Refined Convergence for Genus-Two Pseudo-Holomorphic Maps

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Abstract of the Dissertation

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The moduli spaces of pseudo-holomorphic maps into an almost Kähler manifold are fundamental to Gromov-Witten theory. It has been speculated since the early days of Gromov-Witten theory that these moduli spaces contain natural closed subspaces, whenever the genus is positive, that also give rise to curve-counting invariants. This speculation was confirmed in genus 1 over a decade ago. In this dissertation, we describe a natural subspace of every moduli space of genus 2 pseudo-holomorphic maps that has strata of the correct virtual dimension to give rise to curve-counting invariants. We establish most of the convergence statements for sequences of such maps needed to show that this subspace is closed.

Contents

Acknowledgments

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1 Introduction

The theory of Gromov-Witten invariants arises from Gromov's work [4] on pseudo-holomorphic curves and Witten's work [14] on σ-models in physics and plays prominent roles in symplectic topology and algebraic geometry. These invariants count J-holomorphic maps from closed, possibly nodal, Riemann surfaces to a compact symplectic manifold (X,ω) with a fixed almost complex structure J tamed by ω. The main object of interest in this theory is the moduli space $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ of stable J-holomorphic maps from genus g Riemann surfaces with k marked points in the homology class $A\in H_2(X;\mathbb{Z})$. This space is often singular and is stratified by smooth orbifolds. The subspace $\mathfrak{M}_{g,k}(X,A;J)$ of maps from smooth domains is of special interest because it corresponds to irreducible (and often smooth) curves in X. Even though $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ is conventionally referred to as Gromov's *compactification* of $\mathfrak{M}_{g,k}(X,A;J)$, in general $\mathfrak{M}_{g,k}(X,A;J)$ is *not dense* in $\overline{\mathfrak{M}}_{g,k}(X,A;J)$. The subspace $\mathfrak{M}_{g,k}(X,A;J)$ is dense in $\overline{\mathfrak{M}}_{g,k}(X,A;J)$ if $g=0$ and J is sufficiently regular, but this is not the case if $g\geqslant 1$; see [18, Section 1.1]. In this paper, we investigate this deficiency in the $g=2$ case in detail.

1.1 Background and overview

It was speculated back in [12] that there exists a natural closed subspace

$$\overline{\mathfrak{M}}^0_{g,k}(X,A;J)\subset\overline{\mathfrak{M}}_{g,k}(X,A;J) \tag{1.1}$$

containing $\mathfrak{M}_{g,k}(X,A;J)$ such that $\mathfrak{M}_{g,k}(X,A;J)$ is dense in $\overline{\mathfrak{M}}^0_{g,k}(X,A;J)$ whenever J is sufficiently regular. The naturality condition should include

(N_g1) for every compact almost Kähler submanifold Y of (X,ω,J),

$$\overline{\mathfrak{M}}^0_{g,k}(Y,A;J)=\overline{\mathfrak{M}}^0_{g,k}(X,A;J)\cap\overline{\mathfrak{M}}_{g,k}(Y,A;J); \tag{1.2}$$

(N_g2) the pre-image of $\overline{\mathfrak{M}}^0_{g}(X,A;J)\equiv\overline{\mathfrak{M}}^0_{g,0}(X,A;J)$ under the forgetful morphism

$$\mathfrak{f}\colon\,\overline{\mathfrak{M}}_{g,k}(X,A;J)\longrightarrow\overline{\mathfrak{M}}_{g}(X,A;J)\equiv\overline{\mathfrak{M}}_{g,0}(X,A;J)$$

is $\overline{\mathfrak{M}}^0_{g,k}(X,A;J)$.

The first property is closely related to the existence of a sufficiently nice deformation theory for $\overline{\mathfrak{M}}^0_{g,k}(X,A;J)$ so that this subspace carries a natural virtual fundamental class as in [7]. The second property determines $\overline{\mathfrak{M}}^0_{g,k}(X,A;J)$ with $k>0$ from $\overline{\mathfrak{M}}^0_{g}(X,A;J)$ so that the marked moduli space satisfies (1.2) if the unmarked one does. It is thus sufficient to consider the $k=0$ case of the speculation in [12].

A closed subspace with the desired properties in the $g=1$ case is constructed in [18]. It is described by splitting the elements of $\overline{\mathfrak{M}}_1(X,A;J)$ into two types and describing the conditions for these elements to be inside of $\overline{\mathfrak{M}}^0_1(X,A;J)$. No condition is imposed for one of the types; the corresponding $g=2$ type is Type 0 on page 5. A simple first-order condition is imposed for the other type; its closest $g=2$ analogue is Type 1, though Types 1a and 1b exhibit similar features as well. The construction of $\overline{\mathfrak{M}}^0_1(X,A;J)$ in [18] is the most fundamental step eventually leading to the proof of the mirror symmetry prediction of [1] for the genus 1 Gromov-Witten invariants of the quintic threefold in [20].

In this dissertation, we establish most convergence statements for genus 2 pseudo-holomorphic maps needed to confirm the speculation of [12] in the $g=2$ case; see Section 1.5. This involves splitting the elements of $\overline{\mathfrak{M}}_2(X,A;J)$ into five types at the first stage; see page 5. The first-stage splitting is the direct analogue of the splitting in the $g=1$ case and is based on how the genus of the domain can be split up. All elements of Type 0 are contained in $\overline{\mathfrak{M}}_2^0(X,A;J)$. An element of Type 1, 1a, or 1b is contained in $\overline{\mathfrak{M}}_2^0(X,A;J)$ if and only if it satisfies a first-order condition directly analogous to the condition in the $g=1$ case. An element of Type 2 is contained in $\overline{\mathfrak{M}}_2^0(X,A;J)$ if and only if it satisfies a similar first-order condition *and* a more elaborate second-order condition which depends on how the first-order condition is satisfied. This second-stage splitting takes into account the Weierstrass and conjugate points on genus 2 Riemann surfaces.

While the precise description of $\overline{\mathfrak{M}}_2^0(X,A;J)$ is rather involved, its explicit nature sets the stage for further applications. These include enumerative geometry in genus 2, construction of reduced genus 2 GW-invariants (in the spirit of [19]), comparison between standard and reduced GW-invariants (in the spirit of [17]), Quantum Lefschetz Hyperplane Theorem for reduced genus 2 GW-invariants of complete intersections (in the spirit of [16, 8]), partial desingularization of $\overline{\mathfrak{M}}_{2,k}^0(\mathbb{P}^n,d)$ with specification of the associated equivariant localization data (in the spirit of [13]), and proof of mirror symmetry for genus 2 GW-invariants of complete intersections (in the spirit of [20, 11]). An algebro-geometric approach to studying the structure of $\overline{\mathfrak{M}}_2(\mathbb{P}^n,d)$ is presented in [5, 6].

1.2 Genus 2 curves

Throughout this work, a Riemann surface and complex curve mean a closed connected complex manifold of complex dimension 1 with some disjoint pairs of points identified. If Σ' is a Riemann surface and $\Sigma'\subset\Sigma$ is a union of irreducible components, we denote the closure of $\Sigma-\Sigma'$ in Σ by $(\Sigma')^c$. Given a finite set $\mathcal{D}_i\colon L_i\longrightarrow V$, $i\in S$, of $\mathbb{C}$-linear homomorphisms of vector spaces with the same target and $k\in\mathbb{Z}^{\geqslant 0}$, we write

$$\mathfrak{c}(\{\mathcal{D}_i\}_{i\in S})=k \qquad \text{if} \qquad \dim_{\mathbb{C}}\operatorname{Span}_{\mathbb{C}}\big(\mathcal{D}_i v_i\colon\, v_i\in L_i,\, i\in S\big)\leqslant |S|-k\,.$$

For every smooth (closed, connected) genus 2 Riemann surface Σ, the holomorphic sections of the canonical line bundle $\mathcal{K}_\Sigma$ determine a double cover

$$\Sigma\longrightarrow\mathbb{P}\big(H^0(\Sigma;\mathcal{K}_\Sigma)^*\big),\quad x\longrightarrow\big\{\mu\in H^0(\Sigma;\mathcal{K}_\Sigma)^*\colon\,\mu(\eta)=0\ \ \forall\,\eta\in H^0(\Sigma;\mathcal{K}_\Sigma)\text{ s.t. }\eta|_x=0\big\},\qquad(1.3)$$

with 6 branched points; see [3, p253]. We denote by $\mathrm{WP}(\Sigma)$ the set of these 6 Weierstrass points of Σ. The deck transformation of the double cover $\Sigma\longrightarrow\mathbb{P}^1$ determines a holomorphic involution κ_Σ on Σ. For $x_1,x_2\in\Sigma$, we write $x_1\sim x_2$ if $x_1=\kappa_\Sigma(x_2)$.

For $\ell\in\mathbb{Z}^{\geqslant 0}$, define

$$[\![\ell]\!]=\big\{1,\ldots,\ell\big\}\qquad\text{and}\qquad\mathcal{P}^{\geqslant 2}(\ell)=\big\{I\subset[\![\ell]\!]\colon\,|I|\geqslant 2\big\}.$$

Denote by $\overline{\mathcal{M}}_{2,\ell}$ the Deligne-Mumford moduli space of genus 2 stable curves with ℓ marked points and by $\mathcal{M}_{2,\ell}\subset\overline{\mathcal{M}}_{2,\ell}$ the subspace of smooth curves. Let

$$\mathbb{E},\,\mathbb{L}_i\longrightarrow\overline{\mathcal{M}}_{2,\ell}$$

be the Hodge vector bundle of holomorphic differentials and the universal tangent line bundle for the i-th marked point, respectively. Let

$$\mathbb{F}_\ell=\bigoplus_{i=1}^{\ell}\mathbb{L}_i\longrightarrow\overline{\mathcal{M}}_{2,\ell}\,.$$

For each $i\in[\![\ell]\!]$, denote by $\pi_{\ell;i}\colon\mathbb{F}_\ell\longrightarrow\mathbb{L}_i$ the projection map.

For $\ell\geqslant1$, let

$$\mathcal{W}_{2,\ell}=\big\{[\Sigma;x_1,\ldots,x_\ell]\in\mathcal{M}_{2,\ell}\colon\,x_1\in\mathrm{WP}(\Sigma)\big\},\qquad\overline{\mathcal{W}}_{2,\ell}=\mathrm{Cl}\big(\mathcal{W}_{2,\ell}\big)\subset\overline{\mathcal{M}}_{2,\ell},$$

where Cl denotes the closure. For $\ell\geqslant2$, let

$$\mathcal{C}_{2,\ell}=\big\{[\Sigma,x_1,\ldots,x_\ell]\in\mathcal{M}_{2,\ell}\colon\,x_1\sim x_2\big\},\qquad\overline{\mathcal{C}}_{2,\ell}=\mathrm{Cl}\big(\mathcal{C}_{2,\ell}\big)\subset\overline{\mathcal{M}}_{2,\ell}.$$

For $\ell\geqslant2$, define

$$\pi_{\mathbb{P}F_\ell}\colon\mathbb{P}F_\ell=\mathbb{P}\Big(\bigoplus_{i=1}^{\ell}L_i\Big)\longrightarrow\overline{\mathcal{M}}_{2,\ell},\qquad\mathbb{P}^0F_\ell=\Big\{[v_i]_{i\in[\![\ell]\!]}\in\mathbb{P}F_\ell|_{\mathcal{M}_{2,\ell}-\mathcal{C}_{2,\ell}}\colon v_1,v_2\neq0\Big\}\subset\mathbb{P}F_\ell,$$
$$\varphi_\ell\colon\,\mathbb{P}^0F_\ell\longrightarrow\mathrm{Gr}(2;\ell),\qquad\varphi_\ell\big([v]\equiv[v_i]_{i\in[\![\ell]\!]}\big)=\mathrm{Span}_{\mathbb{C}}\Big\{\big(\eta(v_1),\ldots,\eta(v_\ell)\big)\colon\eta\in\mathbb{E}_{\pi_{\mathbb{P}F_\ell}[v]}\Big\}.$$

The closure of the graph of φ_ℓ,

$$\mathbb{V}_\ell\equiv\mathrm{Cl}\Big\{\big([v],\varphi_\ell[v]\big)\colon\,[v]\in\mathbb{P}^0F_\ell\Big\}\subset\mathbb{P}F_\ell\times\mathrm{Gr}(2;\ell)\,,$$

is a subvariety of $\mathbb{P}F_\ell\times\mathrm{Gr}(2;\ell)$ with a projection map to $\overline{\mathcal{M}}_{2,\ell}$.

1.3 Pseudo-holomorphic maps and derivatives

Let (X,ω,J) be an almost Kähler manifold, i.e. (X,ω) is a symplectic manifold and J is an almost complex structure on X tamed by ω, and ∇ be a connection on (TX,J). For a smooth map $u\colon\Sigma\longrightarrow X$ from a possibly nodal Riemann surface and $m\in\mathbb{Z}^+$, denote by ∇^u the connection on $(T^*\Sigma)^{\otimes m}\otimes u^*TX$ induced by the pull-back of ∇ by u and a fixed connection on Σ. For a smooth point $x\in\Sigma$, let

$$\mathcal{D}_x^{(m)}u\big(v^{\otimes m}\big)=\big\{\{\nabla^u\}^{m-1}\mathrm{d}u\big\}\big(\underbrace{v,\ldots,v}_{m}\big)\in T_{u(x)}X\qquad\forall\;v\in T_x\Sigma\tag{1.4}$$

and write $\mathcal{D}_xu=\mathcal{D}_x^{(1)}u$. For a node $x=(x^+,x^-)$ between the irreducible components Σ_x^+ and Σ_x^- of Σ so that $x^\pm\in\Sigma_x^\pm$, we define $\mathcal{D}_{x^\pm}u$ and $\mathcal{D}_{x^\pm}^{(m)}u$ likewise. If u is J-holomorphic near x and $\mathcal{D}_xu=0$, then neither $\mathcal{D}_x^{(2)}u$ nor the span of $\mathcal{D}_x^{(2)}u$ and $\mathcal{D}_x^{(3)}u$ depends on the choice of the connections. If a finite tuple $(u_i,x_i)_{i\in S}$ of one-marked J-holomorphic maps and $v_i\in T_{x_i}\Sigma_i$ are such that

$$u_i(x_i)=u_j(x_j) \quad \forall\, i,j\in S \qquad \text{and} \qquad \sum_{i\in S}\mathcal{D}_{x_i}u_i(v_i)=0\in T_{u_1(x_1)}X,$$

then

$$\operatorname{Span}_{\mathbb{C}}\bigg(\sum_{i\in S}\mathcal{D}_{x_i}^{(2)}u_i(v_i^{\otimes 2}),\big\{\mathcal{D}_{x_i}u_i(T_{x_i}\Sigma_i)\colon\, i\in S\big\}\bigg)\subset T_{u_1(x_1)}X$$

is also independent of this choice; see Appendix A.

For $A\in H_2(X;\mathbb{Z})-0$, let

$$\mathrm{ev}_1\colon \overline{\mathfrak{M}}_{0,1}(X,A;J)\longrightarrow X, \quad \mathrm{ev}_1\big([\Sigma,u,x_1]\big)=u(x_1)\,, \qquad \text{and} \qquad \mathfrak{L}_1\longrightarrow \overline{\mathfrak{M}}_{0,1}(X,A;J)$$

be the evaluation map and the universal tangent line bundle, respectively, at the marked point. For an element

$$[\mathbf{u}]=[\Sigma,u,x_1]\in\overline{\mathfrak{M}}_{0,1}(X,A;J), \tag{1.5}$$

we denote by $\Sigma_P\subset\Sigma$ the irreducible component containing x_1 and by $\mathcal{D}^{(m)}\mathbf{u}$ and $\mathcal{D}\mathbf{u}$ the homomorphisms $\mathcal{D}_{x_1}^{(m)}u$ and $\mathcal{D}_{x_1}u$, respectively, described above. Let

$$\mathfrak{M}_{0,1}^0(X,A;J)=\big\{[\mathbf{u}]\in\overline{\mathfrak{M}}_{0,1}(X,A;J)\colon u|_{\Sigma_P}\neq\text{const.}\big\}$$

and

$$\mathcal{D}\in\Gamma\big(\overline{\mathfrak{M}}_{0,1}(X,A;J);\mathfrak{L}_1^*\otimes\mathrm{ev}_1^*TX\big)$$

be the bundle section induced by the homomorphisms $\mathcal{D}\mathbf{u}$. If $\mathfrak{U}\longrightarrow\overline{\mathfrak{M}}_{0,1}(X,A;J)$ is the universal curve and $\mathrm{ev}\colon\mathfrak{U}\longrightarrow X$ is the natural evaluation map, then $\mathcal{D}$ is the restriction of $\mathrm{d}_{x_1}\mathrm{ev}$ to the vertical tangent bundle of $\mathfrak{U}$.

For $\ell\in\mathbb{Z}^+$, let

$$\mathfrak{M}_{0,1}^{\ell}=\bigsqcup_{\substack{A_1,\ldots,A_\ell\in H_2(X;\mathbb{Z})-0\\ A_1+\ldots+A_\ell=A}}\big\{\big([\mathbf{u}_1],\ldots,[\mathbf{u}_\ell]\big)\in\prod_{i=1}^{\ell}\mathfrak{M}_{0,1}^0(X,A_i;J)\colon \mathrm{ev}_1[\mathbf{u}_{i_1}]=\mathrm{ev}_1[\mathbf{u}_{i_2}]\ \ \forall\, i_1,i_2\in[\![\ell]\!]\big\},$$

$$\mathrm{ev}\colon\mathfrak{M}_{0,1}^{\ell}\longrightarrow X, \quad \mathrm{ev}\big([\mathbf{u}_1],\ldots,[\mathbf{u}_\ell]\big)=\mathrm{ev}_1(\mathbf{u}_1).$$

For each $i\in[\![\ell]\!]$, let

$$\pi_i\colon\,\mathfrak{M}_{0,1}^{\ell}\longrightarrow\mathfrak{M}_{0,1}^0(X,A_i;J)$$

be the component projection map.

For $\ell\in\mathbb{Z}^+$ and $i\in[\![\ell]\!]$, define

$$\begin{aligned}\mathcal{Z}_{0;\ell;i}^{(2)}&=\big\{([\mathbf{u}_1],\ldots,[\mathbf{u}_\ell])\in\mathfrak{M}_{0,1}^{\ell}\colon\,\mathcal{D}\mathbf{u}_i=0,\ \mathfrak{c}\big(\mathcal{D}\mathbf{u}_1,\ldots,\mathcal{D}\mathbf{u}_\ell,\mathcal{D}^{(2)}\mathbf{u}_i\big)=2\big\}\,,\\ \mathcal{Z}_{0;\ell;i}^{(3)}&=\big\{([\mathbf{u}_1],\ldots,[\mathbf{u}_\ell])\in\mathfrak{M}_{0,1}^{\ell}\colon\,\mathcal{D}\mathbf{u}_i=0,\ \mathfrak{c}\big(\mathcal{D}\mathbf{u}_1,\ldots,\mathcal{D}\mathbf{u}_\ell,\mathcal{D}^{(2)}\mathbf{u}_i,\mathcal{D}^{(3)}\mathbf{u}_i\big)=2\big\}\,,\end{aligned} \tag{1.6}$$

respectively. For $\ell \geqslant 2$, define

$$\mathcal{Z}_{0;\ell;12}^{(2)} = \Big\{([\mathbf{u}_1],\ldots,[\mathbf{u}_\ell])\in\mathfrak{M}_{0,1}^{\ell}:\ \mathfrak{c}(\mathcal{D}\mathbf{u}_1,\mathcal{D}\mathbf{u}_2)=1,\\ \mathfrak{c}\big(\mathcal{D}\mathbf{u}_1,\ldots,\mathcal{D}\mathbf{u}_\ell,\{\mathcal{D}^{(2)}\mathbf{u}_1+\mathcal{D}^{(2)}\mathbf{u}_2\}|_{(\ker\{\mathcal{D}\mathbf{u}_1+\mathcal{D}\mathbf{u}_2\})^{\otimes 2}}\big)=2\Big\}. \tag{1.7}$$

The subspaces $\mathcal{Z}_{0;\ell;1}^{(2)}\subset\mathcal{Z}_{0;\ell;1}^{(3)}, \mathcal{Z}_{0;\ell;12}^{(2)}$ are closed in $\mathfrak{M}_{0,1}^{\ell}$.

Denote by

$$\mathrm{pr}_P, \mathrm{pr}_B : \overline{\mathcal{M}}_{2,\ell}\times\mathfrak{M}_{0,1}^{\ell} \longrightarrow \overline{\mathcal{M}}_{2,\ell}, \mathfrak{M}_{0,1}^{\ell}$$

the component projections. For each $i\in[\![\ell]\!]$, define

$$\mathfrak{D}_{\ell;i}\in\Gamma\big(\overline{\mathcal{M}}_{2,\ell}\times\mathfrak{M}_{0,1}^{\ell};\ \mathrm{pr}_B^*\pi_i^*\mathfrak{L}_1^*\otimes\mathrm{Hom}(\mathbb{C}^\ell,\mathrm{pr}_B^*\mathrm{ev}^*TX)\big),\\ \big\{\mathcal{D}_{\ell;i}(\upsilon_i)\big\}(a_1,\ldots,a_\ell) = a_i\mathcal{D}\upsilon_i.$$

For $\ell\geqslant 2$, let

$$\mathcal{Z}_{2;\ell} = \Big\{\big([\Sigma],[\mathbf{u}]\big)\in\overline{\mathcal{M}}_{2,\ell}\times\mathfrak{M}_{0,1}^{\ell}:\ \sum_{i=1}^{\ell}\mathfrak{D}_{\ell;i}(\upsilon_i)|_{\mathcal{P}}=0\in\mathrm{Hom}\big(\mathcal{P},T_{\mathrm{ev}(\mathbf{u})}X\big)\\ \text{for some } \mathcal{P}\in\mathbb{V}_\ell|_{[\Sigma]},\ (\upsilon_i)_{i\in[\![\ell]\!]}\in\bigoplus_{i=1}^{\ell}(\mathfrak{L}_1|_{\pi_i(\mathbf{u})}-0)\Big\}. \tag{1.8}$$

1.4 A smaller moduli space

For $A\in H_2(X;\mathbb{Z})-0$, we abbreviate

$$\overline{\mathfrak{M}}\equiv\overline{\mathfrak{M}}_{2,0}(X,A;J), \qquad \mathfrak{M}\equiv\mathfrak{M}_{2,0}(X,A;J).$$

An element

$$[\mathbf{u}]\equiv[\Sigma,u]\in\overline{\mathfrak{M}} \tag{1.9}$$

is the equivalence class of a pair consisting of a closed connected, possibly nodal, Riemann surface (or simply a curve) Σ of (arithmetic) genus 2 and a J-holomorphic map $u\colon\Sigma\longrightarrow X$.

