Quaternionic Geometry and Special Holonomy

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Abstract of the Dissertation

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This thesis studies connections with special holonomy group arising from quaternionic manifolds. The focus is on two previously described constructions that produce such connections from positive quaternion-Kähler manifolds with an isometric and quaternionic circle action. The first, due to Hitchin [17], yields quaternionic connections with a preferred complex structure, while the second, due to Haydys [14], yields Kähler metrics. In particular, both constructions produce Kähler metrics in real dimension four, and therefore generalize a theorem of Pontecorvo [38] that produces scalar-flat Kähler metrics from anti-self-dual Hermitian surfaces in real dimension 4.

The goal of this work is to explore the relationship between these two constructions. Although they are superficially similar, the main result of this dissertation shows that they are in fact distinct. This result is obtained by describing a simplification of Haydys's construction that allows for explicit computation of the Levi-Civita connection of the Kähler metric. Hitchin's methods are also generalized to give a construction of quaternionic complex manifolds from quaternionic manifolds without a metric.

Dedication

For Jason

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1 Introduction and Summary of Results

This thesis concerns quaternionic geometry, which we define broadly as the study of connections with holonomy contained in the group $GL(k, \mathbb{H})Sp(1)$. Our study begins with quaternion-Kähler manifolds, defined to be Riemannian manifolds with holonomy contained in the group $Sp(k)Sp(1)$. These manifolds were considered due to Berger's classification of the holonomy groups of irreducible, non-symmetric Riemannian manifolds [8], where they appear as one of the possible groups. Quaternion-Kähler manifolds have a number of geometric properties that make them particular interesting to study. From the perspective of Riemannian geometry, quaternion-Kähler metrics are necessarily Einstein, and therefore provide interesting examples of constant Ricci curvature metrics in higher dimensions. These manifolds also have a fruitful connection to complex geometry and algebraic geometry via their twistor spaces, and in this sense quaternion-Kähler manifolds are natural generalizations of anti-self-dual conformal manifolds in four dimensions.

Our interest in such manifolds arises from two different constructions on quaternion-Kähler manifolds admitting isometric circle actions that preserve the quaternionic structure. The first is due to Hitchin [17], who used such actions to construct connections with holonomy contained in $SL(k, \mathbb{H})U(1)$ on subsets of the underlying manifold. Closely related is a construction due to Haydys [14], who similarly uses circle actions on quaternion-Kähler manifolds to construct Kähler metrics on the same subset of the underlying manifold.

The goal of this thesis is to clarify the relationship between these two constructions. Understanding this relationship is especially important in that both constructions could conceivably be considered as generalizations of a theorem of Pontecorvo [38] that gives the conditions under which the conformal class of an anti-self-dual Hermitian 4-manifold contains a scalar-flat Kähler manifold. In particular, Hitchin's construction in four dimensions in actually yields scalar-flat Kähler manifolds, although the connections he obtains in higher dimensions are not *a priori* connections arising from a metric. Similarly, Haydys's construction yields Kähler metrics, although the curvature properties of such metrics, including any special features of the scalar curvature, are not considered in Haydys's original work. Our main goal is to consider how the special connections arising in either case might generalize the geometric properties of scalar-flat Kähler surfaces.

The most interesting possibility, and the motivation for our investigations, is that the Haydys and Hitchin constructions could actually coincide, in the sense that the connection with holonomy contained in $GL(k, \mathbb{H})U(1)$ considered by Hitchin could in fact be the Levi-Civita connection for the Kähler metric constructed by Haydys. In this case, the holonomy of the connection would in fact be contained in the group $Sp(k)U(1)$, which is not one of the groups on Berger's list, so that the resulting metric would have to be either hyperkähler (with holonomy in $Sp(k)$), locally symmetric, or decomposable as a Riemannian product. Any of these cases would put a strong restriction on the geometry of the resulting manifold which could be used to gain insight into the structure of the original quaternion-Kähler manifold.

Unfortunately, the main result of this thesis is to show that the connections produced by Haydys and Hitchin do not coincide (see Corollary 5.11, as well as the example considered in Section 5.5). Still, the investigations that led to that conclusion have themselves been fruitful, and so along with this negative result we are able to provide further clarifications and generalizations of the Haydys and Hitchin constructions.

The basic structure of this thesis is as follows. We begin with a review of the background material necessary for the thesis. Section 2 gives the necessary definitions of the various geometries studied in this thesis, first in terms of tensor fields and then using the equivalent language of G-structures and holonomy groups. Next, Section 3 reviews the construction and properties of the twistor space for both quaternionic and quaternion-Kähler manifolds.

After these two sections, we present the main results of the thesis. We begin by considering Hitchin's construction in Section 4. After briefly reviewing his construction, we show how his methods can be generalized to give special holonomy connections related to quaternionic manifolds without a metric. Next, we turn to Haydys's construction in Section 5. The first result of this section is an alternate proof of the existence of Kähler metrics on subsets of quaternion-Kähler manifolds with $U(1)$ actions that uses more elementary methods than the original proof given by Haydys. The resulting expressions for the Kähler metric are explicit enough to allow for direct comparison of the Haydys and Hitchin connections, which we demonstrate are distinct. We also verify the difference between the two constructions in a simple example.

2 Quaternionic Geometries

We begin by giving definitions of the various quaternionic geometries considered in this thesis. The definitions can be given in two equivalent and closely related formulations, either in terms of tensor fields preserved by torsion-free connections (Section 2.1), or in terms of G-structures and holonomy groups (Sections 2.2 and 2.3).

A more thorough introduction to the various geometries discussed here is given in Besse [9, Chapter 14], which tends to emphasize the tensor viewpoint. Salamon [41], conversely, develops much of the same material from the perspective of G-structures. Alekseevsky and Marchiafava [2] also develop much of the same material from the perspective of G-structures. In the Riemannian case, Salamon [40] also gives a good introduction to the quaternion-Kähler case, while Hitchin et. al. [18] is the main source for hyperkähler geometry. Most of of the propositions and theorems in this section will be presented without a proof, unless the details of the proof are important for later results. We refer to the sources above for the full details, or will make specific further references when relevant.

2.1 Definitions and Basic Properties

Definition 2.1. Let M be a smooth manifold. Then an *almost quaternionic structure* on M is a rank-three subbundle $\mathcal{Q} \subset \mathrm{End}(TM)$ such that any $x \in M$ has a neighborhood U and a local frame $I, J, K : TM|_U \to TM|_U$ such that I, J, K satisfy the quaternion relations $I^2 = J^2 = K^2 = -\mathrm{Id}, IJ = K = -JI$. Such a choice $\{I, J, K\}$ is called a *local compatible frame* for $\mathcal{Q}$.

Note. We will sometimes write $\{I_1, I_2, I_3\} = \{I, J, K\}$ for a local compatible frame, as it allows for more compact formulas.

A choice of a local compatible frame gives T_xM the structure of a left $\mathbb{H}$-module, for given $X \in T_xM$ and $q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H}$, we can define the action of q on X as

$$q \cdot X = (q_0 + q_1 I + q_2 J + q_3 K)X$$

Note that the choice of a *left* action here is important because of the non-commutativity of the quaternions. We will see in Section 2.2 that it will be better, for our conventions, to consider most modules over the quaternions as *right* modules. We can define a right action by having the right action by $q \in \mathbb{H}$ be equivalent to left action by q^* the quaternionic conjugate, defined by $q^* = q_0 - q_1 i - q_2 j - q_3 k$ for $q = q_0 + q_1 i + q_2 j + q_3 k$. We use q^* instead of the somewhat more common notation $\overline{q}$ to distinguish the conjugation operation on $\mathbb{H}$ from the conjugation operation on $\mathbb{C}$. The quaternionic conjugate has the property that $(qp)^* = p^* q^*$, and so the right action given by $X \cdot q = q^* \cdot X$ is well-defined.

Since $T_x M$ can be given the structure of an $\mathbb{H}$-module, the dimension of an almost quaternionic manifold is always divisible by 4, and so we will write $n = 4k$ for the dimension of an almost quaternionic manifold, where k is the *quaternionic dimension*. Because of various special aspects of these geometries in low dimensions, it is often necessary to distinguish the case of $k = 1$ from the case $k \geq 2$. For the following, we therefore generally assume that $k \geq 2$, and will consider the case of $k = 1$ separately at the end of this section, although some definitions remain valid in the $k = 1$ case.

Whether acting from the left or right, we have that this quaternionic action depends on a choice of local compatible frame for $\mathcal{Q}$, and such a choice is clearly local in nature and also non-canonical. We can consider when such frames are in fact defined globally on M to define an almost hypercomplex structure on a manifold.

Definition 2.2. Let M^{4k} be a smooth manifold. An *almost hypercomplex structure* on M consists of three globally-defined almost complex structures I, J, K on M satisfying the quaternionic relation $IJ = -JI = K$.

A *hypercomplex structure* is an almost hypercomplex structure in which the almost complex structures I, J, K are in fact integrable.

In order to study the geometry of these spaces, we consider connections on these manifolds that preserve these structures.

Definition 2.3. Let $(M^{4k}, \mathcal{Q})$ be an almost quaternionic manifold with $k \geq 2$. We say that this manifold is *weakly quaternionic* if M admits a torsion-free connection ∇ that preserves $\mathcal{Q}$, that is,

$$\nabla S \in \Gamma(T^*M \otimes \mathcal{Q})$$

for $S \in \Gamma(\mathcal{Q})$ a local section. Such a connection is called *quaternionic*.

A *quaternionic manifold* $(M, \mathcal{Q}, \nabla)$ consists of a weakly quaternionic manifold together with a choice of a quaternionic connection.

Implicit in the definition above is that there are many quaternionic connections possible on a weakly quaternionic manifold, which is in fact the case. The structure of the space of quaternionic connections is well understood, and choosing a quaternionic connection essentially amounts to choosing a 1-form on the M, see Lemma 4.1.

We can also consider connections preserving (almost) hypercomplex structures. If (M, J) is any almost complex manifold and ∇ is a connection on M such that $\nabla J = 0$, then the Nijenhuis tensor for J can be associated to a component of the torsion tensor of the connection (see [31]), and so if the connection is torsion-free then the almost complex structure J is in fact integrable. Therefore if an almost hypercomplex manifold admits a torsion-free connection that preserves each of the almost complex structures, we have that these complex structures are integrable and the manifold is in fact hypercomplex. In that case, Obata has shown that this connection is unique.

Theorem 2.4 ([34])**.** *Let (M^{4k}, I, J, K) be a hypercomplex manifold. Then there exists a unique torsion-free connection ∇ on M, called the* Obata connection *such that*

$$\nabla I = \nabla J = \nabla K = 0$$

There is also an intermediate structure in which there is a single parallel almost complex structure.

Definition 2.5. A *quaternionic complex manifold* is a quaternionic manifold together with a globally-defined almost complex structure $I \in \Gamma(\mathcal{Q})$ that is parallel with respect to the quaternionic connection.

Such manifolds have not been well studied, and so this terminology, which is due to Joyce [20], is not widely used.

Given an almost quaternionic or hypercomplex structure on a manifold, we can also consider related geometrical structures on M that are compatible with the quaternionic geometry. The most important additional structure we will consider are volume forms and Riemannian metrics.

Considering volume, we first observe that any almost quaternionic or hypercomplex manifold is orientable. A local compatible frame $\{I, J, K\}$ induces a local orientation by declaring local frames of TM of the form

$$\{X_1, IX_1, JX_1, KX_1, \ldots, X_k, IX_k, JX_k, KX_k\}$$

for $\{X_1, \ldots, X_k\}$ nonvanishing and linearly independent to be positively oriented. This orientation is independent of the choice of local compatible frame since any two choices of a local compatible frame are related by an element of $SO(3)$, see Corollary 2.19. Note that in the hypercomplex case, this orientation is the same as the orientation induced by any of the complex structures on M.

Definition 2.6. A *scale* on a weakly quaternionic manifold is a choice of a fixed non-vanishing volume form μ on M. If M is quaternionic, we say that the scale is compatible with the quaternionic structure if $\nabla\mu = 0$, where ∇ is the quaternionic connection. In this case, we call $(M, \mathcal{Q}, \nabla, \mu)$ a *scaled quaternionic manifold*. Similarly, a *scaled quaternionic complex manifold* is a quaternionic complex manifold for which the connection that preserves the fixed almost complex structure also preserves a volume form.

A scale for a weakly quaternionic manifold induces a unique choice of quaternionic connection, see Proposition 4.2. Evidently one could also define a concept of a scale for hypercomplex manifolds, but in this case the uniqueness of the Obata connection means that scales are no longer associated to a choice of connection.

Finally, we can consider how almost quaternionic and hypercomplex structures interact with Riemannian metrics.

Definition 2.7. A Riemannian metric g on an almost quaternionic manifold $(M, \mathcal{Q})$ is *quaternion-Hermitian* if the metric g is compatible with the almost complex structures I, J, K, for every choice of local compatible frame for $\mathcal{Q}$, that is,

$$g(IX, IY) = g(X, Y)$$

for all tangent vectors X, Y, and similarly for J, K.

Similarly, if (M, I, J, K) is hypercomplex and the metric g is compatible with the complex structures, then we say the manifold is *almost hyperkähler*.

An almost quaternionic manifold admits many quaternion-Hermitian metrics, for if g is an arbitrary metric then the metric

$$\widetilde{g}(X, Y) = g(X, Y) + g(IX, IY) + g(JX, JY) + g(KX, KY)$$

is well-defined (that is, does not depend on the choice of local compatible frame) and evidently quaternion-Hermitian. A quaternion-Hermitian structure gives a natural connection on the almost quaternionic manifold, the Levi-Civita connection, which is torsion-free but in general is not compatible with the quaternionic structure. When this is the case, we arrive at the geometry that is of central importance to this thesis, quaternion-Kähler metrics.

Definition 2.8. Let $(M^{4k}, \mathcal{Q}, g)$ be a quaternion-Hermitian manifold with quaternionic dimension $k \geq 2$. If additionally we have that the Levi-Civita connection is quaternionic, then the metric is called *quaternion-Kähler*.

Let (M, g, I, J, K) be an almost hyperkähler manifold. Then g is called *hyperkähler* if the Levi-Civita connection for g coincides with the Obata connection.

Note that in the quaternion-Kähler case, the Riemannian volume form associated to g is a scale compatible with the quaternionic structure.

Some of the naming conventions around quaternion-Kähler and hyperkähler metrics are unfortunate. First, we note that a hyperkähler manifold is itself quaternion-Kähler, but with the additional property that the bundle of compatible almost complex structures is globally trivialized in the hyperkähler case. (In terms of the holonomy definition, we similarly have that $Sp(k) \subset Sp(k)Sp(1)$, see Proposition 2.20). We adopt the standard convention of using "quaternion-Kähler" to refer only to those metrics that are not additionally hyperkähler. Both quaternion-Kähler and hyperkähler manifolds are Einstein (see, for example, Theorem 5.3), and any quaternion-Kähler manifold with scalar curvature $s = 0$ is then (locally) hyperkähler, so our terminology amounts to assuming that the scalar curvature of a quaternion-Kähler manifold is not zero. We call a quaternion-Kähler manifold "positive" or "negative" according to the sign of the scalar curvature. Second, quaternion-Kähler manifolds are themselves not actually Kähler manifolds in the sense of complex geometry, and many do not even admit globally-defined almost complex structures. Conversely, hyperkähler manifolds are Kähler, and are in fact Kähler with respect to many different complex structures defined on the manifold.

Both quaternion-Kähler and hyperkähler metrics can also be defined using differential forms in a manner analogous to the definition of a Kähler manifold. Given a quaternion-Hermitian metric and a choice $\{I, J, K\}$ of a local compatible basis, one can construct local proto-Kähler 2-forms $\omega_I, \omega_J, \omega_K$ in the standard way, as

$$\omega_I(X, Y) = g(IX, Y)$$

and similarly for ω_J, ω_K. Let $\mathcal{G}$ denote the bundle with local sections spanned by $\omega_I, \omega_J, \omega_K$, so that $\mathcal{G} \subset \Lambda^2 T^*M$ corresponds to $\mathcal{Q} \subset \mathrm{End}(TM)$ under the isomorphism $TM \cong T^*M$ induced by the metric g. Although these 2-forms are only defined locally, they give a globally-defined 4-form, introduced by Kraines [25].

Proposition 2.9 ([25]). *Let $\omega_I, \omega_J, \omega_K$ be the 2-forms defined above for M^{4k} a quaternion-Hermitian manifold. Then the 4-form*

$$\Omega = \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K$$

is independent of the choice of local compatible basis and is therefore globally defined. This form is known as the fundamental *or* Kraines *4-form.*

Moreover, if $k \geq 2$ then the metric is quaternion-Kähler if and only if $\nabla\Omega = 0$, where ∇ is the Levi-Civita connection for the metric.

In the case of an almost hyperkähler manifold, the forms $\omega_I, \omega_J, \omega_K$ are in fact globally defined, and are Kähler forms if the metric is in fact hyperkähler:

Proposition 2.10 ([16])**.** *Let (M, I, J, K, g) be an almost hypercomplex manifold equipped with a quaternion-Hermitian metric. Then the metric is hyperkähler if and only if the 2-forms $\omega_I, \omega_J, \omega_K$ are closed. In particular, if the forms are closed then the almost complex structures are integrable.*

As mentioned earlier, the case of $k = 1$ (real dimension $n = 4$) must be treated separately, and so far we have not defined quaternionic or quaternion-Kähler structures in this case. (The definitions for hypercomplex or hyperkähler manifolds are consistent in all dimensions). The theory of quaternionic geometries in higher dimensions is in many ways a generalization of aspects of 4-dimensional conformal geometry developed by Penrose [37] and later Atiyah, Hitchin, and Singer [4], and so the definitions of quaternionic geometries in dimension 4 are given in those terms.

If M is an oriented 4-manifold with a fixed conformal structure $[g]$, then the Hodge star operator $* : \Lambda^p T^*M \to \Lambda^{4-p} T^*M$ has the property that $*^2 = \mathrm{Id}$ when operating on 2-forms. This leads to the splitting $\Lambda^2 T^*M = \Lambda^+ \oplus \Lambda^-$ into the $+1$ and -1-eigenspaces of $*$, giving the bundles of *self-dual* and *anti-self-dual* 2-forms, respectively. Both are rank 3 bundles, and fixing a metric g in the conformal class gives a method to transform a self-dual 2-form into an orthogonal and orientation-preserving endomorphism of TM, and this related bundle will have local frames of the form I, J, K satisfying the standard quaternionic relations. The bundle Λ^+ on an oriented, conformal 4-manifold is therefore analogous to the bundle $\mathcal{G}$ on a quaternion-Hermitian manifold. In this way any oriented 4-manifold has a natural analog of an almost quaternionic structure obtained by fixing a conformal class. Choosing a torsion-free connection that preserves that conformal class, e.g. the Levi-Civita connection of a metric in that conformal class, then yields an analog of a quaternionic structure.

However, quaternionic manifolds in higher dimensions have some extra properties that do not necessarily hold true for any oriented conformal 4-manifold. In particular the unit sphere bundle of $\mathcal{Q}$ always has a natural complex structure making this space into a complex manifold known as the twistor space (see Proposition 3.4 and Theorem 3.5). In the four-dimensional case, the unit sphere bundle of Λ^+ has a natural *almost* complex structure, and this structure is integrable exactly when the self-dual part of the Weyl curvature of $[g]$ vanishes ([4], see also Proposition 3.4 and following discussion), leading to the following definitions:

Definition 2.11. In real dimension $n = 4$, a *weakly quaternionic manifold* is an oriented manifold equipped with an anti-self-dual conformal structure. A *quaternionic manifold* is a manifold with a chosen anti-self-dual metric. A *quaternionic complex manifold* is an anti-self-dual Kähler surface.

We note that anti-self-dual Kähler surfaces are equivalent to scalar-flat Kähler surfaces [26], which gives an equivalent definition of quaternionic-complex 4-manifolds.

The definition of a quaternionic manifold in real dimension 4 is analogous to that of a quaternion-Kähler manifold in quaternionic dimension $k \geq 2$, in that a quaternionic 4-manifold already has a chosen compatible metric. This fact, however, does not make the category of quaternion-Kähler manifolds redundant in quaternionic dimension $k = 1$. As mentioned earlier, quaternion-Kähler metrics in higher dimension are necessarily Einstein (see Theorem 5.3), and so we include this requirement in defining quaternion-Kähler manifolds in dimension 4.

Definition 2.12. In real dimension $n = 4$, a *quaternion-Kähler manifold* is an oriented manifold equipped with an anti-self-dual Einstein metric.

We end by considering a few examples of quaternion-Kähler and hyperkähler manifolds. For quaternion-Kähler manifolds, much of the interest is focused on the positive case. In that case, Myers's theorem gives that any complete, connected quaternion-Kähler manifold with positive scalar curvature is compact. Positive quaternion-Kähler manifolds are also simply connected, which can be shown by considering the twistor space (see Corollary 3.8). The only known examples of positive quaternion-Kähler manifolds are the so-called *Wolf spaces* [43]. These are all Riemannian symmetric spaces, and are organized into three main families, namely

$$\begin{gathered} \mathbb{HP}_k = \frac{Sp(k+1)}{Sp(k)Sp(1)} \qquad Gr(2,\mathbb{C}^{k+2}) = \frac{SU(k+2)}{S(U(k)\times U(2))} \\ \widetilde{Gr}(4,\mathbb{R}^{k+4}) = \frac{SO(k+4)}{SO(k)SO(4))} \end{gathered} \tag{1}$$

Note that in the case $k = 1$ we have the isomorphisms $\mathbb{HP}_1 \cong \widetilde{Gr}(4,\mathbb{R}^5) \cong S^4$ and $Gr(2,\mathbb{C}^3) \cong \overline{\mathbb{CP}_2}$, where each space has its standard anti-self-dual metric. There are also five Wolf spaces corresponding to exceptional Lie groups, namely

$$\frac{G_2}{SO(4)}, \quad \frac{F_4}{Sp(3)Sp(1)}, \quad \frac{E_6}{SU(6)Sp(1)}, \quad \frac{E_7}{Spin(12)Sp(1)}, \quad \frac{E_8}{E_7Sp(1)}$$

LeBrun and Salamon [30] have shown that in any given dimension there are only finitely many quaternion-Kähler manifolds with positive scalar curvature, which led them to make the conjecture that the Wolf spaces are in fact the only positive quaternion-Kähler manifolds. This conjecture has been verified in low dimensions. For $k = 1$, it is a well-known theorem of Hitchin [15], and for dimension $k = 2$ it was verified by Poon and Salamon [39] (see also [30]). More recently, the cases of $k = 3, 4$ have been verified by Buczyński and Wiśniewski [10]. In the negative case, the non-compact duals of the Wolf spaces described above are symmetric spaces with negative scalar curvature quaternion-Kähler metrics, also considered by Wolf [43], while Alekseevsky [1] has constructed negative quaternion-Kähler metrics that are homogeneous but not symmetric. More generally, LeBrun [28] has shown that the moduli space of negative quaternion-Kähler metrics on $\mathbb{R}^{4k}$ is infinite, so that the landscape of possible negative quaternion-Kähler manifolds is much larger.

For examples of hyperkähler manifolds, the definition gives that hyperkähler four-manifolds are exactly Calabi-Yau surfaces, including for example the K3 surfaces. In higher dimensions, families of compact hyperkähler metrics have been constructed by Mukai [33] and Beauville [6]. Non-compact examples are also in abundance. The most notable is the Calabi metric on $T^*\mathbb{CP}_n$ [11], which was the first example of a metric with holonomy exactly $Sp(k)$. The hyperkähler reduction construction of Hitchin, Karlhede, Lindström, and Rŏcek [18] provides a method to construct a wide variety of hyperkähler metrics. There is also a similar reduction operation in the quaternion-Kähler case, due to Galicki and Lawson [13], that produces quaternion-Kähler orbifolds.

2.2 Representation Theory for Quaternionic Geometries

As we have already mentioned, the various quaternionic geometries defined in Section 2.1 can also be described in terms of G-structures and holonomy groups. In this section we define the groups and representations relevant to this description. Setting some notation for this section, we use q, p to denote generic quaternions, the letters z, w for complex numbers, and x, y for real numbers. We will use boldface to denote k-dimensional vectors of these objects, e.g., $\mathbf{z} \in \mathbb{C}^k$, which we will consider as column vectors so that linear transformations can be represented by matrix multiplication on the left.

In particular, if we wish to identify quaternion-linear transformations of $\mathbb{H}^k$ with left matrix multiplication, we are required to consider $\mathbb{H}^k$ as a *right*

$\mathbb{H}$-module, so that if $A \in M_{k\times k}(\mathbb{H})$ and $p \in \mathbb{H}$ we have

$$(A\mathbf{q}) \cdot p = A\mathbf{q}p = A(\mathbf{q} \cdot p)$$

that is, left matrix multiplication commutes with right scalar multiplication. Note that the non-commutativity of the quaternions requires us to choose whether scalar multiplication acts on $\mathbb{H}^k$ from the left or the right.

As $\mathbb{R} \subset \mathbb{C} \subset \mathbb{H}$, we can restrict scalar multiplication to consider $\mathbb{H}^k$ as a $2k$-dimensional complex vector space or a $4k$-dimensional real vector space. In the real case, this gives the standard identification $\mathbb{H} \cong \mathbb{R}^4$ via $q = x_1 + y_1 i + x_2 j + y_2 \sim (x_1, y_1, x_2, y_3)$, which extends immediately to an identification $\mathbb{H}^k \cong \mathbb{R}^{4k}$ as $\mathbf{q} = \mathbf{x}_1 + \mathbf{y}_1 i + \mathbf{x}_2 j + \mathbf{y}_2 k$. In the complex case, this gives the identification $\mathbb{H} \cong \mathbb{C}^2$ via $q = (x_1 + y_1 i) + j(x_2 - y_2 i) = z_1 + jz_2 \sim (z_1, z_2)$, or more generally $\mathbf{q} = \mathbf{z_1} + j\mathbf{z_2}$. This scheme is somewhat unsatisfying, in that it induces the non-standard identification of $\mathbb{R}^4 \cong \mathbb{C}^2$ as $(x_1, y_1, x_2, y_2) \sim (x_1 + y_1 i, x_2 - y_2 i)$. Some authors therefore use the opposite convention and identify $q = z_1 + z_2 j$. This has the effect that the standard multiplication by i on $\mathbb{C}^2$ agrees with *left* multiplication by the quaternion i, and would therefore necessitate viewing quaternion-linear maps as arising from right-multiplication by matrices acting on row vectors.

If we consider the identification $\mathbb{H} \cong \mathbb{C}^{2k}$ as a complex vector space, then the map $J_{\mathbb{C}} : \mathbb{C}^{2k} \to \mathbb{C}^{2k}$ induced by right multiplication by j is a conjugate-linear map defined by

$$J_{\mathbb{C}}(\mathbf{z_1}, \mathbf{z_2}) = (-\overline{\mathbf{z_2}}, \overline{\mathbf{z_1}}) \tag{2}$$

Similarly, the identification $\mathbb{H} \cong \mathbb{R}^{4k}$ as a real vector space induces $\mathbb{R}$-linear maps $I, J, K : \mathbb{R}^{4k} \to \mathbb{R}^{4k}$ corresponding to right multiplication by i, j, k.

The quaternion conjugate operation induces a norm on $\mathbb{H}$ via

$$|q|^2 = qq^* = q_0^2 + q_1^2 + q_2^2 + q_3^2$$

that is equal to the Euclidean norm on $\mathbb{R}^4$ under the standard identification $\mathbb{H} \cong \mathbb{R}^4$. Let $Sp(1)$ denote the set of quaternions with unit norm. One can easily check from the properties of the norm that $Sp(1)$ is then a group under quaternionic multiplication, with $q^{-1} = q^*$ for $q \in Sp(1)$, and it is straightforward to check that $Sp(1)$ is in fact a Lie group. Note that topologically $Sp(1) \cong S^3$, as it is the unit sphere in $\mathbb{R}^4$. If we let $\mathbb{H}^\times = \mathbb{H} - \{0\}$ denote the set of non-zero quaternions, then evidently $\mathbb{H}^\times \cong Sp(1) \times \mathbb{R}_{>0}$.

Given $q \in \mathbb{H}$, we have that q acts naturally on $\mathbb{R}^4$ by considering $p \in \mathbb{H}$ as an element of $\mathbb{R}^4$ and having q act on p by either left or right multiplication. In fact, it will better match our conventions to consider the right action of

q as right multiplication by the conjugate. With this choice, we define maps $L, R : \mathbb{H}^\times \to GL(4,\mathbb{R})$ by considering $p \in \mathbb{H}$ as an element of $\mathbb{R}^4$ and taking

$$L_q(p) = qp \qquad R_q(p) = pq^*,$$

Note that the map L_q is in fact $\mathbb{H}$-linear and therefore both $\mathbb{C}$ and $\mathbb{R}$-linear. We will use L to denote the embedding into real-linear transformations, and write $L^{\mathbb{C}} : \mathbb{H}^\times \to GL(2,\mathbb{C})$ when we wish to consider the embedding into complex-linear transformations. On the other hand, R_q is in general only $\mathbb{R}$-linear, and is $\mathbb{C}$-linear if an only if $q \in \mathbb{C} \subset \mathbb{H}$. In particular, if we consider $\mathbb{C} \subset \mathbb{H}$ to be the set of quaternions with no j, k part, then R yields the standard action of scalar multiplication on $\mathbb{C}^2$ when restricted to $\mathbb{C}^\times$. The maps L, R, and $L^{\mathbb{C}}$ have useful explicit formulas in terms of matrices, as given below.

Proposition 2.13. *Let $L, R : \mathbb{H}^\times \to GL(4,\mathbb{H})$ and $L^{\mathbb{C}} : \mathbb{H}^\times \to GL(2,\mathbb{C})$ be the maps described above. Then*

1. *The maps L, R, and $L^{\mathbb{C}}$ are Lie group homomorphisms.*
2. *For $q = q_0+q_1 i+q_2 j+q_3 k$, the transformations L_q and R_q are represented by the matrices*

$$L_q = \begin{pmatrix} q_0 & -q_1 & -q_2 & -q_3 \\ q_1 & q_0 & -q_3 & q_2 \\ q_2 & q_3 & q_0 & -q_1 \\ q_3 & -q_2 & q_1 & q_0 \end{pmatrix}, \qquad R_q = \begin{pmatrix} q_0 & q_1 & q_2 & q_3 \\ -q_1 & q_0 & -q_3 & q_2 \\ -q_2 & q_3 & q_0 & -q_1 \\ -q_3 & -q_2 & q_1 & q_0 \end{pmatrix} \tag{3}$$

3. *Writing $q = z_1 + jz_2$, the transformation $L^{\mathbb{C}}_q$ is represented by the matrix*

$$L^{\mathbb{C}}_q = \begin{pmatrix} z_1 & -\overline{z_2} \\ z_2 & \overline{z_1} \end{pmatrix}$$

4. *Restricting L and R to $Sp(1)$ yields Lie group homomorphisms $L, R : Sp(1) \to SO(4)$.*
5. *Restricting $L^{\mathbb{C}}$ to $Sp(1)$ yields a Lie group isomorphism $L^{\mathbb{C}} : Sp(1) \to SU(2)$.*

Proof. That these maps are Lie group homomorphisms follow immediately from their definitions, noting that the conjugation action is necessary in defining R so that $R_{q_1 q_2} = R_{q_1} \circ R_{q_2}$. The matrix expressions for L_q and R_q are easily confirmed by expanding the expressions qp and pq^* in the $\{1, i, j, k\}$ basis for $\mathbb{H} \cong \mathbb{R}^4$, while the formula for $L^{\mathbb{C}}$ follows from the identification $\mathbb{H} \cong \mathbb{C}^2$.

