Classification of gravitational instantons with faster than quadratic curvature decay

A Dissertation presented

by

Gao Chen

to

Stony Brook University

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2017

Stony Brook University

The Graduate School

Gao Chen

We, the dissertation committe for the above candidate for the

Doctor of Philosophy degree, hereby recommend

acceptance of this dissertation

Xiuxiong Chen - Dissertation Advisor
Professor, Department of Mathematics

Blaine Lawson - Chairperson of Defense
Professor, Department of Mathematics

Claude LeBrun
Professor, Department of Mathematics

Martin Roček
Professor, Institute for Theoretical Physics

This dissertation is accepted by the Graduate School

Charles Taber
Dean of the Graduate School

Abstract of the Dissertation

Classification of gravitational instantons with faster than quadratic curvature decay

by

Gao Chen

Doctor of Philosophy

in

Mathematics

Stony Brook University

2017

In this dissertation, a gravitational instanton is defined to be a complete non-compact hyperkähler 4-manifold with curvature decaying fast enough at infinity. Many examples of gravitational instantons have been constructed. The main result of this thesis is the proof that there are no more examples.

Table of Contents

Contents

List of Figures

Acknowledgements

I am deeply indebted to my PhD advisor Professor Xiuxiong Chen. He not only taught me math but also the way to do math. I am also grateful to Professor Sir Simon Donaldson, Samuel Grushevsky, Blaine Lawson, Robert Lazarsfeld, Claude LeBrun, Dennis Sullivan, Song Sun and Aleksey Zinger for their insightful and illuminating courses. Besides them, I would also express my great gratitude to Professor Olivier Biquard, Gilles Carron, Jeff Cheeger, Sergey Cherkis, Kenji Fukaya, Mark Haskins, Hans-Joachim Hein, Nigel Hitchin, Peter Kronheimer, Vincent Minerbe, Martin Roček, Shing-Tung Yau and other equally important pioneers whose work and helpful discussions with me lead me to complete classification of gravitational instantons.

I also want to thank my friends for accompanying me on my math careers. A small part of the full list is: Seyed Ali Aleyasin, Jean-François Arbour, Simone Calamai, Xuemiao Chen, Xujia Chen, Lorenzo Foscolo, Shaosai Huang, Chi Li, Long Li, Vamsi Pingali, Santai Qu, Jingzhou Sun, Yuhan Sun, Zhuang Tao, Selin Taşkent, Alex Waldron, Xiaojie Wang, Yuanqi Wang, Ruijie Yang, Chengjian Yao, Yu Zeng, Dingxin Zhang, Ruobing Zhang and Yizhong Zheng.

Last but not least, I would like to thank my father Qianlin Chen, mother Chahua Cen and grandparents for bring me up as well as my twin sister Yao Chen for a happy childhood.

1 Introduction

A gravitaional instanton is a non-compact, complete hyperkähler 4-manifold with curvature decaying fast enough at infinity. In 1977, gravitational instanton was first introduced by Hawking as building block of the Euclidean quantum gravity theory[36]. Even though physicists expect the faster than quadratic curvature decay at infinity, this seems hasn't been made precise in literatures.

For clarity, we always assume that the curvature satisfies a decay condition

$$|\mathrm{Rm}|(x) \leq r(x)^{-2-\epsilon},$$

where $r(x)$ denotes the metric distance to a base point o in the complex surface and $\epsilon > 0$ is any small positive number, say $< \frac{1}{100}$.

One of our main tools comes from the equivalence between the the Calabi-Yau condition and the hyperkähler condition. Actually, there are three complex structures I, J, K on hyperkähler manifolds. They induce three symplectic forms by

$$\omega_1(X,Y) = g(IX,Y), \omega_2(X,Y) = g(JX,Y), \omega_3(X,Y) = g(KX,Y).$$

The form $\omega^+ = \omega_2 + i\omega_3$ is a I-holomorphic symplectic form. This induces the equivalence of Sp(1) and SU(2). Notice that for any $(a_1, a_2, a_3) \in \mathbb{S}^2$, $a_1 I + a_2 J + a_3 K$ is a Kähler structure. There is a special property of Sp(1): Given any vectors $v, w \in T_p$ which are orthogonal to each other and have same length, there exists an (a_1, a_2, a_3) in $\mathbb{S}^2$ such that $(a_1 I + a_2 J + a_3 K)v = w$. We will use this property to find the best complex structure.

It's quite easy to prove the following theorem:

Theorem 1.1. *For gravitational instanton M, the following conditions are equivalent:*
(1) M is flat;
(2) M has trivial holonomy;
(3) M splits as $\mathbb{R}^{4-k} \times \mathbb{T}^k$, $k = 0, 1, 2, 3$.

It will be proved in Section 2. For simplicity, in this paper, we will exclude the flat gravitational instantons.

Under those conditions, we want to study two fundamental questions:

1. The differential and metric structure of the infinity of these gravitational instantons. Note that this is different from the tangent cone at infinity, especially if the volume growth is sub-Euclidean.

2. Given these end structures, to what extent, do we know these instantons globally and holomorphically? In other words, is gravitational instanton uniquely determined by its end structure?

Both problems seem to be well known to the research community. Since 1977, many examples of gravitational instantons have been constructed[36] [4] [48] [22]. The end structures of these examples are completely known now. According to the volume growth rate, they can be divided into four categories: ALE, ALF, ALG and ALH, where the volume growth are of order 4,3,2 and 1 respectively. For the convenience of readers, we will give a precise definition of these ends in Section 2.1. There is a folklore conjecture that when the curvature decay fast enough, any gravitational instantons must be asymptotic to one of the standard models of ends.

Except the ALE case, the asymptotical volume growth rate is usually hard to control, and may oscillate and may even not be an integer. In an important paper, with additional assumption that the volume growth rate is sub-Euclidean but at least cubic and a slightly weaker curvature decay condition depending on volume growth rate, Minerbe[57][58] proved that it must be ALF. In our paper, we first prove the folklore conjecture.

Theorem 1.2. *Let (M^4, g) be a non-compact, complete, non-flat hyperkähler manifold with curvature decay condition $|\mathrm{Rm}|(x) \le r(x)^{-2-\epsilon}$, then it must be asymptotic to the standard metric of order ϵ. Consequently, it must be one of the four families: ALE,ALF, ALG and ALH.*

For more detail about this theorem, see Theorem 2.14, and Theorem 2.22. We would like to remark that the curvature condition can't be weaken to $|\mathrm{Rm}| = O(r^{-2})$. In 2012, besides the study of ALG and ALH instantons on rational elliptic surfaces, Hein[37] also constructed two new classes of hyperkähler metrics on rational elliptic surfaces with volume growth, injective radius decay, and curvature decay rates $r^{4/3}$, $r^{-1/3}$, r^{-2}, and r^2, $(\log r)^{-1/2}$, $r^{-2}(\log r)^{-1}$, respectively. Note that curvature doesn't satisfy $|\mathrm{Rm}| = O(r^{-2-\epsilon})$ and they don't belong to any of the four families!

Our new contribution lies in ALG and ALH cases; in ALF case, our contribution is to remove the volume growth constraint from Minerbe's work [57].

In fact, Minerbe's volume growth constraint becomes an corollary instead condition of our first main theorem.

We obviously benefit from studying a series of papers by Minerbe [57], [58], and [59]. Although his work seems only valid in ALF-A_k case, we manage to make some modest progress in all cases in the present work.

The next essential step is the improvement of the asymptotic rate.

Theorem 1.3. *Given any non-flat gravitational instanton (M, g), there exist a bounded domain $K \subset M$ and a diffeomorphism $\Phi : E \to M \setminus K$ such that the error term $Err = \Phi^* g - h$ satisfies*

(ALE) $|\nabla^m Err| = O(r^{-4-m}), \forall m \geq 0$.

(ALF-A_k and ALF-D_k) $|\nabla^m Err| = O(r^{-3-m}), \forall m \geq 0$*;*

(ALG) $|\nabla^m Err| = O(r^{-\delta-m}), \forall m \geq 0$, *where* $\delta = \min_{n\in\mathbb{Z}, n<2\beta} \frac{2\beta-n}{\beta}$. *In other words,*

Type	*Regular*	I_0^*	*II*	*II**	*III*	*III**	*IV*	*IV**
β	*1*	$\frac{1}{2}$	$\frac{1}{6}$	$\frac{5}{6}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{3}$	$\frac{2}{3}$
δ	*1*	*2*	*2*	$\frac{4}{5}$	*2*	$\frac{2}{3}$	*2*	$\frac{1}{2}$

(ALH) $|\nabla^m Err| = O(e^{-\delta r}), \forall m \geq 0$, *where* $\delta = 2\pi \min_{\lambda\in\Lambda^*\setminus\{0\}} |\lambda|$*;*

Actually, we will show that the deformation space of hyperkähler 4-manifolds is a subspace of the space of closed anti-self-dual forms. Therefore, the asymptotic rate is at least the decay rate of the first closed anti-self-dual form.

Remark that the ALE part of Theorem 1.3 was done by Bando, Kasue and Nakajima [5]. The ALF-A_k part was done by Minerbe [59]. So we will focus on the other three parts in this paper.

For the ALE part of the second question, after Bando-Kasue-Nakajima's work [5] about the improvement of asymptotic rate, Kronheimer [48] [49] proved that any ALE gravitational instanton must be diffeomorphic to the minimal resolution $\widetilde{\mathbb{C}^2/\Gamma}$ of the quotient singularity $\mathbb{C}^2/\Gamma$, where Γ is a finite subgroup of SU(2). Moreover, the Torelli theorem holds for ALE gravitational instantons.

Theorem 1.4. *(Torelli theorem for ALE gravitational instantons)([48, 49])*

Let M be the smooth 4-manifold which underlies the minimal resolution of $\mathbb{C}^2/\Gamma$. Let $[\alpha^1], [\alpha^2], [\alpha^3] \in H^2(M, \mathbb{R})$ be three cohomology classes which satisfy the nondegeneracy condition:

For each $[\Sigma] \in H_2(M, \mathbb{Z})$ with $[\Sigma]^2 = -2$, there exists $i \in \{1, 2, 3\}$ with $[\alpha^i][\Sigma] \neq 0$.

Then there exists on M an ALE hyperkähler structure such that the cohomology classes of the Kähler forms $[\omega^i]$ are the given $[\alpha^i]$. It's unique up to tri-holomorphic isometries which induce identity on $H_2(M, \mathbb{Z})$.

Moreover, any ALE gravitational instanton must be constructed by this way.

$H_2(\widetilde{\mathbb{C}^2/\Gamma}, \mathbb{Z})$ is generated by holomorphic curves with self intersection number -2. Let k be the number of generators. Then, their intersection patterns can be classified into $A_k(k \geq 1), D_k(k \geq 4), E_k(k = 6, 7, 8)$ Dynkin diagrams. They correspond to different types of Γ.

A crucial point in Kronheimer's work is to understand "the end" holomorphically. In ICM 1978, Yau conjectured that every complete Calabi-Yau manifold can be compactified in the complex analytic sense[79]. There are counterexamples if we only assume the completeness without fast curvature decaying condition[3]. However, when we assume the faster than quadratic curvature decay condition, we can prove Yau's conjecture. In higher dimension $n \geq 3$, assuming the curvature exponentially decay and the metric is asymptotically cylindrical, Haskins, Hein and Nordström [35] constructed a compactification and therefore verified Yau's conjecture in their settings.

Theorem 1.5. *For any ALG or ALH gravitational instanton M, there exist a rational elliptic surface $\bar{M}$ with a meromorphic function $u : \bar{M} \to \mathbb{CP}^1$ whose generic fiber is torus. The fiber $D = \{u = \infty\}$ is regular if M is ALH, while it's of type I_0^*, II, II^*, III, III^*, IV, IV^* if M is ALG. There exist an (a_1, a_2, a_3) in $\mathbb{S}^2$ such that when we use $a_1 I + a_2 J + a_3 K$ as the complex structure, M is biholomorphic to $\bar{M} - D$.*

Remark. The type of D is related to the tangent cone at infinity of M. See the table in Theorem 2.22.

In ALF case, more discussions are needed:

1. In ALF-A_k case, Minerbe[59] proved that any ALF-A_k instanton must be the trivial product or the multi-Taub-NUT metric. In particular, there is no ALF-A_k instantons for $k < -1$.

2. In ALF-D_k case, Biquard and Minerbe [8] proved that there is no ALF-D_k instantons for $k < 0$. For $k \geq 0$, the first example was constructed

by Atiyah and Hitchin[4], where $k = 0$. Ivanov and Roček [42] conjectured a formula for larger k using generalized Legendre transform developed by Lindström and Roček [52]. This conjecture was proved by Cherkis and Kapustin[22] and computed more explicitly by Cherkis and Hitchin [21]. When $k = 2$, it's the Hitchin-Page metric [38] [66]. It's conjectured that any ALF-D_k instanton must be exactly the metric constructed by them. The first step toward this conjecture is the existence of the $\mathcal{O}(4)$ multiplet which plays an important role in the Cherkis-Hitchin-Kapustin-Ivanov-Lindström-Roček construction.

Theorem 1.6. *In the ALF-D_k case, there exists a holomorphic map from the twistor space of M to the total space of the $\mathcal{O}(4)$ bundle over $\mathbb{CP}^1$ which commutes with both the projection to $\mathbb{CP}^1$ and the real structure.*

For the definitions of twistor space and the real structure, see Theorem 3.36 and Theorem 3.37.

With the improved asymptotic rate and existence of $\mathcal{O}(4)$ multiplet, we can prove that any ALF gravitational instanton can be compactified in the complex analytic sense. This confirms Yau's conjecture in ALF case. As Kodaira did in [45], we can then analyze the topology of the compactification. This allows us to give a complete classification of ALF-D_k gravitational instantons.

Theorem 1.7. *Any ALF-D_k gravitational instanton must be the Cherkis-Hitchin-Ivanov-Kapustin-Lindström-Roček metric.*

We will give a precise definition of the Cherkis-Hitchin-Ivanov-Kapustin-Lindström-Roček metric as Example 5.5 in Section 5.

To illustrate our method of proving Theorem 1.7, we will first use the same technique to give a new proof of a theorem of Minerbe [59]:

Theorem 1.8. *(Minerbe [59]) Any ALF-A_k gravitational instanton must be the multi-Taub-NUT metric.*

We will give a precise definition of the multi-Taub-NUT metric as Example 5.1 in Section 5.

Even though Theorem 1.8 has been proved by Minerbe using other methods, our new proof is meaningful because it's a simplification of Theorem 1.7.

As a corollary, we will prove a Torelli-type theorem for ALF gravitational instantons as an analogy of Kronheimer's results [48] [49]:

Corollary 1.9. *(Torelli-type theorem for ALF gravitational instantons)*

Let M be the 4-manifold which underlies an ALF-A_k or ALF-D_k gravitational instanton. Let $[\alpha^1], [\alpha^2], [\alpha^3] \in H^2(M, \mathbb{R})$ be three cohomology classes. Let $L > 0$ be any positive number. Then there exists on M an ALF hyperkähler structure for which the cohomology classes of the Kähler forms $[\omega^i]$ are the given $[\alpha^i]$ and the length of the asymptotic S^1-fiber goes to L at infinity. It's unique up to isometries which respect I,J, and K. Moreover, it's non-singular if and only if $[\alpha^i]$ satisfy the nondegeneracy condition:

For each $[\Sigma] \in H_2(M, \mathbb{Z})$ with $[\Sigma]^2 = -2$, there exists $i \in \{1, 2, 3\}$ with $[\alpha^i][\Sigma] \neq 0$.

In the ALG cases, Hein constructed lots of examples [37]. In this paper, we will slightly modify his construction and then prove that any ALG gravitational instanton must be obtained by the modified Hein's construction:

Theorem 1.10. *(1) Let $(\bar{M}, z)$ be a rational elliptic surface. Suppose the fiber $D = \{z = \infty\}$ has type I_0^*, II, II*, III, III*, IV, or IV*. Let $\omega^+ = \omega^2 + i\omega^3$ be a rational 2-form on $\bar{M}$ with $[D] = \{\omega^+ = \infty\}$. For any Kähler form ω on $\bar{M}$, there exists a real smooth polynomial growth function ϕ on $M = \bar{M} \setminus D$ such that $(M, \omega^1 = \omega + i\partial\bar{\partial}\phi, \omega^2, \omega^3)$ is an ALG gravitational instanton.*

(2) The form $\omega + i\partial\bar{\partial}\phi$ in the first part is uniquely determined by its asymptotic geometry.

(3) Given any ALG gravitational instanton written as $(M, \omega^1, \omega^2, \omega^3)$ after a hyperkähler rotation which replace $a_1 I + a_2 J + a_3 K$ in Theorem 1.5 by I. Then $\omega^+ = \omega^2 + i\omega^3$ is a rational 2-form on $\bar{M}$ with $[D] = \{\omega^+ = \infty\}$. There exist a Kähler form ω on $\bar{M}$ and a real smooth polynomial growth function ϕ on $M = \bar{M} \setminus D$ such that $\omega^1 = \omega + i\partial\bar{\partial}\phi$. When D is of type II, III*, or IV*, we may need a new choice of $\bar{M}$ to achieve this.*

It's interesting to notice that in [8], Biquard and Minerbe constructed ALF-$D_k (k \geq 4)$, ALG (I_0^*, II, III, IV) and ALH gravitational instantons on the minimal resolutions of the quotient of Taub-NUT metric by the binary dihedral group, $(\mathbb{R}^2 \times \mathbb{T}^2)/\mathbb{Z}_k (k = 2, 6, 4, 3)$ or $(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2$, respectively. The three cases ALG-II*, ALG-III*, ALG-IV* are all missing.

When D is of type $I_b (b = 1, 2, ...9)$ or $I_b^* (b = 1, 2, 3, 4)$, Hein [37] also constructed some hyperkähler metrics on $\bar{M} \setminus D$. Since they don't have fast enough curvature decay rates, we exclude them from the discussion. However, they are still very important because Cherkis and Kapustin [23] predicted complete hyperkähler metrics on the moduli space of periodic monopoles, which is a rational elliptic surface minus a fiber of type I_0^*, I_1^*, I_2^*, I_3^*, I_4^*. They are also related to the moduli space of solutions of Hitchin equations on a cylinder. Notice that they are called ALG-D_4, D_3, D_2, D_1, D_0 by Cherkis-Kapustin but certainly their definition is different from our definition. Thus, we suggest the notation ALG* to denote Hein's exceptional examples. In [28] [29], Cherkis-Kapustin's prediction was partially verified by Foscolo. He proved that the moduli space of periodic monopoles is a non-empty hyperkähler manifold. However, it's still unknown whether the metric is complete or whether it's an elliptic surface.

It's worthwhile to notice that Biquard and Boalch [7] proved that the moduli space of meromorphic connections on a curve is a complete hyperkähler manifold. In Boalch's previous work [9], he related such moduli space to the Painlevé equation. Following Okamoto's work [62] [63] [64] [65], Sakai [71] related the Painlevé equation to a rational elliptic surface $\bar{M}$ minus a fiber D. The type of the fiber D is related to a Dynkin diagram:

I_0^*	I_1^*	I_2^*	I_3^*	I_4^*	II	II*	III	III*	IV	IV*
D_4	D_3	D_2	D_1	D_0	E_8	A_0	E_7	A_1	E_6	A_2

It's not known whether the Biquard-Boalch's metric is ALG or ALG*. However, it's known that an open part of Biquard-Boalch's metric is diffeomorphic to the corresponding ALE/ALF gravitational instanton denoted by the same Dynkin diagram. See [10] and [11] for details.

In the ALH case, as a corollary of Theorem 1.5, any ALH gravitational instantons are diffeomorphic to each other. In particular, they are diffeomorphic to the minimal resolution of $(\mathbb{R}\times\mathbb{T}^3)/\mathbb{Z}_2$ by [8]. The torus $\mathbb{T}^3=\mathbb{R}^3/\Lambda$ is determined by the lattice $\Lambda=\mathbb{Z}v_1\oplus\mathbb{Z}v_2\oplus\mathbb{Z}v_3$. It's easy to see that $H_2(\widetilde{(\mathbb{R}\times\mathbb{T}^3)/\mathbb{Z}_2},\mathbb{R})=\mathbb{R}^{11}$ is generated by three faces F_{jk} spanned by v_j and v_k and eight rational curves Σ_j coming from the resolution of eight orbifold points in $(\mathbb{R}\times\mathbb{T}^3)/\mathbb{Z}_2$. Using those notations, we will prove the following classification result of ALH gravitational instantons:

Theorem 1.11. *(Torelli theorem for ALH gravitational instantons)*

Let M be the smooth 4-manifold which underlies the minimal resolution of $(\mathbb{R}\times\mathbb{T}^3)/\mathbb{Z}_2$. Let $[\alpha^1],[\alpha^2],[\alpha^3]\in H^2(M,\mathbb{R})$ be three cohomology classes which satisfy the nondegeneracy conditions:

(1) The integrals f_{ijk} of α^i on the three faces F_{jk} satisfy

$$\begin{vmatrix} f_{123} & f_{131} & f_{112} \\ f_{223} & f_{231} & f_{212} \\ f_{323} & f_{331} & f_{312} \end{vmatrix} > 0;$$

(2) For each $[\Sigma] \in H_2(M, \mathbb{Z})$ with $[\Sigma]^2 = -2$, there exists $i \in \{1, 2, 3\}$ with $[\alpha^i][\Sigma] \neq 0$.

Then there exists on M an ALH hyperkähler structure such that Φ in Theorem 1.3 can be chosen to be the identity map and the cohomology classes of the Kähler forms $[\omega^i]$ are the given $[\alpha^i]$. It's unique up to tri-holomorphic isometries which induce identity on $H_2(M, \mathbb{Z})$.

Moreover, any ALH gravitational instanton must be constructed by this way.

Remark. Recently, Haskins, Hein and Nordström [35] classified asymptotically cylindrical Calabi-Yau manifolds of complex dimension at least 3. In dimension 2, their analytic existence theorem (Theorem 4.1 of [35]) still holds. However, when $\mathbb{T}^3$ doesn't split isometrically as $\mathbb{S}^1 \times \mathbb{T}^2$, their geometric existence theorem (Theorem D of [35]) fails due to the lack of background Kähler form in the cohomology class.

Remark. In [37], Hein proved that the space of ALH gravitational instantons module isometries is 30 dimensional. After adding 3 parameters of hyperkähler rotations, the space of ALH gravitational instantons module triholomorphic isometries which induce identity on $H_2(M, \mathbb{Z})$ is 33 dimensional. Our Theorem 1.11 is consistence with Hein's computation.

It's interesting to compare Theorem 1.11 with the Torelli theorem for ALE gravitational instantons (Theorem 1.4), ALF gravitational instantons (Theorem1.9) as well as K3 surfaces, which was proved by Burns-Rapoport [12], Todorov [77], Looijenga-Peters [54] and Siu [75]. It was reformulated by Besse in Section 12.K of [6]. Anderson [2] also proved a version of Torelli theorem for K3 surfaces which allows orbifold singularities.

Theorem 1.12. *([6])(Torelli theorem for K3 surfaces)*

Let M be the smooth 4-manifold which underlies the minimal resolution of $\mathbb{T}^4/\mathbb{Z}_2$. Let Ω be the space of cohomology classes $([\alpha^1], [\alpha^2], [\alpha^3])$ in $H^2(M, \mathbb{R}) \oplus H^2(M, \mathbb{R}) \oplus H^2(M, \mathbb{R})$ which satisfy the following conditions:

(1) (Integrability)

$$\int_M \alpha^i \wedge \alpha^j = 2\delta_{ij} V.$$

(2) (Nondegeneracy) For any $[\Sigma] \in H_2(M,\mathbb{Z})$ with $[\Sigma]^2 = -2$, there exists $i \in \{1,2,3\}$ with $[\alpha^i][\Sigma] \neq 0$.

Ω has two components Ω^+ and Ω^-. For any $([\alpha^1],[\alpha^2],[\alpha^3]) \in \Omega^+$, there exists on M a hyperkähler structure for which the cohomology classes of the Kähler forms $[\omega^i]$ are the given $[\alpha^i]$. It's unique up to tri-holomorphic isometries which induce identity on $H_2(M,\mathbb{Z})$.

Moreover, any hyperkähler structure on K3 surface must be constructed by this way.

One may ask whether Torelli Theorem holds for ALG gravitational instantons. The answer is false at least when D is of type II*, III*, or IV*.

Theorem 1.13. *When D is of type II*, III*, or IV*, there exist two different ALG gravitational instantons with same $[\omega^i]$.*

2 Asymptotic Fibration

In this section, we will prove the Theorem 1.1 and Theorem 1.2. It's essentially a theorem in Riemannian geometry. The basic tool is to view a ball in the manifold M as a quotient of the ball inside the tangent space equipped with the metric pulled back from exponential map by the group of local covering transforms which correspond to the short geodesic loops in M. In the second subsection, we discuss this picture. In the third subsection, we provide a rough estimate of the parallel transport along short geodesic loops. In the fourth subsection, we use that rough estimate to classify the tangent cone at infinity. In the fifth subsection, we use this information to get a better control of geodesic loops. Finally, we use this better control to prove Theorem 1.2.

2.1 Notations and definitions

First, let's understand the standard models near infinity. The explicit expression of those models are defined in Theorem 2.22. To avoid singularity, a ball B_R is always removed.

Example 2.1. Let (X, h_1) be any manifold of dimension $3-k$ with constant sectional curvature 1 and $C(X)$ its metric cone with standard flat metric $\mathrm{d}r^2 + r^2 h_1$. Let $\mathbb{T}^k$ be a k-dimensional flat torus. Then the $\mathbb{T}^k$ fibration E

over $C(X) - B_R$ with a $\mathbb{T}^k$ invariant metric h provides the standard model near infinity.

1. $C(X) = \mathbb{R}^4/\Gamma$, Γ is a discrete subgroup in SU(2) acting freely on $\mathbb{S}^3$. In this case, $(E, h) = C(X) - B_R$ with the flat metric. It's called ALE.

2. $C(X) = \mathbb{R}^3$, (E, h) is either the trivial product $(\mathbb{R}^3 - B_R) \times \mathbb{S}^1$ or the quotient of the Taub-NUT metric with mass m outside a ball by $\mathbb{Z}_{|e|}$, where $me < 0$. It's called ALF-A_k with $k = -1$ in the first case and $k = -e - 1$ in the second case.

3. $C(X) = \mathbb{R}^3/\mathbb{Z}_2$, (E, h) is either the $\mathbb{Z}_2$ quotient of the trivial product of $\mathbb{R}^3 - B_R$ and $\mathbb{S}^1$ or the quotient of the Taub-NUT metric with mass m outside a ball by the binary dihedral group $D_{4|e|}$ of order $4|e|$, where $me < 0$. It's called ALF-D_k with $k = 2$ for the first case, and $k = -e+2$ for the second case.

4. $C(X)$ is the flat cone $\mathbb{C}_\beta$ with cone angle $2\pi\beta$, (E, h) is a torus bundle over $\mathbb{C}_\beta - B_R$ with a flat metric, where (β, E, h) are in the list of some special values; It's called ALG.

5. $C(X) = \mathbb{R}_+$, (E, h) is the product of $[R, +\infty)$ and a flat 3-torus. It's called ALH.

We may call such fiberation a standard model near infinity. It serves as an asymptotic model in the following sense:

Definition 2.2. A complete Riemannian manifold (M, g) is called asymptotic to the standard model (E, h) of order δ if there exist a bounded domain $K \subset M$, and a diffeomorphism $\Phi : E \to M \setminus K$ such that

$$\Phi^* g = h + O'(r^{-\delta})$$

for some $\delta > 0$.

Any manifold asymptotic to the standard ALE model is called ALE. It stands for asymptotically locally Euclidean. Similarly, any manifold asymptotic to the standard ALF model is called ALF. It means asymptotically locally flat. The ALG and ALH manifold are defined similarly. The letters "G" and "H" don't have any meanings. They are just the letters after "E" and "F".

Notice that our definition of ALH manifold is different from the definition of Hein in [37]. However, Theorem 3.20 implies that there is no essential difference for gravitational instantons.

Notation. o is a fixed point in M. In this section, $r(p) = \mathrm{dist}(o, p)$ is the geodesic distance between o and p. In other sections, E is a fiberation over $C(X) - B_R = \{(r, \theta) : r \geq R, \theta \in X\}$. So the pull back of r by the projection is a function on E. On M, we pull back that function, cut it off by some smooth function, and add 1 to get a smooth function $r \geq 1$. The reader should be careful about the switch of the meanings of r in different sections of our paper.

$O'(r^\alpha)$ means that for any $m \geq 0$, the m-th derivative of the tensor belongs to $O(r^{\alpha - m})$. χ will be a smooth cut-off function from $(-\infty, +\infty)$ to $[0, 1]$ such that $\chi \equiv 1$ on $(-\infty, 1]$ and $\chi \equiv 0$ on $[2, \infty)$. We will always use $\Delta = -\mathrm{Tr}\nabla^*\nabla$ as the Laplacian operator.

2.2 Short geodesic loops and the local covering space

In 1978 Gromov [33] started the research of almost flat manifolds, i.e. manifold with very small curvature. In 1981, Buser and Karcher wrote a book [13] to explain the ideas of Gromov in detail. In 1982 Ruh [70] gave a new way to understand it. Assume p is a point in M. The exponential map $\exp : T_p \to M$ is a local covering map inside the conjugate radius. We can pull back the metric from M using the exponential map inside conjugate radius. There is a lemma about the local geometry on the tangent space:

Lemma 2.3. *Suppose g_{ij} is a metric on $B_1(0) \subset \mathbb{R}^n$ satisfying the following condition:*

(1) The curvature is bounded by Λ^2;

(2) $g_{ij}(0) = \delta_{ij}$;

(3) The line $\gamma(t) = t\mathbf{u}$ is always a geodesic for any unit vector $\mathbf{u}$.

Then there exist constants $\Lambda(n) < \pi/2$ and $C(m, n)$ such that as long as $\Lambda \leq \Lambda(n)$,

(1) Any two points x and y in $B_1(0)$ can be connected by a unique minimal geodesic inside $B_1(0)$;

(2) If the Ricci curvature is identically 0, then the m-th ordinary derivatives $|D^m(g_{ij}(x) - \delta_{ij})| < C(m, n)\Lambda^2$ for all $m \geq 0$ and $x \in B_{1/2}$.

Proof. (1) It was proved by Buser and Karcher as the Proposition 6.4.6 in [13].

(2) Therefore, all the works in [43] apply. We can find functions l_i satisfying
$$|\nabla l_i(x) - e_i(x)| \leq C(n)\Lambda^2$$
and
$$|\nabla^2 l_i(x)| \leq C(n)\Lambda^2$$
for all $x \in B_{1/2}(0)$ as long as $\Lambda(n)$ is small enough, where $e_i(x)$ is a vector field which is parallel along radical geodesics and equals to $\frac{\partial}{\partial x_i}$ at origin. For even smaller $\Lambda(n)$, we can use l_i as coordinate functions in

$$L_{0.9}(0) = \{\sum l_i^2 < (0.9)^2\} \subset B_1(0) = \{\sum x_i^2 < 1\}.$$

In this coordinate,

$$|g_{ij}w^i w^j - |w|^2| \leq C(n)\Lambda^2|w|^2 \leq 0.01|w|^2,$$

$$|\partial_k g_{ij}| < C(n)\Lambda^2 < 1,$$

$$\Delta u = \frac{1}{\sqrt{G}}\frac{\partial}{\partial l_j}(\sqrt{G}g^{ij}\frac{\partial u}{\partial l_i}).$$

What's more $|\Delta l_i| < C(n)\Lambda^2$. By Theorem 9.15 of [31], for all $1 < p < \infty$, there is a unique solution $u_i \in W^{2,p}(L_{0.9}) \cap W_0^{1,p}(L_{0.9})$ such that $\Delta u_i = \Delta l_i$. By Lemma 9.17 of [31], we actually have

$$||u_i||_{W^{2,p}(L_{0.9}(0))} < C(n,p)||\Delta l_i||_{L^p(L_{0.9}(0))} < C(n,p)\Lambda^2.$$

By Sobelev embedding theorem (c.f. Theorem 7.26 of [31]),

$$||u_i||_{C^1(\overline{L_{0.9}(0)})} < C(n)||u_i||_{W^{2,2n}(L_{0.9}(0))} < C(n)\Lambda^2.$$

In particular, when $\Lambda(n)$ is small enough, $h_i = l_i - u_i$ gives a harmonic coordinate in $H_{0.8}(0) := \{\sum h_i^2 < (0.8)^2\} \subset L_{0.9}(0)$. In this harmonic coordinate, $1/1.02|w|^2 < g_{ij}w^i w^j < 1.02|w|^2$. By elliptic regularity, actually all the above functions are smooth. So we can differentiate them to get equations. Since $\Gamma_{ij}^k g^{ij} = 0$, we know that $2\mathrm{Ric}_{mk} = g^{im}R_{ijkl}g^{jl} + g^{ik}R_{ijml}g^{jl}$ satisfies

$$g^{rs}\frac{\partial^2(g_{ij} - \delta_{ij})}{\partial h_r \partial h_s} = -2\mathrm{Ric}_{ij} + Q_{ij}(g, \partial g) + Q_{ji}(g, \partial g),$$

where
$$Q_{mk}(g, \partial g) = g^{jl}\partial_l g_{im}\Gamma_{kj}^i - g^{jl}g_{im}\Gamma_{kj}^h\Gamma_{lh}^i - g_{im}\partial_k g^{jl}\Gamma_{jl}^i.$$

We already know that $||g_{ij} - \delta_{ij}||_{W^{1,p}(H_{0.8}(0))} < C(n)\Lambda^2$ from the $W^{2,p}$ bound of u_i. So $||Q_{ij}(g, \partial g)||_{L^{p/2}(H_{0.8}(0))} < C(n)\Lambda^4$. When the Ricci curvature is identically 0, by Theorem 9.11 of [31], we have

$$||g_{ij} - \delta_{ij}||_{W^{2,p/2}(H_{0.7})} < C(n)(||g_{ij} - \delta_{ij}||_{L^{p/2}(H_{0.8})} + ||Q_{ij}||_{L^{p/2}(H_{0.8})}) < C(n)\Lambda^2.$$

After taking more derivatives, we can get the required bound in the harmonic coordinate. This in turn bounds the Christoffel symbol and gives a bound of the geodesic equation. So when we solve this geodesic equation, we can get the required bound in the geodesic ball. □

The above estimate is an interior estimate. The number 1/2 can be replaced by any number smaller than 1.

To find out the local covering transform, we look at the preimage p_1 of p under the exponential map inside $B_1(0)$. There is a local covering transform F which maps 0 to p_1. The image of the radical geodesic from 0 to p_1 is a geodesic loop based at p. This gives a 1-1 correspondence between short geodesic loops and covering transforms.

Now suppose we have two short enough geodesic loops γ_1 and γ_2 with same base point p. Then they correspond to two local covering transforms F_1 and F_2. The composition $F_1 \circ F_2$ is also a local covering transform. It corresponds to another geodesic loop based at p. It's exactly the product of γ_1 and γ_2 defined by Gromov.

For any q close enough to p, choose an preimage q_0 of q close enough to 0, then $q_1 = F(q_0)$ is another preimage of q which is very close to p_1. The image of the shortest geodesic connecting q_1 and q_2 under the exponential map is a geodesic loop based at q. It's called the sliding of γ. When q moves along a curve α, the sliding of γ becomes a 1-parameter family of curves. It's called the sliding of γ along the curve α.

When we parallel transport any vector v along the geodesic loop γ, we will get another vector $P_\gamma(v)$. P_γ is a map from T_p to itself. For hyperkähler manifold, $P_\gamma \in \mathrm{Sp}(1) = \mathrm{SU}(2)$. Under suitable orthonormal basis, any element in SU(2) can be written as

$$\mathbf{A} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}.$$

So

$$\mathbf{A} - \mathbf{Id} = \begin{pmatrix} e^{i\theta} - 1 & 0 \\ 0 & e^{-i\theta} - 1 \end{pmatrix}, (\mathbf{A} - \mathbf{Id}) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} (e^{i\theta} - 1)v_1 \\ (e^{-i\theta} - 1)v_2 \end{pmatrix}.$$

So $|(\mathbf{A}-\mathbf{Id})\mathbf{v}| = |\mathbf{A}-\mathbf{Id}||\mathbf{v}|$ if we define the norm by

$$|\mathbf{A}-\mathbf{Id}| = |e^{i\theta}-1| = |e^{-i\theta}-1|.$$

This property is also a special property of SU(2). For instance SO(4) doesn't have this property.

In the flat case, local covering transforms are all linear maps. Suppose $T_1(\mathbf{x}) = \mathbf{a}\mathbf{x}+\mathbf{b}$, $T_2(\mathbf{x}) = \mathbf{A}\mathbf{x}+\mathbf{B}$ are two local covering transforms, where $\mathbf{a}, \mathbf{A} \in \mathrm{SO}(n)$ and $\mathbf{b}, \mathbf{B} \in \mathbb{R}^n$. They correspond to two geodesic loops γ_1, γ_2 with same base point p. $\mathbf{A}$, $\mathbf{a}$ are exactly the parallel transports along γ_1 and γ_2 while $|\mathbf{B}|$, $|\mathbf{b}|$ are the same as the length of loops γ_1 and γ_2 respectively.

$$T_1 \circ T_2(\mathbf{x}) = \mathbf{a}(\mathbf{A}\mathbf{x}+\mathbf{B})+\mathbf{b} = \mathbf{a}\mathbf{A}\mathbf{x}+\mathbf{a}\mathbf{B}+\mathbf{b}$$

will correspond to the Gromov product of γ_1 and γ_2. So

$$T_1^{-1}T_2^{-1}T_1T_2(\mathbf{x}) = \mathbf{a}^{-1}\mathbf{A}^{-1}\mathbf{a}\mathbf{A}\mathbf{x} + \mathbf{a}^{-1}\mathbf{A}^{-1}((\mathbf{a}-\mathbf{Id})\mathbf{B}+(\mathbf{Id}-\mathbf{A})\mathbf{b}).$$

The Lie algebra are also linear maps. Taking the derivative in the above expression of the commutator at the origin

$$T_1(\mathbf{x}) = T_2(\mathbf{x}) = \mathrm{Id}(\mathbf{x}) = \mathrm{Id}(\mathbf{x}) + \mathbf{0},$$

the Lie bracket is

$$[\mathbf{a}\mathbf{x}+\mathbf{b}, \mathbf{A}\mathbf{x}+\mathbf{B}] = [\mathbf{a}, \mathbf{A}]\mathbf{x} + (\mathbf{a}\mathbf{B}-\mathbf{A}\mathbf{b}).$$

In general case, we can understand the covering transform in the following way: We start from q_0 in $B_1(0) \subset T_p(M)$. Then exponential map at p maps the point $p_1 \in B_1(0)$ to $p \in M$. The derivative maps the tangent vector at p_1 to the tangent vector at p. Let $\tilde{A}$ be the inverse of the map. Then $F(q_0) = \exp_{p_1}(\tilde{A}q_0)$. In the Ricci flat case, by Lemma 2.3, g_{ij} as well as its m-th derivatives are bounded by $C(m,n)\Lambda^2$. So the Christoffel symbols are also bounded as well as their higher derivatives. By the property of ODE, all the parallel transports and the geodesic equations have the same kind of bound as well as their higher derivatives. In particular, the difference between $\tilde{A}$ and the parallel transport A along the geodesic loop is bounded by $C(n)\Lambda^2$. The difference between $F(q_0)$ and $p_1+\tilde{A}q_0$ is bounded by $C(n)\Lambda^2$. In conclusion, the difference between $F(q_0)$ and p_1+Aq_0 is bounded by $C(n)\Lambda^2$ while the difference between their higher derivatives is bound by $C(m,n)\Lambda^2$.

From now on, we are back to the gravitational instanton M with the point o. We will rescale the ball $B_{\mathrm{dist}(o,p)/2}(p)$ to a ball with radius 1 and apply the theory in this section. In particular, the metric on the local covering space is $\delta_{ij} + O'(r^{-\epsilon})$. The difference between the local covering transform with the linear map given by the length, direction, and the parallel transport along the geodesic loop is $O'(r^{1-\epsilon})$.

For short loops, there is a better control given by Buser and Karcher as Proposition 2.3.1 in [13]. They proved that the rotation (i.e. parallel transport) part of the Gromov's product of γ_1 and γ_2 is given by the calculation in the flat case with error bounded by $Cr^{-2-\epsilon}L(\gamma_1)L(\gamma_2)$, while the error of the translation (i.e. length) part is bounded by $Cr^{-2-\epsilon}L(\gamma_1)L(\gamma_2)(L(\gamma_1)+L(\gamma_2))$.

