.2.3, $\mathscr{A}_\beta(t) \geq \mathcal{F}_t(x^+) = \frac{t}{2}A_U(x^+) + (1-\frac{t}{2})A_L(x^+) = \frac{t}{2}A_U(x^+) + (1-\frac{t}{2})(A(x^+) - A_U(x^+))$.
Now, $A_U(x^+) = \frac{-1+1+lk(U,L)}{2} = \frac{lk(U,L)}{2}$ using the winding number formula (Equation 2.2). Also, $A(x^+) = \frac{tb(\mathcal{L}\cup U) - rot(\mathcal{L}\cup U)+l+1}{2} = \frac{tb(\mathcal{L}) - rot(\mathcal{L}) + l + 2lk(U,L)}{2}$. Therefore,

$$\mathscr{A}_L(t) \geq \frac{lk(U,L)t}{4} + (1-\frac{t}{2})(\frac{tb(\mathcal{L}) - rot(\mathcal{L}) + l + lk(U,L)}{2})$$

We also know that x^- is non-torsion in the same tower hence. Hence, $\mathscr{A}_\beta(t) \geq \mathcal{F}_t(x^-)$ and a similar computation shows that

$$\mathscr{A}_L(t) \geq \frac{lk(U,L)t}{4} + (1-\frac{t}{2})(\frac{tb(\mathcal{L}) + rot(\mathcal{L}) + l + lk(U,L)}{2})$$

.

$\square$

The above lower bound is similar in spirit to lower bound given by Plamenevskaya [32, 34] on τ and Rasmussen's s invariant.

5.2.7 Grid complex of n-cables and $t\mathscr{C}$

For an annular link $U \cup L$ we will build the p-cable by only transform cells in the same row or column of X-markings belonging to L like chapter 2.

We will denote the annular p-cable generated using this construction by $U_r \cup L_{p,q}$. Here, one copy of unknot is replaced by r copies of unknot for any natural number r. Let us consider a subset $\mathcal{K}$ of $\mathscr{C}_{U\cup L_{p,q}}$ which is generated by states that contains intermediate north east corners of X in marking inside the block.

The following proposition is an analog of Proposition 3.2.1 from chapter 3.

Proposition 5.2.19. *$\mathcal{K}$ of is a subcomplex of $\mathscr{C}_{U_r\cup L_{p,q}}$ and is isomorphic to $\frac{r}{p}\mathscr{C}$.*

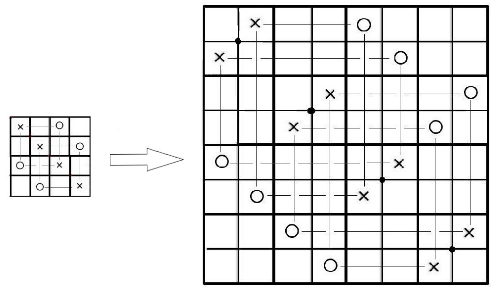


Figure 5.12: The states in subcomplex $\mathcal{K}$ contains black dots

Proof. There is no rectangle coming out the special points(See Fig 5.12) of $\mathcal{K}$. Therefore, any rectangle coming out of a state in $\mathcal{K}$ must join it with another state in $\mathcal{K}$. Hence, its a subcomplex. We can identify the states of $\mathscr{C}_{U_r \cup L_{p,q}}$ with generators of $\frac{r}{p}\mathscr{C}$. It easily follows from the construction that these two complexes are indeed isomorphic.

$\square$

Proposition 5.2.20. *There is a chain map $i : \frac{r}{p}\mathscr{C} \to \mathscr{C}_{U_r \cup L_{p,q}}$ such that $i([\alpha]) = U^k[\alpha']$ where $k \in \mathbb{N}$ and $[\alpha], [\alpha']$ are non torsion elements in the respective complexes satisfying the described property.*

Proof. We know that i sends the distinguished state x^+ to itself. Since $[x^+] \in \mathscr{C}_{U_r \cup L_{p,q}} = U^m[\alpha']$ for some m and $[x^+] \in \frac{r}{p}\mathscr{C} = U^n[\alpha]$ for some n , it follows that $i(U^n[\alpha]) = i([x^+]) = [x^+] = U^m[\alpha']$. This implies $U^n i([\alpha]) = U^m[\alpha']$. So $m \geq n$ as $[\alpha']$ is top of the non torsion tower in $\mathscr{C}_{U \cup L_{2,q}}$. If $m > n$, then $i([\alpha]) = U^k[\alpha']$ where $k = m - n$ is a natural number.