By inspection, we see that the columns of the matrices L_q, R_q are pairwise orthogonal, and that the determinants of these matrices are both

$(q_0^2+q_1^2+q_2^2+q_3^2)^2$, so that restricting to $Sp(1)$ yields maps into $SO(4)$. In the complex expression, if $q \in Sp(1)$ then $|z_1|^2+|z_2|^2=1$, and so $L^{\mathbb{C}}$ maps to $SU(2)$. This map is easily seen to be bijective. □

The above representations extend easily to actions of $\mathbb{H}^\times$ and $Sp(1)$ on $\mathbb{R}^{4k} \cong \mathbb{H}^k$, so that we can consider, in particular, the maps $R : Sp(1) \to \mathbb{H}^k$ where R_q corresponds to right-multiplication by $q^* = q^{-1}$. Let $U(1)$ be the group of unit complex numbers, which we can consider as a subgroup $U(1) \subset Sp(1)$ via the inclusion $\mathbb{C} \subset \mathbb{H}$, so that we can also consider $R(U(1)) \subset GL(4k,\mathbb{R})$ as well. In fact by the above we have that $R(U(1)) \subset GL(2k,\mathbb{C}) \subset GL(4k,\mathbb{R})$. With these action in higher dimensions, we have

Definition 2.14. Consider $R(U(1)) \subset R(Sp(1)) \subset SO(4k) \subset SL(4k,\mathbb{R}) \subset GL(4k,\mathbb{R})$. Then we define the following Lie groups:

1. Let $GL(k,\mathbb{H})$ denote the centralizer of $R(Sp(1))$ in $GL(4k,\mathbb{R})$.
2. Let $SL(k,\mathbb{H})$ denote the centralizer of $R(Sp(1))$ in $SL(4k,\mathbb{R})$
3. Let $Sp(k)$ denote the centralizer of $R(Sp(1))$ in $SO(4k)$.
4. Let $GL(k,\mathbb{H})Sp(1)$, resp. $SL(k,\mathbb{H})Sp(1)$, denote the subgroup product of $GL(k,\mathbb{H})$, resp. $SL(k,\mathbb{H})$, with $R(Sp(1))$ in $GL(4k,\mathbb{R})$.
5. Let $Sp(k)Sp(1)$ denote the subgroup product of $Sp(k)$ and $R(Sp(1))$ in $SO(4k)$.

Note that each of $GL(k,\mathbb{H})$, $SL(k,\mathbb{H})$ and $Sp(k)$ defined above is in fact contained in $GL(2k,\mathbb{C}) \subset GL(4k,\mathbb{R})$ since quaternion-linear transformations are necessarily $\mathbb{C}$-linear, so that we can define two more groups of complex-linear transformations.

6. Let $GL(k,\mathbb{H})U(1)$, resp. $SL(k,\mathbb{H})U(1)$, denote the subgroup product of $GL(k,\mathbb{H})$, resp. $SL(k,\mathbb{H})$, with $R(U(1))$ in $GL(2k,\mathbb{C})$.

The notation we have used for the above groups agrees with the more standard definitions for these objects. For example, an $\mathbb{H}$-linear map on $\mathbb{H}^k \cong \mathbb{R}^{4k}$ is by definition an $\mathbb{R}$-linear transformation that commutes with right quaternion multiplication, which is equivalent to commuting with right multiplication by $Sp(1)$ since $\mathbb{H}^\times = Sp(1) \times \mathbb{R}_{>0}$, and thus the group $GL(k,\mathbb{H})$ defined above does in fact represent the group of bijective $\mathbb{H}$-linear transformations on $\mathbb{H}^k$. Similarly, the standard definition of $Sp(k)$ is as the subgroup of $GL(k,\mathbb{H})$ that preserves the standard quaternion-Hermitian inner product

$$\langle \mathbf{p}, \mathbf{q} \rangle_{\mathbb{H}} = \sum_{i=1}^{k} \overline{p_i} q_i,$$

and so for this reason the group $Sp(k)$ is sometimes also denoted by $U(k,\mathbb{H})$ and considered as the quaternionic unitary group. The real part of this $\mathbb{H}$-valued inner product is exactly the Euclidean inner product on $\mathbb{R}^{4k} \cong \mathbb{H}^k$, and so quaternion-linear transformations that preserve the $\mathbb{H}$-valued inner product are naturally elements of $SO(4k)$, and conversely elements of $SO(4k)$ that commute with $Sp(1)$ and are therefore quaternion-linear are elements of $Sp(k)$. Additionally, this alternate description of $Sp(k) \subset GL(k,\mathbb{H})$ makes clear that $Sp(k)Sp(1) \subset GL(k,\mathbb{H})Sp(1)$.

Note that $SL(k,\mathbb{H})$ has no well-defined meaning independent of an of embedding of $GL(k,\mathbb{H})$ into either $GL(2k,\mathbb{C})$ or $GL(4k,\mathbb{R})$. Although $GL(k,\mathbb{H})$ can be considered as the group of invertible $k \times k$-matrices with quaternionic entries, the non-commutativity of the quaternions means that there is no quaternion-valued determinant function on such matrices, and so $SL(k,\mathbb{H})$ must be defined by reference to complex or real determinants.

We can express the product groups above via the alternate description

$$\begin{aligned} GL(k,\mathbb{H})Sp(1) &\cong (GL(k,\mathbb{H}) \times Sp(1))/\mathbb{Z}_2 \\ SL(k,\mathbb{H})Sp(1) &\cong (SL(k,\mathbb{H}) \times Sp(1))/\mathbb{Z}_2 \\ GL(k,\mathbb{H})U(1) &\cong (GL(k,\mathbb{H}) \times U(1))/\mathbb{Z}_2 \\ SL(k,\mathbb{H})U(1) &\cong (SL(k,\mathbb{H}) \times U(1))/\mathbb{Z}_2 \\ Sp(k)Sp(1) &\cong (Sp(k) \times Sp(1))/\mathbb{Z}_2, \end{aligned} \tag{4}$$

where in each case the $\mathbb{Z}_2$-action is generated by multiplication by $(-1,-1)$

In addition to the Lie group isomorphism $Sp(1) \cong SU(2)$ given in Proposition 2.13, there are two other Lie group isomorphisms relevant in quaternionic geometry.

Proposition 2.15. *We have the following Lie group isomorphisms:*

1. $Sp(1) \times Sp(1) \cong Spin(4)$
2. $Sp(1) \cong Spin(3)$

Proof. Consider the map $\Phi : Sp(1) \times Sp(1) \to SO(4)$ defined by

$$\Phi_{q_1,q_2}(p) = q_1\, p\, q_2^*,$$

or equivalently $\Phi_{q_1,q_2} = R_{q_2} \circ L_{q_1}$. It is straightforward to check that this map is a local diffeomorphism, hence the universal cover as $Sp(1) \times Sp(1)$ is compact and simply connected. Note that the kernel of this map is $(\pm 1, \pm 1)$.

Similarly, we can consider the map $\Psi : Sp(1) \to SO(3) \subset SO(4)$ defined by

$$\Psi(q) = \Phi(q,q)$$

A priori the image of Ψ is in $SO(4)$, but we observe that the action of Ψ preserves the three-dimensional subspace $\mathrm{Im}(\mathbb{H}) = \mathrm{span}_{\mathbb{R}}(\{i,j,k\})$ of purely imaginary quaternions, which is equivalently the set of all $p \in \mathbb{H}$ with $p^* = -p$. That is if $q \in Sp(1)$ and $p \in \mathrm{Im}(\mathbb{H})$, we have

$$(\Psi_q(p))^* = (qpq^*)^* = (q^*)^* p^* q^* = qp^*q^* = -qpq^* = -\Psi_q(p), \tag{5}$$

and so by restricting to $\mathrm{Im}(\mathbb{H})$ we can consider $\Psi(Sp(1)) \subset SO(3)$. It is once again straightforward to check that this map is the universal cover. Note that the kernel here is ± 1. □

By considering $Sp(1) \subset \mathbb{H}$ as the unit sphere, we have that the Lie algebra $\mathfrak{sp}(1) = T_1\mathbb{H}$ is naturally $\mathrm{Im}(\mathbb{H})$, with the Lie bracket being given the commutator with respect to quaternionic multiplication. It then follows easily from the definitions and the computations above that the adjoint representation of $Sp(1)$ on $\mathfrak{sp}(1) = \mathrm{Im}(\mathbb{H})$ is via conjugation, that is, for $q \in Sp(1)$ the adjoint representation $Ad_q : \mathfrak{sp}(1) \to \mathfrak{sp}(1)$ is simply

$$Ad_q(p) = qpq^* \tag{6}$$

for $p \in \mathrm{Im}(\mathbb{H})$. Therefore the representation Ψ defined above is precisely the adjoint representation of $Sp(1)$.

On the other hand, the map Ψ also defines an isomorphism $\mathfrak{sp}(1) \cong \mathfrak{so}(3)$. The standard definition of $\mathfrak{so}(3)$ is as the set of 3×3 skew-symmetric real matrices, with the Lie bracket being the commutator with respect to matrix multiplication. The following proposition relates these two descriptions, and is easily proved by using the explicit matrix representation for the actions L_q, R_q in Equation (3).

Proposition 2.16. *The isomorphism $(d\Psi)_e : \mathfrak{sp}(1) \to \mathfrak{so}(3)$ is given explicitly by*

$$i \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -2 \\ 0 & 2 & 0 \end{pmatrix} \qquad j \mapsto \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{pmatrix} \qquad k \mapsto \begin{pmatrix} 0 & -2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

2.3 Quaternionic Geometries via G-Structures

We defined a variety of quaternionic geometries in Section 2.1 in terms of tensors and connections, while in Definition 2.14 we have defined a number of Lie groups. In this section, we see that our definitions of quaternionic geometries are equivalent to specifying G-structures on manifolds where G is one of the Lie groups we have considered. Besse [9, Chapter 10] and

Joyce [21, Chapters 2-3] give good summaries of the necessary background on principal bundles, G-structures, and holonomy groups.

First, we have that each of the quaternionic geometries we have defined induce a G-structure on M.

Proposition 2.17. *Let M be a smooth manifold. If M is equipped with one of the quaternionic geometries of Section 2.1, then the structure group of the frame bundle of TM can be reduced to a subgroup of $GL(4k, \mathbb{R})$ according to the correspondence:*

Geometry	**Group**
(almost) quaternionic	$GL(k, \mathbb{H})Sp(1)$
(almost) hypercomplex	$GL(k, \mathbb{H})$
Scaled (almost) quaternionic	$SL(k, \mathbb{H})Sp(1)$
Quaternionic complex	$GL(k, \mathbb{H})U(1)$
Scaled quaternionic complex	$SL(k, \mathbb{H})U(1)$
Quaternion-Kähler	$Sp(k)Sp(1)$
Hyperkähler	$Sp(k)$

Proof. We consider only the proof for the reduction of the structure group of an almost quaternionic manifold, as the proofs for the others are similar. Let $(M, \mathcal{Q})$ be an almost quaternionic manifold, and let $\{I, J, K\}$ be a local compatible frame for $\mathcal{Q}_x$. We will call a real-linear bijection $u : \mathbb{H}^k \to T_xM$ a *quaternionic frame* with respect to $\{I, J, K\}$ if u is quaternion-linear with respect to the right $\mathbb{H}$-action on T_xM determined by the choice of $\{I, J, K\}$. More generally, we will say that a real-linear map $u : \mathbb{H}^k \to T_xM$ is an *almost quaternionic frame* if there exists a choice of local frame $\{I, J, K\}$ for $\mathcal{Q}$ that makes u a quaternionic frame. Then a frame u is almost quaternionic if and only if for every $f \in \mathcal{Q}_x$, the transformation of the form $u^{-1} \circ f \circ u : \mathbb{H}^k \to \mathbb{H}^k$ is equivalent to the action on $\mathbb{H}^k$ induced by right-multiplication by a some purely imaginary quaternion [36]. From equation (4), we can consider an element of $G = GL(k, \mathbb{H})Sp(1)$ as a pair $(A, q) \in GL(k, \mathbb{H}) \times Sp(1)$ up to multiplication by $(-1, -1)$, and by Definition 2.14 we have that (A, q) acts on $\mathbb{R}^{4k} \cong \mathbb{H}^k$ by $(A, q) \cdot \mathbf{p} = A\mathbf{p}q^*$. We can therefore define a right action of G on almost quaternionic frames by having (A, q) act on a frame u via composition, that is,

$$(u \cdot (A, q))(\mathbf{p}) = u((A, q) \cdot \mathbf{p}) = u(A\mathbf{p}q^*). \tag{7}$$

This action does in fact preserve almost quaternionic frames, for if we take u an almost quaternionic frame and assume that $u^{-1}fu$ corresponds to right-multiplication by the imaginary quaternion p, then a simple computation shows that $\widetilde{u}^{-1} f \widetilde{u}$ for $\widetilde{u} = u \cdot (A, q)$ acts on $\mathbf{p} \in \mathbb{H}^k$ as right-multiplication by the quaternion $q^* p q$, which is purely imaginary by equation (5), so that $\widetilde{u}$ is again an almost quaternionic frame. One can then verify that this action on frames is free and transitive, so that the collection of almost quaternionic frames is in fact a principal G-subbundle of the frame bundle of TM. □

We can equivalently consider the frames $u : \mathbb{H}^k \to T_x M$ above as choices of a preferred basis for $T_x M$. Taking the standard quaternionic basis $\{e_1, \ldots, e_k\}$ for $\mathbb{H}^k$, we have that the elements $\{e_\ell, ie_\ell, je_\ell, ke_\ell\}_{\ell=1}^k$ give a real basis for $\mathbb{R}^{4k} \cong \mathbb{H}^k$, and thus the map u identifies a corresponding basis for $T_x M$. Thus, given an almost quaternionic manifold and a local compatible frame $\{I, J, K\}$, we can always choose local vector fields $\{X_1, \ldots, X_k\}$ such that the vector fields

$$\{X_1, IX_1, JX_1, KX_1, \ldots, X_k, IX_k, JX_k, KX_k\}$$

form a local frame for TM. If the almost quaternionic manifold has a preferred scale, we can assume that the volume form evaluates this local frame to 1, while if the manifold is in fact quaternion-Kähler we can assume that this basis is orthonormal. Similar statements apply to the hypercomplex geometries by fixing the choice of local compatible frame.

It's also important to point out a special feature of the frames we consider in the quaternionic complex case. In this case, we have a complex structure on M, and so we only want to consider frames $u : \mathbb{H}^k \to T_x M$ that are complex-linear while also being quaternion-linear with respect to the chosen local compatible quaternionic frame. This amounts to assuming that the local quaternionic frames are only of the form $\{I, J, K\}$ where I is no longer arbitrary, but instead the fixed preferred complex structure. The choice of such frames therefore amounts to choosing the remaining local almost complex structures J, K, which must satisfy the standard relations of the quaternion algebra. Under the identification $\mathbb{R}^4 \cong \mathbb{C}^2 \cong \mathbb{H}$ this amounts to choosing the element j as a unit vector in the plane in $\mathbb{R}^4$ orthogonal to $\mathbb{C}$ under the usual Euclidean inner product. Hence all possible choices of J, K are related by a rotation, yielding the $U(1)$ rotation part of the action on frames. (This is also clearer after Corollary 2.19, which shows that all local compatible frames are associated by an $SO(3)$-action, so that the compatible frames with I fixed are all related by a rotation of the J, K part). Equivalently, a frame u is compatible with the quaternionic complex structure if and only if transformations of the form $u^{-1} \circ f \circ u$ are equivalent to the action on $\mathbb{H}^k$ induced by multiplication by a purely imaginary complex number. In this case, the map u is necessarily complex-linear.

We can also consider the converse of the above theorem, and consider how a reduction of the structure group of a smooth manifold gives rise to a quaternionic geometry. In the case of geometries without a preferred connection, this is relatively straightforward.

Proposition 2.18. *Let M be a smooth manifold. Then*

1. *an almost quaternionic structure on M is equivalent to a reduction of the structure group of TM to $GL(k, \mathbb{H})Sp(1)$.*
2. *an almost hypercomplex structure on M is equivalent to a reduction of the structure group of TM to $GL(k, \mathbb{H})$.*

Proof. Again we only consider the almost quaternionic case, as the almost hypercomplex case is very similar. If we assume that TM admits a reduction of its structure group to $G = GL(k, \mathbb{H})Sp(1)$, we can consider the frame bundle F of TM as a principal G-bundle and consider the vector bundle on M associated to the representation of G on $\mathbb{R}^3$ determined by the map

$$(\pm A, \pm q) \to \Psi(q)$$

where $\Psi : Sp(1) \to SO(3)$ is defined in Proposition 2.15. Recall that the kernel of Ψ is ± 1 so that the above representation is well-defined. This representation is essentially equivalent to the adjoint representation on $\mathfrak{sp}(1) = \mathfrak{so}(3) = \mathrm{Im}(\mathbb{H})$, via equation (6), and so the resulting rank-three bundle has a local compatible frame $\{I, J, K\}$ arising from the standard basis i, j, k of $\mathrm{Im}(\mathbb{H}) = \mathfrak{sp}(1)$, and therefore is the bundle $\mathcal{Q}$ giving an almost quaternionic structure. The converse is proved in Proposition 2.17. □

Corollary 2.19. *Let $(M, \mathcal{Q})$ be an almost quaternionic manifold. Let P denote the bundle of local compatible frames $\{I, J, K\}$ of $\mathcal{Q}$. Then P is a principal $SO(3)$-bundle.*

Proof. This is immediate in light of the previous proposition showing that $\mathcal{Q}$ arises from a representation of $SO(3)$. We give a second, more elementary, proof that will be useful when we compute examples.

Assume that $\{I, J, K\}$ and $\{\widetilde{I}, \widetilde{J}, \widetilde{K}\}$ are two different choices of local compatible frame for $\mathcal{Q}_x$. Then we can write the latter uniquely in terms of the former,

$$\begin{aligned}
\widetilde{I} &= a_{11}I + a_{21}J + a_{31}K \\
\widetilde{J} &= a_{12}I + a_{22}J + a_{32}K \\
\widetilde{K} &= a_{13}I + a_{23}J + a_{33}K
\end{aligned}$$

so that the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

gives the change of frame transformation from $\{\widetilde{I}, \widetilde{J}, \widetilde{K}\}$ to $\{I, J, K\}$. The fact that this matrix is an element of $SO(3)$ follows from the fact that both of these frames satisfy the quaternion algebra relations. For example, the fact that $\widetilde{I}^2 = -\operatorname{Id}$ implies that $a_{11}^2 + a_{21}^2 + a_{31}^2 = 1$, and similarly for the other columns of the matrix. We also have that the fact that $\widetilde{I}\,\widetilde{J} = -\widetilde{J}\,\widetilde{I}$ implies that the standard inner product of the first two columns of A vanishes, and similarly can show that this holds for other distinct pairs and therefore that A is an orthogonal matrix. Finally, a simple computation shows that the relation $\widetilde{I}\widetilde{J}\widetilde{K} = -1$ implies that $\det_{\mathbb{R}}(A) = 1$. Conversely, the above arguments show that given an element A of $SO(3)$ and $\{I, J, K\}$ a compatible frame, then the endomorphisms $\widetilde{I}, \widetilde{J}, \widetilde{K}$ defined by the formulas above will yield another local compatible frame. □

We can consider converse statements characterizing the remaining geometries in terms of structure groups as well. In this case, the requirement that there is a connection compatible with the relevant geometry is equivalent to requiring the existence of a principal connection on the reduced frame bundle, which is related to the holonomy of the connection.

Proposition 2.20. *Let ∇ be a torsion-free connection on a smooth manifold M. If the holonomy of ∇ is contained in one of the groups below, then M has the structure of the corresponding quaternionic geometry.*

Holonomy Group	**Geometry**
$GL(k, \mathbb{H})Sp(1)$ with $k \geq 2$	Quaternionic
$GL(k, \mathbb{H})$	Hypercomplex
$SL(k, \mathbb{H})Sp(1)$ with $k \geq 2$	Quaternionic with scale
$GL(k, \mathbb{H})U(1)$ with $k \geq 2$	Quaternionic complex
$SL(k, \mathbb{H})U(1)$ with $k \geq 2$	Quaternionic complex with scale
$Sp(k)Sp(1)$ with $k \geq 2$	Quaternion-Kähler
$Sp(k)$	Hyperkähler

Proof. Let F be the frame bundle for TM, considered as a principal $GL(4k, \mathbb{R})$-bundle. The connection ∇ on TM induces a principal $GL(4k, \mathbb{R})$-connection on the frame bundle F with the same holonomy group. By the Reduction Theorem [23, Theorem 7.1], this connection gives a principal $\operatorname{Hol}(\nabla)$-subbundle of F, and the connection moreover reduces to a principal $\mathrm{Hol}(\nabla)$-connection on this subbundle. The almost quaternionic or hypercomplex structures then arise from Proposition 2.18, and general theory of principal bundles gives that when the holonomy of the connections are additionally included in $SL(4k, \mathbb{R})$ or $SO(4k, \mathbb{R})$, then there are volume forms and metrics on M that will be compatible with the connection and quaternionic structures. Similarly, since $GL(k, \mathbb{H})U(1)$ and $SL(k, \mathbb{H})U(1)$ are both contained in $GL(2k, \mathbb{C})$, the connections in these cases will preserve a complex structure on M, which is therefore integrable. □

Quaternionic, quaternionic complex, and quaternion-Kähler manifolds in four dimensions cannot be defined in terms of holonomy. For example, in the quaternion-Kähler case we have $Sp(1)Sp(1) \cong SO(4)$ (see Proposition 2.15), and so a connection on a 4-manifold with holonomy $Sp(1)Sp(1)$ only gives the structure of an oriented Riemannian manifold. The holonomy condition does not yield the extra requirement that the associated metric be anti-self-dual and Einstein as required in Definition 2.12. Similarly, holonomy $GL(1, \mathbb{H})Sp(1)$, $GL(1, \mathbb{H})U(1)$, and $SL(1, \mathbb{H})U(1)$ structures are equivalent to conformal, conformally Kähler, and Kähler structures, respectively, on the 4-manifold M but do not yield the extra anti-self-duality condition required by Definition 2.11.

3 Twistor Spaces of Quaternionic Geometries

One of the most important techniques for studying quaternionic geometries is via their related *twistor spaces*. These spaces are complex manifolds for which aspects of the holomorphic geometry are closely related to the differential geometric properties of the original quaternionic space. This allows for the study of quaternionic manifolds using techniques of complex and algebraic geometry. In this section, we define twistor spaces for the various geometries we have considered and explain the necessary correspondences.

We begin in Section 3.1 by presenting a convenient method to study quaternionic manifolds via certain complex vector bundles associated to the reduced frame bundle. We use this framework to define the twistor space associated to a quaternionic manifold in Section 3.2, which also describes the additional properties of twistor spaces of quaternion-Kähler manifolds. Section 3.3 then describes the inverse process, which takes a twistor space and recovers the original quaternionic geometry.

The twistor theory of quaternionic geometries was developed independently by Bérard-Bergery [7] and Salamon [40, 41]. Their work built on the the twistor theory of conformal 4-manifolds developed by Atiyah, Hitchin, and Singer [4], who in turn were bringing Penrose's twistor theory of Minkowski space [37] to the Riemannian setting. Our treatment of these results is mostly based on the work of Salamon [40]. Once again, we will not always provide full details except when necessary, and so refer to the above references for rigorous proofs.

3.1 Complex Bundles Associated to Quaternionic Geometries

If a manifold M admits a reduction of its structure group to one of the groups listed in Proposition 2.17, then we can construct a number of associated vector bundles on M using representations of that group. We begin by focusing on the group $G = GL(k, \mathbb{H})Sp(1)$. In this case, there are two basic representations for this group, which we will write as E and H, that yield a number of interesting and relevant bundles on M. This "EH formalism" was developed by Salamon [40, 41], but see also Pederson, Poon, and Swann [36] for a useful generalization. The material of this section is essentially an adaptation of Salamon's arguments in [40] to the quaternionic case.

The group $G = GL(k, \mathbb{H})Sp(1)$ has a double cover, namely the group $\widetilde{G} = GL(k, \mathbb{H}) \times Sp(1)$, and therefore instead of considering G itself we will consider representations of the factors $GL(k, \mathbb{H})$ and $Sp(1)$. If F represents the frame bundle for TM, considered as principal G-bundle, then we can consider a covering bundle $\widetilde{F}$, a principal $\widetilde{G}$-bundle, although in general this lifting is only locally defined over M. We can define complex representations ρ_{E^*} and ρ_{H^*} of $\widetilde{G}$ on $\mathbb{C}^{2k}$ and $\mathbb{C}^2$, respectively, as follows. If $(A, q) \in \widetilde{G}$, then we define ρ_{E^*} by making the identification $\mathbb{C}^{2k} \cong \mathbb{H}^k$, and the action on $\mathbf{p} \in \mathbb{H}^k$ will be given by

$$\rho_{E^*}(A, q)\mathbf{p} = A\mathbf{p},$$

which is complex-linear. The representation ρ_{H^*} will be given by $\overline{L^{\mathbb{C}}} \circ \pi_2$, where $\pi_2 : GL(K, \mathbb{H}) \times Sp(1) \to Sp(1)$ is projection onto the second factor and $\overline{L^{\mathbb{C}}}$ is the conjugate of the representation $L^{\mathbb{C}} : Sp(1) \to SU(2)$ given in Proposition 2.13.

Note that both of these representations are quaternionic, that is, the map $J_{\mathbb{C}}$ defined in Equation (2) corresponding to right multiplication by j is equivariant under the action of the representation. Since ρ_{H^*} is additionally a unitary representation, this implies that it is in fact isomorphic to its dual (hence, conjugate) representation, see also Proposition 3.2.

Let $\mathbf{E}^*, \mathbf{H}^*$ denote the vector bundles over M associated to $\widetilde{F}$ via the representations ρ_{E^*}, ρ_{H^*}. The quaternionic structures for the representations ρ_{E^*}, ρ_{H^*} induced quaternionic structure on $\mathbf{E}^*, \mathbf{H}^*$ that we will denote by J_E, J_H. Each map preserves the fibers of the respective bundle and is conjugate linear on fibers, with $J_E^2 = -\operatorname{Id}_{\mathbf{E}^*}$ and , $J_H^2 = -\operatorname{Id}_{\mathbf{H}^*}$

Just like $\widetilde{F}$, the bundles $\mathbf{E}^*, \mathbf{H}^*$ are generally only locally defined. However, tensor products of these bundles may be globally defined, and in particular we have

Proposition 3.1. *The tensor product bundle* $\mathbf{E}^*\mathbf{H}^* = \mathbf{E}^* \otimes_{\mathbb{C}} \mathbf{H}^*$ *is isomorphic to the complexified tangent bundle* $TM \otimes_{\mathbb{R}} \mathbb{C}$ *of* M*. Under this isomorphism, the real structure on* $TM \otimes_{\mathbb{R}} \mathbb{C}$ *arising from complex conjugation on* $\mathbb{C}$ *corresponds to the map on* $\mathbf{E}^*\mathbf{H}^*$ *induced by quaternionic structures on the factors.*

Proof. In order to obtain the real structure, we observe that the tensor product $J_E \otimes J_H$ of the quaternionic structures on the factors defines a real structure on $\mathbf{E}^*\mathbf{H}^*$, since this map is conjugate-linear on each tensor factor and squares to $(-\operatorname{Id}_{\mathbf{E}^*}) \otimes (-\operatorname{Id}_{\mathbf{H}^*}) = \operatorname{Id}_{\mathbf{E}^*} \otimes \operatorname{Id}_{\mathbf{H}^*}$ since there are an even number of tensor factors.

Checking that the related real bundle obtained as the elements of $\mathbf{E}^*\mathbf{H}^*$ fixed by $J_E \otimes J_H$ is in fact TM is somewhat tedious, but can be carried out directly by considering the actions of $A \in GL(k, \mathbb{H})$ and $q \in Sp(1)$ on the tensor product space using the explicit formulas given in Proposition 2.13. The resulting representation will in fact coincide with the representation of (A, q) on $\mathbb{H}^k \cong \mathbb{R}^{4k}$ that defines TM from the frame bundle as described in equation (7) of Proposition 2.17. □

As an important corollary, we have that the cotangent bundle will be given by

$$T^*M \otimes_{\mathbb{R}} \mathbb{C} \cong \mathbf{EH}$$

More generally, we observe that we can take tensor products of the bundles $\mathbf{E}, \mathbf{H}$, and their duals to obtain a wide variety of vector bundles on a manifold with structure group G for the frame bundle. As in the previous proposition, the quaternionic structures on the bundles $\mathbf{E}, \mathbf{H}$ will induce either quaternionic or real structures on these tensor products, according to whether there are an odd or even number of tensor factors. In the case that a real structure exists, we will use unbolded letters to denote the underlying real bundle, so that, for example, we have

$$T^*M = EH$$

although this should not be interpreted as a tensor product. Moreover, the resulting bundles will be globally defined in these cases, for $(-1,-1)$ will act trivially on tensor products of $\mathbf{E}, \mathbf{H}$ with an even number of factors, and so the relevant representation of $\widetilde{G} = GL(k,\mathbb{H}) \times Sp(1)$ will in fact descend to a representation of $G = GL(k,\mathbb{H})Sp(1)$. The above discussion also applies to symmetric and exterior products of these bundles, so that, for example, the bundle $S^2\mathbf{H}$ is globally defined and yields a real subbundle S^2H, related to the bundle $\mathcal{Q}$, see Proposition 3.2 below.

The above assumed that we were working with an almost quaternionic manifold, so that the bundle $\mathbf{E}$ arises from the defining representation of $GL(k,\mathbb{H})$. If instead we consider scaled almost quaternionic manifolds, then we can consider $\mathbf{E}$ as arising from the defining representation of $SL(k,\mathbb{H})$, and similarly in the quaternion-Kähler case we can take $\mathbf{E}$ to arise from the defining representation of $Sp(k)$. For quaternionic complex manifolds we generally focus on the quaternionic aspect of such objects, and so still consider the $\mathbf{E}, \mathbf{H}$ bundles via the inclusion $GL(k,\mathbb{H})U(1) \subset GL(k,\mathbb{H})Sp(1)$. The preferred complex structure on such manifolds then gives rise to a subbundle of $\mathbf{H}$, see Proposition 3.3.

The bundles $\mathbf{E}, \mathbf{H}$ will also carry extra structure in the case that M has a connection compatible with the reduction of structure group, that is, in the (scaled) quaternionic and quaternion-Kähler cases. In particular, assume that $(M, \mathcal{Q})$ is equipped with a fixed quaternionic connection ∇. This connection induces a principal connection on the frame bundle F, which lifts to $\widetilde{F}$ and therefore induces connections on $\mathbf{E}, \mathbf{H}$. The induced connection on $\mathbf{E}^*\mathbf{H}^* \cong TM \otimes \mathbb{C}$ is then the complexification of the connection on TM.

As mentioned above, the bundles $\mathbf{E}, \mathbf{H}$ can only be globally defined on M when the $GL(k,\mathbb{H})Sp(1)$-frame bundle has a globally-defined double cover.

The obstruction to global existence of these bundles is the second Stiefel-Whitney class $w_2(S^2H)$ of the real rank-3 bundle mentioned above. In the positive quaternion-Kähler case, this obstruction vanishes only for quaternionic projective space $\mathbb{HP}_k$, see [40]. The bundle $S^2\mathbf{H}$ is also important for its relationship to $\mathcal{Q}$.

Proposition 3.2. *Let $(M, \mathcal{Q}, \nabla)$ be a quaternionic manifold with associated local bundles $\mathbf{E}, \mathbf{H}$. Then*

1. *The bundle $\Lambda^2\mathbf{H}^*$ is trivial, and there is a non-vanishing section ω_H of this bundle with $\nabla\omega_H = 0$ that is unique up to multiplication by a constant.*
2. *The section ω_H induces an isomorphism $S^2H \cong \mathcal{Q}$.*

Proof. The details of the isomorphism $S^2H \cong \mathcal{Q}$ will be important to our future work, and so we sketch a proof due to Salamon [40]. The representation yielding the bundle $\mathbf{H}$ is essentially the defining representation of $Sp(1) \cong SU(2)$ on $\mathbb{C}^2$. This representation preserves the standard Hermitian inner product on $\mathbb{C}^2$, and also preserves the quaternionic structure $J_{\mathbb{C}} : \mathbb{C}^2 \to \mathbb{C}^2$ induced by right multiplication by j. We can therefore consider a $\mathbb{C}$-valued 2-form ω_H defined on $\mathbb{C}^2$ by

$$\omega_H(u, v) = \langle J_{\mathbb{C}}u, v\rangle$$

where $\langle -, -\rangle$ denotes the standard Hermitian inner product. It is straightforward to verify that $\omega_H(J_{\mathbb{C}}u, J_{\mathbb{C}}v) = \overline{\omega_H(u, v)}$ and that $\omega_H(u, J_{\mathbb{C}}u) > 0$ for $u \neq 0$. This 2-form is therefore non-degenerate, and induces an isomorphism of $\mathbb{C}^2$ with its dual via

$$u \mapsto (v \mapsto \omega_H(v, u))$$

Now the holonomy representation associated to the connection ∇ on $\mathbf{H}$ is precisely the standard representation of $Sp(1)$ on $\mathbb{C}^2$, and by identifying $\mathbf{H}_x$ with $\mathbb{C}^2$ we have that there exists a 2-form $\omega_H \in \Lambda^2\mathbf{H}_x^*$ that is fixed by the holonomy representation, and therefore by the holonomy principle we can translate this form using parallel transport to obtain a globally-defined form ω_H with the same properties.