2.3 Control of parallel transport along geodesic loops

In this section, we will use ODE comparison to study the sliding of geodesic loops and the variation of the parallel transports along them. First let us recall a well known Jacobi equation

$$J''(t) = (\frac{t}{2})^{-2-\epsilon} J(t).$$

satisfying the following property:

Proposition 2.4. *(c.f. Theorem C of [32])Let J be the solution of the Jacobi equation with*

$$J(2) = 0, \quad J'(2) = 1.$$

Then

$$1 \leq J'(t) \nearrow J'(\infty)(:= \lim_{t\to\infty} J'(t)) \leq \exp \int_2^\infty (t-2)(\frac{t}{2})^{-2-\epsilon} \mathrm{d}t < \infty$$

and

$$t - 2 \leq J(t) \leq J'(\infty)(t-2).$$

Suppose γ is a geodesic loop based at $p \in M$, α is an arc-length parameterized curve passing through p. Suppose $r = \mathrm{dist}(0,p) = r(p) > 3$. As discussed before, we slide γ along α and get a 1-parameter family of geodesic loops γ_t based at $\alpha(t)$. Then their length and parallel transport along them satisfy the following:

Proposition 2.5. *Suppose the length of the geodesic loop γ_t is $L(t)$ and the parallel transport along γ_t is $P(t) : T_{\alpha(t)} \to T_{\alpha(t)}$. Then,*

$$|L'(t)| \leq |P(t) - \mathrm{Id}|$$

and

$$|P(t) - \mathrm{Id}|' \leq L(t) \cdot \max_{x \in \gamma_t} |\mathrm{Rm}|(x).$$

Proof. Let $\gamma(s,t) = \gamma_t(s)$, then $\gamma(0,t) = \gamma(1,t) = \alpha(t)$ and for any fixed t, $\gamma(s,t)$ is a geodesic. So $\partial_s := \gamma_*(\frac{\partial}{\partial s})$ and $\partial_t := \gamma_*(\frac{\partial}{\partial t})$ satisfy

$$\nabla_{\partial_s}\partial_s = 0, [\partial_s, \partial_t] = \nabla_{\partial_s}\partial_t - \nabla_{\partial_t}\partial_s = 0, L(t) = \int_0^1 |\partial_s| \mathrm{d}s.$$

Then

$$\begin{aligned}
\frac{\mathrm{d}L(t)}{\mathrm{d}t}|_{t=t_0} &= \int_0^1 \frac{< \nabla_{\partial_t}\partial_s, \partial_s >}{< \partial_s, \partial_s >^{1/2}} \mathrm{d}s \\
&= \frac{1}{L(t_0)} \int_0^1 < \nabla_{\partial_s}\partial_t, \partial_s > \mathrm{d}s \\
&= \frac{1}{L(t_0)} \int_0^1 \nabla_{\partial_s} < \partial_t, \partial_s > \mathrm{d}s \\
&= \frac{< \partial_t, \partial_s > |_{s=0}^{s=1}}{L(t_0)} = < \alpha'(t_0), \frac{(P - \mathrm{Id})[\partial_s(0,t_0)]}{L(t_0)} > .
\end{aligned}$$

So

$$|L'| \leq |P - \mathrm{Id}|.$$

Moreover, given any unit length vector V at $\gamma(0,t_0)$, we can parallel transport it along $\alpha(t) = \gamma(0,t)$ and then parallel transport it along γ_t. Then $P(t)(V(0,t)) = V(1,t)$. So

$$\begin{aligned}
||P(t) - \mathrm{Id}|'| \leq |\nabla_{\partial_t} V(1,t)| &\leq \int_0^1 |\nabla_{\partial_s}\nabla_{\partial_t} V| \\
&= \int_0^1 |R(\partial_s, \partial_t)V(s,t)| \leq \max_{x \in \gamma_t} |\mathrm{Rm}| L.
\end{aligned}$$

□

Theorem 2.6. *For any geodesic loop based at p with $r = r(p) = d(p, o) > 3$ and length $L \leq C_1 r$, the parallel transport along the loop satisfies*

$$|P - \mathrm{Id}| \leq \frac{J'(r)}{J(r)} L \leq C_2 \frac{L}{r}.$$

Here the constant

$$C_1 = \frac{1}{2} \inf_{t>2} \frac{t}{J(t)} \inf_{t>3} \frac{J(t)}{t}, C_2 = \sup_{t>3} J'(t) \sup_{t>3} \frac{t}{J(t)}.$$

Proof. If we choose $\alpha(t)$ so that $\partial_t = \frac{P-\mathrm{Id}}{|P-\mathrm{Id}|} \frac{\partial_s}{L(t_0)}$, we can get $L'(t) = |P(t) - \mathrm{Id}|$. It's some kind of gradient flow. The other fundamental equation is that $|P(t) - \mathrm{Id}|'$ is bounded by the product of L and the maximal Riemannian curvature along the geodesic loop.

Given p whose distance to origin $r = r(p) = d(p, o) > 3$ and any geodesic loop based at p with length smaller than $C_1 r < \frac{r}{2}$, if $|P - \mathrm{Id}| > \frac{J'(r)}{J(r)} L$, we can slide the curve back along the gradient flow. In other words, we start from $\alpha(r) = p$ and get a curve $\alpha : [t_1, r] \to M$ as well as the corresponding γ_t. Let t_1 be the biggest t_1 such that one of the following happens: (1) $L(t_1) = t_1/2$; (2) $L'(t_1) = |P - \mathrm{Id}| = 0$ or $L(t_1) = 0$; (3) $t_1 = 2$. Then when $t \in (t_1, r)$, we have $0 < L(t) < t/2$ and $t > t_1 \geq 2$. So the distance to the origin is at least $t - L(t) > t/2$. The curvature is bounded by $(t/2)^{-2-\epsilon}$ and the conjugate radius is at least $\pi(\frac{t}{2})^{1+\epsilon/2} > \frac{t}{2} > L(t)$. So the geodesic loop can exist without going out of the conjugate radius. Combining two fundamental equations together,we have

$$L''(t) \leq L(t) \max|\mathrm{Rm}| \leq L(t)(t - L(t))^{-2-\epsilon} < L(t)(\frac{t}{2})^{-2-\epsilon}, \forall t \in (t_1, r).$$

Therefore $(L'J - J'L)' = L''J - J''L < 0$. By our hypothesis $L'(r) > \frac{J'(r)}{J(r)} L(r)$. So $L'(t)J(t) - J'(t)L(t) > 0 \Rightarrow (\frac{L(t)}{J(t)})' > 0 \Rightarrow \frac{L(t)}{J(t)} < \frac{L(r)}{J(r)}, \forall t \in [t_1, r)$. So $L(t_1) < \frac{L(r)}{J(r)} J(t_1) \leq C_1 \frac{r}{J(r)} \frac{J(t_1)}{t_1} t_1 \leq \frac{t_1}{2}$ and $L'(t_1)J(t_1) > J'(t_1)L(t_1) \geq 0$. In other words, $t_1 = 2$. But then $L(2) < \frac{L(r)}{J(r)} J(2) = 0$. It's a contradiction. □

Similarly, we can prove theorem 1.1:

Theorem 2.7. *For gravitational instanton M, the following conditions are equivalent:*

(1) M is flat;
(2) M has trivial holonomy;
(3) M splits as $\mathbb{R}^{4-k} \times \mathbb{T}^k$, $k = 0, 1, 2, 3$.

Proof. Using the method in the proof of Theorem 2.6, it's easy to see that any flat gravitational instanton M must have trivial holonomy. It's well known that M is isometric to the Euclidean space quotient by covering transforms. However, since the holonomy is trivial, any covering transform must be a pure translation. Therefore, M is isometric to the product of the Euclidean space with a flat torus. Conversely, it's trivial that (2) or (3) implies (1). □

For any fixed geodesic ray α starting from o, any number $r > 3$ and any geodesic loop γ based at $p = \alpha(r)$ with length $L \leq C_1 r$, when we slide it along the ray towards infinity, it will always exist i.e. stay within the conjugate radius. This follows from the following rough estimate:

Corollary 2.8. *The length $L(t)$ of the geodesic loop based at $\alpha(t)$ is smaller than $t/2$ for all $t \geq r$.*

Proof. By Proposition 2.5 and Theorem 2.6, we know that $L'(t) \leq \frac{J'(t)}{J(t)} L(t)$. So

$$(\ln L)' \leq (\ln J)' \Rightarrow L(t) \leq \frac{L(r)}{J(r)} J(t) \leq \frac{t}{2}, \forall t \geq r > 3.$$

□

We will derive a better estimate and use it to prove Theorem 1.2.

2.4 Classification of tangent cone at infinity

To under how the length of geodesic loops varies, we first need to understand the structure at infinity. Our assumption of the decay of the curvature means that we are at a manifold with asymptotically nonnegative curvature. The end of such a manifold is well studied and goes back to Kasue [44]. Here, a complete connected noncompact Riemannian manifold M with a base point o is called asymptotically nonnegative curved if there exists a monotone non-increasing function $k : [0, \infty) \to [0, \infty)$ such that the integral $\int_0^\infty t k(t) \mathrm{d}t$ is finite and the sectional curvature of M at any point p is bounded from below by $-k(\mathrm{dist}(o, p))$. Of course, the gravitational instanton M satisfies this condition.

However, Drees [27] pointed out a gap in the argument of [44]. It was smoothed by Mashiko, Nagano and Otsuka [55]. They proved the following theorem:

Theorem 2.9. *(Corollary 0.4 of [55]) For any manifold M with asymptotically nonnegative curvature, there exists a metric cone $C(S(\infty))$ such that $(M, t^{-2}g)$ converges to $C(S(\infty))$ in Gromov-Hausdorff sense when t goes to infinity. In other words, the tangent cone at infinity is unique and must be a metric cone $C(S(\infty))$.*

Assuming the faster than quadratic curvature decay condition, the same theorem was stated without proof by Petrunin and Tuschmann in [67].

The following additional thing is true for gravitational instantons:

Theorem 2.10. *For any non-flat gravitational instanton M, $S(\infty)$ has only one connected component.*

Proof. If $S(\infty)$ has more than one connected components, we can find a large enough ball B_R and two sequences p_i, q_i such that $d(o, p_i) \to \infty$, $d(o, q_i) \to \infty$, and any minimal geodesics connecting p_i and q_i must pass through B_R for any i large enough. By compactness of B_R, the minimal geodesics converge to a line. Notice that M is Ricci-flat, so the splitting theorem [17] implies that M must be isometric to the product of $\mathbb{R}$ and a 3-manifold. The 3-manifold is also Ricci-flat and therefore flat. So M must be flat. □

As a corollary, the following is true:

Corollary 2.11. *Fix a ray γ starting from o. There is a constant C_3 such that for any point p in the large enough sphere $S_{r(p)}$, there is a curve within $B_{1.1r(p)} \setminus B_{0.9r(p)}$ connecting p and $\gamma(r(p))$ with length bounded by $C_3 r(p)$.*

There is more information about the tangent cone at infinity of the gravitational instanton M.

Theorem 2.12. *The tangent cone at infinity $C(S(\infty))$ of the gravitational instanton M must be a flat manifold with only possible singularity at origin.*

Proof. Pick $p \in C(S(\infty)) - \{o\}$, we may find $p_i \in M$ such that $p_i \to p$ in Gromov-Hausdorff sense. Pick some small enough number κ. For i large enough the ball $(B_{\kappa r_i}(p_i), r_i^{-2}g)$ is B_κ/G_i, where B_κ is the ball in the Euclidean space with metric pulled back by exponential map, and G_i is the group of local covering transforms. By Fukaya's result in [30], G_i converge to some Lie group G and B_κ/G_i converge to B_κ/G. So G is a subgroup of $\mathbb{R}^4 \rtimes \mathrm{SU}(2) \leq \mathrm{Iso}(\mathbb{R}^4)$. The action of G on B_κ corresponds to the action of G_i on $B_{\kappa r_i}(p_i)$. So if an element $g \in G - \{\mathrm{Id}\}$ has a fixed point in B_κ, the geodesic loops in $B_{\kappa r_i}(p_i)$ corresponding to the sequence $g_i \in G_i$ converging to g would have large $|P - \mathrm{Id}|$ compared to their lengths by the relationship between geodesic loops and covering transforms. This contradicts Theorem 2.6. So the action of G is free. Therefore it's enough to look at the Lie algebra $\mathfrak{g}$ i.e. the infinitesimal part of G to determine the local geometry. We have the following cases:

(0) $\dim G = 0$. We get $\mathbb{R}^4$ locally.

(1) $\dim G = 1$. Then $\mathfrak{g}$ is generated by $\mathbf{x} \to \mathbf{a}\mathbf{x} + \mathbf{b}$, where $\mathbf{a} \in \mathfrak{su}(2)$. Notice that SU(2) can be naturally identified with the unit sphere of quaternions. Then $\mathfrak{su}(2)$ would be the space of pure imaginary quaternions. So the Lie bracket is exactly twice of the cross product in $\mathbb{R}^3$.

$\mathbf{a}$ must be $\mathbf{O}$ or invertible by the property of quaternions. When $\mathbf{a} = \mathbf{O}$, G consists of pure translations, we get $\mathbb{R}^3$.

Otherwise, $\mathbf{a}\mathbf{x}+\mathbf{b} = \mathbf{a}(\mathbf{x}+\mathbf{a}^{-1}\mathbf{b})$. The fixed point $-\mathbf{a}^{-1}\mathbf{b}$ must be outside B_κ. G is generated by $\mathbf{x} \to \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}(\mathbf{x} + \mathbf{a}^{-1}\mathbf{b}) - \mathbf{a}^{-1}\mathbf{b}$. If we take the 1-1 correspondence $\mathbf{x} \to \mathbf{x} + \mathbf{a}^{-1}\mathbf{b} = (x+iy, z+iw) \to (x+iy, z-iw)$,then G becomes $\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix}$. So it's cone over $\mathbb{S}^3/\mathbb{S}^1$, where $\mathbb{S}^3/\mathbb{S}^1$ is the Hopf fiberation. So it's cone over $\mathbb{S}^2$, i.e.$\mathbb{R}^3$, too.

(2) $\dim G = 2$. Any 2-dimensional Lie algebra has a basis e_1, e_2 satisfying $[e_1, e_2] = ce_1$. For $\mathfrak{g}$, $e_1(\mathbf{x}) = \mathbf{a}\mathbf{x} + \mathbf{b}, e_2(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{B}$ must satisfy

$$[\mathbf{a}, \mathbf{A}]\mathbf{x} + (\mathbf{a}\mathbf{B} - \mathbf{A}\mathbf{b}) = [\mathbf{a}\mathbf{x} + \mathbf{b}, \mathbf{A}\mathbf{x} + \mathbf{B}] = ce_1 = c(\mathbf{a}\mathbf{x} + \mathbf{b}).$$

Here $\mathbf{A}, \mathbf{a} \in \mathfrak{su}(2)$. If $\mathbf{a} = \mathbf{O}$, $\mathbf{A}\mathbf{b} = -c\mathbf{b}$. So $\mathbf{A} = \mathbf{O}$. G consists of pure translations, we get $\mathbb{R}^2$. If $\mathbf{a} \neq \mathbf{O}$, then since $[\mathbf{a}, \mathbf{A}] = c\mathbf{a}$, we must have $\mathbf{a} = \mathbf{A}$, and $c = 0$. So $\mathbf{a}\mathbf{B} = \mathbf{A}\mathbf{b} = \mathbf{a}\mathbf{b} \Rightarrow \mathbf{B} = \mathbf{b}$, contradiction.

(3) $\dim G = 3$. We get $\mathbb{R}^1$. □

Theorem 2.13. *The tangent cone at infinity $C(S(\infty))$ must be the following:*

(ALE) $\mathbb{R}^4/\Gamma$, where Γ is a discrete subgroup of $O(4)$ acting freely on $\mathbb{S}^3$

(ALF-A_k) $\mathbb{R}^3$

(ALF-D_k) $\mathbb{R}^3/\mathbb{Z}_2$=*cone over* $\mathbb{RP}^2$
(ALG) flat cone with angle $\in (0, 2\pi]$
(ALH) $\mathbb{R}_+$

Proof. By Theorem 2.9, the tangent cone at infinity is unique and must be a metric cone $C(S(\infty))$. By Theorem 2.10 and Theorem 2.12, $S(\infty)$ is a connected manifold since we've assumed that M is not flat.

(ALH) If $S(\infty)$ is 0-dimensional, $C(S(\infty))$ must be $\mathbb{R}_+$.

(ALG) If $S(\infty)$ is 1-dimensional, $C(S(\infty))$ is a flat cone. If the cone angle is bigger than 2π, it contains a line, so there is a contradiction from the almost splitting theorem. (c.f. Theorem 6.64 of [15])

(ALF) If $S(\infty)$ is 2-dimensional, $S(\infty)$ must be a 2-manifold with constant positive curvature 1. So its universal cover is the space form $\mathbb{S}^2$. So $S(\infty) = \mathbb{S}^2/\Gamma$, where the group of covering transforms Γ is a subgroup of $\mathrm{Iso}(\mathbb{S}^2) = \mathrm{O}(3)$ acting freely. Now pick any element A in Γ, $A^2 \in \mathrm{SO}(3)$. However, any element in SO(3) has a fixed point, so $A^2 = \mathrm{Id}$. So $A = \pm\mathrm{Id}$. Therefore $S(\infty)$=$\mathbb{S}^2$(the A_k case) or $\mathbb{RP}^2$(the D_k case).

(ALE) If $S(\infty)$ is 3-dimensional, $S(\infty)$ must has constant sectional curvature, too. Its universal cover is the space form, too. So $C(S(\infty)) = \mathbb{R}^4/\Gamma$, where Γ is a discrete subgroup of $O(4)$ acting freely on $\mathbb{S}^3$ □

From now on, we will temporarily use the terminology ALE, ALF, ALG and ALH to distinguish different type of the (unique) tangent cone at infinity. Those terminologies only make sense as in Definition 2.2 after we prove more properties.

Theorem 2.14. *In the ALE case, M has maximal volume growth rate and it's in Kronheimer's list.*

Proof. By Colding's volume convergence theorem [24], M^4 has maximal volume growth rate. Moreover, the faster than quadratic curvature decay condition ensures that $\int_M |\mathrm{Rm}|^2 < \infty$. So by Bando-Kasue-Nakajima's work [5], M is ALE of order 4. So, Kronheimer's works in [48] and [49] apply. □

2.5 Decomposing geodesic loops into basis

Before proceeding, we need a theorem about Lie groups. For any Lie group H, the exponential map exp from a small ball $B_\kappa = B_\kappa(o)$ in its Lie algebra $\mathfrak{h}$ to H is a bijection. We call the inverse of exp to be log. If there is no ambiguity, the length of $g \in H$ will mean $|\log g|$.

Theorem 2.15. *Suppose H is a Lie group, G_i are discrete subgroups of H converging to a k-dimensional closed Lie subgroup G of H. Then for i large enough and κ small enough, there exist k elements $g_{i,j}$ (j=1,2,...,k) such that $|\log g_{i,j}|$ converge to 0 as i goes to infinity and all element in $B_\kappa(\mathrm{Id}) \cap G_i$ is generated by $g_{i,j}$. What's more, for any fixed large enough i, the angle between $\log g_{i,j}$ are bounded from below by a small positive number independent of i. In addition, the commutator $g_{i,a}^{-1} g_{i,b}^{-1} g_{i,a} g_{i,b}$ is generated by $g_{i,c}$, $c = 1, 2, ..., \min\{a, b\} - 1$. In particular, $g_{i,1}$ commutes with others.*

Proof. Let d be the dimension of H. First of all, choose κ small enough such that for all $v, w \in \mathfrak{h} \cap B_{10^{3^{d+5}}\kappa}$,

$$|\log(\exp(v)\exp(w)) - v - w| \leq 10^{-3^{d+6}}\kappa^{-1}|v||w|.$$

Now for all i large enough, we pick the shortest element $g_{i,1}$ in G_i.

Let $G_{i,0} = \{\mathrm{Id}\}$. Suppose inductively, we've picked $g_{i,1}, g_{i,2}, ..., g_{i,j}$ satisfying

$$\lim_{i\to\infty} |\log g_{i,l}| = 0, l = 1, 2, ..., j,$$

$$g_{i,a}^{-1} g_{i,b}^{-1} g_{i,a} g_{i,b} \in G_{i,\min\{a,b\}-1}, a, b = 1, 2, ..., j$$

$$|\log g_{i,l}| \leq 10^{3^l} |\mathrm{Proj}_{V_{i,l-1}^\perp}(\log g_{i,l})|, l = 1, 2, ...j,$$

and

$$|\log g_{i,l_1}| \leq 10^{3^{l_2} - 3^{l_1}} |\log g_{i,l_2}|, 1 \leq l_1 \leq l_2 \leq j,$$

where $V_{i,l} = \mathrm{Span}\{\log g_{i,1}, ... \log g_{i,l}\}$ and $\mathfrak{h} = V_{i,l} \oplus V_{i,l}^\perp$ is the orthogonal decomposition for all $l = 1, 2, ..., j$. Define $G_{i,j}$ be the set

$$G_{i,j} = \{g_{i,1}^{n_1} ... g_{i,j}^{n_j} : \sum_{l=1}^{j} |n_l \log g_{i,l}| \leq 10^{3^{d+5} - 3^{j+4}}\kappa\}.$$

It is a local group in the sense that for all $g_1, g_2 \in B_{10^{3^{d+5}-3^{j+5}+2}\kappa} \cap G_{i,j}$, $g_1 g_2, g_1^{-1} \in G_{i,j}$. Moreover, for all $g_1, g_2, g_3 \in B_{10^{3^{d+5}-3^{j+5}+1}\kappa} \cap G_{i,j}$, there product and inverse $g_1 g_2, g_1^{-1} \in B_{10^{3^{d+5}-3^{j+5}+2}\kappa} \cap G_{i,j}$. They satisfy the relations $(g_1 g_2) g_3 = g_1(g_2 g_3)$ and $g_1^{-1} g_1 = g_1 g_1^{-1} = \mathrm{Id}$.

If

$$G_{i,j} \cap B_{10^{3^{d+5}-3^{j+5}}\kappa} = G_i \cap B_{10^{3^{d+5}-3^{j+5}}\kappa},$$

we stop. Otherwise, let $g_{i,j+1}$ be the element with shortest $\mathrm{Proj}_{V_{i,j}^\perp}(\log g_{i,j+1})$ among all the element $\tilde{g}_{i,j+1} \in G_i \cap B_{10^{3^{d+5}-3^{j+5}}\kappa} \setminus G_{i,j}$ whose length is not longer than any other elements in $\tilde{g}_{i,j+1} G_{i,j}$.

Let $\mathrm{Proj}_{V_{i,j}}(\log g_{i,j+1}) = \sum_{l=1}^{j} c_{i,j,l} \log g_{i,l}$, then there exist integers $n_{i,j,l}$ such that $c_{i,j,l} - n_{i,j,l} \in (-0.5, 0.5]$. If $|\mathrm{Proj}_{V_{i,j}}(\log g_{i,j+1})| \geq 0.7 \sum_{l=1}^{j} |\log g_{i,l}|$, then

$$|\sum_{l=1}^{j} c_{i,j,l} \log g_{i,l}| \geq 1.4 |\sum_{l=1}^{j} (c_{i,j,l} - n_{i,j,l}) \log g_{i,l}|,$$

so

$$|\sum_{l=1}^{j} c_{i,j,l} \log g_{i,l}| - |\sum_{l=1}^{j} (c_{i,j,l} - n_{i,j,l}) \log g_{i,l}| \geq \frac{1}{6} |\sum_{l=1}^{j} n_{i,j,l} \log g_{i,l}|$$

by triangle inequality. By induction assumption,

$$|\sum_{l=1}^{j} n_{i,j,l} \log g_{i,l}| \geq |n_{i,j,j} \mathrm{Proj}_{V_{i,j-1}^{\perp}}(\log g_{i,j})| \geq 10^{-3^j} |n_{i,j,j}| |\log g_{i,j}|.$$

So

$$|\sum_{l=1}^{m} n_{i,j,l} \log g_{i,l}| \leq 10^{\sum_{l=m+1}^{j} (3^l+1)} |\sum_{l=1}^{j} n_{i,j,l} \log g_{i,l}|$$

and

$$|n_{i,j,m} \log g_{i,m}| \leq 10^{3^m + \sum_{l=m+1}^{j} (3^l+1)} |\sum_{l=1}^{j} n_{i,j,l} \log g_{i,l}|.$$

So

$$\sum_{l=1}^{j} |n_{i,j,l} \log g_{i,l}| \leq 10^{3^{j+1}-2} |\sum_{l=1}^{j} n_{i,j,l} \log g_{i,l}|.$$

Let $h_{i,j+1} = |\mathrm{Proj}_{V_{i,j}^{\perp}}(\log g_{i,j+1})|$, then

$$\sqrt{h_{i,j+1}^2 + |\sum_{l=1}^{j} c_{i,j,l} \log g_{i,l}|^2} - \sqrt{h_{i,j+1}^2 + |\sum_{l=1}^{j} (c_{i,j,l} - n_{i,j,l}) \log g_{i,l}|^2}$$

is bounded by the error term in the calculation of $g_{i,j+1}(g_{i,j})^{-n_{i,j}}...(g_{i,1})^{-n_{i,1}}$. So

$$\begin{aligned}
&\frac{10^{-3^{j+1}}\sum_{l=1}^{j}|n_{i,j,l}\log g_{i,l}|}{|\log g_{i,j+1}|}|\sum_{l=1}^{j}c_{i,j,l}\log g_{i,l}| \\
\leq&\frac{1}{2|\log g_{i,j+1}|}(|\sum_{l=1}^{j}c_{i,j,l}\log g_{i,l}|^2-|\sum_{l=1}^{j}(c_{i,j,l}-n_{i,j,l})\log g_{i,l}|^2) \\
\leq&\sqrt{h_{i,j+1}^2+|\sum_{l=1}^{j}c_{i,j,l}\log g_{i,l}|^2}-\sqrt{h_{i,j+1}^2+|\sum_{l=1}^{j}(c_{i,j,l}-n_{i,j,l})\log g_{i,l}|^2} \\
\leq&10^{-3^{j+2}}\sum_{l=1}^{j}|n_{i,j,l}\log g_{i,l}|.
\end{aligned}$$

It follows that the induction assumption is also true for $j+1$ except the statement for the commutator and the length of $\log g_{i,j+1}$.

If $|\text{Proj}_{V_{i,j}}(\log g_{i,j+1})| < 0.7\sum_{l=1}^{j}|\log g_{i,l}|$, then we claim that

$$|\text{Proj}_{V_{i,j}^{\perp}}(\log g_{i,j+1})| > 0.2|\text{Proj}_{V_{i,j-1}^{\perp}}(\log g_{i,j})|.$$

Otherwise, there exists an integer $n_{i,j}$ such that

$$|\text{Proj}_{V_{i,j-1}^{\perp}}(\log g_{i,j+1}g_{i,j}^{-n_{i,j}})| < 0.8|\text{Proj}_{V_{i,j-1}^{\perp}}(\log g_{i,j})|.$$

It's a contradiction with the definition of $g_{i,j}$.

By induction assumption,

$$|\log g_{i,j}| \leq 10^{3^j}|\text{Proj}_{V_{i,j-1}^{\perp}}(\log g_{i,j})|$$

and

$$|\log g_{i,l}| \leq 10^{3^j-3^l}|\log g_{i,j}|, l=1,2,...,j.$$

So

$$|\log g_{i,j+1}| \geq 0.2|\text{Proj}_{V_{i,j-1}^{\perp}}(\log g_{i,j})| \geq \frac{10^{3^l-2(3^j)}}{5}|\log g_{i,l}| \geq 10^{3^l-3^{j+1}}|\log g_{i,l}|,$$

and

$$\begin{aligned}
|\log g_{i,j+1}| \leq& (3.5\sum_{l=1}^{j}10^{2(3^j)-3^l}+1)|\text{Proj}_{V_{i,j}^{\perp}}(\log g_{i,j+1})| \\
&< 10^{3^{j+1}}|\text{Proj}_{V_{i,j}^{\perp}}(\log g_{i,j+1})|.
\end{aligned}$$

Thus, no matter $|\mathrm{Proj}_{V_{i,j}}(\log g_{i,j+1})|$ is smaller than $0.7\sum_{l=1}^{j}|\log g_{i,l}|$ or not, we can achieve the induction assumption for $j+1$ except the proof that $\lim_{i\to\infty}|\log g_{i,j+1}|=0$ and $g_{i,a}^{-1}g_{i,b}^{-1}g_{i,a}g_{i,b}\in G_{i,\min\{a,b\}-1}$. The first statement is true because G_i converges to G. The second statement is true because the length of the commutator is bounded by $C|\log g_{i,a}||\log g_{i,b}|$ and therefore much smaller than both $|\log g_{i,a}|$ and $|\log g_{i,b}|$ if i is large enough.

After iterations, the induction procedure must stop because the dimension of $\mathfrak{h}$ is finite. The number of elements will be exactly the same as the dimension of G because G_i converge to G. □

Remark. If we look at the above proof carefully, we know that we only used elements close enough to identity. Therefore, it's enough to assume that G_i have a local group structure near identity rather than being a group. Actually the theorem is even true if the product of $a,b\in G_i$ contains an error controlled by $C_i|a||b|$, where C_i converge to 0 as i goes to infinity. In particular, the local group G_i in Theorem 2.12 satisfy the Theorem. For those local groups, since the rotation part is bounded by the translation part by Theorem 2.6, the length of the geodesic loop is equivalent to the length in the above theorem.

Now we are ready to go back to study the length of short geodesic loops. In the rest of this section, we fix a geodesic ray α from o to infinity and start doing analysis about geodesic loops based on the ray. $(M,\alpha(t),t^{-2}g)$ converges to $(C(S(\infty)),p_\infty,g_\infty)$ in pointed Gromov-Hausdorff topology.

Theorem 2.16. *In the ALF-A_k or ALF-D_k cases, there is a geodesic loop γ_1 such that when we slide it along the fixed ray to get $\gamma_{r,1}$ based at $\alpha(r)$, its length*

$$L(r):=L(\gamma_{r,1})=L_\infty+O(r^{-\epsilon})$$

and the parellel transport along it satisfies

$$|P-\mathrm{Id}|=O(r^{-1-\epsilon}).$$

What's more, any loop based at $\alpha(r)$ with length smaller than κr is generated by $\gamma_{r,1}$ in the sense of Gromov.

Proof. In this case, $(B_{\kappa r}(\alpha(r)),r^{-2}g)$ converge to $(B_\kappa(p),g_\infty)\subset C(S(\infty))$ by Theorem 2.12. We may make κ even smaller to apply Theorem 2.15. We get $\gamma_{r,1}$ corresponding to $g_{r,1}$ in Theorem 2.15. Then any loop based at $\alpha(r)$ with length smaller than κr is generated by $\gamma_{r,1}$ in the sense of Gromov. There is an ambiguity to choose $\gamma_{r,1}$. The same loop with reverse direction would play the same role. However, we can choose them consistently so that they are the sliding of each other along the ray. By Theorem 2.15,

$$\lim_{r\to\infty}\frac{L(r)}{r}=0.$$

So the parallel transport along the loop converges to identity by Theorem 2.6. It follows that

$$\begin{aligned}|P-\mathrm{Id}|(r)&=|P-\mathrm{Id}|(\infty)-\int_r^\infty |P-\mathrm{Id}|'\mathrm{d}t\\&\le 0+\int_r^\infty CLt^{-2-\epsilon}\mathrm{d}t\\&\le O(r^{-\epsilon}).\end{aligned}$$

by the equation that $||P-\mathrm{Id}|'|<CLr^{-2-\epsilon}$. Plug this back to the equation

$$|L'|\le|P-\mathrm{Id}|,$$

we obtain

$$\begin{aligned}L(r)&=L(r_0)+\int_{r_0}^r L'(t)\mathrm{d}t\le L(r_0)+\int_{r_0}^r|P-\mathrm{Id}|\mathrm{d}t\\&\le L(r_0)+\int_{r_0}^r Ct^{-\epsilon}\mathrm{d}t=L(r_0)+C(r^{1-\epsilon}-r_0^{1-\epsilon}).\end{aligned}$$

In turn $|P-\mathrm{Id}|\le O(r^{-2\epsilon})$, $L\le O(r^{1-2\epsilon})$. $\cdots$ Through finite steps of iterations, we have

$$L=L_\infty+O(r^{-\epsilon})$$

and

$$|P-\mathrm{Id}|\le O(r^{-1-\epsilon}).$$

Claim: The limit length $L_\infty=\lim_{t\to\infty}L_t>0$. Otherwise, since $L=O(r^{-\epsilon})$, after the integration from infinity to r, we can easily obtain

$$|P-\mathrm{Id}|\le O(r^{-1-2\epsilon})$$

After a finite number of iterations, we have

$$L=O(r^{-1-\epsilon})\qquad\text{and}\qquad|P-\mathrm{Id}|\le O(r^{-2-2\epsilon}).$$

Now let

$$f(r)=\sum_{k=0}^{\infty}\frac{2^{(2+\epsilon)k}r^{-k\epsilon}}{\epsilon(\epsilon+1)2\epsilon(2\epsilon+1)...k\epsilon(k\epsilon+1)},$$

Then

$$f''(r)=(\frac{r}{2})^{-2-\epsilon}f(r),f(r)=1+O(r^{-\epsilon}),f'(r)=O(r^{-1-\epsilon}).$$

So for all R large enough, we have

$$L(R)<R^{-1}f(R)$$

and

$$|L'(R)|<R^{-1}|f'(R)|.$$

By ODE comparison, we have $L(r)<R^{-1}f(r)$. Let R go to infinity, $L(r)=0$, this is a contradiction. So $L_\infty>0$. $\square$

Theorem 2.17. *In the ALG case, there are commutative geodesic loops γ_1,γ_2 such that when we slide them along the fixed ray α to get $\gamma_{r,1},\gamma_{r,2}$ based at $\alpha(r)$, their length*

$$L_j(r):=L(\gamma_{r,j})=L_{\infty,j}+O(r^{-\epsilon})$$

and parallel transports along them satisfy

$$|P_{\gamma_{r,j}}-\mathrm{Id}|=O(r^{-1-\epsilon}).$$

What's more,any loop based at $\alpha(r)$ with length smaller than κr is generated by $\gamma_{r,1}$ and $\gamma_{r,2}$ in the sense of Gromov.

Proof. We proceed as in the proof of Theorem 2.16. We get two loops $\gamma_{r,1}$ and $\gamma_{r,2}$ based at $\alpha(r)$. In this case, the ambiguity is as large as $\mathrm{GL}(2,\mathbb{Z})$. In other words, $\gamma_{r,1}$ and $\gamma_{r,2}$ may jump to $\gamma_{r,1}^{100}\gamma_{r,2}^{99}$ and $\gamma_{r,1}^{101}\gamma_{r,2}^{100}$ respectively after the sliding. Actually $\mathrm{GL}(2,\mathbb{Z})$ is a noncompact group, so we can't estimate the length of the geodesic loops obtained by sliding directly. However, we can still get the same conclusion from the fact that $\gamma_{r,1}$ and $\gamma_{r,2}$ commute and that they form a detectable angle.

Suppose the manifold is flat, then the covering transforms corresponding to $\gamma_{r,1}$ and $\gamma_{r,2}$ are linear maps $T_1(\mathbf{x})=\mathbf{ax}+\mathbf{b}$, $T_2(\mathbf{x})=\mathbf{Ax}+\mathbf{B}$, where $\mathbf{a},\mathbf{A}\in\mathrm{SU}(2),\mathbf{b},\mathbf{B}\in\mathbb{C}^2$. So (Note that by the construction $|\mathbf{b}|<C|\mathbf{B}|$)

$$\mathbf{x}=T_1^{-1}T_2^{-1}T_1T_2(\mathbf{x})=\mathbf{a}^{-1}\mathbf{A}^{-1}\mathbf{aAx}+\mathbf{a}^{-1}\mathbf{A}^{-1}((\mathbf{a}-\mathbf{Id})\mathbf{B}+(\mathbf{Id}-\mathbf{A})\mathbf{b}).$$

On the manifold, we need to count the error caused by curvature. So actually

$$|(\mathbf{a}-\mathbf{Id})\mathbf{B}-(\mathbf{A}-\mathbf{Id})\mathbf{b}| \leq Cr^{-2-\epsilon}|\mathbf{b}||\mathbf{B}|^2, |\mathbf{a}^{-1}\mathbf{A}^{-1}\mathbf{a}\mathbf{A}-\mathbf{Id}| < Cr^{-2-\epsilon}|\mathbf{b}||\mathbf{B}|.$$

Now if $|\mathbf{a}-\mathbf{Id}| > r^{-1-\epsilon/3}|\mathbf{b}|$, then

$$|\mathbf{A}-\mathbf{Id}||\mathbf{b}| \geq |\mathbf{a}-\mathbf{Id}||\mathbf{B}| - Cr^{-2-\epsilon}|\mathbf{b}||\mathbf{B}|^2.$$

It follows that $|\mathbf{A}-\mathbf{Id}| > c \cdot r^{-1-\epsilon/3}|\mathbf{B}|$ for some constant c. Thus, if r is large enough, the two vectors $(\mathbf{A}-\mathbf{Id})\mathbf{b}$ and $(\mathbf{a}-\mathbf{Id})\mathbf{B}$ have almost the same angle since their difference has much smaller length. Note that both $\mathbf{A}$ and $\mathbf{a}$ are very close to identity, so $\mathbf{A}-\mathbf{Id}$ and $\mathbf{a}-\mathbf{Id}$ are almost $\log(\mathbf{A})$ and $\log(\mathbf{a})$ respectively. So Theorem 2.15 is reduced to that $(\mathbf{a}-\mathbf{Id}, \mathbf{b})$ form a detectable angle with $(\mathbf{A}-\mathbf{Id}, \mathbf{B})$. Therefore, $\mathbf{A}-\mathbf{Id}$ and $\mathbf{a}-\mathbf{Id}$ also form a detectable angle because $(\mathbf{A}-\mathbf{Id})\mathbf{b}$ has almost the same angle with $(\mathbf{a}-\mathbf{Id})\mathbf{B}$.

Since the Lie algebra in $\mathfrak{su}(2)$ is simply the cross product and all the matrices are very close to identity

$$|\mathbf{a}-\mathbf{Id}||\mathbf{A}-\mathbf{Id}| < C|\mathbf{a}^{-1}\mathbf{A}^{-1}\mathbf{a}\mathbf{A}-\mathbf{Id}| < Cr^{-2-\epsilon}|\mathbf{b}||\mathbf{B}|.$$

This is a contradiction. So

$$|\mathbf{a}-\mathbf{Id}| \leq r^{-1-\epsilon/3}|\mathbf{b}|.$$

Similarly

$$|\mathbf{A}-\mathbf{Id}| \leq r^{-1-\epsilon/3}|\mathbf{B}|.$$

We've proved that for $\gamma_{r,1}$ and $\gamma_{r,2}$, $|P-\mathrm{Id}| \leq r^{-1-\epsilon/3}L$. For any loop with length smaller than κr, we have $|P-\mathrm{Id}| \leq Cr^{-\epsilon/3}$. When we slide $\gamma_{r,j}$ along the fixed ray towards infinity, the parallel transport along the limiting loops must be trivial. The proof in Theorem 2.16 then implies our conclusion. Note that the ambiguity of choosing $\gamma_{r,j}$ now can be removed by requiring that they are the sliding of loops along α. □

Theorem 2.18. *In the ALH case, there are commutative geodesic loops $\gamma_1, \gamma_2, \gamma_3$ such that when we slide them along the fixed ray α to get $\gamma_{r,1}, \gamma_{r,2}, \gamma_{r,3}$ based at $\alpha(r)$, their length $L_j(r) := L(\gamma_{r,j}) = L_{\infty,j}+O(r^{-\epsilon})$ and parallel transports along them satisfy $|P_{\gamma_{r,j}}-\mathrm{Id}| = O(r^{-1-\epsilon})$. What's more, any loop based at $\alpha(r)$ with length smaller than κr is generated by $\gamma_{r,1}$,$\gamma_{r,2}$ and $\gamma_{r,3}$ in the sense of Gromov.*

Proof. We can proceed exactly in the same way as Theorem 2.17. The only thing we need to prove is that $\gamma_{r,2}$ commutes with $\gamma_{r,3}$. It follows from the fact that the length of the commutator converge to 0 since the curvature and therefore the errors converge to 0 as r goes to 0. □

2.6 From geodesic loops to Riemannian fiberation

In [16], Cheeger, Fukaya and Gromov first introduced the N-structure i.e. nilpotent group fiberations of different dimensions patched together consistently. (Torus is the simplest nilpotent group.) In [58], Minerbe followed their method and improved the result for circle fiberations under a strong volume growth condition in ALF case. In their papers they all view $\mathbb{R}^{4-k}\times\mathbb{T}^k$ as the Gromov-Hausdorff approximation of $\mathbb{R}^{4-k}$. In this subsection, we also include the $\mathbb{T}^k$ factor in the analysis. Therefore, we are able to obtain a better estimate without any volume assumptions.

In the last subsection, we get geodesic loops $\gamma_{p,i}$ along a ray. They can be represented by $s \in [0,1] \to \exp_p(sv_i(p))$ for some vectors $v_i(p)$ in the tangent space of the base point p. When p goes to infinity, the vectors $v_i(p)$ converge to some limits $v_i \in \mathbb{R}^4$. Actually, the difference between $v_i(p)$ and v_i is bounded by $O(r^{-\epsilon})$. Define the lattice Λ by $\Lambda = \oplus_{i=1}^k \mathbb{Z}v_i$ and the torus $\mathbb{T}^k = (\oplus_{i=1}^k \mathbb{R}v_i)/\Lambda$ with the induced metric. From the estimates in the last subsection and the estimates in the last paragraph of Section 2.1 (c.f. Proposition 2.3.1 of [13]), it's easy to see that for $\sum_{i=1}^k a_i v_i \in \Lambda \cap B_{\kappa r(p)}$, the translation part of the Gromov product $\prod_{i=1}^k \gamma_{p,i}^{a_i}$ is $\sum_{i=1}^k a_i v_i$ with error bounded by $O(r^{1-\epsilon})$ while the rotation part is bounded by $O(r^{-\epsilon})$. So the lattice Λ almost represent the geodesic loops whose length is smaller than $\kappa r(p)$.

By Proposition 2.5, Corollary 2.11, the estimates in Theorem 2.16, Theorem 2.17 or Theorem 2.18, we can slide the geodesic loops $\gamma_{p,i}$ along a path within $B_{1.1r}(o)\setminus B_{0.9r}(o)$ to get geodesic loops $\gamma_{p,i}$ over the whole manifold M except a compact set K. It satisfies all the above properties. The choice of path is not unique, so after sliding along different paths, $\gamma_{p,i}$ may be different. However, all the differences come from a change of basis in Λ. Locally, we can assume that $\gamma_{p,i}$ are well defined.

Theorem 2.19. *There exists a diffeomorphism from $B_{\kappa r}(p)$ to*

$$B_{\kappa r}(0)\times\mathbb{T}^k \subset \mathbb{R}^{4-k}\times\mathbb{T}^k,$$

such that g = the pull back of the flat metric + $O'(r^{-\epsilon})$.