A (maximal) contracted curve of $\mathbf{u}=(\Sigma,u)$ as in (1.9) is a (maximal) connected union Σ_0 of the irreducible components of Σ so that $u|_{\Sigma_0}$ is constant. Let $\chi(\Sigma_0)$ be the set of nodes connecting Σ_0 with $(\Sigma_0)^c$. A primary contracted curve of $\mathbf{u}$ is a maximal contracted curve Σ_0 with $g_a(\Sigma_0)\geqslant 1$. We denote by $\mathrm{PC}(\mathbf{u})$ the set of of primary contracted curves of $\mathbf{u}$.

We define an element $[\mathbf{u}]$ of $\overline{\mathfrak{M}}$ to be of

- Type 0 if $\mathrm{PC}(\mathbf{u})=\varnothing$;
- Type 1 if $\mathrm{PC}(\mathbf{u})=\{\Sigma_0\}$ and $(\Sigma_0)^c$ contains one connected component of arithmetic genus 1;
- Type 1a if $\mathrm{PC}(\mathbf{u})=\{\Sigma_0\}$ with $g_a(\Sigma_0)=1$ and every connected component of $(\Sigma_0)^c$ is of arithmetic genus 0;

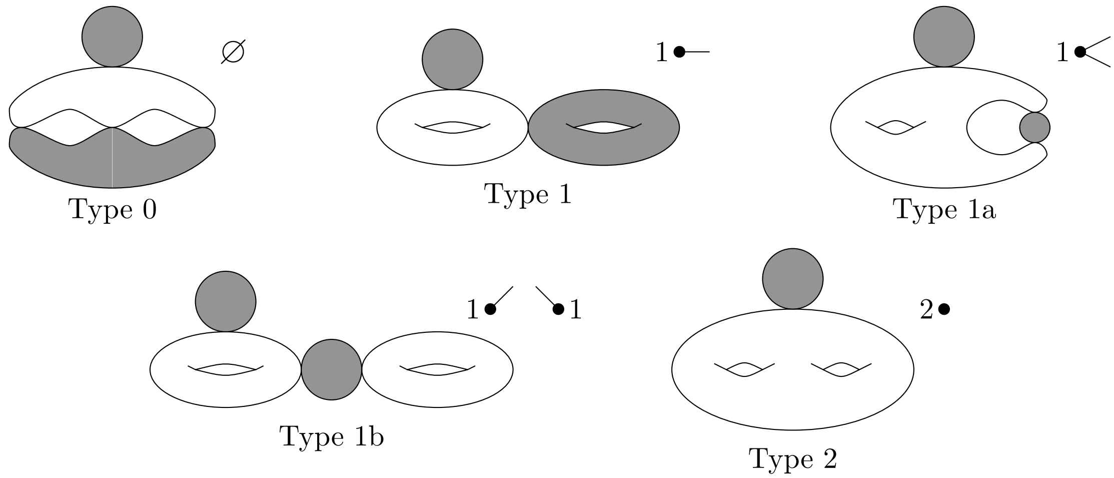

Figure 1: Genus 2 map types and the associated primary contracted graphs

- Type 1b if $\big|\mathrm{PC}(\mathbf{u})\big| = 2$;
- Type 2 if $\mathrm{PC}(\mathbf{u}) = \{\Sigma_0\}$ with $g_a(\Sigma_0) = 2$.

This notion is independent of the choice of representative $\mathbf{u}$ for $[\mathbf{u}]$. The five types are illustrated in Figure 1. The unshaded components correspond to the contracted ones. Shown next to each diagram is the dual graph of the primary contracted component(s), indicating the number of principal nodes on each component; see Section 2.1.

For $\bullet \in \{0, 1, 1a, 1b, 2\}$, let

$$\mathfrak{M}_{\tau\bullet} \equiv \big\{[\mathbf{u}] \in \overline{\mathfrak{M}}\colon\ \mathbf{u} \text{ is of Type } \bullet\big\}.$$

By definition, $\mathfrak{M}_{\tau 0}$ contains $\mathfrak{M}$. The subspaces $\mathfrak{M}_{\tau\bullet}$ are pairwise disjoint, but they are generally not closed, except for $\mathfrak{M}_{\tau 2}$. Figure 2 shows the possible limits of sequences of elements in subspaces. In particular, the intersection of the closures of $\mathfrak{M}_{\tau 1a}$ and $\mathfrak{M}_{\tau 1b}$ is contained in $\mathfrak{M}_{\tau 2}$.

Let $[\mathbf{u}] \in \overline{\mathfrak{M}}$ be as in (1.9) and $\Sigma_0 \in \mathrm{PC}(\mathbf{u})$. For $x \in \chi(\Sigma_0)$, let $u_x\colon \Sigma_x \longrightarrow X$ be the restriction of u to the irreducible component $\Sigma_x \subset (\Sigma_0)^c$ containing x and

$$\mathcal{D}_x\mathbf{u} = \mathcal{D}_x u_x\colon T_x(\Sigma_0)^c \longrightarrow T_{u(x)}X = T_{u(\Sigma_0)}X\,.$$

For $\bullet = 0, 1, 1a, 1b$, define

$$\mathfrak{M}^0_{\tau\bullet} = \big\{[\mathbf{u}] \in \mathfrak{M}_{\tau\bullet}\colon\ \mathfrak{c}\big(\{\mathcal{D}_x\mathbf{u}\colon x \in \chi(\Sigma_0)\}\big) = 1\ \forall\,\Sigma_0 \in \mathrm{PC}(\mathbf{u})\big\} \subset \mathfrak{M}_{\tau\bullet}. \tag{1.10}$$

In particular, $\mathfrak{M}^0_{\tau 0} = \mathfrak{M}_{\tau 0}$. These definitions are analogous to the genus 1 counterparts of these subspaces in [18, Definition 1.1(b)].

For each $\ell \in \mathbb{Z}^+$, let

$$\iota_\ell\colon \overline{\mathcal{M}}_{2,\ell} \times \mathfrak{M}^\ell_{0,1} \longrightarrow \mathfrak{M}_{\tau 2}$$

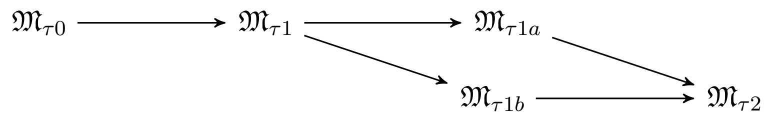

Figure 2: Possible limits of sequences of elements of $\mathfrak{M}_{\tau\bullet}$

be the standard node-identifying map. Define

$$\begin{aligned}\mathfrak{M}^0_{\tau 2} &= \bigcup_{\ell\geqslant 1}\iota_\ell\big(\overline{\mathcal{M}}_{2,\ell}\times\mathcal{Z}^{(2)}_{0;\ell;1}\cup\overline{\mathcal{W}}_{2,\ell}\times\mathcal{Z}^{(3)}_{0;\ell;1}\big)\cup\bigcup_{\ell\geqslant 2}\iota_\ell\big(\overline{\mathcal{C}}_{2,\ell}\times\mathcal{Z}^{(2)}_{0;\ell;12}\cup\mathcal{Z}_{2;\ell}\big),\\ \overline{\mathfrak{M}}^0_{2,0}(X,A;J) &= \mathfrak{M}^0_{\tau 0}\sqcup\mathfrak{M}^0_{\tau 1}\sqcup\mathfrak{M}^0_{\tau 1a}\sqcup\mathfrak{M}^0_{\tau 1b}\sqcup\mathfrak{M}^0_{\tau 2}\subset\overline{\mathfrak{M}}.\end{aligned}\tag{1.11}$$

As described in Section 1.5, most steps needed to establish the following statements are completed in this dissertation.

Theorem 1.1. *If (X,ω) is a compact symplectic manifold, J is an ω-compatible almost complex structure on X, and $A\in H_2(X;\mathbb{Z})-0$, then $\overline{\mathfrak{M}}^0_2(X,A;J)$ is closed in $\overline{\mathfrak{M}}_2(X,A;J)$ and satisfies the $(g,k)=(2,0)$ case of (N_g1) on page 1. Furthermore,*

$$\dim^{\text{vir}}_{\mathbb{R}}\Big(\overline{\mathfrak{M}}^0_2(X,A;J)-\mathfrak{M}_2(X,A;J)\Big) = \dim^{\text{vir}}_{\mathbb{R}}\mathfrak{M}_2(X,A;J)-2.\tag{1.12}$$

Remark 1.2. The assumption that the almost complex structure J is ω-compatible, not just ω-tame, is made for technical reasons and may not be essential. It implies that the torsion tensor of the J-linear connection of the Levi-Civita connection on X is proportional to the Nijenhuis tensor.

1.5 Outline of the proof

We first justify the last statement of Theorem 1.1. Let $2n$ be the (real) dimension of X. For every stratum $\mathfrak{M}_{\mathscr{T}}$ of $\mathfrak{M}^0_{\tau 0}$,

$$\dim^{\text{vir}}_{\mathbb{R}}\mathfrak{M}_{\mathscr{T}} = 2\big(\langle c_1(X),A\rangle+n-3-|\text{Edg}(\mathscr{T})|\big) = \dim^{\text{vir}}_{\mathbb{R}}\mathfrak{M}_2(X,A;J)-2|\text{Edg}(\mathscr{T})|,$$

where $|\text{Edg}(\mathscr{T})|$ is the number of nodes of the maps in $\mathfrak{M}^0_{\tau 0}$. By (1.10),

$$\begin{aligned}\dim^{\text{vir}}_{\mathbb{R}}\mathfrak{M}_{\mathscr{T}} &\leqslant 2\big(\langle c_1(X),A\rangle+n-3-|\text{Edg}(\mathscr{T})|\big)+2n|\text{PC}(\mathbf{u})|-2\sum_{\Sigma_0\in\text{PC}(\mathbf{u})}\big(n+1-|\chi(\Sigma_0)|\big)\\ &= 2\Big(\langle c_1(X),A\rangle+n-3-\Big(|\text{Edg}(\mathscr{T})|-\sum_{\Sigma_0\in\text{PC}(\mathbf{u})}|\chi(\Sigma_0)|\Big)\Big)-2|\text{PC}(\mathbf{u})|\\ &\leqslant \dim^{\text{vir}}_{\mathbb{R}}\mathfrak{M}_2(X,A;J)-2\end{aligned}$$

for any stratum $\mathfrak{M}_{\mathscr{T}}$ of $\mathfrak{M}^0_{\tau 1}$, $\mathfrak{M}^0_{\tau 1a}$, and $\mathfrak{M}^0_{\tau 1b}$ and any element $[\mathbf{u}]\in\mathfrak{M}_{\mathscr{T}}$, *disregarding* the obstruction bundle over the primary contracted curves (i.e. only the non-constant maps are to be deformed). We also disregard this bundle in computing the virtual dimensions below.

By the first equation in (1.6),

$$\begin{aligned}\dim_{\mathbb{R}}^{\mathrm{vir}}\mathfrak{M}_{\mathscr{T}} &\leqslant 2\big(\langle c_1(X),A\rangle+n{-}3-|\mathrm{Edg}(\mathscr{T})|\big)+4n-2\big(n+(n{+}1{-}\ell)\big)\\ &=2\Big(\langle c_1(X),A\rangle+n{-}3-\big(|\mathrm{Edg}(\mathscr{T})|{-}\ell\big)\Big)-2\leqslant \dim_{\mathbb{R}}^{\mathrm{vir}}\mathfrak{M}_2(X,A;J)-2\end{aligned}$$

for any stratum $\mathfrak{M}_{\mathscr{T}}$ of $\iota_\ell\big(\overline{\mathcal{M}}_{2,\ell}\times\mathcal{Z}_{0;\ell;1}^{(2)}\big)$. By the second equation in (1.6),

$$\begin{aligned}\dim_{\mathbb{R}}^{\mathrm{vir}}\mathfrak{M}_{\mathscr{T}} &\leqslant 2\big(\langle c_1(X),A\rangle+n{-}3-|\mathrm{Edg}(\mathscr{T})|\big)+4n-2-2\big(n+(n{-}\ell)\big)\\ &\leqslant \dim_{\mathbb{R}}^{\mathrm{vir}}\mathfrak{M}_2(X,A;J)-2\end{aligned}$$

for any stratum $\mathfrak{M}_{\mathscr{T}}$ of $\iota_\ell\big(\overline{\mathcal{W}}_{2,\ell}\times\mathcal{Z}_{0;\ell;1}^{(3)}\big)$. By (1.7),

$$\begin{aligned}\dim_{\mathbb{R}}^{\mathrm{vir}}\mathfrak{M}_{\mathscr{T}} &\leqslant 2\big(\langle c_1(X),A\rangle+n{-}3-|\mathrm{Edg}(\mathscr{T})|\big)+4n-2-2\big((n{-}1)+(n{+}1{-}\ell)\big)\\ &\leqslant \dim_{\mathbb{R}}^{\mathrm{vir}}\mathfrak{M}_2(X,A;J)-2\end{aligned}$$

for any stratum $\mathfrak{M}_{\mathscr{T}}$ of $\iota_\ell\big(\overline{\mathcal{C}}_{2,\ell}\times\mathcal{Z}_{0;\ell;12}^{(2)}\big)$. By (1.8),

$$\begin{aligned}\dim_{\mathbb{R}}^{\mathrm{vir}}\mathfrak{M}_{\mathscr{T}} &\leqslant 2\big(\langle c_1(X),A\rangle+n{-}3-|\mathrm{Edg}(\mathscr{T})|\big)+4n+2(\ell{-}1)-2(2n-\ell{+}1)-2(\ell{-}1)\\ &\leqslant \dim_{\mathbb{R}}^{\mathrm{vir}}\mathfrak{M}_2(X,A;J)-2\end{aligned}$$

for any stratum $\mathfrak{M}_{\mathscr{T}}$ of $\iota_\ell(\mathcal{Z}_{2;\ell})$. Along with (1.11), this establishes (1.12).

We next turn to the main statement of Theorem 1.1. Since $\mathfrak{M}_{\tau 0}^0=\mathfrak{M}_{\tau 0}$, $\mathfrak{M}_{\tau 0}^0$ is closed in $\mathfrak{M}_{\tau 0}$. By (1.10) and [18, Proposition 5.1], $\mathfrak{M}_{\tau\bullet}^0$ is also closed in $\mathfrak{M}_{\tau\bullet}$ for $\bullet=1,1a,1b$. For the same reasons, the subspaces

$$\begin{gathered}\mathfrak{M}_{\tau 1}^0\cup\mathfrak{M}_{\tau 1a}^0\subset\mathfrak{M}_{\tau 1}\cup\mathfrak{M}_{\tau 1a},\qquad \mathfrak{M}_{\tau 1}^0\cup\mathfrak{M}_{\tau 1b}^0\subset\mathfrak{M}_{\tau 1}\cup\mathfrak{M}_{\tau 1b},\\ \bigcup_{\ell\geqslant 2}\iota_\ell(\mathcal{Z}_{2;\ell}),\ \bigcup_{\ell\geqslant 2}\iota_\ell\big((\overline{\mathcal{C}}_{2,\ell}\times\mathcal{Z}_{0;\ell;12}^{(2)})\cup\mathcal{Z}_{2;\ell}\big),\ \bigcup_{\ell\geqslant 1}\iota_\ell\big(\overline{\mathcal{M}}_{2,\ell}\times\mathcal{Z}_{0;\ell;1}^{(2)}\big)\cup\bigcup_{\ell\geqslant 2}\iota_\ell(\mathcal{Z}_{2;\ell})\subset\mathfrak{M}_{\tau 2},\\ \text{and}\qquad \bigcup_{\ell\geqslant 1}\iota_\ell\big(\overline{\mathcal{W}}_{2,\ell}\times\mathcal{Z}_{0;\ell;1}^{(3)}\big)\cup\bigcup_{\ell\geqslant 2}\iota_\ell\big((\overline{\mathcal{C}}_{2,\ell}\times\mathcal{Z}_{0;\ell;12}^{(2)})\cup\mathcal{Z}_{2;\ell}\big)\subset\mathfrak{M}_{\tau 2}\end{gathered}$$

are closed.

The convergence of sequences of elements of $\mathfrak{M}_{\tau\bullet}$ for $\bullet=1,1a,1b$ is about the behaviors of derivatives of genus 0 and 1 J-holomorphic maps in sequences of such maps. We will establish the necessary statements, which are the analogues of [18, Propositions 5.1, 5.2] in the present situation, in a separate paper. They imply that the subspaces

$$\begin{gathered}\mathfrak{M}_{\tau 1a}^0\cup\bigcup_{\ell\geqslant 1}\iota_\ell\big(\overline{\mathcal{W}}_{2,\ell}\times\mathcal{Z}_{0;\ell;1}^{(3)}\big)\cup\bigcup_{\ell\geqslant 2}\iota_\ell\big((\overline{\mathcal{C}}_{2,\ell}\times\mathcal{Z}_{0;\ell;12}^{(2)})\cup\mathcal{Z}_{2;\ell}\big)\subset\mathfrak{M}_{\tau 1a}\cup\mathfrak{M}_{\tau 2},\\ \mathfrak{M}_{\tau 1b}^0\cup\bigcup_{\ell\geqslant 1}\iota_\ell\big(\overline{\mathcal{M}}_{2,\ell}\times\mathcal{Z}_{0;\ell;1}^{(2)}\big)\cup\bigcup_{\ell\geqslant 2}\iota_\ell\big((\overline{\mathcal{C}}_{2,\ell}\times\mathcal{Z}_{0;\ell;12}^{(2)})\cup\mathcal{Z}_{2;\ell}\big)\subset\mathfrak{M}_{\tau 1b}\cup\mathfrak{M}_{\tau 2},\\ \text{and}\qquad \mathfrak{M}_{\tau 1}^0\cup\mathfrak{M}_{\tau 1a}^0\cup\mathfrak{M}_{\tau 1b}^0\cup\mathfrak{M}_{\tau 2}^0\subset\mathfrak{M}_{\tau 1}\cup\mathfrak{M}_{\tau 1a}\cup\mathfrak{M}_{\tau 1b}\cup\mathfrak{M}_{\tau 2}\end{gathered}$$

are closed.

The next two propositions address the remaining possibilities for the convergence of sequences of elements of $\overline{\mathfrak{M}}^0$ inside of $\overline{\mathfrak{M}}$. Along with the last two paragraphs, they imply the main statement of Theorem 1.1.

Proposition 1.3. *If* $[\mathbf{u}_r]$ *is a sequence of elements of* $\mathfrak{M}^0_{\tau 0}=\mathfrak{M}_{\tau 0}$ *converging to* $[\mathbf{u}]\in\mathfrak{M}_{\tau\bullet}$ *with* $\bullet=1$, $1a$, *or* $1b$, *then* $[\mathbf{u}]\in\mathfrak{M}^0_{\tau\bullet}$.

Proposition 1.4. *If* $[\mathbf{u}_r]$ *is a sequence of elements of* $\mathfrak{M}^0_{\tau 0}=\mathfrak{M}_{\tau 0}$ *converging to* $[\mathbf{u}]\in\mathfrak{M}_{\tau 2}$, *then* $[\mathbf{u}]\in\mathfrak{M}^0_{\tau 2}$.

The proof of Proposition 1.3 is a straightforward adaptation of the proof of [18, Proposition 5.3]; see Section 5.3. The proof of Proposition 1.4 is far more intricate. It sharpens the analytic estimates obtained in [18], obtains estimates on the behavior on holomorphic differentials on families of genus 2 curves that have no parallels in [18], modifies the gluing setup to deal with bubbles attached at conjugate points of a contracted principal component, and introduces a bootstrapping argument to estimate the size of a solution of the $\bar{\partial}$ deformation equation on different regions of the domain.

As the description of $\mathfrak{M}^0_{\tau 2}$ in (1.11) might suggest, the proof of Proposition 1.4 requires considering a significant number of cases. These are distinguished by the type of the limiting map $[\mathbf{u}]\in\mathfrak{M}_{\tau 2}$ *and* by how the sequence $[\mathbf{u}_r]$ approaches $[\mathbf{u}]$. The former distinction is described by the split into Propositions 6.1-6.4. The latter distinction involves the 4 cases on page 69 and further sub-cases described on page 72.

In Section 2, we set up much of the necessary notation and terminology and describe a gluing construction for J-holomorphic maps that smooths out the nodes away from the primary contracted curves. Section 3 smooths out the remaining nodes and obtains estimates on the solutions of the $\bar{\partial}$ deformation equation on different regions of the domain. Section 4 concerns the behavior of holomorphic differentials under smoothings of the domains. Obstructions to completely solving the $\bar{\partial}$ deformation equation are computed in Section 5. The main conclusions of Sections 3-5 are applied in Section 5.3 to obtain Proposition 1.3 and in Section 6 to establish many of the convergence statements behind Proposition 1.4.

The bootstrapping estimate of Proposition 5.5 is used to obtain sharp bounds on the quadratic error term in the $\bar{\partial}$ deformation equation (5.4). This term can be entirely avoided for Kähler targets. Restricting to such targets would significantly simplify the arguments, especially in relation to Section 3, but would still require many of the estimates on holomorphic differentials obtained in Section 4. The Kähler case of Theorem 1.1 would suffice for studying genus 2 Gromov-Witten invariants of complete intersections, but it would say little in regards to the speculation concerning the fundamental structure of $\overline{\mathfrak{M}}_2(X,A;J)$ raised in [12].