$\square$

Proposition 5.2.21. $\tau(L_{p,q} \cup U_r) \geq p\mathcal{A}_L(t) + \frac{(p-1)(p+q-1)}{2}$.

Proof. Like chapter 3, we need to compute the filtered degree of the map i , which is $A(i(x^+)) - p\mathscr{F}_{1/p}(x^+) = A(i(x^+)) - A_U(x^+) - pA_L(x^+)$. So the degree is equal to, $\frac{(p-1)(q-1)}{2} + \frac{(p-1)p}{2} = \frac{(p-1)(p+q-1)}{2}$. Therefore, $\tau(L_{p,q} \cup U) \geq p\mathcal{A}_L(t) + \frac{(p-1)(p+q-1)}{2}$.

$\square$

Proposition 5.2.22. *If $L \cup U$ is a quasi-positive link. then $\mathscr{A}_L(t) = -\mathcal{A}_{m(L)}(t) = \mathscr{F}_t(x^+)$ where x^+ is the distinguished generator in the grid of the annular link $m(L)$.*

Proof. We will show that if L is a quasi-positive then $[x^+]$ is the top of the non-torsion tower in $ctH_*(\mathbf{C})(U \cup m(L))$. First, notice that if $L \cup U$ is quasi-positive, then the n-cable $U_r \cup L_{n,q}$ is quasi-positive for $q \geq 0$. It follows $[x^+]$ is the top of the non-torsion tower in $\mathscr{C}_{U_r \cup m(L_{n,q})}$ since $\mathscr{C}_{U_r \cup m(L_{n,q})} = GC^- \otimes W^{\otimes N}$ for some N and $[x^+]$ is the top of the tower in $GC^-(L)$ for a quasi-positive link L [5]. Now since we have $i(x^+) = x^+$, it follows that x^+ is the top of the tower in $ct\mathbf{C}$. Therefore, $\mathcal{A}_{m(L)}(t) = -\mathscr{F}_t(x^+)$. $\square$

5.3 Braided cobordisms

We can define an invariant of a braid β by considering annular invariant of the annular link $U \cup \beta$ where U acts like the braid axis.

5.3.1 Properties of the annular invariant for braids

A braided cobordism Σ from $\beta_1 = \Sigma \cap (\mathbb{S}^3 \times \{0\})$ to $\beta_2 = \Sigma \cap (\mathbb{S}^3 \times \{1\})$ is braid-orientable if it admits an orientation compatible with the braid-like orientations of β_1 and β_2.

Proposition 5.3.1. *If β_1 and β_2 are braids , and Σ is a braid-orientable braided cobordism from β_2 to β_1 with s split saddles and d deaths then*

$$\mathscr{A}_{\beta_1}(t) - \mathscr{A}_{\beta_2}(t) \leq (s-d)(1-\frac{t}{2}).$$

Proof. From Section 5.1 we know that split moves have filtered degree $1 - \frac{t}{2}$ and death moves have filtered degree $-1 + \frac{t}{2}$. We get the inequality by adding the contributions in the cobordism. $\square$

Now we will study the effect of braid stabilization on the annular invariant.

Proposition 5.3.2. *If β is a n-braid,let β^+ and β^- represent the $n+1$-braids obtained by positively and negatively stabilizing β respectively. Then, we have the following inequalities*

$$\mathscr{A}_{\beta}(t) - \frac{1-t}{2} \leq \mathscr{A}_{\beta^-}(t) \leq \mathscr{A}_{\beta}(t) + \frac{1}{2}$$

and

$$\mathscr{A}_{\beta}(t) + \frac{1-t}{2} \leq \mathscr{A}_{\beta^+}(t) \leq \mathscr{A}_{\beta}(t) + \frac{1}{2}.$$

Proof. The above inequalities follow directly from Proposition 5.2.18.