Proof. First of all we look at the map $\exp : \mathrm{T}_p \to M$. Any $q \in B_{\kappa r}(p)$ has lots of preimages. Choose one preimage q_0, then all the other preimages are $\prod_{i=1}^k F_i^{a_i}(q_0)$, where F_i are the covering transforms corresponding to γ_i, and a_i are integers. We know that $\prod_{i=1}^k F_i^{a_i}(q_0)$ is actually $q_0 + \sum_{i=1}^k a_i v_i$ with error in $O'(r^{1-\epsilon})$. Define

$$f(q) = \pi_{\mathbb{T}^k} \frac{\sum \chi(\frac{10|\prod_{i=1}^k F_i^{a_i}(q_0)|}{\kappa r(p)})(\prod_{i=1}^k F_i^{a_i}(q_0) - \sum_{i=1}^k a_i v_i)}{\sum \chi(\frac{10|\prod_{i=1}^k F_i^{a_i}(q_0)|}{\kappa r(p)})} \in \mathbb{R}^{4-k} \times \mathbb{T}^k,$$

then it's independent of the choice of q_0. It's easy to prove that using f, the metric g= the pull back of the flat metric + $O'(r^{-\epsilon})$. □

Lemma 2.20. *We can find good covers $\{B_{\frac{1}{2}\kappa r(p_i)}(p_i)\}_{i\in I}$ such that I can be divided into $I = I_1 \cup ... \cup I_N$, and if $i, j \in I_l$, $l = 1, 2, ..., N$, the intersection $B_{\kappa r(p_i)}(p_i) \cap B_{\kappa r(p_j)}(p_j) = \emptyset$.*

Proof. This kind of theorem was first proved in [16]. In our situation we can choose maximal $\kappa 2^{l-1}$ nets in $B(2^{l+1}) - B(2^l)$. Then volume comparison implies the property. □

Theorem 2.21. *Outside a compact set K, there is a global fiberation and a $\mathbb{T}^k$ invariant metric $\tilde{g} = g + O'(r^{-\epsilon})$ whose curvature belongs to $O'(r^{-2-\epsilon})$*

Proof. By Lemma 2.20, we can first modify $i \in I_1$ and $j \in I_2$ so that they are compatible. Then modify $i, j \in I_1, I_2$ and $l \in I_3$ to make sure that they are compatible. After N times, we're done. So we start from a map

$$f_{ij} : B_{\kappa r_i}(p_i) \times \mathbb{T}^k \to B_{\kappa r_j}(p_j) \times \mathbb{T}^k.$$

$$f_{ij}(q, \theta) = (f_{ij}^1(q, \theta), f_{ij}^2(q, \theta)) = f_j \circ f_i^{-1}(q, \theta).$$

Average it and get $\tilde{f}_{ij}^1 : B_{\kappa r_i}(p_i) \to B_{\kappa r_j}(p_j)$ by

$$\tilde{f}_{ij}(q) = \frac{1}{\mathrm{Vol}(\mathbb{T}^k)} \int_{\mathbb{T}^k} f_{ij}^1(q, \theta) \mathrm{d}\theta.$$

From the higher derivative control, we know that the distance from origin to $f_{ij}^2(q, \theta) - f_{ij}^2(q, 0) - \theta \in \mathbb{T}^k$ is bounded by $O(r^{-\epsilon})$. (Here we view $\mathbb{T}^k$ as an abelian group.) For r large enough, we can lift it to $\mathbb{R}^k$ while keeping it bounded by $O(r^{-\epsilon})$. Fix q and average it with respect to θ, then project it back to $\mathbb{T}^k$. We get a map $\tilde{f}_{ij}^2 : B_{\kappa r_i}(p_i) \to \mathbb{T}^k$. Define

$$\tilde{f}_{ij} : B_{\kappa r_i}(p_i) \times \mathbb{T}^k \to B_{\kappa r_j}(p_j) \times \mathbb{T}^k$$

by

$$\tilde{f}_{ij}(q,\theta) = (\tilde{f}_{ij}^1(q), \theta + f_{ij}^2(q,0) + \tilde{f}_{ij}^2(q)).$$

It's easy to see that $|\nabla^m \tilde{f}_{ij}| = O(r^{1-m-\epsilon})$. We may glue the common part using $\tilde{f}_{ij}$. Now there are two metrics g_i^{Flat} and g_j^{Flat}. Choose a partition of unity $\chi_i + \chi_j = 1$, $|\nabla^m \chi_i| = O(r^{-m})$. Let $\tilde{g} = \chi_i g_i^{\text{Flat}} + \chi_j g_j^{\text{Flat}}$. It's a $\mathbb{T}^k$ invariant metric with $|\nabla^m \tilde{g}| = O(r^{-m-\epsilon})$. Note that there are still two maps from M to the gluing $B_{\kappa r_i}(p_i) \times \mathbb{T}^k \cup_{\tilde{f}_{ij}} B_{\kappa r_j}(p_j) \times \mathbb{T}^k$: $\tilde{f}_{ij} \circ f_i$ and f_j. However, their distance is bounded by $O(r^{-\epsilon})$. For r large enough, we can find out the unique $\tilde{g}$-minimal geodesic γ satisfying $\gamma(0) = \tilde{f}_{ij} \circ f_i$ and $\gamma(1) = f_j$. Then $\gamma(\chi_j)$ gives a new map from M to $B_{\kappa r_i}(p_i) \times \mathbb{T}^k \cup_{\tilde{f}_{ij}} B_{\kappa r_j}(p_j) \times \mathbb{T}^k$. Call that $\tilde{f}_i \cup \tilde{f}_j$.

In conclusion, we have a $\mathbb{T}^k$-invariant metric h on

$$B_{\kappa r_i}(p_i) \times \mathbb{T}^k \cup_{\tilde{f}_{ij}} B_{\kappa r_j}(p_j) \times \mathbb{T}^k$$

and

$$\tilde{f}_i \cup \tilde{f}_j : M \to B_{\kappa r_i}(p_i) \times \mathbb{T}^k \cup_{\tilde{f}_{ij}} B_{\kappa r_j}(p_j) \times \mathbb{T}^k$$

with both $|\nabla^m h| = O(r^{-m-\epsilon})$ and $|\nabla^m (\tilde{f}_i \cup \tilde{f}_j)| = O(r^{1-m-\epsilon})$.

After repeating everything for $(B_{\kappa r_i}(p_i) \times \mathbb{T}^k \cup_{\tilde{f}_{ij}} B_{\kappa r_j}(p_j) \times \mathbb{T}^k, \tilde{g}, \tilde{f}_i \cup \tilde{f}_j)$ and $(B_{\kappa r_l}(p_l) \times \mathbb{T}^k, g_l^{\text{flat}}, f_l)$, we can get a new big chart. After N times, we are done. □

Theorem 2.22. *Outside K, there is a $\mathbb{T}^k$-fiberation E over $C(S(\infty)) - B_R$ and a standard $\mathbb{T}^k$ invariant metric h such that after the pull back by some diffeomorphism $h = g + O'(r^{-\epsilon})$.*

Proof. The metric $\tilde{g}$ can be written as

$$\sum_{i,j=1}^{4-k} a_{ij}(x) \mathrm{d}x_i \otimes \mathrm{d}x_j + \sum_{l=1}^{k} (\mathrm{d}\theta_l + \sum_{i=1}^{4-k} \eta_{li}(x) \mathrm{d}x_i)^2.$$

The curvature of a_{ij} belongs to $O'(r^{-2-\epsilon})$. By the result of Bando, Kasue and Nakajima [5], there is a coordinate at infinity such that the difference between a_{ij} and the flat metric on $C(S(\infty)) - B_R$ belongs to $O'(r^{-\epsilon})$. So we can assume that $a_{ij} = \delta_{ij}$ without changing the condition $g = \tilde{g} + O'(r^{-\epsilon})$. Similarly, we can also replace $\eta_{lj}(x)$ by any standard connection form. As long as η_{lj} is still in $O'(r^{-\epsilon})$, we still have $h = g + O'(r^{-\epsilon})$. Therefore, we only need to classify the torus fiberations over $C(S(\infty)) - B_R$ topologically and give it a good enough standard metric h.

(ALF-A_k)When $S(\infty) = \mathbb{S}^2$, the circle fiberation must be orientable. It's determined by the Euler class e.

When $e = 0$, we have the trivial product $(\mathbb{R}^3 - B_R) \times \mathbb{S}^1$ as our standard model.

When $e = \pm 1$, we have the Taub-NUT metric with mass $m \neq 0$: Let

$$M_+ = (\{(x_1, x_2, x_3)|x_1^2 + x_2^2 + x_3^2 \geq R^2\} - \{(0, 0, x_3)|x_3 < 0\}) \times \mathbb{S}^1,$$

$$M_- = (\{(x_1, x_2, x_3)|x_1^2 + x_2^2 + x_3^2 \geq R^2\} - \{(0, 0, x_3)|x_3 > 0\}) \times \mathbb{S}^1$$

Identify $(x_1, x_2, x_3, \theta_+)$ in M_+ with $(x_1, x_2, x_3, \theta_- + \mathrm{sign}(m)\arg(x_1 + ix_2))$ in M_-. We get a manifold M.

Let $r = \sqrt{x_1^2 + x_2^2 + x_3^2}, V = 1 + \frac{2m}{r}$,

$$\begin{aligned}\eta &= 4|m|\mathrm{d}\theta_+ + 4m\frac{(x_3 - r)(x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1)}{2(x_1^2 + x_2^2)r} \\ &= 4|m|\mathrm{d}\theta_- + 4m\frac{(x_3 + r)(x_1\mathrm{d}x_2 - x_2\mathrm{d}x_1)}{2(x_1^2 + x_2^2)r}.\end{aligned}$$

Then the Taub-NUT metric with mass m outside the ball $B_R(R >> |m|)$ is

$$\mathrm{d}s^2 = V\mathrm{d}\mathbf{x}^2 + V^{-1}\eta^2$$

with

$$\mathrm{d}x_1 = I^*(V^{-1}\eta) = J^*\mathrm{d}x_2 = K^*\mathrm{d}x_3.$$

There are lots of different conventions in the literatures. We use the convention from [51], but we compute the explicit form of η using the formulas in [40]. When $m > 0$, LeBrun [51] proved that M can be smoothly extended inside B_R and becomes biholomorphic to $\mathbb{C}^2$. For $m < 0$, the metric is only defined outside B_R, but it's enough for our purpose.

There is a natural $\mathbb{Z}_{|e|}$ action on Taub-NUT metric by $\theta_\pm \to \theta_\pm + 2\pi/|e|$ for $e = \pm 1, \pm 2, ...$ The quotient of the Taub-NUT metric with positive mass

m by $\mathbb{Z}_{|e|}$ has Euler class $e < 0$, The quotient of the Taub-NUT metric with negative mass m by $\mathbb{Z}_{|e|}$ has Euler class $e > 0$. Notice that the mass parameter m is essentially a scaling parameter. Only the sign of m determines the topology.

Usually, people let $k = -e - 1$ and call that a standard ALF-A_k metric.

(ALF-D_k) When

$$S(\infty) = \mathbb{RP}^2 = \{(x_1, x_2, x_3) \in \mathbb{S}^2 | x_3 \geq 0\}/(x_1, x_2, 0) \sim (-x_1, -x_2, 0)$$

topologically, the fiberation is the trivial fiberation over the disc after identifying $(\cos t, \sin t, 0, \theta)$ with $(\cos(t + \pi), \sin(t + \pi), 0, f(t) - \theta)$. So

$$f(\pi) - f(0) = -2e\pi.$$

The integer e determines the topological type.

When $e = 0$, we have the trivial product $(\mathbb{R}^3 - B_R) \times \mathbb{S}^1$ after identifying $(\mathbf{x}, \theta)$ with $(-\mathbf{x}, -\theta)$ as our standard model.

When e is nonzero, it's the quotient of the Taub-NUT metric outside B_R by the binary dihedral group $D_{4|e|} = \{\sigma, \tau | \sigma^{2|e|} = 1, \sigma^{|e|} = \tau^2, \tau\sigma\tau^{-1} = \sigma^{-1}\}$ which acts by $\sigma(\mathbf{x}, \theta_\pm) = \sigma(\mathbf{x}, \theta_\pm + \pi/|e|)$ and $\tau(\mathbf{x}, \theta_+) = (-\mathbf{x}, \theta_- = -\theta_+)$ from M_+ to M_- with $\tau(\mathbf{x}, \theta_-) = (-\mathbf{x}, \theta_+ = \pi - \theta_-)$ from M_- to M_+. When the mass is positive, e is negative. When the mass is negative, e is positive.

Usually, people let $k = -e + 2$ and call that a standard ALF-D_k metric.

(ALG)When $S(\infty) = \mathbb{S}^1$, the topological type is determined by the monodramy. In other words, when we travel along $S(\infty)$, there is some rotation but the lattice $\Lambda = \mathbb{Z}|v_1| \oplus \mathbb{Z}\tau|v_1|$ is still invariant. So we have the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix} = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} \begin{pmatrix} 1 \\ \tau \end{pmatrix}.$$

for some

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2, \mathbb{Z}).$$

So

$$0 = \det \begin{pmatrix} a - e^{i\theta} & b \\ c & d - e^{i\theta} \end{pmatrix} = ad - bc - (a + d)e^{i\theta} + (e^{i\theta})^2.$$

Except the case where $e^{i\theta} = \pm 1$,we have $\Delta = (a + d)^2 - 4(ad - bc) < 0$. So $ad - bc > 0$, it must be 1 to make sure the matrix invertible. So $a + d = 0$ or $a + d = \pm 1$. The quadratic equation $e^{i\theta}$ satisfies must be one of the following

$$x^2 + x + 1 = 0, \qquad x^2 - x + 1 = 0, \qquad \text{and} \qquad x^2 + 1 = 0.$$

We can solve $e^{i\theta}$ accordingly:

$$\frac{-1 \pm i\sqrt{3}}{2} = e^{i\frac{2\pi}{3}}, e^{i\frac{4\pi}{3}}; \qquad \frac{1 \pm i\sqrt{3}}{2} = e^{i\frac{\pi}{3}}, e^{i\frac{5\pi}{3}}; \qquad \pm i = e^{i\frac{\pi}{2}}, e^{i\frac{3\pi}{2}}.$$

Therefore, the rotation angle $\theta = 2\pi\beta$ and the lattice $\Lambda = \mathbb{Z}|v_1| \oplus \mathbb{Z}\tau|v_1|$ are in the following list: (We may replace τ by something like $\tau - 1$, but that won't change the lattice at all)

(Regular) $\mathrm{Im}\tau > 0$, $\beta = 1$.
(I_0^*) $\mathrm{Im}\tau > 0$, $\beta = 1/2$.
(II) $\tau = e^{2\pi i/3}$, $\beta = 1/6$.
(II*) $\tau = e^{2\pi i/3}$, $\beta = 5/6$.
(III) $\tau = i$, $\beta = 1/4$.
(III*) $\tau = i$, $\beta = 3/4$.
(IV) $\tau = e^{2\pi i/3}$, $\beta = 1/3$.
(IV*) $\tau = e^{2\pi i/3}$, $\beta = 2/3$.

Note that they all correspond to Kodaira's classification of special fibers of elliptic surface in [45]! If we identify (u, v) with $(e^{2\pi i\beta}u, e^{-2\pi i\beta}v)$ in the space $\{(u, v)|\arg u \in [0, 2\pi\beta], |u| \geq R\} \subset (\mathbb{C} - B_R) \times \mathbb{C}/(\mathbb{Z}|v_1| \oplus \mathbb{Z}\tau|v_1|)$, we have the standard flat hyperkähler metric $h = \frac{i}{2}(du \wedge d\bar{u} + dv \wedge d\bar{v})$. Note that SU(2) is transitive, so we can choose the complex structure $a_1 I + a_2 J + a_3 K$ properly so that $\bar{\partial}_g = \bar{\partial}_h + O(r^{-\epsilon})\nabla_h$.

(ALH)When $C(S(\infty)) = \mathbb{R}_+$, h can be simply chosen to be the product metric of $[R, \infty)$ and a flat 3-torus.

□

3 Weighted Analysis

In this section, we prove Theorem 1.3, Theorem 1.5 and Theorem 1.6. There are two goals in this section: the improvement of asymptotic rate and the construct of global holomorphic functions on gravitational instantons M with prescribed growth order.

To improve the asymptotic rate, we view $(M \setminus K, g)$ as a deformation of (E, h). We will show that the infinitesimal deformation space is a subspace of three copies of anti-self-dual closed forms. The decay rate of such forms can be improved automatically.

To construct holomorphic functions, we start from the construction of holomorphic functions on the standard models (E, h). Then it can be pulled back to (M, g) and cut off to obtain an almost holomorphic function f on M. To get rid of the error, we can solve the $\bar{\partial}$ equation

$$\bar{\partial} g = \bar{\partial} f$$

for g much smaller than f. If successful, then $f - g$ will be the required function. Unfortunately, it is hard for us to solve g directly. So instead, we solve the equation

$$-(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})\phi = \bar{\partial} f.$$

The order of $\bar{\partial}^*\phi$ and $\bar{\partial}\phi$ will be smaller than the order of f if we solve ϕ properly. Notice that there is a covariant constant (0,2)-form ω^-, so the harmonic (0,2)-form $\bar{\partial}\phi$ is essentially a harmonic function. Generally speaking, the order of growth of harmonic functions on M is the same as the harmonic functions on E. So if we get f from the smallest nonconstant harmonic function on E, we expect $\bar{\partial}\phi$ to be 0. Therefore $f + \bar{\partial}^*\phi$ will be the required global holomorphic function on M.

To relate infinitesimal deformation space with deformation space and to solve the Laplacian equation for (0,1)-forms, we need some elliptic estimates. The ALH case requires more care. To obtain a good estimate in ALH case, we need to prove the exponential decay of curvature first. This is feasible after we develop some elliptic estimates for the Riemannian curvature tensor.

Therefore, in the first two subsections, we develop the elliptic estimates for tensors on a manifold M asymptotic to the standard model. We would like to work on both forms and the curvature tensors on general M which may not be hyperkähler. Therefore, we always use the Bochner Laplacian $-\mathrm{Tr}\nabla^*\nabla$ in order to apply the Bochner techniques. For gravitational instantons, the Weitzenböck formula tells us that the Bochner Laplacian equals to the operator $-(\bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial})$ for functions and (0,1)-forms. Then in the third subsection, we use this estimate to prove the exponential decay of curvature of ALH-instantons. This allows us to develop an elliptic estimate with exponential growth weights in the fourth subsection. After the analysis of infinitesimal deformation space in fifth subsection, we will prove theorem 1.3 in sixth subsection. In the seventh subsection, we use the mentioned technique to construct global holomorphic functions on ALF and ALG instantons. Then in the eighth subsection, we use the same method to construct global holomorphic functions on ALH instantons. In the last three subsections, we make use of the global holomorphic functions to prove Theorem 1.5 and Theorem 1.6.

Analysis in weighted Hilbert space is well studied and perhaps some estimates in this section are already known to experts [41] [34] [57]. However, to avoid problems caused by subtle differences between different settings, we instead give a self-contained proof.

3.1 Weighted Hilbert space

In this subsection, we do some technical preparations. We will use the following weighted Hilbert spaces: (Please notice the change of the meaning of r as in the end of Section 2.1.)

Definition 3.1. Define the $L^2_\delta(M)$-norm of a tensor by

$$||\phi||_{L^2_\delta(M)} = \sqrt{\int_M |\phi|^2 r^\delta \mathrm{dVol}}.$$

Let $L^2_\delta(M)$ be the space of tensors with finite $L^2_\delta(M)$-norm. Define $\nabla\phi = \psi$ in the distribution sense if for any $\xi \in \mathrm{C}_0^\infty$, we have $(\phi, \nabla^*\xi) = (\psi, \xi)$. Let $H^2_\delta(M)$ be the space of all tensors ϕ such that

$$\phi \in L^2_\delta(M), \nabla\phi \in L^2_{\delta+2}(M), \nabla^2\phi \in L^2_{\delta+4}(M).$$

We can define the norm in this weighted space by

$$||\phi||_{H^2_\delta(M)} = \sqrt{\int_M |\phi|^2 r^\delta \mathrm{dVol} + \int_M |\nabla\phi|^2 r^{\delta+2} \mathrm{dVol} + \int_M |\nabla^2\phi|^2 r^{\delta+4} \mathrm{dVol}}.$$

The inner product is defined accordingly.

Proposition 3.2. *For any δ, $H^2_\delta(M)$ is a Hilbert space and the space of compactly supported smooth tensors $\mathrm{C}_0^\infty(M)$ is dense.*

Proof. The map $\phi \to \phi r^{\delta/2}$ defines an isometry between $L^2_\delta(M)$ and $L^2(M)$. Since $L^2(M)$ is complete, $L^2_\delta(M)$ is also complete. Now if $|\phi_i - \phi_j|_{H^2_\delta(M)} \to 0$, then both $|\phi_i - \phi_j|_{L^2_\delta(M)}$ and $|\nabla^m\phi_i - \nabla^m\phi_j|_{L^2_{\delta+2m}(M)}$ go to 0, $m = 1, 2$. By completeness, ϕ_i converge to ϕ in $L^2_\delta(M)$, and $\nabla\phi_i$ converge to ψ in $L^2_{\delta+2}(M)$. Now pick any test tensor $\xi \in \mathrm{C}_0^\infty(M)$,

$$(\phi, \nabla^*\xi) = \lim_{i\to\infty}(\phi_i, \nabla^*\xi) = \lim_{i\to\infty}(\nabla\phi_i, \xi) = (\psi, \xi).$$

So $\nabla\phi = \psi$ in the distribution sense. The second derivative is similar. So ϕ_i converge to ϕ in $H^2_\delta(M)$, too.

For the density, let $\chi_R = \chi(r/R)$. Then

$$\begin{aligned}|\phi - \phi\chi(r/R)|_{H^2_\delta(M)} \leq & C(\int_M |(1-\chi_R)\phi|^2 r^\delta + \int_M |(1-\chi_R)\nabla\phi|^2 r^{\delta+2} \\ & + \int_M |\nabla\chi_R||\phi|^2 r^{\delta+2} + \int_M |\nabla^2\chi_R\phi|^2 r^{\delta+4} \\ & + \int_M |(1-\chi_R)\nabla^2\phi|^2 r^{\delta+4} + \int_M |\nabla\chi_R\nabla\phi| r^{\delta+4})\end{aligned}$$

So $\chi(r/R)\phi$ converge to ϕ in $H^2_\delta(M)$ as R goes to infinity since $|\nabla\chi_R| \leq C/R$ and $|\nabla^2\chi_R| \leq C/R^2$. Now the standard convolution method implies the density of $C^\infty_0(M)$.

□

Lemma 3.3. *For any harmonic tensor ϕ in $H^2_\delta(M)$ and any large enough r,*

$$|\phi(y)| \leq C||\phi||_{H^2_\delta(M)} r(y)^{-\delta/2+k/2-2}.$$

When $-\delta/2 + k/2 - 2 < 0$, $\phi = 0$.

Proof. Given $y \in M$, suppose $r(y) = 20R$. Then the ball $B_{2R}(y) \subset M$ is asymptotic to $B_{2R}(0) \times \mathbb{T}^k \subset \mathbb{R}^{4-k} \times \mathbb{T}^k$. Consider the covering space $\mathbb{R}^4$ of $\mathbb{R}^{4-k} \times \mathbb{T}^k$. If we apply Gilbarg and Trudinger's Theorem 9.20 in [31] there, we would get

$$|\phi|^2(y) \leq \frac{C}{|B_{2R}(0)|} \int_{B_{2R}(0)} |\phi|^2.$$

So

$$|\phi(y)| \leq C||\phi||_{H^2_\delta(M)} r(y)^{-\delta/2+k/2-2}.$$

Now the maximal principle implies the last result in the lemma because $\Delta|\phi|^2 = 2|\nabla\phi|^2 \geq 0$. □

Now we need an weighted L^2-estimate.

Lemma 3.4. *For the standard ALF, ALG or ALH metric in Theorem 2.22, suppose ϕ is a smooth form supported in $B_{\tilde{R}} - B_R$. Then as long as R is large enough,*

$$\int_E |\nabla^2\phi|^2 r^{\delta+4} + \int_E |\nabla\phi|^2 r^{\delta+2} \leq C(\int_E |\Delta\phi|^2 r^{\delta+4} + \int_E |\phi|^2 r^\delta)$$

Proof. We only need to prove the same thing on $B_{\kappa r_j}(p_j) \subset E$ uniformly. It's enough to consider the covering $B_{\kappa r_j}(0) \subset \mathbb{R}^4$. Notice that the difference between h and flat metric is in $O'(r^{-1})$. So we can simply use the Theorem 9.11 of [31]. □

3.2 Elliptic estimates with polynomial growth weights

In this subsection, we will prove the main estimate for tensors in the weighted Hilbert space with polynomial growth weights.

We started the estimate for functions on $\mathbb{R}^d$. Then we extend this to $\mathbb{T}^k$ invariant tensors. We can improve it to general tensors on the standard fiberation E. Then we can transfer that estimate back to any manifold M asymptotic to the standard model. This main estimate allows us to prove the solvablity of Bochner Laplacian equation for tensors.

Theorem 3.5. *Suppose f is a real smooth function on $\mathbb{R}^d (d = 1, 2, 3, ...)$ supported in an annulus, δ isn't an integer. Then*

$$\int_{\mathbb{R}^d} |f|^2 r^{\delta} \mathrm{dVol} < C \int_{\mathbb{R}^d} |\Delta f|^2 r^{\delta+4} \mathrm{dVol}.$$

Proof. For the Laplacian on the standard sphere $\mathbb{S}^{d-1}$, it is well know it has eigenfunctions $\phi_{j,l}$ with eigenvalue $-j(d-2+j)$, $l = 1, 2, ..., n_j$. (For $d = 1$,all n_j are 0 except $n_0 = 1$ and $\phi_{j,1} = 1$). We write f in terms of those eigenfunctions

$$f \sim \sum_{j=0}^{\infty} \sum_{l=1}^{n_j} f_{j,l}(r) \phi_{j,l}(\theta),$$

where

$$f_{j,l}(r) = \int_{\mathbb{S}^{d-1}} f(r, \theta) \phi_{j,l}(\theta) \mathrm{dVol}.$$

Then

$$\begin{aligned} \Delta f &\sim \sum_{j=0}^{\infty} \sum_{l=1}^{n_j} (f''_{j,l} + \frac{d-1}{r} f'_{j,l} - \frac{j(d-2+j)}{r^2} f_{j,l}) \phi_{j,l}(\theta) \\ &= \sum_{j=0}^{\infty} \sum_{l=1}^{n_j} r^{-j-d+1} [r^{2j+d-1} (r^{-j} f_{j,l})']' \phi_{j,l}(\theta). \end{aligned}$$

From integral by parts and the Cauchy-Schwartz inequality

$$
\begin{aligned}
(\int_0^\infty g^2 r^\mu \mathrm{d}r)^2 &= \left(\frac{-2}{\mu+1}\int_0^\infty g g' r^{\mu+1} d\,r\right)^2 \\
&\le \frac{4}{(\mu+1)^2}\int_0^\infty g^2 r^\mu \mathrm{d}r \int_0^\infty (g')^2 r^{\mu+2}\mathrm{d}r.
\end{aligned}
$$

So we get the Hardy's inequality

$$
\int_0^\infty g^2 r^\mu \mathrm{d}r \le \frac{4}{(\mu+1)^2}\int_0^\infty (g')^2 r^{\mu+2}\mathrm{d}r.
$$

Therefore

$$
\begin{aligned}
\int_0^\infty f_{j,l}^2 r^\delta r^{d-1}\mathrm{d}r &= \int_0^\infty (r^{-j} f_{j,l})^2 r^{\delta+2j+d-1}\mathrm{d}r \\
&\le \frac{4}{(\delta+2j+d)^2}\int_0^\infty [(r^{-j} f_{j,l})']^2 r^{\delta+2j+d+1}\mathrm{d}r \\
&= \frac{4}{(\delta+2j+d)^2}\int_0^\infty [r^{2j+d-1}(r^{-j} f_{j,l})']^2 r^{\delta-2j-d+3}\mathrm{d}r \\
&\le \frac{16\int_0^\infty (r^{-j-d+1}[r^{2j+d-1}(r^{-j} f_{j,l})']')^2 r^{\delta+4} r^{d-1}\mathrm{d}r}{(\delta+2j+d)^2(\delta-2j-d+4)^2}.
\end{aligned}
$$

By Fubini Theorem and the Hilbert-Schmidt Theorem (When $d = 2$, we get exactly the Fourier series, so the Hilbert-Schmidt theorem is reduced to the Parseval's identity) as long as δ is not an interger, we are done. □

Theorem 3.6. *Given any harmonic function $f \in L^2_\delta(\mathbb{R}^d \setminus B_R)$ for some δ, R, there exist an integer j and some constants $(c_1, ..., c_{n_j}) \neq \mathbf{0}$, such that*

$$
f = r^j \sum_{l=1}^{n_j} c_l \phi_{j,l}(\theta) + O'(r^{j-1})
$$

when $d \ge 3$. There may be additional $\ln r$ term when $d = 2$.

Proof. When $d \ge 3$, write f in terms of eigenfunctions

$$
f \sim \sum_{j=0}^{\infty}\sum_{l=1}^{n_j} f_{j,l}(r)\phi_{j,l}(\theta).
$$

Then the equation is reduced to

$$f''_{j,l} + \frac{d-1}{r} f'_{j,l} - \frac{j(d-2+j)}{r^2} f_{j,l} = 0.$$

So

$$f \sim \sum_{j=0}^{\infty} \sum_{l=1}^{n_j} (a_{j,l} r^j + b_{j,l} r^{2-d-j}) \phi_{j,l}(\theta).$$

By Parseval's identity, the growth condition of u implies that for large enough j, $a_{j,l} = 0$. If one of $a_{j,l}$ is not zero, let j be the largest number such that $a_{j,l} \neq 0$. Let $U = f - r^j \sum_{l=1}^{n_j} a_{j,l} \phi_{j,l}(\theta)$. Parseval's identity again implies that $\int_{B_{2R} \setminus B_R} |U|^2$ has increase rate bounded by $C \int_{B_{2R} \setminus B_R} |r^{j-1}|^2$. By Theorem 9.20 of [31], $U = O'(r^{j-1})$. If all of $a_{j,l}$ are zero, we can do the similar thing for $b_{j,l}$. When $d = 2$, there may be additional $\ln r$ term. □

To generalize them to estimates on E, we still use the decomposition of any tensor ϕ into $\mathbb{T}^k$-invariant part ϕ_1 and the other part ϕ_2 satisfying

$$\int_{\pi^{-1}(x)} \phi_2 = 0$$

as in [57]. Notice that the Laplacian operator $-\mathrm{Tr}\nabla^*\nabla$ and more generally any $\mathbb{T}^k$-invariant operator L preserves this decomposition. Actually, let $\Phi_t(x, \theta) = (x, \theta + t)$ be the diffeomorphism in local coordinates, then L commutes with Φ_t^*. So

$$\Phi_t^*(L\phi_1) = L(\Phi_t^*\phi_1) = L\phi_1,$$

and

$$\int_{t \in \mathbb{T}^k} \Phi_t^*(L\phi_2) = L \int_{t \in \mathbb{T}^k} \Phi_t^* \phi_2 = 0.$$

Theorem 3.7. *Suppose (E, h) is the product of $[R, \infty)$ and $\mathbb{T}^3$, ϕ is a smooth $\mathbb{T}^3$-invariant tensor supported in $B_{\tilde{R}} - B_R$. Then as long as δ is not an integer, for large enough R,*

$$\int_E |\phi|^2 r^\delta \mathrm{dVol} < C \int_E |\Delta\phi|^2 r^{\delta+4} \mathrm{dVol}$$

Proof. Since the tangent bundle is trivial, the estimate of tensors is reduced to the estimate of their coefficients, which has been proved in Theorem 3.5. □

Theorem 3.8. *Suppose (E,h) is the standard ALG metric as in Theorem 2.22, ϕ is a smooth $\mathbb{T}^2$-invariant tensor supported in $B_{\tilde{R}} - B_R$. Then as long as 30δ is not an integer, for large enough R,*

$$\int_E |\phi|^2 r^\delta \mathrm{dVol} < C \int_E |\Delta\phi|^2 r^{\delta+4} \mathrm{dVol}.$$

Proof. Let $\beta = \frac{m}{n}$. Then it's enough to do the same estimate on the n-fold covering $\tilde{E} - B_R$ of $E - B_R$. $\tilde{E} - B_R$ is the isometric product of the m-fold covering of $\mathbb{C} - B_R$ and $\mathbb{T}^2$. So it's enough to prove Theorem 3.5 on the m-fold cover of $\mathbb{C} - B_R$. If we write $f \sim \sum_{j=-\infty}^{\infty} f_j(r) e^{i\theta j/m}$, where $\theta \in [0, 2m\pi]$ then all the works in the proof of Theorem 3.5 go through except that we have to replace j by j/m there. So as long as $m\delta$ isn't an integer, we are done. $(m = 1, 2, 3, 5)$ □

Theorem 3.9. *Suppose (E,h) is the standard ALF metric as in Theorem 2.22, ϕ is a smooth $\mathbb{S}^1$-invariant tensor supported in $B_{\tilde{R}} - B_R$. Then as long as δ is not an integer, for large enough R,*

$$\int_E |\phi|^2 r^\delta \mathrm{dVol} < C \int_E |\Delta\phi|^2 r^{\delta+4} \mathrm{dVol}$$

Proof. By Theorem 2.22, it's enough to consider the trivial product of $\mathbb{R}^3$ and $\mathbb{S}^1$ or the Taub-NUT metric with nonzero mass m. We use 1-forms as example, the proof for general tensors is similar. In the trivial product case, we can write any form as $A\mathrm{d}x_1 + B\mathrm{d}x_2 + C\mathrm{d}x_3 + D\mathrm{d}\theta$. In the remaining cases, any form can be written as $A\mathrm{d}x_1 + B\mathrm{d}x_2 + C\mathrm{d}x_3 + D\eta$. In each case we get 4 functions on $\mathbb{R}^3 - B_R$ which can be filled in by 0 on B_R to get smooth functions on $\mathbb{R}^3$. So we can apply Theorem 3.5 to them. Since the Taub-NUT metric is the flat metric with error $O'(r^{-1})$, while $\eta = \mathrm{d}\theta + O'(r^{-1})$ locally, by Lemma 3.4, we can get our estimate as long as R is large enough. □

Theorem 3.10. *Suppose (E,h) is the standard ALF, ALG, or ALH metric in Theorem 2.22, ϕ is a smooth tensor supported in $B_{\tilde{R}} - B_R$. Then as long as 30δ is not an integer, for large enough R,*

$$\int_E |\phi|^2 r^\delta \mathrm{dVol} < C \int_E |\Delta\phi|^2 r^{\delta+4} \mathrm{dVol}$$

Proof. First average ϕ on each $\mathbb{T}^k$ (k=1,2,3) to get an invariant tensor ϕ_0. Then we only need to get some estimates of the $\phi - \phi_0$ part. It's enough to prove that in each $B_{\kappa r_i}(p_i) \subset E$,

$$\int_{B_{\kappa r_i}(p_i)} |\phi - \phi_0|^2 \mathrm{dVol} < C \int_{B_{\kappa r_i}(p_i)} |\Delta(\phi - \phi_0)|^2 \mathrm{dVol}$$

for a uniform constant C and any tensor ϕ supported in $B_{\kappa r_i}(p_i) \subset E$ because then we can use the partition of unity and move every error term to the left hand side by Lemma 3.4. Again, we may cancel error terms and assume that the metric is flat. So the estimate of forms is reduced to functions which are the coefficients of the forms. Standard Poincaré inequality on torus implies that

$$(\int_{B_{\kappa r} \times \mathbb{T}^k} |f - f_0|^2)^2 \leq C(\int_{B_{\kappa r} \times \mathbb{T}^k} |\nabla_{\mathbb{T}^k}(f - f_0)|^2)^2 \leq C(\int_{B_{\kappa r} \times \mathbb{T}^k} |\nabla(f - f_0)|^2)^2$$

$$= C(\int_{B_{\kappa r} \times \mathbb{T}^k} (f - f_0)\Delta(f - f_0))^2 \leq C \int_{B_{\kappa r} \times \mathbb{T}^k} |f - f_0|^2 \int_{B_{\kappa r} \times \mathbb{T}^k} |\Delta(f - f_0)|^2,$$

where $\nabla_{\mathbb{T}^k}$ means the partial derivative with respect to the fiber direction. So we are done when R is large enough. □

Lemma 3.11. *Suppose X,Y,Z are Banach spaces, $D : X \to Y$, $i : X \to Z$ are bounded linear operators, i is compact. Suppose*

$$||\phi||_X \leq C(||D\phi||_Y + ||i\phi||_Z).$$

Then as long as $\mathrm{Ker}D = \{0\}$*, we have* $||\phi||_X \leq C||D\phi||_Y$.

Proof. If the estimate doesn't hold, then there are ϕ_k satisfying $||\phi_k|| = 1$, but $||D\phi_k|| \to 0$. By the compactness of i, we know that $||i\phi_k - i\phi_l||_Z \to 0$. So

$$||\phi_k - \phi_l||_X \leq C(||D\phi_k - D\phi_l||_Y + ||i\phi_k - i\phi_l||_Z) \to 0.$$

So $\phi_k \to \phi_\infty$. But then $D\phi_k \to D\phi_\infty$, $D\phi_\infty = 0$, $\phi_\infty \in \mathrm{Ker}D$, contradiction. □

Theorem 3.12. *Suppose M is asymptotic to the standard ALF, ALG or ALH model, then for any tensor $\phi \in H^2_\delta(M)$, as long as 30δ is not an integer and $-\delta/2 - 2 + k/2 < 0$, we have*

$$||\phi||_{H^2_\delta(M)} < C \int_M |\Delta\phi|^2 r^{\delta+4} \mathrm{dVol}.$$

Proof. It's enough to prove everything for C_0^∞. Note that

$$\Delta_g\phi = \Delta_h\phi + O(r^{-\epsilon})|\nabla^2\phi| + O(r^{-\epsilon-1})|\nabla\phi| + O(r^{-\epsilon-2})|\phi|.$$

After applying Theorem 3.10 and Lemma 3.4, we know that the estimate holds as long as ϕ is 0 inside a big enough ball B_R. For general ϕ, we can apply the estimate to the form $(1-\chi(r/R))^2\phi$. So

$$||\phi||_{H^2_\delta(M)} < C(\int_M |\Delta\phi|^2 r^{\delta+4}\mathrm{dVol} + ||\phi||_{H^2(B_{2R})})$$

$$< C(\int_M |\Delta\phi|^2 r^{\delta+4}\mathrm{dVol} + \int_{B_{4R}} |\phi|^2)$$

by Theorem 9.11 of [31]. By Lemma 3.11 and Rellich's lemma, it's enough to prove that KerΔ=$\{0\}$. This follows from Lemma 3.3. □

Theorem 3.13. *Suppose* 30δ *is not an integer and* $-\delta/2-2+k/2<0$. *For any* $\phi \in L^2_{-\delta}(M)$, *there exists a tensor* $\psi \in H^2_{-\delta-4}(M)$ *such that* $\Delta\psi = \phi$.

Proof. Consider the Laplacian operator $\Delta : L^2_{-\delta-4}(M) \to L^2_{-\delta}(M)$. The formal adjoint is then $\Delta^*\phi = r^{\delta+4}\Delta(r^{-\delta}\phi)$. Apply Theorem 3.12 to $r^{-\delta}\phi$,

$$C^{-1}||\Delta^*\phi||_{L^2_{-\delta-4}(M)} \le ||\phi||_{H^2_{-\delta}(M)} = ||r^{-\delta}\phi||_{H^2_\delta(M)} \le C||\Delta^*\phi||_{L^2_{-\delta-4}(M)}.$$

So Δ^* has closed range. Now

$$|(\phi,\theta)_{L^2_{-\delta}(M)}| \le ||\phi||_{L^2_{-\delta}(M)}||\theta||_{L^2_{-\delta}(M)} \le C||\phi||_{L^2_{-\delta}(M)}||\Delta^*\theta||_{L^2_{-\delta-4}(M)},$$

so $\Delta^*\theta \to (\phi,\theta)_{L^2_{-\delta}(M)}$ defines a bound linear function in the range of Δ^*. By Riesz representation theorem, there exists $\psi \in \mathrm{Im}(\Delta^*)$ such that

$$(\psi, \Delta^*\theta)_{L^2_{-\delta-4}(M)} = (\phi,\theta)_{L^2_{-\delta}(M)}.$$

Now we get the theorem from the standard elliptic regularity theory. □

We can also solve the Laplacian equation outside a large ball instead of on the whole manifold. Since the maximal principle will not be used globally, we can relax the Laplacian operator to other operators asymptotic to the Laplacian operator.

Theorem 3.14. *For any $\delta \neq 1$, there exists a bounded linear operator*

$$S : L^2_\delta([R,\infty)) \to H^1_{\delta-2}([R,\infty))$$

with $||S|| \leq \frac{2}{|\delta-1|} + 1$ such that $(Sf)' = f$ in the distribution sense.

Proof. Since $(Sf)' = f$, it's enough to control the $L^2_{\delta-2}$-norm of Sf. We can further reduce to prove the same estimate for $f \in C^\infty_0$. So we can assume that $\mathrm{supp}(f) \subset [R_1, R_2]$ with $R < R_1 < R_2 < \infty$. If $\delta > 1$, define

$$Sf(r) = -\int_r^\infty f(t)\mathrm{d}t.$$

So

$$\begin{aligned}
||Sf||^2_{L^2_{\delta-2}([R,\infty))} &= \int_R^\infty [\int_r^\infty f(t)\mathrm{d}t]^2 r^{\delta-2}\mathrm{d}r \\
&= \frac{2}{\delta-1}\int_R^\infty [\int_r^\infty f(t)\mathrm{d}t] f(r) r^{\delta-1}\mathrm{d}r - \frac{R^{\delta-1}}{\delta-1}[\int_R^\infty f(t)\mathrm{d}t]^2 \\
&\leq \frac{2}{\delta-1}\sqrt{\int_R^\infty [\int_r^\infty f(t)\mathrm{d}t]^2 r^{\delta-2}\mathrm{d}r}\sqrt{\int_R^\infty f^2(t) r^\delta \mathrm{d}r} - 0
\end{aligned}$$

by integral by parts and the Cauchy-Schwarz inequality.

If $\delta < 1$, define

$$Sf(r) = \int_R^r f(t)\mathrm{d}t.$$

Then Sf is constant for $r > R_2$, and therefore belongs to $L^2_{\delta-2}$. Moreover, it's 0 for $r < R_1$. Therefore, we can apply the proof of Theorem 3.5 to get the required estimate. □

Lemma 3.15. *Suppose $\phi \in C^\infty(\overline{B_{\tilde{R}} \setminus B_R})$ is a tensor vanishing on the boundary satisfying $\int_{\pi^{-1}(x)} \phi = 0$ for any $x \in B_{\tilde{R}} \setminus B_R$. Suppose*

$$\mathbf{L} = \mathbf{A}^{ij}\nabla_i\nabla_j + \mathbf{B}^i\nabla_i + \mathbf{C}$$

is a $\mathbb{T}^k$-invariant tensor-valued second order elliptic operator with

$$|\mathbf{A}^{ij} - \delta^{ij}\mathbf{Id}| \leq Cr^{-\epsilon}, |\mathbf{B}^i| \leq Cr^{-1-\epsilon}, |\mathbf{C}| \leq Cr^{-2-\epsilon}.$$

Then as long as R is large enough,

$$\int_{B_{\tilde{R}}\setminus B_R} |\phi|^2 r^\delta + \int_{B_{\tilde{R}}\setminus B_R} |\nabla\phi|^2 r^\delta + \int_{B_{\tilde{R}}\setminus B_R} |\nabla^2\phi|^2 r^\delta \leq C\int_{B_{\tilde{R}}\setminus B_R} |L\phi|^2 r^\delta$$

for any $\delta \in \mathbb{R}$, with constant C independent of R and $\tilde{R}$.