In particular, ω_H gives an isomorphism $\mathbf{H} \cong \mathbf{H}^*$ that we use to induce the isomorphism $S^2H \cong \mathcal{Q}$. On the level of representations, taking the dual (equivalently, the conjugate) of the standard representation of $Sp(1) \cong SU(2)$ on $\mathbb{C}^2$ and tensoring yields a representation of $Sp(1)$ on $\mathbb{C}^2 \otimes (\mathbb{C}^2)^* \cong \mathbb{C}^4$ that is the complexification of the representation of $Sp(1)$ on $\mathbb{R}^4$ given by equation (5), and therefore by Proposition 2.18 this representation is the complexification of the representation determining $\mathcal{Q}$.

More concretely, we can use ω_H to consider the embedding of $S^2\mathbf{H}$ into $\mathrm{End}(\mathbf{H})$ given by

$$S^2\mathbf{H} \subset \mathbf{H} \otimes \mathbf{H} \cong \mathbf{H} \otimes \mathbf{H}^* = \mathrm{End}(\mathbf{H})$$

Note that our convention here is to consider elements of $S^2\mathbf{H}$ to be embedded in $\mathbf{H} \otimes \mathbf{H}$ via

$$h_1 \odot h_2 \in S^2\mathbf{H} = (h_1 \otimes h_2 + h_2 \otimes h_2) \in \mathbf{H} \otimes \mathbf{H}$$

This embedding then yields a right action of $S^2\mathbf{H}$ on $TM \otimes \mathbb{C} \cong \mathbf{E}^*\mathbf{H}^*$ by considering

$$\mathbf{E}^* \otimes \mathbf{H}^* \otimes S^2\mathbf{H} \hookrightarrow \mathbf{E}^* \otimes \mathbf{H}^* \otimes \mathbf{H} \otimes \mathbf{H}^* \longrightarrow \mathbf{E}^* \otimes \mathbf{H}^*,$$

where the second arrow is the contraction of the middle pair of dual tensor factors. The isomorphism $\mathbf{H} \cong \mathbf{H}^*$ induces an inner product on $S^2\mathbf{H}$, and if J, K are sections of $S^2\mathbf{H}$ considered as endomorphisms on $TM \otimes \mathbb{C}$, then one has

$$J \circ K + K \circ J = -\langle J, K \rangle \, \mathrm{Id}_{TM \otimes \mathbb{C}} \tag{8}$$

and therefore choosing a basis of real sections of $S^2\mathbf{H}$ that are orthonormal with respect to this inner product yields a real subbundle of $\mathrm{End}(TM)$ satisfying the quaternion algebra relations. □

It will be useful for later work for us to make this isomorphism even more explicit. We can choose a local section h_1 of $\mathbf{H}$ so that $\omega_H(h_1, J_H h_1) = 1$. Let $h_2 = J_H h_1$ for convenience, so that $\{h_1, h_2\}$ is a unitary frame for $\mathbf{H}$. Let h_1^*, h_2^* denote the related elements of $\mathbf{H}^*$ via the isomorphism ω_H, so that for example $h_1^* = \omega_H(-, h_1)$. We therefore have that

$$h_1^*(h_1) = 0 \qquad h_1^*(h_2) = -1 \qquad h_2^*(h_1) = 1 \qquad h_2^*(h_2) = 0$$

With respect to this local unitary frame for $\mathbf{H}$, we consider the sections of $S^2\mathbf{H} \subset \mathbf{H} \otimes \mathbf{H}$ defined by

$$\begin{aligned} I &= i\,(h_1 \otimes h_2 + h_2 \otimes h_1) \\ J &= h_1 \otimes h_1 + h_2 \otimes h_2 \\ K &= i\,(h_1 \otimes h_1 - h_2 \otimes h_2) \end{aligned} \tag{9}$$

These sections give a local compatible frame for $\mathcal{Q}$ under the identification $S^2H \cong \mathcal{Q}$ described above. First, each section is invariant under the real structure on $S^2\mathbf{H}$ induced by J_H on each factor. For example, denoting this real structure by conjugation, we have

$$\overline{I} = \bar{i}\,(J_H h_1 \otimes J_H h_2 + J_H h_2 \otimes J_H h_1) = -i\,(-h_2 \otimes h_1 - h_1 \otimes h_2) = I,$$

where here we use the fact that J_H is conjugate-linear. Using the isomorphism induced by ω_H, we have that I^2 acts on a generic element $e_1^* \otimes h_1^* + e_2^* \otimes h_2^* \in TM \otimes \mathbb{C}$ as

$$\begin{aligned}(e_1^* \otimes h_1^* + e_2^* \otimes h_2^*)\, I^2 &= i\,(e_1^* \otimes h_1^* + e_2^* \otimes h_2^*)\,(h_1 \otimes h_2^* + h_2 \otimes h_1^*)\, I \\ &= i\,(-e_1^* \otimes h_1^* + e_2^* \otimes h_2^*)\, i\,(h_1 \otimes h_2^* + h_2 \otimes h_1^*) \\ &= -\,(e_1^* \otimes h_1^* + e_2^* \otimes h_2^*)\,,\end{aligned}$$

that is, as -1. Similar computations show that the remaining quaternionic relations $J^2 = K^2 = -\operatorname{Id}, IJ = K = -JI$ also hold. Note that with respect to the Hermitian inner product we have defined on $S^2\mathbf{H}$ that each of these sections has norm $\sqrt{2}$, since $\{h_1 \otimes h_1, h_1 \otimes h_2, h_2 \otimes h_1, h_2 \otimes h_2\}$ is by definition a unitary basis for $\mathbf{H} \otimes \mathbf{H}$. This also agrees with equation (8) above. Therefore the identification $S^2H \cong \mathcal{Q}$ defined above does not preserve the Euclidean inner products on these bundles, but they are related by a constant scale factor of $\sqrt{2}$.

Finally, we observe that choosing a different value of $h_1 \in \mathbf{H}_x$ and following the above procedure produces a different local compatible frame for $\mathcal{Q}_x$. As the choice of h_1 is making a choice of a unit vector in $\mathbf{H}_x$, these possible choices correspond to elements of $Sp(1)$. But we can observe from equations (9) that both h_1 and $-h_1$ yield the same local basis, so that the set of real compatible frames is parameterized by $Sp(1)/\mathbb{Z}_2 = SO(3)$, as necessary.

In the case that we are considering a quaternion-Kähler manifold, instead of just a quaternionic manifold, then considering $\mathbf{E}$ as arising from a representation of $Sp(k)$ yields, via a similar argument, that the bundle $\Lambda^2\mathbf{E}^*$ has a parallel, non-vanishing section ω_E. We can consider ω_E, ω_H as sections of $\Lambda^2\mathbf{H}, \Lambda^2\mathbf{E}$ as well using the isomorphisms $\mathbf{E} \cong \mathbf{E}^*$, $\mathbf{H} \cong \mathbf{H}^*$ induced by these forms. We then have that the tensor

$$\omega_E \otimes \omega_H \in \Gamma(\Lambda^2\mathbf{E} \otimes \Lambda^2\mathbf{H}) \subset \Gamma(S^2T^*M) \tag{10}$$

is precisely the complexification of the metric.

Note that in the 4-dimensional case, where the Levi-Civita connection of an oriented Riemannian manifold has holonomy contained in $Sp(1)Sp(1)$, the bundles produced by the process above are the well-known spinor bundles, see [4] and the related discussion following Proposition 3.4.

3.2 Twistor Spaces of Quaternionic and Quaternion-Kähler Manifolds

If $(M, \mathcal{Q}, \nabla)$ is a quaternionic manifold, then, recalling that the bundle $\mathcal{Q}$ has an $SO(3)$-structure under which any local compatible frame $\{I, J, K\}$ is taken to be orthonormal, we can consider the unit sphere bundle Z associated to $\mathcal{Q}$. This is known as the *twistor space* associated to M. If we fix a local compatible frame, writing a local section of $\mathcal{Q}$ as $A = f_1 I + f_2 J + f_3 K$, we observe that

$$A^2 = -(f_1^2 + f_2^2 + f_3^3)\,\mathrm{Id}_{TM}$$

so that A is an almost complex structure exactly when it has unit norm, and therefore local sections of Z are precisely the sections of $\mathcal{Q}$ that are almost complex structures. This space Z can also be conveniently described using the E, H formalism of Section 3.1.

Proposition 3.3 ([40]). *The bundle Z is isomorphic to* $\mathbb{P}(\mathbf{H})$

Proof. Note that although the bundle $\mathbf{H}$ is only locally defined, the projectivization of this bundle is defined globally since projectivizing removes the sign ambiguity that prevents defining $\mathbf{H}$ globally. Therefore proving that $Z = \mathbb{P}(\mathbf{H})$ locally will suffice to give the result globally.

We therefore fix $x \in M$ and work locally around x so that the bundles $\mathbf{E}, \mathbf{H}$ are well-defined. Then an element $h \in \mathbf{H}_x - 0$ defines an almost complex structure J_h on T_xM by declaring the $(1,0)$-forms in $T^*M \otimes \mathbb{C}$ to be those of the form

$$\Lambda_x^{1,0} = \mathbf{E}_x \otimes_{\mathbb{C}} \mathbb{C}h \subset \mathbf{E}_x \otimes_{\mathbb{C}} \mathbf{H}_x = \mathbf{T}_x^* M \otimes \mathbb{C}. \tag{11}$$

One can check that the $(1,0)$-forms given by this method are precisely those determined by the element $J_h \in S^2 H_x \cong \mathcal{Q}_x$ defined by

$$J_h = \frac{i}{\omega_H(h, J_H h)} h \odot (J_H h)\,,$$

using the identifications of Proposition 3.2.

Note from either description of J_h that if we scale h by a non-zero complex number, the resulting almost complex structure remains unchanged. Thus we have that the set $\mathbb{P}(H_x)$ can be identified with the sphere of radius $\sqrt{2}$ in $S^2 H_x$, or equivalently with the unit sphere in $\mathcal{Q}_x$, that is, can be identified with the fiber Z_x. □

The twistor space Z admits a natural complex structure that is most easily understood by considering a related complex structure defined on the total space $\mathbf{H}^\times = \mathbf{H} - \{0\text{-section}\}$ using the isomorphism above.

Proposition 3.4 ([40])**.** *Let M^{4k} be a quaternionic manifold with $k \geq 2$, and consider the (locally defined) bundle* $\mathbf{H}$*. Then the total space of the bundle* $\mathbf{H}^\times$ *admits a natural complex structure.*

Proof. Let $h \in \mathbf{H}_x - \{0\}$ be given. As discussed previously, the quaternionic connection on M induces a connection on the frame bundle of TM, considered as a principal $GL(k, \mathbb{H})Sp(1)$-bundle, that descends to a connection on $\mathbf{H}$. This connection is equivalent to a direct sum decomposition of the tangent space $T_h(\mathbf{H} - \{0\})$ into vertical and horizontal subspaces $V_h \oplus H_h$, where V_h can be identified with the tangent space of the fiber and H_h can be identified with the tangent space of the base. That is, we have

$$T_h(\mathbf{H} - \{0\}) \cong T_h(\mathbf{H}_x - \{0\}) \oplus T_x M \cong \mathbf{H}_x \oplus T_x M$$

As $\mathbf{H}$ is a complex vector bundle, there is an almost complex structure on the summand $\mathbf{H}_x$ given by multiplication by i. On the other summand, we have that $h \in \mathbf{H}_x - \{0\}$ determines an almost complex structure on $T_x M$ by Proposition 3.3. Taking the direct sum yields an almost complex structure on $T_h(\mathbf{H} - \{0\})$, and it is straightforward to check that the resulting almost complex structure depends smoothly on h and thus defines an almost complex structure on the entirety of $\mathbf{H}^\times$.

Using the Newlander-Nirenberg theorem, the integrability of this complex structure is related to the vanishing of a certain component of the curvature related to the quaternionic connection ∇. A direct proof, making use of the curvature properties of quaternionic connections developed in Section 5.1, is given in Besse [9, Proof 14.70]. Salamon gives a more elegant representation-theoretic argument in [41]. His proof showing integrability in the quaternion-Kähler case given in [40] can also be generalized to the quaternionic case. □

This discussion is a generalization of the 4-dimensional case considered by Atiyah, Hitchin, and Singer in [4]. Recall that an almost quaternionic manifold in four dimensions is equivalent to a choice of oriented conformal class on M, as discussed at the end of Section 2.1. Fixing a metric in that conformal class is equivalent to reducing the structure group of TM to $SO(4)$, and so by considering the locally-defined double cover, a $Spin(4) \cong Sp(1) \times Sp(1)$ bundle, we obtain the spinor bundles S_+, S_- that are exactly analogous to $\mathbf{E}, \mathbf{H}$. In particular, we have $S_+ \otimes S_- \cong T^*M \otimes \mathbb{C}$, just as for $\mathbf{E}, \mathbf{H}$, and so the same method that defines the almost complex structure on $\mathbf{H}^\times$ described above yields an almost complex structure on $S_+ - \{0\text{-section}\}$. In this case, though, the integrability of these almost complex structures is not automatic, and instead requires that the conformal structure be anti-self-dual, explaining the extra condition in Definition 2.11. Note that we have fixed a metric in the conformal class for concreteness, but Atiyah et. al. moreover show that this construction and the resulting complex structure are independent of this choice.

The action of multiplication by a non-zero complex number on $\mathbf{H}^\times$ is holomorphic with respect to this complex structure by definition, and therefore the quotient $Z = \mathbb{P}(\mathbf{H})$, is a complex manifold. Note that, Z has real dimension $4k+2$, as it is a sphere bundle over M^{4k}, and therefore has complex dimension $2k+1$. We will use $p : Z \to M$ to denote the sphere bundle fibration, which is also known as the twistor projection.

There a few more important features of this fibration. If we work locally over an open subset $U \subset M$, so that we can assume that $\mathbf{H}$ exists globally over U, then the projectivization $\mathbb{P}(\mathbf{H})$ admits a tautological line bundle we will call L^{-1}. Explicitly, if $z \in p^{-1}(x)$ is the equivalence class $[h]$ for $h \in \mathbf{H}_x$, the fiber $(L^{-1})_z$ is exactly the complex span of h in $\mathbf{H}_x$. This gives an inclusion of L^{-1} into $p^*\mathbf{H}$ over $p^{-1}(U)$. If we let $T_F = \ker dp : TZ \to TM$ denote the subbundle of the holomorphic tangent bundle of Z that is tangent to the fibers of the projection, then these bundles fit together into the standard Euler exact sequence for the projectivization of a vector bundle,

$$0 \longrightarrow L^{-1} \longrightarrow p^*\mathbf{H} \longrightarrow L^{-1} \otimes T_F \longrightarrow 0$$

Considering determinant bundles, we then have the isomorphism $\Lambda^2 p^*\mathbf{H} \cong L^{-2} \otimes T_F$, which yields $L^2 \cong T_F$ since $\Lambda^2\mathbf{H}$ is trivialized by the section ω_H. Note that although L^{-1} is defined only locally, this shows even tensor powers of this bundle are in fact defined globally. Moreover, since $Z = \mathbb{P}(\mathbf{H})$, we have that the fibers $p^{-1}(x)$ of the twistor projection, which we know to be spheres from the description of Z as a unit sphere bundle, are in fact complex submanifolds of Z biholomorphic to $\mathbb{CP}_1$, which are therefore called real twistor lines. In particular, restricting the isomorphism $L^2 \cong T_F$ to one of these fibers yields that $L^2|_{\mathbb{CP}_1} \cong T\mathbb{CP}_1 = \mathcal{O}(2)$, which partially explains our choice of notation.

The bundle L^2 is also related to the canonical bundle of Z. To see this, we can work locally, so that L^{-1} exists, and choose a standard basis h_1, h_2 for $\mathbf{H}$ associated to the 2-form ω_H as in the discussion following Proposition 3.2. Let z^1, z^2 be the related coordinate functions, so that an element in $\mathbf{H}_x$ is given by $z^1 h_1(x) + z^2 h_2(x)$. If we write

$$\nabla h_i = \sigma_i^j h_j$$

for the local connection 1-forms for the bundle $\mathbf{H}$, then we can consider the forms

$$\beta^i = dz^i + z^j \pi^* \sigma^i_j$$

defined on $\mathbf{H}^\times$, where $\pi : \mathbf{H} \to M$ is the vector bundle projection. These are in fact the $(1,0)$-forms associated to the vertical part of the complex structure defined on $\mathbf{H}^\times$, so that the form

$$\alpha = z^2 \beta^1 - z^1 \beta^2$$

is a $(1,0)$-form on $\mathbf{H}^\times$. The same curvature properties of the quaternionic connection that insure the integrability of the complex structure on the twistor space imply that in fact α is a holomorphic 1-form. Note that α is quadratic in z^1, z^2, and that β^1, β^2 are pullbacks of $(1,0)$-forms on Z, so that in fact α descends to a holomorphic 1-form θ on Z with values in the bundle L^2. Considering this as an inclusion of L^{-2} into $\Omega^1(Z)$ the set of holomorphic 1-forms on Z, we have a short exact sequence of holomorphic vector bundles on Z

$$0 \longrightarrow L^{-2} \xrightarrow{\theta} \Omega^1(Z) \longrightarrow L^{-1} \otimes p^*\mathbf{E} \longrightarrow 0 \qquad (12)$$

since the quotient is exactly the bundle of $(1,0)$ forms at $z \in p^{-1}(x)$ in the horizontal direction, which are given by the forms

$$p^*(T^{1,0}_x M) = p^*(\mathbf{E}_x \otimes \mathbb{C}h) = p^*\mathbf{E}_x \otimes (L^{-1})_z$$

where $z = [h]$ for $h \in \mathbf{H}_x$. Note that the above defines a holomorphic structure for the bundle $p^*\mathbf{E}$ over Z. Considering determinant bundles for this short exact sequence yields an isomorphism

$$K_Z \cong L^{-2(k+1)} \otimes \Lambda^{2k} p^*\mathbf{E}$$

Note that if we assume that our quaternionic manifold has a chosen scale then the bundle $\Lambda^{2k} p^*\mathbf{E} = p^*\Lambda^{2k}\mathbf{E}$ is canonically trivial as a smooth vector bundle, so that in a scaled quaternionic manifold we have that L^2 is a $(k+1)$-root of the anticanonical bundle of Z.

We can also consider the normal bundle of each fiber. If we fix $x \in M$, and consider the fiber $\mathbf{H}_x - \{0\}$ as a submanifold of $\mathbf{H}^\times$, then the splitting $T^*_h \mathbf{H} - \{0\} \cong \mathbf{H}_x \oplus T_x M$ and the definition of the complex structure on $\mathbf{H}^\times$ gives that the holomorphic conormal bundle of $\mathbf{H}_x - \{0\}$ at h consists of the $(1,0)$-forms on $T_x M$ determined by h. These $(1,0)$-forms are those in the space $\mathbf{E}_x \otimes \mathbb{C}h$ by equation (11). When we projectivize, we have that the conormal bundle of $p^{-1}(x)$ in Z is isomorphic to $\mathbf{E}_x \otimes L^{-1}|_{\mathbb{CP}_1} = \mathbf{E}_x \otimes \mathcal{O}(-1)$, and taking the dual therefore yields that the normal bundle of a twistor fiber is $\mathbb{C}^{2k} \otimes \mathcal{O}(1)$.

Finally, considering Z as the unit sphere bundle in the $SO(3)$-bundle $\mathcal{Q}$ gives that each fiber of Z has a natural antipodal map induced by multiplication by -1 on the fiber of $\mathcal{Q}$. We can therefore define a map $\sigma : Z \to Z$ that acts on the fibers of p by this antipodal map. Describing Z as $\mathbb{P}(\mathbf{H})$, we have that σ sends the equivalence class of $h \in \mathbf{H}$ to the equivalence class of $J_H h$. Evidently $\sigma^2 = \mathrm{Id}_Z$, and we observe that σ has no fixed points, although it does preserve the fibers of p. This map is also antiholomorphic, for if we consider a point in $Z = \mathbb{P}(\mathbf{H})$ as the almost complex structure J_h on $T_{p(h)}M$, then σ sends J_h to $-J_h$. We therefore have that $d\sigma : TZ \to TZ$ changes the sign of the complex structure defined on Z via Proposition 3.4, so that σ is antiholomorphic. An antiholomorphic involution on a complex manifold is also known as a *real structure.*

Gathering these various properties of the twistor space into a theorem, we have

Theorem 3.5. *Let M^{4k} be a quaternionic manifold. Then the twistor space Z is a complex manifold of dimension $2k+1$ with the following properties:*

(i) There exists a smooth fibration $p : Z \to M$ giving Z the structure of a smooth S^2-bundle over M.

(ii) The fibers $p^{-1}(x)$ are complex submanifolds of Z biholomorphic to $\mathbb{CP}_1$, with normal bundle isomorphic to $\mathbb{C}^{2k} \otimes O(1)$.

(iii) The manifold Z admits an antiholomorphic involution σ that acts as the antipodal map on the fibers of p.

(iv) The manifold Z admits a holomorphic line bundle L^2 that is isomorphic as a smooth vector bundle to $T_F = \ker dp$, while $L^{2(k+1)}$ is isomorphic as a holomorphic vector bundle to $-K_Z$, the anticanonical bundle of Z.

Although the above theorem is stated for quaternionic manifolds, it holds for weakly quaternionic manifolds if we discard property (iv). That is, given a weakly quaternionic manifold, we can fix a quaternionic connection to define the complex structure on the twistor space, and the properties of (i) - (iii) above are in fact invariant under the choice of connection. The isomorphism $L^2 \cong T_F$, however, depends on the form ω_H and therefore will depend on the choice of connection. This is similar to the conformal invariance in the four-dimensional case discussed above. This independence is partially demonstrated in Section 3.3 when we consider the inverse of the twistor construction. A direct proof that the twistor construction is independent of the chosen quaternionic connection has also been given by Alekseevsky, Marchiafava, and Pontecorvo [3]

Unsurprisingly, if M is in fact quaternion-Kähler then the additional structure of a Riemannian metric gives additional structure to Z, in the form of a complex contact structure.

Theorem 3.6 ([7, 40])**.** *Let M^{4k} be quaternion-Kähler manifold. Then the twistor space Z is a complex manifold of dimension $2k+1$ with the following properties:*

(i) There exists a smooth fibration $p : Z \to M$ giving Z the structure of a smooth S^2-bundle over M.

(ii) The fibers $p^{-1}(x)$ are complex submanifolds of Z biholomorphic to $\mathbb{CP}_1$, with normal bundle isomorphic to $\mathbb{C}^{2k} \otimes O(1)$.

(iii) The manifold Z admits an antiholomorphic involution σ that acts as the antipodal map on the fibers of p.

(iv) The manifold Z admits a complex contact structure, a holomorphic 1-form θ with values in the bundle L^2 with the property that the contact distribution $D = \ker \theta$ is transverse to the fibers $p^{-1}(x)$ for $x \in M$, with $d\theta|_D$ a non-degenerate 2-form.

Proof. The first three properties above are the properties of the twistor space of a quaternionic manifold, which we repeat here only for completeness. We only need to show the existence of the complex contact structure on Z.

The isomorphism of determinant bundles induced by the short exact sequence (12) yields an isomorphism $K_Z \cong L^{-2(n+1)}$, since the determinant bundle of $\mathbf{E}$ is canonically trivial from the scale. Then considering the inclusion θ in that short exact sequence as an L^2-valued 1-form, we can use this isomorphism to treat $\theta \wedge (d\theta)^n$ as a holomorphic section of the bundle $L^{2(n+1)} \otimes K_Z \cong \mathcal{O}$, that is, as essentially a holomorphic function. This expression will not depend on the choice of local parallel section of L^2 used to compute the exterior derivative. Again the curvature properties give that this holomorphic function is a positive multiple of the scalar curvature, so as long as the scalar curvature is non-zero (which we assume in our terminology for a quaternion-Kähler manifold) the form θ yields a complex contact structure.

Considered as a line-bundle valued 1-form, we have that θ acts as projection onto $L^2 \subset TZ$. Recalling that $L^2|_{p^{-1}(x)} = T\mathbb{CP}_1$ when restricted to twistor fibers, we have that the kernel of θ is transverse to the fibers, while the fact that θ yields an isomorphism $L^{-2(n+1)} \cong K_Z$ gives that $d\theta$ is nondegenerate on the kernel. □

Finally, in the case that M is in fact a positive quaternion-Kähler manifold, we have

Theorem 3.7 ([40])**.** *Let M be a quaternion-Kähler manifold with scalar curvature $s > 0$. Then Z admits a complete Kähler-Einstein metric of positive scalar curvature.*

Proof. Again we work locally over an open set $U \subset M$ where we can assume that the bundle $\mathbf{H}$ is well defined, so that we have the tautological bundle L^{-1} defined on $p^{-1}(U)$. This line bundle has a Hermitian inner product induced by the inclusion $L^{-1} \subset p^*\mathbf{H}$ and the Hermitian inner product already present on $\mathbf{H}$. Letting $\|s\|^2$ denote the induced norm of a holomorphic section of the dual L, the curvature of the Chern connection on this bundle is given by the $(1,1)$-form $\overline{\partial}\partial \log \|s\|^2$, which is a well-defined form on all of Z, and multiplying by i yields a positive-definite $(1,1)$ form. Proving the positive definite property of this connection is where the hypothesis that $s > 0$ is necessary.

To see that this metric is Kähler-Einstein, we recall that on a Kähler manifold the Ricci form is i times the curvature of the connection on the canonical bundle induced by the Levi-Civita connection of the Kähler metric. We also have the isomorphism $K_Z \cong L^{-2(n+1)}$ from above, and so the Chern connection on L also induces a connection on K_Z, which will in fact coincide from the uniqueness of the Chern connection. Therefore the Kähler form on Z is a multiple of the Ricci form. □

Corollary 3.8. *A positive quaternion-Kähler manifold is simply connected.*

Proof. Myers's theorem gives that the existence of a complete Einstein metric with positive scalar curvature on Z implies that Z is compact. A theorem of Kobayashi [22] then gives that Z is simply connected. Since Z is an S^2-bundle over M, this implies M is simply connected as well. □

We end by briefly describing the twistor spaces associated to the Wolf spaces. Wolf [43] showed that the twistor spaces of the symmetric quaternion-Kähler spaces described in equation (1) are obtained by replacing the second factor in the quotient group by $U(1)$. For example, the twistor space of the the quaternionic projective space $\mathbb{HP}_k$ is

$$\mathbb{CP}_{2k+1} = \frac{Sp(k+1)}{Sp(k) \times U(1)}$$

In particular, if we consider $[h] \in \mathbb{HP}_k$ as the quaternionic line in $\mathbb{H}^{k+1}$ spanned by some element $h \in \mathbb{H}^{k+1}$, then the twistor fiber $Z_{[h]}$ is the *complex* line in $\mathbb{H}^{k+1} \cong \mathbb{C}^{2(k+1)}$ spanned by h. The quaternionic line in $\mathbb{H}^{k+1}$ can be identified with $\mathbb{C}^2$, and so the fiber over $[h]$ in the twistor space is the projectivizaztion of this $\mathbb{C}^2$, namely $\mathbb{CP}_1$, and moreover the set of all twistor lines is the set of all lines in $\mathbb{CP}_1$, which are well-known to have the appropriate normal bundle. The real structure is induced by quaternionic multiplication by j on $\mathbb{H}^{k+1}$. In homogeneous coordinates, it is given by

$$[z_0, z_1, \ldots, z_{2k}, z_{2k+1}] \mapsto [-\overline{z_1}, \overline{z_0}, \ldots, -\overline{z_{2k+1}}, \overline{z_{2k}}]$$

The contact and Kähler-Einstein structures on $\mathbb{CP}_{2k+1}$ are the canonical such structures on these spaces. In particular, the Kähler-Einstein metric is the Fubini-Study metric, and the contact structure on $\mathbb{CP}_{2k+1}$ is the one induced from the canonical complex symplectic form on $\mathbb{C}^{2k+2}$, see [29].

3.3 The Inverse Twistor Construction

The construction yielding a twistor space from either a quaternionic or quaternion-Kähler manifold can also be inverted, that is, a complex manifold with the same holomorphic properties as a twistor space can be used to produce a weakly quaternionic or quaternion-Kähler manifold. The inversion in quaternionic dimension $k \geq 2$ is due to Pedersen and Poon [35], while in inversion in the quaternion-Kähler case was proved independently by LeBrun [27]. In both cases the process is a generalization of the inverse construction in quaternionic dimension $k = 1$ demonstrated by Atiyah et. al. [4].

Theorem 3.9. *Let Z be a complex manifold of dimension $2k+1$, with the additional properties that*
(i) Z admits an real structure σ without fixed points.
(ii) Z is fibered by a family of non-singular rational curves that are invariant under σ, with normal bundle $\mathbb{C}^{2k} \otimes \mathcal{O}(1)$
Then Z is the twistor space of a weakly quaternionic manifold.

Proof. Again the curves described in (ii) above are known as real twistor lines, while generically a curve with the specified normal bundle that is not necessarily fixed by σ is simply a twistor line. If we fix $C \in Z$ such a twistor line, then the normal bundle is $N = \mathbb{C}^{2k} \otimes \mathcal{O}(1)$ and therefore we have the vanishing of the cohomology group

$$H^1(C, N) \cong H^1(\mathbb{CP}_1, \mathbb{C}^{2k} \otimes \mathcal{O}(1)) = \mathbb{C}^{2k} \otimes H^1(\mathbb{CP}_1, \mathcal{O}(1)) = 0,$$

Therefore the methods of Kodaira [24] give that the parameter space of twistor lines in Z is a complex manifold. Moreover, if we call this family X, then, then for $x \in X$ corresponding to the curve $C_x \subset Z$ we have

$$T_xX \cong H^0(C_x, N) \cong H^0(\mathbb{CP}_1, \mathbb{C}^{2k} \otimes \mathcal{O}(1)) \cong H^0(\mathbb{CP}_1, \mathbb{C}^{2k}) \otimes H^0(\mathbb{CP}_1, \mathcal{O}(1))$$

Denoting these vector spaces by $\mathbf{E}^*_x$ and $\mathbf{H}^*_x$, respectively, for reasons that will become clear momentarily, we observe that these vector spaces have dimensions $2k$ and 2, respectively, so that X is a complex manifold of dimension $4k$. Considering a neighborhood of $x \in X$, we then obtain locally-defined bundles $\mathbf{E}^*, \mathbf{H}^*$ on X, along with the isomorphism $TX \cong \mathbf{E}^* \otimes \mathbf{H}^*$.

Let $M \subset X$ denote the set of real twistor lines, which is then a real submanifold of X of dimension $4k$. We claim that M has the structure of a quaternionic manifold, with associated twistor space Z. Let $p : Z \to M$ denote the fibration that sends a point $z \in Z$ to the real twistor line containing it.

To construct the weakly quaternionic structure on M, fix $x \in M \subset X$ representing the curve C_x. Since $\sigma(C_x) = C_x$ by assumption, we have that the derivative σ_* preserves the normal bundle of C_x, and since σ is anti-holomorphic we have a real structure on $T_xX \cong H^0(C_x, N)$, with the related real subspace being T_xM.