$\square$

Proposition 5.3.3. *$\mathscr{A}_{\beta}(2) = \frac{n}{2}$ for any n-braid β.*

Proof. To see this, we can consider a strong braided cobordism from β to Id_n. Then, it follows from Proposition 5.3.1.

$\square$

Proposition 5.3.4. *If β has 1 component then, $\mathscr{A}_{\beta}(0) = \tau(\beta) + \frac{n}{2}$.*

Proof. We can think of $-\mathscr{A}_{\beta}(0)$ as the max A_β grading of the x^+ tower in $0\mathbf{C}(m(\beta)) \cong HFK^{-,2}(m(\beta))$. Now from the inclusion isomorphism in [25], it is clear that the non-torsion tower is taken to the non-torsion tower in $HFK^-(m(\beta))$ and the grading shift is $-\frac{n}{2}$. The conclusion follows. $\square$

5.3.2 Bounds on band rank

We also get the following lower bound on band rank from the annular invariant.

Theorem 5.3.1. *Let β be an n-braid with l components and Id_n be the identity n-braid. Then $\mathscr{A}_\beta(t) - \mathscr{A}_{Id_n}(t) \leq \frac{rk_n(\beta)+l-n}{2}(1-\frac{t}{2})$.*

Proof. Recall that if a braid β has band rank $rk_n(\beta)$, then it can be written as

$$\beta = \prod_{j=1}^{rk_n(\beta)} \omega_j \sigma_{i_j}^{\pm 1} \omega_j^{-1}$$

Therefore, there is a cobordism from Id_n to β that has rk_n number of saddles. Now, $rk_n = s + m$, where s is the number of split and m is the number of merge cobordism componenets in that cobordism. Also we have, $s - m = l - n$. Hence, the inequality follows from Proposition 5.3.1. $\square$

$K \subset S$ is called a ribbon knot if it bounds a smoothly embedded disk in B^4, Morse, with no interior maxima. Rudolph's theorem (2.5.1) tells us, if K is ribbon then it has a closed n-braid representative σ with $rk_n(\sigma) = n-1$. So if a closed n-braid representative β of a some slice knot K satisfies $|\mathscr{A}_\beta(t) - \mathscr{A}_{Id_n}(t)| > (n-1)(1-\frac{t}{2})$ and this inequality is preserved under (de)stabilization then that will provide a counterexample to slice-ribbon conjecture.

Given a braid β we can transform it to a Legendrian link (See Figure 5.13)which is call Legendrization of the braid. The following inequality establishes an interesting relationship between band rank and classical Legendrian invariants.

Theorem 5.3.2. *If β is a n-braid with l componenets and $\mathcal{L}$ its Legendrization then,*

$$\frac{rk_n(\beta)+l-n}{2} \geq \frac{tb(\mathcal{L}) + |rot(\mathcal{L})| + l}{2}.$$

Proof. For all $t \in [0,2]$, the inequality

$$\frac{rk_n(\beta)+l-n}{2}(1-\frac{t}{2}) \geq \frac{nt}{4} + (1-\frac{t}{2})\frac{tb(\mathcal{L}) + |rot(\mathcal{L})| + l + n}{2} - \frac{n}{2}$$

follows easily by combining Theorem 5.3.1 and Theorem 5.2.4. In particular for $t = 0$, we get

$$\frac{rk_n(\beta)+l-n}{2} \geq \frac{tb(\mathcal{L}) + |rot(\mathcal{L})| + l}{2}.$$

$\square$

5.3.3 Right veering and transverse properties

The annular braid invariant also has right veering and transverse properties properties analagous to Grigsby-Wehrli-Licata invariant. Before proving those results, let us first compute the invariant for quasi-positive braids.