Proof. It's easy to see that

$$\int_{B_{\tilde{R}}\setminus B_R} |\nabla^2\phi|^2 r^\delta + \int_{B_{\tilde{R}}\setminus B_R} |\nabla\phi|^2 r^\delta \leq C(\int_{B_{\tilde{R}}\setminus B_R} |\phi|^2 r^\delta + \int_{B_{\tilde{R}}\setminus B_R} |L\phi|^2 r^\delta)$$

by Theorem 9.11 of [31]. By Theorem 3.10, if ϕ is compactly supported in $B_{\kappa r_i(p_i)}$,

$$\int_{(B_{\tilde{R}}\setminus B_R)\cap B_{\kappa r_i(p_i)}} |\phi|^2 \leq C \int_{(B_{\tilde{R}}\setminus B_R)\cap B_{\kappa r_i(p_i)}} |\nabla\phi|^2 \leq C \int_{(B_{\tilde{R}}\setminus B_R)\cap B_{\kappa r_i(p_i)}} |\Delta\phi|^2.$$

Therefore

$$\begin{aligned}\int |\phi|^2 r^\delta + \int |\nabla\phi|^2 r^\delta \leq & C \sum (\int |\chi_i\phi|^2 r^\delta + \int |\nabla(\chi_i\phi)|^2 r^\delta) \\ \leq & C \sum \int |\Delta(\chi_i\phi)|^2 r^\delta \\ \leq & C \sum (\int \chi_i^2 |\Delta\phi|^2 r^\delta + \int |\nabla\chi_i|^2 |\nabla\phi|^2 r^\delta \\ & + \int |\Delta\chi_i|^2 |\phi|^2 r^\delta).\end{aligned}$$

Here, the first inequality holds because R is large enough and by Section 2, we can choose the charts properly so that the number of charts overlapping at any given point is uniformly bounded.

Notice that $\nabla\chi_i = O(r^{-1})$ and $\nabla^2\chi_i = O(r^{-2})$. By canceling terms, we can prove the theorem for $L = \Delta = -\mathrm{Tr}\nabla^*\nabla$. By the same reasons, it can be generalized to more general operator L whose coefficients equal to the Laplacian operator plus small error terms. □

From the approximation by $\phi_n = \phi\chi(r-n)$, the condition in Lemma 3.15 that $\phi \in C^\infty(\overline{B_{\tilde{R}} \setminus B_R})$ vanishes on the boundary can be replaced by the condition that $\phi \in C^\infty(\overline{B_R^c})$ vanishes on ∂B_R and $\phi, \nabla\phi, \nabla^2\phi \in L^2_\delta$.

Notice that the estimate in Lemma 3.15 doesn't scale correctly. Therefore, we must have the following fact:

Theorem 3.16. *If $\phi \in H^2_\delta(B^c_R)$ satisfies $L\phi = 0$. Then ϕ is $\mathbb{T}^k$-invariant plus exponentially decay term.*

Proof. We can assume $\int_{\pi^{-1}(x)} \phi = 0$ and prove that ϕ decay exponentially. For any R large enough, we can apply Lemma 3.15 to $(1 - \chi(r - R))\phi$. Therefore,

$$\begin{aligned}\int_{r>R+2} |\phi|^2 r^\delta &\le C \int_{r>R} |L((1-\chi(r-R))\phi)|^2 r^\delta \\ &\le C \int_{R+1<r<R+2} (|\nabla^2\phi|^2 + |\nabla\phi|^2 + |\phi|^2) r^\delta \\ &\le C \int_{R<r<R+3} |\phi|^2 r^\delta\end{aligned}$$

for some constant C independent of R. The last inequality holds by Theorem 9.11 of [31]. So $\int_{r>R} |\phi|^2 r^\delta$ decay exponentially. ϕ also decay exponentially in L^∞ norm by Theorem 9.20 of [31]. □

Now we are able to prove the following generalization of Lemma 4 of Minerbe's paper [57].

Theorem 3.17. *As long as 30δ is not an integer, there exists a bounded linear operator $G_L : L^2_\delta(B^c_R) \to H^2_{\delta-4}(B^c_R)$ such that $L(G_L\phi) = \phi$.*

Proof. It's enough to prove the same thing for $L = \Delta = -\mathrm{Tr}\nabla^*\nabla$ and for smooth tensor ϕ on E. For $\mathbb{T}^k$-invariant part, we can use Theorem 3.14 instead of Theorem 3.5 and go through the proof of Theorem 3.7, Theorem 3.8 and Theorem 3.9. For the other part, we can solve the equation $\Delta\psi = \phi$ in $B_{\tilde{R}} \setminus B_R$ and $\psi = 0$ on $\partial(B_{\tilde{R}} \setminus B_R)$. It's solvable because we can solve it in H^1_0 first, i.e $(\nabla\psi, \nabla\xi) = (\phi, \xi)$. Then Theorem 8.13 of [31] implies that $\psi \in C^\infty(\overline{B_{\tilde{R}} \setminus B_R})$ and vanishes on the boundary. After throwing away the $\mathbb{T}^k$-invariant part, we can apply Lemma 3.15. Now let $\tilde{R}$ goes to infinity. We can get a sequence of $\psi_{\tilde{R}}$. A subsequence converges to a function ψ_∞ in $H^1(B_{\tilde{R}} \setminus B_R)$ for any $\tilde{R}$ by Rellich lemma and the diagonal argument. ψ_∞ is a generalized solution since we define derivatives in distribution sense. Notice that actually $\psi_\infty, \nabla\psi_\infty, \nabla^2\psi_\infty \in L^2_\delta(B^c_R)$. ψ_∞ also lies in $C^\infty(\overline{B^c_R})$ and equals to 0 on ∂B_R by Theorem 8.13 of [31]. Therefore, we can apply Lemma 3.15 to ϕ_∞. In particular, the difference of two ψ_∞ must be 0. In other words, ψ_∞ is independent of the choice of subsequence. We call that $G_L\phi$. □

3.3 Exponential decay of curvature of ALH gravitational instantons

For ALH instantons, there is a self-improvement forcing the curvature to decay exponentially. Therefore, the metric must converge to the flat one exponentially.

Proposition 3.18. *If the Ricci curvature is 0, then*

$$\Delta R_{ijkl} = Q(\mathrm{Rm}).$$

Proof.

$$\begin{aligned}\Delta R_{ijkl} = R_{ijkl,m}{}^{m} &= -R_{ijlm,k}{}^{m} - R_{ijmk,l}{}^{m} \\ &= -R_{ijlm}{}^{,m}{}_{k} - R_{ijmk}{}^{,m}{}_{l} + Q(\mathrm{Rm})\end{aligned}$$

By Bianchi identity and the vanishing of the Ricci curvature,

$$R_{ijlm}{}^{,m} = R_{lmij}{}^{,m} = -R_{lmj}{}^{m}{}_{,i} - R_{lm}{}^{m}{}_{i,j} = 0.$$

Similarly

$$R_{ijmk}{}^{,m} = 0.$$

So we get the conclusion. □

Theorem 3.19. *In the ALH case, there exists a constant μ such that the Riemannian curvature at p is bounded by $Ce^{-\mu r(p)}$.*

Proof. Pull back the Riemannian curvature tensor of g to $([R,\infty)\times\mathbb{T}^3, h)$, where h is the standard flat metric , we get a tensor $\mathbf{T}$ satisfying the equation $\mathbf{DT} = 0$, where $\mathbf{D} = \mathbf{A}^{ij}\nabla_i\nabla_j + \mathbf{B}^i\nabla_i + \mathbf{C}$ is a tensor-valued second order elliptic operator such that

$$|\mathbf{A}^{ij} - \delta^{ij}\mathbf{Id}| \le Cr^{-\epsilon}, |\mathbf{B}^i| \le Cr^{-1-\epsilon}, |\mathbf{C}| \le Cr^{-2-\epsilon}.$$

By Theorem 2.3,

$$|\mathbf{T}| = O(r^{-2-\epsilon}), |\nabla\mathbf{T}| = O(r^{-3-\epsilon}), |\nabla^{\mathbf{2}}\mathbf{T}| = O(r^{-4-\epsilon}),$$

so $\mathbf{T} \in H^2_\delta$ for all $\delta < 3 + 2\epsilon$. By Theorem 3.12 and the interior L^2 estimate (c.f. Theorem 9.11 of [31]), for any large enough R,

$$\int_{[R+2,\infty)\times\mathbb{T}^3} |\mathbf{T}|^2 \le \int_{[R,\infty)\times\mathbb{T}^3} (r-R)^\epsilon(1-\chi(r-R))^2|\mathbf{T}|^2$$

$$\leq C\int_{[R,\infty)\times\mathbb{T}^3}(r-R)^{\epsilon+4}|\mathbf{D}((1-\chi(r-R))\mathbf{T})|^2$$

$$\leq C||\mathbf{T}||^2_{H^2([R+1,R+2]\times\mathbb{T}^3)}\leq C\int_{[R,R+3]\times\mathbb{T}^3}|\mathbf{T}|^2.$$

So

$$\int_{[R,\infty)\times\mathbb{T}^3}|\mathbf{T}|^2\geq(1+1/C)\int_{[R+3,\infty)\times\mathbb{T}^3}|\mathbf{T}|^2.$$

In other words, the Riemannian curvature decays exponentially in L^2 sense. The improvement to L^∞ bound is simply Gilbarg and Trudinger's Theorem 9.20 in [31]. □

From this better control of curvature, the parallel transport along the loops $\gamma_{r,i}$ in Theorem 2.18 can be improved to $|P-\mathrm{Id}|<Ce^{-\mu r}$. Therefore, we are able to prove the following theorem:

Theorem 3.20. *For any ALH gravitational instanton (M,g), there exist a positive number μ, a compact subset $K\subset M$, and a diffeomorphism*

$$\Phi:[R,\infty)\times\mathbb{T}^3\to M-K$$

such that

$$|\nabla^m(\Phi^*g-h)|_h\leq C(m)e^{-\mu r}$$

for any $m=0,1,2,...,$ where $h=\mathrm{d}r^2\oplus h_1$ for some flat metric h_1 on $\mathbb{T}^3$.

3.4 Elliptic estimates with exponential growth weights

In this subsection, we are trying to prove the elliptic estimates for weighted Hilbert spaces with exponential growth weights.

We first look at the Laplacian operator on $\mathbb{T}^3=\mathbb{R}^3/\Lambda$. Define the dual lattice Λ^* by

$$\Lambda^*=\{\lambda\in\mathbb{R}^3|<\lambda,v>\in\mathbb{Z},\forall v\in\Lambda\}.$$

Then Δ has eigenvalues $-4\pi^2|\lambda|^2$ with eigenvectors $e^{2\pi i<\lambda,\theta>}$ for all $\lambda\in\Lambda^*$. We call δ critical if $\delta=4\pi|\lambda|$ for some $\lambda\in\Lambda^*$. So Theorem 3.5 is replaced by the following theorem on $[R,\infty)\times\mathbb{T}^3$.

Theorem 3.21. *Suppose f is a real smooth function on $[0,\infty)\times\mathbb{T}^3$ supported in $[R,R']\times\mathbb{T}^3$, δ isn't critical. Then*

$$\int_{[0,\infty)\times\mathbb{T}^3}|f|^2e^{\delta r}\mathrm{dVol}<C\int_{[0,\infty)\times\mathbb{T}^3}|\Delta f|^2e^{\delta r}\mathrm{dVol}.$$

Proof. We write f in terms its Fourier series

$$f\sim\sum_{\lambda\in\Lambda^*}f_\lambda(r)e^{2\pi i<\lambda,\theta>}.$$

Then

$$\begin{aligned}\Delta f&\sim\sum_{\lambda\in\Lambda^*}(f_\lambda''(r)-4\pi^2|\lambda|^2f_\lambda(r))e^{2\pi i<\lambda,\theta>}\\&=\sum_{\lambda\in\Lambda^*}(\frac{\mathrm{d}}{\mathrm{d}r}-2\pi|\lambda|)(\frac{\mathrm{d}}{\mathrm{d}r}+2\pi|\lambda|)f_\lambda(r)e^{2\pi i<\lambda,\theta>}.\end{aligned}$$

This time the Hardy's inequality is

$$\int_0^\infty g^2e^{\nu r}\mathrm{d}r\le\frac{4}{\nu^2}\int_0^\infty(g')^2e^{\nu r}\mathrm{d}r.$$

Therefore

$$\begin{aligned}\int_0^\infty f_\lambda^2e^{\delta r}\mathrm{d}r&=\int_0^\infty(e^{2\pi|\lambda|r}f_\lambda)^2e^{(\delta-4\pi|\lambda|)r}\mathrm{d}r\\&\le\frac{4}{(\delta-4\pi|\lambda|)^2}\int_0^\infty[(e^{2\pi|\lambda|r}f_\lambda)']^2e^{(\delta-4\pi|\lambda|)r}\mathrm{d}r\\&=\frac{4}{(\delta-4\pi|\lambda|)^2}\int_0^\infty[(\frac{\mathrm{d}}{\mathrm{d}r}+2\pi|\lambda|)f_\lambda]^2e^{\delta r}\mathrm{d}r\\&\le\frac{16\int_0^\infty[(\frac{\mathrm{d}}{\mathrm{d}r}-2\pi|\lambda|)(\frac{\mathrm{d}}{\mathrm{d}r}+2\pi|\lambda|)f_\lambda(r)]^2e^{\delta r}\mathrm{d}r}{(\delta+4\pi|\lambda|)^2(\delta-4\pi|\lambda|)^2}.\end{aligned}$$

So as long as δ isn't critical, we are done. □

Now we define $\underline{L}^2_\delta(M)$ by

$$||\phi||_{\underline{L}^2_\delta(M)}=\int_M|\phi|^2e^{\delta r}\mathrm{dVol},$$

and $\underline{H}^2_\delta(M)$ by

$$||\phi||_{\underline{H}^2_\delta(M)} = \sqrt{\int_M |\phi|^2 e^{\delta r}\mathrm{dVol} + \int_M |\nabla\phi|^2 e^{\delta r}\mathrm{dVol} + \int_M |\nabla^2\phi|^2 e^{\delta r}\mathrm{dVol}.}$$

Then Theorem 3.13 is replaced by

Theorem 3.22. *Suppose δ isn't critical and $\delta < 0$. For any $\phi \in \underline{L}^2_\delta(M)$, there exists a tensor $\psi \in \underline{H}^2_\delta(M)$ such that $\Delta\psi = \phi$.*

3.5 Deformation of hyperkähler 4-manifolds

It's well known that in real dimension 4, the hyperkähler condition is equivalent to the Calabi–Yau condition. So we can study the deformation theory by viewing them as Calabi–Yau manifolds. However, to keep track of the symmetry between three complex structures, we prefer a more direct approach inspired by the lecture of Sir Simon Donaldson in the spring of 2015 at Stony Brook University.

Lemma 3.23. *A 4-manifold is hyperkähler if and only if there exist three closed 2-forms ω^i satisfying*

$$\omega^i \wedge \omega^j = 2\delta_{ij} V,$$

where V is a nowhere vanishing 4-form.

Proof. Given three 2-forms, we can call the linear span of them the "self-dual" space. The orthogonal complement of the "self-dual" space under wedge product is called "anti-self-dual" space. These two spaces determine a star operator. It's well known that the star operator determine a conformal class of metric. We can then determine the conformal factor by requiring V to be the volume form. Using this metric and the three forms ω^i, we can determine three almost complex structures I, J and K. It's easy to see that $IJ = K$ or $IJ = -K$. Since the two cases are disconnected, we can without loss of generality assume that the first case happens. By Lemma 6.8 of [39], I,J,K are parallel. □

Therefore, given a family of hyperkähler metrics $\omega^i(t)$ on a fixed manifold M, the deformations $\theta^i = \frac{\mathrm{d}}{\mathrm{d}t}\omega^i(t)|_{t=0}$ satisfy

$$\omega^i \wedge \theta^j + \omega^j \wedge \theta^i = 0, i \neq j;$$
$$\omega^1 \wedge \theta^1 = \omega^2 \wedge \theta^2 = \omega^3 \wedge \theta^3.$$

Notice that the anti-self-dual components of θ^i don't affect the equation, so we only need to look at the self-dual components. Let

$$V = \{\theta \in \Lambda^+ \oplus \Lambda^+ \oplus \Lambda^+ : \omega^i \wedge \theta^j + \omega^j \wedge \theta^i = 0, i \neq j; \omega^1 \wedge \theta^1 = \omega^2 \wedge \theta^2 = \omega^3 \wedge \theta^3\}.$$

Then V is generated by the following basis:

$$\begin{aligned}
e_1 &: \theta^1 = -\omega^1, \theta^2 = -\omega^2, \theta^3 = -\omega^3;\\
e_2 &: \theta^1 = 0, \theta^2 = \omega^3, \theta^3 = -\omega^2;\\
e_3 &: \theta^1 = -\omega^3, \theta^2 = 0, \theta^3 = \omega^1;\\
e_4 &: \theta^1 = \omega^2, \theta^2 = -\omega^1, \theta^3 = 0.
\end{aligned}$$

Now we look at the action of diffeomorphism group. The infinitesimal diffeomorphism group X acts simply by $\theta^i = L_X\omega^i = \mathrm{d}(X\rfloor\omega^i)$. The projection of $L_X\omega^i$ to V defines an operator $\mathcal{D} : \mathrm{Vect}(M) \to V$. Notice that $\mathcal{D}$ is canonically determined by ω^i. In particular, if there is a symmetry group G, then $\mathcal{D}$ is also invariant under G.

On $\mathbb{R}^4$,

$$\begin{aligned}
\omega^1 &= \mathrm{d}x^1 \wedge \mathrm{d}x^2 + \mathrm{d}x^3 \wedge \mathrm{d}x^4,\\
\omega^2 &= \mathrm{d}x^1 \wedge \mathrm{d}x^3 + \mathrm{d}x^4 \wedge \mathrm{d}x^2,\\
\omega^3 &= \mathrm{d}x^1 \wedge \mathrm{d}x^4 + \mathrm{d}x^2 \wedge \mathrm{d}x^3.
\end{aligned}$$

It's easy to compute that

$$\begin{aligned}
&\mathcal{D}(f^1\frac{\partial}{\partial x^1} + f^2\frac{\partial}{\partial x^2} + f^3\frac{\partial}{\partial x^3} + f^4\frac{\partial}{\partial x^4})\\
&\quad = -(\frac{\partial f^1}{\partial x^1} + \frac{\partial f^2}{\partial x^2} + \frac{\partial f^3}{\partial x^3} + \frac{\partial f^4}{\partial x^4})e_1 + (\frac{\partial f^1}{\partial x^2} - \frac{\partial f^2}{\partial x^1} + \frac{\partial f^3}{\partial x^4} - \frac{\partial f^4}{\partial x^3})e_2\\
&\qquad + (\frac{\partial f^1}{\partial x^3} - \frac{\partial f^2}{\partial x^4} - \frac{\partial f^3}{\partial x^1} + \frac{\partial f^4}{\partial x^2})e_3 + (\frac{\partial f^1}{\partial x^4} + \frac{\partial f^2}{\partial x^3} - \frac{\partial f^3}{\partial x^2} - \frac{\partial f^4}{\partial x^1})e_4.
\end{aligned}$$

It looks like the Dirac operator on $\mathbb{R}^4$ [50].

In fact, define

$$\cdot : V \times TM \to TM$$

by

$$(a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4) \cdot v = (a_1 + a_2I + a_3J + a_4K)v.$$

Define the dual operator

$$\cdot : TM \times TM \to V$$

by

$$(w \cdot v, c)_V = -(w, c \cdot v),$$

where $(e_i, e_j)_V = \delta_{ij}$ on V. Then

$$\mathcal{D}X = \sum_{i,j=1}^{4} g^{ij} \nabla_{\frac{\partial}{\partial x_j}} X \cdot \frac{\partial}{\partial x_i}$$

on $\mathbb{R}^4$. It is also true on M. Its formal adjoint

$$\mathcal{D}^* Y = \sum_{i,j=1}^{4} g^{ij} \nabla_{\frac{\partial}{\partial x_j}} Y \cdot \frac{\partial}{\partial x_i}.$$

So $\mathcal{D}\mathcal{D}^* = -\Delta$ on M.

On the general hyperkähler 4-manifold, if $\mathcal{D}$ has full image, then we can without loss of generally assume that θ^i are all anti-self-dual. Notice that they must be closed as the variation of closed forms. They must also be co-closed since $\mathrm{d}(*\theta^i) = -\mathrm{d}\theta^i = 0$. In other words, the deformation space of hyperkähler 4-manifolds is a subspace of three copies of the space of anti-self-dual harmonic 2-forms. There may be further reductions if $(L_X\omega^1, L_X\omega^2, L_X\omega^3)$ is anti-self-dual harmonic for some vector field X.

3.6 Asymptotic behavior of gravitational instantons

In this section, we use the principles in the previous section to prove Theorem 1.3.

To prove that $\mathcal{D}$ has full image, we instead prove that $-\mathcal{D}\mathcal{D}^*$ has full image. It's enough to apply Lemma 4 of Minerbe's paper [57] in ALF case and its generalization in ALG case.

We are ready to prove Theorem 1.3.

Theorem 3.24. *Any ALF-D_k gravitational instanton (M, g) is ALF of order 3 in the sense of Definition 2.2*

Proof. We already proved that M is asymptotic to (E, h) with error $O'(r^{-\epsilon})$. We will improve the decay rate slightly and iterate the improvement. The decay rates are in L^∞ sense. However, they are also in weighted L^2 sense after choosing correct weights. To be convenient, we transfer the weighted L^2 estimates back into L^∞ estimates using standard elliptic theory. During this process, the weights are usually slightly changed. Therefore, we will choose irrational $\delta_1 < \epsilon$ arbitrarily close to ϵ and irrational $\delta_2 < \delta_1$ arbitrary close to δ_1.

Let ω_g be $(\omega_g^1, \omega_g^2, \omega_g^3)$ and ω_h accordingly. Then

$$\omega_g - \omega_h = O'(r^{-\epsilon}).$$

The difference is small. So we can write it as infinitesimal difference plus some quadratic term. In other words, if we use h to distinguish self-dual and anti-self-dual forms, then the self dual part

$$\omega_g^+ - \omega_h = \theta + O'(r^{-2\epsilon}),$$

where $\theta = O'(r^{-\epsilon}) \in V$.

The operator $-\mathcal{D}\mathcal{D}^*$ satisfies all the conditions of Theorem 3.17. So there exists $G_{-\mathcal{D}\mathcal{D}^*}$ such that

$$\theta = -\mathcal{D}\mathcal{D}^* G_{-\mathcal{D}\mathcal{D}^*}\theta.$$

Let $X = \mathcal{D}^* G_{-\mathcal{D}\mathcal{D}^*}\theta$, then $X = O'(r^{1-\delta_1})$. Let $\Phi_t = \exp(-tX)$ be the 1-parameter subgroup of diffeomorphisms generated by X. Then

$$\Phi_t^*(L_X\omega_h) - L_X\omega_h = O'(r^{-2\delta_1}), \forall t \in [0, 1].$$

Therefore,

$$\Phi_1^*\omega_h - \omega_h - L_X\omega_h = \int_{t=0}^1 (\Phi_t^*(L_X\omega_h) - L_X\omega_h)\mathrm{d}t = O'(r^{-2\delta_1}).$$

So

$$(\Phi_1^*\omega_g)^+ - \omega_g^+ - \mathcal{D}X = (\Phi_1^*\omega_g - \omega_g - L_X\omega_h)^+ = O'(r^{-2\delta_1})$$

because $\omega_g - \omega_h = O'(r^{-\epsilon})$. After replacing ω_g by $\Phi_1^*\omega_g$, we can assume that

$$\omega_g^+ - \omega_h = O'(r^{-2\delta_1}).$$

We also have $\omega_g^- = O'(r^{-\epsilon})$. Write it as $\omega_g^- = \phi + \psi$ with ϕ $\mathbb{T}^k$-invariant and $\int_{\pi^{-1}(x)} \psi = 0$. Since $-\mathrm{d} * \omega_g^- = \mathrm{d}\omega_g^- = -\mathrm{d}\omega_g^+ = O'(r^{-2\delta_1 - 1})$,

$$(\mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d})\omega_g^- = O'(r^{-2\delta_1 - 2}).$$

In particular,

$$\tilde{\psi} = \psi + G_{-(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})}(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\psi$$

is harmonic and $\psi - \tilde{\psi} = O'(r^{-2\delta_2})$. By Theorem 3.16, $\tilde{\psi}$ decay exponentially. Therefore, $\psi = O'(r^{-2\delta_2})$.

Now, we write ϕ as $\phi = \alpha \wedge \eta - V *_{\mathbb{R}^3} \alpha$ for $\alpha \in \Lambda^1(\mathbb{R}^3)$. Then

$$\mathrm{d}\phi = \mathrm{d}\alpha \wedge \eta - \alpha \wedge \mathrm{d}\eta - \mathrm{d}V \wedge *_{\mathbb{R}^3}\alpha - V\mathrm{d}(*_{\mathbb{R}^3}\alpha) = O'(r^{-2\delta_1 - 1})$$

Let $\delta_3 = \min\{2\delta_2, \delta_2 + 1\}$. Then $\mathrm{d}\alpha = O'(r^{-\delta_3 - 1})$ and $\mathrm{d}(*_{\mathbb{R}^3}\alpha) = O'(r^{-\delta_3 - 1})$. Therefore $\tilde{\alpha} = \alpha + G_{-(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})}(\mathrm{dd}^* + \mathrm{d}^*\mathrm{d})\alpha$ is a harmonic 1-form on $\mathbb{R}^3$. What's more $\alpha - \tilde{\alpha} = O'(r^{-\delta_4})$ for all irrational $\delta_4 < \delta_3$.

Using the method in the proof of Theorem 3.6, we know that $\tilde{\alpha} = O'(r^{-1})$.

Combining everything together, the decay rate of $\omega_g - \omega_h$ can be improved to $\min\{\delta_4, 1\}$ when we start from ϵ, where the irrational number δ_4 can be arbitrarily close to $\min\{2\epsilon, \epsilon + 1\}$. After finite times of iterations, the decay rate of $\omega_g - \omega_h$ can be improved to 1. Moreover, the decay rate of $\mathrm{d}\tilde{\alpha}$ can be arbitrarily close to 3. Notice that the coefficients of $\mathrm{d}x_i$ in $\tilde{\alpha}$ is even, so

$$\tilde{\alpha} = \frac{a\mathrm{d}x^1 + b\mathrm{d}x^2 + c\mathrm{d}x^3}{r} + O'(r^{-3})$$

for some constants a, b and c. It's easy to deduce that $a = b = c = 0$ from the decay rate of $\mathrm{d}\tilde{\alpha}$. So $\tilde{\alpha} = O'(r^{-3})$ instead. More iterations yield that the asymptotic rate, i.e the decay rate of $\omega_g - \omega_h$ can be improved to 3. $\square$

Remark. It's known [74] that up to some exponentially decay term, the Cherkis–Hitchin–Ivanov–Kapustin–Lindström–Roček metric outside a compact set can be written as the $\mathbb{Z}_2$-quotient of a Gibbons–Hawking ansatz whose V can be written as

$$V = 1 - \frac{16m}{|\mathbf{x}|} + \sum_{\alpha=1}^{k}(\frac{4m}{|\mathbf{x} - \mathbf{x}_\alpha|} + \frac{4m}{|\mathbf{x} + \mathbf{x}_\alpha|}) = 1 + \frac{8m(k-2)}{r} + O'(r^{-3}).$$

Therefore, our estimate is optimal. In ALF-A_k case, the coefficients of $\mathrm{d}x_i$ in $\tilde{\alpha}$ are not necessarily even. So the asymptotic rate is only 2 here. Later, we will use this estimate to give a new proof of Theorem 1.8. Notice that the asymptotic rate of the multi-Taub-NUT metric is actually 3.

Theorem 3.25. *Any ALG gravitational instanton (M, g) is ALG of order $\min_{n \in \mathbb{Z}, n < 2\beta} \frac{2\beta - n}{\beta}$ in the sense of Definition 2.2.*

Proof. The proof of Theorem 3.24 go through until the analysis of the $\mathbb{T}^2$-invariant closed anti-self-dual form ϕ. Following the notations of our first paper, the basis of anti-self-dual forms can be written as

$$\xi^1 = \mathrm{d}u \wedge \mathrm{d}\bar{v}, \xi^2 = \mathrm{d}\bar{u} \wedge \mathrm{d}v, \xi^3 = \mathrm{d}u \wedge \mathrm{d}\bar{u} - \mathrm{d}v \wedge \mathrm{d}\bar{v}$$

When (u, v) become $(e^{2\pi i\beta}u, e^{-2\pi i\beta}v)$, the corresponding forms (ξ^1, ξ^2, ξ^3) become $(e^{4\pi i\beta}\xi^1, e^{-4\pi i\beta}\xi^2, \xi^3)$. Notice that ϕ can be decomposed into combinations of $u^{-\delta}\xi^1$, $\bar{u}^{-\delta}\xi^1$, $u^{-\delta}\xi^3$ and their conjugates. Only the first one and its conjugate are closed. To make $u^{-\delta}\xi^1$ well defined, $-2\pi\beta\delta + 4\pi\beta$ must be in $2\pi\mathbb{Z}$. Therefore, δ must be $\min_{n\in\mathbb{Z},n<2\beta} \frac{2\beta-n}{\beta}$. □

Remark. In Theorem 1.5 of [37], Hein constructed lots of ALG gravitational instantons of order $\min_{n\in\mathbb{Z},n<2\beta} \frac{2\beta-n}{\beta}$ whose tangent cone at infinity has cone angle $2\pi\beta < 2\pi$. Therefore, our estimate of asymptotic rate is optimal in ALG case if $\beta < 1$.

It's not hard to extend our method to ALH gravitational instanton using exponential growth weights and therefore complete the proof of Theorem 1.3. We will omit the details.

3.7 Holomorphic functions on ALF and ALG gravitational instantons

In this subsection, we will prove the existence of global holomorphic functions on both ALF and ALG instatons.

Our first theorem deal with the growth order of harmonic functions on M.

Theorem 3.26. *Suppose M is ALF or ALG, g is a Ricci-flat metric on M. Given any harmonic function $f \in L^2_\delta(M)$ for some δ, there exist an γ such that f is $O(r^\gamma)$ but not $o(r^\gamma)$. What's more, in the ALG case, when $C(S(\infty)) = \mathbb{C}_\beta$, $\beta\gamma$ must be an integer. In the ALF-A_k case, γ must be an integer. In the ALF-D_k case, γ must be an even number.*

Proof. f also belongs to $L^2_{\delta'}(M)$ for other δ'. Without of loss of generality, assume δ is bigger than the superior of those δ' minus ϵ. The superior exists because if $f \in L^2_{\delta'}(M)$ for some δ' satisfying $\frac{-\delta'}{2} + \frac{k}{2} - 2 < 1$, then f must be a constant by Lemma 3.4, Theorem 3.3 and Cheng-Yau's gradient estimate [20]. By Lemma 3.4, $f \in H^2_\delta(M)$. Cut off f so that it vanish inside a large ball B_R. Move this function to E. Then $\Delta(f(1-\chi(r/R)) \in L^2_{\delta+4+\epsilon}$. Decompose $f(1-\chi(r/R))$ into $\mathbb{T}^k$-invariant part f_0 and the perpendicular part f_1.

Then f_1 is much smaller than the growth rate of $f(1-\chi)$. Without loss of generality, we can assume that $f(1-\chi(r/R))$ is invariant.

Now again, we can transfer this invariant function to the tangent cone at infinity $C(S(\infty))$. When $C(S(\infty)) = \mathbb{R}^3/\mathbb{Z}_2$ (ALF-D_k), we get a function $\tilde{f}$ on its double cover $\mathbb{R}^3$ naturally. When $C(S(\infty)) = \mathbb{C}_\beta$ (ALG), we get a function $\tilde{f}(z)$ on $\mathbb{C} = \mathbb{R}^2$ defined by $\tilde{f} = (f(1-\chi(r/R)))(z^\beta)$. Again the growth rate of $\Delta(\tilde{f})$ is at most the growth rate of $\tilde{f}$ minus two then minus ϵ, so we can find out a function ψ with growth rate the rate of $\tilde{f}$ minus ϵ such that $\Delta\psi = \Delta(\tilde{f})$. So $\tilde{f} - \psi$ becomes a harmonic function on $\mathbb{R}^3$ or $\mathbb{R}^2$. The gradient estimate implies that after taking derivatives for some times, we get 0. In other word $\tilde{f} - \psi$ must be a polynomial. So the growth rate must be integer. For the $C(S(\infty)) = \mathbb{R}^3/\mathbb{Z}_2$(ALF-$D_k$) case, we may replace $\psi(x)$ by $(\psi(x) + \psi(-x))/2$ so that it's invariant under the $\mathbb{Z}_2$ action. So the polynomial must have even degree. □

Now we can prove the existence of global holomorphic function on ALG gravitational instantons.

Theorem 3.27. *There exists a global holomorphic function on any ALG gravitational instanton M such that any far enough fiber is biholomorphic to a complex torus.*

Proof. In this case $k = 2$. By theorem 2.22, the metric near infinity is asymptotic to the elliptic surface (E, h). For (E, h), $u^{1/\beta}$ is a well defined holomorphic function outside B_R. Now if we pull back $u^{1/\beta}$ from the elliptic surface, cut it off and fill in with 0 inside K, we obtain a function f satisfying

$$\bar{\partial}_g f = \phi = O(r^{1/\beta - 1 - \epsilon}).$$

Pick any small positive number $\delta \in (\max\{-2, 2/\beta - 2\epsilon\}, 2/\beta - \epsilon)$, such that 30δ is not an integer. Thus, $\phi \in L^2_{-\delta}(M)$. By Theorem 3.13, there exists $\psi \in H^2_{-\delta-4}(M)$ such that

$$\phi = \Delta\psi = -(\bar{\partial}^*\bar{\partial} + \bar{\partial}\bar{\partial}^*)\psi$$

in the distribution sense. Elliptic regularity implies that ψ is a smooth $(0,1)$-form. Take $\bar{\partial}$ on both side of this equation. Notice that $\bar{\partial}\phi = 0$. Thus

$$0 = -\bar{\partial}\bar{\partial}^*(\bar{\partial}\psi) = \Delta(\bar{\partial}\psi).$$

By Lemma 3.3, $\bar{\partial}\psi = O(r^{1/\beta-\epsilon/2})$. We can write this (0,2) form as $\xi\omega^+$, where ω^+ is the parallel (0,2)-form. Then ξ is a harmonic function. By Theorem 3.26, it's constant. Therefore $\bar{\partial}(f + \bar{\partial}^*\psi) = 0$. So $f + \bar{\partial}^*\psi$ is a global holomorphic function. After analyzing the growth rate, we can also show that $|\mathrm{d}\bar{\partial}^*\psi| << |\mathrm{d}f|$ for large r. So the fiber far from origin is an compact Riemann surface with genus 1. It must be a complex torus by the uniformization theorem. □

Similarly, we can prove

Theorem 3.28. *There exists a global holomorphic function on any ALF-D_k gravitational instanton M.*

Proof. M is asymptotic to a fiberation over $\mathbb{R}^3/\mathbb{Z}_2 = \mathbb{R}^3/\mathbf{x} \sim -\mathbf{x}$. The function $(x_2 + ix_3)^2$ is well defined over E. The proof of the last theorem will produce a global holomorphic function in ALF-D_k case. □

The existence of global holomorphic function on any ALF-A_k gravitational instanton M can also be proved by the same way. Actually, Minerbe had a simpler proof in [59]. It's an essential step in his classification of ALF-A_k instantons.

3.8 Holomorphic functions on ALH gravitational instantons

To go through all the steps in ALF and ALG cases, we first need to control the growth rate of harmonic functions:

Lemma 3.29. *Suppose (N, h) is a smooth manifold such that outside a compact set, it's exactly $[R, \infty) \times \mathbb{T}^3$ with flat metric. Then any smooth function u on N harmonic outside a large enough ball with at most exponential growth rate can be written as linear combinations of 1, r, $e^{2\pi|\lambda|r}e^{2\pi i<\lambda,\theta>}$ and an exponential decay function , where r and θ are the coordinate functions on $[R, \infty) \times \mathbb{T}^3$ pulled back by the diffeomorphism.*

Proof. Write u as its Fourier series $\sum_{\lambda\in\Lambda^*} u_\lambda(r)e^{2\pi i<\lambda,\theta>}$. Then the equation is reduced to $u''_\lambda = 4\pi^2|\lambda|^2 u_\lambda$. So

$$u \sim a_0 + b_0 r + \sum_{\lambda\in\Lambda^*-\{0\}} a_\lambda e^{2\pi|\lambda|r}e^{2\pi i<\lambda,\theta>} + \sum_{\lambda\in\Lambda^*-\{0\}} b_\lambda e^{-2\pi|\lambda|r}e^{2\pi i<\lambda,\theta>}.$$

By Parserval's identity, the growth condition of u implies that the first sum has finite terms. For the second sum U, Parseval's identity again implies that $\int_{[R,R+1]\times\mathbb{T}^3}|U|^2$ decay exponentially. By Theorem 9.20 of [31], U also decay exponentially in L^∞ sense. □

Now we can still find the global holomorphic function on ALH instanton (M,g)

Theorem 3.30. *In the ALH case, there exists a global holomorphic function on M such that any far enough fiber is biholomorphic to a complex torus.*

Proof. As before, let $[R,\infty)\times\mathbb{T}^3=\{(r,\theta)|r\geq R,\theta=(\theta_1,\theta_2,\theta_3)\in\mathbb{R}^3/\Lambda\}$. Let Λ^* be the dual lattice. Choose $\lambda\in\Lambda^*-\{0\}$ with minimal length. Choose $(a_1,a_2,a_3)\in\mathbb{S}^2$ such that

$$(a_1I^*+a_2J^*+a_3K^*)\mathrm{d}r=-\frac{\lambda_1\mathrm{d}\theta_1+\lambda_2\mathrm{d}\theta_2+\lambda_3\mathrm{d}\theta_3}{|\lambda|}.$$

Use $a_1I+a_2J+a_3K$ as the complex structure. Then the function $e^{2\pi|\lambda|r}e^{2\pi i<\lambda,\theta>}$ is holomorphic. The growth rate of this function is exactly $O(e^{2\pi|\lambda|r})$.

Now we pull back this function from $[R,\infty)\times\mathbb{T}^3$ to M, cut it off and fill in with 0 inside K, we obtain a function f satisfying

$$\bar\partial_g f=\phi=O(e^{(2\pi|\lambda|-\mu)r}),$$

where μ is the constant in Theorem 3.19. So $\phi\in\underline{L}^2_{-2\delta}$ for any non-critical positive number $\delta\in(2\pi|\lambda|-\mu,2\pi|\lambda|)$. By Theorem 3.22, there exists a solution $\psi\in\underline{H}^2_{-2\delta}$ to

$$\phi=\Delta\psi=-(\bar\partial^*\bar\partial+\bar\partial\bar\partial^*)\psi$$

in the distribution sense. Elliptic regularity implies that ψ is a smooth $(0,1)$-form. As before, $\bar\partial\psi=\xi\omega^+$ is a harmonic (0,2)-form. So ξ is a harmonic function of order $O(e^{\delta r})$.

Now we use a cut-off function and the diffeomorphism to average g and the pull back of h. We get a smooth metric g' on M which is identically the pull back of h outside a very big ball. Now let ν be the inferior of positive ν' such that ξ is bounded by $O(e^{\nu' r})$. If $\nu>0$, then $\Delta_{g'}\xi\in\underline{L}^2_{-2\delta'}$ for any positive $\nu>\delta'>\nu-\mu$. It follows that there exists a function in $\underline{L}^2_{-2\delta'}$ whose

Laplacian $\Delta_{g'}$ is $\Delta_{g'}\xi$. The difference of those two functions is a g'-harmonic function. By Lemma 3.29, it must have at most linear growth rate since the growth rate is below the first nonlinear harmonic function. It follows that ξ must lie in $O(e^{\delta' r})$, a contradiction. So $\nu = 0$. Therefore, ξ is bounded by any exponential growth function.

So $\Delta_{g'}\xi$ decay exponentially. In particular, it's in $L^2_{1-\epsilon}$. By Theorem 3.13, we can find out a function in $H^2_{-3-\epsilon}$ whose $\Delta_{g'}$ is $\Delta_{g'}\xi$. Therefore, we know that ξ is actually in $O(r^{1+\epsilon})$. Of course, $\bar{\partial}\psi = \xi\omega^+$ has the same estimate.