2 Analytic setup

For the remainder of this dissertation, we assume that (X,ω) is a symplectic manifold, J is an ω-*compatible* almost complex structure on X, $g_X(\cdot,\cdot)=\omega(\cdot,J\cdot)$, and ∇^J is the J-linear connection induced by the Levi-Civita connection ∇ of g_X. The ω-compatibility of J implies that the torsion tensor T_{∇^J} of ∇^J and the Nijenhuis tensor N_J satisfy

$$T_{\nabla^J}\big(\xi(x),\xi'(x)\big)\equiv\big(\nabla^J_\xi\xi'-\nabla^J_{\xi'}\xi-[\xi,\xi']\big)\big|_x=-\frac{1}{4}N_J\big(\xi(x),\xi'(x)\big)\quad\forall\,x\in X,\ \xi,\xi'\in\Gamma(X;TX); \tag{2.1}$$

see [10, Section 2.1]. We denote by exp the exponential map induced by ∇^J. For every $x\in X$ and $v\in T_xX$, let Π_v be the parallel transport with respect to the connection ∇^J along the geodesic

$$\gamma_v:[0,1]\longrightarrow X,\qquad t\mapsto \exp_x(tv).$$

A genus g X-valued map, or simply a genus g map, is a tuple

$$\mathbf{u}=(\Sigma,u) \tag{2.2}$$

consisting of a curve Σ of arithmetic genus g and a smooth map $u:\Sigma\longrightarrow X$. For every irreducible component Σ_w of Σ, let

$$\mathbf{u}_w=\big(\Sigma_w,u_w=u|_{\Sigma_w}\big).$$

Given a constant $p>2$, define

$$\Gamma(\mathbf{u})\equiv L^p_1(\Sigma;u^*TX),\qquad \Gamma^{0,1}(\mathbf{u})\equiv L^p\big(\Sigma;(T^*\Sigma)^{0,1}\otimes_{\mathbb{C}}u^*TX\big).$$

By [10, Section 3.1],

$$D_{\mathbf{u}_w}:\ \Gamma(\mathbf{u}_w)\longrightarrow\Gamma^{0,1}(\mathbf{u}_w),\qquad D_{\mathbf{u}_w}\xi=\frac{1}{2}\Big(\nabla^{J,u_w}\xi+J\circ\nabla^{J,u_w}\xi\circ\mathfrak{j}_{\Sigma_w}\Big)+\frac{1}{4}N_J(\partial_Ju_w,\xi), \tag{2.3}$$

is a linearization of the $\bar\partial_J$-operator at $\mathbf{u}_w$ with respect to the connection ∇^J, the exponential map exp, and the parallel transport Π_v. Let

$$D_{\mathbf{u}}:\ \Gamma(\mathbf{u})\longrightarrow\Gamma^{0,1}(\mathbf{u}) \tag{2.4}$$

be the corresponding linearization at $\mathbf{u}$.

2.1 Maps and dual graphs

A graph (Ver, Edg) is pair consisting of a finite set Ver of vertices and an element

$$\mathrm{Edg}\in\mathrm{Sym}^m(\mathrm{Sym}^2\mathrm{Ver})$$

for some $m\in\mathbb{Z}^{\geqslant0}$. We will view Edg as a collection of two-element subsets of Ver, called edges, but some of these subsets may contain the same element of Ver twice and Edg may contain several copies of the same two-element subset. An edge is called separating if its removal from the graph (Ver, Edg) disconnect the graph; otherwise it called non-separating. Hereafter we use w and e to denote vertices and edges, respectively. Every edge e will be oriented and written as (w_e^+,w_e^-).

An S-marked decorated graph or simply decorated graph

$$\mathcal{T}=\big(\mathrm{Ver},\mathrm{Edg},S,\mathfrak{g},\mathfrak{m}\big) \tag{2.5}$$

consists of a graph (Ver, Edg), a finite set S, and maps

$$\mathfrak{g}\colon \mathrm{Ver} \longrightarrow \mathbb{Z}^{\geqslant 0} \qquad \text{and} \qquad \mathfrak{m}\colon S \longrightarrow \mathrm{Ver}.$$

We define the arithmetic genus $g_a(\mathcal{T})$ of $\mathcal{T}$ as in (2.5) by

$$g_a(\mathcal{T}) = p_a(\mathcal{T}) + \sum_{w\in \mathrm{Ver}} \mathfrak{g}(w),$$

where $p_a(\mathcal{T})$ is the arithmetic genus of the graph (Ver, Edg).

A decorated subgraph or simply subgraph of a decorated graph $\mathcal{T}$ as in (2.5) is a decorated graph

$$\mathcal{T}' = (\mathrm{Ver}', \mathrm{Edg}', S', \mathfrak{g}', \mathfrak{m}') \tag{2.6}$$

such that $\mathrm{Ver}' \subset \mathrm{Ver}$ and

- $\mathrm{Edg}' \subset \mathrm{Edg}$ is the subcollection of the edges with both vertices in Ver',
- S' is the disjoint union of $\mathfrak{m}^{-1}(\mathrm{Ver}')$ and the subcollection $\mathrm{Edg}'_{\bullet} \subset \mathrm{Edg}$ of the edges with one vertex in Ver' and the other in $\mathrm{Ver}-\mathrm{Ver}'$,
- $\mathfrak{g}' = \mathfrak{g}|_{\mathrm{Ver}'}$,
- $\mathfrak{m}'|_{\mathfrak{m}^{-1}(\mathrm{Ver}')} = \mathfrak{m}|_{\mathfrak{m}^{-1}(\mathrm{Ver}')}$ and $\mathfrak{m}'(e) = w$ if $e\in \mathrm{Edg}'_{\bullet}$ and $e\cap \mathrm{Ver}' = \{w\}$.

Thus, we choose the vertices $\mathrm{Ver}' \subset \mathrm{Ver}$ to be contained in $\mathcal{T}'$ and then cut the edges connecting Ver' with $\mathrm{Ver}-\mathrm{Ver}'$ in half and thus convert them to marked points. We define the complement of a decorated subgraph $\mathcal{T}'$ as in (2.6) to be the decorated subgraph

$$(\mathcal{T}')^c = (\mathrm{Ver}'^c, \mathrm{Edg}'^c, S'^c, \mathfrak{g}'^c, \mathfrak{m}'^c) \tag{2.7}$$

of $\mathcal{T}$ with $\mathrm{Ver}'^c = \mathrm{Ver}-\mathrm{Ver}'$.

Let $\mathcal{T}$ be a connected decorated graph as in (2.5) with $g_a(\mathcal{T}) \geqslant 1$. The principal subgraph

$$\mathcal{T}_P = \big(\mathrm{Ver}_P, \mathrm{Edg}_P, S_P, \mathfrak{g}_P, \mathfrak{m}_P\big) \tag{2.8}$$

of $\mathcal{T}$ is the minimal connected decorated subgraph satisfying $g_a(\mathcal{T}_P) = g_a(\mathcal{T})$. Elements of Ver_P and Edg_P are called principal vertices and principal edges, respectively. Every principal vertex w with $\mathfrak{g}_P(w) = 0$ belongs to at least 2 principal edges.

For a curve Σ with marked points labeled by a finite set S, its dual graph $\mathcal{T}_\Sigma$ is a decorated graph. A vertex and an edge correspond to an irreducible component and a node, respectively. A separating (resp. non-separating) node corresponds to a separating (resp. non-separating) edge. For $e = (w_e^+, w_e^-)$, let $\Sigma_e^\pm = \Sigma_{w_e^\pm}$ and denote by $x_e^\pm \in \Sigma_e^\pm$ the corresponding nodal points. The map $\mathfrak{m}$ assigns the marked point labeled by $s\in S$ to the irreducible component $\Sigma_{\mathfrak{m}(s)}$ of Σ containing it, while the map $\mathfrak{g}$ labels each vertex with the genus of the corresponding component. The arithmetic genus $g_a(\Sigma)$ of Σ is $g_a\big(\mathcal{T}_\Sigma\big)$. If $g_a(\Sigma) \geqslant 1$, the principal curve Σ_P of Σ is the union of the irreducible components of Σ whose dual graph is $\mathcal{T}_{\Sigma;P}$.

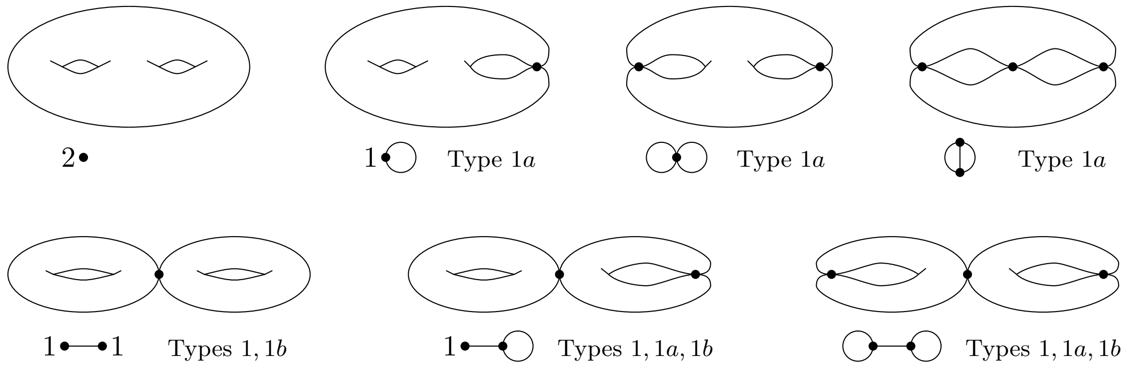

Figure 3: Genus 2 principal curves, their dual graphs, and compatible bubble types other than 0 and 2; each node can be replaced with a chain of spheres

For every $g\in\mathbb{Z}^{\geqslant 0}$ and every finite set S satisfying $2g+|S|\geqslant 3$, we denote by $\overline{\mathcal{M}}_{g,S}$ the Deligne-Mumford moduli space of S-marked genus g curves. If $\mathcal{T}$ is the dual graph of an element of $\overline{\mathcal{M}}_{g,S}$, let

$$\mathcal{M}_{\mathcal{T}} \equiv \big\{ [\Sigma]\in\overline{\mathcal{M}}_{g,S}\colon \mathcal{T}_\Sigma=\mathcal{T} \big\} \subset \overline{\mathcal{M}}_{g,S}\,.$$

Such $\mathcal{M}_{\mathcal{T}}$'s decompose $\mathcal{M}_{g,S}$ into strata.

Let Σ be a connected curve, not necessarily stable. If $g_a(\Sigma)=1$, then Σ_P is either a smooth torus with a complex structure or a circle of spheres. If $g_a(\Sigma)\geqslant 2$, however, the topological structure of Σ_P can be rather complicated. A (maximal) chain $\mathcal{C}$ of spheres is a (maximal) connected union of irreducible components of Σ_P so that $g_a(\mathcal{C})=0$ and each irreducible component of $\mathcal{C}$ contains exactly 2 nodes of Σ_P. In Figure 3, the topological types of unmarked genus 2 principal curves and their dual graphs are listed; the labeling $\mathfrak{g}(w)$ is omitted if it is 0. Each node in Figure 3 can be replaced with a chain of spheres.

The general structure of genus g maps is described by bubble types

$$\mathscr{T} = \big(\mathcal{T}\,;\,\mathfrak{d}\colon\mathrm{Ver}\to H_2(X;\mathbb{Z})\,\big), \tag{2.9}$$

where $\mathcal{T}$ is a decorated graph as in (2.5). A (maximal) contracted subgraph of a bubble type $\mathscr{T}$ is a (maximal) connected decorated subgraph

$$\mathcal{T}_0 \equiv \big(\mathrm{Ver}_0, \mathrm{Edg}_0, S_0, \mathfrak{g}_0, \mathfrak{m}_0\big) \tag{2.10}$$

of $\mathcal{T}$ such that $\mathfrak{d}(w)=0$ for all $w\in\mathrm{Ver}_0$. Let $\chi(\mathcal{T}_0)$ be the set of the edges connecting $\mathcal{T}_0$ with its complement $\mathcal{T}_0^c$ in $\mathcal{T}$. We assume the edges of $\mathcal{T}$ are oriented so that

$$w_e^-\in\mathrm{Ver}_0 \quad \forall\, e\in\chi(\mathcal{T}_0). \tag{2.11}$$

Define

$$\widehat{\mathrm{Ver}}_0 \equiv \mathrm{Ver}_0 \sqcup \big\{w_e^+\colon\; e\in\chi(\mathcal{T}_0)\big\}\,, \qquad \widehat{\mathrm{Edg}}_0 \equiv \mathrm{Edg}_0 \sqcup \chi(\mathcal{T}_0)\,. \tag{2.12}$$

A primary contracted subgraph of $\mathscr{T}$ is a maximal contracted subgraph $\mathcal{T}_0$ with $g_a(\mathcal{T}_0) \geqslant 1$. We denote by $\mathrm{PC}(\mathscr{T})$ the set of primary contracted subgraphs of $\mathscr{T}$. For each $\mathcal{T}_0 \in \mathrm{PC}(\mathscr{T})$ as in (2.10), let

$$\mathcal{T}_{0;P} \equiv \big(\mathrm{Ver}_{0;P}, \mathrm{Edg}_{0;P}, S_{0;P}, \mathfrak{g}_{0;P}, \mathfrak{m}_{0;P}\big) \tag{2.13}$$

be the principal subgraph of $\mathcal{T}_0$ and $\chi(\mathcal{T}_{0;P})$ be the set of the edges connecting $\mathcal{T}_{0;P}$ with its complement in $\mathcal{T}$.

We define a genus 2 bubble type $\mathscr{T}$ to be of

- Type 0 if $\mathrm{PC}(\mathscr{T}) = \emptyset$;
- Type 1 if $\mathrm{PC}(\mathscr{T}) = \{\mathcal{T}_0\}$ with $g_a(\mathcal{T}_0) = 1$ and $\big|\chi(\mathcal{T}_0) \cap \mathrm{Edg}_P\big| = 1$;
- Type 1a if $\mathrm{PC}(\mathscr{T}) = \{\mathcal{T}_0\}$ with $g_a(\mathcal{T}_0) = 1$ and $\big|\chi(\mathcal{T}_0) \cap \mathrm{Edg}_P\big| = 2$;
- Type 1b if $\big|\mathrm{PC}(\mathscr{T})\big| = 2$;
- Type 2 if $\mathrm{PC}(\mathscr{T}) = \{\mathcal{T}_0\}$ with $g_a(\mathcal{T}_0) = 2$.

Every curve in Figure 3 can be the principal curve of some maps of Types 0 and 2. We note below each principal curve in Figure 3 the remaining possibilities for the compatible bubble types.

For a genus g X-valued map $\mathbf{u} = (\Sigma_{\mathbf{u}}, u)$, the dual graph $\mathcal{T}_{\mathbf{u}}$ of $\Sigma_{\mathbf{u}}$ together with the map

$$\mathfrak{d}_{\mathbf{u}} \colon \mathrm{Ver}_{\mathbf{u}} \longrightarrow H_2(X;\mathbb{Z}), \qquad \mathfrak{d}_{\mathbf{u}}(w) = \{u_w\}_*[\Sigma_{\mathbf{u};w}],$$

determines a bubble type $\mathscr{T}_{\mathbf{u}}$. A (primary) contracted curve $\Sigma_{\mathbf{u};0}$ of $\mathbf{u}$ corresponds to a (primary) contracted subgraph $\mathcal{T}_0 = \mathcal{T}_{\Sigma_{\mathbf{u};0}}$ of $\mathscr{T}_{\mathbf{u}}$. For $\bullet = 0$, 1, $1a$, $1b$, and 2, an element $[\mathbf{u}]$ of $\overline{\mathfrak{M}}$ belongs to the subspace $\mathfrak{M}_{\mathcal{T}\bullet}$ if and only if $\mathscr{T}_{\mathbf{u}}$ is of Type $\bullet$.

2.2 A two-step pregluing construction

We call a triple $(\mathscr{T}, \mathcal{T}_0, \mathcal{T}_{0;P})$ a graph framing if

- $\mathscr{T} = (\mathcal{T}, \mathfrak{d})$ is a genus g bubble type as in (2.9),
- $\mathcal{T}_0$ is a contracted subgraph of $\mathcal{T}$ as in (2.10),
- $\mathcal{T}_{0;P}$ is the principal subgraph of $\mathcal{T}_0$ as in (2.13) if $g_a(\mathcal{T}_0) \geqslant 1$ and is a connected subgraph of $\mathcal{T}_0$ if $g_a(\mathcal{T}_0) = 0$,

and $\mathfrak{m}(S) \subset \mathrm{Ver}_{0;P}$. Let $(\mathscr{T}, \mathcal{T}_0, \mathcal{T}_{0;P})$ be a graph framing. We describe a two-step pregluing construction. We first smooth out the nodes contained in $\mathcal{T}_0^c$. After this unobstructed step, we obtain J-holomorphic maps from the original contracted curve Σ_0 with the remaining irreducible components attached directly to Σ_0. In the second step, we smooth out the remaining nodes and obtain an approximately J-holomorphic map.

Let $\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be the configuration space of genus g X-valued maps whose bubble types are $\mathscr{T}$ (not of the equivalence classes of such maps). For $\mathbf{u} \in \widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$, we denote by $\Sigma_{\mathbf{u};0;P} \subset \Sigma_{\mathbf{u};0}$ the unions of irreducible components of $\Sigma_{\mathbf{u}}$ corresponding to $\mathcal{T}_{0;P}$ and $\mathcal{T}_0$, respectively, and by

$$\mathbf{u}_{0;P} = \big(\Sigma_{\mathbf{u};0;P}, u_{0;P}\big) \qquad \text{and} \qquad \mathbf{u}_0 = \big(\Sigma_{\mathbf{u};0}, u_0\big)$$

the restrictions of $\mathbf{u}$ to these domains. For $e\in\mathrm{Edg}$, we denote by $\Sigma^{\pm}_{\mathbf{u};e}$ the irreducible components of the domain $\Sigma_{\mathbf{u}}$ of $\mathbf{u}$ corresponding to $w_e^{\pm}$. Let

$$\widetilde{L}_e \longrightarrow \widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$$

be the complex line bundle so that

$$\widetilde{L}_e|_{\mathbf{u}} = T_{x_e^+}\Sigma^+_{\mathbf{u};e}\otimes T_{x_e^-}\Sigma^-_{\mathbf{u};e} \qquad \forall\ \mathbf{u}\in\widetilde{\mathfrak{X}}_{\mathscr{T}}(X).$$

Let

$$\pi_{\widetilde{\mathcal{F}\mathscr{T}}}\colon \widetilde{\mathcal{F}\mathscr{T}} = \bigoplus_{e\in\mathrm{Edg}}\widetilde{L}_e \longrightarrow \widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$$

be the bundle of gluing parameters. Define

$$|\upsilon|=\max\big\{|\upsilon_e|\colon e\in\mathrm{Edg}\big\} \quad \forall\,\upsilon=(\upsilon_e)_{e\in\mathrm{Edg}}\in\widetilde{\mathcal{F}\mathscr{T}}, \qquad \widetilde{\mathcal{F}\mathscr{T}}^{\varnothing}=\big\{\upsilon\in\widetilde{\mathcal{F}\mathscr{T}}\colon |\upsilon_e|>0\ \ \forall\,e\in\mathrm{Edg}\big\}.$$

For every subset E of Edg, define

$$\widetilde{\mathcal{F}\mathscr{T}}(E) = \bigoplus_{e\in E}\widetilde{L}_e \subset \widetilde{\mathcal{F}\mathscr{T}}. \tag{2.14}$$

In particular, $\widetilde{\mathcal{F}\mathscr{T}}=\widetilde{\mathcal{F}\mathscr{T}}(\mathrm{Edg})$. For every $\upsilon\in\widetilde{\mathcal{F}\mathscr{T}}$, we denote by $\upsilon(E)$ its projection to $\widetilde{\mathcal{F}\mathscr{T}}(E)$. For each $e\in\mathrm{Edg}$, we define $\upsilon_e^c=\upsilon(\mathrm{Edg}-\{e\})$. Let

$$\widetilde{\mathcal{F}\mathscr{T}}_0=\widetilde{\mathcal{F}\mathscr{T}}\big(\mathrm{Edg}_0\big), \qquad \widetilde{\mathcal{F}\mathscr{T}}_1=\widetilde{\mathcal{F}\mathscr{T}}\big(\mathrm{Edg}_0^c\big), \qquad \widetilde{\mathcal{F}\mathscr{T}}_{01}=\widetilde{\mathcal{F}\mathscr{T}}\big(\mathrm{Edg}_0\sqcup\mathrm{Edg}_0^c\big). \tag{2.15}$$

For every $\upsilon\in\widetilde{\mathcal{F}\mathscr{T}}$, we denote by υ_0, υ_1, and υ_{01} the projections of υ onto $\widetilde{\mathcal{F}\mathscr{T}}_0$, $\widetilde{\mathcal{F}\mathscr{T}}_1$, and $\widetilde{\mathcal{F}\mathscr{T}}_{01}$, respectively.

Let $\mathbf{u}=(\Sigma_{\mathbf{u}},u)$ be an element of $\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ so that u is J-holomorphic. We choose finite-dimensional linear subspaces

$$\begin{aligned}&\widetilde{\Gamma}^{0,1}_-(\mathbf{u};P)\subset\Gamma\big(\Sigma_{\mathbf{u};0;P}\times X;\pi_1^*(T^*\Sigma_{\mathbf{u};0;P})^{0,1}\otimes_{\mathbb{C}}\pi_2^*TX\big) \qquad \text{and}\\ &\widetilde{\Gamma}^{0,1}_-(\mathbf{u};w)\subset\Gamma\big(\Sigma_{\mathbf{u};w}\times X;\pi_1^*(T^*\Sigma_{\mathbf{u};w})^{0,1}\otimes_{\mathbb{C}}\pi_2^*TX\big) \qquad \forall\,w\in\mathrm{Ver}_0^c\end{aligned} \tag{2.16}$$

with the following properties:

(B1) every element of $\widetilde{\Gamma}^{0,1}_-(\mathbf{u};P)$ vanishes on a neighborhood of every nodal point of $\Sigma_{\mathbf{u}}$ contained in $\Sigma_{\mathbf{u};0;P}$ and

$$\Gamma^{0,1}(\mathbf{u}_{0;P})=\big\{D_{\mathbf{u}_{0;P}}\xi\colon \xi\in\Gamma(\mathbf{u}_{0;P})\big\}\oplus\big\{\{\mathrm{id}\times u_{0;P}\}^*\eta\colon \eta\in\widetilde{\Gamma}^{0,1}_-(\mathbf{u};P)\big\}; \tag{2.17}$$

(B2) every element of $\widetilde{\Gamma}^{0,1}_-(\mathbf{u};w)$ with $w\in\mathrm{Ver}_0^c$ vanishes on a neighborhood of every nodal point contained in $\Sigma_{\mathbf{u};w}$ and

$$\Gamma^{0,1}(\mathbf{u}_w)=\big\{D_{\mathbf{u}_w}\xi\colon \xi\in\Gamma(\mathbf{u}_w)\ \text{s.t.}\ \xi\big(x_e\big)=0\ \ \forall\,e\ni w\big\}+\big\{\{\mathrm{id}\times u_w\}^*\eta\colon \eta\in\widetilde{\Gamma}^{0,1}_-(\mathbf{u};w)\big\}. \tag{2.18}$$

Let

$$\widetilde{\Gamma}_-^{0,1}(\mathbf{u}) \subset \Gamma\big(\Sigma\times X;\, \pi_1^*(T^*\Sigma)^{0,1}\otimes_{\mathbb{C}}\pi_2^*TX\big) \tag{2.19}$$

be the finite-dimensional subspace obtained by extending the elements of $\widetilde{\Gamma}_-^{0,1}(\mathbf{u};P)$ and $\widetilde{\Gamma}_-^{0,1}(\mathbf{u};w)$ with $w\in\mathrm{Ver}_0^c$ by zero outside of $\Sigma_{\mathbf{u};0;P}$ and $\Sigma_{\mathbf{u};w}$, respectively.