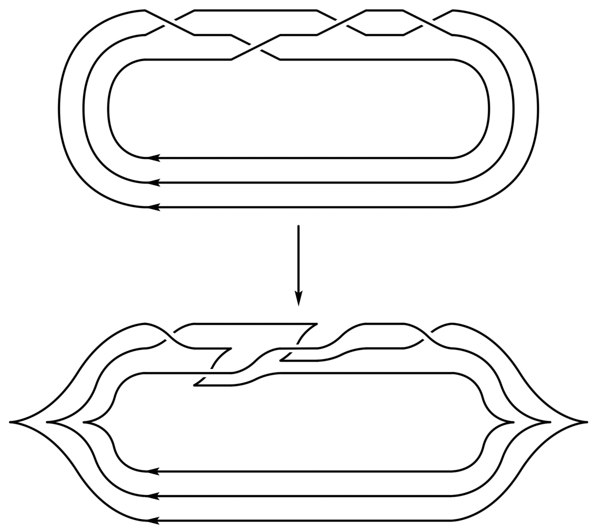


Figure 5.13: Legendrization of a braid

Proposition 5.3.5. *If β is a quasi-positive braid of index n with l componenents. Then,* $\mathscr{A}_\beta(t) = t\frac{-wr(\beta)-l+n}{4} + \frac{wr(\beta)+l}{2}$.

Proof. By Proposition 5.2.22, $\mathscr{A}_\beta(t) = \mathcal{F}_t(x^+) = \frac{t}{2}A_U(x^+) + (1-\frac{t}{2})A_L(x^+) = (-1+t)A_U(x^+) + (1-\frac{t}{2})A(x^+)$.
Now $A_U(x^+) = \frac{-1+1+lk(U,L)}{2} = \frac{n}{2}$ and $A(x^+) = \frac{-(n+1)+(wr(\beta)+2n)+l+1}{2} = \frac{wr(\beta)+n+l}{2}$. Hence, $\mathcal{A}_\beta(t) = (t-1)\frac{n}{2} + \frac{wr(\beta)+n+l}{2}(1-\frac{t}{2})$. □

For quasi-positive braids with one component, we recover the τ to be $\frac{sl(\beta)+1}{2}$ since $\mathscr{A}_\beta(0) = \tau(\beta) + \frac{n}{2}$.

Recall that we defined slope function $m_t(\beta)$ and y-value $y_t(\beta)$ associated to $\mathscr{A}_\beta(t)$. Let us define a related function.

Definition 5.3.1. *For a braid β and $t \in [0,2]$*

$$M_t(\beta) := 2m_t(\beta) + y_t(\beta).$$

Then by Proposition 5.1.7, $M_t(\beta) = A_U(x_0)$ for some generator x_0 at each $t \in [0,2]$. The function M_t also has an alternative formulation in the $ct\mathbf{C}$ complex. By Theorem 5.2.3, $\mathcal{A}_L(t) = -\mathscr{F}_t([\alpha])$ where α is a maximum non-torsion element in $[x^+]$ tower in $ct\mathbf{C}$. Then again by Proposition 5.1.7, $M_t(\beta) = -A_U(\alpha)$.

Notice that $M_t(\beta) = \frac{n}{2}$ for all t in the case of quasi-positive braids. The following proposition gives a more general criteria in terms of right veering.

Proposition 5.3.6. *If $M_t(\beta) = \frac{n}{2}$ for some $0 < t < 2$ then β is right veering.*

Proof. Suppose β is non-right veering. Then we know that $\hat{\theta}$ and $\tilde{\theta}$ vanishes. Consider the short exact sequence

$$0 \longrightarrow t\mathscr{C}(\beta) \xrightarrow{V} t\mathscr{C}(\beta) \longrightarrow \widetilde{GC}(m(\beta)) \xrightarrow{p} 0$$

In the induced long exact sequence, p_* takes $[x^+]$ to $\tilde{\theta}$. It follows that $[x^+]$ is V-image in $t\mathscr{C}$. But this implies $M_t(\beta) < A_U(x^+) = \frac{n}{2}$.