Considering the tensor factors $T_xX \cong \mathbf{E}_x^* \otimes \mathbf{H}_x^*$, the derivative σ_* can be factored into maps σ_*^E, σ_*^H acting on each tensor factor, which are conjugate linear and must satisfy $(\sigma_*^E)^2 = \pm \mathrm{Id}$, $(\sigma_*^H)^2 = \pm \mathrm{Id}$, with the sign being identical in each case so that $\sigma_*^2 = (\sigma_*^E \otimes \sigma_*^H)^2 = \mathrm{Id}_* = 1$, since $\sigma^2 = \mathrm{Id}$ is a real structure. In order to chose the sign, we observe that we can identify $\mathbb{P}(\mathbf{H}_x) = C_x$ by identifying a section $s \in H^0(\mathbb{CP}_1, \mathcal{O}(1))$ with the set $s^{-1}(0)$, which consists of a single point since s is a holomorphic section of $\mathcal{O}(1)$. Under this identification, the map σ_*^H is then identified with the antipodal map by assumption. This implies that $(\sigma_*^H)^2 = -\mathrm{Id}$, for if $(\sigma_*^H)^2 = \mathrm{Id}$ then ± 1 would be eigenvalues of σ_*^H, and so the induced map on the projectivizations would have fixed points, which the antipodal map does not have. The maps σ_*^E, σ_*^H therefore determine quaternionic structures on the bundles $\mathbf{E}^*, \mathbf{H}^*$, by declaring multiplication by j to act by these maps.

Combining all of the above, we have that as $TM \otimes \mathbb{C} \cong TX|_M$, we have a (locally defined) isomorphism

$$TM \otimes \mathbb{C} \cong \mathbf{E}^* \otimes \mathbf{H}^*,$$

where the bundles on the right-hand side have structure groups $GL(k, \mathbb{H})$ and $GL(1, \mathbb{H})$, respectively. Restricting to the real subbundle, we have that TM has structure group $GL(k, \mathbb{H})GL(1, \mathbb{H}) \cong GL(k, \mathbb{H})Sp(1)$ via the isomorphism

$$Aq \in GL(k, \mathbb{H})GL(1, \mathbb{H}) \leftrightarrow (A|q|)\left(\frac{q}{|q|}\right) \in GL(k, \mathbb{H})Sp(1)$$

Therefore M has the structure of an almost quaternionic manifold by Proposition 2.18, and that the bundle $\mathbf{E}^*, \mathbf{H}^*$ are precisely the bundles associated to that almost quaternionic manifold via the construction defined in Section 3.1.

It remains to show that the almost quaternionic structure on M is in fact weakly quaternionic, that is, the $GL(k, \mathbb{H})Sp(1)$-structure on M above is preserved by some torsion-free connection. Using Proposition 3.4 the desired connections on M are related to choices of complex structure on $\mathbf{H}^\times$ via the choice of a connection on the bundle **H** itself. We therefore fix a connection on **H**, and use this connection to define an almost complex structure on $\mathbb{P}(\mathbf{H})$. Once again the identification $\mathbb{P}(\mathbf{H}_x) \cong C_x$ discussed above yields a diffeomorphism from Z to $\mathbb{P}(\mathbf{H})$ as a smooth bundle map. This diffeomorphism can in fact be shown to be holomorphic with respect to the complex structures on each space, regardless of the choice of connection made on **H**. In particular, all of the complex structures so defined on $\mathbb{P}(\mathbf{H})$ are integrable, and therefore the exterior derivative of any $(1,1)$-form on $\mathbb{P}(\mathbf{H})$ will have no $(0,3)$-part. Thus, if we have a 2-form α on M that is of type $(1,1)$ with respect to all of the possible almost complex structures associated to the $GL(k,\mathbb{H})Sp(1)$-structure on M, this pulls back to a $(1,1)$ form on $\mathbb{P}(\mathbf{H})$ so that $d(p^*\alpha) = p^*(d\alpha)$ has no $(0,3)$-part. Salamon [41] has shown that the obstruction to the existence of torsion-free quaternionic connections on a $GL(k,\mathbb{H})Sp(1)$-structure is precisely this $(0,3)$-part associated to the exterior derivatives of a $(1,1)$ form. Therefore M has a weakly quaternionic structure. □

As in the case of quaternionic manifolds, the twistor construction for quaternion-Kähler manifolds is also invertible, as was shown by Pedersen and Poon [35], and independently by LeBrun [27]. We will not need this inversion in our work, and therefore will not state it precisely here, but the process is essentially to carry out the inversion described in Theorem 3.9 to obtain a weakly quaternionic manifold, and then show that the contact structure on Z then induces a quaternion-Kähler metric. If θ is the contact form, then $d\theta$ is nondegenerate when restricted to the contact distribution, and this nondegenerate 2-form induces the 2-form ω_E on the fibers of **E**. The form ω_H can be chosen on **H**, as in the inversion for quaternionic manifolds, so that the symmetric complex 2-form $\omega_E \otimes \omega_H$ yields a complexified metric, as in equation (10), that will define a Riemannian metric when taking real parts. Note that in this case the contact structure then determines a preferred quaternionic connection from the many possible connections available in the weakly quaternionic structure, namely the Levi-Civita connection.

4 Construction of Quaternionic Complex Manifolds

This section focuses on the construction of quaternionic complex manifolds, which are by our definition quaternionic manifolds with a preferred parallel complex structure. In principle, there are two methods for constructing such manifolds. Beginning with a quaternionic manifold, we can search for sections of $\mathcal{Q}$ that are parallel with respect to the given connection. Conversely, one could begin with a weakly quaternionic manifold and a fixed complex structure, and attempt to find a quaternionic connection that preserves that complex structure. Note that in either case the resulting complex structure may only be locally defined, since quaternionic manifolds need not admit any global almost complex structure.

Our method is the latter one, and so to begin we study the space of quaternionic connections on a weakly quaternionic manifold in Section 4.1. In Section 4.2 we summarize Hitchin's construction of quaternionic complex manifolds, which uses Killing fields on quaternion-Kähler manifolds to produce the candidate complex structure. Our main result is to give a generalization of this procedure that can produce candidate complex structures on any quaternionic manifold, not just one with a metric. This result is presented in a local, coordinate-dependent treatment in Section 4.3, and then reconsidered in the more general framework of paraconformal structures in Section 4.4. Finally, we end by giving a twistorial interpretation of these results in Section 4.5.

4.1 The Space of Quaternionic Connections

Definition 2.3 of weakly quaternionic and quaternionic manifolds implies that a weakly quaternionic manifold can admit many quaternionic connections. Given a fixed quaternionic connection ∇, it is well known that any other (not necessarily quaternionic) connection on TM can be written as $\widehat{\nabla}_X Y = \nabla_X Y + A(X,Y)$ for some $(2,1)$-tensor A. The following lemma gives necessary and sufficient conditions on the tensor A to ensure that $\widehat{\nabla}$ is also a quaternionic connection.

Lemma 4.1 ([2, 36])**.** *Let M^{4k} be a weakly quaternionic manifold, and let ∇ be a fixed quaternionic connection on TM. Then $\widehat{\nabla}$ is a quaternionic connection on M if and only if it is of the form $\widehat{\nabla}_X Y = \nabla_X Y + A_\Upsilon(X,Y)$, where*

$\Upsilon \in \Omega^1(M)$ *is an arbitrary* 1*-form and the (2,1)-tensor* A_Υ *is defined by*

$$A_\Upsilon(X,Y) = \frac{1}{2}\left(\Upsilon(X)Y + \Upsilon(Y)X - \sum_{i=1}^{3}(\Upsilon(I_iX)I_iY + \Upsilon(I_iY)I_iX)\right) \quad (13)$$

for any local compatible frame $\{I_1, I_2, I_3\} = \{I, J, K\}$.

Proof. It is well-known that if ∇ is torsion free then $\widehat{\nabla}_XY = \nabla_XY + A(X,Y)$ will be torsion-free if and only if $A(X,Y) = A(Y,X)$, and we observe that A_Υ as defined above is in fact symmetric. The main condition to consider, though, is the requirement that the new connection $\widehat{\nabla}$ preserves sections of $\mathcal{Q}$.

If S is any local section of $\mathcal{Q}$, then we have

$$(\widehat{\nabla}_XS)(Y) = (\nabla_XS)Y + A_\Upsilon(X,SY) - SA_\Upsilon(X,Y) \quad (14)$$

Since ∇_XS is by assumption a local section of $\mathcal{Q}$, we have that $(\widehat{\nabla}_XS)$ will be a local section of $\mathcal{Q}$ if an only if the the endomorphism defined by $Y \mapsto A_\Upsilon(X,SY) - SA_\Upsilon(X,Y)$ is a local section of $\mathcal{Q}$. A direct computation using the definition of A_Υ above and choosing $S = I, J, K$ yields

$$\begin{aligned} A_\Upsilon(X,IY) - IA_\Upsilon(X,Y) &= -\Upsilon(KX)JY + \Upsilon(JX)KY \\ A_\Upsilon(X,JY) - JA_\Upsilon(X,Y) &= \Upsilon(KX)IY - \Upsilon(IX)KY \\ A_\Upsilon(X,KY) - KA_\Upsilon(X,Y) &= -\Upsilon(JX)IY + \Upsilon(IX)JY \end{aligned} \quad (15)$$

In each case we see that the transformation on the right-hand side, considered as an endomorphism acting on Y, is local section of $\mathcal{Q}$, and so by linearity any connection $\widehat{\nabla}$ determined by the expression above is in fact quaternionic. The converse statement is more tedious to justify, so we will not give it here and instead refer to the citations above. □

This variability in the space of quaternionic connections is greatly reduced when we consider quaternionic manifolds with scale.

Proposition 4.2 ([2, 17]). *Let* $(M^{4k}, \mathcal{Q})$ *with* $k \geq 2$ *be a weakly quaternionic manifold. If* μ *is any volume form on* M*, then there exists a unique quaternionic connection* ∇ *such that* $\nabla\mu = 0$.

Let $(M^{4k}, \mathcal{Q}, \nabla, \mu)$ *be a scaled quaternionic manifold for* $k \geq 2$*. If* $\widehat{\nabla}$ *is a different choice of quaternionic* 1*-form related to* ∇ *by the* 1*-form* Υ*, then* $\widehat{\nabla}$ *preserves a scale if and only if* Υ *is exact.*

Proof. We begin by understanding how changing a connection via a choice of 1-form Υ changes the connection when acting on volume forms. Let E_i for

$i = 1, \dots 4k$ be a local quaternionic frame as in the discussion following Proposition 2.17, that is, the basis is generated by k linearly independent vector fields and the endomorphisms I, J, K so that, for example, $E_2 = IE_1, E_3 = JE_1$, etc. Then we have

$$(\widehat{\nabla}_{E_j}\mu)(E_1, \dots, E_{4k}) = E_j \cdot \mu(E_1, \dots, E_{4k}) - \sum_{i=1}^{4k} \mu\left(E_1, \dots, \widehat{\nabla}_{E_j}E_i, \dots, E_{4k}\right)$$
$$= E_j \cdot \mu(E_1, \dots, E_{4k}) - \sum_{i=1}^{4k} \mu\left(E_1, \dots, \nabla_{E_j}E_i + A_\Upsilon(E_j, E_i), \dots, E_{4k}\right)$$
$$= (\nabla_{E_j}\mu)(E_1, \dots, E_{4k}) - \sum_{i=1}^{4k} \mu\left(E_1, \dots, A_\Upsilon(E_j, E_i), \dots, E_{4k}\right)$$

From the definition of A_Υ and the fact that μ is a volume form and therefore alternating, we have that only the E_i component of $A_\Upsilon(E_j, E_i)$ will appear in this sum.

To simplify the argument, assume for now that $j \equiv 1 \pmod 4$, so that $E_{j+1} = IE_j, E_{j+2} = JE_j$, and $E_{j+3} = KE_j$. Considering the sum over i above, if $i \neq j, j+1, j+2, j+3$, that is, if E_j and E_i are not in the same quaternionic subspace, then the E_i component of $A_\Upsilon(E_j, E_i)$ is exactly $\frac{1}{2}\Upsilon(E_j)E_i$. For $i = j, j+1, j+2, j+3$, the E_i components of $A_\Upsilon(E_j, E_i)$ are all $\Upsilon(E_j)$, by a direct computation. Therefore if $j \equiv 1 \pmod 4$ we have that

$$\widehat{\nabla}_{E_j}\mu = \nabla_{E_j}\mu - \left(\frac{4(k-1)}{2} + 4\right)\Upsilon(E_j)\mu = \nabla_{E_j}\mu - 2(k+1)\Upsilon(E_j)\mu$$

Similar arguments hold for the other possible equivalence classes of j, so that by linearity we ultimately have

$$\widehat{\nabla}\mu = \nabla\mu - 2(k+1)\Upsilon \otimes \mu$$

Turning to the first part, as M is weakly quaternionic we can begin by choosing an arbitrary quaternionic connection ∇. Then as μ is a volume form we have that there exists a 1-form η such that $\nabla\mu = \eta \otimes \mu$. Then if we set $\Upsilon = \frac{1}{2(k+1)}\eta$, we have

$$\widehat{\nabla}\mu = \eta \otimes \mu - 2(k+1)\frac{1}{2(k+1)}\eta \otimes \mu = 0$$

giving the desired connection, and evidently this is the only possible choice of Υ, giving uniqueness.

For the second part, if we now consider ∇ to be the connection compatible with a fixed volume form μ, then we have for any other quaternionic connection $\widehat{\nabla}$ and any smooth function $f > 0$ that

$$\begin{aligned}\widehat{\nabla}(f\mu) &= df \otimes \mu + f\widehat{\nabla}\mu \\ &= df \otimes \mu + f\left(\nabla\mu - 2(k+1)\Upsilon \otimes \mu\right) \\ &= (df - 2(k+1)f\Upsilon) \otimes \mu\end{aligned}$$

Therefore if $\widehat{\nabla}$ preserves the scale $f\mu$, we have that $df - 2(k+1)f\Upsilon = 0$, that is,

$$\Upsilon = \frac{1}{2(k+1)} d(\log f), \tag{16}$$

so that Υ is exact. Conversely, if Υ is exact, we can write $\Upsilon = dh$, and if we set

$$f = \exp(2(k+1)h), \tag{17}$$

we have that $df = 2(k+1)f dh$ and therefore $\widehat{\nabla} f\mu = 0$, so that $\widehat{\nabla}$ preserves the scale $f\mu$. □

In much of the work in the following sections we will restrict to an open subset of a quaternionic manifold M with a fixed compatible frame $\{I, J, K\}$ for $\mathcal{Q}$, and so we will need the following lemmas to describe quaternionic connections with respect to this local frame.

Lemma 4.3. *Let $\{I, J, K\}$ be a compatible frame for a quaternionic manifold $(M, \mathcal{Q}, \nabla)$. With respect to this local frame, the local connection 1-forms for ∇, considered as a connection on $\mathcal{Q}$, are of the form*

$$\begin{array}{rcl}\nabla_X I &=& a(X)J - b(X)K \\ \nabla_X J &=& -a(X)I + c(X)K \\ \nabla_X K &=& b(X)I - c(X)J\end{array} \tag{18}$$

Proof. From Corollary 2.19, we have that the structure group of $\mathcal{Q}$ is $SO(3)$ and the connection ∇ is compatible with this structure, so that the connection 1-forms a, b, c can be considered as an $\mathfrak{so}(3)$-valued 1-form

$$\mathfrak{a}(X) = \begin{pmatrix} 0 & -a(X) & b(X) \\ a(X) & 0 & -c(X) \\ -b(X) & c(X) & 0 \end{pmatrix}$$

where this matrix acts on $\mathcal{Q}$ via standard matrix multiplication with respect to the chosen basis $\{I, J, K\}$. Note the signs here are chosen to be compatible with the isomorphism of Proposition 2.16.

This can also be verified more directly by observing that $\nabla_X I, \nabla_X J, \nabla_X K$ are by assumption sections of $\mathcal{Q}$, and so each section can be expressed in terms of the compatible frame $\{I, J, K\}$. The quaternionic relations among I, J, K then induces the relationships between the coefficients for these derivatives with respect to that frame, which are precisely the relationships given above. □

Lemma 4.4. *Let a, b, c be the connection 1-forms for ∇ a quaternionic connection with respect to the compatible frame $\{I, J, K\}$. Let $\widehat{\nabla}$ be the quaternionic connection associated to the 1-form Υ via Lemma 4.1, and let $\widehat{a}, \widehat{b}, \widehat{c}$ be the associated connection 1-forms for $\widehat{\nabla}$ with respect to the same compatible frame. Then*

$$\begin{aligned}\widehat{a} &= a - \Upsilon \circ K \\ \widehat{b} &= b - \Upsilon \circ J \\ \widehat{c} &= c - \Upsilon \circ I\end{aligned}$$

Proof. Combining equation (14) with $S = I$ and the first equation of (15) above yields the local expression

$$\begin{aligned}\widehat{a}(X)J - \widehat{b}(X)K &= \widehat{\nabla}_X I \\ &= (\nabla_X I) - \Upsilon(KX)J + \Upsilon(JX)K \\ &= (a(X) - \Upsilon(KX))\, J - (b(X) - \Upsilon(JX))\, K\end{aligned}$$

The equality for $\widehat{c}$ is proved similarly. □

With this framework for understanding how to change the quaternionic connection on a weakly quaternionic manifold, we can now state conditions under which a given almost complex structure I is preserved by a quaternionic connection.

Lemma 4.5. *Let $(M, \mathcal{Q}, \nabla)$ be a quaternionic manifold, and let $\{I, J, K\}$ be a local compatible frame for $\mathcal{Q}$. There exists a quaternionic connection $\widehat{\nabla}$ with $\widehat{\nabla} I = 0$ if and only if*

$$a \circ K = b \circ J$$

where a, b, c are the local connection 1-forms of Lemma 4.3. In this case, the connection is unique and given by the 1-form

$$\Upsilon = -(a \circ K) = -(b \circ J)$$

Proof. From equation (18) we have that $\widehat{\nabla} I = 0$ if $\widehat{a} = \widehat{b} = 0$, and so by Lemma 4.4 this is equivalent to being able to choose Υ such that $a = \Upsilon \circ K$ and $b = \Upsilon \circ J$. Precomposing with K, J, respectively, this requires that $a \circ K = -\Upsilon$ and $b \circ J = -\Upsilon$. □

Of course for a general almost complex structure $I \subset \Gamma(\mathcal{Q})$, the conditions of the above proposition will not hold, as for example, it is necessary that I is in fact integrable. The following sections give methods for producing sections of $\mathcal{Q}$ satisfying the above properties.

4.2 Quaternionic Complex Manifolds from Quaternion-Kähler Manifolds

The first method to produce a compatible complex structure I fitting the requirements of Lemma 4.5 is due to Hitchin [17], who has shown that such a complex structure can arise from the momentum section associated to a quaternionic $U(1)$ action on a quaternion-Kähler manifold. We review this construction in this section.

Definition 4.6. We call a vector field X_0 on M a *quaternionic Killing field* if $\mathcal{L}_{X_0} g = 0$ and $\mathcal{L}_{X_0} \Omega = 0$, where Ω is the fundamental 4-form of Proposition 2.9.

Quaternionic Killing fields therefore generate isometries that preserve the quaternionic structure by preserving Ω. Given a quaternionic Killing field X_0 and a choice of local compatible frame $\{I, J, K\}$, we can construct a $\mathcal{G}$-valued 1-form on M by

$$\Theta_{X_0} = i_{X_0}\omega_I \otimes \omega_I + i_{X_0}\omega_J \otimes \omega_J + i_{X_0}\omega_K \otimes \omega_K \tag{19}$$

This form does not in fact depend on the choice of local compatible frame, and therefore defines a global $\mathcal{G}$-valued 1-form on M. Moreover, this Θ_{X_0} is exact in the sense that it can be obtained by taking the covariant derivative of a unique section of the bundle $\mathcal{G}$, which can be obtained as follows. As X_0 is a Killing field, it defines a 2-form on M by $g(\nabla_X X_0, Y) = -g(\nabla_Y X_0, X)$. Let α_{X_0} denote the orthogonal projection of this 2-form to $\mathcal{G} \subset \Lambda^2 T^* M$.

Theorem 4.7 ([12, 13, 42])**.** *Let X_0 be a quaternionic Killing field on M a quaternion-Kähler manifold. Then there exists a unique section ρ_{X_0} of $\mathcal{G}$, called the* momentum section*, such that*

$$\nabla \rho_{X_0} = \Theta_{X_0},$$

given by

$$\rho_{X_0} = \frac{4k(k+2)}{s}\alpha_{X_0}$$

We note the constant here arises from the fact that the curvature operator $R : \Lambda^2 TM \to \Lambda^2 TM$ acts as $\lambda\,\mathrm{Id}$ when restricted to $\mathcal{G}$ for $\lambda = s/(4k(k+2))$ a constant positive multiple of the Einstein constant of the metric (see the discussion following Corollary 5.4), and so the constant here is $1/\lambda$.

This notion of a quaternion-Kähler momentum map, developed by Galicki [12] and Galicki and Lawson [13] can be applied in the more general context of compact Lie group actions on M preserving both the metric and quaternionic structure, although here we only make use of 1-dimensional $U(1)$ actions generated by a Killing field.

Under the correspondence $\mathcal{Q} \cong \mathcal{G}$ induced by g, the momentum section $\rho_{X_0} \in \Gamma(\mathcal{G})$ corresponds to a section $\widetilde{\rho}_{X_0} \in \Gamma(\mathcal{Q})$. If we choose any local compatible frame $\{I, J, K\}$, then in local coordinates these sections are expressed as

$$\rho_{X_0} = \rho_1\omega_I + \rho_2\omega_J + \rho_3\omega_K \qquad \widetilde{\rho}_{X_0} = \rho_1 I + \rho_2 J + \rho_3 K$$

We can then observe that

$$\widetilde{\rho}_{X_0} \circ \widetilde{\rho}_{X_0} = -(\rho_1^2 + \rho_2^2 + \rho_3^2)\,\mathrm{Id}_{TM}$$

so that the momentum section determines a compatible almost complex structure

$$I_{X_0} = \frac{1}{\sqrt{\rho_1^2 + \rho_2^2 + \rho_3^2}}\widetilde{\rho}_{X_0}$$

on the open set $M_0 = M - \{\rho_{X_0} = 0\}$. This set is in fact dense, although we defer the proof of this fact to Theorem 4.24 so that we may use twistor methods.

Theorem 4.8 ([17])**.** *Let (M, g) be a quaternion-Kähler manifold admitting a quaternionic Killing field X_0 with momentum section ρ_{X_0}.*

Then there exists a unique quaternionic connection $\widehat{\nabla}$ on the open, dense submanifold $M_0 = M - \{\rho_{X_0} = 0\}$ preserving the almost complex structure I_{X_0}, that is, $(M_0, \mathcal{Q}|_{M_0}, \widehat{\nabla}, I_{X_0})$ is a quaternionic complex manifold.

Proof. We can assume that a local compatible frame $\{I, J, K\}$ for $\mathcal{Q}|_{M_0}$ has been chosen so that $I = I_{X_0}$. This is equivalent to assuming that $\rho_{X_0} = \rho_1\omega_I$. Let ∇ be the Levi-Civita connection associated to g, and let a, b, c be the connection 1-forms for ∇ with respect to the local compatible frame $\{I, J, K\}$. Then

$$\begin{aligned}\nabla \rho_{X_0} &= \nabla(\rho_1 \omega_I) = d\rho_1 \otimes \omega_I + \rho_1 \nabla \omega_I \\ &= d\rho_1 \otimes \omega_I + \rho_1 \left(a \otimes \omega_J - b \otimes \omega_K\right).\end{aligned}$$

Now ρ_{X_0} is a momentum section, so that $\nabla \rho_{X_0} = \Theta_{X_0}$ and therefore by equation (19) we have that

$$d\rho_1 = i_{X_0}\omega_I \qquad \rho_1 a = i_{X_0}\omega_J \qquad -\rho_1 b = i_{X_0}\omega_K. \tag{20}$$

We then observe

$$\begin{aligned}(a \circ K)(X) &= \rho_1^{-1} g(JX_0, KX) = \rho_1^{-1} g(IJX_0, IKX) \\ &= -\rho_1^{-1} g(KX_0, JX) = (b \circ J)(X),\end{aligned}$$

so that the conditions of Lemma 4.5 are satisfied. Therefore the connection $\widehat{\nabla}$ associated to the 1-form $\Upsilon = -(a \circ K)$ is the unique quaternionic connection that preserves $I = I_{X_0}$, yielding a quaternionic complex structure on M_0. $\square$

Lemma 4.9. *The 1-form Υ defined in the proof of Theorem 4.8 is*

$$\Upsilon = -\frac{1}{\rho_1} d\rho_1 = -d(\log \rho_1) \tag{21}$$

Proof. Computing directly from the definitions and equation (20), we have

$$\begin{aligned}\Upsilon(X) &= -(a \circ K)(X) = -\frac{1}{\rho_1} g(JX_0, KX) \\ &= -\frac{1}{\rho_1} g(IX_0, X) = -\frac{1}{\rho_1}(i_{X_0}\omega_I)(X) \\ &= -\frac{1}{\rho_1} d\rho_1(X)\end{aligned}$$

$\square$

Corollary 4.10. *If μ denotes the Riemannian volume form for M, the connection $\widehat{\nabla}$ preserves the volume form*

$$\widehat{\mu} = \rho_1^{-2(k+1)} \mu$$

Proof. Since Υ is evidently exact by equation (21), we have from Proposition 4.2 that it preserves a volume form, the formula for which is given by equation (17). $\square$

Thus Hitchin's construction in fact yields a scaled quaternionic complex manifold, as the connection preserves an almost quaternionic structure, a volume form, and a complex structure. Equivalently, the holonomy group for ∇ is contained in $SL(k,\mathbb{H})U(1)$, at least in quaternionic dimension $k \geq 2$. In the case of quaternionic dimension $k = 1$, Hitchin [17] also observes that the formula (13) for the change in connection is equivalent to the change in the Levi-Civita connection induced by the conformal change of the metric to $\widehat{g} = \rho_1^{-2} g$ on M_0. Thus the process yields an anti-self-dual Kähler metric on M_0, which is our definition of a quaternionic complex 4-manifold.

4.3 Quaternionic Complex Manifolds from Quaternionic Manifolds

As we have discussed, we can construct connections with holonomy contained in $GL(k,\mathbb{H})U(1)$ by finding a local almost complex structure on a quaternionic manifold M that satisfies the requirements of Lemma 4.5, and Hitchin's construction in the previous section gives a way to produce such almost complex structures from positive quaternion-Kähler manifolds with $U(1)$-symmetries. Many aspects of Hitchin's construction, however, do not truly require the quaternion-Kähler metric structure, and can be generalized to make reference only to the underlying scaled quaternionic structure. In this section, we carry out this generalization to give a method to construct connections with holonomy contained in $GL(k,\mathbb{H})U(1)$ on open subsets of generic quaternionic manifolds.

The generalization relaxes two aspects of Hitchin's argument. In his argument, the desired almost complex structure arose from the momentum section, a 2-form on M that was related to $\mathcal{Q}$ via the metric. Without a metric, we must instead search for sections of $\mathcal{Q}$ directly. The second generalization is to make use of the identification of $S^2H \cong \mathcal{Q}$ described in Section 3.1 to instead search for real sections of the bundle $S^2\mathbf{H}$.

We therefore need to understand conditions on sections of the bundle $S^2\mathbf{H}$ that are equivalent to the necessary conditions for the related complex structure described in Lemma 4.5. To this end, we first recast that proposition in terms of sections of the bundle $S^2\mathbf{H}$.

Lemma 4.11. *Let $(M, \mathcal{Q}, \nabla)$ be a quaternionic manifold, and let I, J, K be real sections of $S^2\mathbf{H}$ associated to a choice of local compatible frame for $\mathcal{Q}$, and choose a unitary frame $\{h_1, h_2\}$ for $\mathbf{H}$ so that I, J, K have the expressions given in equation* (9).

Let a, b, c be the connection 1-forms with respect to this frame, which can be decomposed into $\mathbf{E}, \mathbf{H}$ parts as

$$a = a^1 \otimes h_1 + a^2 \otimes h_2 \qquad b = b^1 \otimes h_1 + b^2 \otimes h_2 \qquad c = c^1 \otimes h_1 + c^2 \otimes h_2$$

where a^1, a^2, etc. are local sections of $\mathbf{E}$.

Then there exists a quaternionic connection on M that preserves the almost complex structure associated to I if and only if

$$b^1 = -ia^1 \text{ and } b^2 = ia^2 \tag{22}$$

Proof. The equation is simply a restatement of the requirement of Lemma 4.5 that $a \circ K = b \circ J$. For, using the expressions for I, J, K in terms of the unitary basis h_1, h_2, we have that the evaluation of $a \circ K$ on an arbitrary vector $X = e_1^* \otimes h_1^* + e_2^* \otimes h_2^*$ is given by

$$\begin{aligned} a(KX) &= -(a^1 \otimes h_1 + a^2 \otimes h_2)(e_1^* \otimes h_1^* + e_2^* \otimes h_2^*) i\,(h_1 \otimes h_1^* - h_2 \otimes h_2^*) \\ &= -i(a^1 \otimes h_1 + a^2 \otimes h_2)(e_1^* \otimes h_2^* + e_2^* \otimes h_1^*) \\ &= -i(e_1^* a^1 - e_2^* a^2), \end{aligned}$$

while the evaluation of $b \circ J$ is

$$\begin{aligned} b(JX) &= -(b^1 \otimes h_1 + b^2 \otimes h_2)(e_1^* \otimes h_1^* + e_2^* \otimes h_2^*)\,(h_1 \otimes h_1^* + h_2 \otimes h_2^*) \\ &= -(b^1 \otimes h_1 + b^2 \otimes h_2)(-e_1^* \otimes h_2^* + e_2^* \otimes h_1^*) \\ &= e_1^* b^1 + e_2^* b^2 \end{aligned}$$

Then $a(KX) = b(JX)$ for arbitrary X, that is for arbitrary e_1^* and e_2^*, if and only if the the equalities of (22) hold. Note the extra negative signs above appear as we have J, K act on the left, hence by the quaternion conjugate of the right action. □

We note for future reference that the computation for a above shows that the 1-form $(a \circ K)$ decomposes as

$$i(a^2 \otimes h_1 + a^1 \otimes h_2) \tag{23}$$

Sections of $S^2\mathbf{H}$ meeting the above condition can be obtained by studying the twistor operator on M.

Definition 4.12. Let $(M, \mathcal{Q}, \nabla)$ be a quaternionic manifold. The *twistor operator* is the differential operator $D : S^2\mathbf{H} \to \mathbf{E} \otimes S^3\mathbf{H}$ given by the composition

$$S^2\mathbf{H} \xrightarrow{\nabla} T^*M \otimes \mathbb{C} \otimes S^2\mathbf{H} = \mathbf{E} \otimes \mathbf{H} \otimes S^2\mathbf{H} \longrightarrow E \otimes S^2\mathbf{H}$$

where the final map is symmetrization on the three $\mathbf{H}$ factors.

Theorem 4.13. *Let $(M, \mathcal{Q}, \nabla)$ be a quaternionic manifold. Assume that $\varphi \in S^2\mathbf{H}$ is a real section with $D\varphi = 0$.*

Then there exists a unique quaternionic connection $\widehat{\nabla}$ on the open, dense submanifold $M_0 = M - \{\varphi = 0\}$ preserving the almost complex structure associated to φ, that is, the connection $\widehat{\nabla}$ has holonomy contained in $GL(k, \mathbb{H})U(1)$.

Proof. As in the proof of Theorem 4.8, we can assume that we have chosen our fixed unitary frame of $\mathbf{H}$, which in turn gives a local compatible frame $\{I, J, K\}$ using equation (9), so that the section $\varphi \in S^2\mathbf{H}$ with $D\varphi = 0$ is of the form $\varphi = s_1 I$. Note that as φ is assumed to be real we have that the function s_1 is real-valued as well, while $M_0 = \{s_1 \neq 0\}$. We again defer the proof of the density of the set M_0 until Theorem 4.24.