Let $\widetilde{\mathfrak{U}}_{\mathscr{T}}\longrightarrow\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be the universal curve and $\mathfrak{V}\subset\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be a small neighborhood of $\mathbf{u}$. For $w\in\mathrm{Ver}_0^c$, the vector space $\widetilde{\Gamma}_-^{0,1}(\mathbf{u};w)$ can be extended to $\widetilde{\mathfrak{U}}_{\mathscr{T}}|_{\mathfrak{V}}$ so that every element of $\widetilde{\Gamma}_-^{0,1}(\mathbf{u};w)$ still vanishes on a neighborhood of all nodal points of the irreducible component labeled by w. Therefore, we obtain subspaces

$$\widetilde{\Gamma}_-^{0,1}(\mathbf{u}') \subset \Gamma\big(\Sigma_{\mathbf{u}'}\times X;\, \pi_1^*(T^*\Sigma_{\mathbf{u}'})^{0,1}\otimes_{\mathbb{C}}\pi_2^*TX\big), \qquad \mathbf{u}'\in\mathfrak{V}. \tag{2.20}$$

Set

$$\widetilde{\mathcal{U}}_{\mathscr{T}}(X;J)\equiv\big\{\,\mathbf{u}'=(\Sigma_{\mathbf{u}'},u')\in\widetilde{\mathfrak{X}}_{\mathscr{T}}(X):\ \bar\partial_J u'\in\{\mathrm{id}\times u'\}^*\widetilde{\Gamma}_-^{0,1}(\mathbf{u}')\,\big\}\subset\widetilde{\mathfrak{X}}_{\mathscr{T}}(X).$$

By the Implicit Function Theorem, $\widetilde{\mathcal{U}}_{\mathscr{T}}(X;J)$ is a smooth manifold near $\mathbf{u}$. For every $\mathbf{u}'\in\widetilde{\mathcal{U}}_{\mathscr{T}}(X;J)$, the subspace of $T_{\mathbf{u}'}\widetilde{\mathcal{U}}_{\mathscr{T}}(X;J)$ corresponding to the deformation with $\Sigma_{\mathbf{u}'}$ fixed is described by

$$\Gamma_-(\mathbf{u}')=\big\{\xi\in\Gamma(\mathbf{u}'):\ D_{\mathbf{u}'}\xi\in\{\mathrm{id}\times u'\}^*\widetilde{\Gamma}_-^{0,1}(\mathbf{u}')\,\big\}\,. \tag{2.21}$$

By (2.17), $u'_{0;P}\equiv u'|_{\Sigma_{\mathbf{u}';0;P}}$ is a constant map.

Fix a small precompact open neighborhood $\widetilde{U}$ of $\mathbf{u}$ in $\widetilde{\mathcal{U}}_{\mathscr{T}}(X;J)$. Let Δ be a small neighborhood of the zero section of $\widetilde{\mathcal{F}\mathscr{T}}|_{\widetilde{U}}$, $\mathfrak{U}\longrightarrow\Delta$ be a family of deformations of the domains of the elements of $\widetilde{U}$, and $\Delta^{\varnothing}=\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}^{\varnothing}$. The fiber of $\mathfrak{U}$ over $\upsilon\in\Delta$ is a nodal curve Σ_υ. Its bubble type $\mathscr{T}_\upsilon$ is obtained from $\mathscr{T}$ by deleting the edges corresponding to the non-zero components of υ and identifying the vertices of each deleted edge. We fix a Riemannian metric on $\mathfrak{U}$ and denote its restriction to Σ_υ by g_υ. For any smooth map $f:\Sigma_\upsilon\longrightarrow X$, the metrics g_υ and g_X determine the modified Sobolev norms $\|\cdot\|_{\upsilon,p,1}$ on $\Gamma(\Sigma_\upsilon,f)$ and $\|\cdot\|_{\upsilon,p}$ on $\Gamma^{0,1}(\Sigma_\upsilon,f)$ as in [7, Section 3]. By [15, Lemma 3.5(3)], there exists a constant $C=C(f)$ such that

$$\|\xi\|_{\upsilon,C^0}\leqslant C\|\xi\|_{\upsilon,p,1}\qquad\forall\,\xi\in\Gamma(\Sigma_\upsilon,f). \tag{2.22}$$

There is a collection

$$q_{E_1;E_2}:\ \mathfrak{U}|_{\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}(E_2)}\longrightarrow\mathfrak{U}|_{\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}(E_1)},\qquad E_1\subset E_2\subset\mathrm{Edg},$$

of continuous fiber-preserving retractions such that the diagrams

$$\begin{array}{ccccc}
\mathfrak{U} & \xrightarrow{q_{E_2;\mathrm{Edg}}} & \mathfrak{U}|_{\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}(E_2)} & \xrightarrow{q_{E_1;E_2}} & \mathfrak{U}|_{\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}(E_1)}\\
\downarrow & & \downarrow & & \downarrow\\
\Delta & \longrightarrow & \Delta\cap\widetilde{\mathcal{F}\mathscr{T}}(E_2) & \longrightarrow & \Delta\cap\widetilde{\mathcal{F}\mathscr{T}}(E_1)
\end{array}
\qquad (\text{with } q_{E_1;\mathrm{Edg}}:\mathfrak{U}\longrightarrow\mathfrak{U}|_{\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}(E_1)}) \tag{2.23}$$

commute. For every $\upsilon\in\Delta$, we denote by

$$q_{\upsilon(E_1);\upsilon(E_2)}:\ \Sigma_{\upsilon(E_2)}\longrightarrow\Sigma_{\upsilon(E_1)} \tag{2.24}$$

the restriction of $q_{E_1;E_2}$ to $\Sigma_{\upsilon(E_2)}$ and call it a basic gluing map. If $E_2=\mathrm{Edg}$ and $E_1=\emptyset$, the corresponding basic gluing map is simply denoted by q_υ.

For every $\delta\in\mathbb{R}$, $e\in\mathrm{Edg}$, $\mathbf{u}'\in\widetilde{U}$, and $\upsilon\in\Delta|_{\mathbf{u}'}$, denote by $\Sigma_{\mathbf{u}';e}(\delta)\subset\Sigma_{\mathbf{u}'}$ the closed $g_{\mathbf{u}'}$-ball of radius δ centered at $x_e(\mathbf{u}')$ and let

$$\Sigma_{\upsilon;e}(\delta)=q_\upsilon^{-1}\big(\Sigma_{\mathbf{u}';e}(\delta)\big),\qquad \Sigma^\pm_{\upsilon;e}(\delta)=\Sigma_{\upsilon;e}(\delta)\cap q_\upsilon^{-1}\big(\Sigma^\pm_{\mathbf{u}';e}\big).$$

If $\upsilon_e=0$, let

$$x_e(\upsilon)=\big(x_e^+(\upsilon),x_e^-(\upsilon)\big)\in q_\upsilon^{-1}\big(\Sigma^+_{\mathbf{u}';e}\times\Sigma^-_{\mathbf{u}';e}\big)$$

be the node corresponding to e.

After possibly shrinking Δ, the Riemannian metric on $\mathfrak{U}$ and the maps $q_{E_1;E_2}$ can be chosen so that

(q1) every map $q_{E_1;E_2}$ is smooth on each stratum of $\mathfrak{U}|_{\Delta\cap\widetilde{\mathcal{FT}}(E_2)}$;

(q2) for every $\upsilon\in\Delta$, $q_{\upsilon(E_1);\upsilon(E_2)}$ is biholomorphic on the complement of the subspaces

$$q^{-1}_{\upsilon(E_1);\upsilon(E_2)}\big(\Sigma_{\upsilon(E_1);e}(2\sqrt{|\upsilon_e|})\big)\subset\Sigma_{\upsilon(E_2)}$$

with $e\in E_2-E_1$ and $\upsilon_e\neq0$;

(q3) there exists a constant $\delta_q=\delta_q(\mathbf{u})\in\mathbb{R}^+$ such that every restriction (2.24) is a $\big(g_{\upsilon(E_2)},g_{\upsilon(E_1)}\big)$-isometry on the complement of the subspaces $q^{-1}_{\upsilon(E_1);\upsilon(E_2)}\big(\Sigma_{\upsilon(E_1);e}(\delta_q)\big)$ with $e\in E_2-E_1$ and $\upsilon_e\neq0$;

(q4) there exist a constant $\delta_{\bar\partial}=\delta_{\bar\partial}(\mathbf{u})\in\mathbb{R}^+$ and holomorphic functions

$$z^+_{e;\upsilon},z^-_{e;\upsilon}\colon\Sigma_{\upsilon;e}(8\delta_{\bar\partial})\longrightarrow\mathbb{C},\qquad e\in\mathrm{Edg},\ \mathbf{u}'\in\widetilde{U},\ \upsilon\in\Delta|_{\mathbf{u}'},$$

such that $z^\pm_{e;\mathbf{u}'}$ are unitary coordinates centered at $x_e^\pm(\mathbf{u}')$ and

$$\begin{gathered}\big(z^+_{e;\upsilon}z^-_{e;\upsilon}\big)\big|_{\Sigma_{\upsilon;e}(8\delta_{\bar\partial})}\frac{\partial}{\partial z^+_{e;\mathbf{u}'}}\bigg|_{x_e^+(\mathbf{u}')}\otimes\frac{\partial}{\partial z^-_{e;\mathbf{u}'}}\bigg|_{x_e^-(\mathbf{u}')}=\upsilon_e\,,\qquad\big|z^+_{e;\upsilon}|_{q_\upsilon^{-1}(x_e(\mathbf{u}'))}\big|=\big|z^-_{e;\upsilon}|_{q_\upsilon^{-1}(x_e(\mathbf{u}'))}\big|\,,\\ z^\pm_{e;\upsilon}\big|_{\Sigma^\pm_{\upsilon;e}(8\delta_{\bar\partial})-\Sigma_{\upsilon;e}(2\sqrt{|\upsilon_e|})}=z^\pm_{e;\mathbf{u}'}\circ q_\upsilon\big|_{\Sigma^\pm_{\upsilon;e}(8\delta_{\bar\partial})-\Sigma_{\upsilon;e}(2\sqrt{|\upsilon_e|})};\end{gathered}\tag{2.25}$$

(q5) if $g\big(\Sigma^\pm_{\mathbf{u}';e}\big)=0$, the function $z^\pm_{e;\mathbf{u}'}$ extends to a meromorphic function on $\Sigma^\pm_{\mathbf{u}';e}$ such that $x_e^\pm(\mathbf{u}')$ is the only zero of $z^\pm_{e;\mathbf{u}'}$.

By (2.25), (2.23), (q2), and $z^\pm_{e;\upsilon}$ being holomorphic,

$$\begin{gathered}z^\pm_{e;\upsilon(E_2)}(x)=z^\pm_{e;\upsilon(E_1)}\big(q_{\upsilon(E_1);\upsilon(E_2)}(x)\big)\\ \forall\ x\in\begin{cases}\Sigma^\pm_{\upsilon(E_2);e}(8\delta_{\bar\partial})-\Sigma_{\upsilon(E_2);e}(2\sqrt{|\upsilon_e|})&\text{if }e\in E_2-E_1\text{ and }\upsilon_e\neq0;\\ \Sigma_{\upsilon(E_2);e}(8\delta_{\bar\partial})&\text{otherwise.}\end{cases}\end{gathered}\tag{2.26}$$

for all $e\in\mathrm{Edg}$, $E_1\subset E_2\subset\mathrm{Edg}$, and $\upsilon\in\Delta$. A choice of the functions $z_e^{\pm}$ associates every element $\upsilon\in\Delta|_{\mathbf{u}'}$ with a tuple

$$\big(\upsilon_e\big)_{e\in\mathrm{Edg}}\in\mathbb{C}^{\mathrm{Edg}} \qquad \text{s.t.} \qquad \upsilon_e=\upsilon_e\frac{\partial}{\partial z_{e;\mathbf{u}'}^+}\bigg|_{x_e^+(\mathbf{u}')}\otimes\frac{\partial}{\partial z_{e;\mathbf{u}'}^-}\bigg|_{x_e^-(\mathbf{u}')}. \tag{2.27}$$

We can assume that $1+64\delta_{\bar\partial}$ is less than the minimal distance between the nodal and marked points of $\Sigma_{\mathbf{u}'}$ for every $\mathbf{u}'\in\widetilde{U}$. We can also assume that $\delta_q<\delta_{\bar\partial}<1$,

$$16|\upsilon|<\delta_{\bar\partial}^2 \qquad \forall\,\upsilon\in\Delta\,, \tag{2.28}$$

and every element of $\widetilde\Gamma_-^{0,1}(\mathbf{u})$ vanishes on $\Sigma_{\mathbf{u}';e}(32\delta_{\bar\partial})$ for every $e\in\mathrm{Edg}$ and every $\mathbf{u}'\in\widetilde{U}$.

For $\mathbf{u}'\in\widetilde{U}$, let $\Sigma_{\mathbf{u}';0}$ be the contracted curve corresponding to $\mathcal{T}_0$. For $\upsilon\in\Delta|_{\mathbf{u}'}$ and $\delta\in\mathbb{R}$, let

$$\Sigma_{\upsilon;0}=q_\upsilon^{-1}\big(\Sigma_{\mathbf{u}';0}\big), \qquad \Sigma_{\upsilon;0}(\delta)=\Sigma_{\upsilon;0}\cup\bigcup_{e\in\chi(\mathcal{T}_0)}\Sigma_{\upsilon;e}(\delta), \qquad \mathrm{ev}_0(\upsilon)=\mathbf{u}'\big(\Sigma_{\mathbf{u}';0}\big)\in X. \tag{2.29}$$

With υ_1 as defined immediately after (2.15), let

$$\Gamma(\upsilon_1)=\Gamma\big(\Sigma_{\upsilon_1},u'\circ q_{\upsilon_1}\big).$$

By (2.18), the family of the deformations

$$\mathfrak{U}|_{\Delta\cap\widetilde{\mathcal{F}\mathcal{T}}_1}\longrightarrow\Delta\cap\widetilde{\mathcal{F}\mathcal{T}}_1$$

of the domains of elements of $\widetilde{U}$ extends to a continuous family of maps

$$\widetilde{u}_{\upsilon;1}\colon\Sigma_{\upsilon_1}\longrightarrow X,\quad \upsilon\in\Delta,\quad \text{s.t.}\quad \widetilde{u}_{\upsilon;1}\big(\Sigma_{\upsilon_1;0}\big)=\mathrm{ev}_0(\upsilon),\quad \bar\partial_J\widetilde{u}_{\upsilon;1}\in\big\{q_{\upsilon_1}\times\widetilde{u}_{\upsilon;1}\big\}^*\,\widetilde\Gamma_-^{0,1}(\mathbf{u}')\,. \tag{2.30}$$

For every $\upsilon\in\Delta|_{\mathbf{u}'}$, the map $\widetilde{u}_{\upsilon;1}$ determines an element

$$\zeta_{\upsilon;1}\in\Gamma(\upsilon_1) \qquad \text{s.t.} \qquad \zeta_{\upsilon;1}|_{\Sigma_{\upsilon_1;0}}=0 \quad\text{and}\quad \widetilde{u}_{\upsilon;1}=\exp_{u'\circ q_{\upsilon_1}}\zeta_{\upsilon;1} \tag{2.31}$$

and a smooth map

$$\widetilde\xi_{\upsilon;1}\colon\,\Sigma_{\upsilon_1;0}(8\delta_{\bar\partial})\longrightarrow T_{\mathrm{ev}_0(\upsilon)}X \qquad\text{by}\qquad \widetilde{u}_{\upsilon;1}=\exp_{\mathrm{ev}_0(\upsilon)}\widetilde\xi_{\upsilon;1}\,. \tag{2.32}$$

By continuity of the family of $\widetilde{u}_{\upsilon;1}$, there exist a continuous function $\epsilon=\epsilon_{\mathbf{u}}\colon(\mathbb{R},0)\longrightarrow(\mathbb{R},0)$ and $C\in\mathbb{R}^+$ such that

$$\|\zeta_{\upsilon;1}\|_{\upsilon_1,p,1}\leqslant\epsilon\big(|\upsilon_1|\big), \qquad \|\mathrm{d}\widetilde{u}_{\upsilon;1}\|_{\upsilon_1,p}\leqslant C \qquad \forall\,\upsilon\in\Delta\,. \tag{2.33}$$

Let $\beta\colon\mathbb{R}^+\longrightarrow[0,1]$ be a smooth cutoff function such that

$$\beta(r)=\begin{cases}1, & \text{if } r\leqslant 1/2;\\ 0, & \text{if } r\geqslant 1;\end{cases} \qquad \beta'(r)<0 \quad \forall\,r\in\big(\tfrac12,1\big). \tag{2.34}$$

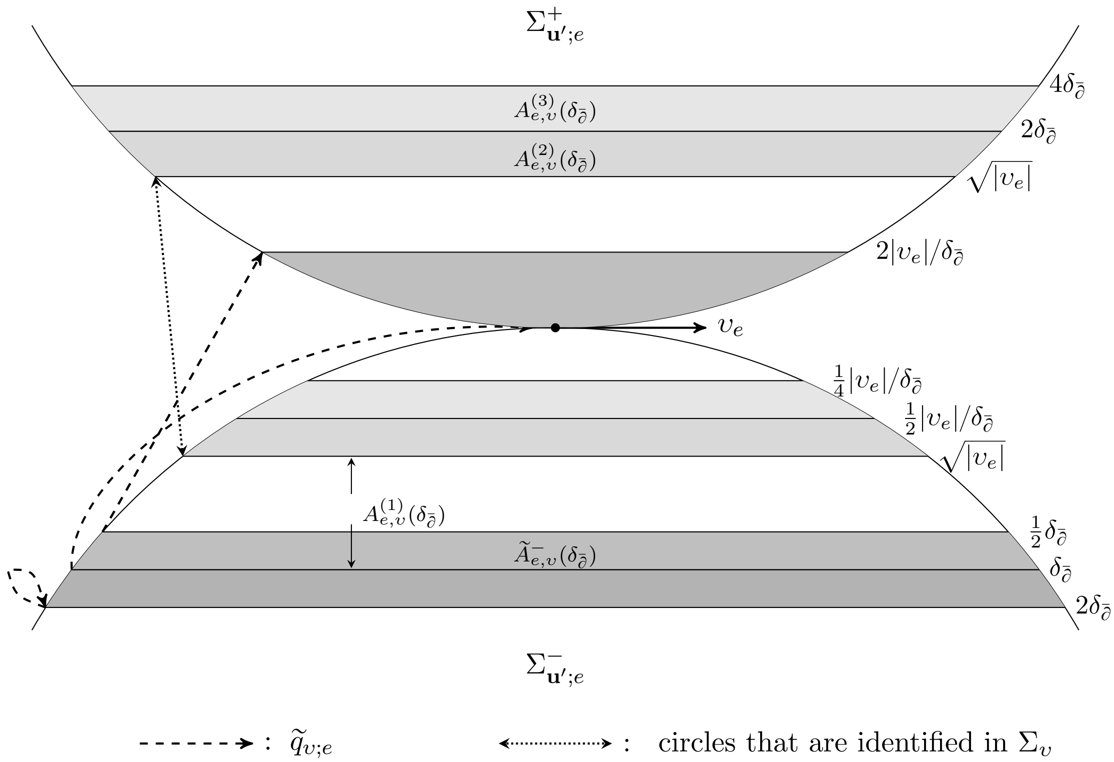

Figure 4: The modified gluing map $\widetilde{q}_{\upsilon;e}$ for $\upsilon\in\Delta|_{\mathbf{u}'}$ and $e\in\mathrm{Edg}$

For $\delta\in\mathbb{R}^+$, we define $\beta_\delta\in C^\infty(\mathbb{R};\mathbb{R})$ by $\beta_\delta(r)=\beta(r/\delta)$. Let

$$\beta_\upsilon\colon \Sigma_\upsilon \longrightarrow [0,1], \qquad \beta_\upsilon(x) = \begin{cases} 1, & \text{if } x\in\Sigma_{\upsilon;0}-\bigcup_{e\in\chi(\mathcal{T}_0)}\Sigma_{\upsilon;e}(\delta_{\bar\partial}); \\ \beta_{4\delta_{\bar\partial}}\big(|z^+_{e;\upsilon}(x)|\big), & \text{if } x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial}),\ e\in\chi(\mathcal{T}_0); \\ 0, & \text{otherwise.} \end{cases} \tag{2.35}$$

for every $\upsilon\in\Delta$.

For $e\in\mathrm{Edg}$, $\mathbf{u}'\in\widetilde{U}$, and $\upsilon\in\Delta|_{\mathbf{u}'}$, define

$$\begin{gathered} \widetilde{q}_{\upsilon;e}\colon \Sigma_{\upsilon;e}(2\delta_{\bar\partial}) \longrightarrow \Sigma^1_{\upsilon^c_e;e}(2\delta_{\bar\partial}) \qquad \text{by} \\ \begin{cases} z^+_{e;\upsilon^c_e}\big(\widetilde{q}_{\upsilon;e}(x)\big) = \beta_{\delta_{\bar\partial}}\big(|z^-_{e;\upsilon}(x)|\big)z^+_{e;\upsilon}(x), & \text{if } |z^-_{e;\upsilon}(x)|\leqslant\delta_{\bar\partial}\,; \\ z^-_{e;\upsilon^c_e}\big(\widetilde{q}_{\upsilon;e}(x)\big) = \Big(1-\beta_{\delta_{\bar\partial}}\big(|z^-_{e;\upsilon}(x)|/2\big)\Big)z^-_{e;\upsilon}(x), & \text{if } |z^-_{e;\upsilon}(x)|\geqslant\delta_{\bar\partial}\,. \end{cases} \end{gathered} \tag{2.36}$$

The map $\widetilde{q}_{\upsilon;e}$ is equivalent to the modified gluing map $\widetilde{q}_{\upsilon_0;2}$ of [18, Section 4.2] near $x_e(\upsilon^c_e)$; see Figure 4. Let

$$\widetilde{A}^-_{e,\upsilon}(\delta_{\bar\partial}) = \big\{x\in\Sigma_{\upsilon;e}(2\delta_{\bar\partial})\colon\ \tfrac{1}{2}\delta_{\bar\partial}\leqslant|z^-_e(x)|\leqslant\delta_{\bar\partial}\big\} \subset \Sigma_\upsilon\,. \tag{2.37}$$

Lemma 2.1. *The map* $\widetilde{q}_{\upsilon;e}$ *is biholomorphic on* $\Sigma_{\upsilon;e(2\delta_{\bar\partial})}-\widetilde{A}^-_{e;\upsilon}(\delta_{\bar\partial})$ *for every* $e\in\mathrm{Edg}$. *For every* $q\in[1,2)$, *there exists* $C_q\in\mathbb{R}$ *such that*

$$\|\mathrm{d}\widetilde{q}_{\upsilon;e}\|_{C^0}\leqslant C_1, \qquad \|\mathrm{d}\widetilde{q}_{\upsilon;e}\|_{C^0(\widetilde{A}^-_{e,\upsilon}(\delta_{\bar\partial}))}\leqslant C_1|\upsilon_e|, \tag{2.38}$$

$$\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|\mathrm{d}\widetilde{q}_{\upsilon;e}|\big(|\upsilon_e|+|z^-_{e;\upsilon}|\big)|\mathrm{d}z^-_{e;\upsilon}|\leqslant C_1|\upsilon|, \qquad \bigg(\int_{\Sigma_{\upsilon;e}(8\delta_{\bar\partial})}|z^+_{e;\upsilon}|^q|\mathrm{d}z^-_{e;\upsilon}|^q\bigg)^{\frac{1}{q}}\leqslant C_q|\upsilon_e|, \tag{2.39}$$

$$\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|\mathrm{d}\widetilde{q}_{\upsilon;e}||\mathrm{d}z^-_{e;\upsilon}|\leqslant C_1|\upsilon_e|\ln\big(|\upsilon_e|\big) \tag{2.40}$$

for all $\upsilon\in\Delta$.