By Lemma 3.4, the harmonic (0,1)-form $\bar{\partial}^*\bar{\partial}\psi = \bar{\partial}(f + \bar{\partial}^*\psi)$ is in $O(r^\epsilon)$ and its covariant derivative is in $O(r^{-1+\epsilon})$. The Weitzenböck formula implies that $\nabla^*\nabla(\bar{\partial}^*\bar{\partial}\psi) = 0$. Therefore

$$\int_M |\nabla(\bar{\partial}^*\bar{\partial}\psi)|^2\chi \leq \int_M |\bar{\partial}^*\bar{\partial}\psi||\nabla(\bar{\partial}^*\bar{\partial}\psi)||\nabla\chi|$$

for any smooth compactly supported χ. Let $\chi = \chi(r - R)$, the right hand side converges to 0. Therefore $\bar{\partial}^*\bar{\partial}\psi$ is a covariant constant (0,1)-form. If this form is non-zero, it would be invariant under the holonomy group. However, elements in SU(2) have no fixed point except the identity matrix. So (M, g) must have trivial holonomy. Therefore, it's $\mathbb{R}^{4-k} \times \mathbb{T}^k$ with flat metric. It's a contradiction with our non-flat assumption. So actually $\bar{\partial}^*\bar{\partial}\psi$ is identically 0. $f + \bar{\partial}^*\psi$ is a global holomorphic function on M. □

3.9 Compactification of ALG and ALH gravitational instantons

In Theorem 3.27 and 3.30, we proved the existence of global holomorphic function u in ALG and ALH cases such that any far enough fiber is biholomorphic to a complex torus. Notice that $\mathrm{d}u$ is never zero on far enough fiber. Define a holomorphic vector field X by $\omega^+(Y, X) = \mathrm{d}u(Y)$. Then since $X(u) = \mathrm{d}u(X) = \omega^+(X, X) = 0$, X is well defined when it's restricted to each far enough fiber. On each fixed far enough fiber, there exists a unique holomorphic form ϕ such that $\phi(X) = 1$. Locally

$$\omega^+ = f(u, v)\mathrm{d}u \wedge \mathrm{d}v, X = f^{-1}(u, v)\frac{\partial}{\partial v}, \phi = f(u, v)\mathrm{d}v.$$

Notice that each far enough fiber is topologically a torus. So we can integrate the form ϕ to get a holomorphic function $v \in \mathbb{C}/\mathbb{Z}\tau_1(u) \oplus \mathbb{Z}\tau_2(u)$

up to a constant. We can fix this constant locally by choosing a holomorphic section of u as the base point. Therefore M is biholomorphic to $U \times \mathbb{C}/(u,v) \sim (u, v + m\tau_1(u) + n\tau_2(u))$, where $\tau_1(u)$ and $\tau_2(u)$ are locally defined holomorphic functions. Actually, they are the integral of ϕ in the basis of H_1 of each fiber. This gives a holomorphic torus fiberation locally.

Recall that there is a diffeomorphism from M minus a large compact set to the standard fiberation. Denote the inverse image of the zero section by s. s is again a section outside large compact set because $\mathrm{d}u$ differ with the standard one by a decaying error. Write $\bar{\partial} s$ as $e(u)\mathrm{d}\bar{u} \otimes X$, then e is a function defined on the inverse of the punctured disc with polynomial growth rate. So there is an at most polynomial growth function E on the inverse of punctured disc such that $\bar{\partial} E(u) = e(u)\mathrm{d}\bar{u}$. Now we apply the flow $-E(u)X$ to the section s to get a holomorphic section s_0 on the neighborhood of infinity. View s_0 as the zero section, we know that M minus a large compact set is biholomorphic to $(\mathbb{C} - B_R) \times \mathbb{C}/(u,v) \sim (u, v + m\tau_1(u) + n\tau_2(u))$ globally, where $\tau_1(u)$ and $\tau_2(u)$ are multi-valued holomorphic functions.

As proved in Kodaira's paper [45], there exists an (unique) elliptic fiberation B over $B_{R^{-1}}$ with a section such that B minus the central fiber D is biholomorphic to $(B_{R^{-1}} - \{0\}) \times \mathbb{C}/(\tilde{u}, v) \sim (\tilde{u}, v + m\tau_1(\tilde{u}^{-1}) + n\tau_2(\tilde{u}^{-1}))$. We can naturally identity points and get a compactification $\bar{M}$ of M. So $\bar{M}$ is a compact complex surface with a meromorphic function $u = \tilde{u}^{-1}$. Now since the subvariety of critical points $\{\mathrm{d}u = 0\}$ is a finite union of irreducible curves (On those irreducible curves, u is of course constant) and points, we know that except finite critical values in $\mathbb{CP}^1$, any fiber of u has no intersection with $\{\mathrm{d}u = 0\}$. Therefore, a generic fiber has genus 1 and must be an elliptic curve. In other words, $\bar{M}$ is a compact elliptic surface. In conclusion, we've proved the second main theorem.

3.10 The topology of ALG and ALH gravitational instantons

In this section, we will study the topology of ALG and ALH gravitational instantons.

Therefore, as mentioned in the introduction, we will assume that any gravitational instanton is non-flat.

Theorem 3.31. *The first betti number of any ALG or ALH gravitational instanton must be 0. Moreover, in the ALG case, D can't be regular.*

Proof. In the ALH case, Melrose's theory [56] works. In particular, the first cohomology group $H^1(M, \mathbb{R})$ is a subspace of the space of bounded harmonic 1-forms [56]. By Weitzenböck formula, any bounded $(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)$-harmonic 1-form ϕ is also $\nabla^*\nabla$-harmonic. By Melrose's theory, $\nabla\phi$ decays exponentially. After integration by parts,

$$\int_M |\nabla\phi|^2\chi(r-R) \leq \int_M |\nabla\phi||\nabla\chi(r-R)| \to 0,$$

as $R \to \infty$. Therefore, ϕ is a parallel 1-form. If it's nonzero, the holonomy group must be trivial since the action of $\mathrm{Sp}(1)$ is free on $\mathbb{R}^4 \setminus \{0\}$. It contradicts the non-flat assumption.

In the ALG case, (after hyperkähler rotation) if D is regular, i.e. if $\beta = 1$, the I-holomorphic function z on M is asymptotic to the function u on E. $\nabla \mathrm{d}u = 0$ on E, so when we go through the construction of z in Theorem 3.27, it's easy to see that $|\nabla \mathrm{d}z| = O(r^{-1-\epsilon})$ for any small enough ϵ. So

$$\int_M |\nabla \mathrm{d}z|^2\chi(\frac{r}{2R}-1) \leq \int_M |\nabla \mathrm{d}z||\nabla\chi(\frac{r}{2R}-1)| \to 0,$$

as $R \to \infty$. As before, it contradicts the non-flat assumption.

Therefore $\beta < 1$. Inspired by Lemma 6.11 of [56] ,we define

$$f(r) = (\frac{1}{r} + \frac{1}{4R}\chi(\frac{2R}{r} - \frac{3}{2}))^{-1}.$$

Then f is increasing. $f(r) = r$ when $r \leq R$ and $\lim_{r\to\infty} f(r) = 4R$. Let $u = re^{i\theta}$. The map $F(re^{i\theta}, v) = (f(r)e^{i\theta}, v)$ on M is homotopic to the identity. Therefore, any smooth closed 1-form ϕ is cohomologous to $F^*\phi$. It's easy to see that $F^*\phi = O(r)$.

By Theorem 3.13, for any small positive ϵ, there exists a smooth 1-form ψ such that

$$\int_M |\psi|^2 r^{-8-\epsilon} + |\nabla\psi|^2 r^{-6-\epsilon} + |\nabla^2\psi|^2 r^{-4-\epsilon} < \infty$$

and

$$F^*\phi = \mathrm{d}\mathrm{d}^*\psi + \mathrm{d}^*\mathrm{d}\psi.$$

Since $F^*\phi$ is closed, it's easy to see that $\mathrm{d}^*\mathrm{d}\psi$ is a closed harmonic 1-form.

Similar to Theorem 3.25, the leading term of $\mathrm{d}^*\mathrm{d}\psi$ can be written as $au^\delta \mathrm{d}u + b\bar{u}^\delta \mathrm{d}\bar{u}$ for some $\delta \leq 1$. To make it well-defined, $(\delta+1)\beta$ must be an integer. The first available choice is $\delta = 1/\beta - 1$ if $\beta \geq 1/2$ or $\delta = -1$ if $\beta < 1/2$. In the first case, $\mathrm{d}^*\mathrm{d}\psi - a\beta\mathrm{d}z - b\beta\mathrm{d}\bar{z}$ is a much smaller closed harmonic 1-form on M. Its order is also at most r^{-1}. However, by maximal principle and the Ricci flatness, any decaying harmonic 1-form on M must be 0. In conclusion, ϕ must be exact. In other words, $H^1(M) = 0$ for any ALG gravitational instanton. □

Definition 3.32. Let F, G be two linearly independent cubic homogenous polynomials on $\mathbb{CP}^2$. $\{F = 0\}$ and $\{G = 0\}$ intersect at 9 points with multiplicity. Let $\bar{M}$ be the blow up of $\mathbb{CP}^2$ on these 9 points, if needed repeatedly. Then $z = F/G$ is a well-defined meromorphic function on $\bar{M}$ whose generic fiber has genus 1. $(\bar{M}, z)$ is called the rational elliptic surface. It's well known that it has a global section σ corresponding to any exceptional curve in the blowing up construction.

Theorem 3.33. *$\bar{M}$ is a rational elliptic surface.*

Proof. Choose a tubular neighborhood T of D. Then $\bar{M} = M \cup T$. The Mayer-Vietoris sequence is

$$H^0(M) \oplus H^0(T) \to H^0(M \cap T) \to H^1(\bar{M}) \to H^1(M) \oplus H^1(T).$$

The first map is surjective, so the second map is 0. So the third map is injective. Notice that $H^1(T) = H^1(D) = 0$ because D is of type I_0^*, II, II^*, III, III^*, IV, or IV^*. $H^1(M)$ also vanishes by Theorem 3.31. So $H^1(\bar{M}) = 0$.

A careful examination of our construction of $\bar{M}$ in Section 3.9 and Kodaira's paper [45] yields that ω^+ can be extended to a meromorphic 2-form on $\bar{M}$ with a pole D. In other words, D is the anti-canonical divisor of $\bar{M}$. Since D is homologous to another fiber of z, the self intersection number of D is 0. In other words $c_1^2(\bar{M}) = c_1^2(-K) = [D]^2 = 0$. It's also very easy to see that $H^0(\bar{M}, mK) = 0$ for any $m > 0$. In particular the geometric genus $p_g = \dim H^0(\bar{M}, K) = 0$.

By Kodaira's classification of complex surfaces [46], since the first betti number of $\bar{M}$ is even and $p_g = 0$, $\bar{M}$ must be algebraic. By Castelnuovo theorem, $\bar{M}$ must be rational because $H^{0,1}(\bar{M}) = H^0(\bar{M}, 2K) = 0$. By Kodaira's Equation 13 in [46], $c_1^2 + \dim H^{0,1} + b^- = 10p_g + 9$. So $b^- = 9$. By Theorem 3 of [46], $b^+ = 1 + 2p_g = 1$. Therefore the second betti number b_2 of $\bar{M}$ equals to 10. It's standard [60] to prove that $(\bar{M}, z)$ is a rational elliptic surface defined in Definition 3.32. □

Theorem 3.34. *Given any ALH gravitational instantons M, there exists a diffeomorphism from the minimal resolution of* $(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2$ *to M whose restriction on $[R, \infty) \times \mathbb{T}^3$ is Φ in Theorem 1.3.*

Proof. The divisor D is smooth by our construction of the compactification. So for any small enough deformation in the coefficients of F and G, the diffeomorphism type of $M = \bar{M} \setminus D$ is invariant. For generic choice of coefficients of F and G, $\{G = 0\}$ is smooth and $\{F = 0\}$ intersects $\{G = 0\}$ in distinct points. Since the non-generic parameters have real codimension 2, generic points can be connected by paths inside the set of generic points. Therefore, it's easy to see that any ALH gravitational instantons are diffeomorphic to each other. In particular, they are diffeomorphic to the specific example of Biquard and Minerbe [8] on the minimal resolution of $(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2$. □

3.11 Twistor space of ALF-D_k instantons

On ALF-D_k gravitational instantons, we've found quadratic growth holomorphic functions for each compatible complex structure. A natural question is, is there any relationship between those functions? Before going ahead, let's recall the definition of twistor space of hyperkähler manifolds.

Definition 3.35. (c.f. [40]) Let (M, g, I, J, K) be a hyperkähler manifold. Then the twistor space Z of M is the product manifold $M \times \mathbb{S}^2$ equipped with an integrable complex structure

$$\underline{I} = (\frac{1 - \zeta\bar{\zeta}}{1 + \zeta\bar{\zeta}} I - \frac{\zeta + \bar{\zeta}}{1 + \zeta\bar{\zeta}} J + i \frac{\zeta - \bar{\zeta}}{1 + \zeta\bar{\zeta}} K, I_0),$$

where $\zeta \in \mathbb{C} \subset \mathbb{C} \cup \{\infty\} = \mathbb{CP}^1 = \mathbb{S}^2$ is the coordinate function, and I_0 is the standard complex structure on $\mathbb{CP}^1$.

Notice that our definition is different from [40] to correct a sign error. We will briefly rewrite Page 554-557 of their paper with a correct sign.

Let ϕ be a (1,0)-form of I. Then $I^*\phi = i\phi$, where $(I^*\phi)(X) = \phi(IX)$. Set $\theta = \phi + \zeta K^*\phi$, then

$$(1 + \zeta\bar{\zeta})\underline{I}^*\theta = ((1 - \zeta\bar{\zeta})I^* - (\zeta + \bar{\zeta})J^* + i(\zeta - \bar{\zeta})K^*)\theta = i(1 + \zeta\bar{\zeta})\theta,$$

because we have relationships like $J^*I^* = K^*$. (In [40], their was a sign error caused by the wrong statement $I^*J^* = K^*$.)

Now if the form $\omega^+ = \omega_2 + i\omega_3$ can be written as

$$\frac{1}{2}\omega^+ = \sum_{i=1}^{n} \phi_i \wedge \phi_{n+i}$$

for some (1,0)-forms of I. Then we can define a form on the twistor space by

$$\omega = 2\sum_{i=1}^{n}(\phi_i + \zeta K^*\phi_i) \wedge (\phi_{n+i} + \zeta K^*\phi_{n+i}) = (\omega_2 + i\omega_3) + 2\zeta\omega_1 - \zeta^2(\omega_2 - i\omega_3).$$

It's a holomorphic section of the vector bundle $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$, where F means the fiber of Z which is diffeomorphic to M. We also have a real structure $\tau(p,\zeta) = (p, -1/\bar{\zeta})$. It takes the complex structure $\underline{I}$ to its conjugate $-\underline{I}$. In [40], they proved the following theorem:

Theorem 3.36. *Let Z^{2n+1} be a complex manifold such that*

(i) Z is a holomorphic fiber bundle $\pi : Z \to \mathbb{CP}^1$ over the projective line;

(ii) The bundle admits a family of holomorphic sections each with normal bundle isomorphic to $\mathbb{C}^{2n} \otimes \mathcal{O}(1)$;

(iii) There exists a holomorphic section ω of $\Lambda^2 T_F^* \otimes \mathcal{O}(2)$ defining a symplectic form on each fiber;

(iv) Z has a real structure compatible with (i),(ii),(iii) and inducing the antipodal map on $\mathbb{CP}^1$.

Then the parameter space of real sections is a $4n$-dimensional manifold with a natural hyperkähler metric for which Z is the twistor space.

Return to the gravitational instantons for which n in the above theorems equals to 1. Recall that we've found the holomorphic function on M by modifying the pull back of the standard function on the standard model. So let's look at the standard model (E, h, I, J, K) first. It's the quotient of the Taub-NUT metric outside a compact set by $D_{4|e|}$. Recall that the Taub-NUT metric is (c.f. Theorem 2.22)

$$\mathrm{d}s^2 = V\mathrm{d}\mathbf{x}^2 + V^{-1}\eta^2$$

with

$$\mathrm{d}x_1 = I^*(V^{-1}\eta) = J^*\mathrm{d}x_2 = K^*\mathrm{d}x_3.$$

So

$$\left(\frac{1-\zeta\bar{\zeta}}{1+\zeta\bar{\zeta}}I^* - \frac{\zeta+\bar{\zeta}}{1+\zeta\bar{\zeta}}J^* + i\frac{\zeta-\bar{\zeta}}{1+\zeta\bar{\zeta}}K^* - i\right)(2\zeta\mathrm{d}x_1 - i(1+\zeta^2)\mathrm{d}x_2 + (1-\zeta^2)\mathrm{d}x_3) = 0.$$

Therefore, $(-x_3 + ix_2 - 2x_1\zeta - (-x_3 - ix_2)\zeta^2)^2$ is a holomorphic function on the twistor space of E. So the holomorphic function on $M \times \{\zeta\} \in Z$ is asymptotic to $(-x_3 + ix_2 - 2x_1\zeta - (-x_3 - ix_2)\zeta^2)^2$ with error $O'(r^{2-\epsilon})$

Notice that any harmonic function has even integer growth rate, so the holomorphic function is unique up to the adding of constant. We may fix this ambiguity by requiring the value at the fixed base point o to be 0. We will prove that after the modification the holomorphic functions have some simple relationship.

Actually, we have a I-holomorphic ($\zeta = 0$) function $u_1 + iv_1$ asymptotic to $(-x_3 + ix_2)^2 = (x_3^2 - x_2^2) - 2ix_2x_3$, J-holomorphic ($\zeta = -1$) function u_2+iv_2 asymptotic to $(2x_1+2ix_2)^2 = 4(x_1^2-x_2^2)+8ix_1x_2$, and K-holomorphic ($\zeta = -i$) function u_3+iv_3 asymptotic to $(-2x_3+2ix_1)^2 = 4(x_3^2-x_1^2)-8ix_3x_1$. Notice that $u_2 + u_3 - 4u_1$ is a harmonic function asymptotic to 0, i.e. in $O'(r^{2-\epsilon})$, so it must be 0. Similarly the harmonic function

$$z(p, \zeta) = (u_1+iv_1) - \frac{1}{2}(v_3+iv_2)\zeta + \frac{1}{2}(u_2-u_3)\zeta^2 + \frac{1}{2}(v_3-iv_2)\zeta^3 + (u_1-iv_1)\zeta^4$$

is asymptotic to $(-x_3 + ix_2 - 2x_1\zeta - (-x_3 - ix_2)\zeta^2)^2$ and therefore must be the holomorphic one. In conclusion, we've proved the following theorem:

Theorem 3.37. *In the ALF-D_k case, there exist 6 harmonic functions u_i,v_i with $4u_1 = u_2 + u_3$ such that*

$$z(p, \zeta) = (u_1+iv_1) - \frac{1}{2}(v_3+iv_2)\zeta + \frac{1}{2}(u_2-u_3)\zeta^2 + \frac{1}{2}(v_3-iv_2)\zeta^3 + (u_1-iv_1)\zeta^4$$

is a $\underline{I}$-holomorphic map from the twistor space of M to the total space of the $\mathcal{O}(4)$ bundle over $\mathbb{CP}^1$.

There is a real structure on the $\mathcal{O}(4)$ bundle $(\zeta, \eta) \to (-1/\bar{\zeta}, \bar{\eta}/\bar{\zeta}^4)$. It's easy to see that the map z commutes with the real structure.

4 ALE gravitational instantons

It's well known that there is a deep relationship between discrete subgroups of SU(2), ADE Dynkin diagrams, platonic solids, ADE singularities and the ALE gravitational instantons. The main contributors of this well-known relationship are Arnold, Cartan, Coxeter, du Val, Dynkin, Killing, Klein, Kodaira, Kronheimer, Lie, McKay, Milnor, Plato, Weyl and other

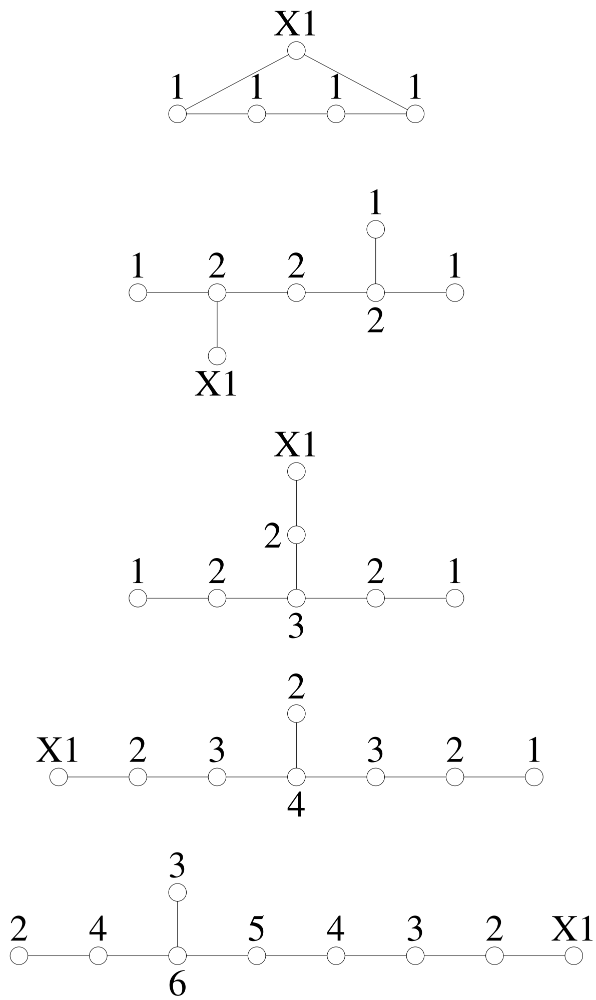

Figure 1: Extended A_4, D_6, E_6, E_7 and E_8 diagrams.

mathematicians. Since many original references were not written in English, we will not list them here. Moreover, many statements in this section will be copied from related textbooks or websites. The Figure 1 is provided by Alexey Spiridonov's homepage. Since none of the materials in this section is new, we will omit the proof and only state the well-known statements.

(1) Extended ADE Dynkin diagram.

By definition, it is a connected diagram with integers associated to each vertex such that 1 is associated to a marked vertex and two times the number associated to each vertex equals to the sum of numbers associated to adjacent vertices. Notice that except the extended-A_1 diagram with two vertices and two edges between them and 1 associated to them, all other diagrams have at most one edge between two vertices.

They can be classified into extended $A_k(k = 1, 2, ...)$, $D_k(k = 4, 5...)$, $E_k(k = 6, 7, 8)$ diagrams. $k + 1$ is the number of vertices.

(2) ADE Dynkin diagram

If we erase the marked point and the numbers in an extended ADE Dynkin diagram, then we get an ADE Dynkin diagram.

(3) ADE root system

Let V be a finite-dimensional Euclidean vector space, with the standard Euclidean inner product denoted by $(\cdot, \cdot)$. A root system in V is a finite set Φ of non-zero vectors (called roots) that satisfies the following conditions:

- The roots span V.
- The only scalar multiples of a root $x \in \Phi$ that belong to Φ are x itself and $-x$.
- For any two roots x and y, the element $\sigma_x(y) = y - 2\frac{(x,y)}{(x,x)}x \in \Phi$
- For any two roots x and y, the number $\langle y, x\rangle := 2\frac{(x,y)}{(x,x)}$ is an integer.

It's called reducible if $\Phi = \Phi_1 \cup \Phi_2$ with $\Phi_1 \perp \Phi_2$ for some $\Phi_1, \Phi_2 \neq \emptyset$. An irreducible root system is called an ADE roots system if the number $\langle y, x\rangle := 2\frac{(x,y)}{(x,x)}$ is either 0 or ± 1.

Given a root system Φ we can always choose (in many ways) a set of positive roots. This is a subset Φ^+ of Φ such that:

For each root $\alpha \in \Phi$, exactly one of the roots $\alpha, -\alpha$ is contained in Φ^+. For any two distinct $\alpha, \beta \in \Phi^+$ such that $\alpha + \beta$ is a root, $\alpha + \beta \in \Phi^+$. If a set of positive roots $\Phi^+$is chosen, elements of $-\Phi^+$ are called negative roots.

An element of Φ^+ is called a simple root if it cannot be written as the sum of two elements of Φ^+.

If Φ is an ADE root system, then let $\theta_1, ..., \theta_k$ be the simple roots. We write $\theta_0 = -\sum_{i=1}^k n_i\theta_i$ for the negative of the highest root. Let $n_0 = 1$, then $\sum_{i=0}^k n_i\theta_i = 0$. Now let's draw a diagram with n_i associated to each vertex θ_i. Draw an edge between θ_i and θ_j if $(\theta_i, \theta_j) = -\frac{1}{2}(\theta_i, \theta_i) = -\frac{1}{2}(\theta_j, \theta_j)$. Draw two edges between θ_0 with θ_1 if $k = 1$ and $\theta_0 = -\theta_1$. Then it must be an extended ADE Dynkin diagram. Moreover, θ_0 is the marked vertex.

Thus we've obtained the classification of ADE root systems:

(A_k, $k = 1, 2, ...$) $\Phi = \{\alpha \in \mathbb{Z}^{k+1} : \sum_{i=1}^{k+1} \alpha_i^2 = 2, \sum_{i=1}^{k+1} \alpha_i = 0.\}$.

(D_k, $k = 4, 5, ...$) $\Phi = \{\alpha \in \mathbb{Z}^k : \sum_{i=1}^{k+1} \alpha_i^2 = 2.\}$.

(E_6) $\Phi = \{\alpha \in \mathbb{Z}^6 \cup (\mathbb{Z} + \frac{1}{2})^6 : \sum_{i=1}^6 \alpha_i^2 + 2\alpha_1^2 = 2, \sum_{i=1}^8 \alpha_i + 2\alpha_1 \in 2\mathbb{Z}\}$.

(E_7) $\Phi = \{\alpha \in \mathbb{Z}^7 \cup (\mathbb{Z} + \frac{1}{2})^7 : \sum_{i=1}^7 \alpha_i^2 + \alpha_1^2 = 2, \sum_{i=1}^7 \alpha_i + \alpha_1 \in 2\mathbb{Z}\}$.

(E_8) $\Phi = \{\alpha \in \mathbb{Z}^8 \cup (\mathbb{Z} + \frac{1}{2})^8 : |\alpha|^2 = \sum_{i=1}^8 \alpha_i^2 = 2, \sum_{i=1}^8 \alpha_i \in 2\mathbb{Z}\}$.

(4) Lie group

Given a compact Lie group, the adjoint action of the maximal torus on the Lie algebra produces a root system. In particular, the A_k Dynkin diagram is related to the projective special unitary group PSU$(k+1)$, the D_k Dynkin diagram is related to the projective special orthogonal group PSO$(2k)$, the E_k Dykin diagrams are related to exceptional Lie groups.

(5) Platonic solids

In three-dimensional space, a Platonic solid is a regular, convex polyhedron. It is constructed by congruent regular polygonal faces with the same number of faces meeting at each vertex. Five solids meet those criteria:

- Tetrahedron
- Cube
- Octahedron
- Dodecahedron
- Icosahedron

(6) Finite subgroup of SU(2).

Given a finite subgroup Γ of SU(2), we can look at all of its irreducible representations $R_0, R_1, ...R_k$, where R_0 is the trivial representation on $\mathbb{C}^1$. Let Q be the canonical representation on $\mathbb{C}^2$. Let n_i be the rank of each representations. Draw a graph with n_i associated to each vertex R_i. Draw l edge between R_i and R_j if $(\mathrm{Hom}_\Gamma Q\otimes R_i, R_j)=\mathbb{C}^l$. Then we get an extended ADE Dynkin diagram.

Γ can be classified into the cyclic group, binary dihedral group, binary tetrahedral group, binary octahedral group, and binary icosahedral group. A more precise definition will be given in the ADE singularity part. We can also look at the image of Γ in SO(3). The image of binary dihedral group is dihedral group. The image of binary tetrahedral group is the orientation-preserving symmetry group of the tetrahedron which is isomorphic to the alternating group A_4. The image of binary octahedral group is the orientation-preserving symmetry group of the octahedron or cube which is isomorphic to the symmetric group S_4. The image of binary icosahedral group is the orientation-preserving symmetry group of the icosahedron or dodecahedron which is isomorphic to the alternating group A_5.

(7) ADE singularity

Let Γ be a finite subgroup of SU(2), then $\mathbb{C}^2/\Gamma$ is an ADE singularity. It can be resolved by blowing-ups. After case-by-case check, the singular fiber $\pi^{-1}(0)$ in the resolution $\pi:\widetilde{\mathbb{C}^2/\Gamma}\to\mathbb{C}^2/\Gamma$ consists of non-singular rational curves whose intersection pattern is the same as the ADE Dynkin diagram.

$\mathbb{C}^2/\Gamma$ can be expressed as the following:

(A_k) Let u, v be the coordinates of $\mathbb{C}^2$. Define the action of the cyclic group

$$\mathbb{Z}_{k+1}=<\sigma|\sigma^{k+1}=1>$$

by

$$\sigma(u,v)=(e^{2i\pi/(k+1)}u, e^{-2i\pi/(k+1)}v).$$

Then

$$x=uv, y=u^{k+1}, z=v^{k+1}$$

are invariant under the action with the relationship $x^{k+1}-yz=0$.

(D_k) Let u, v be the coordinates of $\mathbb{C}^2$. Define the action of the binary dihedral group

$$D_{4(k-2)}=<\sigma,\tau|\sigma^{2k-4}=1,\sigma^{k-2}=\tau^2,\tau\sigma\tau^{-1}=\sigma^{-1}>$$

by

$$\tau(u,v)=(v,-u),\sigma(u,v)=(e^{i\pi/(k-2)}u, e^{-i\pi/(k-2)}v).$$

Then

$$x=uv(u^{2k-4}-v^{2k-4})/2, y=(u^{2k-4}+v^{2k-4})/2, z=u^2v^2$$

are invariant under the action with the relationship $x^2 - zy^2 = -z^{k-1}$.

(E_6) Let u, v be the coordinates of $\mathbb{C}^2$. Define the action of the binary tetrahedral group

$$2T_{24} =< \sigma, \tau | (\sigma\tau)^2 = \sigma^3 = \tau^3 >$$

by

$$\sigma(u, v) = \frac{1}{2}((1+i)u + (-1-i)v, (1-i)u + (1-i)v),$$
$$\tau(u, v) = \frac{1}{2}((1+i)u + (-1+i)v, (1+i)u + (1-i)v).$$

Then

$$x = uv(u^4 - v^4),$$
$$y = (u^4 + 2\sqrt{3}iu^2v^2 + v^4)(u^4 - 2\sqrt{3}iu^2v^2 + v^4),$$
$$z = (u^4 + 2\sqrt{3}iu^2v^2 + v^4)^3 + (u^4 - 2\sqrt{3}iu^2v^2 + v^4)^3,$$

are invariant under the action with the relationship

$$432x^4 - 4y^3 + z^2 = 0.$$

(E_7) Let u, v be the coordinates of $\mathbb{C}^2$. Define the action of the binary octahedral group

$$2O_{48} =< \sigma, \tau | (\sigma\tau)^2 = \sigma^3 = \tau^4 >$$

by

$$\sigma(u, v) = \frac{1}{2}((1+i)u + (-1-i)v, (1-i)u + (1-i)v),$$
$$\tau(u, v) = \frac{1}{\sqrt{2}}((1+i)u, (1-i)v).$$

Then

$$x = u^2v^2(u^4 - v^4)^2,$$
$$y = u^8 + 14u^4v^4 + v^8,$$
$$z = uv(u^{12} - 33u^8v^4 - 33u^4v^8 + v^{12})(u^4 - v^4),$$

are invariant under the action with the relationship

$$108x^3 - xy^3 + z^2 = 0.$$

(E_8) Let u, v be the coordinates of $\mathbb{C}^2$. Define the action of the binary icosahedral group

$$2l_{120} =< \sigma, \tau | (\sigma\tau)^2 = \sigma^3 = \tau^5 >$$

by

$$\sigma(u,v) = \frac{1}{2}((1+i)u + (-1-i)v, (1-i)u + (1-i)v),$$
$$\tau(u,v) = \frac{1}{2}((\phi + \phi^{-1}i)u - v, u + (\phi - \phi^{-1}i)v),$$

where $\phi = \frac{\sqrt{5}+1}{2}$. Let

$$(U,V) = (-\frac{(2+\sqrt{2(5+\sqrt{5})}}{1+\sqrt{5}}iu + v, -u + \frac{(2+\sqrt{2(5+\sqrt{5})}}{1+\sqrt{5}}iv)$$

Then

$$x = iUV(U^{10} + 11iU^5V^5 + V^{10}),$$
$$y = -U^{20} - V^{20} + 228i(U^{15}V^5 + U^5V^{15}) + 494U^{10}V^{10},$$
$$z = U^{30} - V^{30} + 522i(U^{25}V^5 - U^5V^{25}) + 10005(U^{20}V^{10} - U^{10}V^{20}),$$

are invariant under the action with the relationship

$$-1728x^5 + y^3 + z^2 = 0.$$

(8) Singular fibers in elliptic surface.

Kodaira [45] classified singular fibers in elliptic surfaces. Some of them only consist of non-singular rational curves $\sum_{i=0}^{k} n_i\Theta_i$ with self-intersection number -2 such that one of n_i is 1. They will satisfy

$$2n_i = -n_i(\Theta_i)^2 = \sum_{j \neq i} n_j(\Theta_i\Theta_j).$$

So they must be an extended ADE Dynkin diagram.

I_b, b=2,...	I_b^* b=0,...	II^*	III	III^*	IV	IV^*
A_{b-1}	D_{b+4}	E_8	A_1	E_7	A_2	E_6

There is an elliptic surface with a pair of singular fibers (II,II*), (III,III*) or (IV,IV*). Roughly speaking, this explains the relationship between ALG gravitational instantons and ALE gravitational instantons.

(9) In [48], Kronheimer constructed ALE hyperkähler metrics on $\mathbb{C}^2/\Gamma$ by the following way:

Given a finite subgroup Γ of SU(2), we can look at all of its irreducible representations $R_0, R_1, ...R_k$, where R_0 is the trivial representation on $\mathbb{C}^1$. Let Q be the canonical representation on $\mathbb{C}^2$. Let n_i be the rank of each representations. Then the regular representation

$$R = \oplus_{i=0}^{k} \mathbb{C}^{n_i} \otimes R_i.$$

Let $V = \mathrm{Hom}_\Gamma(R, Q \otimes R)$. Let $G = S(\times_{i=0}^{k} \mathrm{U}(n_i))$. Then the hyperkähler quotient of V by G provides an ALE hyperkähler metric on $M = \widetilde{\mathbb{C}^2/\Gamma}$.

(10) After the construction of ALE gravitational instantons, Kroheimer [49] proved the following theorem:

Theorem 4.1. *(Torelli theorem for ALE gravitational instantons)*

Let M be the smooth 4-manifold which underlies the minimal resolution of $\mathbb{C}^2/\Gamma$. Let $[\alpha^1], [\alpha^2], [\alpha^3] \in H^2(M, \mathbb{R})$ be three cohomology classes which satisfy the nondegeneracy condition:

For each $[\Sigma] \in H_2(M, \mathbb{Z})$ with $[\Sigma]^2 = -2$, there exists $i \in \{1, 2, 3\}$ with $[\alpha^i][\Sigma] \neq 0$.

Then there exists on M an ALE hyperkähler structure such that the cohomology classes of the Kähler forms $[\omega^i]$ are the given $[\alpha^i]$. It's unique up to tri-holomorphic isometries which induce identity on $H_2(M, \mathbb{Z})$.

Moreover, any ALE gravitational instanton must be constructed by this way.

We will prove the analogy in ALF and ALH cases.

5 ALF gravitational instantons

5.1 Definitions and Notations

In this section, we follow the notations of previous sections. Now let's define the multi-Taub-NUT metric and more general Gibbons–Hawking ansatz:

Example 5.1. Let

$$V(\mathbf{x}) = 1 + \sum_{\alpha=1}^{k+1} \frac{2m}{|\mathbf{x} - \mathbf{x}_\alpha|}.$$

Let $\pi : M_0 \to \mathbb{R}^3 \setminus \{\mathbf{x}_\alpha\}$ be the $\mathbb{S}^1$-bundle of Euler class -1 around each $\mathbf{x}_\alpha$. Let η be the connection form with curvature $\mathrm{d}\eta = *\mathrm{d}V$. Then

$$g = V\mathrm{d}\mathbf{x}^2 + V^{-1}\eta^2$$

gives a metric on M_0. Let $M = M_0 \cup \{p_\alpha\}$ be the completion. Then M is called the multi-Taub-NUT metric with total mass $(k+1)m$. When $k = -1$, M is the trivial product of $\mathbb{S}^1$ and $\mathbb{R}^3$.

More generally, as long as V is harmonic, we can do the similar construction and call M the Gibbons-Hawking ansatz. It has complex structures satisfying

$$\mathrm{d}x^1 = I^*(V^{-1}\eta) = J^*\mathrm{d}x^2 = K^*\mathrm{d}x^3.$$

Now let's recall the holomorphic structure of the multi-Taub-NUT metric proved by Claude LeBrun [51]

Theorem 5.2. *(LeBrun)(M, I) is biholomorphic to the manifold*

$$uv = \prod_{\alpha=1}^{k+1}(z - (-x_\alpha^3 + ix_\alpha^2))$$

if $-x_\alpha^3 + ix_\alpha^2$ are distinct or the minimal resolution of it otherwise.

Proof. The function $z = -x^3 + ix^2$ is an I-holomorphic function on M. We can define a holomorphic vector field X by

$$\omega^+(X, Y) = -i\mathrm{d}z(Y).$$

The action of X gives $\mathbb{C}^*$-orbits of M. For $z \neq -x_\alpha^3 + ix_\alpha^2$, there is only one $\mathbb{C}^*$-orbit. If $-x_\alpha^3 + ix_\alpha^2$ are distinct, each $\{z = -x_\alpha^3 + ix_\alpha^2\}$ is divided into three $\mathbb{C}^*$-orbits: $\{x^1 < x_\alpha^1\}$,$\{x^1 = x_\alpha^1\}$, and $\{x^1 > x_\alpha^1\}$. Let

$$M^- = \{z \neq -x_\alpha^3 + ix_\alpha^2\} \cup \cup_\alpha \{z = -x_\alpha^3 + ix_\alpha^2, x^1 < x_\alpha^1\},$$
$$M^+ = \{z \neq -x_\alpha^3 + ix_\alpha^2\} \cup \cup_\alpha \{z = -x_\alpha^3 + ix_\alpha^2, x^1 > x_\alpha^1\}.$$

Then both M^+ and M^- are biholomorphic to $\mathbb{C} \times \mathbb{C}^*$. On the overlap, we have

$$(z, v)_{M^+} \sim (z, \frac{v}{\prod(z - (-x^3_\alpha + ix^2_\alpha))})_{M^-}.$$

If $-x^3_\alpha + ix^2_\alpha$ are not distinct. For example, suppose we have two points $(0, 0, 0)$ and $(1, 0, 0)$. Then we can define

$$\begin{aligned} M_1 &= \{z \neq 0\} \cup \{z = 0, x^1 < 0\}, \\ M_2 &= \{z \neq 0\} \cup \{z = 0, 0 < x^1 < 1\}, \\ M_3 &= \{z \neq 0\} \cup \{z = 0, x^1 > 1\}. \end{aligned}$$

On the overlap, we have $(z, v)_{M_1} \sim (z, \frac{v}{z})_{M_2} \sim (z, \frac{v}{z^2})_{M_3}$. In other words, it's the minimal resolution of $uv = z^2$ in the sense that we replace the point $\{u = v = z = 0\}$ by $\mathbb{CP}^1 = \{z = 0, 0 \leq x^1 \leq 1\}$. It's similar in general case. □

Using Theorem 5.2, we can get the twistor description of the multi-Taub-NUT metric as in [22].

Example 5.3. Let U be the affine variety in $\mathbb{C}^4$ with coordinates (ζ, z, ρ, ξ) defined by

$$\rho\xi = \prod_{\alpha=1}^{k+1}(z - P_\alpha(\zeta))$$

or the minimal resolution of it, where

$$P_\alpha(\zeta) = a_\alpha \zeta^2 + 2b_\alpha \zeta - \bar{a}_\alpha$$

with parameters $a_\alpha \in \mathbb{C}$ and $b_\alpha \in \mathbb{R}$. Take two copies of U and glue them together over $\zeta \neq 0, \infty$ by

$$\begin{aligned} \tilde{\zeta} &= \zeta^{-1}, \\ \tilde{z} &= \zeta^{-2} z, \\ \tilde{\rho} &= e^{-z/\zeta} \zeta^{-k-1} \rho, \\ \tilde{\xi} &= e^{z/\zeta} \zeta^{-k-1} \xi. \end{aligned}$$

Then ζ lies in $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ and z is a section in $\mathcal{O}(2)$. Define

$$\omega = 4i \mathrm{d} \log \rho \wedge \mathrm{d}z = i\mathrm{d}z \wedge \mathrm{d}\chi.$$

Define the real structure τ by

$$\tau(\zeta, z, \rho, \xi) = (-1/\bar{\zeta}, -\bar{z}/\bar{\zeta}^2, e^{\bar{z}/\bar{\zeta}}(1/\bar{\zeta})^{k+1}\bar{\xi}, e^{-\bar{z}/\bar{\zeta}}(-1/\bar{\zeta})^{k+1}\bar{\rho}).$$

The gluing of U and $\tilde{U}$ is the twistor space of the multi-Taub-NUT metric up to rescaling.

Notice that our convention is slightly different from [22]. We use the real form $i\partial\bar{\partial}K$ as the Kähler form but they use $\partial\bar{\partial}K$ following the convention of [42]. The other difference is that they use the scaling parameter μ but we rescale our metric to make $\mu = 1$.

Similarly, we can define the twistor space of a hyperkähler 8-manifold:

Example 5.4. Let U be the subvariety with coordinates $(\zeta, w, z, \rho_0, \rho_1, \xi_0, \xi_1)$ defined by

$$(\rho_0 + \rho_1\eta)(\xi_0 + \xi_1\eta) = \prod_{\alpha=1}^{k}(\eta - P_\alpha(\zeta)) \mod \eta^2 - w\eta - z = 0,$$

where

$$P_\alpha(\zeta) = a_\alpha\zeta^2 + 2b_\alpha\zeta - \bar{a}_\alpha$$

with parameters $a_\alpha \in \mathbb{C}$ and $b_\alpha \in \mathbb{R}$.

Take two copies of U and glue them together over $\zeta \neq 0, \infty$ by

$$\begin{aligned}
\tilde{\zeta} &= \zeta^{-1},\\
\tilde{w} &= \zeta^{-2}w,\\
\tilde{z} &= \zeta^{-4}z\\
(\tilde{\rho}_0 + \tilde{\rho}_1\tilde{\eta}) &= e^{-\eta/\zeta}\zeta^{-k}(\rho_0 + \rho_1\eta) \mod \eta^2 - w\eta - z = 0,\\
(\tilde{\xi}_0 + \tilde{\xi}_1\tilde{\eta}) &= e^{\eta/\zeta}\zeta^{-k}(\xi_0 + \xi_1\eta) \mod \eta^2 - w\eta - z = 0,
\end{aligned}$$

Then ζ lies in $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$ and z is a section in $\mathcal{O}(4)$. Define

$$\omega = 4i\sum_{j=1}^{2}\frac{(\mathrm{d}\rho_0 + \beta_j\mathrm{d}\rho_1) \wedge \mathrm{d}\beta_j}{\rho_0 + \beta_j\rho_1},$$

where β_1, β_2 are the two roots of $\eta^2 - w\eta - z = 0$. Define the real structure by

$$\tau(\zeta, z, \rho, \xi) = (-1/\bar{\zeta}, \bar{z}/\bar{\zeta}^4, e^{\bar{\eta}/\bar{\zeta}}(1/\bar{\zeta})^k\bar{\xi}, e^{-\bar{\eta}/\bar{\zeta}}(-1/\bar{\zeta})^k\bar{\rho}),$$

where $\rho = \rho_0 + \rho_1\eta$ and $\xi = \xi_0 + \xi_1\eta$. We can realize the gluing of U and $\tilde{U}$ as the twistor space of a complete hyperkähler 8-manifold.