Let a, b, c be the connection 1-forms with respect to the local compatible frame I, J, K, with decomposition into $\mathbf{E}, \mathbf{H}$ parts as described in Lemma 4.11. Then we can find $D\varphi$ explicitly. We first have

$$\nabla\varphi = ds_1 I + s_1 \nabla I = ds_1 I + s_1 aJ - s_1 bK$$

Expanding the 1-form ds_1 as $e^1 \otimes h_1 + e^2 \otimes h_2$, while using the decomposition of the remaining terms into $\mathbf{E}, \mathbf{H}$ parts that has already been described, we have, dropping the tensor product symbols for brevity,

$$\begin{aligned}
\nabla\varphi &= \left(e^1 h_1 + e^2 h_2\right) i \left(h_1 h_2 + h_2 h_1\right) + s_1(a^1 h_1 + a^2 h_2)(h_1 h_1 + h_2 h_2) \\
&\quad - s_1(b^1 h_1 + b^2 h_2) i \left(h_1 h_1 - h_2 h_2\right) \\
&= s_1(a^1 - ib^1) h_1 h_1 h_1 + ie^1(h_1 h_1 h_2 + h_1 h_2 h_1) + s_1(a^2 - ib^2) h_2 h_1 h_1 \\
&\quad + ie^2(h_2 h_2 h_1 + h_2 h_1 h_2) + s_1(a^1 + ib^1) h_1 h_2 h_2 + s_1(a^2 + ib^2) h_2 h_2 h_2
\end{aligned}$$

Symmetrizing on the h factors therefore yields

$$\begin{aligned}
D\varphi &= s_1(a^1 - ib^1) h_1 h_1 h_1 + (2ie^1 + s_1(a^2 - ib^2))(h_1 \odot h_1 \odot h_2) \\
&\quad + (2ie^2 + s_1(a^1 + ib^1))(h_2 \odot h_2 \odot h_1) + s_1(a^2 + ib^2) h_2 h_2 h_2
\end{aligned}$$

Working on M_0 so that $s_1 \neq 0$, we have that $D\varphi = 0$ implies that $a^1 - ib^1 = 0$ and $a^2 + ib^2 = 0$ by inspecting the first and fourth components, which is equivalent to the required equations (22). □

Further inspecting the expression for $D\varphi$, we have that if $D\varphi = 0$ then $2ie^1 + s_1(a^2 - ib^2) = 0$ and $2ie^2 + s_1(a^1 + ib^1) = 0$, which we can rewrite using (22) as

$$\frac{1}{s_1} e^1 = ia^2 \qquad \frac{1}{s_1} e^2 = ia^1$$

Then, using equation (23), we have that the 1-forms $(a \circ K)$ and $s_1^{-1} ds_1$ are equal, and therefore Lemma 4.5 gives that the new connection given by Theorem 4.13 is obtained from the original quaternionic connection ∇ via the 1-form $\Upsilon = -d(\log s_1)$, c.f. Lemma 4.9. Evidently this form is closed, and so in the case that the original quaternionic conneciton was compatible with a scale we have that the resulting connection will also preserve a scale by Proposition 4.2.

4.4 Quaternionic Complex Manifolds and Paraconformal Structures

One weakness of the discussion in Section 4.3 is that it relies on local coordinate expressions, and we have not carefully shown that the results are independent of the various choices made when using these local expressions. In this section, we rectify this by using a global, coordinate-free framework, the theory of *complex paraconformal manifolds* developed by Bailey and Eastwood [5]. A complex paraconformal manifold is a complex manifold along with a splitting of the holomorphic tangent bundle as a tensor product, and therefore these structures are a complexification of the splitting $T^*M \otimes \mathbb{C} \cong \mathbf{E} \otimes \mathbf{H}$ in the theory of quaternionic geometries.

In order to avoid choosing local frames and coordinates, we will adopt abstract index notation for this section. The conventions will be as follows. Lower-case indices will be used to represent tensors involving $TM \otimes \mathbb{C}$ and its dual. Superscripts will be used to denote sections of $TM \otimes \mathbb{C}$, while subscripts will be used to denote sections of $T^*M \otimes \mathbb{C}$. For example, X^a represents a complex vector field, α_a represents a complex-valued 1-form, and $\chi_{ab}{}^c$ represents a complex $(2,1)$-tensor. Repeated indices are used to indicate tensor contractions and evaluations, and we use the "south-east" conventions for such contractions, so that, for example, $X^a \alpha_a$ represents the function $\alpha(X)$ obtained by evaluating the 1-form α on the vector field X.

We use uppercase letters to denote sections of the bundle $\mathbf{E}$, and uppercase primed letters to represent sections of $\mathbf{H}$. We choose the subscripts and superscripts to be compatible with the identification $T^*M \otimes \mathbb{C} \cong \mathbf{E} \otimes \mathbf{H}$ on a quaternionic manifold. Therefore $\beta_{A'}$ represents a section of $\mathbf{H}$, while ν^A represents a section of $\mathbf{E}^*$. We can also use the identification to rewrite lower-case indices as pairs of unprimed and primed uppercase indices, e.g., $\alpha_a = \alpha_{AA'}$.

We will use parentheses and square brackets, respectively, to denote symmetrization and antisymmetrization operations, so that

$$\alpha_{(A_1\cdots A_k)} = \frac{1}{k!}\sum_{\sigma\in S_k}\alpha_{A_{\sigma(1)}\cdots A_{\sigma(k)}}$$
$$\alpha_{[A_1\cdots A_k]} = \frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma\alpha_{A_{\sigma(1)}\cdots A_{\sigma(k)}},$$

where S_k denotes the symmetric group acting on the indices $\{1,\ldots,k\}$. Finally, we will use δ to denote the identity, so that, for example, $\nu^A{\delta_A}^B = \nu^B$ represents the action of the identity map on the section $\nu \in \Gamma(\mathbf{E}^*)$.

Let $(M,\mathcal{Q})$ be a weakly quaternionic manifold. Since M is orientable, we can for convenience make any choice of volume form μ on M and therefore fix a quaternionic connection ∇ using Proposition 4.2 to obtain a scaled quaternionic manifold. The quaternionic connection then has holonomy contained in $SL(k,\mathbb{H})Sp(1)$, so we can then consider $\mathbf{E}^*$ and $\mathbf{H}^*$ as bundles with structure group $SL(k,\mathbb{H})$ and $Sp(1)$, respectively, so that $\mathbf{E}^*$ admits the volume form $\varepsilon\in\Gamma(\Lambda^{2k}\mathbf{E})$ and $\mathbf{H}^*$ admits the form $\omega_H\in\Gamma(\Lambda^2\mathbf{H})$ with $\nabla\varepsilon = 0$ and $\nabla\omega_H = 0$ for the connections on $\mathbf{E},\mathbf{H}$ induced by ∇. In order to avoid confusion with the abstract index notation we have used, we will write ω instead of ω_H, or also $\omega_{A'B'}$ when using abstract index notation. (In particular, we do not want to confuse ω_H with a section of $\mathbf{E}$.) Note that in the case that the manifold is actually quaternion-Kähler, we have that $\mathbf{E}^*$ also admits a parallel 2-form ω_E with $\varepsilon = \omega_E^{\wedge k}$.

We remark that this notation is not traditional. It is more common, especially in the context of spinors in real dimension 4, to use the letter ε to denote the top-degree forms on both tensor factors $\mathbf{E},\mathbf{H}$, and rely on indices for disambiguation. We use ω instead for the top-degree form on $\mathbf{H}$ to remain consistent with the notation introduced in Section 3.1.

We can also consider the converse process. That is, if we vary the connections on $\mathbf{E},\mathbf{H}$, we will produce new connections on $TM\otimes\mathbb{C}$. The result will not in general be a quaternionic connection, but this can be guaranteed by placing some limitations on how the connections on $\mathbf{E},\mathbf{H}$ are modified (c.f. [5, Section 2.2])

Lemma 4.14. *Let $(M,\mathcal{Q},\nabla,\mu)$ be a scaled quaternionic manifold, with related forms ε,ω, respectively. We also use ∇ to denote the induced connections on $\mathbf{E},\mathbf{H}$ as above. Let $f>0$ be a strictly positive real-valued function on M, and let $\Upsilon = d(\log f) = f^{-1}df$.*

Consider the connections $\widehat{\nabla}$ defined on $\mathbf{E}, \mathbf{H}$ by

$$\widehat{\nabla}_{AA'}\nu_B = \nabla_{AA'}\nu_B - \Upsilon_{A'B}\nu_A$$
$$\widehat{\nabla}_{AA'}\sigma_{B'} = \nabla_{AA'}\sigma_{B'} - \Upsilon_{AB'}\sigma_{A'}$$

Then the induced connection $\widehat{\nabla}$ on $TM \otimes \mathbb{C}$ is the complexification of a quaternionic connection on TM. Moreover, this connection preserves the scale $\widehat{\mu} = f^{2(k+1)}\mu$.

Proof. Changing the connections defined on $\mathbf{E}, \mathbf{H}$ as above induces a change of connection on the bundles $\Lambda^{2k}\mathbf{E}$ and $\Lambda^2\mathbf{H}$ as well. For example, we could represent a section $\psi_{B'C'} = \psi_{[B'C']}$ of $\Lambda^2\mathbf{H}$ as $\sigma_{B'}\nu_{C'} - \sigma_{C'}\nu_{B'}$, and use the above change of connection formulas to derive the formula

$$\widehat{\nabla}_{AA'}\psi_{B'C'} = \nabla_{AA'}\psi_{B'C'} + 2\Upsilon_{A[B'}\psi_{C']A'} \tag{24}$$

In particular, we have that $\Lambda^2\mathbf{H}$ has the preferred section $\omega_{B'C'}$ that is parallel with respect to ∇, and using this we observe that

$$\begin{aligned}
\widehat{\nabla}_{AA'}\left(f\omega_{B'C'}\right) &= \left(\widehat{\nabla}_{AA'}f\right)\omega_{B'C'} + f\left(\widehat{\nabla}_{AA'}\omega_{B'C'}\right)\\
&= ff^{-1}(\nabla_{AA'}f)\omega_{B'C'} + f\left(\nabla_{AA'}\omega_{B'C'} + 2\Upsilon_{A[B'}\omega_{C']A'}\right)\\
&= f\left(\Upsilon_{AA'}\omega_{B'C'} + \Upsilon_{AB'}\omega_{C'A'} - \Upsilon_{AC'}\omega_{B'A'}\right)\\
&= 3f\Upsilon_{A[A'}\omega_{B'C']} = 0
\end{aligned}$$

since $\mathbf{H}$ has dimension 2 and therefore $\Lambda^3\mathbf{H} = 0$. Thus the connection $\widehat{\nabla}$ preserves the form $\widehat{\omega} = f\omega$ in $\Gamma(\Lambda^2\mathbf{H})$.

The above argument makes no use of the special unitary properties of the bundle $\mathbf{H}^*$, simply the fact that ω is a volume form for that bundle, and so a similar argument for $\mathbf{E}^*$ yields that the connection $\widehat{\nabla}$ also preserves the volume form $\widehat{\varepsilon} = f\varepsilon$ on $\mathbf{E}^*$. Moreover since f is a real-valued function, the connection $\widehat{\nabla}$ will also preserve the quaternionic structure on the bundles $\mathbf{E}, \mathbf{H}$ as well.

Therefore, if we change the connections on $\mathbf{E}, \mathbf{H}$ by the proscription in the lemma, we see that the resulting connections on these bundles yield a connection on $T^*M \otimes \mathbb{C}$ that preserves the $SL(k, \mathbb{H})Sp(1)$ structure, which will moreover be the complexification of a real connection on TM since f is a real-valued function and therefore $\widehat{\nabla}$ will preserve real sections of $\mathbf{E} \otimes \mathbf{H}$. (This also follows from the formula for the change of connection on $T^*M \otimes \mathbb{C}$ below).

To show that $\widehat{\nabla}$ is quaternionic we must additionally show that it is torsion-free. To this end, we can express a 1-form as a tensor product $\nu_B\sigma_{B'}$ and apply the change of connection formulas on $\mathbf{E}, \mathbf{H}$ to show that the change of connection induced on the cotangent bundle is given by

$$\widehat{\nabla}_a \alpha_b = \nabla_a \alpha_b - \left(\Upsilon_{BA'} \delta_A{}^C \delta_{B'}{}^{C'} + \Upsilon_{AB'} \delta_B{}^C \delta_{A'}{}^{C'} \right) \alpha_c$$

where in the above we make free use of the relations $a = AA'$, etc., among indices. Next, the definition of the torsion tensor of a connection, given in abstract index notation, is

$$T_{ab}{}^c \nabla_c f = 2\nabla_{[a} \nabla_{b]} f$$

Letting $\widehat{T}$ denote the torsion tensor associated to $\widehat{\nabla}$, we therefore have

$$\begin{aligned}
\widehat{T}_{ab}{}^c \widehat{\nabla}_c f &= \widehat{\nabla}_a (\widehat{\nabla}_b f) - \widehat{\nabla}_b (\widehat{\nabla}_a f) = \widehat{\nabla}_a (\nabla_b f) - \widehat{\nabla}_b (\nabla_a f) \\
&= \nabla_a (\nabla_b f) - \nabla_b (\nabla_a f) - \left(\Upsilon_{BA'} \delta_A{}^C \delta_{B'}{}^{C'} + \Upsilon_{AB'} \delta_B{}^C \delta_{A'}{}^{C'} \right) \nabla_c f \\
&\quad + \left(\Upsilon_{AB'} \delta_B{}^C \delta_{A'}{}^{C'} + \Upsilon_{BA'} \delta_A{}^C \delta_{B'}{}^{C'} \right) \nabla_c f \\
&= T_{ab}{}^c \widehat{\nabla}_c f
\end{aligned}$$

so that the torsion tensors of $\widehat{\nabla}$ and ∇ are identical. Since ∇ is torsion free, we have $\widehat{\nabla}$ is as well.

We have that $\widehat{\nabla}$ preserves $\widehat{\varepsilon}$ and $\widehat{\omega}$, and therefore preserves a scale on M as well. In order to identify the related volume form, we use the isomorphism

$$\Lambda^{4k}(T^*M \otimes \mathbb{C}) \cong \Lambda^{4k}(\mathbf{E} \otimes \mathbf{H}) \cong \left(\Lambda^{2k}\mathbf{E}\right)^{\otimes 2} \otimes \left(\Lambda^2 \mathbf{H}\right)^{\otimes 2k}$$

to observe that the $4k$-form $\varepsilon^2 \otimes \omega^{2k}$ is parallel with respect to the Levi-Civita connection, and therefore we have $\mu = \varepsilon^2 \otimes \omega^{2k}$ by the uniqueness in Proposition 4.2. Then the volume form

$$\widehat{\mu} = \widehat{\varepsilon}^2 \otimes \widehat{\omega}^{2k} = (f^2 \varepsilon^2) \otimes (f^{2k} \omega^{2k}) = f^{2k+2} \varepsilon^2 \otimes \omega^{2k} = f^{2(k+1)} \mu$$

is evidently parallel with respect to $\widehat{\nabla}$, and is a real multiple of the real form μ, hence is the real volume form preserved by $\widehat{\nabla}$ by uniqueness. □

The above lemma therefore allows us to recast the rescaling of a quaternionic manifold from the original setting described in Proposition 4.2 to the setting of rescaling volume forms on the related bundles $\mathbf{E}, \mathbf{H}$. Note this introduces some ambiguity in terminology, which we will address by calling the choice of ε, ω a choice of *paraconformal scale*, while calling the choice of μ on M a choice of *quaternionic scale*. These notions correspond, as we see that a scale on a quaternionic manifold determines a paraconformal scale and vice-versa. However, changing the paraconformal scale by a factor of f has the effect of changing the quaternionic scale by $f^{2(k+1)}$.

Next, we can reinterpret the identification between sections of $\mathcal{Q}$ and real sections of the bundle $S^2\mathbf{H}$ in Proposition 3.2 in terms of paraconformal structures. This identification depends on the choice of quaternionic connection, while Lemma 4.14 gives that a choice of quaternionic connection can be considered as a choice of a paraconformal scale. The appropriate identification is then formalized using the so-called *paraconformal weight* bundles (see [5]).

Let $\mathcal{P}[-1]$ denote the bundle $\Lambda^2\mathbf{H}$. A quaternionic connection on M induces a connection on $\Lambda^2\mathbf{H}$ as well as a non-vanishing section $\omega_{B'C'}$ of this bundle that is parallel with respect to this connection, which we can also view as a chosen trivialization of the line bundle. This in turn induces trivializations of the tensor power bundles $\mathcal{P}[-k] = (\mathcal{P}[-1])^{\otimes k}$, and again the trivializations will be parallel with respect to the induced connections on the bundles. Since $\omega_{B'C'}$ also yields an isomorphism between $\mathbf{H} \cong \mathbf{H}^*$, we also have a parallel section $\omega^{B'C'}$ of $\mathcal{P}[1] = \Lambda^2\mathbf{H}^*$, with similar preferred trivializations for the tensor power bundles $\mathcal{P}[k] = (\mathcal{P}[1])^{\otimes k}$. If we consider $\omega_{B'C'}$ and $\omega^{B'C'}$ as elements of $\mathbf{H} \otimes \mathbf{H}$ and $\mathbf{H}^* \otimes \mathbf{H}^*$, respectively, using the identifications of Proposition 3.2, then we have

$$\omega^{B'C'}\omega_{B'C'} = 2$$

Whether k is positive or negative, we call $\mathcal{P}[k]$ the bundle of *paraconformal weight k*.

More generally, the isomorphism $\mathbf{H} \cong \mathbf{H}^*$ given in Proposition 3.2 can be written in abstract index notation as mapping $\nu_{A'}$, a section of $\mathbf{H}$, to

$$\nu_{A'} \mapsto \omega^{B'A'}\nu_{A'} = \nu^{B'}$$

Therefore we have a way to raise and lower primed indices, but at the cost of changing the paraconformal weight. Raising an index, as we have done above, adds one to the paraconformal weight. Put another way, we have that a connection induces an isomorphism $\mathbf{H}^* \cong \mathbf{H} \otimes \mathcal{P}[1]$. We can also lower indices, via

$$\sigma^{B'} \mapsto \sigma^{B'}\omega_{B'A'} = \sigma_{B'},$$

which subtracts one from the paraconformal weight and gives an identification $\mathbf{H} \cong \mathbf{H}^* \otimes \mathcal{P}[-1]$. Raising and lowering indices also agrees with the correspondence between $\omega = \omega^{B'A'}$ considered as a section of $\Lambda^2\mathbf{H}^* = \mathcal{P}[1]$ and $\omega = \omega_{B'A'}$ considered as a section of $\Lambda^2\mathbf{H} = \mathcal{P}[-1]$, in that we can obtain the latter from the former by lowering two indices, which subtracts 2 from the paraconformal weight.

We note that when we use this method to raise a single index to yield $\omega_{B'}{}^{C'} = \omega^{C'A'}\omega_{B'A'}$, the resulting endomorphism of $\mathbf{H}^*$ is given by

$$\nu^{C'} \mapsto \nu^{B'}\omega_{B'}{}^{C'} = \nu^{B'}\omega^{C'A'}\omega_{B'A'} = \omega^{C'A'}\nu_{A'} = \nu^{C'},$$

that is, $\omega_{B'}{}^{C'} = \delta_{B'}{}^{C'}$ is the identity.

Using paraconformal weight bundles, the identification of $S^2H \cong \mathcal{Q}$ for quaternionic manifolds is equivalent to raising an index. If $\varphi_{B'C'} = \varphi_{(B'C')}$ denotes a section of $S^2\mathbf{H}$, then the related endomorphism of TM described in Proposition 3.2 is obtained by raising the second index. That is, the endomorphism $J_b{}^c$ associated to $\varphi_{B'C'}$ is

$$J_b{}^c = \varphi_{B'}{}^{C'}\delta_B{}^C = \omega^{C'A'}\varphi_{B'A'}\delta_B{}^C$$

where on the right-hand-side φ with the raised index is then a quantity with paraconformal weight 1. The norm on $S^2\mathbf{H}$ as defined in equation (8) also depends on the isomorphism $\mathbf{H} \cong \mathbf{H}^*$. In particular, given $\varphi_{B'C'}$, the inner product of φ with itself is given by

$$\varphi^{B'C'}\varphi_{B'C'} = \omega^{B'D'}\omega^{C'E'}\varphi_{D'E'}\varphi_{B'C'}$$

so that the squared norm of a section of $S^2\mathbf{H}$ is a quantity of paraconformal weight 2.

Note that in our discussion of the isomorphism $S^2H \cong \mathcal{Q}$ in Proposition 3.2 and following, we observed that the real sections of $S^2\mathbf{H}$ with norm $\sqrt{2}$ were the sections that corresponded to almost complex structures on the quaternionic manifold. This can also be observed in the abstract index notation, without choosing a local frame for $\mathbf{H}$.

Lemma 4.15. *Let $\alpha_{A'B'} = \alpha_{[A'B']}$ be skew symmetric. Then with respect to any choice of scale $\omega_{A'B'}$, we have*

$$\alpha_{A'B'} = \frac{1}{2}\alpha_{C'}{}^{C'}\omega_{A'B'} \tag{25}$$

Proof. Since $\mathbf{H}$ is two-dimensional, a skew 2-form on $\mathbf{H}$ must be a multiple of ω, and one simply needs to find this multiple. Writing $\alpha_{A'B'} = f\omega_{A'B'}$ for some function f to be determined, we can contract by $\omega^{A'B'}$ on both sides to obtain

$$\begin{aligned}
\omega^{A'B'}\alpha_{A'B'} &= f\omega^{A'B'}\omega_{A'B'} \\
\alpha_{A'}{}^{A'} &= f\omega_{A'}{}^{A'} = f\delta_{A'}{}^{A'} = 2f \\
f &= \frac{1}{2}\alpha_{A'}{}^{A'}
\end{aligned}$$

and therefore changing the dummy index gives the formula above. □

Lemma 4.16. *Let $\varphi_{B'C'}$ be a section of $S^2\mathbf{H}$, and assume the scale ω is such that $\varphi^{B'C'}\varphi_{B'C'} = 2$. Then*

$$\varphi_{A'B'}\varphi^{B'}{}_{C'} = \omega_{A'C'},$$

so raising an index yields

$$\varphi_{A'B'}\varphi^{B'C'} = \omega_{A'}{}^{C'} = \delta_{A'}{}^{C'}$$

Proof. Considering the first expression, we have

$$\varphi_{A'B'}\varphi^{B'}{}_{C'} = \varphi_{A'B'}\omega^{B'D'}\varphi_{D'C'}$$

We see that this expression is symmetric in the pairs of indices $A'B'$ and $D'C'$, so the fact that ω is skew in B', D' implies that $\varphi_{A'B'}\varphi_{D'C'}$ is also skew in A', C'. Therefore equation (25) gives

$$\varphi_{A'B'}\omega^{B'D'}\varphi_{D'C'} = \frac{1}{2}\varphi_{E'B'}\omega^{B'D'}\varphi_{D'}{}^{E'}\omega_{A'C'} = \frac{1}{2}\varphi_{E'B'}\varphi^{B'E'}\omega_{A'C'} = \omega_{A'C'}$$

as desired. □

Proposition 4.17. *Let $\varphi_{B'C'}$ be a section of $S^2\mathbf{H}$, and assume the scale ω is such that $\varphi^{B'C'}\varphi_{B'C'} = 2$. Then the tensor $J_b{}^c = \varphi_{B'}{}^{C'}\delta_B{}^C$ is an almost complex structure in the sense that $J_a{}^b J_b{}^c = -\delta_a{}^c$.*

Proof. We have

$$\begin{aligned} J_a{}^b J_b{}^c &= \varphi_{A'}{}^{B'}\delta_A{}^B\varphi_{B'}{}^{C'}\delta_B{}^C = \varphi_{A'}{}^{B'}\varphi_{B'}{}^{C'}\delta_A{}^C \\ &= -\varphi_{A'B'}\varphi^{B'C'}\delta_A{}^C = -\delta_{A'}{}^{C'}\delta_A{}^C = -\delta_a{}^c \end{aligned}$$

using Lemma 4.16. □

If $\varphi \in \Gamma(S^2\mathbf{H})$ is a real section and is parallel with respect to a given connection, then it will have constant length in the scale associated to that connection and therefore, by rescaling by a constant if necessary, we have that the complex structure related to φ by Proposition 4.17 will be parallel as well. Conversely, if φ is an arbitrary real section, then we can choose a scale in which φ will have constant length $\sqrt{2}$, see equation (26) in the second proof of Theorem 4.13 below. We will still need an extra condition, however, to ensure that this section is also parallel. As we have already seen in the earlier proof of Theorem 4.13, this extra condition is the vanishing of the section under the twistor operator, which has a particularly simple expression in abstract index notation as

$$D_{AA'}\varphi_{B'C'} = \nabla_{A(A'}\varphi_{B'C')}$$

That is, if φ has constant norm with respect to a chosen scale and is also in the kernel of the twistor operator associated to that scale, then in fact φ is parallel with respect to the connection associated to that scale.

Proposition 4.18. *Let $\varphi_{B'C'}$ be a section of $S^2\mathbf{H}$, and assume the scale ω is such that $\varphi^{B'C'}\varphi_{B'C'} = 2$. Then if $D_{AA'}\varphi_{B'C'} = 0$ we in fact have that $\nabla_{AA'}\varphi_{B'C'} = 0$.*

Proof. Since $\varphi^{B'C'}\varphi_{B'C'} = 2$, taking the derivative yields

$$\begin{aligned} 0 &= \nabla_{AA'}(\varphi^{B'C'}\varphi_{B'C'}) = (\nabla_{AA'}\varphi^{B'C'})\varphi_{B'C'} + \varphi^{B'C'}(\nabla_{AA'}\varphi_{B'C'}) \\ &= 2(\nabla_{AA'}\varphi_{B'C'})\varphi^{B'C'} \end{aligned}$$

by raising and lowering indices on the first term and rearranging

Taking the expression $D_{AA'}\varphi_{B'C'} = 0$ and contracting it with the form $\varphi^{A'B'}$ yields

$$\begin{aligned} 0 &= \left(\nabla_{AA'}\varphi_{B'C'} + \nabla_{AB'}\varphi_{C'A'} + \nabla_{AC'}\varphi_{A'B'}\right)\varphi^{A'B'} \\ &= \left(\nabla_{AA'}\varphi_{B'C'}\right)\varphi^{A'B'} + \left(\nabla_{AB'}\varphi_{C'A'}\right)\varphi^{A'B'} \\ &= \nabla_{AA'}(\varphi_{B'C'}\varphi^{A'B'}) - \varphi_{B'C'}\nabla_{AA'}\varphi^{A'B'} + \nabla_{AB'}(\varphi_{C'A'}\varphi^{A'B'}) \\ &\quad - \varphi_{C'A'}\nabla_{AB'}\varphi^{A'B'} \\ &= -\varphi_{B'C'}\nabla_{AA'}\varphi^{A'B'} - \varphi_{C'A'}\nabla_{AB'}\varphi^{A'B'} \end{aligned}$$

using Lemma 4.16. We can further rewrite this expression by changing dummy indices, which yields

$$\begin{aligned} 0 &= \varphi_{B'C'}\nabla_{AA'}\varphi^{A'B'} + \varphi_{C'A'}\nabla_{AB'}\varphi^{A'B'} \\ &= \varphi_{D'C'}\nabla_{AE'}\varphi^{E'D'} + \varphi_{C'D'}\nabla_{AE'}\varphi^{D'E'} \\ &= 2\varphi_{D'C'}\nabla_{AE'}\varphi^{E'D'} \end{aligned}$$

using the fact that φ is symmetric. We can then contract with $\varphi^{B'C'}$ to obtain

$$0 = \varphi^{B'C'}\varphi_{D'C'}\nabla_{AE'}\varphi^{E'D'} = \omega^{B'}{}_{D'}\nabla_{AE'}\varphi^{E'D'},$$

again using Lemma 4.16, and therefore

$$\nabla_{AE'}\varphi^{E'D'} = 0$$

since $\omega^{B'}{}_{D'} = -\delta^{B'}{}_{D'}$ is nondegenerate.

We now consider the expression $\nabla_{A[A'}\varphi_{B']C'}$. Since this expression is skew-symmetric in the indices A', B', this expression can be rewritten in terms of the element $\omega_{A'B'}$, so that

$$\nabla_{A[A'}\varphi_{B']C'} = \frac{1}{2}\nabla_{AD'}\varphi^{D'}{}_{C'}\omega_{A'B'} = \frac{1}{2}\nabla_{AD'}\varphi^{D'E'}\omega_{E'C'}\omega_{A'B'} = 0$$

by the above.

We therefore have that the expression $\nabla_{AA'}\varphi_{B'C'}$ is symmetric in A' and B', and we already have that it is symmetric in B' and C'. It is therefore equal to its own symmetrization on the indices A', B', C', that is,

$$\nabla_{AA'}\varphi_{B'C'} = \nabla_{A(A'}\varphi_{B'C')} = D_{AA'}\varphi_{B'C'} = 0$$

as required. □

Proposition 4.18 shows that in order to produce a quaternionic complex structure from a given section $\varphi \in S^2\mathbf{H}$, we need to be able to choose a paraconformal scale in which φ both has constant length and is also in the kernel of the twistor operator. The difficulty is that the twistor operator is dependent on the connection, and therefore dependent on the choice of paraconformal scale.

However, the twistor operator will be paraconformally invariant if we give it a proper paraconformal weight. To see this, recall from the definition of the paraconformal weight bundles that the choice of a quaternionic connection gives the bundles $\mathcal{P}[k]$ preferred trivializations that are parallel with respect to the induced connection, via the form $\omega_{B'C}$. Changing the paraconformal scale by a factor f then leads to new preferred trivializations $\widehat{\omega}_{B'C'} = f\omega_{B'C}$ for $\mathcal{P}[-1] = \Lambda^2\mathbf{H}$ and $\widehat{\omega}^{B'C'} = f^{-1}\omega^{B'C'}$ for $\mathcal{P}[1] = \Lambda^2\mathbf{H}^*$. Note the factors are such that

$$\widehat{\omega}^{B'C'}\widehat{\omega}_{B'C'} = f^{-1}\omega^{B'C'}f\omega_{B'C} = \omega^{B'C'}\omega_{B'C} = 2,$$

and

$$\widehat{\omega}_{B'}{}^{C'} = \widehat{\omega}^{C'A'}\widehat{\omega}_{B'A'} = f^{-1}\omega^{C'A'}f\omega_{B'A'} = \delta_{B'}{}^{C'}$$

that is, the special properties of $\omega_{B'C'}$ when raising and lowering indices are independent of paraconformal scale.

We can then observe how the induced connections on the paraconformal weight bundles change as we vary the paraconformal scale.

Lemma 4.19. *Let σ be a section of $\mathcal{P}[k]$. Then, under a change of paraconformal scale by a factor of f as in Lemma 4.14, we have that*

$$\widehat{\nabla}_a \sigma = \nabla_a \sigma + k \Upsilon_a \sigma$$

Proof. We can first observe that the proposition holds for $\omega_{B'C'}$, the preferred section of $\mathcal{P}[-1] = \Lambda^2 \mathbf{H}$ with respect to the connection ∇. Considering the right-hand side of the desired equation, we have

$$\nabla_a \omega_{B'C'} - \Upsilon_a \omega_{B'C'} = f^{-1} \left(\nabla_a f \right) \omega_{B'C'}$$

since ω is parallel with respect to ∇. On the other hand, the fact that $\widehat{\omega}_{B'C'}$ is parallel with respect to $\widehat{\nabla}$ gives that

$$\begin{aligned} \widehat{\nabla}_a \omega_{B'C'} &= \widehat{\nabla}_a \left(f^{-1} f \omega_{B'C'} \right) \\ &= \left(\widehat{\nabla}_a f^{-1} \right) f \omega_{B'C'} + f^{-1} \widehat{\nabla}_a \left(f \omega_{B'C'} \right) \\ &= -f^{-2} \left(\nabla_a f \right) f \omega_{B'C'} = -f^{-1} \left(\nabla_a f \right) \omega_{B'C'}, \end{aligned}$$

giving the desired equality when $\sigma = \omega_{B'C'}$. Then the Leibniz rule implies that the proposition holds for general sections of $\mathcal{P}[-1]$, and the general formula follows from considering tensor powers. □

Proposition 4.20. *Let $(M, \mathcal{Q}, \nabla, \mu)$ be a scaled quaternionic manifold. Then the twistor operator is scale-invariant when acting on sections of $S^2\mathbf{H}$ with paraconformal weight 2, that is, on sections of the bundle $S^2\mathbf{H} \otimes \mathcal{P}[2]$.*

Proof. Let $f > 0$ be the paraconformal scale factor and write $\Upsilon_a = f^{-1} \nabla_a f$ as usual. Then just as we did for the change of connection on the bundle $\Lambda^2 \mathbf{H}$ in the proof of Lemma 4.14, we can express a section $\varphi_{B'C'} = \varphi_{(B'C')}$ of $S^2\mathbf{H}$ as $\sigma_{B'} \nu_{C'} + \sigma_{C'} \nu_{B'}$, and use the change of connection formulas to derive the formula

$$\widehat{\nabla}_{AA'} \varphi_{B'C'} = \nabla_{AA'} \varphi_{B'C'} - \Upsilon_{AC'} \varphi_{B'A'} - \Upsilon_{AB'} \varphi_{C'A'}$$

for the change of connection on $S^2\mathbf{H}$ under paraconformal rescaling.