□

Proposition 5.3.7. *If $M_{t_0}(\beta) = \frac{n}{2}$ for some $t_0 \in [0, 2)$ then $M_t(\beta) = \frac{n}{2}$ for $2 \geq t > t_0$.*

Proof. By Proposition 5.1.7, we can easily see that For $t > t_0$ $\Delta M_t \geq 0$. Since $\frac{n}{2}$ is the maximum possible value, it follows that $M_t(\beta) = \frac{n}{2}$ for $2 \geq t > t_0$. □

Proposition 5.3.8. *Suppose β_+ and β_- are obtained from β by addition of a positive and negative crossing respectively then*

$$M_{\beta_-}(t) \geq M_\beta(t) \geq M_{\beta_+}(t).$$

Proof. It is easy to see that $m(\beta)$ can be obtained from $m(\beta_-)$ by addition of a negative crossing. So we can consider the crossing change map $c_- : ct\mathbf{C}(m(\beta_-)) \to ct\mathbf{C}(m(\beta))$. Now let $\alpha \in t\mathbf{C}(m(\beta_-))$ non-torsion in x^+ tower with $A_U(\alpha) = -M_{\beta_-}(t)$. Then $c_-(\alpha)$ is also non-torsion in the x^+ tower with $A_U(c_-(\alpha)) = -M_{\beta_-}(t)$. Hence, it follows that $-M_\beta(t) \geq -M_{\beta_-}(t)$. Similarly, we obtain the other inequality.

□

Let

$$\mathscr{M}_t := \{\beta \mid \beta \text{ has index } n, M_t(\beta) = \frac{n}{2}\}.$$

Let $t_1, t_2 \cdots t_n$ be real numbers satisfying $0 \leq t_1 \leq \cdots \leq t_n < 2$. Then we clearly have,

$$QP \subseteq \mathscr{M}_{t_1} \subseteq \cdots \subseteq \mathscr{M}_{t_n} \subseteq RV.$$

Where QP and RV denotes the monoids of Quasi-positive and right-veering braids respectively.

Theorem 5.3.3. *Membership in $\mathscr{M}_t$ is a transverse invariant and furthermore, $\mathscr{M}_t$ is a monoid.*

Proof. Suppose $\beta \in \mathscr{M}_t$, then there is a non-torsion element α in the x^+ tower in $ct\mathbf{C}(m(\beta))$ with $A_U(\alpha) = -\frac{n}{2}$. Now, we consider the negative stabilization map $NS_- : ct\mathbf{C}(m(\beta)) \to ct\mathbf{C}(m(\beta_{+stab}))$. Then, from Proposition 5.2.16 we have, $A_U(NS_-(\alpha)) = -\frac{n}{2} - \frac{1}{2} = -\frac{n+1}{2}$. This implies that max non torsion element in $t\mathbf{C}(m(\beta_{+stab}))$ also has A_U equal to $-\frac{n+1}{2}$ since its the minimum possible value. Therefore, $\beta_{+stab} \in \mathscr{M}_t$.

Let us take an index N- braid $\beta_1 \in \mathscr{M}_t$ and an index M- braid $\beta_2 \in \mathscr{M}_t$. To prove that $\mathscr{M}_t$ is a monoid, we need to show that $\beta_1\beta_2 \in \mathscr{M}_t$. Firs, we observe that $M_t(\beta_1 \sqcup \beta 2) = M_t(\beta_1) + M_t(\beta_2) = \frac{N}{2} + \frac{M}{2} = \frac{M+N}{2}$. So $\beta_1 \sqcup \beta_2 \in \mathscr{M}_t$. Now Baldwin [1] showed that $\beta_1\beta_2$

is transversely isotopic to $\beta_1 \sqcup \beta_2$ after adding negative crossing. Since M_t is non decreasing for addition of negative crossing and membership in $\mathscr{M}_t$ is a transverse invariant, it follows that $\beta_1\beta_2 \in \mathscr{M}_t$.

$\square$

Chapter 6

Some remarks about the τ invariant

In this chapter, we will use techniques of Chapter 3 to derive an inequality of τ for cables. Then, we will re-interpret τ as filtration level of a distinguished element in a filtered complex. This is analogous to the interpretation of Rasmussen's s-invariant as the filtration level of distinguished element in Khovanov homology. This interpretation might make computation of τ more feasible in some cases. A similar formula can also be used to reinterpret the annular invariant. This re-interpretation might make computational implementation more feasible in some cases.