Proof. The first statement follows from (2.36), (2.34), and the property (q4) of the basic gluing maps. The inequalities in (2.38) are the analogues of [18, (4.6),(4.7)] in the current setting. The inequalities in (2.39) and (2.40) follow from (2.36) and (q4). □

Lemma 2.2. *Let* $e\in\mathrm{Edg}$, $\upsilon\in\Delta$, *and*

$$\widetilde{f}\colon\ \Sigma^+_{\upsilon^c_e;e}(\delta_{\bar\partial})\longrightarrow X$$

be J*-holomorphic. Then* $f\equiv\widetilde{f}\circ\widetilde{q}_{\upsilon;e}:\Sigma_{\upsilon;e}(\delta_{\bar\partial})\longrightarrow X$ *is* J*-holomorphic on* $\Sigma_{\upsilon;e}(\delta_{\bar\partial})-\widetilde{A}^-_{e;\upsilon}(\delta_{\bar\partial})$. *Moreover, there exists a constant* $C\in\mathbb{R}$ *such that*

$$\big\|\bar\partial_J f|_{\widetilde{A}^-_{e;\upsilon}}\big\|_{\upsilon,p}\leqslant C\,\big\|\mathrm{d}\widetilde{f}|_{\Sigma^+_{\upsilon^c_e;e}(2|\upsilon_e|/\delta_{\bar\partial})}\big\|_{\upsilon^c_e,p}\,|\upsilon_e|^{\frac{p-2}{p}}$$

for all $\upsilon\in\Delta$.

Lemma 2.2 is a restatement of the second bound in [18, (4.8)].

Given a collection E of (oriented) edges, the maps $\widetilde{q}_{\upsilon;e}$ with $e\in E$ determine a single map

$$\begin{aligned}\widetilde{q}_{\upsilon;E}\colon\ \Sigma_\upsilon\longrightarrow\Sigma_{\upsilon(\mathrm{Edg}-E)} \qquad &\text{s.t.}\\ \widetilde{q}_{\upsilon;E}(z)=q_{\upsilon(\mathrm{Edg}-E);\upsilon}(z)\qquad &\forall\,z\in\Sigma_\upsilon-\bigsqcup_{e\in E}\Sigma_{\upsilon;e}(2\delta_{\bar\partial}).\end{aligned} \tag{2.41}$$

We call $\widetilde{q}_{\upsilon;E}$ a modified gluing map. Let

$$\widetilde{q}_\upsilon=\widetilde{q}_{\upsilon;\chi(\mathcal{T}_0)}\colon\ \Sigma_\upsilon\longrightarrow\Sigma_{\upsilon_{01}}.$$

For $\mathbf{u}'\in\widetilde{U}$ and $\upsilon\in\Delta|_{\mathbf{u}'}$, define

$$\mathbf{u}_\upsilon=\big(\Sigma_\upsilon,\,u_\upsilon=\widetilde{u}_{\upsilon;1}\circ q_{\upsilon_1;\upsilon_{01}}\circ\widetilde{q}_\upsilon\big)\,,\qquad \Gamma(\upsilon)=\Gamma(\mathbf{u}_\upsilon),\qquad \Gamma^{0,1}(\upsilon)=\Gamma^{0,1}(\mathbf{u}_\upsilon). \tag{2.42}$$

By the first bound in [18, (4.8)] and (2.33), there exists a constant $C\in\mathbb{R}$ such that

$$\|\mathrm{d}u_\upsilon\|_{\upsilon,p}\leqslant C\qquad\forall\,\upsilon\in\Delta. \tag{2.43}$$

Since $\widetilde{u}_{\upsilon;1}|_{\Sigma_{\upsilon_1;0}}$ is constant,

$$u_\upsilon=\text{const.}\qquad\text{on}\qquad\Sigma_{\upsilon;0}-\bigsqcup_{e\in\chi(\mathcal{T}_0)}\Sigma_{\upsilon;e}(\delta_{\bar\partial}). \tag{2.44}$$

We denote by $\nabla^\upsilon\equiv\nabla^{J,u_\upsilon}$ the J-linear connection on u_υ^*TX induced by ∇^J and by D_υ the linearization D_{u_υ} of the $\bar\partial_J$-operator at u_υ with respect to the connection ∇^J.

3 Multi-step gluing constructions

We call a graph framing $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ a pseudo-tree framing if every connected component $\mathcal{T}^c_{0;e'}$ of $\mathcal{T}_0^c$ contains a unique element e' of $\chi(\mathcal{T}_0)$. If a pseudo-tree framing $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ satisfies $|S|=1$, $g_a(\mathcal{T}_0)=0$, and $|\mathrm{Ver}_{0;P}|=1$, then $\mathrm{Ver}_{0;P}$ consists of a unique vertex

$$w_0=\mathfrak{m}(S).$$

We denote such $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ by $(\mathscr{T},\mathcal{T}_0)$ and call it a tree framing.

For every tree framing $(\mathscr{T},\mathcal{T}_0)$, the graph $\big(\widehat{\mathrm{Ver}}_0,\widehat{\mathrm{Edg}}_0\big)$ as in (2.12) is a tree; we call $w_0\in\mathrm{Ver}_{0;P}$ its root. There is no obstruction to completing the pregluing construction of Section 2.2 for a tree framing. We do so in this section by smoothing out the nodes inductively towards the root.

For every pseudo-tree framing $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ and $e\in\chi(\mathcal{T}_{0;P})$, let

$$\mathscr{T}_e=\Big(\mathcal{T}_e=\big(\mathrm{Ver}_e,\mathrm{Edg}_e,S_e,\mathfrak{g}_e,\mathfrak{m}_e\big),\,\mathfrak{d}|_{\mathcal{T}_e}\Big)$$

be the bubble type associated with the connected component of the complement of $\mathcal{T}_{0;P}$ in $\mathcal{T}$. Let

$$\mathcal{T}_{e;0}=\big(\mathrm{Ver}_{e;0},\mathrm{Edg}_{e;0},S_{e;0},\mathfrak{g}_{e;0},\mathfrak{m}_{e;0}\big)$$

be the contracted subgraph containing a marked point in $\chi(\mathcal{T}_0)$ if $e\in\chi(\mathcal{T}_{0;P})-\chi(\mathcal{T}_0)$. The corresponding pair $(\mathscr{T}_e,\mathcal{T}_{e;0})$ is then a tree framing. If $e\in\chi(\mathcal{T}_{0;P})\cap\chi(\mathcal{T}_0)$, define $\mathcal{T}_{e;0}$ to be the empty subgraph. We first smooth out the nodes of each $\mathcal{T}_{e;0}$ as in the previous paragraph, and then smooth out the remaining nodes in various ways according to the topological types of the principal curves. The key properties for the resulting nearly J-holomorphic maps are summarized in Section 3.1 and proved in Section 3.3.

3.1 Main statements

Let $(\mathscr{T},\mathcal{T}_0)$ be a tree framing. For $w\in\widehat{\mathrm{Ver}}_0-\{w_0\}$, let

$$e_w\in\widehat{\mathrm{Edg}}_0 \tag{3.1}$$

be the edge containing w and separating it from w_0 (i.e. the removal of e_w from the tree $(\widehat{\mathrm{Ver}}_0,\widehat{\mathrm{Edg}}_0)$ splits it into two connected components with the vertices w_0 and w lying on different components). We define e_{w_0} to be the unique element of $S_0-\chi(\mathcal{T}_0)$ and

$$w^+_{e_{w_0}}\equiv w_0.$$

Denote by $\prec$ the prefix order on $\widehat{\mathrm{Ver}}_0$ so that $w\prec w'$ if $w\neq w'$ and either $w=w_0$ or e_w separates w' from w_0. The prefix (i.e. the minimal element) of $\widehat{\mathrm{Ver}}_0$ is the root w_0. We assume that the chosen orientations of the edges in $\widehat{\mathrm{Edg}}_0$ satisfy

$$w^-_e\prec w^+_e\qquad\forall\ e\in\widehat{\mathrm{Edg}}_0.$$

Such a choice of orientations is consistent with (2.11). For every $w\in\mathrm{Ver}_0$, let

$$E^+_w=\big\{e\in\widehat{\mathrm{Edg}}_0:\ w^-_e=w\big\},\qquad V^+_w=\big\{w'\in\widehat{\mathrm{Ver}}_0:\ e_{w'}\in E^+_w\big\}. \tag{3.2}$$

For $\mathbf{u}'\in\widetilde{U}$ and $\delta\in\mathbb{R}$, denote by $\Sigma_{\mathbf{u}';e_{w_0}}(\delta)\subset\Sigma_{\mathbf{u}';w_0}$ the $g_{\mathbf{u}'}$-ball of radius δ centered at the marked point $x_0(\mathbf{u}')\equiv x_{e_{w_0}}(\mathbf{u}')$. We choose a continuous family of meromorphic functions $z^+_{e_{w_0};\mathbf{u}'}$ on $\Sigma_{\mathbf{u}';w_0}$ with $\mathbf{u}'\in\widetilde{U}$ so that $x_0(\mathbf{u}')$ is the only zero of $z^+_{e_{w_0};\mathbf{u}'}$ and this zero is simple. For $\upsilon\in\Delta|_{\mathbf{u}'}$, let

$$\Sigma_{\upsilon;e_{w_0}}(\delta)=q_\upsilon^{-1}\big(\Sigma_{\mathbf{u}';e_{w_0}}(\delta)\big),\qquad z^+_{e_{w_0};\upsilon}=z^+_{e_{w_0};\mathbf{u}'}\circ q_\upsilon\colon\,\Sigma_{\upsilon;e_{w_0}}(8\delta_{\bar\partial})\longrightarrow\mathbb{C},$$
$$\Sigma^{\mathrm{mn}}_{\upsilon;w}=q_\upsilon^{-1}\Bigg(\Sigma_{\mathbf{u}';w}-\bigsqcup_{\substack{e\in\widehat{\mathrm{Edg}}_0\\ e\ni w}}\Sigma_{\mathbf{u}';e}(\delta_{\bar\partial})\Bigg)\qquad\forall\,w\in\mathrm{Ver}_0. \tag{3.3}$$

We assume that the functions $z^\pm_{e;\mathbf{u}'}$ of (q4) in Section 2.2 with $e\in\mathrm{Edg}_0$ and $z^+_{e_{w_0};\mathbf{u}'}$ satisfy

$$z^-_{e;\mathbf{u}'}(x)=(z^+_{e_w;\mathbf{u}'})^{-1}(x)-(z^+_{e_w;\mathbf{u}'})^{-1}\big(x^-_e(\mathbf{u}')\big)\qquad\forall\;w\in\mathrm{Ver}_0,\;e\in E^+_w,\;x\in\widetilde{A}^-_{e,\mathbf{u}'}(\delta_{\bar\partial}). \tag{3.4}$$

By (2.18), the family of the deformations $\mathfrak{U}\longrightarrow\Delta$ of the domains of elements of $\widetilde{U}$ extends to a continuous family of maps

$$\widetilde{u}_\upsilon\colon\,\Sigma_\upsilon\longrightarrow X,\quad \upsilon\in\Delta,\qquad\text{s.t.} \tag{3.5}$$
$$\widetilde{u}_\upsilon\big(x_0(\upsilon)\big)=\mathrm{ev}_0(\upsilon),\qquad \bar\partial_J\widetilde{u}_\upsilon\in\big\{q_\upsilon\times\widetilde{u}_\upsilon\big\}^*\,\widetilde{\Gamma}^{0,1}_-(\mathbf{u}')\quad\forall\,\mathbf{u}'\in\widetilde{U},\;\upsilon\in\Delta|_{\mathbf{u}'}\,.$$

For every $\upsilon\in\Delta|_{\mathbf{u}'}$, the map $\widetilde{u}_\upsilon$ determines an element $\xi_\upsilon\in\Gamma(\upsilon)$ by

$$\widetilde{u}_\upsilon=\exp_{u_\upsilon}\xi_\upsilon. \tag{3.6}$$

Let $\widetilde{u}_{\upsilon;1}$, $\zeta_{\upsilon;1}$, and T_{∇^J} be as in (2.30), (2.31), and (2.1), respectively.

Proposition 3.1. *Let $(\mathscr{T},\mathcal{T}_0)$ be a tree framing and $\mathbf{u}\in\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be a J-holomorphic map. If Δ is a sufficiently small neighborhood of $\mathbf{u}$ in $\widetilde{\mathcal{F}\mathscr{T}}$, the family (3.6) can be chosen so that there exist $C\in\mathbb{R}^{\geqslant0}$,*

$$\varepsilon_{\upsilon;w}\in\mathbb{R}^{\geqslant0},\quad \mathcal{D}_{\upsilon;w}\in T_{\mathrm{ev}_0(\upsilon)}X\quad\forall\;w\in\widehat{\mathrm{Ver}}_0,\;\upsilon\in\Delta,\qquad\text{with}\quad\mathcal{D}_{\upsilon;w_0}=\mathrm{d}_{x_0(\upsilon)}\widetilde{u}_\upsilon\Big(\frac{\partial}{\partial z^+_{e_{w_0};\upsilon}}\Big), \tag{3.7}$$
$$\gamma_{\upsilon;e}\in C^0\big(\Sigma_{\upsilon;e}(8\delta_{\bar\partial})\big)\quad\forall\;e\in\mathrm{Edg}_0\sqcup\{e_{w_0}\},\;\upsilon\in\Delta,$$

such that

(M1) for all $w\in\widehat{\mathrm{Ver}}_0-\mathrm{Ver}_0$,

$$\varepsilon_{\upsilon;w}=1,\quad\mathcal{D}_{\upsilon;w}=\mathrm{d}_{x^+_{e_w}(\upsilon_1)}\widetilde{u}_{\upsilon;1}\Big(\frac{\partial}{\partial z^+_{e_w;\upsilon_1}}\Big),\quad\big\|\xi_\upsilon|_{\Sigma_{\upsilon;e_w}(8\delta_{\bar\partial})}\big\|_{\upsilon,p,1}\leqslant C\sum_{w'<w}\varepsilon_{\upsilon;w'}\,;$$

(M2) for all $w\in\mathrm{Ver}_0$,

$$\varepsilon_{\upsilon;w}\leqslant C\sum_{w'\in V^+_w}\big(|\mathcal{D}_{\upsilon;w'}||\upsilon_{e_{w'}}|+\varepsilon_{\upsilon;w'}|\upsilon_{e_{w'}}|^2\big)\,,\qquad\Big|\mathcal{D}_{\upsilon;w}-\sum_{w'\in V^+_w}\upsilon_{e_{w'}}\mathcal{D}_{\upsilon;w'}\Big|\leqslant C\,\varepsilon_{\upsilon;w}\sum_{w'\in V^+_w}|\upsilon_{e_{w'}}|\,,$$
$$\big\|\xi_\upsilon|_{\Sigma^{\mathrm{mn}}_{\upsilon;w}}\big\|_{\upsilon,p,1}\leqslant C\sum_{w'\preceq w}\varepsilon_{\upsilon;w'}\,;$$

(M3) for all $e\in\mathrm{Edg}_0\sqcup\{e_{w_0}\}$ *and* $x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial})$,

$$|\xi_\upsilon|_x\leqslant C\Big(|\mathcal{D}_{\upsilon;w_e^+}||z_{e;\upsilon}^+(x)|+\varepsilon_{\upsilon;w_e^+}|z_{e;\upsilon}^+(x)|^2+\sum_{w'\prec w_e^+}\varepsilon_{\upsilon;w'}\Big),$$

$$\big|\mathrm{d}_x\xi_\upsilon\big|\leqslant C\Big(\big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}|z_{e;\upsilon}^+(x)|\big)|\mathrm{d}_x\widetilde{q}_{\upsilon;e}|+\gamma_{\upsilon;e}(x)\Big),\qquad \|\gamma_{\upsilon;e}\|_{\upsilon,p}\leqslant\sum_{w'\prec w_e^+}\varepsilon_{\upsilon;w'},$$

and

$$\begin{aligned}|T_{\nabla^J}(\xi_\upsilon(x),\mathrm{d}_x\xi_\upsilon)|\leqslant C\Bigg(&\Big(\big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}|z_{e;\upsilon}^+(x)|\big)|\mathrm{d}_x\widetilde{q}_{\upsilon;e}|+\gamma_{\upsilon;e}\Big)\sum_{w'\prec w_e^+}\varepsilon_{\upsilon;w'}\\&+|\mathcal{D}_{\upsilon;w_e^+}|\Big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}|\mathrm{d}_x\widetilde{q}_{\upsilon;e}|\Big)|z_{e;\upsilon}^+(x)|^2+\varepsilon_{\upsilon;w_e^+}^2|z_{e;\upsilon}^+(x)|^3\\&+\Big(|\mathcal{D}_{\upsilon;w_e^+}||z_{e;\upsilon}^+(x)|+\varepsilon_{\upsilon;w_e^+}|z_{e;\upsilon}^+(x)|^2\Big)\gamma_{\upsilon;e}\Bigg).\end{aligned}$$

Let $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ be a graph framing. Define

$$\begin{aligned}&\mathrm{Ver}_{0;P}^c=\mathrm{Ver}_0-\mathrm{Ver}_{0;P}=\bigsqcup_{e\in\chi(\mathcal{T}_{0;P})}\mathrm{Ver}_{e;0}\subset\mathrm{Ver}_0, &&\mathrm{Edg}_{0;P}^c=\bigsqcup_{e\in\chi(\mathcal{T}_{0;P})}\mathrm{Edg}_{e;0}\subset\mathrm{Edg}_0,\\&\widetilde{\mathrm{Edg}}_{0;P}^c=\mathrm{Edg}_0-\mathrm{Edg}_{0;P}=\mathrm{Edg}_{0;P}^c\sqcup\big(\chi(\mathcal{T}_{0;P})-\chi(\mathcal{T}_0)\big), &&\widehat{\mathrm{Edg}}_{0;P}^c=\widetilde{\mathrm{Edg}}_{0;P}^c\sqcup\chi(\mathcal{T}_0)\subset\mathrm{Edg}.\end{aligned}\tag{3.8}$$

The prefix orders $\prec$ on $\mathrm{Ver}_{e;0}$ for $e\in\chi(\mathcal{T}_{0;P})$ determine a prefix order $\prec$ on $\mathrm{Ver}_{0;P}^c$ and extend to a prefix order $\prec$ on $\{P\}\sqcup\mathrm{Ver}_{0;P}^c$ so that P is the minimal element. Define

$$\langle w,w'\rangle=\max\big\{w''\in\{P\}\sqcup\widehat{\mathrm{Ver}}_{0;P}^c\colon\, w''\preceq w,w'\big\}\tag{3.9}$$

for all $w,w'\in\{P\}\sqcup\mathrm{Ver}_{0;P}^c$.

We assume that the chosen orientations of the edges satisfy

$$w_e^-\prec w_e^+\qquad\forall\, e\in\widehat{\mathrm{Edg}}_{0;P}^c.$$

The assumption is consistent with (2.11). The prefix order $\prec$ on $\mathrm{Ver}_{0;P}^c$ induces a prefix order on $\widehat{\mathrm{Edg}}_{0;P}^c$ so that

$$e\prec e'\iff w_e^+,w_{e'}^-\in\mathrm{Ver}_{0;P}^c,\ w_e^+\preceq w_{e'}^-\qquad\forall\, e,e'\in\widehat{\mathrm{Edg}}_{0;P}^c.$$

For every $e\in\widehat{\mathrm{Edg}}_{0;P}^c$, we denote by $\langle e\rangle$ the element of $\chi(\mathcal{T}_{0;P})$ satisfying $\langle e\rangle\preceq e$. For all $e,e'\in\widehat{\mathrm{Edg}}_{0;P}^c$ with $\langle e\rangle=\langle e'\rangle$, we define

$$\langle e,e'\rangle=\min\big\{e''\in\widehat{\mathrm{Edg}}_{0;P}^c\colon\, e''\preceq e,e'\big\}.$$

With $\langle w_e^+,w_{e'}^+\rangle$ as in (3.9),

$$\langle e,e'\rangle=e_{\langle w_e^+,w_{e'}^+\rangle}\qquad\forall\, e,e'\in\widehat{\mathrm{Edg}}_{0;P}^c\ \text{ s.t. }\ \langle e\rangle=\langle e'\rangle.$$

Suppose the graph framing $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ is in addition a pseudo-tree framing. Define

$$\widehat{\mathrm{Ver}}^c_{0;P}=\mathrm{Ver}^c_{0;P}\sqcup\big\{w_e^+:\ e\in\chi(\mathcal{T}_0)\big\}.$$

The prefix order $\prec$ extends from $\{P\}\sqcup\mathrm{Ver}^c_{0;P}$ to $\{P\}\sqcup\widehat{\mathrm{Ver}}^c_{0;P}$ so that the elements of $\widehat{\mathrm{Ver}}^c_{0;P}-\mathrm{Ver}^c_{0;P}$ are maximal. Define

$$E_P^+=\chi(\mathcal{T}_{0;P}),\qquad V_P^+=\big\{w\in\widehat{\mathrm{Ver}}^c_{0;P}:\ e_w\in E_P^+\big\}.$$

For a graph framing $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$, define

$$\widetilde{\rho}_e(\upsilon)\equiv\prod_{\langle e\rangle\prec e'\preceq e}\upsilon_{e'}\in\mathbb{C},\qquad \rho_e(\upsilon)\equiv\upsilon_{\langle e\rangle}\widetilde{\rho}_e(\upsilon)\in\mathbb{C}. \tag{3.10}$$

for all $\upsilon\in\Delta$ and $e\in\widehat{\mathrm{Edg}}^c_{0;P}$ and

$$\rho_w(\upsilon)=\begin{cases}\rho_{e_w}(\upsilon) & \text{if } w\in\mathrm{Ver}^c_{0;P};\\ 1 & \text{if } w\in\mathrm{Ver}_{0;P}.\end{cases} \tag{3.11}$$

for all $\upsilon\in\Delta$ and $w\in\mathrm{Ver}_0$. If in addition, $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ is a pseudo-tree framing, we define

$$\rho_{w,w'}(\upsilon)=\prod_{w\prec w''\preceq w'}\upsilon_{e_{w''}}\in\mathbb{C},\qquad \rho_w(\upsilon)=\rho_{P,w}(\upsilon)\in\mathbb{C} \tag{3.12}$$

for all $\upsilon\in\Delta$ and $w,w'\in\{P\}\sqcup\widehat{\mathrm{Ver}}^c_{0;P}$. In particular, $\rho_{w,w}(\upsilon)=1$. The second definition in (3.12) is consistent with (3.11).