If we now consider $\varphi_{B'C'}$ as a quantity with paraconformal weight 2 we instead have

$$\widehat{\nabla}_{AA'} \varphi_{B'C'} = \nabla_{AA'} \varphi_{B'C'} - \Upsilon_{AC'} \varphi_{B'A'} - \Upsilon_{AB'} \varphi_{C'A'} + 2\Upsilon_{AA'} \varphi_{B'C'}$$

by Lemma 4.19.

If we symmetrize on A', B', C', we therefore have

$$\begin{aligned}\widehat{D}_{AA'}\varphi_{B'C'} &= D_{AA'}\varphi_{B'C'} - \Upsilon_{A(C'}\varphi_{B'A')} - \Upsilon_{A(B'}\varphi_{C'A')} + 2\Upsilon_{A(A'}\varphi_{B'C')} \\ &= D_{AA'}\varphi_{B'C'} - \Upsilon_{A(A'}\varphi_{B'C')} - \Upsilon_{A(A'}\varphi_{B'C')} + 2\Upsilon_{A(A'}\varphi_{B'C')} \\ &= D_{AA'}\varphi_{B'C'}\end{aligned}$$

so that the twistor operator is scale-invariant when it is considered with this paraconformal weight. □

Using this, we can present a second proof of Theorem 4.13, which we reproduce here for convenience.

Theorem 4.13. *Let $(M, \mathcal{Q}, \nabla)$ be a quaternionic manifold. Assume that $\varphi \in S^2\mathbf{H}$ is a real section with $D\varphi = 0$.*

Then there exists a unique quaternionic connection $\widehat{\nabla}$ on the open, dense submanifold $M_0 = M - \{\varphi = 0\}$ preserving the almost complex structure associated to φ, that is, the connection $\widehat{\nabla}$ has holonomy contained in $GL(k, \mathbb{H})U(1)$.

Proof. Let $\omega_{B'C}$ be the form associated to the connection ∇. Using this, we can raise indices to consider $\varphi^{B'C'}$ as an element of $S^2\mathbf{H}^*$ using the identification $S^2\mathbf{H} \otimes \mathcal{P}[2] \cong_\omega S^2\mathbf{H}^*$. We also let $f = \varphi^{B'C'}\varphi_{B'C'}$ be the squared norm of φ induced by $\omega_{B'C'}$.

Let $\widehat{\nabla}$ be the connection obtained on M_0 by changing the paraconformal scale on the $\mathbf{H}$ bundle by a factor of $(f/2)^{-1/2}$. The form $\widehat{\omega}_{B'C'} = (f/2)^{-1/2}\omega_{B'C'}$ is then parallel with respect to this form, and this form also induces a new isomorphism $S^2\mathbf{H} \otimes \mathcal{P}[2] \cong_{\widehat{\omega}} S^2\mathbf{H}^*$. Let $\widehat{\varphi}_{B'C'}$ be the section of $S^2\mathbf{H}$ associated to $\varphi^{B'C'}$ under this new isomorphism. That is, we have $\widehat{\varphi}^{B'C'} = \varphi^{B'C'}$, with the important caveat that we raise indices of the first using $\widehat{\omega}$ and the indices of the second with ω, so that $\widehat{\varphi}_{B'C'} \neq \varphi_{B'C'}$.

Using Proposition 4.20, we have that $\widehat{D}_{AA'}\widehat{\varphi}_{B'C'} = 0$. We also have that $\widehat{\varphi}$ has norm $\sqrt{2}$ with respect to the inner product on $S^2\mathbf{H}$ induced by $\widehat{\nabla}$, as

$$\begin{aligned}\widehat{\varphi}^{B'C'}\widehat{\varphi}_{B'C'} &= \widehat{\varphi}^{B'C'}\widehat{\varphi}^{D'E'}\widehat{\omega}_{D'B'}\widehat{\omega}_{E'C'} \\ &= (f/2)^{-1}\varphi^{B'C'}\varphi^{D'E'}\omega_{D'B'}\omega_{E'C'} \\ &= 2f^{-1}\varphi^{B'C'}\varphi_{B'C'} = 2\end{aligned} \tag{26}$$

Therefore Proposition 4.18 gives that $\widehat{\nabla}\widehat{\varphi} = 0$, and so the almost complex structure associated to $\widehat{\varphi}$ via Proposition 4.17 is parallel with respect to $\widehat{\nabla}$, completing the proof. Once again we defer until Theorem 4.24 the proof that M_0 is dense. □

We note that if we assume further that the connection ∇ above is associated to a choice of quaternionic scale, then in the proof above we are changing the paraconformal scale by a factor of $(f/2)^{-1/2}$. This means we are changing the quaternionic scale by $(f/2)^{-(k+1)}$ from Lemma 4.14, and so by equation (16) we have that the new connection arises from choosing the 1-form

$$\Upsilon = \frac{1}{2(k+1)} d\log\left((f/2)^{-(k+1)}\right) = \frac{-1}{2} d\log(f/2) = -\frac{1}{2} d\log f = -d\log(\sqrt{f})$$

Note that this gives a new interpretation of the change of connection obtained by Hitchin in Theorem 4.8. In that case, the connection arose from the momentum section $\rho = \rho_1\omega_I$ by choosing $\Upsilon = -d\log\rho_1$ (see equation (21)). But the norm of the momentum section is exactly $\sqrt{\rho_1^2} = \rho_1$. Thus Hitchin's construction can be more generally interpreted as picking the scale for which the momentum section has norm $\sqrt{2}$ in that scale.

We end by considering the what occurs in the quaternion-Kähler case. In this case, we have that the bundle E admits a form $\omega_E \in \Gamma(\Lambda^2\mathbf{E})$ as well, with $\nabla\omega_E = 0$ for the Levi-Civita connection. This form has the property that $\omega_E^{\wedge k} = \varepsilon$, where ε is the volume form on $\mathbf{E}$ associated to the scale for the quaternion-Kähler structure.

Then, given a real section $\varphi_{B'C'} \in \Gamma(S^2\mathbf{H})$ in the kernel of the twistor operator, the above theorem gives that we can change the paraconformal scale to yield a parallel complex structure on M_0. Under this change of scale, we produce new forms $\widehat{\omega}$ and $\widehat{\varepsilon}$ as in Lemma 4.14.

Our ultimate goal is to compare the Hitchin and Haydys constructions. So far, we have shown that the Hitchin construction in general only produces non-metric $SL(k,\mathbb{H})Sp(1)$ connections in the quaternion-Kähler case, but the Haydys construction produces metrics. In attempting to determine if the Haydys and Hitchin constructions coincide, it is necessary to see if the Hitchin construction does in fact produce a metric, which would require the new connection to preserve a skew-symmetric form on $\mathbf{E}^*$ analogous to ω_E, which would give a metric via equation (10). This problem is what inspired our use of paraconformal structures to study the Hitchin construction.

Since the new form $\widehat{\omega}_E$ should also be compatible with the new volume form $\widehat{\varepsilon} = f\varepsilon = f\omega_E^{\wedge k}$ under a change of paraconformal scale, the most obvious candidate for the form $\widehat{\omega}_E$ is $f^{1/k}\omega_E$. However, we can see that this form is not parallel with respect to the new connection $\widehat{\nabla}$ obtained from paraconformal rescaling by f.

To avoid confusion in the abstract index notation, we denote the form ω_E

by $\widetilde{\omega} = \widetilde{\omega}_{AB} = \widetilde{\omega}_{[AB]}$. We then have

$$\begin{aligned}\widehat{\nabla}_{AA'}(f^{1/k}\widetilde{\omega}_{BC}) &= \frac{1}{k}f^{(1/k)-1}\left(\widehat{\nabla}_{AA'}f\right)\widetilde{\omega}_{BC} + f^{1/k}\widehat{\nabla}_{AA'}\widetilde{\omega}_{BC} \\ &= \frac{1}{k}f^{1/k}\Upsilon_{AA'}\widetilde{\omega}_{BC} + f^{1/k}\left(\nabla_{AA'}\widetilde{\omega}_{BC} + 2\Upsilon_{A'[B}\widetilde{\omega}_{C]A}\right) \\ &= f^{1/k}\left(\frac{1}{k}\Upsilon_{A'A}\widetilde{\omega}_{BC} + \Upsilon_{A'B}\widetilde{\omega}_{CA} - \Upsilon_{A'C}\widetilde{\omega}_{BA}\right)\end{aligned}$$

Note that we use the expression of the new connection $\widehat{\nabla}$ in terms of the old connection ∇ obtained from equation (24) by exchanging primed and unprimed indices.

Evidently we cannot expect this expression to vanish for $k \geq 2$. In that case, the expression we obtain is essentially a section of $\mathbf{H} \otimes \Lambda^3\mathbf{E}$, but if $k \geq 2$ then the bundle $\Lambda^3\mathbf{E}$ has nonzero rank and so we cannot expect the right-hand side to be zero (c.f. the proof of Lemma 4.14). This does explain, however, the existence of the metric in the case of quaternionic dimension $k = 1$ in Hitchin's construction. For if $k = 1$ we do have $\Lambda^3\mathbf{E} = 0$, and the right-hand side is exactly $f\Upsilon_{A'[A}\widetilde{\omega}_{BC]} = 0$, so that there is a parallel skew form on $\mathbf{E}$.

It may still be possible that there are other choices for a parallel skew 2-form on $\mathbf{E}$ that yield a compatible metric structure in particular cases. However, these forms will not be multiples of the original form ω_E arising from the quaternion-Kähler structure, and so it is unclear how such a structure could be found. Even if such a metric were found, the metric would be different from the one obtained by Haydys, as we shall see in Section 5.4.

4.5 Twistorial Interpretation of Quaternionic Complex Manifolds

The constructions of Sections 4.2, 4.3, and 4.4 provide sufficient conditions for finding complex structures satisfying the hypothesis of Lemma 4.5 and therefore yield quaternionic complex manifolds. However, we still need methods to produce either Killing fields on a quaternion-Kähler manifold or real sections of $S^2\mathbf{H}$ that vanish under the twistor operator if these constructions are to be useful. This is actually fairly straightforward, as both can be produced using the holomorphic geometry of the twistor space using well-known methods.

Lemma 4.21 ([40]). *Let $(M, \mathcal{Q}, \nabla)$ be a quaternionic manifold. Then the kernel of the twistor operator is isomorphic to the space $H^0(Z, \mathcal{O}(L^2))$ of holomorphic sections of the bundle L^2.*

Proof. If we fix $x \in M$ and consider the fiber $p^{-1}(x) \subset Z$ as $\mathbb{P}(\mathbf{H}_x)$, then recall that the bundle L^2 when restricted to $p^{-1}(x)$ is isomorphic to $\mathcal{O}(2)$ so that $H^0(p^{-1}(x), \mathcal{O}(L^2)) \cong H^0(\mathbb{CP}_1, \mathcal{O}(2))$. Moreover, the standard description of holomorphic sections of $\mathcal{O}(2)$ gives that we can consider $H^0(\mathbb{CP}_1, \mathcal{O}(2))$ as the space of homogeneous polynomials of degree 2 acting on $\mathbf{H}_x$, which is itself identified with $S^2\mathbf{H}_x^*$. Now the chosen quaternionic connection on M induces a section $\omega_H \in \Lambda^2\mathbf{H}^*$ that yields an isomorphism $\mathbf{H} \cong \mathbf{H}^*$ and therefore an isomorphism $S^2\mathbf{H}_x \cong S^2\mathbf{H}_x^*$.

Putting all of this together, the quaternionic connection on M induces an isomorphism

$$H^0(p^{-1}(x), \mathcal{O}(L^2)) \cong S^2\mathbf{H}_x$$

for all x in M. We therefore have an induced homomorphism

$$\varphi : H^0(Z, \mathcal{O}(L^2)) \to \Gamma(S^2\mathbf{H}) \tag{27}$$

that arises from restricting sections to each twistor fiber. We claim that the image of this homomorphism is the kernel of the twistor operator.

To see this, we can choose as in Section 3.1 a local frame $h_1, h_2 = J_H h_1$ of $\mathbf{H}$ such that $\omega_H(h_1, h_2) = 1$, and let z_1, z_2 be the related coordinate functions on $\mathbf{H}$ so that an element of $\mathbf{H}$ in the fiber over $x \in M$ can be expressed as $z_1 h_1(x) + z_2 h_2(x)$. In order to simplify the computations, we also can assume without loss of generality that $\nabla h_1|_x = 0 = \nabla h_2|_x$ at a fixed point where we make all our computations.

Recalling that $L^{-1} \subset p^*\mathbf{H}$ is the tautological bundle for the projectivization $\mathbf{H} - \{0\} \to \mathbb{P}(\mathbf{H})$, we have that a smooth section of $s \in \Gamma(L^2)$ can be considered as a homogeneous degree 2 polynomial defined on $\mathbf{H} - \{0\}$ of the form

$$\widehat{s} = s_1 z_1^2 + 2 s_2 z_1 z_2 + s_3 z_2^2$$

where s_1, s_2, s_3 are smooth functions on M. Moreover the section $s \in \Gamma(L^2)$ is holomorphic when the related function $\widehat{s}$ defined on $\mathbf{H}^\times$ is holomorphic with respect to the complex structure defined in Proposition 3.4. More specifically, if we consider the decomposition of the forms ds_1, ds_2, ds_3 into $\mathbf{E}, \mathbf{H}$ parts as $ds_i = e_i^1 h_1 + e_i^2 h_2$, then the 1-form $d\widehat{s}$ can be expressed as

$$\begin{aligned} d\widehat{s} &= z_1^2 ds_1 + 2 s_1 z_1 dz_1 + 2 z_1 z_2 ds_2 + 2 s_2 z_1 dz_2 + 2 s_2 z_2 dz_1 + z_2^2 ds_3 + 2 s_3 z_2 dz_2 \\ &= z_1^2 (e_1^1 h_1 + e_1^2 h_2) + 2 z_1 z_2 (e_2^1 h_1 + e_2^2 h_2) + z_2^2 (e_3^1 h_1 + e_3^2 h_2) \\ &\quad + 2(s_1 z_1 + s_2 z_2) dz_1 + 2(s_2 z_1 + s_3 z_2) dz_2 \\ &= (z_1^2 e_1^1 + 2 z_1 z_2 e_2^1 + z_2^2 e_3^1) h_1 + (z_1^2 e_1^2 + 2 z_1 z_2 e_2^2 + z_2^2 e_3^2) h_2 \\ &\quad + 2(s_1 z_1 + s_2 z_2) dz_1 + 2(s_2 z_1 + s_3 z_2) dz_2 \end{aligned}$$

We have assumed for simplicity that, over $x \in M$, the covariant derivatives of h_1, h_2 vanish. Since the complex structure on $\mathbf{H}^\times$ is defined to be the complex structure on $\mathbf{H}_x$ in the vertical direction, this gives that the vertical fiber coordinates z_1, z_2 are holomorphic with respect to the complex structure. Therefore the dz_1, dz_2 parts of $d\widehat{s}$ above are of type $(1,0)$. Then $d\widehat{s}$ is a holomorphic form precisely when the remaining terms also give a $(1,0)$ form.

For this, we see from equation (11) that $(1,0)$ forms in the horizontal direction at $h = z_1 h_1(x) + z_2 h_2(x) \in \mathbf{H}$ are precisely those that can be expressed as $e \otimes h$ for some $e \in \mathbf{E}_x$, and so we have that the remaining part terms in $d\widehat{s}$ will be of type $(1,0)$ if and only if they can be combined into the form $e \otimes (z_1 h + z_2 h_2)$. which requires

$$z_1 e_1^1 + 2z_2 e_2^1 + \frac{z_2^2}{z_1} e_3^1 = \frac{z_1^2}{z_2} e_1^2 + 2z_1 e_2^2 + z_2 e_3^2$$

that is,

$$e_1^1 = 2e_2^2 \qquad e_3^2 = 2e_2^1 \qquad e_3^1 = e_1^2 = 0 \tag{28}$$

If s is any general section as described above, we have that $\varphi(s)$ is the section

$$\varphi(s) = s_1(h_2 \otimes h_2) - s_2(h_1 \otimes h_2 + h_2 \otimes h_1) + s_3(h_1 \otimes h_1) \tag{29}$$

Again computing at $x \in M$ where we assume the covariant derivatives of h_1, h_2 vanish, we have that

$$\begin{aligned} \nabla\varphi(s)|_x = (e_1^1 h_1 + e_1^2 h_2) h_2 \otimes h_2 - (e_2^1 h_1 + e_2^2 h_2)(h_1 \otimes h_2 + h_2 \otimes h_1) \\ + (e_3^1 h_1 + e_3^2 h_2)(h_1 \otimes h_1) \end{aligned}$$

Gathering terms and symmetrizing on the h terms yields

$$\begin{aligned} D\varphi(s)|_x = e_3^1 h_1 h_1 h_1 + (e_3^2 - 2e_2^1) h_1 \odot h_1 \odot h_2 \\ + (e_1^1 - 2e_2^2) h_1 \odot h_2 \odot h_2 + e_1^2 h_2 h_2 h_2 \end{aligned}$$

and we observe that the condition that $D\varphi(s) = 0$ is equivalent to the equations (28) necessary for $\widehat{s}$ to be holomorphic. □

We note that as originally stated by Salamon in [40] this lemma had the additional assumtion that M is a quaternion-Kähler, although the proof given there, which is identical to the proof given above except for some changes in notation, clearly generalizes to the quaternionic case. Salamon's work also proved a related lemma that shows that if (M, g) is a quaternion-Kähler manifold, then the kernel of the twistor operator is isomorphic to the vector space of Killing fields on M. It was these lemmas of Salamon's that lead to the generalization of Hitchin's work obtained in the previous sections, as they imply that the Killing fields considered by Hitchin can be reinterpreted as sections in the kernel of the twistor operator.

The crucial point of the above lemma is that the isomorphism φ of equation (27) depends on the form ω_H, and therefore on the quaternionic connection on M, even though the holomorphic structure of Z and therefore the space of sections $H^0(Z, \mathcal{O}(L^2))$ are both independent of the choice of connection. We will write $\varphi = \varphi^\nabla$ when we need to make the dependence of this map on ∇ explicit. Similarly, the identification of $S^2\mathbf{H}$ with the complexification of $\mathcal{Q}$ also depends on ω_H, so that given a holomorphic section of L^2 there are many possible sections of $\mathcal{Q}$ that are associated to it, varying depending on the choice of quaternionic connection on M.

Our earlier work then implies that if we have a section of $H^0(Z, \mathcal{O}(L^2))$ that corresponds to a *real* section of $S^2\mathbf{H}$ under some choice of connection, then we should be able to choose a suitable quaternionic connection so that this section can be identified with a complex structure on M that is parallel with respect to that connection. In this direction we can first observe that the requirement that a section of $H^0(Z, \mathcal{O}(L^2))$ correspond to a real section of $S^2\mathbf{H}$ is independent of the choice of connection used to identify S^2H and $H^0(Z, \mathcal{O}(L^2))$.

Proposition 4.22. *Let $s \in \Gamma(L^2)$ be any section, and assume that there is a choice of quaternionic connection ∇ on M so that $\varphi^\nabla(s) \in \Gamma(S^2H)$ is a real section. Then $\varphi^{\widehat{\nabla}}(s)$ is a real section for any other quaternionic connection $\widehat{\nabla}$ on M. In this case, we say that the section s is a* real section *of L^2.*

Proof. Let s be given and fix any choice of connection ∇. This yields a local frame h_1, h_2 for $\mathbf{H}$ in the usual way, and allows us to express $\varphi^\nabla(s)$ in coordinates using equation (29). The real structure on $S^2\mathbf{H}$ is induced from the quaternionic action $h_1 \mapsto h_2, h_2 \mapsto -h_1$, so we have that $\varphi^\nabla(s)$ and its conjugate are given by

$$\begin{aligned}\varphi^\nabla(s) &= s_1(h_2 \otimes h_2) - s_2(h_1 \otimes h_2 + h_2 \otimes h_1) + s_3(h_1 \otimes h_1)\\ \overline{\varphi^\nabla(s)} &= \overline{s_1}(h_1 \otimes h_1) + \overline{s_2}(h_2 \otimes h_1 + h_1 \otimes h_1) + \overline{s_3}(h_2 \otimes h_2),\end{aligned}$$

so that $\varphi^\nabla(s)$ is real if and only if

$$s_1 = \overline{s_3} \qquad \overline{s_2} = -\overline{s_2}$$

Then assuming that $\varphi^\nabla(s)$ is real, let $\widehat{\nabla}$ be any other quaternionic connection. As we have seen in Lemma 4.14, this connection arises by a change of paraconformal scale, so that $\widehat{\nabla}$ is the connection that preserves that section $\widehat{\omega}_H = f\omega_H \in \Gamma(\Lambda^2\mathbf{H}^*)$ for f a non-vanishing, real-valued function. Taking the square root, we see that the new connection $\widehat{\nabla}$ gives $\widehat{h_1} = f^{-1/2}h_1, \widehat{h_2} = f^{-1/2}h_2$ as the basis for $\mathbf{H}$ compatible with $\widehat{\omega_H}$.

If we write $\widehat{s_1}, \widehat{s_2}, \widehat{s_3}$ as the coordinate functions for the section s with respect to this new basis, then evidently $\widehat{s_i} = fs_i$ for $i = 1, 2, 3$, and therefore if $\varphi^{\nabla}(s)$ is real then $\varphi^{\widehat{\nabla}}(s)$ is real as well. □

Since the reality of a section s of L^2 is independent of the choice of connection it can be defined purely in terms of the geometry of the twistor space Z.

Proposition 4.23. *A section $s \in \Gamma(L^2)$ is real if and only if the set $X = \{s = 0\}$ is preserved by the real structure $\sigma : Z \to Z$.*

Proof. Fix any quaternionic connection on M so that we can consider a local frame h_1, h_2 for $\mathbf{H}$ in the usual way and consider a section $s \in \Gamma(L^2)$ as a homogeneous degree 2 polynomial $\widehat{s}$ on $\mathbf{H}^{\times}$ with the expression $\widehat{s} = s_1z_1^2 + 2s_2z_1z_2 + s_3z_2^2$ as in the proof of Lemma 4.21.

The real structure σ on $Z = \mathbb{P}(\mathbf{H})$ is induced from the quaternionic structure on the bundle $\mathbf{H}$ that maps $h_1 \mapsto h_2$ and $h_2 \mapsto -h_1$. This action therefore pulls back to an action on sections of $\Gamma(L^2)$ considered as functions, so that

$$\sigma^*\widehat{s} = s_1\overline{z_2}^2 - 2s_2\overline{z_2}\,\overline{z_1} + s_3\overline{z_1}^2$$

Consider the zero set $0 = s_1z_1^2 + 2s_2z_1z_2 + s_3z_2^2$. If the zero set is invariant under the real structure, we then have that $0 = s_1\overline{z_2}^2 - 2s_2\overline{z_2z_1} + s_3\overline{z_1}^2$ as well. Taking the complex conjugate on both sides, we have that

$$0 = \overline{s_1}z_2^2 - 2\overline{s_2}z_1z_2 + \overline{s_3}z_1^2$$

as well. Comparing coefficients, we have that if the zero set is invariant under the real structure then $\overline{s_1} = s_3$ and $s_2 = -\overline{s_2}$, giving reality. By proposition 4.22 this result is independent of the choice of the quaternionic connection. Conversely, if $\overline{s_1} = s_3$ and $s_2 = -\overline{s_2}$, reversing the above argument shows that the zero set the section is invariant under the real structure. □

Theorem 4.24. *Let M be a quaternionic manifold with twistor space Z, and let $s \in H^0(Z, \mathcal{O}(L^2))$ be a real, nontrivial section.*

Then there exists a unique quaternionic connection $\widehat{\nabla}$ on an open, dense submanifold $M_0 \subset M$ which has holonomy contained in $GL(k, \mathbb{H})U(1)$.

Proof. By Lemma 4.21 and Proposition 4.22, we have that the section s corresponds to a real section of the bundle $S^2\mathbf{H}$ in the kernel of the twistor operator associated to some quaternionic connection on M. Therefore Theorem 4.13 gives the desired quaternionic complex structure, and all that remains is to show that the set M_0 is dense.

Let X denote the subset of Z on which s vanishes. Given any $x \in M$, we can consider the intersection $p^{-1}(x) \cap X$. Recalling that the restriction of L^2 to any real twistor line is isomorphic to the bundle $\mathcal{O}(2)$, we have that $X \cap p^{-1}(x)$ is the divisor corresponding to the vanishing of a section of $\mathcal{O}(2)$ on $\mathbb{CP}_1$. Therefore this intersection consists of either two points with multiplicity 1, a single point of multiplicity 2, or the entire twistor line in the case that the restriction of s to $p^{-1}(x)$ is trivial.

We have by assumption that the section s is real, and therefore the set X is preserved by the real structure σ. Moreover, this real structure restricts to the antipodal map on twistor fibers, and therefore implies that if $z \in X \cap p^{-1}(x)$, then the antipodal point $\sigma(z)$ is in this intersection as well. This implies that the intersection $X \cap p^{-1}(x)$ cannot consist of a single point, and therefore we have that either $X \cap p^{-1}(x) = p^{-1}(x)$ or $X \cap p^{-1}(x) = \{z, \sigma(z)\}$, a pair of antipodal points.

Let M_0 be the set of points in $x \in M$ such that $X \cap p^{-1}(x)$ is a pair of points. If we assume by contradiction that M_0 is not dense, then there exists some open set $U \subset M$ that does not intersect M_0. We then have by assumption that the section s vanishes identically on the open set $p^{-1}(U)$, but then s vanishes on all of Z since s is holomorphic. Therefore we must have that M_0 is dense. $\square$

The discussion in the above proof also allows us to interpret the complex structure obtained on M_0 using the twistor space. We can observe that $X|_{M_0}$ is a double cover, and so in the case that this cover consists of two connected components, for example, in the case that M_0 is simply-connected, we can call these components $\Sigma, \overline{\Sigma}$. Each of these components is diffeomorphic to M_0 and exchanged by the real structure σ. A choice of either connected component is by definition a section of the bundle $Z|_{M_0} \to M_0$. Since $Z|_{M_0}$ is evidently the twistor space of the quaternionic manifold M_0, this section gives a choice of a compatible almost complex structure on M_0. Moreover the complex structure is in fact integrable, since the choice of section embeds M_0 with that almost complex structure into Z as the complex submanifold Σ or $\overline{\Sigma}$. Our work above shows that there is then a unique connection on M that identifies the section of $S^2\mathbf{H}$ associated to s with one of these complex structures.

This twistorial interpretation is well-understood in the case of quaternion-Kähler manifolds in dimension four. Hitchin [17] has explicitly computed some examples, which we discuss further in Section 5.5. Those examples themselves are special cases of previous work in the four-dimensional by Pontecorvo [38], who studied real sections of the $\mathcal{O}(2)$ bundle on $\mathbb{CP}_3$, the twistor space of $S^4 \cong \mathbb{HP}_1$, in order to classify conformally flat hermitian surfaces.

It remains to see whether this more general construction can be fruitfully applied to find new examples of interesting $GL(k, \mathbb{H})Sp(1)$-connections in other contexts. The examples of quaternionic manifolds that are easiest to work with explicitly are the weakly quaternionic manifolds underlying the quaternion-Kähler Wolf spaces, and in this case the resulting connections can be analyzed in terms of Killing fields via Hitchin's methods. It is a goal of future work to see what further insights into quaternionic complex manifolds can be gained when they are constructed from quaternionic manifolds that are not themselves quaternion-Kähler, the situation in which our generalization becomes necessary.

5 Kähler Metrics Associated to $U(1)$ Actions

In this section, we explore a very similar construction of special connections on quaternion-Kähler manifolds that arise from isometric circle actions, this time arising from the work of Haydys [14]. We begin by reviewing some special properties of the curvature tensor of a quaternionic connection in Section 5.1. With this background, we show in Section 5.2 how these curvature properties can be used to construct Kähler metrics on subsets of positive quaternion-Kähler manifolds using quaternionic Killing fields. These metrics have already been shown to exist by Haydys, but we give a more direct proof that involves properties only of the quaternion-Kähler manifold itself, instead of the more involved construction carried out by Haydys. We show in Section 5.3 that our metrics are in fact identical to those produced by Haydys. Our simplified expression is especially useful in that it allows us to directly compare the Levi-Civita connection of the resulting Kähler metric, which has holonomy contained in $U(2m)$, to the holonomy $GL(k,\mathbb{H})Sp(1)$ connections constructed from $U(1)$ actions by Hitchin discussed in Section 4.2. This comparison is made in Section 5.4, where we prove our main result showing that these two connections are not identical. We end by considering an example in Section 5.5.

5.1 Curvature of Quaternionic Geometries

If we work locally with a fixed compatible frame $\{I, J, K\}$, then we can obtain simple and concrete expressions for the connection and curvature forms on a quaternionic or quaternion-Kähler manifold, which we derive in the following propositions. These derivations are taken, with some modifications in notation, from Besse [9, Chapter 14], which is in turn adapted from Ishihara [19].

Given a quaternionic manifold $(M, \mathcal{Q}, \nabla)$, let $\mathcal{R} : \Omega^2(M) \to \mathrm{End}(\mathcal{Q})$ denote the curvature operator of the connection ∇ considered as a connection on $\mathcal{Q}$. Similarly, let $R : \Omega^2(M) \to \mathrm{End}(TM)$ denote the usual curvature operator for ∇ when considered as a connection on TM. We then have the following relations:

Lemma 5.1. *Let $\{I, J, K\}$ be a compatible frame for a quaternionic manifold $(M, \mathcal{Q}, \nabla)$, and let a, b, c be the connection 1-forms for ∇ acting on $\mathcal{Q}$ in this* *compatible frame as is Lemma 4.3 Then*

$$\begin{array}{rcl} [R(X,Y),I] &=& \alpha(X,Y)J - \beta(X,Y)K \\ [R(X,Y),J] &=& -\alpha(X,Y)I + \gamma(X,Y)K \\ [R(X,Y),K] &=& \beta(X,Y)I - \gamma(X,Y)J \end{array} \quad (30)$$

where

$$\alpha = da - b \wedge c$$
$$\beta = db - c \wedge a$$
$$\gamma = dc - a \wedge b$$

Proof. It is easy to verify from the definition of curvature that, for any $S \in \Gamma(\mathcal{Q}) \subset \Gamma(\mathrm{End}(TM))$, one has

$$[R(X,Y),S] = \mathcal{R}(X,Y)S \quad (31)$$

where the bracket on the left is the commutator with respect to composition of $R(X,Y)$ and S as sections of $\mathrm{End}(TM)$. Thus the lemma amounts to computing the local curvature 2-forms for $\mathcal{R}$ with respect to the chosen frame $\{I,J,K\}$. □

In the case that M is a quaternion-Kähler manifold, we can relate the local curvature forms α, β, γ to the Ricci form.