6.1 The concordance invariant τ for cables

Ozsváth and Szabó defined the concordance invariant $\tau(K)$ for a knot K. It can be showed that $-\tau(K)$ is maximal Alexander grading of a non-torsion element in $GH^-(K)$. The following proposition relates it with the collapsed complex.

Proposition 6.1.1. *Let D be the grid diagram of a knot K. Let $x \in H_*(\mathscr{C}_D)$ be the non-torsion element with maximal Alexander grading. Then $\tau(K) = -A(x)$.*

Proof. We know from Proposition 2.4.4, $H_*(\mathscr{C}_D) \cong GH^-(K) \otimes W^{\otimes n-1} \cong (\mathbb{F}_2[V] \oplus Tor) \otimes W^{\otimes n-1}$. Therefore, the free part is isomorphic to $\mathbb{F}_2[V]_{(-2\tau(K),-\tau(K))} \otimes W^{\otimes n-1}$. Hence, the conclusion follows. □

We can derive the following inequality by looking at the free part of the decomposition in Theorem 3.2.1.

Proposition 6.1.2. *Let K be a knot and $K_{p,q}$ its cable knot. Then,*

$$p\tau(K) + \frac{(p-1)(q+1)}{2} \geq \tau(K_{p,q}) \geq p\tau(K) + \frac{(p-1)(q-1)}{2}$$

holds for all $p \geq 2, q \in \mathbb{Z}$.

Proof. Assume that D from earlier discussion represent $m(K)$. Then, D_p represents $m(K)_{p,-q} = m(K_{p,q})$. Lets take a non-torsion element in $x \in H_*(\mathscr{C}_D)$. Then by, Theorem 3.2.1, $i(x) \in H_*(\mathscr{C}(D_p))$ is non-torsion. Using Proposition 2.4.6, we can conclude that $-p\tau(m(K)) +$

$\frac{(p-1)(q-1)}{2} \leq -\tau(m(K_{p,q}))$. Since mirroring changes the sign of τ, the lower bound follows. We can also take D to represent K, then D_p represents $K_{p,q}$. However, in this case we can modify Proposition 3.2.2 to see that i is graded of degree $-\frac{(p-1)(q+1)}{2}$. Now a similar argument gives the upper bound. □

Remark. *J. Hom proved (See [15]) that*

$$\tau(K_{p,q}) = p\tau(K) + \frac{(p-1)(q-\epsilon(K))}{2} \quad when\ \epsilon(K) \neq 0$$

and

$$\tau(K_{p,q}) = \tau(T_{p,q}) \quad when\ \epsilon(K) = 0$$

where $\epsilon(K)$ is a concordance invariant valued -1, 0 or 1. Proposition 6.1.2 is an obvious corollary of Hom's theorem. However, it will be interesting to see if one can further analyze the $\mathcal{R}$ summand in Theorem 3.2.1 to extract information about $\epsilon(K)$.

6.2 Computing τ as a filtration level

Proposition 6.2.1. *Let K be a knot represented by grid D. K^- denotes the same knot with the orientation reversed represented by grid D^- obtained by reversal of roles of $\mathbb{X}$ and $\mathbb{O}$. There is an isomorphism $\chi : \frac{GC^-(D)}{V_1=V_2=...=V_n=1} \to \widetilde{\mathcal{GC}}(D^-)$.*

Proof. The differential $\partial^- x := \sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r\cap\mathbb{X}=\phi} V_1^{O_1(r)}...V_m^{O_m(r)} y$ in $GC^-(K)$ becomes the differential $\sum_{y \in S(D)} \sum_{r \in Rect^o(x,y), r\cap\mathbb{X}=\phi} y$ which happens to be the differential of $\widetilde{\mathcal{GC}}(D^-)$, since the reversal of roles of $\mathbb{X}$ and $\mathbb{O}$ changes the orientation of the knot. Therefore, the natural identification of states gives the stated isomorphism.

□

We know that $\tau(K)$ can be defined as -1 times the maximal Alexander grading of a non torsion element $[\alpha] \in GH^-(K)$ (We can assume α is homogeneous in $GC^-(D)$). Let us consider the projection onto quotient, $\phi : GC^-(D) \to \frac{GC^-(D)}{V_1=V_2=...=V_n=1}$. A class $[\beta]$ is non torsion if and only if $H(\phi)([\beta]) \neq 0$.