Let $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ be a pseudo-tree framing and $\mathbf{u}\in\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be a J-holomorphic map. For $\upsilon\in\Delta$, define

$$\Sigma^{\mathrm{mn}}_{\upsilon;P}=q_\upsilon^{-1}\Big(\Sigma_{\mathbf{u}';0;P}-\bigsqcup_{e\in\chi(\mathcal{T}_{0;P})}\Sigma_{\mathbf{u}';e}(\delta_{\bar\partial})\Big).$$

By (2.17) and (2.18), the family of the deformations $\mathfrak{U}\longrightarrow\Delta$ of the domains of elements of $\widetilde{U}$ extends to a continuous family of maps

$$\widetilde{u}_\upsilon\colon\Sigma_\upsilon\longrightarrow X,\quad \upsilon\in\Delta\qquad\text{s.t.}\qquad \bar\partial_J\widetilde{u}_\upsilon\in\big\{q_\upsilon\times\widetilde{u}_\upsilon\big\}^*\,\widetilde{\Gamma}^{0,1}_-(\mathbf{u}')\quad\ \forall\,\mathbf{u}'\in\widetilde{U},\ \upsilon\in\Delta|_{\mathbf{u}'}\,. \tag{3.13}$$

For every $\upsilon\in\Delta|_{\mathbf{u}'}$, the map $\widetilde{u}_\upsilon$ determines an element $\xi_\upsilon\in\Gamma(\upsilon)$ by

$$\widetilde{u}_\upsilon=\exp_{u_\upsilon}\xi_\upsilon. \tag{3.14}$$

Corollary 3.2. *Let $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ be a pseudo-tree framing and $\mathbf{u}\in\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be a J-holomorphic map. If Δ is a sufficiently small neighborhood of $\mathbf{u}$ in $\widetilde{\mathcal{F}\mathscr{T}}$, the family (3.14) can be chosen so that there exist $C\in\mathbb{R}^{\geqslant0}$,*

$$\varepsilon_{\upsilon;w}\in\mathbb{R}^{\geqslant0}\quad\forall\ w\in\{P\}\sqcup\widehat{\mathrm{Ver}}^c_{0;P},\ \upsilon\in\Delta,\qquad\quad \mathcal{D}_{\upsilon;w}\in T_{\mathrm{ev}_0(\upsilon)}X\quad\forall\ w\in\widehat{\mathrm{Ver}}^c_{0;P},\ \upsilon\in\Delta,$$
$$\gamma_{\upsilon;e}\in C^0\big(\Sigma_{\upsilon;e}(8\delta_{\bar\partial})\big)\quad\forall\ e\in\widetilde{\mathrm{Edg}}^c_{0;P},\ \upsilon\in\Delta,$$

such that (M1) holds for all $w\in\widehat{\mathrm{Ver}}_0-\mathrm{Ver}_0$, the first and last inequalities in (M2) hold for all $w\in\{P\}\sqcup\mathrm{Ver}_{0;P}^c$, the second inequality in (M2) holds for all $w\in\mathrm{Ver}_{0;P}^c$, and (M3) holds for all $e\in\widetilde{\mathrm{Edg}}_{0;P}^c$.

By Corollary 3.2, the first two equations in (M1) of Proposition 3.1, and the continuity of the family $\widetilde{u}_{\upsilon;1}$ in (2.30), there exists a constant $C\in\mathbb{R}$ such that

$$|\mathcal{D}_{\upsilon;w}|,\ \varepsilon_{\upsilon;w}\leqslant C\qquad\forall\,w\in\widehat{\mathrm{Ver}}_{0;P}^c-\mathrm{Ver}_{0;P}^c.$$

By Corollary 3.2, the first two statements in (M2) of Proposition 3.1, and induction, there thus exists a constant $C\in\mathbb{R}$ such that

$$|\mathcal{D}_{\upsilon;w}|,\ \varepsilon_{\upsilon;w}\leqslant C\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ e\ge e_w}}\big|\rho_{w,w_e^+}(\upsilon)\big|\quad\forall\,w\in\widehat{\mathrm{Ver}}_{0;P}^c,\qquad\varepsilon_{\upsilon;P}\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\big|\rho_e(\upsilon)\big|.\tag{3.15}$$

In particular,

$$\sum_{w'\preceq w}\varepsilon_{\upsilon;w'}\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\big|\rho_{\langle w,w_e^+\rangle,w_e^+}(\upsilon)\big|\qquad\forall\,w\in\widehat{\mathrm{Ver}}_{0;P}^c.\tag{3.16}$$

By the last inequality in (M1) in Proposition 3.1 and (3.15), there exists a constant $C\in\mathbb{R}$ such that

$$\big\|\xi_\upsilon|_{\Sigma_{\upsilon;e}(8\delta_{\bar\partial})}\big\|_{\upsilon,p,1}\leqslant C\Big(\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ e'>\langle e\rangle}}\big|\rho_{\langle w_e^-,w_{e'}^+\rangle,w_{e'}^+}(\upsilon)\big|+\varepsilon_{\upsilon;P}\Big)\qquad\forall\,e\in\chi(\mathcal{T}_0).\tag{3.17}$$

By (3.12), (3.15)-(3.17), and (2.22), there exists a constant $C\in\mathbb{R}$ such that

$$\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ e'>e}}\big|\rho_{w_e^+,w_{e'}^+}(\upsilon)\big|,\ \sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ e'>\langle e\rangle}}\big|\rho_{\langle w_e^-,w_{e'}^+\rangle,w_{e'}^+}(\upsilon)\big|,\ \varepsilon_{\upsilon;P},\ \|\xi_\upsilon\|_{\upsilon,p,1},\ \|\xi\|_{C^0}\ \leqslant\ C|\upsilon|\tag{3.18}$$

for all $\upsilon\in\Delta$ and $e\in\widetilde{\mathrm{Edg}}_{0;P}^c$.

3.2 Further implications

Suppose $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ is a pseudo-tree framing so that $\mathcal{T}_{0;P}$ contains connected subgraphs $\mathcal{T}_{P^+}$, $\mathcal{T}_{P^-}$, and $\mathcal{T}_{\mathcal{C}}$ satisfying

$$\mathrm{Ver}_{0;P}=\mathrm{Ver}_{P^-}\sqcup\mathrm{Ver}_{\mathcal{C}}\sqcup\mathrm{Ver}_{P^+},\qquad g_a(\mathcal{T}_{P^+})+g_a(\mathcal{T}_{P^-})=g_a(\mathcal{T}_{0;P}).\tag{3.19}$$

The subgraph $\mathcal{T}_{\mathcal{C}}$ may be empty. For $\mathbf{u}$ as in Corollary 3.2, $\Sigma_{\mathbf{u};0;P}$ thus consists of two principal curves $\Sigma_{\mathbf{u};P^\pm}$ and a maximal chain of spheres $\mathcal{C}$; see Figure 5. Let $e^\pm\in\mathrm{Edg}_{0;P}$ be so that

$$S_{P^\pm}\cap S_{\mathcal{C}}=\{e^\pm\}\quad\text{if }\mathrm{Ver}_{\mathcal{C}}\neq\varnothing;\qquad S_{P^+}\cap S_{P^-}=\{e^+\}=\{e^-\}\quad\text{if }\mathrm{Ver}_{\mathcal{C}}=\varnothing.$$

Define

$$\begin{aligned}\mathrm{Ver}^b&\equiv\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}\sqcup\mathrm{Ver}_{0;P}^c, &\mathrm{Edg}^b&\equiv\{e^+,e^-\}\sqcup\mathrm{Edg}_{\mathcal{C}}\sqcup\widetilde{\mathrm{Edg}}_{0;P}^c,\\ \widehat{\mathrm{Ver}}^b&\equiv\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}\sqcup\widehat{\mathrm{Ver}}_{0;P}^c, &\widehat{\mathrm{Edg}}^b&\equiv\{e^+,e^-\}\sqcup\mathrm{Edg}_{\mathcal{C}}\sqcup\widehat{\mathrm{Edg}}_{0;P}^c.\end{aligned}\tag{3.20}$$

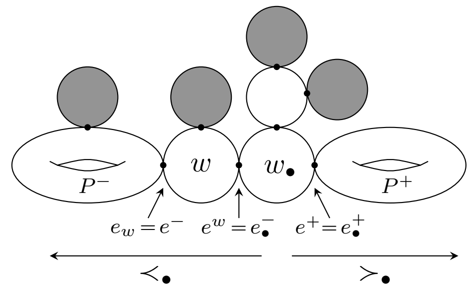

Figure 5: A pseudo-tree framing satisfying the sentence containing (3.19)

The prefix orders $\prec$ extend from $\widehat{\mathrm{Ver}}^c_{0;P}$ and $\widehat{\mathrm{Edg}}^c_{0;P}$ to $\widehat{\mathrm{Ver}}^b$ and $\widehat{\mathrm{Edg}}^b$, respectively, so that P^- is the minimal element of $\widehat{\mathrm{Ver}}^b$ and all minimal elements of $\widehat{\mathrm{Edg}}^b$ contain P^-. We assume that $w_e^-\prec w_e^+$ for every $e\in\mathrm{Edg}^b$. For $w\in\{P^+\}\sqcup\mathrm{Ver}_{\mathcal C}$, define $e_w\in\mathrm{Edg}^b$ as in (3.1).

Choose

$$\widetilde\Gamma^{0,1}_-(\mathbf u;P^\pm)\subset\Gamma\big(\Sigma_{\mathbf u;P^\pm}\times X;\pi_1^*(T^*\Sigma_{\mathbf u;P^\pm})^{0,1}\otimes_{\mathbb C}\pi_2^*TX\big) \tag{3.21}$$

so that every element of $\widetilde\Gamma^{0,1}_-(\mathbf u;P^\pm)$ vanishes on a neighborhood of every nodal point of $\Sigma_{\mathbf u;P^\pm}$ and

$$\Gamma^{0,1}(\mathbf u_{P^\pm})=\big\{D_{\mathbf u_{P^\pm}}\xi:\ \xi\in\Gamma(\mathbf u_{P^\pm})\big\}\oplus\big\{\{\mathrm{id}\times u_{P^\pm}\}^*\eta:\,\eta\in\widetilde\Gamma^{0,1}_-(\mathbf u;P^\pm)\big\}.$$

These two spaces determine $\widetilde\Gamma_-(\mathbf u;P)$ as in (2.16) so that (B1) is satisfied.

Corollary 3.3. *Let $(\mathscr T,\mathcal T_0,\mathcal T_{0;P})$ be as in the sentence containing (3.19) and $\mathbf u\in\widetilde{\mathfrak X}_{\mathscr T}(X)$ be a J-holomorphic map. If Δ is a sufficiently small neighborhood of $\mathbf u$ in $\widetilde{\mathcal F\mathscr T}$, the family (3.14) can be chosen so that there exist $C\in\mathbb R^{\geqslant0}$,*

$$\varepsilon_{\upsilon;w}\in\mathbb R^{\geqslant0}\quad\forall\ w\in\widehat{\mathrm{Ver}}^b,\ \upsilon\in\Delta,\qquad\quad \mathcal D_{\upsilon;w}\in T_{\mathrm{ev}_0(\upsilon)}X\quad\forall\ w\in\widehat{\mathrm{Ver}}^b-\{P^-\},\ \upsilon\in\Delta,$$
$$\gamma_{\upsilon;e}\in C^0\big(\Sigma_{\upsilon;e}(8\delta_{\bar\partial})\big)\quad\forall\ e\in\mathrm{Edg}^b,\ \upsilon\in\Delta,$$

such that (M1) in Proposition 3.1 holds for all $w\in\widehat{\mathrm{Ver}}_0-\mathrm{Ver}_0$, the first and last inequalities in (M2) hold for all $w\in\mathrm{Ver}^b$, the second inequality in (M2) holds for all $w\in\mathrm{Ver}^b-\{P^+,P^-\}$,

$$|\mathcal D_{\upsilon;P^+}|\leqslant C\,\varepsilon_{\upsilon;P^+}\qquad\forall\ \upsilon\in\Delta, \tag{3.22}$$

and (M3) holds for all $e\in\mathrm{Edg}^b$.

Suppose $(\mathscr T,\mathcal T_0,\mathcal T_{0;P})$ is a pseudo-tree framing satisfying the sentence containing (3.19). Suppose in addition $\mathcal C$ contains a distinguished vertex $w_\bullet$. Denote by $e_\bullet^+$ and $e_\bullet^-$ the edges containing $w_\bullet$ so that the removal of $e_\bullet^\pm$ separates $w_\bullet$ from $\mathcal T_{P^\pm}$; see Figure 5.

The prefix order $\prec$ on $\widehat{\mathrm{Ver}}^c_{0;P}$ extends to a partial order $\prec_\bullet$ on $\widehat{\mathrm{Ver}}^b$ so that P^+,P^- are the minimal elements of $\widehat{\mathrm{Ver}}^b$, $w_\bullet$ is the maximal element of $\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal C}$, and the restriction of $\prec_\bullet$ to each of the two connected components of $\mathrm{Ver}_{\mathcal C}-\{w_\bullet\}$ is a prefix order. For each $w\in\widehat{\mathrm{Ver}}^b$, let

$$\langle w\rangle=\max\big\{w'\in\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal C}:\ w'\prec_\bullet w\big\}.$$

The prefix order $\prec$ on $\widehat{\mathrm{Edg}}^c_{0;P}$ extends to a partial order $\prec_\bullet$ on $\widehat{\mathrm{Edg}}^b$ similarly. We assume that $w_e^- \prec_\bullet w_e^+$ for every $e\in\widehat{\mathrm{Edg}}^b$. Define

$$\begin{aligned} &e_w\in\mathrm{Edg}^b \quad \text{by} \quad w=w^+_{e_w} \qquad \forall\, w\in\mathrm{Ver}_{\mathcal{C}}-\{w_\bullet\},\\ &e^w\in\mathrm{Edg}^b \quad \text{by} \quad w=w^-_{e^w} \qquad \forall\, w\in\{P^+,P^-\}\sqcup\big(\mathrm{Ver}_{\mathcal{C}}-\{w_\bullet\}\big). \end{aligned}$$

The restriction of $\prec_\bullet$ to $\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}$ extends to a total order. Along with the original partial order $\prec_\bullet$ on $\widehat{\mathrm{Ver}}^b$, this determines a new partial order $<_\bullet$ on $\widehat{\mathrm{Ver}}^b$.

For $w\in\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}$, $\mathbf{u}'\in\widetilde{U}$, and $\upsilon\in\Delta|_{\mathbf{u}'}$, let

$$\Sigma^{(w)}_{\upsilon_1;0}=\Sigma_{\upsilon_1;0}(8\delta_{\bar\partial})\cap q_{\upsilon_1}^{-1}\Big(\bigcup_{\substack{w'\in\widehat{\mathrm{Ver}}^b\\ \langle w'\rangle=w}}\Sigma_{\mathbf{u}';w'}\Big),$$

$$\Sigma^{(w_\bullet);s}_{\upsilon_1;0}=\Sigma^{(w_\bullet)}_{\upsilon_1;0}-\big(\Sigma^+_{\upsilon_1;e^+_\bullet}(4\delta_{\bar\partial})\sqcup\Sigma^-_{\upsilon_1;e^+_\bullet}(4\delta_{\bar\partial})\big),\qquad \Sigma^{(P^\pm);s}_{\upsilon_1;0}=\Sigma^{(P^\pm)}_{\upsilon_1;0}\cup\Sigma^+_{\upsilon_1;e^\pm}(4\delta_{\bar\partial}),$$

$$\Sigma^{(w);s}_{\upsilon_1;0}=\big(\Sigma^{(w)}_{\upsilon_1;0}\cup\Sigma^+_{\upsilon_1;e^w}(4\delta_{\bar\partial})\big)-\Sigma^+_{\upsilon_1;e_w}(4\delta_{\bar\partial})\quad\forall\, w\in\mathrm{Ver}_{\mathcal{C}}.$$

Corollary 3.4. *Let $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ be as in the sentence containing (3.19) and $\mathbf{u}\in\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be a J-holomorphic map. If Δ is a sufficiently small neighborhood of $\mathbf{u}$ in $\widetilde{\mathcal{F}\mathscr{T}}$, the family (3.14) can be chosen so that there exist*

$$\begin{aligned} &C\in\mathbb{R}^{\geqslant0}, && \xi^s_\upsilon\in\Gamma(\Sigma_{\upsilon_1};\widetilde{u}^*_{\upsilon;1}TX)\quad\forall\,\upsilon\in\Delta|_{\mathbf{u}'},\ \mathbf{u}'\in\widetilde{U},\\ &\varepsilon_{\upsilon;w}\in\mathbb{R}^{\geqslant0}\quad\forall\, w\in\widehat{\mathrm{Ver}}^b,\ \upsilon\in\Delta, && \mathcal{D}_{\upsilon;w}\in T_{\mathrm{ev}_0(\upsilon)}X\quad\forall\, w\in\widehat{\mathrm{Ver}}^c_{0;P},\ \upsilon\in\Delta,\\ &|\mathcal{D}_{\upsilon;e}|\in\mathbb{R}^{\geqslant0}\quad\forall\, e\in\{e^+,e^-\}\sqcup\mathrm{Edg}_{\mathcal{C}},\ \upsilon\in\Delta, && \gamma_{\upsilon;e}\in C^0\big(\Sigma_{\upsilon;e}(8\delta_{\bar\partial})\big)\quad\forall\, e\in\mathrm{Edg}^b,\ \upsilon\in\Delta, \end{aligned}$$

such that

(M′0) for all $\upsilon\in\Delta|_{\mathbf{u}'}$, ξ^s_υ is locally constant on $\Sigma_{\upsilon_1;0}$ outside of $\Sigma^+_{\upsilon_1;e}(\delta_{\bar\partial})$ with $e\in\{e^+,e^-\}\sqcup\mathrm{Edg}_{\mathcal{C}}$ and satisfies

$$\|\xi^s_\upsilon\|_{C^1(\Sigma^{(w);s}_{\upsilon_1;0})}\leqslant\sum_{\substack{w'\in\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}\\ w'>_\bullet w}}\varepsilon_{\upsilon;w'}\qquad\forall\, w\in\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}};$$

(M′1) for all $w\in\widehat{\mathrm{Ver}}_0-\mathrm{Ver}_0$,

$$\varepsilon_{\upsilon;w}=1,\qquad\Big|\mathcal{D}_{\upsilon;w}-\mathrm{d}_{x^+_{e_w}(\upsilon_1)}\widetilde{u}_{\upsilon;1}\Big(\frac{\partial}{\partial z^+_{e_w;\upsilon_1}}\Big)\Big|\leqslant C\sum_{w'<_\bullet w}\varepsilon_{\upsilon;w'}\,,$$

$$\big\|(\xi_\upsilon-\xi^s_\upsilon\circ q_{\upsilon_1;\upsilon_{01}}\circ\widetilde{q}_\upsilon)|_{\Sigma_{\upsilon;e_w}(8\delta_{\bar\partial})}\big\|_{\upsilon,p,1}\leqslant C\sum_{w'<_\bullet w}\varepsilon_{\upsilon;w'}\,;$$

(M′2) for all $w\in\mathrm{Ver}^b$,

$$\varepsilon_{\upsilon;w}\leqslant C\bigg(\sum_{w'\in V^+_w\cap\widehat{\mathrm{Ver}}^c_{0;P}}\big(|\mathcal{D}_{\upsilon;w'}||\upsilon_{e_{w'}}|+\varepsilon_{\upsilon;w'}|\upsilon_{e_{w'}}|^2\big)+\sum_{e\in E^+_w\cap\mathrm{Edg}_{0;P}}\Big(|\mathcal{D}_{\upsilon;e}||\upsilon_e|+|\upsilon_e|^2\sum_{w<_\bullet w'\leqslant_\bullet w^+_e}\varepsilon_{\upsilon;w'}\Big)\bigg),\tag{3.23}$$

$$\big\|(\xi_\upsilon-\xi^s_\upsilon\circ q_{\upsilon_1;\upsilon_{01}}\circ\widetilde{q}_\upsilon)|_{\Sigma^{\mathrm{mn}}_{\upsilon;w}}\big\|_{\upsilon,p,1}\leqslant C\sum_{w'\leqslant_\bullet w}\varepsilon_{\upsilon;w'}\,,$$

and the second inequality in (M2) of Proposition 3.1 holds if $w\in\mathrm{Ver}^c_{0;P}$;

(M′3) *for all* $e\in\widetilde{\mathrm{Edg}}^c_{0;P}$, *the inequalities in (M3) of Proposition 3.1 hold with* $(\xi_v,\prec)$ *replaced by* $\big((\xi_v-\xi_v^s\circ q_{v_1;v_{01}}\circ\widetilde{q}_v),\prec_\bullet\big)$;

(M′4) *for all* $e\in\{e^+,e^-\}\sqcup\mathrm{Edg}_{\mathcal{C}}$,

$$|\mathcal{D}_{v;e}|\leqslant C\sum_{w_e^-\prec_\bullet w'\leqslant_\bullet w_e^+}\varepsilon_{v;w'}\tag{3.24}$$

and the first three inequalities in (M3) of Proposition 3.1 hold with $\big(|\mathcal{D}_{v;w_e^+}|,\xi_v,\prec\big)$ *replaced by* $\big(|\mathcal{D}_{v;e}|,(\xi_v-\xi_v^s\circ q_{v_1;v_{01}}\circ\widetilde{q}_v),\prec_\bullet\big)$.