Lemma 5.2. *Let M be a quaternion-Kähler manifold with quaternionic dimension $k \geq 2$, with local compatible frame $\{I,J,K\}$ and local curvature 2-forms α, β, γ as in Lemma 5.1. Then*

$$\alpha(X,Y) = \frac{1}{k+2} r(X,KY)$$
$$\beta(X,Y) = \frac{1}{k+2} r(X,JY)$$
$$\gamma(X,Y) = \frac{1}{k+2} r(X,IY)$$

Proof. Taking the third equality from (30), we apply the endomorphism to Z and then take the inner product of the result with JZ to obtain, after some rearranging

$$\gamma(X,Y)g(Z,Z) = g(R(X,Y)Z,IZ) + g(R(X,Y)JZ,KZ) \quad (32)$$

making use of the compatibility of I,J,K with g as well as the symmetries of the Riemann curvature tensor. Next, we can choose a local orthonormal and quaternionic frame E_j for $j = 1, \dots, 4k$ as in the proof of Proposition 4.2, that is we can take $E_2 = IE_1, E_3 = JE_1$, etc. Substituting each basis element in place of Z in equation (32) and taking the sum yields

$$\begin{aligned} 4k\gamma(X,Y) &= \sum_{j=1}^{4k} g(R(X,Y)E_j, IE_j) + g(R(X,Y)JE_j, KE_j) \\ &= 2\sum_{j=1}^{4k} g(R(X,Y)E_j, IE_j), \end{aligned}$$

since the collections of pairs (E_j, IE_j) and (JE_j, KE_j) are the same, for example, $(JE_1, KE_1) = (JE_1, IJE_1) = (E_3, IE_3)$. Using the first Bianchi identity and the symmetries of the curvature tensor, the right-hand side can be rewritten to yield

$$2k\gamma(X,Y) = \sum_{j=1}^{4k} g(R(X,E_j)Y, IE_j) - g(R(X,IE_j)Y, E_j)$$

We similarly have that the collections of pairs (E_j, IE_j) and $(IE_j, -E_j)$ are the same, for example, $(IE_1, -E_1) = (IE_1, I^2E_1) = (E_2, IE_2)$, and so we have

$$k\gamma(X,Y) = \sum_{j=1}^{4k} g(R(X,E_j)Y, IE_j) = -\sum_{j=1}^{4k} g(IR(X,E_j)Y, E_j)$$

Taking the first equality from (30), we have

$$\begin{aligned} k\gamma(X,Y) &= \sum_{j=1}^{4k} -g(R(X,E_j)IY, E_j) + g(\alpha(X,E_j)JY, E_j) - g(\beta(X,E_j)KY, E_j) \\ &= r(X,IY) + \alpha(X,JY) - \beta(X,KY) \end{aligned}$$

Finally, replacing Y with IY and rearranging we have

$$k\gamma(X,IY) + \alpha(X,KY) + \beta(X,JY) = -r(X,Y)$$

We can repeat the above computations for the 2-forms α and β to yield the equalities

$$\begin{aligned} k\gamma(X,IY) + \alpha(X,KY) + \beta(X,JY) &= -r(X,Y) \\ \gamma(X,IY) + k\alpha(X,KY) + \beta(X,JY) &= -r(X,Y) \\ \gamma(X,IY) + \alpha(X,KY) + k\beta(X,JY) &= -r(X,Y) \end{aligned}$$

If we subtract the second equation from the first, we have the equality

$$(k-1)\gamma(X, IY) = (k-1)\alpha(X, KY)$$

and therefore we have $\gamma(X, IY) = \alpha(X, KY)$ as long as $k \geq 2$. The same argument with the latter two equations gives that $\alpha(X, KY) = \beta(X, JY)$, and therefore we have

$$\alpha(X, KY) = \beta(X, JY) = \gamma(X, IY) = \frac{-1}{k+2} r(X, Y)$$

Replacing Y with KY, JY, and IY therefore gives the desired equalities. □

Theorem 5.3. *Let M be a quaternion-Kähler manifold with quaternionic dimension $k \geq 2$. Then M is Einstein.*

Proof. From the result of the previous lemma, along with equation (32), we have

$$\begin{aligned} r(X, X)g(Z, Z) &= (k+2)\gamma(X, IX)g(Z, Z) \\ &= (k+2)\left(g(R(X, IX)Z, IZ) + g(R(X, IX)JZ, KZ)\right) \end{aligned}$$

for any vectors X, Z. Replacing X with JX yields

$$\begin{aligned} r(JX, JX)g(Z, Z) &= (k+2)\gamma(JX, IJX)g(Z, Z) \\ &= (k+2)\left(g(R(JX, KX)Z, IZ) + g(R(JX, KX)JZ, KZ)\right) \end{aligned}$$

On the other hand, $r(X, X) = r(JX, JX)$, as

$$\begin{aligned} r(JX, JX) &= -(k+2)\beta(JX, J^2X) = (k+2)\beta(JX, X) \\ &= -(k+2)\beta(X, JX) = r(X, X), \end{aligned}$$

so adding the above equations yields that

$$\begin{aligned} r(X, X)g(Z, Z) = 2(k+2)\Big(&g(R(X, IX)Z, IZ) + g(R(X, IX)JZ, KZ) \\ &+ g(R(JX, KX)Z, IZ) + g(R(JX, KX)JZ, KZ)\Big) \end{aligned}$$

for any X, Z. Using the symmetries of the Riemann curvature tensor, we observe that the right-hand side is symmetric in X, Z, and therefore we have

$$r(X, X)g(Z, Z) = r(Z, Z)g(X, X) \Rightarrow r(X, X) = \frac{r(Z, Z)}{g(Z, Z)} g(X, X)$$

assuming Z is non-vanishing. Therefore Schur's lemma gives that the metric is Einstein. □

Corollary 5.4. *Let M be a quaternion-Kähler manifold with scalar curvature s, which is constant by Theorem 5.3 for $k \geq 2$. Then given a local compatible frame $\{I, J, K\}$ with local connection 1-forms a, b, c, we have*

$$\begin{aligned} da - b \wedge c &= \frac{-s}{4k(k+2)} \omega_K \\ db - c \wedge a &= \frac{-s}{4k(k+2)} \omega_J \\ dc - a \wedge b &= \frac{-s}{4k(k+2)} \omega_I \end{aligned} \tag{33}$$

where $\omega_I, \omega_J, \omega_K$ are the local proto-Kähler forms associated to the local compatible frame.

Proof. Considering the first equation, we have

$$\begin{aligned} (da - b \wedge c)(X, Y) = \alpha(X, Y) = \frac{1}{k+2} r(X, KY) = \frac{s}{4k(k+2)} g(X, KY) \\ = \frac{-s}{4k(k+2)} \omega_K(X, Y) \end{aligned}$$

The first equality is Lemma 5.1, the second is Lemma 5.2, and the third equality follows as quaternion-Kähler manifolds are Einstein. The equalities for ω_J, ω_K are proved similarly. □

The equations (33) of Corollary 5.4 above can be reinterpreted slightly. The curvature $\mathcal{R}$ of the bundle $\mathcal{Q}$ is by definition a mapping $\mathcal{R} : \Lambda^2 TM \to \mathrm{End}(\mathcal{Q})$. The connection is compatible with the $SO(3)$ structure on the bundle, so in fact $\mathcal{R}$ takes values in the set of skew-symmetric endomorphisms of $\mathcal{Q}$. Moreover, the Euclidean inner product on $\mathcal{Q}$ allows us to identify $\mathrm{SkewEnd}(\mathcal{Q}) \cong \mathcal{Q}$ via the cross product, while $\mathcal{Q} \cong \mathcal{G}$ via the metric, so that we can consider the curvature as an operator $\mathcal{R} : \Lambda^2 TM \to \mathcal{G}$. On the other hand, we can consider the curvature R of the bundle TM as an operator $R : \Lambda^2 TM \to \Lambda^2 TM$ as well, and the fact that the connection preserves the bundle $\mathcal{G}$ implies that the restriction of R to $\mathcal{G}$ still has image contained in $\mathcal{G}$, while equation (31) implies that that $R|_{\mathcal{G}}$ is the same as map $\mathcal{R}|_{\mathcal{G}}$. That is, the equations of Corollary 5.4 are equivalent to stating that $R|_{\mathcal{G}} = \lambda \, \mathrm{Id}_{\mathcal{G}}$, where $\lambda = \frac{s}{4k(k+2)}$ (c.f. Galicki and Lawson, [13]).

This essentially gives a second characterization of quaternion-Kähler manifolds in dimensions $k \geq 2$, as those quaternionic-Hermitian manifolds for which $R|_{\mathcal{G}} = \lambda \, \mathrm{Id}_{\mathcal{G}}$. This definition in fact generalizes to quaternionic dimension $k = 1$ as well, under the identification $\mathcal{G} = \Lambda^+$ identifying the quaternionic structure with the self-dual 2-forms.

Lemma 5.5. *Let (M^4, g) be a Riemannian manifold. Then M is quaternion-Kähler if and only if $R|_{\Lambda^+} = \frac{s}{12}\operatorname{Id}_{\Lambda^+}$. Moreover, if M is quaternion-Kähler then the equalities* (33) *still hold.*

Proof. For an arbitrary 4-dimensional Riemannian manifold we can consider the curvature tensor as a self-adjoint operator on 2-forms, and the decomposition $\Lambda^2 T^*M = \Lambda^+ \oplus \Lambda^-$ into the self-dual and anti-self-dual 2-forms gives a block-diagonal decomposition of the curvature operator into

$$R = \begin{pmatrix} W^+ + \frac{s}{12}\operatorname{Id} & \mathring{r} \\ \mathring{r}^t & W^- + \frac{s}{12}\operatorname{Id} \end{pmatrix}$$

where W^+, W^- are the self-dual and anti-self-dual portions of the Weyl tensor, $\mathring{r}$ is the trace-free Ricci curvature, and s is the scalar curvature. Thus R restricts to a multiple of the identity on Λ^+ if and only if $W^+ = 0$ and $\mathring{r} = 0$, that is, if and only if the metric is anti-self-dual and Einstein, the definition of a quaternion-Kähler manifold in dimension 4. The discussion above shows that the equations of Lemma 5.2 are equivalent to the stated curvature property. □

We end by noting a few more properties of the equations (33). The left-hand side expression here is invariant under constant rescalings of the metric, in that a, b, c depend only on the choice of quaternionic connection for the underlying weakly quaternionic manifold, and rescaling a metric by a constant factor does not change the associated Levi-Civita connection. The right-hand side is therefore invariant under constant rescalings as well. We can also observe this directly. If we rescale the original metric g by a positive constant C, then the new metric $\widetilde{g} = Cg$ remains quaternion-Kähler with scalar curvature $\widetilde{s} = s/C$, and for any given quaternionic almost complex structure the associated proto-Kähler form is also rescaled by c. Applying the proposition to this new metric yields, for example, that

$$da - b \wedge c = \frac{-\widetilde{s}}{4k(k+2)}\widetilde{\omega_K} = \frac{-s}{C \cdot 4k(k+2)}C\omega_K = \frac{-s}{4k(k+2)}\omega_K$$

We will therefore generally assume via rescaling that any quaternion-Kähler metric we consider has constant scalar curvature $s = \pm 4k(k+2)$ so that the above equations can be simplified to remove the extraneous constants. In particular, in the positive scalar curvature case we obtain the simplifications

$$\begin{aligned} da - b \wedge c &= -\omega_K \\ db - c \wedge a &= -\omega_J \\ dc - a \wedge b &= -\omega_I \end{aligned} \tag{34}$$

5.2 Construction of the Kähler Metrics

In this section, we make the same hypothesis and assumptions as those in Theorem 4.8. In particular, let (M, g) be a quaternion-Kähler manifold admitting a quaternionic Killing field X_0 with momentum section ρ_{X_0}. Let $M_0 = M - \{\rho_{X_0} = 0\}$, and choose a local compatible frame $\{I, J, K\}$ for $\mathcal{Q}$ over M_0 so that I is the complex structure associated to ρ_{X_0}, and fix the 1-form Υ defined by equation (21) on M_0. For convenience, we note the following properties of the forms a, b, Υ:

Lemma 5.6. *Let $\rho = \rho_1 \omega_I$ be the momentum section associated to the quaternionic Killing field X_0. Then the forms a, b, Υ satisfy the equations*

$$\begin{array}{rclrcl} a(X) & = & \rho_1^{-1}\omega_J(X_0, X) & b(X) & = & -\rho_1^{-1}\omega_K(X_0, X) \\ a(IX) & = & b(X) & b(IX) & = & -a(X) \\ a(JX) & = & \rho_1^{-1}g(X_0, X) & b(JX) & = & -\Upsilon(X) \\ a(KX) & = & -\Upsilon(X) & b(KX) & = & -\rho_1^{-1}g(X_0, X) \end{array} \tag{35}$$

In particular, evaluating on the Killing field X_0 gives

$$\begin{array}{rclrcl} a(X_0) & = & 0 & b(X_0) & = & 0 \\ a(IX_0) & = & 0 & b(IX_0) & = & 0 \\ a(JX_0) & = & \rho_1^{-1}g(X_0, X_0) & b(JX_0) & = & 0 \\ a(KX_0) & = & 0 & b(KX_0) & = & -\rho_1^{-1}g(X_0, X_0) \end{array} \tag{36}$$

Proof. The first row of (35) above is a restatement of equation (20), and the remaining entries in that table follow from the definitions and the properties of the orthogonal transformations I, J, K, for example,

$$a(IX) = \frac{1}{\rho_1} g(JX_0, IX) = \frac{-1}{\rho_1} g(KX_0, X) = -b(X)$$

□

With this setup, we consider the 2-form $W = -dc$.

Theorem 5.7. *If M is a positive quaternion-Kähler manifold, then W is a Kähler form with respect to the complex structure I on the open, dense set $M_0 \subset M$.*

Proof. If is immediate that W is closed, as $dW = -d^2c = 0$. It remains to check that the related bilinear form $\widetilde{g}(X, Y) = W(X, IY)$ is a Riemannian metric.

We have that $\widetilde{g}$ is symmetric if $W(IX, IY) = W(X, Y)$ for all X, Y, for this implies that

$$\widetilde{g}(X, Y) = W(X, IY) = W(IX, I^2Y) = -W(IX, Y) = W(Y, IX) = g(Y, X)$$

We have from Corollary 5.4 that W can be expressed as

$$W = \frac{s}{4k(k+2)}\omega_I - a \wedge b$$

Considering the first summand, the fact that I and g are compatible gives

$$\omega_I(IX, IY) = g(I^2X, IY) = g(IX, Y) = \omega_I(X, Y),$$

while the equalities from (35) give that

$$\begin{aligned}(a \wedge b)(IX, IY) &= ((a \circ I) \wedge (b \circ I))\,(X, Y) \\ &= (b \wedge (-a))(X, Y) = (a \wedge b)(X, Y),\end{aligned}$$

so that $W(IX, IY) = W(X, Y)$ and $\widetilde{g}$ is symmetric.

To see that $\widetilde{g}$ is positive definite, we have

$$\begin{aligned}W(X, IX) &= \frac{s}{4k(k+2)}\omega_I(X, IX) - (a \wedge b)(X, IX) \\ &= \frac{s}{4k(k+2)}g(IX, IX) - a(X)b(IX) + a(IX)b(X) \\ &= \frac{s}{4k(k+2)}g(X, X) + a(X)^2 + b(X)^2\end{aligned}$$

so the assumption that $s > 0$ gives that $W(X, IX) \geq 0$, while the fact that g is positive-definite implies that $W(X, IX)$ is as well. □

Corollary 5.8. *The metric*

$$\widetilde{g} = \frac{s}{4k(k+2)}g + a \otimes a + b \otimes b$$

is a Kähler metric on M_0*.*

5.3 Equivalence to Kähler Metrics Constructed by Haydys

Theorem 5.7 gives a way to construct a Kähler metric on a subset of a quaternion-Kähler manifold from an isometric and quaternionic $U(1)$-action.

Haydys [14, Section 5] has already given such a construction, and in this section we show that the metrics obtained through Theorem 5.7 are identical to those obtained via Haydys's method.

To do so, we first review the construction as carried out by Haydys. Given a quaternion-Kähler manifold M, there exists a natural $\mathbb{H}^\times/\{\pm 1\}$-fibration over M known as the Swann bundle [42], which we denote by $\mathcal{U}(M)$. This bundle always has a natural pseudo-hyperkähler structure, which is in fact hyperkähler in the case that M is a positive quaternion-Kähler manifold. An isometric and quaternionic $U(1)$-action on M then lifts to a hyperkähler action on the Swann bundle $\mathcal{U}(M)$, that is, an isometric action that preserves the Kähler forms associated to the hyperkähler structure.

Using the work of Hitchin et. al. [18], this action yields a hyperkähler momentum map $\varrho : \mathcal{U}(M) \to \mathrm{Im}(\mathbb{H})$, analogous to the quaternionic momentum map of Galicki and Lawson described in Theorem 4.7. If the j, k component of this map is considered as a complex-valued function ϱ_c, then ϱ_c is holomorphic with respect to the complex structure on $\mathcal{U}(M)$ associated to the i component, so that $\varrho_C^{-1}(0)$ is a Kähler submanifold of $\mathcal{U}(M)$.

In addition to the $U(1)$ action on $\mathcal{U}(M)$ lifted from M, the Swann bundle of a positive quaternion-Kähler manifold also admits a natural, isometric $Sp(1)$ action that permutes complex structures in the sense that, if the Kähler forms of the hyperkähler bundle $\mathcal{U}(M)$ are gathered into an $\mathrm{Im}(\mathbb{H})$-valued 2-form $\omega = \omega_I i + \omega_J j + \omega_K k$, then the left action of $q \in Sp(1)$ pulls back ω as $L_q^* \omega = q\omega q^*$. In particular, there is a $U(1)$ action on $\mathcal{U}(M)$, distinct from the one lifted from M, obtained from $U(1) \subset Sp(1)$ as the subset of unit complex numbers. By the permuting property this action preserves the Kähler form ω_I, and so, after making some mild assumptions about the compatibility of the actions of $Sp(1)$ and the lifted $U(1)$ on $\mathcal{U}(M)$, we obtain a Kähler metric on a subset of $M_0 \subset M$ by considering a Kähler reduction of $\varrho_C^{-1}(0)$ with respect to a non-zero value of the Kähler momentum map for the second $U(1)$ action. For details of hyperkähler moment maps and Kähler reductions we refer to [18, 32].

Haydys also remarks on an alternative way to obtain the Kähler form associated to the above metric, as the curvature of a principal circle bundle over M_0 constructed from $\mathcal{Q}$. We describe this procedure below, and show that it produces the Kähler form W of Theorem 5.7. That theorem therefore reproves Haydys's result on the existence of Kähler metric on subsets of quaternion-Kähler manifolds, but without making use of the more complicated machinery of the Swann bundle.

We again use the notation and assumptions of Section 5.2, letting M be a positive quaternion-Kähler manifold with quaternionic Killing field X_0, taking

$M_0 = M - \{\rho_{X_0} = 0\}$ as the set where the associated momentum section is nonzero, and choosing a local compatible frame $\{I, J, K\}$ so that I is the complex structure associated to ρ_{X_0}.

Next, we recall from Corollary 2.19 that the bundle P of local compatible frames for $\mathcal{Q}$ is a principal $SO(3)$-bundle. The choice of local frame $\{I, J, K\}$ for $\mathcal{Q}$ over M_0 allows us to consider the local connection 1-forms a, b, c associated to ∇ on $\mathcal{Q}$, which we can consider collectively as an $\mathfrak{so}(3)$-valued 1-form denoted by $\mathfrak{a}$ as in Lemma 4.3.

The local frame $\{I, J, K\}$ for $\mathcal{Q}$ over M_0 is equivalent to a local section of P over M_0, which we can also consider as a trivialization $P|_{M_0} \cong M_0 \times SO(3)$. The Levi-Civita connection ∇, considered as a connection on $\mathcal{Q}$, therefore lifts to a principal $SO(3)$-connection on P. By definition such a connection is an $\mathfrak{so}(3)$-valued 1-form $\mathfrak{b}$ on P that pulls back under the principal action via the adjoint representation and acts as the identity on vectors that are vertical with respect to the principal bundle projection using the standard identification of vertical vectors with the Lie algebra. The fact that this connection is a lift of the connection on $\mathcal{Q}$ implies that if we consider the local frame $\{I, J, K\}$ as a section $s : M_0 \to P$, then $s^*\mathfrak{b} = \mathfrak{a}$. These properties uniquely determine $\mathfrak{b}$, which we can see is given by the formula

$$\mathfrak{b}_{(m,g)}(X, q \cdot g) = g^{-1} \left(\mathfrak{a}(X) + q\right) g$$

where $m \in M_0$, $X \in T_m M_0$, $g \in SO(3)$, and $q \in \mathfrak{so}(3)$ so that $q \cdot g$ represents the right translation of $q \in \mathfrak{so}(3) = T_1 SO(3)$ to $T_g SO(3)$ via g. By definition the kernel of this 1-form defines the horizontal distribution for the connection, so that a vector $(X, q \cdot g)$ will be horizontal if and only if $q = -\mathfrak{a}(X)$.

Although we have fixed a local compatible frame, only the complex structure I is determined by the momentum section ρ_{X_0}, and so in fact we have a freedom to choose the remaining almost complex structures J, K in the frame. In terms of the $SO(3)$ structure, the collection of all possible frames including I is then an $U(1)$-subbundle of $P|_{M_0}$ that we will call $\mathcal{I}$, consisting of frames of the form $\{I, \widetilde{J}, \widetilde{K}\}$ where $\widetilde{J}, \widetilde{K}$ are a rotation of $\{J, K\}$. In terms of the trivialization determined by $\{I, J, K\}$, we have

$$\mathcal{I} \cong M_0 \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}$$

for $\theta \in \mathbb{R}$ the angle of rotation from the fixed choice of J, K. We write the second element of the product here as $R(\theta)$ for the matrix of rotation θ, noting that $R(\theta)^{-1} = R(-\theta)$. This leads to an inclusion of $U(1) \subset SO(3)$, which after differentiating gives an inclusion $\mathfrak{u}(1) \subset \mathfrak{so}(3)$ on the level of Lie algebras via

$$i\theta \mapsto \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\theta \\ 0 & \theta & 0 \end{pmatrix} \tag{37}$$

Using this we see that $T_{(m,R(\theta))}\mathcal{I} \subset T_{(m,R(\theta))}P$ consists of vectors of the form $(X, q \cdot R(\theta))$, where q is a matrix of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -q_1 \\ 0 & q_1 & 0 \end{pmatrix}$$

Therefore the restriction of $\mathfrak{b}$ to $\mathcal{I}$ is the $\mathfrak{so}(3)$-valued form

$$\mathfrak{b}_{(m,R(\theta))}(X, q_1 \cdot R(\theta)) = R(\theta)\,(\mathfrak{a}(X) + q)\,R(-\theta),$$

The right-hand side of the above, written explicitly as a matrix, is

$$\begin{pmatrix} 0 & -a(X)\cos\theta - b(X)\sin\theta & b(X)\cos\theta - a(X)\sin\theta \\ a(X)\cos\theta + b(X)\sin\theta & 0 & -c(X) - q_1 \\ -b(X)\cos\theta + a(X)\sin\theta & c(X) + q_1 & 0 \end{pmatrix}$$

Note that the part of this matrix associated to $\mathfrak{u}(1)$ via equation (37) does not depend on the $U(1)$ action, and therefore determines a $\mathfrak{u}(1)$-valued 1-form on $\mathcal{I}$, which we will call $\mathfrak{c}$, by

$$\mathfrak{c}_{(m,R(\theta))}(X, q_1 \cdot R(\theta)) = c(X) + q_1$$

Here we are considering $\mathfrak{u}(1) \cong \mathbb{R}$, instead of the more usual $i\mathbb{R}$.

Proposition 5.9. *We have the following properties of the* 1*-form* $\mathfrak{c}$

1. $\mathfrak{c}$ *is a connection* 1*-form for a principal* $U(1)$*-connection on* $\mathcal{I}$*.*
2. *The curvature of* $\mathfrak{c}$*, considered as a* 2*-form* $\mathcal{W}$ *on* M_0*, is dc.*

Proof. Since $U(1)$ is an abelian Lie group with Lie algebra $\mathbb{R}$, a connection 1-form is simply a standard $\mathbb{R}$-valued 1-form on $\mathcal{I}$ with the additional property that it acts as the identify on vertical vectors, which is evidently the case for $\mathfrak{c}$, as

$$\mathfrak{c}_{(m,R(\theta))}(0, q_1 \cdot R(\theta)) = q_1$$

and so the first property is immediately verified.

For the second, given $X, Y \in T_m M$, the 2-form on M_0 determined by the curvature of $\mathfrak{c}$ is by definition the form

$$\mathcal{W}(X, Y) = d\mathfrak{c}(\widetilde{X}, \widetilde{Y}),$$

where $\widetilde{X}, \widetilde{Y}$ are horizontal lifts of X, Y to $\mathcal{I}$ with respect to the connection determined by $\mathfrak{c}$. This horizontal lift is given by

$$\widetilde{X}_{(m,R(\theta))} = (X_m, -c(X_m) \cdot R(\theta)) \in T_m M \times T_{R(\theta)} U(1)$$

We observe that

$$d\mathfrak{c}(\widetilde{X}, \widetilde{Y}) = \widetilde{X} \cdot \mathfrak{c}(\widetilde{Y}) - \widetilde{Y} \cdot \mathfrak{c}(\widetilde{X}) - \mathfrak{c}([\widetilde{X}, \widetilde{Y}]) = -\mathfrak{c}([\widetilde{X}, \widetilde{Y}])$$

since $\widetilde{X}, \widetilde{Y}$ are horizontal and therefore $\mathfrak{c}(\widetilde{X}), \mathfrak{c}(\widetilde{Y}) \equiv 0$. The Lie bracket on the right hand side can be decomposed as

$$\begin{aligned}
[\widetilde{X}, \widetilde{Y}] &= [(X, -c(X)), (Y, -c(Y))] \\
&= [(X, 0), (Y, 0)] + [(X, 0), (0, -c(Y))] + [(0, -c(X)), (Y, 0)] \\
&\quad + [(0, -c(X)), (0, -c(Y))]
\end{aligned}$$

For the first and last terms in this expression, we have

$$\begin{aligned}
[(X, 0), (Y, 0)] &= ([X, Y], 0) \\
[(0, c(X)), (0, c(Y))] &= (0, [c(X), c(Y)]) = (0, 0)
\end{aligned}$$

For the middle terms, we have

$$[(X, 0), (0, -c(Y))] = (0, -X \cdot c(Y))$$

and so a similar argument for $[(0, -c(X)), (Y, 0)]$ gives that

$$[\widetilde{X}, \widetilde{Y}] = ([X, Y], 0) + (0, -X \cdot c(Y)) + (0, Y \cdot c(X))$$

and therefore

$$\begin{aligned}
\mathcal{W}(X, Y) &= -\mathfrak{c}\Big(([X, Y], 0) + (0, -X \cdot c(Y)) + (0, Y \cdot c(X)\Big) \\
&= -c([X, Y]) + X \cdot c(Y) - Y \cdot c(X) \\
&= dc(X, Y)
\end{aligned}$$

□

This gives the relationship $\mathcal{W} = -W$ between the curvature of the connection on the principal bundle $\mathcal{I}$ and the Kähler form W we found in Section 5.2. This relationship is, up to changes in conventions, the relationship obtained by Haydys, see Remark 15 in [14]. The connection considered by Haydys actually arises from a $U(1)$ bundle associated to the double cover $Sp(1)$ of $SO(3)$ via the Swann bundle, and therefore the curvature form he considers is a multiple of the form we consider by 1/2. The above proposition can therefore be considered as an alternate proof of Remark 15 in [14] without making reference to the Swann bundle.

5.4 Comparison of Haydys and Hitchin Connections

We now arrive at the central problem that motivated the studies in this thesis. As we have reviewed above, Haydys and Hitchin each have a procedure that takes a quaternionic killing field X_0 on M a (positive) quaternion-Kähler manifold and produces a connection on the subset M_0 where the momentum section is non-zero. Hitchin constructs a quaternionic connection $\widehat{\nabla}$ with holonomy contained in $SL(k, \mathbb{H})U(1)$, while Haydys's procedure gives the Levi-Civita connection $\widetilde{\nabla}$ of $\widetilde{g}$ a Kähler metric, which therefore has holonomy contained in $U(2k)$. Both connections preserve the same complex structure I that is determined by the momentum section associated to the Killing field, and so both give an additional geometric structure on the same underlying complex manifold M_0.

Were these two connections to coincide, the holonomy group would therefore be contained in the intersection $SL(k, \mathbb{H})U(1) \cap U(2k) = Sp(k)U(1)$, which is not one of the holonomy groups on Berger's list. This would imply that the Kähler metric obtained by Haydys is either hyperkähler, with holonomy contained in $Sp(k)$, or is a product or symmetric space, which could lead to further interesting insights into the structure of the original quaternion-Kähler manifold. Unfortunately, these connections are distinct, as evidenced by the following proposition.

Proposition 5.10. *Let (M, g) be a quaternion-Kähler manifold with constant scalar curvature $s = 4k(k+2)$ and quaternionic Killing field X_0, and let $M_0 = M - \{\rho_{X_0} = 0\}$ be the subset on which the momentum map associated to X_0 is nonvanishing. Let $\widetilde{g}$ be the Kähler metric on M_0 obtained via Theorem*

5.7 and Corollary 5.8. Then the Levi-Civita connection $\widetilde{\nabla}$ for $\widetilde{g}$ is given by

$$\widetilde{\nabla}_X Y = \nabla_X Y - \frac{1}{2}\big(a(X)KY + a(Y)KX + b(X)JY + b(Y)JX\big) + \frac{1}{2}\big(B(X,Y)JX_0 - B(X,IY)KX_0\big),$$

where ∇ is the Levi-Civita connection for the original quaternion-Kähler metric g on M and B is a symmetric $(2,0)$-tensor defined by the formula

$$B(X,Y) = \frac{1}{(\rho_1 + \rho_1^{-1} g(X_0, X_0))}\Big(-2a\,(\nabla_X Y) + 2\big(X \cdot a(Y)\big) + 2b(Y)c(X) + g(KX,Y) + a(X)a(KY) + a(Y)a(KX) + b(X)a(JY) + b(Y)a(JX)\Big) \tag{38}$$

The proof of this proposition is a lengthly and tedious, if straightforward, application of the Koszul formula for the Levi-Civita connection which we relegate to the appendix in Section A.1.

The assumption that $s = 4k(k+2)$ is made to simplify the expression of the Levi-Civita connection by removing extraneous constant factors. Following the remarks after Corollary 5.4, we can always assume that a given positive quaternion-Kähler metric has this scalar curvature after rescaling, and in this case the Kähler form and metric on M_0 are

$$\begin{aligned} W &= \omega_I - a \wedge b \\ \widetilde{g} &= g + a \otimes a + b \otimes b \end{aligned} \tag{39}$$

Evidently the change of connection is not of the form described in Lemma 4.1, and so the connection $\widetilde{\nabla}$ is not quaternionic and therefore cannot coincide with the connection $\widehat{\nabla}$ obtained by Hitchin. We can easily compute their difference.

Corollary 5.11. *Let $\widetilde{\nabla}$ be the connection for the Kähler metric associated to X_0 via Theorem 5.7, and let $\widehat{\nabla}$ be the quaternionic connection that preserves I associated to X_0 via Theorem 4.8. Then we have*

$$\widetilde{\nabla}_X Y = \widehat{\nabla}_X Y - \frac{1}{2}\Big(\Upsilon(X)Y + \Upsilon(Y)X - \Upsilon(IX)IY - \Upsilon(IY)IX - B(X,Y)JX_0 + B(X,IY)KX_0\Big) \tag{40}$$

Therefore $\widetilde{\nabla}_X Y$ and $\widehat{\nabla}_X Y$ never coincide.

Proof. We can write the difference between the two connections as

$$\widetilde{\nabla}_X Y - \widehat{\nabla}_X Y = (\widetilde{\nabla}_X Y - \nabla_X Y) - (\widehat{\nabla}_X Y - \nabla_X Y)$$

and use Proposition 5.10 and Lemma 4.1 to expand the two terms on the right-hand side, which yields the desired equation after some cancelations, using the equations of (35) to relate the form Υ to a and b. □

To be sure that there are no hidden cancelations in equation (40) or some set of very special circumstances in which $\widetilde{\nabla}$ and $\widehat{\nabla}$ may actually coincide, we can consider this equation in the particular case of $X = Y = X_0$, the given Killing field. Then using the evaluations in equation (36), we have in particular that $\Upsilon(X_0) = 0$ and $\Upsilon(IX_0) = -\rho_1^{-1} g(X_0, X_0)$, so that

$$\begin{aligned}\widetilde{\nabla}_{X_0} X_0 - \widehat{\nabla}_{X_0} X_0 = &-\rho_1^{-1} g(X_0, X_0) IX_0 \\ &+ \frac{1}{2}\left(B(X_0, X_0) JX_0 - B(X_0, IX_0) KX_0\right)\end{aligned}$$

In particular, the IX_0 component is nonzero, and so the difference between the two connections will never be identically zero.

5.5 Examples Arising from $\mathbb{HP}_1 \cong S^4$

We end by considering an explicit example of the special holonomy connections and Kähler metrics obtained in the previous sections by considering $U(1)$ actions on $\mathbb{HP}_1 \cong S^4$ with its round metric.