The hyperkähler quotient of the previous example is the Cherkis–Hitchin–Ivanov–Kapustin–Lindström–Roček metric. When $k = 0$, it's the famous Atiyah–Hitchin metric. Actually, the Atiyah–Hitchin metric [4] provided the first example of ALF-D_k gravitational instantons. Later, Ivanov and Roček [42] conjectured a formula for positive k using generalized Legendre transform developed by Lindström and Roček [52]. Cherkis and Kapustin [22] confirmed this formula. This metric was computed more explicitly by Cherkis and Hitchin [21].

Example 5.5. In the previous example, we look at the $\mathbb{C}^*$ action by $\rho_j \to \lambda\rho_j$ and $\xi_j \to \lambda^{-1}\xi_j$. The moment map is w. To get the hyperkähler quotient, we set $w = 0$ and take the $\mathbb{C}^*$ quotient.

The submanifold $w = 0$ in U can be written as

$$\begin{aligned} \rho_0\xi_0 + z\rho_1\xi_1 &= p(z), \\ \rho_1\xi_0 + \rho_0\xi_1 &= q(z), \end{aligned}$$

where

$$\prod_\alpha(\eta - P_\alpha) = p(z) + \eta q(z) \mod \eta^2 - z = 0.$$

The $\mathbb{C}^*$-quotient can be obtained by using the $\mathbb{C}^*$-invariant coordinates

$$\begin{aligned} x &= i^k[\rho_1\xi_0 - \rho_0\xi_1], \\ y &= i^k[-2\rho_1\xi_1 + r(z)], \end{aligned}$$

where

$$p(z) = zr(z) + \prod_\alpha(-P_\alpha).$$

Thus

$$\begin{aligned} \rho_1\xi_0 &= \frac{q(z) + (-i)^k x}{2}, \\ \rho_0\xi_1 &= \frac{q(z) - (-i)^k x}{2}, \\ \rho_1\xi_1 &= \frac{r(z) - (-i)^k y}{2}, \\ \rho_0\xi_0 &= \frac{zr(z) + (-i)^k zy}{2} + \prod_\alpha(-P_\alpha). \end{aligned}$$

The equation

$$(\rho_0\xi_0)(\rho_1\xi_1) = (\rho_0\xi_1)(\rho_1\xi_0)$$

is reduced to

$$x^2 - zy^2 = \frac{1}{-z}(\prod_\alpha (z - P_\alpha^2) - \prod_\alpha (-P_\alpha^2)) + 2\prod_\alpha (-iP_\alpha)y.$$

Moreover,

$$\omega = i\mathrm{d}(\frac{1}{\sqrt{z}}\log(\frac{yz + \prod_\alpha(-iP_\alpha) + \sqrt{z}x}{yz + \prod_\alpha(-iP_\alpha) - \sqrt{z}x})) \wedge \mathrm{d}z.$$

This is the twistor space of Cherkis–Hitchin–Ivanov–Kapustin–Lindström–Roček metric.

When $P_\alpha(\zeta)$ and $-P_\alpha(\zeta)$ are distinct, the manifold is non-singular. Otherwise, the CHIKLR metric is the minimal resolution of the singular manifold which will be discussed later. It's interesting to notice that [14] when two P_α equal to 0, the singular manifold is the $\mathbb{Z}_2$-quotient of the multi-Taub-NUT metric. Moreover, when $k \geq 3$, if all of P_α equal to 0, the singular manifold is exactly the quotient of the Taub-NUT metric by the binary dihedral group $D_{4(k-2)}$ because of the following calculation:

Example 5.6. It's well known that the Taub-NUT metric is biholomorphic to $\mathbb{C}^2$. Let u, v be the coordinates of $\mathbb{C}^2$. Define the action of the binary dihedral group

$$D_{4(k-2)} =< \sigma, \tau | \sigma^{2k-4} = 1, \sigma^{k-2} = \tau^2, \tau\sigma\tau^{-1} = \sigma^{-1} >$$

by

$$\tau(u, v) = (v, -u), \sigma(u, v) = (e^{i\pi/(k-2)}u, e^{-i\pi/(k-2)}v).$$

Then

$$x = uv(u^{2k-4} - v^{2k-4})/2, y = (u^{2k-4} + v^{2k-4})/2, z = u^2v^2$$

are invariant under the action with the relationship $x^2 - zy^2 = -z^{k-1}$. This is exactly the previous example with all $P_\alpha = 0$.

5.2 Rigidity of multi-Taub-NUT metric

In this subsection, we analyze the ALF-A_k gravitational instantons as a warm up of Theorem 1.7. We will use the twistor space method as in [22]. An important step in our approach is a compactification in the complex analytic sense and the analysis of topology of this compactification following Kodaira's work [45].

We start from the compactification.

Theorem 5.7. *Any ALF-A_k gravitational instanton (M, I) can be compactified in the complex analytic sense.*

Proof. By the remark after Theorem 3.24, M is asymptotic to the standard model E with error $O'(r^{-2})$. E is either the trivial product $(\mathbb{R}^3 \setminus B_R) \times \mathbb{S}^1$ or the quotient of the Taub-NUT metric outside a ball by $\mathbb{Z}_{k+1}$. In any case, there exist two I-holomorphic functions z_E and ρ_E satisfying

$$\omega^+ = 4i \mathrm{d} \log \rho_E \wedge \mathrm{d}z_E.$$

We are mostly interested in the behaviors when x^1 goes to $-\infty$. It corresponds to

$$\mathbb{C} \times (\mathbb{C}^* \cap B_{e^{-R}}) \cong \mathbb{C} \times (B^c_{e^R}) = \{(z_E, \rho_E) : |\rho_E| > e^R\}.$$

We are also interested in the corresponding part of M.

On M, there exists an I-holomorphic function $z = z_E + O'(r^{-\delta})$ for any $\delta < 1$. As in Section 3.9, we can define a holomorphic vector field X by $\omega^+(X, Y) = -i\mathrm{d}z(Y)$. On each fixed fiber, there exists a unique holomorphic form ϕ such that $\phi(X) = 1$. Locally

$$\omega^+ = ic(z, v)\mathrm{d}z \wedge \mathrm{d}v, X = \frac{1}{c(z, v)} \frac{\partial}{\partial v}, \phi = c(z, v)\mathrm{d}v.$$

Each fiber in the interesting part of M is topologically $\mathbb{C}^* \cap B_{e^{-R}}$. So on each fiber, we can integrate the form ϕ to get a holomorphic function $\chi \in \mathbb{C}/\mathbb{Z}\tau(u)$ up to a function of z. We can fix this ambiguity by requiring that $\chi - \Phi^*(-4 \log \rho_E)$ goes to 0 when χ becomes negative infinity, where Φ is the map from M to E. $\tau(u) = 8\pi i$ since M is asymptotic to E and it's true on E. We can fix this ambiguity by writing χ as $\chi = -4 \log \rho$. Therefore we get a part of M biholomorphic to

$$\mathbb{C} \times (\mathbb{C}^* \cap B_{e^{-R}}) \cong \{(z, \rho) : |\rho| > e^R\}$$

with $\omega^+ = 4i\mathrm{d}\log\rho \wedge \mathrm{d}z$. Similarly, the part of M where x^1 goes to $+\infty$ is biholomorphic to

$$\mathbb{C} \times (\mathbb{C}^* \cap B_{e^{-R}}) \cong \{(z, \xi) : |\xi| > e^R\}$$

with

$$\omega^+ = 4i\mathrm{d}z \wedge \mathrm{d}\log\xi.$$

Now we can add the divisors $D_- = \{\rho = \infty\}$ and $D_+ = \{\xi = \infty\}$ to compactify the two parts. We can get a manifold with a holomorphic function z whose generic fiber is $\mathbb{CP}^1$. Adding $D_\infty = \mathbb{CP}^1 = \{z = \infty\}$, we can get a compact manifold $\bar{M}$ with a meromorphic function $z : \bar{M} \to \mathbb{CP}^1$ whose generic fiber is $\mathbb{CP}^1$. □

It's easy to see that $-K = \{\omega^+ = \infty\} = D_- + D_+ + 2D_\infty$ is the anti-canonical divisor. Any generic fiber is a non-singular rational curve $C = \mathbb{CP}^1$ with $(-KC) = 2$ and $(C^2) = 0$. Following the work of Kodaira [45], we can classify singular fibers.

Theorem 5.8. *Any singular fiber C can be written as the sum of non-singular rational curves*

$$C = \Theta_0 + ... + \Theta_m, m = 1, 2, 3, ...,$$

with

$$\begin{aligned}(\Theta_i\Theta_j) &= \delta(|j-i|-1),\\ (\Theta_i^2) &= -2 + \delta(0) + \delta(m),\\ (-K\Theta_i) &= \delta(0) + \delta(m),\end{aligned}$$

where $\delta(n) = 1$ if $n = 0$, and $\delta(n) = 0$ otherwise.

Proof. Let $C = \sum n_i\Theta_i$. The main tools are Kodaira's identities [45]

$$\begin{aligned}2\pi'(\Theta_i) - 2 - (\Theta_i^2) &= (K\Theta_i),\\ (C\Theta_i) = 0 &= n_i(\Theta_i^2) + \sum_{j\neq i} n_j(\Theta_i\Theta_j),\end{aligned}$$

where the virtual genus $\pi'(\Theta_i)$ is non-negative and $\pi'(\Theta_i)$ vanishes if and only if Θ_i is a non-singular rational curve.

If there is only one curve $C = \Theta_0$, then $(K\Theta_0) = -2$. Notice that $(\Theta_0^2) = 0$ by the second identity. So $\pi'(\Theta_0) = 0$ by the first identity. So Θ_0 is a non-singular rational curve. In other words, the fiber is regular.

Otherwise, from the information near $D_\pm$, there exist two curves Θ_0 and Θ_m satisfying $(-K\Theta_0) = (-K\Theta_m) = 1$ and $n_0 = n_m = 1$. All other curves don't intersect $-K$. From the second identity and the fact that C is connected [45], we know that $(\Theta_i^2) < 0$. Therefore, $\pi'(\Theta_i)$ must be 0, i.e. each Θ_i is a non-singular rational curve. It follows that

$$(\Theta_i^2) = -2 + \delta(0) + \delta(m).$$

Now the second identity becomes

$$1 = \sum_{j\neq 0} n_j(\Theta_0\Theta_j).$$

If $(\Theta_0\Theta_m) = 1$, we are done with $m = 1$. Otherwise, suppose $(\Theta_0\Theta_1) = 1$. Then $n_1 = 1$, so

$$2 = 1 + \sum_{j\neq 0, j\neq 1} n_j(\Theta_1\Theta_j).$$

If $(\Theta_1\Theta_m) = 1$, we are done with $m = 2$. Otherwise, we can continue. After several steps, we must stop because the number of curves is finite. Therefore, the singular fibers must have the required properties. □

Remark. That's exactly the picture in Theorem 5.2.

Remark. Suppose each fiber is regular except the fiber $\{z = 0\}$, M is biholomorphic to the minimal resolution of $xy = z^{k+1}$. In this case, the central fiber has $k = m - 1$ non-singular rational curves Θ_1,...,Θ_{m-1} whose intersection diagram is called the A_k Dynkin diagram. That's the reason why we call M ALF-A_k.

Now we are able to give a new proof of the following theorem. It was first proved by Minerbe in [59] using the existence of Killing vector fields. However, since there is no Killing vector field on ALF-D_k gravitational instantons, we prefer a new proof of this theorem using the twistor space.

Theorem 5.9. *(Minerbe) Any ALF-A_k gravitational instanton must be the multi-Taub-NUT metric.*

Proof. First of all, let's look at the slice $\zeta = 0$. In other words, we use I as the complex structure. ω^+ as the holomorphic symplectic form. By Theorem 5.7, there exist ρ and ξ such that

$$\omega^+ = 4i\mathrm{d}\log\rho \wedge \mathrm{d}z = 4i\mathrm{d}z \wedge \mathrm{d}\log\xi.$$

So $\rho\xi$ is a holomorphic function of z satisfying

$$\lim_{z\to\infty} \rho\xi/z^{k+1} = 1.$$

It's completely determined by its zeros. By Theorem 5.8 and Theorem 5.2, it's easy to see that (M, I) is biholomorphic to $\rho\xi = \prod_{\alpha=1}^{k+1}(z - P_\alpha)$ or the minimal resolution of it.

Now we may vary $\zeta \neq \infty$. We can still get

$$\rho\xi = \prod_{\alpha=1}^{k+1}(z - P_\alpha(\zeta))$$

with

$$\omega = 4i\mathrm{d}\log\rho \wedge \mathrm{d}z.$$

Similarly, for $\zeta \neq 0$, we may use $\tilde{\zeta} = \zeta^{-1}$ instead. Then $\tilde{\omega} = \zeta^{-2}\omega$ and $\tilde{z} = \zeta^{-2}z$ are non-singular. So we can get $\tilde{\omega} = 4i\mathrm{d}\log\tilde{\rho} \wedge \mathrm{d}\tilde{z}$ instead. The difference $\tilde{\rho}/\rho$ is a holomorphic function of ζ and z. It equals to $e^{-z/\zeta}\zeta^{-k-1}$ on E, so $\tilde{\rho}/\rho$ must be $e^{-z/\zeta}\zeta^{-k-1}$ on M. Similarly, $\tilde{\xi} = e^{z/\zeta}\zeta^{-k-1}\xi$.

Since $\tilde{\rho}\tilde{\xi} = \prod_{\alpha=1}^{k+1}(\tilde{z} - \tilde{P}_\alpha(\tilde{\zeta}))$. It's easy to see that $P_\alpha(\zeta) = \zeta^2\tilde{P}_\alpha(\tilde{\zeta})$. So $P_\alpha(\zeta)$ must be a degree two polynomial of ζ.

Now let's look at the action of the real structure. When ζ becomes $-1/\bar{\zeta}$, M becomes exactly its own conjugation. Since z is invariant under the action $(\zeta, z) \to (-1/\bar{\zeta}, -\bar{z}/\bar{\zeta}^2)$, P_α must have the same property under the real structure. In other words, $P_\alpha(\zeta) = a_\alpha\zeta^2 + 2b_\alpha\zeta - \bar{a}_\alpha$ for some $a_\alpha \in \mathbb{C}$ and $b_\alpha \in \mathbb{R}$. It's easy to see that the real structure τ must act by

$$\tau(\zeta, z, \rho, \xi) = (-1/\bar{\zeta}, -\bar{z}/\bar{\zeta}^2, e^{\bar{z}/\bar{\zeta}}(1/\bar{\zeta})^{k+1}\bar{\xi}, e^{-\bar{z}/\bar{\zeta}}(-1/\bar{\zeta})^{k+1}\bar{\rho}).$$

It's well known [40] that the form ω and the real structure on the twistor space determine the metric on M. So M must be the multi-Taub-NUT metric. □

5.3 Classification of ALF-D_k gravitational instantons

In this section we prove Theorem 1.7 as we did for the ALF-A_k gravitational instantons in the previous subsection.

We still start from the compactification.

Theorem 5.10. *Any ALF-D_k gravitational instanton (M, I) can be compactified in the complex analytic sense.*

Proof. We already know that outside a compact set, M is up to $O'(r^{-3})$, the $\mathbb{Z}_2$-quotient of a standard $\mathbb{S}^1$-fiberation E over $\mathbb{R}^3 - B_R$. Moreover, there is an I_M-holomorphic function $z = (-x^3 + ix^2)^2 + O'(r^{-1})$ on M.

Recall that there is a part of (E, I_E) biholomorphic to

$$\mathbb{C} \times (\mathbb{C}^* \cap B_{e^{-R}}) = \{(a_E, b_E) : a_E = -x^3 + ix^2 \in \mathbb{C}, b_E \in \mathbb{C}^* \cap B_{e^{-R}}\}.$$

Now we claim that the corresponding part of (M, I_M) is also biholomorphic to

$$\mathbb{C} \times (\mathbb{C}^* \cap B_{e^{-R}}) = \{(a, b) : a \in \mathbb{C}, b \in \mathbb{C}^* \cap B_{e^{-R}}\}$$

What's more, under this diffeomorphism, $\omega_M^+ = -4i\mathrm{d}\log b \wedge \mathrm{d}a$.

It's hard to solve a, b as functions of a_E, b_E directly. However, following the idea of Newlander and Nirenberg [61], we can instead solve a_E, b_E as functions of a, b and apply the inverse function theorem.

Let

$$a = u + iv, \log b = t + i\theta, a_E = a + z^1, b_E = be^{z^2}.$$

Let

$$\partial_1 = \frac{\partial}{\partial a} = \frac{1}{2}(\frac{\partial}{\partial u} - i\frac{\partial}{\partial v}), \bar{\partial}_1 = \frac{\partial}{\partial \bar{a}}, \partial_2 = b\frac{\partial}{\partial b} = \frac{1}{2}(\frac{\partial}{\partial t} - i\frac{\partial}{\partial \theta}), \bar{\partial}_2 = \bar{b}\frac{\partial}{\partial \bar{b}}.$$

Then the equation is reduced to

$$\bar{\partial}_j z^k + \phi_l^k(u + iv + z^1, t + i\theta + z^2)(\bar{\partial}_j \bar{z}^l + \delta_j^l) = 0,$$

where

$$|\nabla^m \phi(u + iv, t + i\theta)| < C(m)(u^2 + v^2 + t^2)^{(-3-m)/2}$$

for all $m \geq 0$ and all $t < -R$ if R is large enough.

Instead of the space $\tilde{C}^{n+\alpha}$ in [61], we prefer weighted Hilbert spaces.

Define

$$||f||_{L^2_{\alpha,\beta}} = \int_{t<-R} |f|^2(1 + u^2 + v^2)^{\alpha/2}|t|^\beta,$$

and

$$||f||_{H^m_{\alpha,\beta}} = \sqrt{\sum_{i+j+k+l\leq m} ||\partial_u^i \partial_v^j \partial_t^k \partial_\theta^l f||^2_{L^2_{\alpha+2i+2j,\beta+2k+2l}}},$$

then we can find an operator

$$T_1 : L^2_{\alpha,\beta} \to L^2_{\alpha-2,\beta}$$

satisfying

$$\bar{\partial}_1 T_1 f = f$$

in the distribution sense if $\alpha < 2$ and α isn't an integer. Actually, by Theorem 3.13, we can find G_1 such that $4\partial_1\bar{\partial}_1 G_1 f = f$ in the distribution sense. So $T_1 f = 4\partial_1 G_1 f$.

Similarly, by Theorem 3.17, we can find an operator

$$T_2 : L^2_{\alpha,\beta} \to L^2_{\alpha,\beta-2}$$

satisfying

$$\bar{\partial}_2 T_2 f = f$$

in the distribution sense if β isn't an integer. Since both T_1 and T_2 are canonically defined, T_1 commutes with ∂_2 and $\bar{\partial}_2$ while T_2 commutes with ∂_1 and $\bar{\partial}_1$. By the work of Newlander and Nirenberg [61], the integrability condition implies that it's enough to solve the equation

$$z^i = T^1 f_1^i + T^2 f_2^i - \frac{1}{2} T^1 \bar{\partial}_1 T^2 f_2^i - \frac{1}{2} T^2 \bar{\partial}_2 T^1 f_1^i,$$

where

$$f_j^i = -\phi_l^i(u + iv + z^1, t + i\theta + z^2)(\bar{\partial}_j \bar{z}^l + \delta_j^l).$$

It has a unique solution in $H^{10}_{-2\epsilon,-2\epsilon}$ for any $0 < \epsilon < 1/2$ if R is large enough. By Sobolev embedding theorem, $|z^i| \leq C(1 + u^2 + v^2)^{(-1+\epsilon)/2}|t|^{-1+\epsilon}$.

In conclusion, we've solved a_E and b_E in terms of a and b. We can invert them to get a and b in terms of a_E and b_E. By the arguments similar to Theorem 5.7, we can slightly modify b such that $\lim_{b\to 0}(b/b_E) = 1$ and $\omega_M^+ = -4i\mathrm{d}\log b \wedge \mathrm{d}a$.

Therefore, we can add the divisor $D = \{b = 0\}$ to compactify this part. On $M \cup D$, the condition $z = a_E^2 + O'(r^{-1})$ is reduced to $z(a, 0) = a^2$. Near $a = \infty$, let $c = 1/a$, then $M \cup D$ is locally biholomorphic to

$$((\mathbb{C}^* \cap B_{1/R}) \times \mathbb{CP}^1)/\mathbb{Z}_2 = \{(c, b) : 0 < |c| < 1/R, b \in \mathbb{CP}^1\}/(c, b) \sim (-c, 1/b).$$

As Kodaira did in [45], we can add $\{(0,b)\}/(0,b) \sim (0,1/b)$ and then replace the neighborhoods of two singular points $(0,1)$ and $(0,-1)$ by two copies of N_{+2}.(See page 583 of [45]). As in page 586 of [45], $D_\infty = 2\Theta + \Theta_0 + \Theta_1$ with $(\Theta\Theta_i) = (D\Theta) = 1$, $(\Theta_1\Theta_2) = (D\Theta_i) = 0$, $(\Theta^2) = -1$ and $(\Theta_i^2) = -2$. Therefore, we get a compact manifold $\bar{M} = M \cup D \cup D_\infty$ with a meromorphic function $z : \bar{M} \to \mathbb{CP}^1$ whose generic fiber is a non-singular rational curve. □

On $\bar{M}$, the anti-canonical divisor $-K = \{\omega^+ = \infty\} = D + D_\infty$. Any generic fiber is a non-singular rational curve $C = \mathbb{CP}^1$ with $(-KC) = 2$ and $(C^2) = 0$. Any singular fiber $\{z = z_0\}$ must belong to the list in Theorem 5.8 if $z_0 \neq 0, \infty$. So we only need to classify the fiber $\{z = 0\}$. The main property is that $-K = D + D_\infty$ intersects C at only one point.

Theorem 5.11. *The fiber $C = \{z = 0\}$ can be written as the sum of non-singular rational curves. There are three cases:*

(i) $C = \Theta$, $(\Theta^2) = 0$, $(-K\Theta) = (D\Theta) = 2$, but D intersects Θ at one point with multiplicity 2.

(ii) $C = \Theta_0 + \Theta_1$, $(\Theta_0^2) = (\Theta_1^2) = -1$, three curves Θ_0, Θ_1, D intersect at same point.

(iii) $C = 2\Theta_0 + ... + 2\Theta_m + \Theta_{m+1} + \Theta_{m+2}, m = 0, 1, ...,$

$$(\Theta_0\Theta_1) = ... = (\Theta_{m-1}\Theta_m) = (\Theta_m\Theta_{m+1}) = (\Theta_m\Theta_{m+2}) = 1,$$
$$(\Theta_0^2) = -1, (\Theta_1^2) = ... = (\Theta_{m+2}^2) = -2, (-K\Theta_0) = (D\Theta_0) = 1,$$

and all other intersection numbers are 0.

Proof. Let $C = \sum n_i\Theta_i$. We still use Kodaira's two identities

$$2\pi'(\Theta_i) - 2 - (\Theta_i^2) = (K\Theta_i),$$
$$(C\Theta_i) = 0 = n_i(\Theta_i^2) + \sum_{j\neq i} n_j(\Theta_i\Theta_j)$$

and the fact the C is connected. By the second identity,

$$(\Theta_i^2) = -\frac{1}{n_i}\sum_{j\neq i} n_j(\Theta_i\Theta_j) \leq 0.$$

Since $(DC) = (-KC) = 2$, but D intersects C at only one point, there are only three possibilities.

(i) Θ_0 intersects D at one point with multiplicity 2. By Kodaira's first identity, $(\Theta_0^2) = 0$ and $\pi'(\Theta_0) = 0$. Therefore, there are no other curves at all. It's the first case.

(ii) Θ_0 and Θ_1 intersect D at same point. So $(\Theta_i^2) = -1$ and $\pi'(\Theta_i) = 0$. There are still no other curves at all. It's the second case.

(iii) Θ_0 intersects D at one point but $n_0 = 2$. In this case, $(\Theta_0^2) = -1$ and $\pi'(\Theta_0) = 0$ by Kodaira's first identity. As in Theorem 5.8, since any other curve has no intersection with D, it must be a non-singular rational curve with self intersection number -2.

Therefore, either two different curves Θ_1 and Θ_2 intersect Θ_0 or one curve Θ_1 intersects Θ_0 but $n_1 = 2$. In the first case, we are done. In the second case, we can continue the same kind of analysis. After finite steps, we are done since there are only finitely many curves. □

Remark. If each fiber is regular except the fiber $\{z = 0\}$, M is biholomorphic to the minimal resolution of $x^2 - zy^2 = -z^{k-1}$. In this case, the central fiber has $k = m + 2$ non-singular rational curves Θ_1,...,Θ_{m+2} whose intersection diagram is called the D_k Dynkin diagram. That's the reason why we call M ALF-D_k.

Now, we are able to prove Theorem 1.7.

Theorem 5.12. *Any ALF-D_k gravitational instanton must be the Cherkis-Hitchin-Ivanov-Kapustin-Lindström-Roček metric.*

Proof. We still start from the slice $\zeta = 0$. We already know that the double cover $\tilde{M}$ of $M \setminus \{z = 0\}$ is asymptotic to E. $\sqrt{z} \approx a$ is well defined on $\tilde{M}$. As before, we can define a holomorphic function f on $\tilde{M}$ by $\omega^+ = 4i\mathrm{d}\log f \wedge \mathrm{d}\sqrt{z}$ and $\lim_{b\to 0} fb = 1$. The composition of f and the covering transform is called f'.

Now we are interested in the behavior near $a = 0$ and $b = 0$. We can write z as

$$z = e^{bf_1(a,b)}(a^2 + bf_2(b)a + bf_3(b)).$$

The function $f_4(b) = -(bf_2(b))^2/4 + bf_3(b)$ can be written as $cb^m(1 + bf_5(b))$, where $m = 1, 2, \infty$ depending on the type of $\{z = 0\}$. Change the coordinates by

$$a' = e^{bf_1(a,b)/2}(a + bf_2(b)/2),$$
$$b' = be^{bf_1(a,b)/m}(1 + bf_5(b))^{1/m}.$$

Then

$$z = a'^2 + cb'^m,$$

with

$$\omega^+ = -4i\mathrm{d}\log b' \wedge (1 + b'f_6(a,b))\mathrm{d}a' = 4i\mathrm{d}\log f \wedge \mathrm{d}\sqrt{z}.$$

(i) In the first case of Theorem 5.11, $m = 1$. So

$$\log f = -\int \frac{\sqrt{z}\mathrm{d}b'}{a'b'} = -\log b' + \log\frac{(\sqrt{z}+a')^2}{4z} = \log\frac{a'+\sqrt{z}}{a'-\sqrt{z}} + \log\frac{-c}{4z}$$

if we ignore the term $b'f_6(a,b)$. However, the contribution from the term $b'f_6(a,b)$ is bounded by $C\int_0^{e^{-R}} |\frac{\sqrt{|z|}}{\sqrt{||z|-|c|t|}}|\mathrm{d}t \le C\sqrt{|z|}$. So $\lim_{z\to 0} fz = -c/4$. Similarly $\lim_{z\to 0} f'z = -c/4$. Therefore, we can write f, f' as

$$fz = P + \sqrt{z}Q, f'z = P - \sqrt{z}Q.$$

Away from $\{z = 0\}$, the picture is similar to the A_{2k-5} case, so

$$P^2 - zQ^2 = (P + \sqrt{z}Q)(P - \sqrt{z}Q) = \prod_{\alpha=1}^{k}(z - P_\alpha^2).$$

Notice that

$$\lim_{z\to 0} P = -c/4 = \prod_\alpha(-iP_\alpha),$$

so we can write

$$P = yz + \prod_\alpha(-iP_\alpha)$$

and write $Q = x$. A simple calculation yields

$$x^2 - zy^2 = \frac{1}{-z}(\prod_\alpha(z - P_\alpha^2) - \prod_\alpha(-P_\alpha^2)) + 2\prod_\alpha(-iP_\alpha)y$$

and

$$\begin{aligned}\omega^+ =& i\mathrm{d}(\frac{1}{\sqrt{z}}\log(\frac{P+\sqrt{z}Q}{P-\sqrt{z}Q}))\wedge\mathrm{d}z\\ =& i\mathrm{d}(\frac{1}{\sqrt{z}}\log(\frac{yz+\prod_\alpha(-iP_\alpha)+\sqrt{z}x}{yz+\prod_\alpha(-iP_\alpha)-\sqrt{z}x}))\wedge\mathrm{d}z.\end{aligned}$$

(ii) In the second case of Theorem 5.11, $m = 2$. So

$$\log f = -\int \frac{\sqrt{z}\mathrm{d}b'}{a'b'} = -\log b' + \log\frac{\sqrt{z}+a'}{2\sqrt{z}} = \frac{1}{2}\log\frac{a'+\sqrt{z}}{a'-\sqrt{z}} + \frac{1}{2}\log\frac{-c}{4z}$$

if we ignore the term $b'f_6(a,b)$. In this case, the contribution from the term $b'f_6(a,b)$ is bounded by

$$C\int_0^{e^{-R}} |\frac{\sqrt{|z|}}{\sqrt{||z|-|c|t^2|}}|\mathrm{d}t \le C\sqrt{|z|}\log(1/\sqrt{|z|}).$$

So $\lim_{z\to 0} f\sqrt{z} = \sqrt{-c}/2$. It's also true that $\lim_{z\to 0} -f'\sqrt{z} = \sqrt{-c}/2$. So we can write f, f' as

$$f\sqrt{z} = x + \sqrt{z}y, -f'\sqrt{z} = x - \sqrt{z}y.$$

It's easy to see that

$$x^2 - zy^2 = (x+\sqrt{z}y)(x-\sqrt{z}y) = -\prod_{\alpha=1}^{k-1}(z - P_\alpha^2).$$

Notice that $\lim_{z\to 0} x = \sqrt{-c}/2$ on Θ_0, but $\lim_{z\to 0} x = -\sqrt{-c}/2$ on Θ_1, so we can no longer reduce x and y. However, let $P_k = 0$, then

$$x^2 - zy^2 = \frac{1}{-z}(\prod_{\alpha=1}^{k}(z-P_\alpha^2) - \prod_{\alpha=1}^{k}(-P_\alpha^2)) + 2\prod_{\alpha=1}^{k}(-iP_\alpha)y$$

and

$$\omega^+ = 4i\mathrm{d}\log f \wedge \mathrm{d}\sqrt{z} = i\mathrm{d}(\frac{1}{\sqrt{z}}\log(\frac{yz + \prod_{\alpha=1}^{k}(-iP_\alpha) + \sqrt{z}x}{yz + \prod_{\alpha=1}^{k}(-iP_\alpha) - \sqrt{z}x})) \wedge \mathrm{d}z.$$

It's convenient to write $P = yz$ and $Q = x$. So $fz = P + \sqrt{z}Q$ still holds.

(iii) In the third case of Theorem 5.11, $m = \infty$. Just as we did in Theorem 5.10 near $z = \infty$, the manifold becomes the minimal resolution of the $\mathbb{Z}_2$-quotient of multi-Taub-NUT metric. $\mathbb{Z}_2$ acts by interchanging ρ and ξ. So $f = \rho$ and $f' = \xi$. They satisfy $ff' = \prod_{\alpha=1}^{k-2}(z - P_\alpha^2)$. Let $x = \sqrt{z}(f - f')/2$ and $y = (f + f')/2$. Then $x^2 - zy^2 = -z\prod_{\alpha=1}^{k-2}(z - P_\alpha^2)$. Let $P_{k-1} = P_k = 0$. Then (M, I) is biholomorphic to the minimal resolution of

$$x^2 - zy^2 = \frac{1}{-z}(\prod_{\alpha=1}^{k}(z-P_\alpha^2) - \prod_{\alpha=1}^{k}(-P_\alpha^2)) + 2\prod_{\alpha=1}^{k}(-iP_\alpha)y$$

and

$$\omega^+ = 4i\mathrm{d}\log f \wedge \mathrm{d}\sqrt{z} = i\mathrm{d}(\frac{1}{\sqrt{z}}\log(\frac{yz + \prod_{\alpha=1}^k(-iP_\alpha) + \sqrt{z}x}{yz + \prod_{\alpha=1}^k(-iP_\alpha) - \sqrt{z}x})) \wedge \mathrm{d}z.$$

Let $P = yz$ and $Q = x$. Then $fz = P + \sqrt{z}Q$ still holds.

In conclusion, we always have the correct biholomorphic type and correct ω^+. The only difference is how many P_α's equal to 0. Now we may vary $\zeta \neq \infty$. We can still get similar pictures. For $\zeta \neq 0$, we may use $\tilde{\zeta} = \zeta^{-1}$ instead. Then $\tilde{\omega} = \zeta^{-2}\omega$ and $\tilde{z} = \zeta^{-4}z$ are non-singular. So we can get $\tilde{f}$, $\tilde{f}'$ $\tilde{x}$, $\tilde{y}$, $\tilde{P}$, and $\tilde{Q}$ instead. The difference $\tilde{f}/f$ transfer as $\tilde{\rho}/\rho$ in the A_{2k-5} case. Therefore $\tilde{f}/f = e^{-\sqrt{z}/\zeta}\zeta^{-2k+4}$. So $(\tilde{P} + \sqrt{\tilde{z}}\tilde{Q}))/(P + \sqrt{z}Q) = e^{-\sqrt{z}/\zeta}\zeta^{-2k}$.

It's conventional to rescale the metric. Therefore, we actually have

$$(\tilde{P} + \sqrt{\tilde{z}}\tilde{Q}))/(P + \sqrt{z}Q) = e^{-2\sqrt{z}/\zeta}\zeta^{-2k}$$

as our transition function instead. In other words,

$$\begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix} = \zeta^{-2k} \begin{pmatrix} \cosh(2\sqrt{z}/\zeta) & -\sqrt{z}\sinh(2\sqrt{z}/\zeta) \\ -\zeta^2\sinh(2\sqrt{z}/\zeta)/\sqrt{z} & \zeta^2\cosh(2\sqrt{z}/\zeta) \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix}.$$

As before, $P_\alpha(\zeta)$ must be a degree two polynomial in ζ. When we look at the action of the real structure, it's easy to see that actually

$$P_\alpha(\zeta) = a_\alpha\zeta^2 + 2b_\alpha\zeta - \bar{a}_\alpha$$

for some $a_\alpha \in \mathbb{C}$ and $b_\alpha \in \mathbb{R}$. Moreover, the real structure τ must act by

$$\tau(\zeta, z, P, Q) = (\tilde{\zeta} = -\bar{\zeta}, \tilde{z} = \bar{z}, \tilde{P} = \bar{P}, \tilde{Q} = -\bar{Q}).$$

We can transfer those expressions into expressions in terms of x and y by the fact that $P = yz + \prod_\alpha(-iP_\alpha(\zeta))$, $Q = x$ and $\tilde{P} = \tilde{y}\tilde{z} + \prod_\alpha(-i\tilde{P}_\alpha(\tilde{\zeta}))$, $\tilde{Q} = \tilde{x}$.

It's well known [40] that the form ω and the real structure on the twistor space determine the metric on M. So M must be the Cherkis–Hitchin–Ivanov–Kapustin–Lindström–Roček metric. □

5.4 A Torelli-type theorem for ALF gravitational instantons

In this subsection we prove the Torelli-type theorem for ALF gravitational instantons as an analogy of Kronheimer's results [48] [49].

First of all, we can rescale the metric to make the scaling parameter $\mu = 1$.

In the ALF-A_k case, for each $\alpha \neq \beta$, π^{-1} of the segment connecting $\mathbf{x}_\alpha$ and $\mathbf{x}_\beta$ is a sphere $S_{\beta,-\alpha}$. They generate $H_2(M,\mathbb{Z})$. It's easy to see that they are the only roots, i.e. homology classes with self-intersection number -2. The simple roots can be chosen as $S_{2,-1}, S_{3,-2}, ..., S_{k+1,-k}$. They form an A_k root system.

By a simple calculation,

$$\int_{S_{\beta,-\alpha}} \omega = \int_{S_{\beta,-\alpha}} 4i \mathrm{d}\log\rho \wedge \mathrm{d}z = 8\pi \int_{\pi(S_{\beta,-\alpha})} \mathrm{d}z = 8\pi(P_\beta - P_\alpha).$$

So $(\int_{S_{\beta,-\alpha}} \omega^1, \int_{S_{\beta,-\alpha}} \omega^2, \int_{S_{\beta,-\alpha}} \omega^3)$ equals to $(b_\beta - b_\alpha, \mathrm{Re}(a_\beta - a_\alpha), \mathrm{Im}(a_\beta - a_\alpha))$ up to a constant multiple. Since the ALF hyperkähler structure is completely determined by the parameters $(a_\beta - a_\alpha, b_\beta - b_\alpha)$, it's also determined by three cohomology classes $[\omega^i]$.

The ALF-A_k gravitational instanton is singular if and only if there exist $\alpha \neq \beta$ such that $(a_\alpha, b_\alpha) = (a_\beta, b_\beta)$. It's equivalent to the vanishing of $[\omega^i]$ on some root.

The ALF-D_k case is similar. When $k \geq 2$, the roots $S_{\pm\beta,\pm\alpha}, \alpha \neq \beta$ generate $H_2(M,\mathbb{Z})$. The simple roots can be chosen as $S_{+2,+1}, S_{+2,-1}, S_{+3,-2}$, $S_{+4,-3}..., S_{+k,-(k-1)}$. They form a D_k root system. The integrals on them $(\int_{S_{\pm\beta,\pm\alpha}} \omega^1, \int_{S_{\pm\beta,\pm\alpha}} \omega^2, \int_{S_{\pm\beta,\pm\alpha}} \omega^3)$ are $(\pm b_\beta \pm b_\alpha, \mathrm{Re}(\pm a_\beta \pm a_\alpha), \mathrm{Im}(\pm a_\beta \pm a_\alpha))$ up to a constant multiple, too. The ALF-D_k gravitational instanton is singular if and only if there exist $\alpha \neq \beta$ such that $(a_\alpha, b_\alpha) = \pm(a_\beta, b_\beta)$. So the Torelli-type theorem also holds in this case.

When $k = 1$, $H_2(M,\mathbb{Z})$ is generated by $S_{+1,-1}$, a sphere with one ordinary double point. Its self-intersection number is 0. It's easy to see that the Torelli-type theorem holds, too.

When $k = 0$, $H_2(M,\mathbb{Z}) = 0$ and there is only one ALF-D_0 gravitational instanton. The Torelli-type theorem holds trivially.

6 ALG gravitational instantons

In this section, we will slightly modify Hein's result in [37] to get Theorem 1.10.

Let ω be any Kähler form on $\bar{M}$. Let a be the area of each regular fiber with respect to ω. Recall that for any section σ' of z on $\Delta^* = \{|z|^\beta \geq R\}$, Hein [37] wrote down some explicit formula of the semi-flat Calabi-Yau metric $\omega_{\mathrm{sf},a}[\sigma']$ on $M|_{\Delta^*}$ whose area of each regular fiber is also a:

Definition 6.1. ([37]) Using σ' as the zero section, $M|_{\Delta^*}$ is locally biholomorphic to

$$M|_U = (U \times \mathbb{C})/(z, v) \sim (z, v + m\tau_1(z) + n\tau_2(z))$$

for some holomorphic functions τ_1 and τ_2. So locally, $\omega^+ = g(z)\mathrm{d}z \wedge \mathrm{d}v$ for some holomorphic function $g : U \to \mathbb{C}$. Then locally

$$\omega_{\mathrm{sf},a}[\sigma'] = i|g|^2 \frac{\mathrm{Im}(\bar{\tau}_1\tau_2)}{a} \mathrm{d}z \wedge \mathrm{d}\bar{z} + \frac{i}{2} \frac{a}{\mathrm{Im}(\bar{\tau}_1\tau_2)} (\mathrm{d}v - \Gamma \mathrm{d}z)(\mathrm{d}\bar{v} - \bar{\Gamma}\mathrm{d}\bar{z}),$$

where

$$\Gamma(z, v) = \frac{1}{\mathrm{Im}(\bar{\tau}_1\tau_2)} (\mathrm{Im}(\bar{\tau}_1 v) \frac{\mathrm{d}\tau_2}{\mathrm{d}z} - \mathrm{Im}(\bar{\tau}_2 v) \frac{\mathrm{d}\tau_1}{\mathrm{d}z}).$$

It's easy to check that $\omega_{\mathrm{sf},a}[\sigma']$ is actually a globally well-defined form.

After that, the following theorem is essential:

Theorem 6.2. *There exist a real smooth polynomial growth function ϕ_1 on $M|_{\Delta^*}$ and a polynomial growth holomorphic section σ' of z over Δ^* such that $\omega_{\mathrm{sf},a}[\sigma'] = \omega + i\partial\bar{\partial}\phi_1$.*

Remark. Compared to Hein's Claim 1 in page 382 of [37], the key improvement in our paper is that both σ' and ϕ_1 grow at most polynomially.

Proof. As Hein did in [37], there exists a real smooth 1-form ζ on $M|_{\Delta^*}$ such that $\mathrm{d}\zeta = \omega_{\mathrm{sf},a}[\sigma] - \omega$. Choose the map F as in Theorem 3.31. By Cartan's formula, the homotopy between F and the identity map implies that $F^*\omega - \omega = \mathrm{d}\zeta_1$ and $F^*\omega_{\mathrm{sf},a}[\sigma] - \omega_{\mathrm{sf},a}[\sigma] = \mathrm{d}\zeta_2$ for some real polynomial growth 1-forms ζ_1 and ζ_2. However, $\mathrm{d}F^*\zeta = F^*\omega_{\mathrm{sf},a}[\sigma] - F^*\omega$ for some polynomial growth 1-form $F^*\zeta$. In conclusion, we can without loss of generality assume that ζ grows polynomially.