Suppose $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ is a pseudo-tree framing so that $\mathcal{T}_{0;P}$ contains connected subgraphs $\mathcal{T}_{P_+}$ and $\mathcal{T}_{\mathcal{C}_+}$ satisfying

$$\mathrm{Ver}_{0;P}=\mathrm{Ver}_{P_+}\sqcup\mathrm{Ver}_{\mathcal{C}_+},\qquad g_a(\mathcal{T}_{P_+})+1=g_a(\mathcal{T}_{0;P}),\qquad g_a(\mathcal{T}_{\mathcal{C}_+})=0,\tag{3.25}$$

and $\mathcal{T}_{\mathcal{C}_+}$ is a maximal subgraph of $\mathcal{T}_{0;P}$ so that (3.25) holds. Let $\mathcal{T}_{P_-}$ be the principal subgraph of $\mathcal{T}_{P_+}$. Since $\mathcal{T}_{\mathcal{C}_+}$ is maximal, the complement $\mathcal{T}_{\mathcal{C}_-}$ of $\mathcal{T}_{P_-}$ in $\mathcal{T}_{P_+}$ is connected and may be empty. For $\mathbf{u}$ as in Corollary 3.2, $\Sigma_{\mathbf{u};0;P}$ thus contains a maximal non-separating chain of spheres $\mathcal{C}_+$; see Figure 6. Let $e_1^+,e_2^+\in\mathrm{Edg}_{0;P}$ be so that

$$S_{P_+}\cap S_{\mathcal{C}_+}=\{e_1^+,e_2^+\}.$$

Define

$$\begin{aligned}\mathrm{Ver}^a&\equiv\{P_-\}\sqcup\mathrm{Ver}_{\mathcal{C}_-}\sqcup\mathrm{Ver}_{\mathcal{C}_+}\sqcup\mathrm{Ver}^c_{0;P}, & \mathrm{Edg}^a&\equiv\mathrm{Edg}_0-\mathrm{Edg}_{P_-},\\ \widehat{\mathrm{Ver}}^a&\equiv\{P_-\}\sqcup\mathrm{Ver}_{\mathcal{C}_-}\sqcup\mathrm{Ver}_{\mathcal{C}_+}\sqcup\widehat{\mathrm{Ver}}^c_{0;P}, & \widehat{\mathrm{Edg}}^a&\equiv\widehat{\mathrm{Edg}}_0-\mathrm{Edg}_{P_-}.\end{aligned}\tag{3.26}$$

Suppose in addition $\mathcal{C}_+$ contains a distinguished vertex $w_\bullet$. Let $e_{\bullet,1},e_{\bullet,2}\in\mathrm{Edg}_{0;P}$ be the edges containing $w_\bullet$ so that either $e_{\bullet,i}=e_i^+$ or $e_{\bullet,i}$ separates $w_\bullet$ from e_i^+ for $i=1,2$.

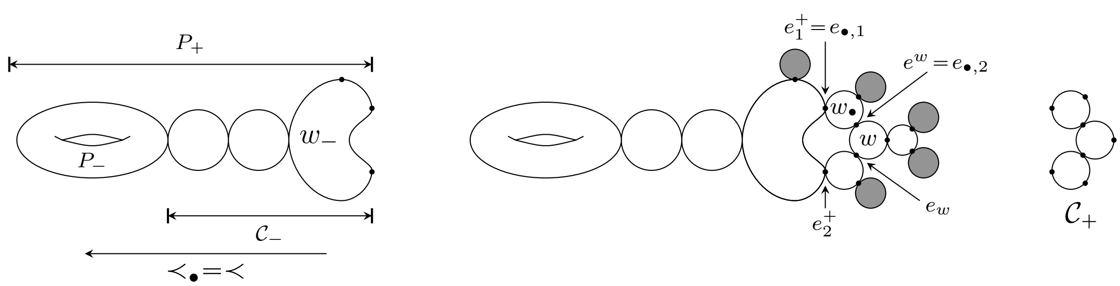

Figure 6: A pseudo-tree framing satisfying the sentence containing (3.25)

The prefix order $\prec$ on $\widehat{\mathrm{Ver}}^c_{0;P}$ extends to a partial order $\prec$ on $\{P_-\}\sqcup\mathrm{Ver}_{\mathcal{C}_-}\sqcup\widehat{\mathrm{Ver}}^c_{0;P}$ so that P_- is the minimal element. Denote by w_- the maximal element of $\{P_-\}\sqcup\mathrm{Ver}_{\mathcal{C}_-}$. This order further extends to a partial order $\prec_\bullet$ on $\widehat{\mathrm{Ver}}^a$ so that $w_\bullet$ is the maximal element of $\{P_-\}\sqcup\mathrm{Ver}_{\mathcal{C}_-}\sqcup\mathrm{Ver}_{\mathcal{C}_+}$ and the restriction of $\prec_\bullet$ to

$$\{P_-\}\sqcup\mathrm{Ver}_{\mathcal{C}_-}\sqcup\big(\mathrm{Ver}_{\mathcal{C}_+}-\{w_\bullet\}\big)$$

is a prefix order. The prefix order $\prec$ on $\widehat{\mathrm{Edg}}^c_{0;P}$ extends to a partial order $\prec_\bullet$ on $\widehat{\mathrm{Edg}}^a$ similarly. We assume that $w_e^- \prec_\bullet w_e^+$ for every $e \in \widehat{\mathrm{Edg}}^a$ and define

$$\begin{aligned} e_w &\in \mathrm{Edg}^a \quad \text{by} \quad w = w^+_{e_w} \qquad \forall\, w \in \mathrm{Ver}_{\mathcal{C}_-} \sqcup \big(\mathrm{Ver}_{\mathcal{C}_+} - \{w_\bullet\}\big),\\ e^w &\in \mathrm{Edg}^a \quad \text{by} \quad w = w^-_{e^w} \qquad \forall\, w \in \mathrm{Ver}_{\mathcal{C}_+} - \{w_\bullet\}. \end{aligned}$$

The restriction of $\prec_\bullet$ to $\mathrm{Ver}_{\mathcal{C}_+}$ extends to a total order. Along with the original partial order $\prec_\bullet$ on $\widehat{\mathrm{Ver}}^a$, this determines a new partial order $<_\bullet$ on $\widehat{\mathrm{Ver}}^a$.

Choose $\widetilde{\Gamma}^{0,1}_-(\mathbf{u};P_-)$ as in the sentence containing (3.21). Choose

$$\widetilde{\Gamma}^{0,1}_-(\mathbf{u};\mathcal{C}_+) \subset \Gamma\big(\Sigma_{\mathbf{u};\mathcal{C}_+} \times X; \pi_1^*(T^*\Sigma_{\mathbf{u};\mathcal{C}_+})^{0,1} \otimes_{\mathbb{C}} \pi_2^* TX\big) \tag{3.27}$$

so that every element of $\widetilde{\Gamma}^{0,1}_-(\mathbf{u};\mathcal{C}_+)$ is supported on $\Sigma_{\mathbf{u};w_\bullet}$ away from the nodal points of $\Sigma_{\mathbf{u};w_\bullet}$ and

$$\Gamma^{0,1}(\mathbf{u}_{\mathcal{C}_+}) = \big\{D_{\mathbf{u}_{\mathcal{C}_+}}\xi\colon \xi \in \Gamma(\mathbf{u}_{\mathcal{C}_+}),\ \xi(x_{e_i^+}(\mathbf{u})) = 0,\ i = 1,2\big\} \oplus \big\{\{\mathrm{id} \times u_{\mathcal{C}_+}\}^*\eta\colon \eta \in \widetilde{\Gamma}^{0,1}_-(\mathbf{u};\mathcal{C}_+)\big\}.$$

These two spaces determine $\widetilde{\Gamma}_-(\mathbf{u};P)$ as in (2.16) so that (B1) is satisfied.

Corollary 3.5. *Let $(\mathscr{T}, \mathcal{T}_0, \mathcal{T}_{0;P})$ be as in the sentence containing (3.25) and $\mathbf{u} \in \widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be a J-holomorphic map. If Δ is a sufficiently small neighborhood of $\mathbf{u}$ in $\widetilde{\mathcal{F}\mathscr{T}}$, the family (3.14) can be chosen so that there exist $C \in \mathbb{R}^{\geqslant 0}$,*

$$\begin{aligned} &\varepsilon_{\upsilon;w} \in \mathbb{R}^{\geqslant 0} \quad \forall\, w \in \widehat{\mathrm{Ver}}^a,\ \upsilon \in \Delta, && \mathcal{D}_{\upsilon;w} \in T_{\mathrm{ev}_0(\upsilon)}X \quad \forall\, w \in \widehat{\mathrm{Ver}}^c_{0;P},\ \upsilon \in \Delta,\\ &\gamma_{\upsilon;e} \in C^0\big(\Sigma_{\upsilon;e}(8\delta_{\bar\partial})\big) \quad \forall\, e \in \mathrm{Edg}^a,\ \upsilon \in \Delta, && \mathcal{D}_{\upsilon;e} \in T_{\mathrm{ev}_0(\upsilon)}X \quad \forall\, e \in \mathrm{Edg}_{0;P} - \mathrm{Edg}_{P_-},\ \upsilon \in \Delta, \end{aligned}$$

such that

- *(M1) in Proposition 3.1 with $\prec$ replaced by $<_\bullet$ holds for all $w \in \widehat{\mathrm{Ver}}_0 - \mathrm{Ver}_0$,*
- *the first and last inequalities in (M2) with $\prec$ replaced by $<_\bullet$ hold for all $w \in \mathrm{Ver}^c_{0;P} \sqcup \{w_\bullet\}$,*
- *the second inequality in (M2) holds for all $w \in \mathrm{Ver}^c_{0;P}$,*
- *(M3) with $\prec$ replaced by $<_\bullet$ holds for all $e \in \widetilde{\mathrm{Edg}}^c_{0;P}$,*
- *for all $w \in \{P_-\} \sqcup \mathrm{Ver}_{\mathcal{C}_-} \sqcup \mathrm{Ver}_{\mathcal{C}_+} - \{w_\bullet\}$, (3.23) and the last inequality in (M2) with $\prec$ replaced by $<_\bullet$ hold, and*
- *for all $e \in \mathrm{Edg}_{0;P} - \mathrm{Edg}_{P_-}$, (3.24) and (M3) with $\big(|\mathcal{D}_{\upsilon;w_e^+}|, \prec\big)$ replaced by $\big(|\mathcal{D}_{\upsilon;e}|, <_\bullet\big)$ hold.*

3.3 Proofs of main statements

Proof of Proposition 3.1. We first smooth out the nodes corresponding to Edg^c_0 and obtain the maps $\widetilde{u}_{\upsilon;1}$ and u_υ as in (2.30) and (2.42), respectively. We then continue to smooth out the remaining nodes as described below.

For $\mathbf{u}'\in\widetilde{U}$, $\upsilon\in\Delta|_{\mathbf{u}'}$, and $w\in\widehat{\mathrm{Ver}}_0$, let

$$\chi_w=\{e\in\chi(\mathcal{T}_0)\colon w_e^+\ge w\},\quad V_w^{\ge}=\{w'\in\widehat{\mathrm{Ver}}_0\colon w'\ge w\}\cup\bigcup_{e\in\chi_w}\mathrm{Ver}^c_{0;e}\,,\quad E_w^{\succ}=\{e\in\mathrm{Edg}\colon w_e^-\in V_w^{\ge}\},$$

$$\upsilon_w^{\succ}=\upsilon(E_w^{\succ}),\qquad \Sigma_{\upsilon;w}^{\ge}=\bigcup_{w'\in V_w^{\ge}}q_{\upsilon_w^{\succ}}^{-1}(\Sigma_{\mathbf{u}';w'}),\qquad \Sigma_{\upsilon;w}^{c}=(\Sigma_{\upsilon_w^{\succ}}-\Sigma_{\upsilon;w}^{\ge})\cup\{x_0(\upsilon_w^{\succ})\},\tag{3.28}$$
$$\Gamma_B(\upsilon_w^{\succ})=\{\zeta\in\Gamma(\upsilon_w^{\succ})\colon\ \zeta|_{\Sigma_{\upsilon;w}^c}=0\}$$

with $\Gamma(\upsilon)$ as in (2.42). For $w\in\mathrm{Ver}_0$, we define

$$\upsilon_w^{+,\succ}=\upsilon(E_w^{\succ}-E_w^+),\quad \widehat{\Gamma}_B(\upsilon_w^{+,\succ})=\Big\{\zeta\in\Gamma(\upsilon_w^{+,\succ})\colon\ \zeta(x)=0\ \ \forall\,x\in q_{\upsilon_w^{+,\succ};\upsilon_w^{\succ}}(\Sigma_{\upsilon;w}^c)\cup\Sigma_{\upsilon_w^{+,\succ};w}\Big\}.\tag{3.29}$$

There is a natural injective homomorphism

$$\Gamma_B(\upsilon_{w'}^{\succ})\longrightarrow\widehat{\Gamma}_B(\upsilon_w^{+,\succ})\tag{3.30}$$

for every $w'\in V_w^+$.

For $\mathbf{u}'\in\widetilde{U}$, let $\widetilde{\Gamma}_-^{0,1}(\mathbf{u}')$ be as in (2.20). For $\upsilon\in\Delta|_{\mathbf{u}'}$ and $w\in\widehat{\mathrm{Ver}}_0$, we will inductively construct

$$\begin{gathered}\xi_{\upsilon;w}\in\Gamma_B(\upsilon_w^{\succ}),\ \varepsilon_{\upsilon;w}\in\mathbb{R}^{\geqslant0}\qquad\text{s.t.}\qquad\|\xi_{\upsilon;w}\|_{\upsilon_w^{\succ},p,1}\leqslant C|\upsilon|,\\ \bar\partial_J\exp_{u_{\upsilon_w^{\succ}}}\xi_{\upsilon;w}\in\{q_{\upsilon_w^{\succ}}\times\exp_{u_{\upsilon_w^{\succ}}}\xi_{\upsilon;w}\}^*\widetilde{\Gamma}_-^{0,1}(\mathbf{u}').\end{gathered}\tag{3.31}$$

Every such $\xi_{\upsilon;w}$ determines

$$\widetilde{u}_{\upsilon;w}=\exp_{u_{\upsilon_w^{\succ}}}\xi_{\upsilon;w}\colon\ \Sigma_{\upsilon_w^{\succ}}\longrightarrow X,\qquad\mathcal{D}_{\upsilon;w}=\mathrm{d}_{x^+_{e_w}(\upsilon_w^{\succ})}\widetilde{u}_{\upsilon;w}\Big(\frac{\partial}{\partial z^+_{e_w;\upsilon_w^{\succ}}}\Big)\in T_{\mathrm{ev}_0(\upsilon)}X.\tag{3.32}$$

We will construct $\xi_{\upsilon;w}$ so that $\widetilde{u}_{\upsilon;w}$ depends on υ continuously. When the induction terminates, we obtain a vector field

$$\xi_\upsilon\equiv\xi_{\upsilon;w_0}\in\Gamma(\upsilon)\qquad\text{s.t.}\qquad\xi_\upsilon(x_0(\upsilon))=0.\tag{3.33}$$

This vector field ξ_υ determines a family $\widetilde{u}_\upsilon\equiv\widetilde{u}_{\upsilon;w_0}$ as in (3.6) satisfying Proposition 3.1.

Define

$$\xi_{\upsilon;w}=0,\quad\varepsilon_{\upsilon;w}=1\qquad\forall w\in\widehat{\mathrm{Ver}}_0-\mathrm{Ver}_0.\tag{3.34}$$

By (2.30), this $\xi_{\upsilon;w}$ satisfies (3.31). Suppose $w\in\mathrm{Ver}_0$ and for every $w'\in V_w^+$ we have constructed $\xi_{\upsilon;w'}\in\Gamma_B(\upsilon_{w'}^{\succ})$ and $\varepsilon_{\upsilon;w'}\in\mathbb{R}^+$ so that $\widetilde{u}_{\upsilon;w'}$ depends on υ continuously and (3.34), (3.31), and Lemmas 3.7 and 3.8 with w replaced by w' are satisfied. By (3.30), every such $\xi_{\upsilon;w'}$ can be treated as an element of $\widehat{\Gamma}_B(\upsilon_w^{+,\succ})$. Define

$$\xi_{\upsilon;w}^+=\sum_{w'\in V_w^+}\xi_{\upsilon;w'}\in\widehat{\Gamma}_B(\upsilon_w^{+,\succ}),\qquad\widetilde{u}_{\upsilon;w}^+=\exp_{u_{\upsilon_w^{+,\succ}}}\xi_{\upsilon;w}^+.\tag{3.35}$$

By the first equation in (3.35), the first property in (3.31) for $w'\in V_w^+$, and (2.22),

$$\|\xi_{\upsilon;w}^+\|_{\upsilon_w^{\succ},p,1},\ \|\xi_{\upsilon;w}^+\|_{C^0}\leqslant C|\upsilon|\tag{3.36}$$

for some $C\in\mathbb{R}$. Since $\widetilde{u}_{\upsilon;w'}$ satisfies (3.31) for every $w'\in V_w^+$,

$$\bar{\partial}_J\widetilde{u}_{\upsilon;w}^+ \in \big\{q_{\upsilon_w^+,>}\times\widetilde{u}_{\upsilon;w}^+\big\}^*\widetilde{\Gamma}_-^{0,1}(\mathbf{u}'). \tag{3.37}$$

Let

$$\widetilde{q}_{\upsilon;w}=\widetilde{q}_{\upsilon_w^>;E_w^+}\colon \Sigma_{\upsilon_w^>}\longrightarrow\Sigma_{\upsilon_w^+,>}, \qquad \widehat{u}_{\upsilon;w}=\widetilde{u}_{\upsilon;w}^+\circ\widetilde{q}_{\upsilon;w}. \tag{3.38}$$

We note that

$$u_{\upsilon_w^>},\ \widehat{u}_{\upsilon;w}=\text{const.} \qquad \text{on} \qquad \Sigma_{\upsilon;w}^c\cup q_{\upsilon_w^>}^{-1}\Big(\Sigma_{\mathbf{u}';w}-\bigsqcup_{e\in E_w^+}\Sigma_{\mathbf{u}';e}(\delta_{\bar\partial})\Big), \tag{3.39}$$

$$u_{\upsilon_w^>}=u_{\upsilon_w^+,>}\circ\widetilde{q}_{\upsilon;w}, \qquad \widehat{u}_{\upsilon;w}=\exp_{u_{\upsilon_w^>}}\big(\xi_{\upsilon;w}^+\circ\widetilde{q}_{\upsilon;w}\big). \tag{3.40}$$

With $\Gamma_-(\mathbf{u}')$ as in (2.21), define

$$\Gamma_{B,-}(\mathbf{u}')=\big\{\xi\in\Gamma_-(\mathbf{u}')\colon\ \xi|_{x_0(\mathbf{u}')}=0\big\}\,. \tag{3.41}$$

Since $u'|_{\Sigma_{\mathbf{u}';0}}$ is constant, every element of $\Gamma_{B,-}(\mathbf{u}')$ vanishes on $\Sigma_{\mathbf{u}';0}$. Let

$$\Gamma_B\big(\widehat{u}_{\upsilon;w}\big)=\big\{\zeta\in\Gamma(\widehat{u}_{\upsilon;w})\colon\ \zeta|_{\Sigma_{\upsilon;w}^c}=0\big\}, \qquad \Gamma_B^{0,1}\big(\widehat{u}_{\upsilon;w}\big)=\big\{\eta\in\Gamma^{0,1}(\widehat{u}_{\upsilon;w})\colon\ \eta|_{\Sigma_{\upsilon;w}^c}=0\big\},$$

$$\Gamma_{B,-}\big(\widehat{u}_{\upsilon;w}\big)=\bigg\{\Big(\Pi_{\xi_{\upsilon;w}^+}\Big(\big(\Pi_{\zeta_{\upsilon;1}}(\xi\circ q_{\upsilon_1})\big)\circ q_{\upsilon_w^+,>}\Big)\Big)\circ\widetilde{q}_{\upsilon;w}\colon\ \xi\in\Gamma_{B,-}(\mathbf{u}')\bigg\}\subset\Gamma_B\big(\widehat{u}_{\upsilon;w}\big). \tag{3.42}$$

We denote by $\Gamma_{B,+}\big(\widehat{u}_{\upsilon;w}\big)$ the L^2-orthogonal complement of $\Gamma_{B,-}\big(\widehat{u}_{u;w}\big)$ in $\Gamma_B\big(\widehat{u}_{\upsilon;w}\big)$.

For $\zeta\in\Gamma_B(\widehat{u}_{\upsilon;w})$, let

$$\Gamma_{B,-}^{0,1}(\widehat{u}_{\upsilon;w};\zeta)=\Pi_\zeta^{-1}\Big(\big\{q_{\upsilon_w^>}\times\exp_{\widehat{u}_{\upsilon;w}}\zeta\big\}^*\widetilde{\Gamma}_-^{0,1}(\mathbf{u}')\Big)\subset\Gamma_B^{0,1}\big(\widehat{u}_{\upsilon;w}\big). \tag{3.43}$$

Let $\Gamma_{B,+}^{0,1}\big(\widehat{u}_{\upsilon;w};\zeta\big)$ be the L^2-orthogonal complement of $\Gamma_{B,-}^{0,1}(\widehat{u}_{\upsilon;w};\zeta)$ in $\Gamma_B^{0,1}\big(\widehat{u}_{\upsilon;w}\big)$. We denote by

$$\pi_{B;\upsilon;w;\zeta}^{0,1;\pm}\colon\ \Gamma_B^{0,1}\big(\widehat{u}_{\upsilon;w}\big)\longrightarrow\Gamma_{B,\pm}^{0,1}\big(\widehat{u}_{\upsilon;w};\zeta\big) \tag{3.44}$$

the associated L^2-projections. It is straightforward that there exists a constant $C\in\mathbb{R}$ such that

$$\big\|\pi_{B;\upsilon;w;\zeta}^{0,1;+}(\eta)-\pi_{B;\upsilon;w;\zeta'}^{0,1;+}(\eta)\big\|_{\upsilon_w^>,p}\leqslant C\|\zeta-\zeta'\|_{\upsilon_w^>,p,1}\|\eta\|_{\upsilon_w^>,p} \tag{3.45}$$

for all $\eta\in\Gamma_{B,-}^{0,1}\big(\widehat{u}_{\upsilon;w};\zeta\big)$ and sufficiently small $\zeta,\zeta'\in\Gamma_B\big(\widehat{u}_{\upsilon;w}\big)$.