In order to be explicit, we work on $\mathbb{R}^4 \cong S^4 - \{pt\}$ using stereographic projection, although we will use a somewhat non-standard coordinate system, following the example of Hitchin [17]. Using $\{x_0, x_1, x_2, x_3\}$ for the standard coordinates on $\mathbb{R}^4$, the round metric is given by

$$g = \frac{48}{(1 + x_0^2 + x_1^2 + x_2^2 + x_3^2)^2}\left(dx_0^2 + dx_1^2 + dx_2^2 + dx_3^3\right) \tag{41}$$

Note that we have rescaled this metric from the standard unit sphere so that we have scalar curvature $s = 12$ as assumed above, so that we can express the resulting Kähler metric on M_0 as without any extraneous constants as in equation (39). This metric is well-known to be conformally flat, hence anti-self-dual, and has constant sectional curvature, so is a quaternion-Kähler manifold using our definitions. The quaternionic structure can be given by identifying $T_x\mathbb{R}^4 \cong \mathbb{R}^4 \cong \mathbb{H}$ in the usual way and taking the local almost complex structures I, J, K defined by left multiplication by the quaternions i, j, k on $\mathbb{H}$.

Making a first change of coordinates, we can instead write the metric by writing $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ and considering polar coordinates on each factor, with

$$\begin{array}{rclcrcl} x_0 & = & \rho\cos\varphi & & x_1 & = & \rho\sin\varphi \\ x_2 & = & \sigma\cos\theta & & x_3 & = & \sigma\sin\theta \end{array} \tag{42}$$

In this coordinate system, the metric is given by

$$g = \frac{4}{(1+\rho^2+\sigma^2)^2}\left(d\rho^2 + \rho^2 d\varphi^2 + d\sigma^2 + \sigma^2 d\theta^2\right) \tag{43}$$

This makes it easier to express the isometric $U(1)$-action we will consider, which is the standard rotation action on the second $\mathbb{R}^2$ factor. This is generated by the Killing field $X_0 = \frac{\partial}{\partial\theta}$. To find the momentum section, Hitchin observes that the dual 1-form associated to X_0 has the form $X_0^\flat = 4\sigma^2 d\theta/(1+\rho^2+\sigma^2)^2$. In order to simplify this expression, Hitchin makes the change of coordinates

$$u = \sigma/(1+\rho^2+\sigma^2), \qquad v = (\rho^2+\sigma^2-1)/\rho, \tag{44}$$

so that $X_0^\flat$ can be written more conveniently as $4u^2 d\theta$ after making a coordinate change. In the (u, θ, v, φ) coordinate system we then have that the metric on 4-sphere is given by

$$g = \frac{4}{1-4u^2}du^2 + 4u^2 d\theta^2 + \frac{4(1-4u^2)}{(v^2+4)^2}dv + \frac{4(1-4u^2)}{v^2+4}d\varphi^2$$

Hitchin's construction then leads to the following metric.

Proposition 5.12 ([17])**.** *The metric on $S^4 - S^1$ defined by*

$$\widehat{g} = \frac{4}{(1-4u^2)^2}du^2 + \frac{4u^2}{1-4u^2}d\theta^2 + \frac{4}{(v^2+4)^2}dv + \frac{4}{v^2+4}d\varphi^2$$

is a scalar-flat Kähler metric, and therefore defines a quaternionic complex structure on $S^4 - S^1$.

A few remarks about this metric are in order. It is related to the original round metric on S^4 via the conformal factor $(1-4u^2)^{-1}$, just as required (see the discussion of the four-dimensional case of Hitchin's construction following Corollary 4.10). This metric is defined on the set where $1 - 4u^2 \neq 0$, or, in polar coordinates, away from the circle $\rho = 0, \sigma = 1$. Hitchin's construction gives that this metric is anti-self-dual Kähler, equivalently scalar-flat Kähler, but this can also be observed directly from the formula for the metric. The metric is in fact the product of the constant scalar curvature $s = -2$ metric on the first, (u, θ), factor, considered as the hyperbolic plane, and the constant scalar curvature $s = 2$ metric on the second, (v, φ), factor, considered as the 2-sphere. Both of these factor metrics are Kähler, and so the product metric is Kähler, with scalar curvature $-2 + 2 = 0$.

We can also consider the Kähler metric that arises from the same Killing field using Haydy's construction using the explicit methods for computation we derived in Section 5.2. These computations are not difficult, but are somewhat tedious, and so we give the details in Appendix A.2. The final result of those computations is the following proposition.

Proposition 5.13. *The Kähler metric on $S^4 - S^1$ obtained by the Haydys construction in this case is given explicitly by*

$$\widetilde{g} = \frac{4}{1-4u^2}du^2 + 4u^2 d\theta^2 + \frac{4}{(4+v^2)^2}dv^2 + \frac{4}{4+v^2}d\varphi^2 \tag{45}$$

This metric is very similar to the metric obtained by Hitchin, with some important differences. It is also product metric, with the second factor identical to the second factor of Hitchin's metric. However the first factor of $\widetilde{g}$ is distinct, and is also a metric with positive constant scalar curvature 2, although it is not defined on the circle where $1 - 4u^2 = 0$. The two factors are conformally related by the factor $1 - 4u^2$, though. We therefore have that the Haydys metric is Kähler, but with constant scalar curvature $s = 4$ instead of the scalar-flat metric obtained by Hitchin.

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Appendix

A.1 Proof of Proposition 5.10

This section is devoted to the proof of Proposition 5.10, which we reproduce below for convenience:

Proposition 5.10. *Let (M, g) be a quaternion-Kähler manifold with constant scalar curvature $s = 4k(k+2)$ and quaternionic Killing field X_0, and let $M_0 = M - \{\rho_{X_0} = 0\}$ be the subset on which the momentum map associated to X_0 is nonvanishing. Let $\widetilde{g}$ be the Kähler metric on M_0 obtained via Theorem 5.7 and Corollary 5.8. Then the Levi-Civita connection $\widetilde{\nabla}$ for $\widetilde{g}$ is given by*

$$\begin{aligned}\widetilde{\nabla}_X Y = \nabla_X Y &- \frac{1}{2}\big(a(X)KY + a(Y)KX + b(X)JY + b(Y)JX\big)\\ &+ \frac{1}{2}\big(B(X,Y)JX_0 - B(X,IY)KX_0\big),\end{aligned}$$

where ∇ is the Levi-Civita connection for the original quaternion-Kähler metric g on M and B is a symmetric $(2,0)$-tensor defined by the formula

$$\begin{aligned}B(X,Y) = \frac{1}{(\rho_1 + \rho_1^{-1} g(X_0, X_0))}\Big(&-2a\left(\nabla_X Y\right) + 2\big(X \cdot a(Y)\big) + 2b(Y)c(X)\\ &+ g(KX, Y) + a(X)a(KY) + a(Y)a(KX) + b(X)a(JY) + b(Y)a(JX)\Big)\end{aligned}$$

Proof. The Levi-Civita connection for a Riemannian metric can be obtained using the well-known Koszul formula,

$$\begin{aligned}2\widetilde{g}\left(\widetilde{\nabla}_X Y, Z\right) &= X \cdot \widetilde{g}(Y,Z) + Y \cdot \widetilde{g}(X,Z) - Z \cdot \widetilde{g}(X,Y)\\ &\quad + \widetilde{g}([X,Y],Z) - \widetilde{g}([X,Z],Y) - \widetilde{g}([Y,Z],X)\end{aligned}$$

We can expand each term on the right-hand side using the definition in equation (39). The terms involving g are exactly the Koszul formula for the Levi-Civita connection ∇ associated to g, and so after expanding we have

$$\begin{aligned}2\widetilde{g}\left(\widetilde{\nabla}_X Y, Z\right) = 2g\left(\nabla_X Y, Z\right) &+ X \cdot (a(Y)a(Z) + b(Y)b(Z))\\ &+ Y \cdot (a(X)a(Z) + b(X)b(Z)) - Z \cdot (a(X)a(Y) + b(X)b(Y))\\ &+ a([X,Y])a(Z) + b([X,Y])b(Z) - a([X,Z])a(Y)\\ &- b([X,Z])b(Y) - a([Y,Z])a(X) - b([Y,Z])b(X)\end{aligned}$$

We can expand the first three non-metric terms involving derivatives in the X, Y, Z directions using the Leibniz rule, to obtain

$$\begin{aligned}&\big(X\cdot a(Y)\big)a(Z)+a(Y)\big(X\cdot a(Z)\big)+\big(X\cdot b(Y)\big)b(Z)+b(Y)\big(X\cdot b(Z)\big)\\&\quad+\big(Y\cdot a(X)\big)a(Z)+a(X)\big(Y\cdot a(Z)\big)+\big(Y\cdot b(X)\big)b(Z)+b(X)\big(Y\cdot b(Z)\big)\\&\quad-\big(Z\cdot a(X)\big)a(Y)-a(X)\big(Z\cdot a(Y)\big)-\big(Z\cdot b(X)\big)b(Y)-b(X)\big(Z\cdot b(Y)\big)\end{aligned}$$

These terms can be combined with the terms involving the Lie bracket of vector fields in our expansion of the Koszul formula, for example,

$$a(Y)\big(X\cdot a(Z)\big)-a(Y)\big(Z\cdot a(X)\big)-a(Y)a([X,Z])=a(Y)da(X,Z)$$

Using similar arguments for db, we have that

$$\begin{aligned}2\widetilde{g}\left(\widetilde{\nabla}_XY,Z\right)&=2g\left(\nabla_XY,Z\right)+2\big(X\cdot a(Y)-da(X,Y)\big)a(Z)\\&\quad+2\big(X\cdot b(Y)-db(X,Y)\big)b(Z)+a(Y)da(X,Z)\\&\quad+a(X)da(Y,Z)+b(Y)db(X,Z)+b(X)db(Y,Z)\end{aligned}$$

We can simplify the 2-forms da, db using the equations of (34), for example

$$\begin{aligned}da(X,Y)&=-g(KX,Y)+b(X)c(Y)-b(Y)c(X)\\db(X,Y)&=-g(JX,Y)+c(X)a(Y)-c(Y)a(X)\end{aligned}$$

Using these formulas, making appropriate cancellations, and gathering like term yields the formula

$$\begin{aligned}2\widetilde{g}\left(\widetilde{\nabla}_XY,Z\right)&=2g\left(\nabla_XY,Z\right)\\&\quad-g\big(a(X)KY+a(Y)KX+b(X)JY+b(Y)JX,Z\big)\\&\quad+\Big(2\big(X\cdot a(Y)\big)+2b(Y)c(X)+g(KX,Y)\Big)a(Z)\\&\quad+\Big(2\big(X\cdot b(Y)\big)-2a(Y)c(X)+g(JX,Y)\Big)b(Z)\end{aligned}$$

Considering the two terms in the above involving the metric g, both can be rewritten in terms of the metric $\widetilde{g}$ using equation (39). For the first, we have

$$\widetilde{g}\left(\nabla_XY,Z\right)=g\left(\nabla_XY,Z\right)+a\left(\nabla_XY\right)a(Z)+b\left(\nabla_XY\right)b(Z),$$

while for the second we have

$$\begin{aligned}&\widetilde{g}\big(a(X)KY+a(Y)KX+b(X)JY+b(Y)JX,Z\big)\\&\quad=g\big(a(X)KY+a(Y)KX+b(X)JY+b(Y)JX,Z\big)\\&\qquad+\big(a(X)a(KY)+a(Y)a(KX)+b(X)a(JY)+b(Y)a(JX)\big)a(Z)\\&\qquad+\big(a(X)b(KY)+a(Y)b(KX)+b(X)b(JY)+b(Y)b(JX)\big)b(Z)\end{aligned}$$

Substituting these two equalities into the expression and rearranging gives that $2\widetilde{g}\left(\widetilde{\nabla}_X Y, Z\right)$ is equal to

$$
\begin{aligned}
2\widetilde{g}\left(\nabla_X Y, Z\right) - \widetilde{g}\big(a(X)KY + a(Y)KX + b(X)JY + b(Y)JX, Z\big) \\
+ \Big(- 2a\left(\nabla_X Y\right) + 2\big(X \cdot a(Y)\big) + 2b(Y)c(X) + g(KX, Y) \\
+ a(X)a(KY) + a(Y)a(KX) + b(X)a(JY) + b(Y)a(JX)\Big)a(Z) \\
+ \Big(- 2b\left(\nabla_X Y\right) + 2\big(X \cdot b(Y)\big) - 2a(Y)c(X) + g(JX, Y) \\
+ a(X)b(KY) + a(Y)b(KX) + b(X)b(JY) + b(Y)b(JX)\Big)b(Z)
\end{aligned}
$$

Finally, the tensor $B(X,Y)$ is defined precisely so that the remaining $a(Z), b(Z)$ terms in this expression can be rewritten as

$$\widetilde{g}\left(B(X,Y)JX_0 - B(X,IY)KX_0, Z\right)$$

To check this, expanding this expression using the definition of $\widetilde{g}$ yields

$$
\begin{aligned}
&\widetilde{g}(B(X,Y)JX_0 - B(X,IY)KX_0, Z) \\
&\quad = B(X,Y)g(JX_0, Z) - B(X,IY)g(KX_0, Z) \\
&\qquad + B(X,Y)a(JX_0)a(Z) - B(X,IY)a(KX_0)a(Z) \\
&\qquad + B(X,Y)b(JX_0)b(Z) - B(X,IY)b(KX_0)b(Z) \\
&\quad = B(X,Y)\rho_1 a(Z) + B(X,IY)\rho_1 b(Z) \\
&\qquad + B(X,Y)\rho_1^{-1}g(X_0,X_0)a(Z) + B(X,IY)\rho_1^{-1}g(X_0,X_0)b(Z) \\
&\quad = \big(\rho_1 + \rho_1^{-1}g(X_0,X_0)\big)\, B(X,Y)a(Z) + \big(\rho_1 + \rho_1^{-1}g(X_0,X_0)\big)\, B(X,IY)b(Z)
\end{aligned}
$$

using the equalities of (20) and (36). Therefore defining $B(X,Y)$ as in equation (38) gives that the $a(Z)$ term in this expansion exactly matches the $a(Z)$ term that remains to be simplified in our expression for $2\widetilde{g}\left(\widetilde{\nabla}_X Y, Z\right)$.

To check the $b(Z)$ terms, we have that $(\rho_1 + \rho_1^{-1}g(X_0,X_0))B(X,IY)$ is given by the expression

$$
\begin{aligned}
&-2a\left(\nabla_X(IY)\right) + 2\big(X \cdot a(IY)\big) + 2b(IY)c(X) + g(KX, IY) \\
&\quad + a(X)a(KIY) + a(IY)a(KX) + b(X)a(JIY) + b(IY)a(JX) \\
&\quad = -\,2a\left((\nabla_X I)Y\right) - 2a\left(I\nabla_X Y\right) + 2\big(X \cdot b(Y)\big) - 2a(Y)c(X) + g(JX, Y) \\
&\quad + a(X)a(JY) + b(Y)a(KX) - b(X)a(KY) - a(Y)a(JX) \\
&\quad = -\,2a(X)a(JY) + 2b(X)a(KY) - 2b\left(\nabla_X Y\right) + 2\big(X \cdot b(Y)\big) - 2a(Y)c(X) \\
&\quad + g(JX, Y) + a(X)a(JY) + b(Y)b(JX) - b(X)a(KY) + a(Y)b(KX) \\
&\quad = -\,2b\left(\nabla_X Y\right) + 2\big(X \cdot b(Y)\big) - 2a(Y)c(X) + g(JX, Y) \\
&\quad - a(X)a(JY) + b(Y)b(JX) + b(X)a(KY) + a(Y)b(KX) \\
&\quad = -\,2b\left(\nabla_X Y\right) + 2\big(X \cdot b(Y)\big) - 2a(Y)c(X) + g(JX, Y) \\
&\quad + a(X)b(KY) + b(Y)b(JX) + b(X)b(JY) + a(Y)b(KX)
\end{aligned}
$$

where we make use of the equalities in (35) in order to exchange a and b. This expression gives exactly $b(Z)$ terms that remain to be simplified.

Combining all of our simplifications, we have

$$
\begin{aligned}
2\widetilde{g}(\widetilde{\nabla}_X Y, Z) = 2\widetilde{g}\left(\nabla_X Y\right) \\
&- \widetilde{g}\big(a(X)KY + a(Y)KX + b(X)JY + b(Y)JX, Z\big) \\
&+ \widetilde{g}\big(B(X,Y)JX_0 - B(X, IY)KX_0, Z\big)
\end{aligned}
$$

Dividing by 2 and gathering terms gives

$$
\begin{aligned}
\widetilde{g}\left(\widetilde{\nabla}_X Y, Z\right) = \widetilde{g}\Big(\nabla_X Y - \frac{1}{2}\big(a(X)KY + a(Y)KX + b(X)JY + b(Y)JX\big) \\
+ \frac{1}{2}\big(B(X,Y)JX_0 - B(X, IY)KX_0\big), Z\Big),
\end{aligned}
$$

and so we have the desired formula as $\widetilde{g}$ is non-degenerate. □

As a check of this expression, we can note that the complex structure I is in fact parallel with respect to the connection $\widetilde{\nabla}$ determined by this formula, as we have

$$
\begin{aligned}
(\widetilde{\nabla}_X I)Y &= \widetilde{\nabla}_X(IY) - I\widetilde{\nabla}_X Y \\
&= \nabla_X(IY) - \frac{1}{2}\big(a(X)KIY + a(IY)KX + b(X)JIY + b(IY)JX\big) \\
&\qquad + \frac{1}{2}\big(B(X,IY)JX_0 - B(X,I^2Y)KX_0\big) \\
&\quad - I\nabla_X Y + \frac{1}{2}\big(a(X)IKY + a(Y)IKX + b(X)IJY + b(Y)IJX\big) \\
&\qquad - \frac{1}{2}\big(B(X,Y)IJX_0 - B(X,IY)IKX_0\big) \\
&= (\nabla_X I)Y - \frac{1}{2}\big(a(X)JY + b(Y)KX - b(X)KY - a(Y)JX\big) \\
&\qquad + \frac{1}{2}\big(B(X,IY)JX_0 + B(X,Y)KX_0\big) \\
&\qquad + \frac{1}{2}\big(-a(X)JY - a(Y)JX + b(X)KY + b(Y)KX\big) \\
&\qquad - \frac{1}{2}\big(B(X,Y)KX_0 + B(X,IY)JX_0\big) \\
&= (\nabla_X I)Y - a(X)JY + b(X)KY \\
&= a(X)JY - b(X)KY - a(X)JY + b(X)KY = 0.
\end{aligned}
$$

A.2 Proof of Proposition 5.13

In this section, we explicitly carry out the computations in Section 5.2 required to obtain the metric described in Proposition 5.13. Although we give enough details so that the results can be verified by hand, most of the actual computations in this section were performed using Mathematica.

We begin by describing the quaternion-Kähler structure on S^4. The round metric on S^4 is conformally flat and has constant curvature, hence is Einstein and therefore by definition quaternion-Kähler. The quaternionic structure arises from the isomorphism $\mathbb{HP}_1 \cong S^4$ given by stereographic projection. In particular, if we consider the diffeomorphism $S^4 - \{pt\} \cong \mathbb{R}^4$ given by stereographic projection, then we can choose a natural compatible frame for the quaternionic structure by identifying $T_x\mathbb{R}^4 \cong \mathbb{R}^4 \cong \mathbb{H}$ and letting I, J, K act as i, j, k. This can be expressed as matrices by using standard Cartesian coordinates $\{x_0, x_1, x_2, x_3\}$ for $\mathbb{R}^4$ and taking the matrix expressions for left quaternion multiplication given by equation (3). The resulting transformations are then orthogonal with respect to the metric given by equation (41).

Alternately, we could use the fact that for a quaternion-Kähler 4-manifold, the bundle $\mathcal{Q}$ is related to the bundle Λ^+ of self-dual 2-forms, so that choosing the compatible basis I, J, K for $\mathcal{Q}$ is equivalent to choosing a basis for Λ^+. The basis for the self-dual forms associated to the choice of I, J, K described above is then

$$\begin{aligned}\omega_I &= \frac{4}{(1+x_0^2+x_1^2+x_2^2+x_3^2)^2}\,(dx_0\wedge dx_1 + dx_2\wedge dx_3)\\ \omega_J &= \frac{4}{(1+x_0^2+x_1^2+x_2^2+x_3^2)^2}\,(dx_0\wedge dx_2 - dx_1\wedge dx_3)\\ \omega_K &= \frac{4}{(1+x_0^2+x_1^2+x_2^2+x_3^2)^2}\,(dx_0\wedge dx_3 + dx_1\wedge dx_2)\end{aligned}$$

As discussed in Section 5.5, though, it will be more convenient to use a coordinate system adapted to the Killing field we will consider. We first change to the double polar coordinates given by equation (42), for which the metric has the expression already given in equation (43). With respect to these coordinates, we have that the basis for the self-dual forms considered above is

$$\begin{aligned}\omega_I &= \frac{4}{(1+\rho^2+\sigma^2)^2}\,(\rho d\rho\wedge d\varphi + \sigma d\sigma\wedge d\theta)\\ \omega_J &= \frac{4}{(1+\rho^2+\sigma^2)^2}\Big(\cos(\varphi+\theta)d\rho\wedge d\sigma - \rho\sigma\cos(\varphi+\theta)d\varphi\wedge d\theta\\ &\quad - \rho\sin(\varphi+\theta)d\varphi\wedge d\sigma - \sigma\sin(\varphi+\theta)d\rho\wedge d\theta\Big)\\ \omega_K &= \frac{4}{(1+\rho^2+\sigma^2)^2}\Big(\sin(\varphi+\theta)d\rho\wedge d\sigma - \rho\sigma\sin(\varphi+\theta)d\varphi\wedge d\theta\\ &\quad + \rho\cos(\varphi+\theta)d\varphi\wedge d\sigma + \sigma\cos(\varphi+\theta)d\rho\wedge d\theta\Big)\end{aligned}$$

The Kähler metric we ultimately obtain is expressed via equation (39) in terms of the proto-Kähler form ω_I as well as the connection 1-forms a, b associated to the action of the connection on $\mathcal{Q}$. In order to obtain expressions for these 1-forms in local coordinates, we can first find the Christoffel symbols associated to the metric in our given coordinate system to compute the action of the connection on vector fields, and then use the definition $(\nabla_X I)(Y) = \nabla_X(IY) - I\nabla_X Y$ to find the connection 1-forms. The resulting forms are

$$
\begin{aligned}
a &= \frac{2}{1+\rho^2+\sigma^2}\Big(\sigma\sin(\varphi+\theta)d\rho + \rho\sigma\cos(\varphi+\theta)d\varphi - \rho\sin(\varphi+\theta)d\sigma \\
&\qquad\qquad - \rho\sigma\cos(\varphi+\theta)d\theta\Big) \\
b &= \frac{2}{1+\rho^2+\sigma^2}\Big(\sigma\cos(\varphi+\theta)d\rho - \rho\sigma\sin(\varphi+\theta)d\varphi - \rho\cos(\varphi+\theta)d\sigma \\
&\qquad\qquad + \rho\sigma\sin(\varphi+\theta)d\theta\Big) \\
c &= \frac{2}{1+\rho^2+\sigma^2}\left(-\rho^2 d\varphi - \sigma^2 d\theta\right)
\end{aligned}
$$

With these explicit expressions, it is straightforward to verify that the equalities of (34) hold.

Of course, the above expressions all depend on the choice of compatible basis $\{I, J, K\}$, and the setup of Theorem 5.7 assumes that the choice of compatible basis is such that I is the complex structure related to a momentum section. In order to compute explicit formulas for our metric, we will need to choose a different local compatible basis, where the first member is related to the momentum section of a Killing field, and then find new expressions for the forms $\omega_I, \omega_J, \omega_K, a, b, c$ with respect to this new basis.

Generically speaking, this process can be carried out as follows. After specifying the Killing field, find the associated momentum section by taking the self-dual part of the exterior derivative of the 1-form dual to the Killing field. Normalize this momentum section to produce the almost complex structure $\widetilde{I}$ that will ultimately be parallel with respect to the new connection, and then complete this to a local frame $\{\widetilde{I}, \widetilde{J}, \widetilde{K}\}$. This yields a change of basis matrix, as in Corollary 2.19, that can then be used to transform the proto-Kähler forms $\omega_I, \omega_J, \omega_K$ with respect to the original basis into $\widetilde{\omega}_I, \widetilde{\omega}_J, \widetilde{\omega}_K$ with respect to the new basis. This change of basis matrix also then gives the change of connection forms in the usual way. Alternately, once could directly recompute the connection 1-forms from the Christoffel symbols. Finally, expressions in coordinates for the Kähler form and metric can then be read off directly from the result using equation (39).

To complete the proof of Proposition 5.13, we carry out this procedure with the Killing field $X_0 = \frac{\partial}{\partial\theta}$. The exterior derivative of the dual 1-form related to this field is then

$$
dX_0^\flat = \frac{8\sigma}{(1+\rho^2+\sigma^2)^3}\left(-2\rho\sigma d\rho\wedge d\theta + (1+\rho^2-\sigma^2)d\sigma\wedge d\theta\right)
$$

We then need to take the self-dual part of this 2-form, which can be accomplished by taking the inner product of the form above with $\omega_I, \omega_J, \omega_K$. The result is that the momentum section ρ_{X_0} is given by $\rho_{X_0} = \rho_1\omega_I + \rho_2\omega_J + \rho_3\omega_K$, where the coordinate functions are

$$\rho_1 = \frac{1+\rho^2-\sigma^2}{1+\rho^2+\sigma^2} \qquad \rho_2 = \frac{2\rho\sigma\sin(\varphi+\theta)}{1+\rho^2+\sigma^2} \qquad \rho_3 = \frac{-2\rho\sigma\cos(\varphi+\theta)}{1+\rho^2+\sigma^2}$$

These functions allow us to identify the set $M_0 = M - \{\rho_{X_0} = 0\}$. We observe that the $\sin(\varphi+\theta), \cos(\varphi+\theta)$ terms in ρ_2, ρ_3 will never simultaneously vanish, and so $\rho_2 = 0 = \rho_3$ only when either $\rho = 0$, $\sigma = 0$, or both. However, we see that if $\sigma = 0$ then ρ_1 will never vanish. Conversely, if $\rho = 0$ the we have that $\rho_1 = 0$ only if $\sigma = 1$. Therefore in current coordinate system, we have that $M_0 = M - \{\rho = 0 \text{ and } \sigma = 1\}$. Note that the set we remove is mapped to a circle in S^4 under the stereographic projection.

The desired almost complex structure on M_0 and its related 2-form are then given explicitly in coordinates by

$$\widetilde{I} = \frac{1}{\sqrt{\rho_1^2+\rho_2^2+\rho_3^2}}\left(\rho_1 I + \rho_2 J + \rho_3 K\right)$$
$$\widetilde{\omega}_I = \frac{1}{\sqrt{\rho_1^2+\rho_2^2+\rho_3^2}}\left(\rho_1\omega_I + \rho_2\omega_J + \rho_3\omega_K\right)$$

Using the Gram-Schmidt process on the basis $\{\widetilde{I}, J, K\}$ for $\mathcal{Q}$ yields a new local compatible frame $\{\widetilde{I}, \widetilde{J}, \widetilde{K}\}$. Alternately, given two vectors in the compatible basis, one could also use the cross product to obtain the third. With our choices, we take the compatible frame $\{\widetilde{I}, \widetilde{J}, \widetilde{K}\}$ given by $\widetilde{I}$ above, with

$$\widetilde{K} = \frac{2\rho\sigma\cos(\varphi+\theta)I + (1+\rho^2-\sigma^2)K}{\sqrt{\rho^4+2\rho^2+(\sigma^2-1)^2+2\rho^2\sigma^2\cos(2(\varphi+\theta))}}$$

and $\widetilde{J} = \widetilde{K}\times\widetilde{I}$, when these are considered as coordinated vectors with respect to the $\{I, J, K\}$ frame. It is straightforward if tedious to check that this is in fact a compatible basis.

With respect to this basis, we can once again compute the connection 1-forms, which we now call $\widetilde{a}, \widetilde{b}, \widetilde{c}$. Before giving expressions for these forms, we set some notation that will allow us to represent our formulas more compactly. Let

$$C_1 = 1+\rho^2+\sigma^2$$
$$C_2 = 1+\rho^2-\sigma^2$$
$$C_3 = (\rho_1^2+\rho_2^2+\rho_3^2)C_1^2 = \rho^4+(\sigma^2-1)^2+2\rho^2(1+\sigma^2)$$
$$C_4 = \sqrt{\rho^4+2\rho^2+(\sigma^2-1)^2+2\rho^2\sigma^2\cos(2(\varphi+\theta))}$$

Using these, we have the following expressions for $\widetilde{a}, \widetilde{b}, \widetilde{c}$, the connection 1-forms in the local compatible frame $\{\widetilde{I}, \widetilde{J}, \widetilde{K}\}$.

$$
\begin{aligned}
\widetilde{a} &= \frac{4\sigma}{C_1 C_4}\left(\frac{C_2^2}{C_3}\sin(\varphi+\theta)d\rho + \rho\cos(\varphi+\theta)d\varphi + \frac{2C_2}{C_3}\rho\sigma\sin(\varphi+\theta)d\sigma\right)\\
\widetilde{b} &= \frac{4\sigma}{C_1 C_4\sqrt{C_3}}\left(C_2\cos(\varphi+\theta)d\rho - C_2\rho\sin(\varphi+\theta)d\varphi + 2\rho\sigma\cos(\varphi+\theta)d\sigma\right)\\
\widetilde{c} &= \frac{2}{C_4^2\sqrt{C_3}}\Big(-\rho\sigma^2(\rho^2+\sigma^2-1)\sin(2(\varphi+\theta))d\rho\\
&\quad -\rho^2(1+\rho^4-\sigma^2+\rho^2(2+\sigma^2)+\sigma^2(\rho^2+\sigma^2-1)\cos(2(\varphi+\theta)))d\varphi\\
&\quad + C_1\rho^2\sigma\sin(2(\varphi+\theta))d\sigma + \frac{\sigma^2 C_3}{C_1}(\sigma^2-1+\rho^2\cos(2(\varphi+\theta)))d\theta\Big)
\end{aligned}
$$

With these expressions, we have an explicit formula for the Kähler form $W = \widetilde{\omega}_I - \widetilde{a}\wedge\widetilde{b}$ on M_0, which, along with the explicit formula for $\widetilde{I}$ given above, we can use to obtain a formula for the Kähler metric $\widetilde{g}$

$$
\begin{aligned}
\widetilde{g} &= \frac{4(C_1^2C_2^2 + 4\rho^2\sigma^2 C_3)}{C_1^2C_3^2}d\rho^2 + \frac{4\rho^2}{C_3}d\varphi^2\\
&\quad + \frac{4}{C_1^2C_3^2}\Big(4\left(\rho^2-1\right)\sigma^6 + 6\left(\rho^2+1\right)^2\sigma^4 + 4\left(\rho^2-1\right)\left(\rho^2+1\right)^2\sigma^2 +\\
&\quad + \left(\rho^2+1\right)^4 + \sigma^8\Big)d\sigma^2 + \frac{4\sigma^2}{C_1^2}d\theta^2 + \frac{32C_2\rho\sigma^3}{C_1^2C_3^2}(d\rho\otimes d\sigma + d\sigma\otimes d\rho)
\end{aligned}
$$

Although this expression is correct, and can be used to obtain further information about the metric (in particular, one can compute the scalar curvature from this expression), it is clearly not the most enlightening way to write the metric. Following Hitchin, we can instead make the coordinate change to (u,θ,v,φ) using equation (44). Note from these formulas that the domain of the coordinate u is $0 \le u < \frac{1}{2}$, as

$$
\frac{\sigma}{1+\rho^2+\sigma^2} < \frac{\sigma}{1+\sigma^2} \le \frac{1}{2},
$$

The inverse expression giving ρ, σ in terms of u, v is then

$$
\begin{aligned}
\rho &= \frac{\sqrt{(1-4u^2)\left(v^2+4\right)} + v(1-4u^2)}{2u^2v^2+2}\\
\sigma &= \frac{uv\sqrt{(1-4u^2)\left(v^2+4\right)} + u(v^2+4)}{2u^2v^2+2}
\end{aligned}
$$

In these coordinates, the set where the metric is not defined is the set where $1 - 4u^2 = 0$. We can pull back the expression for $\widetilde{g}$ presented above using this coordinate change, and the result is the formula for the metric $\widetilde{g}$ in the (u, θ, v, φ) coordinates presented in equation (45), completing the proof of Proposition 5.13.