Proposition 6.2.2. *$H(\phi \circ \chi)([\alpha]) = [x^+]_{\widetilde{\mathcal{GH}}}$ where x^+ is the distinguished state consisting of north east corners of Os in the grid D^- (or north east corners of Xs in the grid D) .*

Proof. We know that $[\alpha]U^k + [\beta] = [x^+]_{GH^-}$ for some non-negative integer k and some torsion class $[\beta]$. Therefore, the conclusion follows. In particular, we have $\widetilde{\mathcal{GH}}(K^-) \ni [x^+]_{\widetilde{\mathcal{GH}}} \neq 0$ since $[\alpha]$ is non-torsion.

□

Suppose A_1 and A_2 are Alexander gradings in the grid complexes of D and D^- respectively. Then we know that $A_1(x) + A_2(x) = 1 - n$ for a state x in the grids of K or K^- with grid number n. Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be the filtration induced by A_1 and A_2 in their respective filtered grid complexes. Since orientation reversal induces filtered quasi-isomorphism in the grid complex, we have $\tau(K^-) = min\{s | H_0(\mathcal{F}_2^s(\widehat{\mathcal{GC}}(D^-))) \neq 0\} = \tau(K)$

Theorem 6.2.1. $\tau(K) = \mathcal{F}_2([x^+]_{\widetilde{\mathcal{GH}}}) + (n-1)$.

Proof. Let $W \cong \mathbb{F}_{(0,0)} \oplus \mathbb{F}_{(-1,-1)}$ be the bigraded two dimensional vector space. We know that there is a filtered quasi-isomorphism $f : \widehat{\mathcal{GC}}(D^-) \otimes W^{\otimes(n-1)} \to \widetilde{\mathcal{GC}}(D^-)$. Let $k = \mathcal{F}_2([x^+]_{\widetilde{\mathcal{GH}}})$ We can explicitly write ,

$$\mathcal{F}_2^{k+n-1)}(\widehat{\mathcal{GC}}(D^-) \otimes W^{\otimes(n-1)}) = (\mathcal{F}_2^{k+n-1}(\widehat{\mathcal{GC}}(D^-)_0 \otimes {\mathbb{F}_{(-1,-1)}}^{\otimes(n-1)}) \oplus .... \oplus (\mathcal{F}^k(\widehat{\mathcal{GC}}(D^-)_{n-1} \otimes {\mathbb{F}_{(0,0)}}^{\otimes(n-1)})$$

.

Since Maslov grading of $[x^+]_{\widetilde{\mathcal{GH}}}$ is $1-n$, we have,

$$[x^+]_{\widetilde{\mathcal{GH}}} \in H_0(\mathcal{F}_2^{k+(n-1)}(\widehat{\mathcal{GC}}(D^-)) \otimes {\mathbb{F}_{(-1,-1)}}^{\otimes(n-1)})$$

(Since homology of $\widehat{\mathcal{GC}}$ is zero for positive Maslov grading). Therefore ,$H_0(\mathcal{F}_2^{k+n-1}(\widehat{\mathcal{GC}}(D^-)) \neq 0$. Hence $\tau(K) = \tau(K^-) \leq k + n - l = \mathcal{F}_2([x^+]_{\widetilde{\mathcal{GH}}}) + n - 1$.

On the other hand we have, $\tau(K) = -A_1(\alpha)$. We also have $A_1((\phi \circ \chi)(x)) \geq A_1(x)$ for any $x \in GC^-(D)$. Under the identification , we obtain

$$\mathcal{F}_2([(\phi\circ\chi)(\alpha)]) \leq A_2((\phi\circ\chi)(\alpha)) = -A_1((\phi\circ\chi)(\alpha)) + 1 - n \leq -A_1(\alpha) + 1 - n\tau(K) + (1-n)$$

Therefore , $\mathcal{F}_2([x^+]_{\widetilde{\mathcal{GH}}}) \leq \tau(K) + (1-n)$. Hence, the equality follows.

□

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