Using σ as the zero section, any section σ' of z can be written as $v = \sigma'(z)$ in local coordinates. Hein calculated that there exists a real 1-form $\tilde{\zeta}$ such that $\omega_{\mathrm{sf},a}[\sigma'] - \omega_{\mathrm{sf},a}[\sigma] = \mathrm{d}\tilde{\zeta}$ and the (0,1)-part $\tilde{\xi}$ of $\tilde{\zeta}$ can be written as

$$\tilde{\xi} = -\frac{i}{2}\frac{a}{\mathrm{Im}(\bar{\tau}_1\tau_2)}[\sigma'(z)(\mathrm{d}\bar{v} - \bar{\Gamma}(z,v)\mathrm{d}\bar{z}) - \frac{1}{2}\bar{\Gamma}(z,\sigma'(z))\mathrm{d}\bar{z}].$$

Choose σ' so that $\frac{i}{2}\frac{a}{\mathrm{Im}(\bar{\tau}_1\tau_2)}\sigma'$ equals to the average of the coefficient of $\mathrm{d}\bar{v}$ term of the (0,1)-part ξ of ζ on each fiber. Then σ' and $\tilde{\xi}$ grow polynomially. Moreover, the average of the coefficient of $\mathrm{d}\bar{v}$ term of $\xi + \tilde{\xi}$ on each fiber vanishes. So on each fiber, $\xi + \tilde{\xi}$ can be written as $i\bar{\partial}\phi_2$ by solving the $\bar{\partial}$-equation on each fiber. It's easy to see that ϕ_2 also grows polynomially. So the (0,1)-form $\xi + \tilde{\xi} - i\bar{\partial}\phi_2$ can be written as $f(z,v)\mathrm{d}\bar{z}$. However, it's $\bar{\partial}$-closed, so $f(z,v) = f(z)$. By solving the $\bar{\partial}$-equation on Δ^*, $\xi + \tilde{\xi} - i\bar{\partial}\phi_2 = i\bar{\partial}\phi_3$ for some polynomial growth function $\phi_3(z)$. In conclusion

$$\omega_{\mathrm{sf},a}[\sigma'] - \omega = \mathrm{d}[i\bar{\partial}(\phi_2 + \phi_3) - i\partial(\bar{\phi}_2 + \bar{\phi}_3)] = i\partial\bar{\partial}(\phi_2 + \bar{\phi}_2 + \phi_3 + \bar{\phi}_3).$$

□

Theorem 6.3. *There exists a real smooth polynomial growth function ϕ_4 such that $\omega + i\partial\bar{\partial}\phi_4$ is ALG and*

$$(\omega + i\partial\bar{\partial}\phi_4)^2 = \frac{1}{2}\omega^+ \wedge \bar{\omega}^+$$

Remark. Compared to Hein's Theorem 1.3 of [37], the key improvements in our paper are that ϕ_4 grows polynomially and that we obtain $\frac{1}{2}\omega^+ \wedge \bar{\omega}^+$ instead of $\frac{\alpha}{2}\omega^+ \wedge \bar{\omega}^+$ for large enough α.

Proof. To achieve this, we still introduce a real positive bump function b on $\mathbb{C}$ supported in $\{R \leq |z|^\beta \leq 4R\}$ such that $b = 1$ on $\{2R \leq |z|^\beta \leq 3R\}$. The involution with the Green function provides a real at most polynomial growth function ϕ_5 on $\mathbb{C}$ such that $i\partial\bar{\partial}\phi_5 = ib(z)\mathrm{d}z \wedge \mathrm{d}\bar{z}$.

Now let's look at the form $\omega + i\partial\bar{\partial}((1 - \chi(\frac{r}{R} - \frac{5}{2}))\phi_1)$. It equals to ω when $r \leq 2R$ and $\omega_{\mathrm{sf},a}[\sigma']$ when $r \geq 3R$. On the part $2R \leq r \leq 3R$, this form may not be positive. However, as Hein did in Claim 3 of [37],

$$\omega_t = \omega + i\partial\bar{\partial}((1 - \chi(\frac{r}{R} - \frac{5}{2}))\phi_1 + t\phi_5)$$

is positive for large enough t.

To achieve the integrability condition $\int_M(\omega_1^2 - \frac{1}{2}\omega^+ \wedge \bar{\omega}^+) = 0$, we start from choosing large enough R and t such that $\omega_{\mathrm{sf},a}[\sigma']$ is close enough to the standard ALG model $\frac{i}{2}\frac{a}{\mathrm{Im}\tau}(\mathrm{d}z^\beta \wedge \mathrm{d}\bar{z}^\beta + \mathrm{d}v \wedge \mathrm{d}\bar{v})$ and $\int_M(\omega_t^2 - \frac{1}{2}\omega^+ \wedge \bar{\omega}^+) > 0$. Then we consider

$$\omega_{s,t} = \omega_t - \frac{i}{4}\frac{a}{\mathrm{Im}\tau}(1 - \chi(\frac{r}{R} - 6))\chi(\frac{r}{R} - s)\beta^2|z|^{2\beta-2}\mathrm{d}z \wedge \mathrm{d}\bar{z}.$$

It's easy to see that for any $s \geq 5$, $\omega_{s,t}$ must be positive. What's more, since $\int_M(\omega_{s,t}^2 - \frac{1}{2}\omega^+ \wedge \bar{\omega}^+)$ decreases to negative infinity when s goes to infinity, by intermediate value theorem, there exists s such that the integrability condition is achieved. By the work of Tian-Yau [76], there exists a real smooth bounded function ϕ_6 such that $(\omega_{s,t} + i\partial\bar{\partial}\phi_6)^2 = \frac{1}{2}\omega^+ \wedge \bar{\omega}^+$. By Proposition 2.9 of [37], the solution $\omega_{s,t} + i\partial\bar{\partial}\phi_6$ is actually ALG. □

Thus, the first part of Theorem 1.10 has been proved. The second part is quite simple:

Theorem 6.4. *Suppose there exist two ALG metrics $\omega_j = \omega + i\partial\bar{\partial}\phi_j, j = 7, 8$, satisfying $\omega_7^2 = \omega_8^2 = \frac{1}{2}\omega^+ \wedge \bar{\omega}^+$, $|\nabla^m(\omega_7 - \omega_8)| = O(r^{-m-\delta})$ and $|\phi_j| = O(r^N)$ for all $j = 7, 8, m \geq 0$ and some $\delta, N > 0$. Then $\omega_7 = \omega_8$.*

Proof. It's easy to see that $\tilde{\omega} = \omega + i\partial\bar{\partial}(\phi_7 + \phi_8)/2$ also defines a Kähler metric which is asymptotic to the standard ALG model. Since

$$\tilde{\omega} \wedge i\partial\bar{\partial}(\phi_7 - \phi_8) = 0,$$

$\phi_7 - \phi_8$ is harmonic with respect to $\tilde{\omega}$. We can't directly apply Theorem 3.26 because $\tilde{\omega}$ may not be Ricci-flat. However, using Theorem 3.17, we can transfer the $\tilde{\omega}$-harmonic function to a h-harmonic function on E. By Theorem 3.16, it's asymptotic to a $\mathbb{T}^k$-invariant h-harmonic function on E. It can be reduced to a harmonic function on $\mathbb{R}^2 \setminus B_R$. So it's asymptotic to $a_n z^n + b_n \bar{z}^n$ for some constants $(a_n, b_n) \neq \mathbf{0}$ or $c \log|z|$ for some nonzero-constant c by Theorem 3.6. In the second case, the boundary term in the integration $\int_{B_R} \Delta_{\tilde{\omega}}(\phi_7 - \phi_8)$ will be non-zero. It's a contradiction. In the first case, the difference $\phi_7 - \phi_8 - a_n z^n + b_n \bar{z}^n$ has smaller order. Repeat the procedure until the order is smaller than 0. The maximal principle implies that the difference is 0. In other words,

$$\phi_7 - \phi_8 = \sum_{k=0}^{n} a_k z^k + b_k \bar{z}^k.$$

□

Theorem 6.5. *The third part of Theorem 1.10 holds.*

Proof. Let $(M, \omega_{\mathrm{ALG}}, \omega^2, \omega^3)$ be the ALG gravitational instanton in the third part of Theorem 1.10. Let a be the area of each regular fiber with respect to ω_{ALG}. Now pick a Kähler metric ω_0 on $\bar{M}$ whose area of each regular fiber is a. Then $\omega_{\mathrm{sf},a}[\sigma'] = \omega_0 + i\partial\bar{\partial}\phi_9$ for some holomorphic section σ' on $\{|z|^\beta \geq R\}$ and some real function ϕ_9. It's easy to see that Theorem 6.2 also holds for ω_{ALG}, i.e. $\omega_{\mathrm{sf},a}[\sigma''] = \omega_{\mathrm{ALG}} + i\partial\bar{\partial}\phi_{10}$ for some holomorphic section σ'' on $\{|z|^\beta \geq R\}$.

When D is of type I_0^*, II, III, or IV, i.e. $\beta \leq 1/2$, our goal is to show that the action $T(z, v) = (z, v + \sigma''(z) - \sigma'(z))$ as well as its inverse can be extended across D. If it's true, then

$$\omega_{\mathrm{ALG}} = \omega_{\mathrm{sf},a}[\sigma''] - i\partial\bar{\partial}\phi_{10} = (T^{-1})^*\omega_{\mathrm{sf},a}[\sigma'] - i\partial\bar{\partial}\phi_{10} = (T^{-1})^*\omega_0 - i\partial\bar{\partial}\phi_{11}.$$

So $\omega = \omega_{\mathrm{ALG}} + i\partial\bar{\partial}((1-\chi(\frac{r}{R}-\frac{5}{2}))\phi_{11} + t\phi_5)$ will be the required Kähler form on $\bar{M}$ for large enough t.

To understand the structure near D, we start from the elliptic surface over $\Delta = \{|\tilde{u}| \leq R^{-1}\}$ constructed by $(\Delta \times \mathbb{C})/(\tilde{u}, v) \sim (\tilde{u}, v + m\tau_1(\tilde{u}) + n\tau_2(\tilde{u}))$. Take the quotient by $(\tilde{u}, v) \sim (e^{2\pi i\beta}\tilde{u}, e^{2\pi i\beta}v)$, then there are several orbifold points in the central fiber. As Kodaira did in [45], those orbifold points can be resolved by replacing the neighborhoods by the non-singular models N_{+m} constructed in page 583 of [45]. Then blow down exceptional curves if they exist. $M|_{\Delta^*} \cup D$ is biholomorphic to such model by the relationship $z = u^{1/\beta} = \tilde{z}^{-1} = \tilde{u}^{-1/\beta}$.

In those coordinates, if T is given by $T(\tilde{u}, v) = (\tilde{u}, v + f(\tilde{u})\tilde{u})$, then by the proof of Theorem 6.2, $\frac{i}{2}\frac{af(\tilde{u})\tilde{u}}{\mathrm{Im}(\bar{\tau}_1\tau_2)}$ will be the average of the coefficient of $\mathrm{d}\bar{v}$ term of the (0,1)-part ξ of ζ on each fiber, where ζ is a real smooth polynomial growth 1-form satisfying $\mathrm{d}\zeta = \omega_{\mathrm{sf},a}[\sigma'] - \omega_{\mathrm{ALG}}$. By Theorem 1.3, the difference between the two ALG metrics ω_{ALG} and $\omega_{\mathrm{sf},a}[\sigma']$ is bounded by $|u|^{-2}$. By Theorem 3.17, $\zeta = (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\psi_1$ and $\omega_{\mathrm{sf},a}[\sigma'] - \omega_{\mathrm{ALG}} = (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)\psi_2$ for some ψ_1 and ψ_2 on $M|_{\Delta^*}$. So $(\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)(\mathrm{d}^*\mathrm{d}\psi_1 - \mathrm{d}^*\psi_2) = 0$. Therefore, the leading term of $\mathrm{d}^*\mathrm{d}\psi_1 - \mathrm{d}^*\psi_2$ must be the linear combinations of $u^\delta \mathrm{d}u$, $\bar{u}^\delta \mathrm{d}u$, $u^\delta \mathrm{d}v$, $\bar{u}^\delta \mathrm{d}v$ and their conjugates. However, $\mathrm{d}(\mathrm{d}^*\mathrm{d}\psi_1 - \mathrm{d}^*\psi_2) = \mathrm{d}^*\mathrm{d}\psi_2$ has small order. So if δ is large, the leading term of $\mathrm{d}^*\mathrm{d}\psi_1 - \mathrm{d}^*\psi_2$ must be the linear combinations of $u^\delta \mathrm{d}u$ and its conjugate. However, such kind of term can be written as the linear combinations of $\mathrm{d}z^m$ and its conjugate for some integer

m. We can then subtract the leading term from $d^*d\psi_1 - d^*\psi_2$ and repeat the process. Finally, it's easy to see that $f(\tilde{z}) = f(\tilde{u})$ is bounded by $|\tilde{z}|^{-\epsilon}$ for any small positive ϵ. By removal of singularity theorem of holomorphic functions on the punctured disc Δ^*, $f(\tilde{z})$ can be extended to a holomorphic function on Δ.

Therefore, the induced map of T on the resolution is holomorphic outside the central curves in N_{+m} and continuous across those curves. By removal of singularity theorem, it can be extended holomorphically. Then, in the blow down procedure, the induced map is holomorphic outside the blow down of the exceptional curves. By Hartog's theorem, it can be extended on $\bar{M}|_\Delta$. Similarly, T^{-1} can also be extended.

When D is of type II*, III*, IV*, i.e. $1/2 < \beta < 1$, the arguments above fail because the meromorphic function f may have a pole at $\{\tilde{z} = 0\}$ corresponding to the term $u^{1/\beta-2}du \wedge d\bar{v}$ in the difference $\omega_{\mathrm{sf},a}[\sigma'] - \omega_{\mathrm{ALG}}$. However, recall that in Section 3.9, we used the section σ as zero section to compactify M into $\bar{M}$. If we use the section $\sigma + \sigma'' - \sigma'$ instead, then we may get a different $\bar{M}$. For this new choice of $\bar{M}$, the form $\omega_{\mathrm{sf},a}[\sigma''] + i\partial\bar{\partial}\phi_{12} = \omega_{\mathrm{ALG}} + i\partial\bar{\partial}\phi_{13}$ is a smooth Kähler form on $\bar{M} \cap \{|\tilde{z}| \leq R^{-1/\beta}\}$ for some real smooth polynomial growth functions ϕ_{12}, ϕ_{13} on $M \cap \{|\tilde{z}| \leq R^{-1/\beta}\}$. So $\omega = \omega_{\mathrm{ALG}} + i\partial\bar{\partial}((1 - \chi(\frac{r}{R} - \frac{5}{2}))\phi_{13} + t\phi_5)$ will be the required Kähler form on the new choice of $\bar{M}$ for large enough t. □

7 ALH gravitational instantons

7.1 Gluing of ALH gravitational instantons

In this subsection, we will prove that the gluing of any ALH gravitational instanton with itself is a K3 surface. We learned this idea as well as some initial set-ups of the gluing construction [26] from the lecture of Sir Simon Donaldson in the spring of 2015 at Stony Brook University. We will also construct a counterexample of ALG Torelli Theorem when D is of type II*, III*, or IV*.

We will use the notations of Kovalev and Singer [47] in order to apply their results. Pick two copies of M. Define $t_1 = r$ on the first copy M_1. Define $t_2 = -r$ on the second copy M_2. For any gluing parameters (ρ, Θ) in $[R + 8, \infty) \times \mathbb{T}^3$, the gluing manifold $M_{\rho,\Theta}$ is defined by truncating the two manifolds at $t_j = \pm\rho$ and identifying the boundary points $(\rho, \theta) \in M_1$

with the points $(-\rho, \Theta-\theta) \in M_2$. On $M_{\rho,\Theta}$, the function t is defined by $t = t_1 - \rho = t_2 + \rho$. The picture can be found in page 10 of [47].

Our metric on $M_{\rho,\Theta}$ is slightly different from [47]. In fact, there are three Kähler forms ω^1, ω^2 and ω^3 on M. The closed forms

$$\omega^i - \omega^i_{\text{flat}} = a^i_j(r,\theta)\mathrm{d}r \wedge \mathrm{d}\theta^j + b^i_{jk}(r,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k$$

are very small on $\{\rho - 1 \leq r \leq \rho + 1\}$ by Theorem 1.3. Now define

$$\phi^i = [\int_\rho^r a^i_j(s,\theta)\mathrm{d}s]\mathrm{d}\theta^j.$$

Then

$$\begin{aligned}\mathrm{d}\phi^i &= a^i_j(r,\theta)\mathrm{d}r \wedge \mathrm{d}\theta^j + [\int_\rho^r \frac{\partial}{\partial\theta^k} a^i_j(s,\theta)\mathrm{d}s]\mathrm{d}\theta^k \wedge \mathrm{d}\theta^j \\ &= a^i_j(r,\theta)\mathrm{d}r \wedge \mathrm{d}\theta^j + [\int_\rho^r \frac{\partial}{\partial r} b^i_{jk}(s,\theta)\mathrm{d}s]\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k \\ &= a^i_j(r,\theta)\mathrm{d}r \wedge \mathrm{d}\theta^j + b^i_{jk}(r,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k - b^i_{jk}(\rho,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k.\end{aligned}$$

Therefore

$$\omega^i - \omega^i_{\text{flat}} = \mathrm{d}\phi^i + b^i_{jk}(\rho,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k$$

are cohomologous to the forms $b^i_{jk}(\rho,\theta)\mathrm{d}\theta^j \wedge \mathrm{d}\theta^k$ on $\mathbb{T}^3$.

Notice that any closed form on $\mathbb{T}^3$ can be cohomologous to a form with constant coefficients and any 2-form with constant coefficients is invariant under the map $\theta \to \Theta - \theta$. Therefore, when we glue $\{\rho - 1 \leq t_1 \leq \rho + 1\}$ on M_1 with $\{\rho - 1 \leq -t_2 \leq \rho + 1\}$ on M_2, the difference $\omega^i_{M_2} - \omega^i_{M_1} = \mathrm{d}\psi^i$ for some small ψ^i. Now define the forms $\omega^i_{\rho,\Theta}$ on $M_{\rho,\Theta}$ by

$$\omega^i_{\rho,\Theta} = \omega^i_{M_1} + \mathrm{d}((1-\chi(t))\psi^i) = \omega^i_{M_2} - \mathrm{d}(\chi(t)\psi^i).$$

Then $\omega^i_{\rho,\Theta}$ are three closed forms and $|\nabla^m(\omega^i_{\rho,\Theta} - \omega^i_{M_j})| = O(e^{-\lambda_1\rho})$ for all $m \geq 0$.

Now we can call the linear span of $\omega^i_{\rho,\Theta}$ the "self-dual" space. The orthogonal complement of the "self-dual" space under wedge product is called the "anti-self-dual" space. These two spaces determine a star operator. It's well known that the star operator determines a conformal class of metrics. The the conformal factor is determined by requiring the volume form to be $\frac{1}{2}(\det(\omega^i_{\rho,\Theta} \wedge \omega^j_{\rho,\Theta}))^{1/3}$. The resulting metric is called $g_{\rho,\Theta}$. It's slightly different from [47], but it satisfies all the properties needed in [47].

Now we define three operators on the space of self-dual 2-forms by

$$\begin{aligned} P_1\phi &= e^{-\delta t_1}(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(e^{\delta t_1}\phi),\\ P_2\phi &= e^{-\delta t_2}(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(e^{\delta t_2}\phi),\\ P_{\rho,\Theta}\phi &= e^{-\delta t}(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(e^{\delta t}\phi), \end{aligned}$$

where $\delta < \lambda_1/100$ is a small positive number. It's easy to prove the following theorem:

Theorem 7.1. *(1) $P_1, P_2, P_{\rho,\Theta}$ are Fredholm operators from $W^{k+2,2}$ to $W^{k,2}$ for any $k \geq 0$. In other words, the kernels are finite dimensional and the cokernels, i.e. the kernels of $P_1^*, P_2^*, P_{\rho,\Theta}^*$ are also finite dimensional. The range is the L^2-orthogonal complement of the cokernel. The operator from the L^2-orthogonal complement of the kernel to the range is an isomorphism.*

(2) $\ker P_1 = \mathrm{span}\{e^{-\delta t_1}\omega^i\}$, $\mathrm{coker} P_1 = \{0\}$,
$\ker P_2 = \{0\}$, $\mathrm{coker} P_1 = \mathrm{span}\{e^{\delta t_2}\omega^i\}$,
$\mathrm{span}\{e^{-\delta t}\omega^i_{\rho,\Theta}\} \subset \ker P_{\rho,\Theta}$, $\mathrm{span}\{e^{\delta t}\omega^i_{\rho,\Theta}\} \subset \mathrm{coker} P_{\rho,\Theta}$.

Proof. The first part was proved by Lockhart and McOwen in [53]. As for the second part, on any Kähler manifold with Kähler form ω, define the operator L by $L\phi = \phi \wedge \omega$. Then by Kähler identities, $[L, \bar\partial] = 0$ and $[L, \bar\partial^*] = -i\partial$. Therefore

$$[L, \mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*] = 2[L, \bar\partial^*\bar\partial+\bar\partial\bar\partial^*] = 2[L,\bar\partial^*]\bar\partial + 2\bar\partial[L,\bar\partial^*] = -2i(\partial\bar\partial+\bar\partial\partial) = 0.$$

In particular for any function f,

$$(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(f\omega) = (-\Delta f)\omega,$$

where $\Delta f = -(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)f$ is the ordinary Laplacian operator on functions. On hyperkähler manifolds M_1 and M_2, there are three Kähler structures I, J and K. Therefore,

$$\sum_{i=1}^{3}(\mathrm{d}^*\mathrm{d}+\mathrm{d}\mathrm{d}^*)(f_i\omega^i) = -\sum_{i=1}^{3}(\Delta f_i)\omega^i.$$

In other words, the Laplacian on the self-dual forms is exactly the Laplacian on the coefficients. On $M_{\rho,\Theta}$, even though the metric is not hyperkähler, $\omega^i_{\rho,\Theta}$ are still harmonic since they are closed self-dual forms. The second part follows directly from the two facts above. □

The most important result of [47] is the following theorem: (Proposition 4.2 in their paper)

Theorem 7.2. *There exists $\rho_* > 0$ such that for all $\rho \geq \rho_*$, the induced map $P''_{\rho,\Theta}$ from the L^2-orthogonal complement of* $\mathrm{span}\{\chi(t_1 - \rho/2)e^{-\delta t_1}\omega^i_{\rho,\Theta}\}$ *in $W^{k+2,2}$ to the L^2-orthogonal complement of* $\mathrm{span}\{\chi(-t_2 - \rho/2)e^{\delta t_2}\omega^i_{\rho,\Theta}\}$ *in $W^{k,2}$ is an isomorphism and the operator norm of $[P''_{\rho,\Theta}]^{-1}$ is bounded independent of ρ and Θ.*

It's easy to prove the following lemmas in functional analysis:

Lemma 7.3. *(1) Suppose $V = \mathrm{span}\{v_1, ...v_m\}$ is a finite dimensional subspace in $W^{k,2}$ for some $k \geq 0$. Let $V^\perp$ be the L^2-orthogonal complement of V in $W^{k,2}$, then $W^{k,2} = V \oplus V^\perp$ and*

$$||f + g||_{W^{k,2}} \leq ||f||_{W^{k,2}} + ||g||_{W^{k,2}} \leq (1 + 2C_1)||f + g||_{W^{k,2}}$$

for all $f \in V$ and $g \in V^\perp$, where $C_1 = \sup_{f \in V\setminus\{0\}} \frac{||f||_{W^{k,2}}}{||f||_{L^2}}$.

(2) Suppose $W = \mathrm{span}\{w_1, ...w_m\}$ is another subspace. If the matrix $A = \{a_{ij}\} = \{(w_i, v_j)_{L^2}\}$ is invertible with $A^{-1} = \{a^{ij}\}$, then the composition of the inclusion and the projection maps $P = \mathrm{Proj}_{W^\perp} \circ i : V^\perp \to W^\perp$ is an isomorphism. What's more

$$(1 + C_2)^{-1}||Pf||_{W^{k,2}} \leq ||f||_{W^{k,2}} \leq C_3||Pf||_{W^{k,2}},$$

where

$$C_2 = \sup_{f \in W\setminus\{0\}} \frac{||f||_{W^{k,2}}}{||f||_{L^2}}, C_3 = 1 + C(m)||A^{-1}|| \max ||v_i||_{L^2} \max ||w_j||_{W^{k,2}}.$$

Proof. The proof is quite obvious. The only thing to notice is that

$$P^{-1}f = f - \sum_{i,j=1}^{m} a^{ij}(f, v_i)_{L^2} w_j.$$

□

The following corollary of Theorem 7.2 and Lemma 7.3 provides the main estimate of this subsection:

Corollary 7.4. *There exists $\rho_* > 0$ such that for all $\rho \geq \rho_*$, the space of harmonic self-dual 2-forms on $M_{\rho,\Theta}$ equals to $\mathcal{H}^+ = \mathrm{span}\{\omega^i_{\rho,\Theta}\}$. The Laplacian operator $\Delta_{\rho,\theta} = \mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*$ from the L^2-orthogonal complement of $\mathcal{H}^+$ in $W^{k+2,2}(\Lambda^+)$ to the L^2-orthogonal complement of $\mathcal{H}^+$ in $W^{k,2}(\Lambda^+)$ is an isomorphism and the operator norm of $G_{\rho,\theta} = \Delta^{-1}_{\rho,\theta}$ is bounded by $Ce^{2\delta\rho}$ for some constant C independent of ρ and Θ.*

Proof. The isomorphism map in Theorem 7.2 can be decomposed into the composition of following maps:

$$(\chi(t_1 - \rho/2)e^{-\delta t_1}\omega^i_{\rho,\Theta})^\perp \to (e^{-\delta t}\omega^i_{\rho,\Theta})^\perp \to (\ker P_{\rho,\Theta})^\perp$$
$$\to (\mathrm{coker} P_{\rho,\Theta})^\perp \to (e^{\delta t}\omega^i_{\rho,\Theta})^\perp \to (\chi(-t_2 - \rho/2)e^{\delta t_2}\omega^i_{\rho,\Theta})^\perp.$$

The first and the fifth maps are isomorphisms by Lemma 7.3. Therefore, the second map must be injective and the fourth map must be surjective. In other words, $\ker P_{\rho,\Theta} = \mathrm{span}\{e^{-\delta t}\omega^i_{\rho,\Theta}\}$ and $\mathrm{coker} P_{\rho,\Theta} = \mathrm{span}\{e^{\delta t}\omega^i_{\rho,\Theta}\}$. So all the maps are actually isomorphisms. By Theorem 7.2 and Lemma 7.3, the operator norm of the inverse of the map $P_{\rho,\Theta} : (\ker P_{\rho,\Theta})^\perp \to (\mathrm{coker} P_{\rho,\Theta})^\perp$ is bounded. It's straight forward to switch this estimate into the estimate of the Laplacian operator. □

We are ready for the main theorem of this subsection:

Theorem 7.5. *Fix $k \geq 3$. For large enough ρ_* and any $\rho \geq \rho_*$, there exists a hyperkähler structure $\tilde{\omega}^i_{\rho,\Theta}$ on $M_{\rho,\Theta}$ such that $||\tilde{\omega}^i_{\rho,\Theta} - \omega^i_{\rho,\Theta}||_{W^{k,2}} \leq Ce^{(-\lambda_1+2\delta)\rho}$ for some constant C independent of ρ and Θ.*

Proof. Fix the volume form $V = \frac{1}{2}\det(\omega^i_{\rho,\Theta} \wedge \omega^j_{\rho,\Theta})^{1/3}$ on $M_{\rho,\Theta}$. When two symmetric matrices $A = \frac{\omega^i_{\rho,\Theta} \wedge \omega^j_{\rho,\Theta}}{2V}$ and B are close enough to the identity matrix, the equation $CAC^{\mathrm{T}} = B$ has a solution $C = B^{1/2}A^{-1/2}$. Define $F^i(B)$ by $F^i(B) = C_{ij}\omega^j_{\rho,\Theta}$. Then $F^i(B) \wedge F^j(B) = 2b_{ij}V$.

Recall that the map $G_{\rho,\Theta}$ on $(\mathcal{H}^+)^\perp \subset \Lambda^+$ satisfies

$$\begin{aligned}\psi &= (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)G_{\rho,\Theta}\psi = -(*\mathrm{d} * \mathrm{d} + \mathrm{d} * \mathrm{d}*)G_{\rho,\Theta}\psi \\ &= -(* + \mathrm{Id})\mathrm{d} * \mathrm{d}G_{\rho,\Theta}\psi = \mathrm{d}^+(-2 * \mathrm{d}G_{\rho,\Theta}\psi).\end{aligned}$$

So if $\phi^i \in \Lambda^1$ satisfy the equation

$$\phi^i = -2 * \mathrm{d}G_{\rho,\Theta}\mathrm{Proj}_{(\mathcal{H}^+)^\perp} F^i(\delta_{\alpha\beta} - \frac{\mathrm{d}^-\phi^\alpha \wedge \mathrm{d}^-\phi^\beta}{2V}),$$

then the closed forms $\tilde\omega^i_{\rho,\Theta}=\mathrm{d}\phi^i+\mathrm{Proj}_{\mathcal{H}^+}F^i(\delta_{\alpha\beta}-\frac{\mathrm{d}^-\phi^\alpha\wedge\mathrm{d}^-\phi^\beta}{2V})$ will satisfy the required equation $\tilde\omega^i_{\rho,\Theta}\wedge\tilde\omega^j_{\rho,\Theta}=2\delta_{ij}V$.

We will solve the equation by iterations

$$\phi^i_0=0,$$

$$\phi^i_{n+1}=-2*\mathrm{d}G_{\rho,\Theta}\mathrm{Proj}_{(\mathcal{H}^+)^\perp}F^i(\delta_{\alpha\beta}-\frac{\mathrm{d}^-\phi^\alpha_n\wedge\mathrm{d}^-\phi^\beta_n}{2V}).$$

Since $W^{k,2}$ embeds into C^0, if $||\phi^i_n||_{W^{k+1,2}}\le e^{-\lambda_1\rho/2}$ and $\rho\ge\rho_*$, then

$$||\frac{\omega^i_{\rho,\Theta}\wedge\omega^j_{\rho,\Theta}}{2V}-\delta_{ij}||_{C^0}+||\frac{\mathrm{d}^-\phi^i_n\wedge\mathrm{d}^-\phi^j_n}{2V}||_{C^0}\le Ce^{-\lambda_1\rho}$$

can be arbitrarily small if ρ_* is large. So

$$||\phi^i_{n+1}||_{W^{k+1,2}}\le Ce^{2\delta\rho}||\mathrm{Proj}_{(\mathcal{H}^+)^\perp}F^i(\delta_{\alpha\beta}-\frac{\mathrm{d}^-\phi^\alpha_n\wedge\mathrm{d}^-\phi^\beta_n}{2V})||_{W^{k,2}}\le Ce^{(-\lambda_1+2\delta)\rho}.$$

As long as ρ_* is large enough, the above estimate holds by induction. It follows that

$$||\phi^i_{n+2}-\phi^i_{n+1}||_{W^{k+1,2}}\le Ce^{(-\lambda_1+4\delta)\rho}||\phi^i_{n+1}-\phi^i_n||_{W^{k+1,2}}.$$

As long as ρ_* is large enough, $\phi^i=\lim_{n\to\infty}\phi^i_n$ will be the solution. □

Corollary 7.6. *For any ALH gravitational instanton M, $\int_M|Rm|^2=96\pi^2$.*

Proof. It's easy to deduce this conclusion from the well known fact that for K3 surface $M_{\rho,\Theta}$, $\int_{M_{\rho,\Theta}}|Rm(\tilde\omega^i_{\rho,\Theta})|^2=8\pi^2\chi(M_{\rho,\Theta})=192\pi^2$. □

Theorem 7.7. *Suppose α^i are three 2-forms satisfying the following conditions:*

(1)

$$\int_{M_{\rho,\Theta}}\alpha^2\wedge\alpha^3=\int_{M_{\rho,\Theta}}\alpha^3\wedge\alpha^1=\int_{M_{\rho,\Theta}}\alpha^1\wedge\alpha^2=0,$$

$$\int_{M_{\rho,\Theta}}\alpha^1\wedge\alpha^1=\int_{M_{\rho,\Theta}}\alpha^2\wedge\alpha^2=\int_{M_{\rho,\Theta}}\alpha^3\wedge\alpha^3;$$

(2)

$$||\alpha^i-\tilde\omega^i_{\rho,\Theta}||_{L^2}\le e^{-3\lambda_1\rho/4}.$$

Then there exists a hyperkähler structure ω^i on $M_{\rho,\Theta}$ such that $\omega^i\in[\alpha^i]$ and $||\omega^i-\tilde\omega^i_{\rho,\Theta}||_{W^{k,2}}\le Ce^{2\delta\rho}||\alpha^i-\tilde\omega^i_{\rho,\Theta}||_{L^2}$.

Proof. Using $\tilde{\omega}^i_{\rho,\Theta}$ as the background hyperkähler structure, we can choose harmonic representatives β^i from the cohomology classes $[\alpha^i]$. Therefore,

$$C^{-1}||\beta^i - \tilde{\omega}^i_{\rho,\Theta}||_{W^{k,2}} \leq ||\beta^i - \tilde{\omega}^i_{\rho,\Theta}||_{L^2} \leq ||\alpha^i - \tilde{\omega}^i_{\rho,\Theta}||_{L^2} \leq e^{-3\lambda_1\rho/4}.$$

After replacing $\omega^i_{\rho,\Theta}$ by β^i in the proof of Theorem 7.5, we can find a hyperkähler structure ω^i_0 such that $\mathrm{span}\{[\omega^i_0]\} = \mathrm{span}\{[\alpha^i]\}$. By the condition (1) of α^i, the hyperkähler structure ω^i can be chosen to be some rescaling and hyperkähler rotation of ω^i_0. □

Now we will use the method in this subsection to construct a counterexample of Torelli Theorem in ALG case:

Theorem 7.8. *When D is of type II*, III*, or IV*, there exist two different ALG gravitational instantons with same $[\omega^i]$.*

Proof. In Example 3.1 of [37], Hein explains how the pairs (IV,IV*) occur in rational elliptic surfaces $\bar{M}$ birational to $(\mathbb{P}^1 \times \mathbb{T}^2)/\Gamma$ with $\Gamma = \mathbb{Z}_3$. Let D be the fiber of type IV*. Then, the construction in [37] provides an ALG gravitational instanton ω^i on $M = \bar{M} \setminus D$. Moreover, the asymptotic rate is $2 + \frac{1}{\beta}$. In particular, $|\mathrm{Rm}| = O(r^{-\frac{1}{\beta}-4})$. There is a similar example when D is of type II* or III*.

By Theorem 3.13, there exists a harmonic (0,1) form h on M asymptotic to $\frac{1}{\frac{1}{\beta}-1}u^{1/\beta-1}\mathrm{d}\bar{v}$. So $\mathrm{d}(\mathrm{Re}h)$ is an exact harmonic form asymptotic to $\mathrm{Re}(u^{1/\beta-2}\mathrm{d}u\wedge\mathrm{d}\bar{v})$. Moreover, it's anti-self-dual because the coefficients of its self-dual part are decaying harmonic and thus 0.

Let's use the notations in Section 3. For example,

$$||\phi||_{H^2_\delta} = \sqrt{\int_M |\phi|^2 r^\delta \mathrm{dVol} + \int_M |\nabla\phi|^2 r^{\delta+2}\mathrm{dVol} + \int_M |\nabla^2\phi|^2 r^{\delta+4}\mathrm{dVol}}.$$

Then by Theorem 3.13, for $k \geq 5$ and small positive ϵ, there exists a map $G : H^k_{6-\frac{4}{\beta}-\epsilon}(\Lambda^+) \to H^{k+2}_{2-\frac{4}{\beta}-\epsilon}(\Lambda^+)$ such that $\psi = (\mathrm{d}^*\mathrm{d} + \mathrm{d}\mathrm{d}^*)G\psi$. We still define $F^i : \Gamma(\mathbb{R}^{3\times3}) \to \Lambda^+$ as in Theorem 7.5 so that $F^i(B)\wedge F^j(B) = 2b_{ij}V$. Then we do the iteration

$$\phi^1_0 = \phi^3_0 = 0, \phi^2_0 = t\mathrm{Re}h,$$

$$\phi^i_{n+1} = -2 * \mathrm{d}G(F^i(\delta_{\alpha\beta} - \frac{\mathrm{d}^-\phi^\alpha_n \wedge \mathrm{d}^-\phi^\beta_n}{2V}) - \omega^i) + \delta_{i2}t\mathrm{Re}h$$

When t is small enough, $(\phi_n^1, \phi_n^2 - t\mathrm{Re}h, \phi_n^3) \to (\phi^1, \phi^2 - t\mathrm{Re}h, \phi^3) \in H^{k+1}_{4-\frac{4}{\beta}-\epsilon}$. Then $\omega^i + \mathrm{d}\phi^i$ will be an ALG gravitational instanton. By direct computation, the curvature of the metric corresponding to $(J, \omega^2 + t\mathrm{d}(\mathrm{Re}h))$ is proportional to $r^{\frac{1}{\beta}-4}$. It's also true for the metric corresponding to $\omega^i + \mathrm{d}\phi^i$ because their difference is in $H^k_{6-\frac{4}{\beta}-\epsilon}$. In particular, the metric corresponding to $\omega^i + \mathrm{d}\phi^i$ is not isometric to the metric corresponding to ω^i. □

7.2 Uniqueness of ALH gravitational instantons

In this subsection, we will prove the uniqueness part of Theorem 1.11. We start from the understanding of the cross section:

Theorem 7.9. *The integrals of ω^i on the three faces determine the torus $\mathbb{T}^3$.*

Proof. On the flat model, recall that

$$\mathrm{d}r = I^*\mathrm{d}\theta^1 = J^*\mathrm{d}\theta^2 = K^*\mathrm{d}\theta^3.$$

So

$$\begin{aligned}\omega^1 &= \mathrm{d}r \wedge \mathrm{d}\theta^1 + \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3,\\ \omega^2 &= \mathrm{d}r \wedge \mathrm{d}\theta^2 + \mathrm{d}\theta^3 \wedge \mathrm{d}\theta^1,\\ \omega^3 &= \mathrm{d}r \wedge \mathrm{d}\theta^3 + \mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2.\end{aligned}$$

The torus $\mathbb{T}^3 = \mathbb{R}^3/\Lambda$ is determined by the lattice $\Lambda = \mathbb{Z}v_1 \oplus \mathbb{Z}v_2 \oplus \mathbb{Z}v_3$. Let v_{ij} be the $\frac{\partial}{\partial\theta^j}$ components of v_i. Then

$$\begin{pmatrix} f_{123} & f_{131} & f_{112} \\ f_{223} & f_{231} & f_{212} \\ f_{323} & f_{331} & f_{312} \end{pmatrix} = \begin{pmatrix} v_{22}v_{33} - v_{23}v_{32} & v_{32}v_{13} - v_{33}v_{12} & v_{12}v_{23} - v_{13}v_{22} \\ v_{23}v_{31} - v_{21}v_{33} & v_{33}v_{11} - v_{31}v_{13} & v_{13}v_{21} - v_{11}v_{23} \\ v_{21}v_{32} - v_{22}v_{31} & v_{31}v_{12} - v_{32}v_{11} & v_{11}v_{22} - v_{12}v_{21} \end{pmatrix}$$

is exactly the adjunct matrix $\mathrm{adj}(A)$ of

$$A = \begin{pmatrix} v_{11} & v_{12} & v_{13} \\ v_{21} & v_{22} & v_{23} \\ v_{31} & v_{32} & v_{33} \end{pmatrix}.$$

Since $\mathrm{adj}(A)A = \det(A)I$, it's easy to see that $\det(\mathrm{adj}(A)) = (\det(A))^2$. Thus, $A = (\det(\mathrm{adj}(A)))^{-1/2}\mathrm{adj}(\mathrm{adj}(A))$ is determined by $\mathrm{adj}(A)$. On M, the hyperkähler structure is asymptotic to the flat model. So we can get the same conclusion. □

Theorem 7.10. *ALH gravitational instantons are uniquely determined by their three Kähler classes $[\omega^i]$ up to tri-holomorphic isometry which induces identity on $H_2(M,\mathbb{Z})$.*

Proof. Suppose two ALH hyperkähler structures $\omega^{k,i}, k = 1,2$ on M satisfy $[\omega^{1,i}] = [\omega^{2,i}]$. By the results in the previous subsection, we have two families of K3 surfaces $(M_{\rho_k,\Theta_k}, \tilde{\omega}^{k,i}_{\rho_k,\Theta_k})$. To understand the relationship between M and M_{ρ_k,Θ_k}, let's start from the flat orbifold $(\mathbb{R}\times\mathbb{T}^3)/\mathbb{Z}_2$. Take two copies of it. On the first copy, define $t_1 = r$. On the second copy, define $t_2 = -r$. Now we glue them by truncating the two manifolds at $t_j = \pm\rho$ and identifying the boundary points (ρ,θ) with the points $(-\rho,\Theta-\theta)$. An alternating way to describe the gluing is to start from $[0,2\rho]\times\mathbb{T}^3$, and then identify $(0,\theta)$ with $(0,-\theta)$ and identify $(2\rho,\theta)$ with $(2\rho,2\Theta-\theta)$. Let $\mathbb{T}^4 = (\mathbb{R}\times\mathbb{T}^3)/\mathbb{Z}(4\rho,2\Theta)$. Then it's easy to see that the gluing is actually the orbifold $\mathbb{T}^4/\mathbb{Z}_2$.

The resolution of this picture provides the topological picture of the construction of M_{ρ_k,Θ_k}. The second homology group $H_2(M_{\rho_k,\Theta_k},\mathbb{R}) = \mathbb{R}^{22}$ is generated by 16 curves Σ_α corresponding to 16 orbifold points, 3 faces $F_{\alpha\beta}$ spanned by v_α and v_β and 3 faces F_α spanned by $(4\rho,2\Theta)$ and $(0,v_\alpha)$. Any hyperkähler structure ω^i on the K3 surface determines 48 integrals $c_{i\alpha}$ on Σ_α, 9 integrals $f_{i\alpha\beta}$ on $F_{\alpha\beta}$ and 9 integrals $f_{i\alpha}$ on F_α. The integrability condition $\int_M \omega^i\wedge\omega^j = 2\delta_{ij}V$ is equivalent to

$$-\frac{1}{2}\sum_{\alpha=1}^{16} c_{i\alpha}c_{j\alpha} + \frac{1}{2}\sum_{(\alpha,\beta,\gamma)=(1,2,3),(2,3,1),(3,1,2)} f_{i\alpha}f_{j\beta\gamma} + f_{j\alpha}f_{i\beta\gamma} = 2\delta_{ij}V.$$

If $c_{i\alpha}$, $f_{i\alpha\beta}$ are given, it's a rank 5 linear system in 9 variables $f_{i\alpha}$.

By the construction of $\tilde{\omega}^{k,i}_{\rho_k,\Theta_k}$ on M_{ρ_k,Θ_k}, the differences $c^{1,\rho_1,\Theta_1}_{i\alpha} - c^{2,\rho_2,\Theta_2}_{i\alpha}$ and $f^{1,\rho_1,\Theta_1}_{i\alpha\beta} - f^{2,\rho_2,\Theta_2}_{i\alpha\beta}$ are all bounded by $Ce^{(-\lambda_1+2\delta)\rho_1} + Ce^{(-\lambda_1+2\delta)\rho_2}$ for large enough ρ_k. However $f^{1,\rho_1,\Theta_1}_{i\alpha} - f^{2,\rho_2,\Theta_2}_{i\alpha}$ may be very large. Fortunately, we are free to change the 8 parameters ρ_k,Θ_k. When ρ_k and Θ_k are changed by adding $\delta\rho_k$ and $\delta\Theta_k$, the integrals $f^{k,\rho_k,\Theta_k}_{i\alpha}$ are changed by adding the almost linear terms $L(\delta\rho_k,\delta\Theta_k) + O(e^{(-\lambda_1+3\delta)\rho_1} + e^{(-\lambda_1+3\delta)\rho_2})$, where

$$L(\delta\rho_k,\delta\Theta_k) = 4\delta\rho_k(v_\alpha,\frac{\partial}{\partial\theta^i}) + 2(\delta\Theta_k\times v_\alpha,\frac{\partial}{\partial\theta^i})$$

are determined by the cross section $\mathbb{R}^3/(\mathbb{Z}v_1\oplus\mathbb{Z}v_2\oplus\mathbb{Z}v_3)$, $\delta\rho_k$ and $\delta\Theta_k$. The image of L is exactly the linear space

$$\{(f_{i\alpha})|\exists C \text{s.t.} \sum_{(\alpha,\beta,\gamma)=(1,2,3),(2,3,1),(3,1,2)} f_{i\alpha}f_{j\beta\gamma} + f_{j\alpha}f_{i\beta\gamma} = 2\delta_{ij}C\},$$

where $f_{i\alpha\beta} = (v_\alpha \times v_\beta, \frac{\partial}{\partial\theta^i})$. Therefore, after increasing the gluing parameters ρ_1 or ρ_2 and changing the parameters Θ_k,

$$\min_{\phi \in [\tilde{\omega}^{1,i}_{\rho_1,\Theta_1} - \tilde{\omega}^{2,i}_{\rho_2,\Theta_2}]} ||\phi||_{L^2} \leq Ce^{(-\lambda_1+4\delta)\rho_1} + Ce^{(-\lambda_1+4\delta)\rho_2}.$$

By Theorem 7.7, there exists a hyperkähler structure ω^i on M_{ρ_2,Θ_2} such that $[\omega^i] = [\tilde{\omega}^{1,i}_{\rho_1,\Theta_1}]$ and $||\omega^i - \tilde{\omega}^{2,i}_{\rho_2,\Theta_2}||_{W^{k,2}} \leq Ce^{(-\lambda_1+6\delta)\rho_1} + Ce^{(-\lambda_1+6\delta)\rho_2}$. By Theorem 1.12, ω^i and $\tilde{\omega}^{1,i}_{\rho_1,\Theta_1}$ are tri-holomorphically isometric to each other. Moreover the isometry induces identity on $H_2(M_{\rho_k,\Theta_k}, \mathbb{Z})$. Notice that the long neck regions are almost flat but by Corollary 7.6, the compact parts are not flat. So all the isometrics must map compact parts to compact parts. In particular, we can apply the Arzela-Ascoli theorem and the diagonal argument to get a limiting tri-holomorphic isometry on the original manifold M which induces identity on $H_2(M, \mathbb{Z})$ when ρ_k go to infinity. □

Theorem 7.11. *The Kähler classes* $[\omega^i]$ *satisfy the two conditions in Theorem 1.11.*

Proof. The first condition is a trivial consequence of $\det(\text{adj}(A)) = (\det(A))^2$ and $\det(A) \neq 0$ in the proof of Theorem 7.9. As for the second condition, any ALH gravitational instanton M can be glued with itself to obtain a K3 surface. By Theorem 7.7, we can modify the hyperkähler metric on the K3 surface so that the integrals of ω^i on the 11 cycles are unchanged in the gluing process. For any $[\Sigma] \in H_2(M, \mathbb{Z})$ such that $[\Sigma]^2 = -2$, we can find a corresponding element in the second homology group of the K3 surface. By Theorem 1.12, there exists i such that $[\omega^i][\Sigma] \neq 0$. Since the integrals of ω^i on the K3 surface are the same as the integrals on M, the second condition must be satisfied. □

7.3 Existence of ALH gravitational instantons

In this subsection, we will use the continuity method to prove the existence part of Theorem 1.11. Given any three classes $[\alpha_1^i]$ satisfying two conditions in Theorem 1.11, the cross section $\mathbb{T}^3$ is determined by Theorem 7.9. By the work of Biquard and Minerbe [8], there exists an ALH hyperkähler structure ω_0^i on $(\mathbb{R} \times \mathbb{T}^3)/\mathbb{Z}_2$. Now we are going to connect $[\alpha_1^i]$ with $[\alpha_0^i] = [\omega_0^i]$. We require that along the path, the cross section $\mathbb{T}^3$, i.e. the integrals on the faces F_{jk} are invariant.

We already know that for any $k = 0, 1$, and any $[\Sigma] \in H_2(M, \mathbb{Z})$ with $[\Sigma]^2 = -2$, there exists $i \in \{1, 2, 3\}$ with $[\alpha_k^i][\Sigma] \neq 0$. After a hyperkähler rotation, we can assume that $[\alpha_k^i][\Sigma] \neq 0$ for any $k = 0, 1$, any $i = 1, 2, 3$ and any $[\Sigma] \in H_2(M, \mathbb{Z})$ with $[\Sigma]^2 = -2$.

Now we can connect $[\alpha_0^i]$ with $[\alpha_1^i]$ by several pieces of segments. Along each segment, two of $[\alpha^i]$ are fixed while the remaining one is varying. We require that the actions of the two fixed $[\alpha^i]$ on any $[\Sigma] \in H_2(M, \mathbb{Z})$ with $[\Sigma]^2 = -2$ are nonzero. Therefore along the path, the two conditions of Theorem 1.11 are always satisfied.

So we only need to consider each segment. Without loss of generality, we can assume that there is only one segment and $[\alpha^2]$, $[\alpha^3]$ are fixed along the segment. Actually, we can assume that I, ω^2 and ω^3 are invariant along the continuity path. Only $[\alpha^1]$, i.e. the I- Kähler class is varying. We denote the original $\omega_0^1 \in [\alpha_0^1]$ by ω_0. We will use it as the background metric.

By Proposition 6.16 of [56], the second cohomology group $H^2(M, \mathbb{R})$ is naturally isomorphic to the space of bounded harmonic forms which are asymptotic to the linear combinations of $\mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3$, $\mathrm{d}\theta^3 \wedge \mathrm{d}\theta^1$ and $\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2$ with respect to ω_0. We only care about the forms whose integrals on F_{jk} are 0. Such kind of forms must decay exponentially. By the calculation in Theorem 7.1 and the maximal principle, the self-dual part of any decaying harmonic form vanishes. It's well known that any anti-self-dual form must be (1,1).

Thus, we can add linear combinations of those exponential decay anti-self-dual harmonic forms to change the Kähler class. However, the integrability condition $\int_M ((\alpha^1)^2 - (\omega^2)^2) = 0$ may not be satisfied. Fortunately, there is an exponential decay exact form $\mathrm{d}((1 - \chi(r - R - 2))I^* \mathrm{d}r)$ on M. Moreover, it's (1,1) since in local coordinates

$$\mathrm{d}((1 - \chi(r - R - 2))I^* \mathrm{d}r) = \mathrm{d}((1 - \chi(r - R - 2))(ir_j \mathrm{d}z^j - ir_{\bar{j}} \mathrm{d}\bar{z}^j))$$

$$= -2i(1 - \chi(r - R - 2))r_{j\bar{k}} \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k + 2i\chi'(r - R - 2)r_j r_{\bar{k}} \mathrm{d}z^j \wedge \mathrm{d}\bar{z}^k.$$

If we add this term with α^1, then

$$\int_M (\alpha^1 + a\mathrm{d}((1 - \chi(r - R - 2))I^* \mathrm{d}r))^2 - (\alpha^1)^2 = 2a \lim_{R \to \infty} \int_{r=R} I^* \mathrm{d}r \wedge \alpha^1.$$

The integral $\int_{r=R} I^* \mathrm{d}r \wedge \alpha^1$ on M converges to the term $\int_{\mathbb{T}^3} -\mathrm{d}\theta^1 \wedge \mathrm{d}\theta^2 \wedge \mathrm{d}\theta^3$ on the flat model, which is non-zero. So we can choose a suitable a to achieve the integrability condition. We call the resulting (1,1) form α_t. It satisfies the following conditions:

(1) For any $m \geq 0$, $||e^{\lambda_1 r}\nabla^m_{\omega_0}(\alpha_t - \alpha_T)||_{C^0}$ converges to 0 when t goes to T. In particular, $||e^{\lambda_1 r}\nabla^m_{\omega_0}(\alpha_t - \omega_0)||_{C^0}$ is uniformly bounded.

(2) $\int_M(\alpha_t^2 - \omega_0^2) = 0$.

Remark. α_t is positive in far enough region. However, it may not be positive in the compact part. That's the reason why the geometric existence part of [35] fails.

Now define I as the set

$$\{t \in [0,1] | \exists \phi_t s.t. \forall m \geq 0, |\nabla^m_{\omega_0}\phi_t| = O(e^{-\lambda_1 r}), \omega_t = \alpha_t + i\partial\bar{\partial}\phi_t > 0, \omega_t^2 = \omega_0^2\}.$$

It's trivial that $0 \in I$.

Theorem 7.12. *I is open*

Proof. Suppose $T \in I$, then as long as t is close enough to T, $\alpha_t + i\partial\bar{\partial}\phi_T$ is positive. It satisfies the integrability condition

$$\int_M((\alpha_t + i\partial\bar{\partial}\phi_T)^2 - \omega_0^2) = \int_M((\alpha_t + i\partial\bar{\partial}\phi_T)^2 - \alpha_t^2) + \int_M(\alpha_t^2 - \omega_0^2) = 0.$$

By Theorem 4.1 of [35], $(\alpha_t + i\partial\bar{\partial}\phi_T + i\partial\bar{\partial}\phi)^2 = \omega_0^2$ has a solution ϕ. So $t \in I$ with $\phi_t = \phi_T + \phi$. □

Now we are going to show that I is closed. Assume that $\{t_i\} \in I$ converge to T. To make the notation simpler, we will use α_i, ω_i and ϕ_i to denote α_{t_i}, ω_{t_i} and ϕ_{t_i}.

We start from an estimate:

Theorem 7.13.

$$\int_M(\mathrm{tr}_{\omega_0}\omega_i - 2)\frac{\omega_0^2}{2} \leq C.$$

Moreover

$$\int_M(\mathrm{tr}_{\omega_j}\omega_i - 2)\frac{\omega_j^2}{2} \to 0$$

as $i, j \to \infty$.

Proof.

$$\int_M (\mathrm{tr}_{\omega_0}\omega_i - 2)\frac{\omega_0^2}{2} = \int_M \omega_0 \wedge \omega_i - \omega_0^2 = \int_M \omega_0 \wedge (\alpha_i - \omega_0) \leq C.$$

Moreover

$$\int_M (\mathrm{tr}_{\omega_j}\omega_i - 2)\frac{\omega_j^2}{2} = \int_M \omega_j \wedge \omega_i - \omega_j^2 = \int_M \alpha_j \wedge (\alpha_i - \alpha_j) \to 0$$

as $i, j \to \infty$. □

Remark. By mean inequality, both $\mathrm{tr}_{\omega_0}\omega_i - 2$ and $\mathrm{tr}_{\omega_j}\omega_i - 2$ are non-negative since $\omega_0^2 = \omega_i^2 = \omega_j^2$.

Theorem 7.14. *Let U_N be the sets $\{N \leq r \leq N+1\}$ in the sense of ω_0. Then for all large enough N, there exist subsets $V_{Ni} \subset U_N$ such that the volume $\mathrm{Vol}(V_{Ni}) \geq \mathrm{Vol}(U_N)/2 \geq C$ and for any $y_1, y_2 \in V_{Ni}$, $d_{\omega_i}(y_1, y_2) \leq C_1$.*

Proof. It was proved by Demailly, Peternell and Schneider as Lemma 1.3 of [25] from the bound in Theorem 7.13. □

By the volume comparison theorem on Ricci flat manifolds, if we pick any point $p_{Ni} \in V_{Ni}$, then the volume of radius R ball centered at p_{Ni} in the sense of ω_i has a uniform lower bound depending on R.

Theorem 7.15. *For any fixed number R, the ω_i-curvature in $B_{\omega_i}(p_{Ni}, R)$ is uniformly bounded. Moreover, the ω_i-holomorphic radius in $B_{\omega_i}(p_{Ni}, R)$ is uniformly bounded below.*

Proof. Suppose on the contrary, the ω_i-curvature goes to infinity. Then we can rescale the metric so that the largest curvature equals to 1. By Theorem 4.7 of [18], the volume lower bound and the curvature bound imply the lower bound on the injectivity radius. Then, by Lemma 4.3 of [68], the holomorphic radius has a lower bound. By Page 483 of [1], the bound on the L^2-norm of curvature, the lower bound on the volume and the harmonic radius imply that the rescaled metric converges to an Einstein ALE space M_∞. Replacing the harmonic radius by the holomorphic radius, we can show that M_∞ is actually Kähler. Moreover, before taking limit, the manifold has a parallel holomorphic symplectic form $\omega^2 + i\omega^3$. Thus, on M_∞, there exists a parallel holomorphic symplectic form, too. In other words, M_∞ is actually hyperkähler.

By Bando-Kasue-Nakajima [5] and Kronheimer [48] [49], the (non-flat) ALE-gravitational instanton M_∞ contains a curve Σ_∞ with self intersection number -2. Before rescaling, the integrals of ω_i, ω^2 and ω^3 on Σ_i converge to 0.

Recall that $H_2(M,\mathbb{R})$ is generated by 8 curves Σ_α and 3 faces F_{23}, F_{31}, F_{12}. In fact, similar to Section B of [73], any element in $H_2(M,\mathbb{Z})$ can be represented by half integral linear combinations of Σ_α and $F_{\alpha\beta}$. Let

$$[\Sigma_i] = \frac{1}{2}\sum(m_{i\alpha}[\Sigma_\alpha] + m_{i\alpha\beta}[F_{\alpha\beta}]).$$

Then $[\Sigma_i]^2 = -2 = \frac{-2}{4}\sum m_{i\alpha}^2$. So there are only finitely many possibilities of $m_{i\alpha}$. By condition (1) of Theorem 1.11, the actions of $\lim_{i\to\infty}[\omega_i]$, $[\omega^2]$ and $[\omega^3]$ on $F_{\alpha\beta}$ are linearly independent. Since the integrals of ω_i, ω^2 and ω^3 on Σ_i converge to 0, we know that $m_{i\alpha\beta}$ also has a uniform bound. In other words, the homology class $[\Sigma_i]$ only has finitely many possibilities. Taking a subsequence where the homology class of Σ_i are same, we obtain a contradiction to the condition (2) of Theorem 1.11.

We've obtained a bound on the curvature. Theorem 4.7 of [18] and Lemma 4.3 of [68] now provide a lower bound on the holomorphic radius. □

Let D_0 be the upper bound on the diameter of U_N with respect to ω_0. We are interested in the function $e(t_i) = \mathrm{tr}_{\omega_i}\omega_0 = \mathrm{tr}_{\omega_0}\omega_i$ on $B_{\omega_i}(p_{Ni}, 10D_0)$. We start from a theorem:

Theorem 7.16. *There exists a constant C_2 such that if*

$$\gamma^{-2}\int_{B_{\omega_i}(p,\gamma)} e(t_i) \le C_2$$

for some ball $B_{\omega_i}(p,\gamma) \subset B_{\omega_i}(p_{Ni}, 10D_0)$, then

$$\sup_{\sigma\in[0,\frac{2}{3}\gamma]} \sigma^2 \sup_{B_{\omega_i}(p,\frac{2}{3}\gamma-\sigma)} e(t_i) \le \frac{1}{C_2}\gamma^{-2}\int_{B_{\omega_i}(p,\gamma)} e(t_i) \le 1.$$

Proof. It's well known that in the $C^{1,\alpha}$-holomorphic radius, there are higher derivative bounds automatically. Thus, the constant in Proposition 2.1 of [68] is uniform. (There are several errors in [68]. After correcting them, we can only get "$\frac{2}{3}\gamma$" instead of "γ" in the statement) □

Theorem 7.17. *For each i, there exists a set $A_i \subset \{0, 1, 2...\}$ such that for all $N \notin A_i$, $\sup_{B_{\omega_i}(p_{Ni}, 4D_0)} e(t_i) \leq 2.5$ and the number of elements in A_i is bounded.*

Proof. For any point $p \in M$, the number of N such that $p \in B_{\omega_i}(p_{Ni}, 10D_0)$ is bounded. Actually, by Theorem 7.14, for all such N, the set V_{Ni} is contained in $B_{\omega_i}(p, 10D_0 + C_1)$. The volume of $B_{\omega_i}(p, 10D_0 + C_1)$ has an upper bound by volume comparison, while the volumes of the disjoint sets V_{Ni} have a lower bound.

Therefore, by Theorem 7.13

$$\sum_N \int_{B_{\omega_i}(p_{Ni}, 10D_0)} (e(t_i) - 2) \leq C.$$

Let γ be a constant smaller than D_0 such that $\gamma^2 \leq \frac{C_2}{2\pi^2}$. Then by the volume comparison theorem, the volume of $B(R)$ on a Ricci flat 4-manifold is bounded by $\frac{\pi^2}{2}R^4$. So

$$\int_{B_{\omega_i}(p,\gamma)} 2 = 2\mathrm{Vol}(B_{\omega_i}(p, \gamma)) \leq \frac{C_2}{2}\gamma^2.$$

Therefore, as long as

$$\int_{B_{\omega_i}(p_{Ni}, 10D_0)} (e(t_i) - 2) \leq \frac{C_2}{2}\gamma^2,$$

we can get

$$\int_{B_{\omega_i}(p,\gamma)} e(t_i) \leq C_2\gamma^2$$

for all $p \in B_{\omega_i}(p_{Ni}, 8D_0)$. So $e(t_i) = \mathrm{tr}_{\omega_i}\omega_0 \leq \frac{9}{4}\gamma^{-2}$ in $B_{\omega_i}(p_{Ni}, 8D_0)$ by Theorem 7.16. In particular, $B_{\omega_i}(p_{Ni}, 8D_0) \subset B_{\omega_0}(p_{Ni}, 18\gamma^{-2}D_0)$.

For any $\epsilon \leq \frac{C_2}{2}\gamma^2$, let $A_{i,\epsilon}$ denote the set of N such that the integral $\int_{B_{\omega_i}(p_{Ni}, 10D_0)}(e(t_i) - 2) > \epsilon$ or $\sup_{B_{\omega_0}(p_{Ni}, 18\gamma^{-2}D_0)} |Rm(\omega_0)| > \epsilon$. Then the number of elements in $A_{i,\epsilon}$ is bounded. For all $N \notin A_{i,\epsilon}$, it's well known [68] that

$$-\Delta_{\omega_i} e(t_i) \leq C|Rm(\omega_0)|e(t_i)^2 \leq C|Rm(\omega_0)|$$

in $B_{\omega_i}(p_{Ni}, 8D_0)$. So both $\sup_{B_{\omega_i}(p_{Ni}, 8D_0)} -\Delta_{\omega_i} e(t_i)$ and $\int_{B_{\omega_i}(p_{Ni}, 8D_0)}(e(t_i) - 2)$ are bounded by $C\epsilon$. By Theorem 9.20 of [31], $\sup_{B_{\omega_i}(p_{Ni}, 4D_0)}(e(t_i) - 2) \leq C\epsilon$ for all $N \notin A_{i,\epsilon}$. After a suitable choice of ϵ, we can make it smaller than 1/2. □

Lemma 7.18. *There exists a constant R such that $M \subset \cup_{N \notin A_i} \overline{B_{\omega_i}(p_{Ni}, R)}$.*

Proof. For all $N \notin A_i$, $\frac{1}{2}\omega_i \le \omega_0 \le 2\omega_i$ in $B_{\omega_i}(p_{Ni}, 4D_0)$ by Theorem 7.17. In particular,

$$U_N \subset \overline{B_{\omega_0}(p_{Ni}, D_0)} \subset B_{\omega_i}(p_{Ni}, 4D_0).$$

Let $U_{A_i} = \cup_{N \in A_i} U_N$. Suppose $R = \sup_{q \in U_{A_i}} \inf_{N \notin A_i} d_{\omega_i}(p_{Ni}, q)$ is achieved by q_i and N_i. Then we will use the argument similar to Theorem 3.1 of [69] and Theorem 3.1 of [78] proved by applying Theorem I.4.1 of [72]. Actually, if $R > 10D_0$, by Theorem 7.17, it's easy to see that $B_{\omega_i}(q_i, R - 10D_0) \subset U_{A_i}$ and $U_{N_i} \subset B_{\omega_i}(q_i, R + 10D_0) \setminus B_{\omega_i}(q_i, R - 10D_0)$. Since the volume of U_{A_i} is bounded from above and the volume of U_{N_i} is bounded from below, it's easy to get a bound on R by the Bishop-Gromov volume comparison theorem. □

Theorem 7.19. *$e(t_i) = \mathrm{tr}_{\omega_0}\omega_i = \mathrm{tr}_{\omega_i}\omega_0$ is uniformly bounded on M.*

Proof. By Theorem 7.15 and Lemma 7.18, the ω_i-holomorphic radius is bounded from below. So the constant in Theorem 7.16 is uniform if we replace ω_0 by ω_j in the statement of Theorem 7.16. By Theorem 7.13,

$$\int_M (\mathrm{tr}_{\omega_j}\omega_i - 2)\frac{\omega_j^2}{2} \to 0$$

as $i, j \to \infty$. So for large enough i and j, $\mathrm{tr}_{\omega_j}\omega_i$ is uniformly bounded on M. Fix j and let i go to infinity. Since $C_j^{-1}\omega_0 \le \omega_j \le C_j\omega_0$, the bound on $\mathrm{tr}_{\omega_j}\omega_i$ automatically implies a bound on $\mathrm{tr}_{\omega_0}\omega_i$. □

Now we are ready to use the arguments in [35] to prove Theorem 1.11. Let N be a large constant such that when $r \ge N$, $\frac{1}{2}\omega_0 \le \alpha_i \le 2\omega_0$. We start from a theorem which can be easily deduced from Proposition 4.21 of [35]:

Theorem 7.20. *Let $w = \frac{e^{-\delta r}}{\int_M e^{-\delta r}}$ be a weight function. Define the weighted norm $||u||_{L^p_w}$ by $||u||_{L^p_w} = (\int |u|^p w)^{1/p}$, then for all $u \in C_0^\infty$,*

$$||u||_{L^4_w(\{r \ge N\})} \le C||\nabla u||_{L^2(\{r \ge N\})} + C||u||_{L^4(\{N \le r \le N+1\})}.$$

It's easy to see that for all $1 \le p \le q \le \infty$, $||u||_{L^p_w} \le ||u||_{L^q_w}$ by Hölder's inequality.

Theorem 7.21. *I is closed.*

Proof. Let $\phi_{ai} = \frac{\int_M \phi_i e^{-2\delta r}}{\int_M e^{-2\delta r}}$ be the weighted average of ϕ_i. By the standard Lockhart-McOwen theory [53], since constant is the only harmonic function less than $e^{\delta r}$, we can obtain a bound on $||e^{-\delta r}(\phi_i - \phi_{ai})||_{W^{2,2}}$ from the L^2 bound of $e^{-\delta r}\Delta_{\omega_0}\phi_i = e^{-\delta r}(\mathrm{tr}_{\omega_0}\omega_i - \mathrm{tr}_{\omega_0}\alpha_i)$.

Let $u_i = \phi_i - \phi_{ai}$. We already obtain a bound on $||u_i||_{W^{2,2}(\{r\leq N+4\})}$ and $||\Delta_{\omega_0}u_i||_{L^\infty(M)}$. So $||u_i||_{W^{2,p}(\{r\leq N+3\})}$ is bounded for any $p\in(1,\infty)$ by Theorem 9.11 of [31].

The $C^{2,\alpha}$-estimate for real Monge-Ampère equation was done by Evans-Krylov-Trudinger. See Section 17.4 of [31] for details. Now we are in the complex case. However, the arguments in Section 17.4 of [31] still work. An alternative way to achieve the bound on $[\partial\bar\partial u_i]_{C^\alpha(\{r\leq N+2\})}$ for all $0<\alpha<1$ was done by Theorem 1.5 of [19] using the rescaling argument. Now it's standard to get a C^∞ bound of u_i on $\{r\leq N+1\}$ through Schauder estimates.

As in [35],

$$\begin{aligned}&\int_{r\geq N}|\nabla|u_i|^{\frac{p}{2}}|^2\alpha_i^2\\ \leq&\frac{p^2}{p-1}[\int_{r\geq N}u_i|u_i|^{p-2}(\omega_i^2-\alpha_i^2)-\frac{1}{2}\int_{r=N}u_i|u_i|^{p-2}\mathrm{d}^c u_i\wedge(\omega_i+\alpha_i)].\end{aligned}$$

Therefore, for $p\geq 2$,

$$\int_{r\geq N}|\nabla|u_i|^{\frac{p}{2}}|^2\leq Cp(\int_{r\geq N}|u_i|^{p-1}w+C_3^{p-1})\leq Cp(C_3\int_{r\geq N}|u_i|^{p-1}w+C_3^p),$$

where C_3 is a bound on $\sup_{\{r\leq N+1\}}|u_i|$. By Young's inequality,

$$\int_{r\geq N}|\nabla|u_i|^{\frac{p}{2}}|^2\leq Cp^2(||u_i||^p_{L_w^{p-1}(\{r\geq N\})}+C_3^p)\leq Cp^2(||u_i||^p_{L_w^p(\{r\geq N\})}+C_3^p).$$

Apply Theorem 7.20 to $|u_i|^{p/2}$. Then

$$||u_i||^{2p}_{L_w^{2p}(\{r\geq N\})}\leq C_4p^4(||u_i||^{2p}_{L_w^p(\{r\geq N\})}+C_3^{2p}).$$

We already know that $||u_i||_{L_w^2(\{r\geq N\})}\leq C_5$. That's our starting point. We are going to obtain a bound on $||u_i||_{L^\infty(\{r\geq N\})}=\lim_{j\to\infty}||u_i||_{L_w^{2^j}(\{r\geq N\})}$ by Moser iteration.

(1) If $||u_i||_{L_w^{2^j}(\{r\geq N\})}\leq C_3$ for all $j\geq 1$, then $||u_i||_{L^\infty(\{r\geq N\})}\leq C_3$.

(2) If $||u_i||_{L^{2^j}_w(\{r\geq N\})} \leq C_3$ for all $1 \leq j \leq k$ but $||u_i||_{L^{2^j}_w(\{r\geq N\})} > C_3$ for all $j > k$, then

$$||u_i||_{L^\infty(\{r\geq N\})} \leq (2C_4)^{\sum_{j=k}^\infty 2^{-j-1}} 2^{\sum_{j=k}^\infty 2^{-j+1} j} C_3 \leq C$$

(3) If $||u_i||_{L^{2^j}_w(\{r\geq N\})} > C_3$ for all $j \geq 1$, then

$$||u_i||_{L^\infty(\{r\geq N\})} \leq (2C_4)^{\sum_{j=1}^\infty 2^{-j-1}} 2^{\sum_{j=1}^\infty 2^{-j+1} j} C_5 \leq C$$

The L^∞ bound on $u_i = \phi_i - \phi_{ai}$ implies a bound on ϕ_{ai} since ϕ_i decay exponentially. Therefore, we actually have a L^∞ bound on ϕ_i. Then we can obtain a global C^∞ bound as before. Finally, we can go through the Step 3 and Step 4 in [35] to get the C^∞ bound on $e^{\lambda_1 r}\phi_i$. We are done by taking the limit of some subsequence of $\{\phi_i\}$. □

References

[1] *M. T. Anderson*, Ricci curvature bounds and Einstein metrics on compact manifolds, J. Amer. Math. Soc. **2** (1989), no.3, 455–490.

[2] *M. T. Anderson*, The L^2 structure of moduli spaces of Einstein metrics on 4-manifolds, Geom. Funct. Anal. **2** (1992), no.1, 29–89.

[3] *M. T. Anderson*, *P. B. Kronheimer* and *C. LeBrun*, Complete Ricci-flat Kähler manifolds of infinite topological type, Comm. Math. Phys. 125 (1989), no.**4**, 637–642.

[4] *M. Atiyah* and *N. J. Hitchin*, The geometry and dynamics of magnetic monopoles, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 1988. viii+134 pp.

[5] *S. Bando*, *A. Kasue* and *H. Nakajima*, On a construction of coordinate at infinity on manifolds with fast curvature decay and maximal volume growth, Invent. Math. **97** (1989), no.2, 313–349.

[6] *A. L. Besse*, Einstein manifolds, **10**, Springer-Verlag, Berlin, 1987.

[7] *O. Biquard* and *P. Boalch*, Wild non-abelian Hodge theory on curves, Compos. Math. **140** (2004), no.1, 179–204.

[8] *O. Biquard* and *V. Minerbe* A Kummer construction for gravitational instantons, Comm. Math. Phys. **308** (2011), no.3, 773–794.

[9] *P. Boalch*, Symplectic manifolds and isomonodromic deformations, Adv. Math. **163** (2001), no.2, 137–205.

[10] *P. Boalch*, Quivers and difference Painlevé equations. Groups and symmetries, 25–51, CRM Proc. Lecture Notes, **47**, Amer. Math. Soc., Providence, RI, 2009.

[11] *P. Boalch*, Hyperkahler manifolds and nonabelian Hodge theory of (irregular) curves, preprint 2012, `http://arxiv.org/abs/1203.6607`.

[12] *D. Burns, Jr.* and *M. Rapoport*, On the Torelli problem for kählerian K3 surfaces, Ann. Sci. École Norm. Sup. (4) **8** (1975), no.2, 235–273.

[13] *P. Buser* and *H. Karcher*, Gromov's almost flat manifolds, Astérisque, **81**. Société Mathématique de France, Paris, 1981.

[14] *G. Chalmers*, *M. Roček* and *S. Wiles*, Degeneration of ALF D_n metrics, J. High Energy Phys. 1999, no.1, Paper 9–14.

[15] *J. Cheeger* and *T. H. Colding*, Lower bounds on Ricci curvature and the almost rigidity of warped products. Ann. of Math. (2) **144** (1996), no.1, 189–237.

[16] *J. Cheeger*, *K. Fukaya* and *M. Gromov*, Nilpotent structures and invariant metrics on collapsed manifolds, J. Amer. Math. Soc. **5** (1992), no.2, 327–372.

[17] *J. Cheeger* and *D. Gromoll*, The splitting theorem for manifolds of nonnegative Ricci curvature, J. Differential Geometry **6** (1971/72), 119–128.

[18] *J. Cheeger*, *M. Gromov* and *M. Taylor*, Finite propagation speed, kernel estimates for functions of the Laplace operator, and the geometry of complete Riemannian manifolds, J. Differential Geom. **17**(1982), no.1, 15–53.

[19] *X. X. Chen* and *Y. Q. Wang*, $C^{2,\alpha}$-estimate for Monge-Ampère equations with Hölder-continuous right hand side. Ann. Global Anal. Geom. **49** (2016), no.2, 195–204.

[20] *S. Y. Cheng* and *S. T. Yau*, Differential equations on Riemannian manifolds and their geometric applications, Comm. Pure Appl. Math. **28** (1975), no.3, 333–354.

[21] *S. A. Cherkis* and *N. J. Hitchin*, Gravitational instantons of type D_k, Comm. Math. Phys. **260** (2005), no.2, 299–317.

[22] *S. A. Cherkis* and *A. Kapustin*, Singular monopoles and gravitational instantons, Comm. Math. Phys. **203** (1999), no.3, 713–728.

[23] *S. A. Cherkis* and *A. Kapustin*, Hyper-Kähler metrics from periodic monopoles, Phys. Rev. D (3) **65** (2002), no.8.

[24] *T. H. Colding*, Ricci curvature and volume convergence, Ann. of Math. (2) **145** (1997), no.3, 477–501.

[25] *J. P. Demailly*, *T. Peternell* and *M. Schneider*, Kähler manifolds with numerically effective Ricci class. Compositio Math. **89** (1993), no.2, 217–240.

[26] *S. K. Donaldson*, Two-forms on four-manifolds and elliptic equations, Inspired by S. S. Chern, 153–172, Nankai Tracts Math., 11, World Sci. Publ., Hackensack, NJ, 2006.

[27] *G. Drees*, Asymptotically flat manifolds of non-negative curvature, Differential Geom. Appl. **4** (1994), 77–90.

[28] *L. Foscolo*, A gluing construction for periodic monopoles, preprint 2014, `https://arxiv.org/abs/1411.6951`.

[29] *L. Foscolo*, Deformation theory of periodic monopoles (with singularities), Comm. Math. Phys. **341** (2016), no.1, 351–390.

[30] *K. Fukaya*, A boundary of the set of the Riemannian manifolds with bounded curvatures and diameters. J. Differential Geom. **28** (1988), no.1, 1–21.

[31] *D. Gilbarg* and *N. S. Trudinger*, Elliptic partial differential equations of second order, Reprint of the 1998 edition, Springer-Verlag, Berlin, 2001.

[32] *R. E. Greene* and *H. Wu*, Function theory on manifolds which possess a pole, Lecture Notes in Mathematics, **699**, Springer, Berlin, 1979.

[33] *M. Gromov*, Almost flat manifolds, J. Differential Geom. **13** (1978), no.2, 231–241.

[34] *T. Hausel*, *E. Hunsicker* and *R. Mazzeo*, Hodge cohomology of gravitational instantons, Duke Math. J. **122** (2004), no.3, 485–548.

[35] *M. Haskins*, *H. -J. Hein* and *J. Nordström*, Asymptotically cylindrical Calabi-Yau manifolds, J. Differential Geom. **101** (2015), no.2, 213–265.

[36] *S. Hawking*, Gravitational instantons, Phys. Lett. **60A** (1977), 81–83.

[37] *H. -J. Hein*, Gravitational instantons from rational elliptic surfaces, J. Amer. Math. Soc. **25** (2012), no.2, 355–393.

[38] *N. J. Hitchin*, Twistor construction of Einstein metrics, Global Riemannian geometry (Durham, 1983), 115–125, Ellis Horwood Ser. Math. Appl., Horwood, Chichester, 1984.

[39] *N. J. Hitchin*, The self-duality equations on a Riemann surface, Proc. London Math. Soc. (3) **55** (1987), no.1, 59–126.

[40] *N. J. Hitchin*, *A. Karlhede*, *U. Lindström* and *M. Roček*, Hyper-Kähler metrics and supersymmetry, Comm. Math. Phys. **108** (1987), no.4, 535–589.

[41] *L. Hörmander*, An introduction to complex analysis in several variables, Third edition, North-Holland Publishing Co., Amsterdam, 1990.

[42] *I. T. Ivanov* and *M. Roček*, Supersymmetric σ-models, twistors, and the Atiyah-Hitchin metric, Comm. Math. Phys. **182** (1996), no.2, 291–302.

[43] *J. Jost* and *H. Karcher*, Geometrische Methoden zur Gewinnung von a-priori-Schranken für harmonische Abbildungen, Manuscripta Math. **40** (1982), no.1, 27–77.

[44] *A. Kasue*, A compactification of a manifold with asymptotically nonnegative curvature, Ann. Sci. École Norm. Sup. (4) **21** (1988), no.4, 593–622.

[45] *K. Kodaira*, On compact analytic surface II, Ann. of Math. (2) **77** (1963), 563–626.

[46] *K. Kodaira*, On the structure of compact complex analytic surfaces. I. Amer. J. Math. **86** (1964), 751–798.

[47] *A. Kovalev* and *M. Singer*, Gluing theorems for complete anti-self-dual spaces, Geom. Funct. Anal. **11** (2001), no.6, 1229–1281.

[48] *P. B. Kronheimer*, The construction of ALE spaces as hyper-Kähler quotients, J. Differential Geom. **29** (1989), no.3, 665–683.

[49] *P. B. Kronheimer*, A Torelli-type theorem for gravitational instantons, J. Differential Geom. **29** (1989), no.3, 685–697.

[50] *H. B. Lawson, Jr.* and *M. -L. Michelsohn*, Spin geometry, Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989. xii+427 pp.

[51] *C. LeBrun*, Complete Ricci-flat Kähler metrics on C^n need not be flat, Several complex variables and complex geometry, Part 2 (Santa Cruz, CA, 1989), 297–304, Proc. Sympos. Pure Math., 52, Part 2, Amer. Math. Soc., Providence, RI, 1991.

[52] *U. Lindström* and *M. Roček*, New hyper-Kähler metrics and new supermultiplets, Comm. Math. Phys. **115** (1988), no. 1, 21–29.

[53] *R. B. Lockhart* and *R. C. McOwen*, Elliptic differential operators on noncompact manifolds, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **12** (1985), no.3, 409–447.

[54] *E. Looijenga* and *C. Peters*, Torelli theorems for Kähler K3 surfaces, Compositio Math. **42** (1980/81), no.2, 145–186.

[55] *Y. Mashiko*, *K. Nagano* and *K. Otsuka*, The asymptotic cones of manifolds of roughly non-negative radial curvature., J. Math. Soc. Japan **57** (2005), no.1, 55–68.

[56] *R. B. Melrose*, The Atiyah-Patodi-Singer index theorem, Research Notes in Mathematics, **4**, A K Peters, Ltd., Wellesley, MA, 1993.

[57] *V. Minerbe*, A mass for ALF manifolds, Comm. Math. Phys. **289** (2009), no.3, 925–955.

[58] *V. Minerbe*, On the asymptotic geometry of gravitational instantons. Ann. Sci. Éc. Norm. Supér. (4) **43** (2010), no.6, 883–924.

[59] *V. Minerbe*, Rigidity for multi-Taub-NUT metrics, J. Reine Angew. Math. **656** (2011), 47–58.

[60] *I. Naruki*, Configurations related to maximal rational elliptic surfaces, Complex analytic singularities, 315–347, Adv. Stud. Pure Math., 8, North-Holland, Amsterdam, 1987.

[61] *A. Newlander* and *L. Nirenberg*, Complex analytic coordinates in almost complex manifolds, Ann. of Math. (2) **65** (1957), 391–404.

[62] *K. Okamoto*, Studies on the Painlevé equations. I, Ann. Mat. Pura Appl. (4) **146** (1987), 337–381.

[63] *K. Okamoto*, Studies on the Painlevé equations. II, Japan. J. Math. (N.S.) **13** (1987), no.1, 47–76.

[64] *K. Okamoto*, Studies on the Painlevé equations. III, Math. Ann. **275** (1986), no.2, 221–255.

[65] *K. Okamoto*, Studies on the Painlevé equations. IV, Funkcial. Ekvac. **30** (1987), no.2–3, 305–332.

[66] *D. N. Page*, A periodic but nonstationary gravitational instanton, Phys. Lett. B **100** (1981), no.4, 313–315.

[67] *A. Petrunin* and *W. Tuschmann*, Asymptotic flatness and cone structure at infinity, Math. Ann. **321** (2001), no.4, 775–788.

[68] *W. D. Ruan*, On the convergence and collapsing of Kähler metrics, J. Differential Geom. **52** (1999), no.1, 1–40.

[69] *W. D. Ruan* and *Y. G. Zhang*, Convergence of Calabi-Yau manifolds, Adv. Math. **228** (2011), no.3, 1543–1589.

[70] *E. A. Ruh*, Almost flat manifolds, J. Differential Geom. **17** (1982), no.1, 1–14.

[71] *H. Sakai*, Rational surfaces associated with affine root systems and geometry of the Painlevé equations, Comm. Math. Phys. **220** (2001), no.1, 165–229.

[72] *R. M. Schoen* and *S. T. Yau*, Lectures on differential geometry, Conference Proceedings and Lecture Notes in Geometry and Topology, I. International Press, Cambridge, MA, 1994.

[73] *M. B. Schulz* and *E. F. Tammaro*, M-theory/type IIA duality and K3 in the Gibbons-Hawking approximation, preprint 2012, `https://arxiv.org/abs/1206.1070`

[74] *A. Sen*, A note on enhanced gauge symmetries in M- and string theory, J. High Energy Phys. 1997, no.**9**, Paper 1–10.

[75] *Y. T. Siu*, A simple proof of the surjectivity of the period map of K3 surfaces, Manuscripta Math. **35** (1981), no.3, 311–321.

[76] *G. Tian* and *S. T. Yau*, Complete Kähler manifolds with zero Ricci curvature. I, J. Amer. Math. Soc. **3** (1990), no.3, 579–609.

[77] *A. N. Todorov*, Applications of the Kähler-Einstein-Calabi-Yau metric to moduli of K3 surfaces. Invent. Math. **61** (1980), no.3, 251–265.

[78] *V. Tosatti*, Limits of Calabi-Yau metrics when the Kähler class degenerates. J. Eur. Math. Soc. **11** (2009), no.4, 755–776.

[79] *S. T. Yau*, The role of partial differential equations in differential geometry, Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp.237–250, Acad. Sci. Fennica, Helsinki, 1980.