We denote by $\nabla^{\upsilon;w}\equiv\nabla^{J,\widehat{u}_{u;w}}$ the J-linear connection on $\widehat{u}_{\upsilon;w}^*TX$ and by

$$D_{B;\upsilon;w}\colon\ \Gamma_B\big(\widehat{u}_{\upsilon;w}\big)\longrightarrow\Gamma_B^{0,1}\big(\widehat{u}_{\upsilon;w}\big)$$

the restriction of the linearization $D_{\widehat{u}_{\upsilon;w}}$ of the $\bar\partial_J$-operator at $\widehat{u}_{\upsilon;w}$ with respect to ∇^J. Similar to [15, Lemma 3.16(1)], there exists a constant $C\in\mathbb{R}$ such that

$$\|\zeta\|_{\upsilon_w^>,p,1}\leqslant C\|D_{B,\upsilon;w}\zeta\|_{\upsilon_w^>,p} \qquad \forall\ \upsilon\in\Delta,\ \zeta\in\Gamma_{B,+}\big(\widehat{u}_{\upsilon;w}\big). \tag{3.46}$$

Let $\widetilde{\Gamma}^{0,1}_{B,+}(\widehat{u}_{\upsilon;w})$ be the image of $\Gamma_{B,+}(\widehat{u}_{\upsilon;w})$ under $D_{B;\upsilon;w}$. For sufficiently small $\zeta\in\Gamma_{B,+}(\widehat{u}_{\upsilon;w})$, the restriction of $\pi^{0,1;+}_{B;\upsilon;w;\zeta}$ to $\widetilde{\Gamma}^{0,1}_{B,+}(\widehat{u}_{\upsilon;w})$ is an isomorphism. The norm of its inverse

$$f^{+}_{B;\upsilon;w;\zeta}\colon \Gamma^{0,1}_{B,+}(\widehat{u}_{\upsilon;w};\zeta)\longrightarrow \widetilde{\Gamma}^{0,1}_{B,+}(\widehat{u}_{\upsilon;w}) \tag{3.47}$$

is bounded by a constant $C\in\mathbb{R}$; see [15, Lemma 3.16(3)]. Let

$$\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\zeta}=f^{+}_{B;\upsilon;w;\zeta}\circ\pi^{0,1;+}_{B;\upsilon;w;\zeta}\colon \Gamma^{0,1}_{B}(\widehat{u}_{\upsilon;w})\longrightarrow \widetilde{\Gamma}^{0,1}_{B,+}(\widehat{u}_{\upsilon;w}).$$

By the boundedness of (3.44) and of (3.47), there exists a constant $C\in\mathbb{R}$ such that

$$\big\|\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\zeta}\eta\big\|_{\upsilon^{>}_{w},p}\leqslant C\|\eta\|_{\upsilon^{>}_{w},p}\qquad \forall\ \eta\in\Gamma^{0,1}_{B}(\widehat{u}_{\upsilon;w}). \tag{3.48}$$

The restriction of $\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\zeta}$ to $\widetilde{\Gamma}^{0,1}_{B,+}(\widehat{u}_{\upsilon;w})$ is the identity map by definition.

The Contraction Principle as used in the proof of [15, Lemma 3.18] implies that the equation

$$\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\zeta}\Big(\Pi_{\zeta}^{-1}\big(\bar{\partial}_J\exp_{\widehat{u}_{\upsilon;w}}\zeta\big)\Big)=0 \tag{3.49}$$

has a unique small solution

$$\widehat{\zeta}_{\upsilon;w}\in\Gamma_{B,+}(\widehat{u}_{\upsilon;w}); \tag{3.50}$$

see the proof of Lemma 3.6 for more details. We define $\varepsilon_{\upsilon;w}$ as in (3.31) and $\xi_{\upsilon;w}$ by

$$\varepsilon_{\upsilon;w}=\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon^{>}_{w},p,1},\qquad \exp_{u_{\upsilon^{>}_{w}}}\xi_{\upsilon;w}=\exp_{\widehat{u}_{\upsilon;w}}\widehat{\zeta}_{\upsilon;w}. \tag{3.51}$$

By the second equation in (3.51), (3.39), and (3.50), $\xi_{\upsilon;w}\in\Gamma_B(\upsilon^{>}_{w})$. By (3.49) with $\zeta=\widehat{\zeta}_{\upsilon;w}$ and (3.43), $\xi_{\upsilon;w}$ satisfies (3.31). By the continuity argument in [15, Section 4.1], $\widehat{\zeta}_{\upsilon;w}$ and $\xi_{\upsilon;w}$ depend on υ continuously in the sense of [15, (4.2)].

The two equalities in (M1) follow from (3.34) and (3.32), respectively. The last property in (M1) follows from (3.33) and Lemma 3.7 below. The first property in (M2) follows from (3.51) and Lemma 3.6 below. By the proof of the $r=1$ case of [18, Lemma 3.5(2a)],

$$\mathcal{D}_{\upsilon;w}=\sum_{w'\in V^{+}_{w}}\big(\mathcal{D}_{\upsilon;w'}+\breve{\epsilon}_{\upsilon;w'}\big)v_{e_{w'}}\qquad\text{with}\qquad |\breve{\epsilon}_{\upsilon;w'}|\leqslant C\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon^{>}_{w},p,1} \tag{3.52}$$

for some $C\in\mathbb{R}$. Along with (3.51), this implies the second property in (M2). The last property in (M2) follows from (3.33) and Lemma 3.7.

For $e\in\mathrm{Edg}_0\sqcup\{e_{w_0}\}$, we define

$$\gamma_{\upsilon;e}\colon \Sigma_{\upsilon;e}(8\delta_{\partial})\longrightarrow\mathbb{R},\qquad \gamma_{\upsilon;e}(x)=\sum_{w'<w^{+}_{e}}\big|\nabla^{\upsilon;w'}\widehat{\zeta}_{\upsilon;w'}\big|_x. \tag{3.53}$$

By the first equation in (3.51), $\gamma_{\upsilon;e}$ satisfies the third property in (M3). The remaining three properties in (M3) follow from (3.33) and Lemma 3.8 below. □

Proof of Corollary 3.2. Denote by υ_2 and υ_{02} the projections of $\upsilon\in\Delta$ to

$$\Delta_2\equiv\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}\Big(\mathrm{Edg}-\big(\mathrm{Edg}_{0;P}\sqcup\chi(\mathcal{T}_{0;P})\big)\Big),\qquad \Delta_{02}\equiv\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}\big(\mathrm{Edg}-\chi(\mathcal{T}_{0;P})\big), \tag{3.54}$$

respectively. By Proposition 3.1, the family of the deformations $\mathfrak{U}|_{\Delta_2}\longrightarrow\Delta_2$ of the domains of elements of $\widetilde{U}$ extends to a continuous family of maps

$$\widetilde{u}_{\upsilon;2}=\exp_{u_{\upsilon_2}}\xi_{\upsilon;2}\colon\Sigma_{\upsilon_2}\longrightarrow X,\quad \upsilon_2\in\Delta_2, \tag{3.55}$$

such that

$$\widetilde{u}_{\upsilon;2}\big(\Sigma_{\upsilon_2;0;P}\big)=\mathrm{ev}_0(\upsilon_2),\qquad \bar\partial_J\widetilde{u}_{\upsilon;2}\in\big\{q_{\upsilon_2}\times\widetilde{u}_{\upsilon;2}\big\}^*\,\widetilde{\Gamma}_-^{0,1}(\mathbf{u}')\quad\forall\,\mathbf{u}'\in\widetilde{U},\ \upsilon_2\in\Delta_2|_{\mathbf{u}'},$$

and for every $e\in\chi(\mathcal{T}_{0;P})$ the restriction $\widetilde{u}_{\upsilon;2}^{(e)}$ of $\widetilde{u}_{\upsilon;2}$ to the curve $\Sigma_{\upsilon_2;e}$ corresponding to $(\mathscr{T}_e,\mathcal{T}_{e;0})$ satisfies the conclusions of Proposition 3.1.

The objects associated in Proposition 3.1 with all such $\widetilde{u}_{\upsilon;2}^{(e)}$ determine

$$\varepsilon_{\upsilon_2;w},\,\mathcal{D}_{\upsilon_2;w}\quad\forall\,w\in\widehat{\mathrm{Ver}}_{0;P}^c,\qquad \gamma_{\upsilon_2;e}\quad\forall\,e\in\widetilde{\mathrm{Edg}}_{0;P}^c,$$

satisfying Corollary 3.2 with $\upsilon=\upsilon_2$. Define

$$\widetilde{\xi}_{\upsilon;2}\colon\Sigma_{\upsilon_2;e}(8\delta_{\bar\partial})\longrightarrow T_{\mathrm{ev}_0(\upsilon_2)}X\quad\text{s.t.}\quad \widetilde{u}_{\upsilon;2}=\exp_{\mathrm{ev}_0(\upsilon_2)}\widetilde{\xi}_{\upsilon;2}\qquad\forall\,e\in\chi(\mathcal{T}_{0;P}). \tag{3.56}$$

We note that there exists a constant $C\in\mathbb{R}$ such that

$$\|\widetilde{\xi}_{\upsilon;2}\|_{\upsilon_2,p,1}\leqslant C\,\varepsilon_{\upsilon_2;w_e^+}\qquad\forall\,e\in\chi(\mathcal{T}_{0;P}). \tag{3.57}$$

If $e\in\chi(\mathcal{T}_{0;P})\cap\chi(\mathcal{T}_0)$, this equality holds by the continuity of the family $\widetilde{u}_{\upsilon;2}^{(e)}$ and the first property in (M1) in Proposition 3.1. If $e\in\chi(\mathcal{T}_{0;P})-\chi(\mathcal{T}_0)$, then the restriction of u_{υ_2} to $\Sigma_{\upsilon_2;e}(8\delta_{\bar\partial})$ is constant and hence the restrictions of $\widetilde{\xi}_{\upsilon;2}$ and $\xi_{\upsilon;2}$ are the same. The inequality in (3.57) then holds by the last bound in (M2) in Proposition 3.1.

For every $\upsilon\in\Delta$, let

$$\begin{gathered}\upsilon_{P2}=\upsilon\big(\mathrm{Edg}-\chi(\mathcal{T}_{0;P})\big),\qquad \widehat{u}_{\upsilon;2}=\widetilde{u}_{\upsilon;2}\circ q_{\upsilon_2;\upsilon_{P2}}\circ\widetilde{q}_{\upsilon;\chi(\mathcal{T}_{0;P})}\colon\Sigma_\upsilon\longrightarrow X,\\ \Gamma_-\big(\widehat{u}_{\upsilon;2}\big)=\left\{\bigg(\Pi_{\xi_{\upsilon;2}}\Big(\Big(\Pi_{\zeta_{\upsilon;1}}\big(\xi\circ q_{\upsilon_1}\big)\Big)\circ q_{\upsilon_2}\Big)\bigg)\circ q_{\upsilon_2;\upsilon_{P2}}\circ\widetilde{q}_{\upsilon;\chi(\mathcal{T}_{0;P})}\colon\,\xi\in\Gamma_-(\mathbf{u}')\right\}\subset\Gamma\big(\widehat{u}_{\upsilon;2}\big),\\ \Gamma_-^{0,1}\big(\widehat{u}_{\upsilon;2};\zeta\big)=\Pi_\zeta^{-1}\Big(\big\{q_{\upsilon_2}\times\exp_{\widehat{u}_{\upsilon;2}}\zeta\big\}^*\widetilde{\Gamma}_-^{0,1}(\mathbf{u}')\Big)\subset\Gamma^{0,1}\big(\widehat{u}_{\upsilon;2}\big)\qquad\forall\,\zeta\in\Gamma\big(\widehat{u}_{\upsilon;2}\big).\end{gathered} \tag{3.58}$$

Denote by $\Gamma_+\big(\widehat{u}_{\upsilon;2}\big)$ and $\Gamma_+^{0,1}\big(\widehat{u}_{\upsilon;2};\zeta\big)$ the L^2-orthogonal complements of $\Gamma_-\big(\widehat{u}_{\upsilon;2}\big)$ and $\Gamma_-^{0,1}\big(\widehat{u}_{\upsilon;2};\zeta\big)$ in $\Gamma\big(\widehat{u}_{\upsilon;2}\big)$ and $\Gamma^{0,1}\big(\widehat{u}_{\upsilon;2}\big)$, respectively. Let

$$\pi_{B;\upsilon;\zeta}^{0,1;\pm}\colon\,\Gamma^{0,1}\big(\widehat{u}_{\upsilon;2}\big)\longrightarrow\Gamma_\pm^{0,1}\big(\widehat{u}_{\upsilon;2};\zeta\big)$$

be the associated L^2-projections.

By the same reasoning as in the proof of Lemma 3.6, the equation

$$\widetilde{\pi}^{0,1;+}_{B;\upsilon;\zeta}\Big(\Pi_\zeta^{-1}\big(\bar{\partial}_J \exp_{\widehat{u}_{\upsilon;2}}\zeta\big)\Big) = 0 \tag{3.59}$$

has a unique small solution $\widehat{\zeta}_{\upsilon;2}\in\Gamma_+\big(\widehat{u}_{\upsilon;2}\big)$ and

$$\big\|\widehat{\zeta}_{\upsilon;2}\big\|_{\upsilon,p,1} \leqslant C \sum_{e\in\chi(\mathcal{T}_{0;P})}\Big(|\mathcal{D}_{\upsilon;w_e^+}||\upsilon_e|+\varepsilon_{\upsilon;w_e^+}|\upsilon_e|^2\Big). \tag{3.60}$$

We take

$$\widetilde{u}_\upsilon = \exp_{\widehat{u}_{\upsilon;2}}\widehat{\zeta}_{\upsilon;2}, \qquad \gamma_{\upsilon;e} = \gamma_{\upsilon_2;e} + \big|\nabla^{\widehat{u}_{\upsilon;2}}\widehat{\zeta}_{\upsilon;2}\big| \quad \forall\, e\in\widetilde{\mathrm{Edg}}^c_{0;P}, \tag{3.61}$$

$$\mathcal{D}_{\upsilon;w} = \mathcal{D}_{\upsilon_2;w} \quad \forall\, w\in\widehat{\mathrm{Ver}}^c_{0;P}, \qquad \varepsilon_{\upsilon;w} = \begin{cases} \big\|\widehat{\zeta}_{\upsilon;2}\big\|_{\upsilon,p,1}, & \text{if } w=P;\\ \varepsilon_{\upsilon_2;w}, & \text{if } w\in\widehat{\mathrm{Ver}}^c_{0;P}. \end{cases} \tag{3.62}$$

Since $\widetilde{u}_\upsilon$ solves (3.59), it satisfies (3.13). The two equalities in (M1) and the second inequality in (M2) for $w\in\mathrm{Ver}^c_{0;P}$ follow from the corresponding statements for $(\mathscr{T}_e,\mathcal{T}_{e;0})$. The last properties in (M1) and (M2) are obtained as in the proof of Lemma 3.7. The first property in (M2) follows from (3.60) as in the proof of Lemma 3.6. The last property in (M3) holds by the definitions of $\gamma_{\upsilon;e}$ and $\varepsilon_{\upsilon;w'}$ and by the corresponding statements for $(\mathscr{T}_e,\mathcal{T}_{e;0})$. The first two properties in (M3) are obtained as in the proof of Lemma 3.8. □

Proof of Corollary 3.3. We apply the proof of Corollary 3.2 with $\mathcal{T}_{0;P}$ replaced by $\mathcal{T}_{P^-}$. One of the tree framings $(\mathscr{T}_e,\mathcal{T}_{e;0})$ to which we apply the proof of Proposition 3.1 in this case contains $\Sigma_{\mathbf{u};P^+}$, a union of irreducible components of $\Sigma_{\mathbf{u}}$. We treat it in a single inductive step as it corresponds to a single element of $\widehat{\mathrm{Ver}}^b$. The only statement in the proof of Proposition 3.1 that does not apply to this vertex is (3.52); it yields the second bound in (M2). On the other hand,

$$\widetilde{u}_{\upsilon;P^+} \equiv \exp_{u_{\upsilon^>_{P^+}}}\xi_{\upsilon;P^+} \equiv \exp_{\widehat{u}_{\upsilon;P^+}}\widehat{\zeta}_{\upsilon;P^+}$$

is J-holomorphic on $\Sigma^+_{\upsilon^>_{P^+};e^+}(8\delta_{\bar{\partial}})$ and $\widehat{u}_{\upsilon;P^+}$ is constant there. Thus, the bound in (3.22) on

$$\mathcal{D}_{\upsilon;P^+} \equiv \mathrm{d}_{x^+_{e^+}(\upsilon^>_{P^+})}\widetilde{u}_{\upsilon;P^+}\Big(\frac{\partial}{\partial z^+_{e^+;\upsilon^>_{P^+}}}\Big)$$

follows from Lemma A.8 and the first equation in (3.91) with $w=P^+$. □

Proof of Corollary 3.4. The following proof applies as long as there is a total order on a partition of the irreducible components of $\Sigma_{\mathbf{u}';0;P}$ into connected unions. In particular, this proof applies to the setting of Corollary 3.3, although it would lead to weaker conclusions.

For $w\in\{P^+,P^-\}\sqcup\big(\mathrm{Ver}_{\mathcal{C}}-\{w_\bullet\}\big)$, let $w+1\in\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}$ be the immediate successor of w with respect to $<_\bullet$. For each $w\in\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}$, let $(\mathscr{T}_w;\mathcal{T}_{w;0})$ be the tree framing determined by the vertex w and the sets $V^{\gtrless}_{w'}$ with $w'\in\widehat{\mathrm{Ver}}^c_{0;P}$ and $e_{w'}\ni w$; see (3.28) for the notation. We denote the corresponding graphs of $\mathscr{T}_w$ and $\mathcal{T}_{w;0}$ by (V_w,E_w) and $(V_{w;0},E_{w;0})$, respectively. Denote by $\mathscr{T}^{\geqslant_\bullet}_w$ the bubble type corresponding to the graphs of $\mathscr{T}_{w'}$ with $w'\in\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}$ and $w'\geqslant_\bullet w$, by $E_w^{\geqslant\bullet}\subset\mathrm{Edg}$ its edges, and by $\Sigma^{\geqslant\bullet}_{\mathbf{u}';w}\subset\Sigma_{\mathbf{u}'}$ the union of the irreducible components corresponding to $\mathscr{T}_w^{\geqslant\bullet}$. For $\upsilon\in\Delta$, let

$$\upsilon_w^{\geqslant\bullet}=\upsilon(E_w^{\geqslant\bullet}),\qquad \Sigma^{\geqslant\bullet}_{\upsilon;w}=q^{-1}_{\upsilon_w^{\geqslant\bullet}}\big(\Sigma^{\geqslant\bullet}_{\mathbf{u}';w}\big)\subset\Sigma_{\upsilon_w^{\geqslant\bullet}},\qquad u^{\geqslant\bullet}_{\upsilon;w}=u_{\upsilon_w^{\geqslant\bullet}}|_{\Sigma^{\geqslant\bullet}_{\upsilon;w}}\colon\Sigma^{\geqslant\bullet}_{\upsilon;w}\longrightarrow X\quad\forall\,\upsilon\in\Delta.$$

Let $\mathbf{u}_{(w)}=(\Sigma_{(w)},u_{(w)})$ be the component of

$$\mathbf{u}\in\widetilde{U}\subset\prod_{w\in\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}}\widetilde{\mathcal{U}}_{\mathscr{T}_w}(X,J)$$

corresponding to $\mathscr{T}_w$ and $\widetilde{U}_{(w)}$ be a small neighborhood of $\mathbf{u}_{(w)}$ in $\widetilde{\mathcal{U}}_{\mathscr{T}_w}(X,J)$. By (2.17) and (2.18), the evaluation map

$$\mathrm{ev}_0\colon\widetilde{U}_{(w)}\longrightarrow X$$

at the primary contracted component $\mathcal{T}_{w;0}$ of $\mathscr{T}_w$ is a submersion. Thus, there exist $\delta\in\mathbb{R}^+$ and a smooth map

$$\begin{aligned}&\mathrm{ev}_0^*TX\longrightarrow\widetilde{\mathcal{U}}_{\mathscr{T}_w}(X,J),\qquad Y\mapsto\mathbf{u}'_Y\\ \text{s.t.}\qquad&\mathbf{u}'_{\mathbf{u}'}=\mathbf{u}'\quad\forall\,\mathbf{u}'\in\widetilde{U}_{(w)},\qquad\mathrm{ev}_0(\mathbf{u}'_Y)=\exp_{\mathrm{ev}_0(\mathbf{u}')}Y\quad\forall\,Y\in T_{\mathrm{ev}_0(\mathbf{u}')}X,\ |Y|<\delta.\end{aligned}\tag{3.63}$$

For every $\mathbf{u}'_Y$, we smooth out the nodes of $\mathscr{T}_w$ contained in Edg_0^c and obtain a map $\widetilde{u}_{Y;\upsilon;1}$ as in (2.30).

Applying Corollary 3.2 with $P=w_\bullet$ to $(\mathscr{T}_{w_\bullet},\mathcal{T}_{w_\bullet;0})$, we obtain a continuous family of nearly J-holomorphic maps

$$\widetilde{u}^{\geqslant\bullet}_{\upsilon;w_\bullet}=\exp_{u^{\geqslant\bullet}_{\upsilon;w_\bullet}}\xi^{\geqslant\bullet}_{\upsilon;w_\bullet}\colon\Sigma^{\geqslant\bullet}_{\upsilon;w_\bullet}\longrightarrow X$$

as in (3.6). It satisfies all requirements of Corollary 3.4 for $(\mathscr{T}^{\geqslant\bullet}_{w_\bullet},\mathcal{T}^{\geqslant\bullet}_{w_\bullet;0})=(\mathscr{T}_{w_\bullet},\mathcal{T}_{w_\bullet;0})$.

Suppose $w\in\{P^+,P^-\}\sqcup\big(\mathrm{Ver}_{\mathcal{C}}-\{w_\bullet\}\big)$ and we have constructed a continuous family of maps

$$\widetilde{u}^{\geqslant\bullet}_{\upsilon;w+1}=\exp_{u^{\geqslant\bullet}_{\upsilon;w+1}}\xi^{\geqslant\bullet}_{\upsilon;w+1}\colon\Sigma^{\geqslant\bullet}_{\upsilon;w+1}\longrightarrow X\tag{3.64}$$

as in (3.6) satisfying all requirements of Corollary 3.4 for $(\mathscr{T}^{\geqslant\bullet}_{w+1},\mathcal{T}^{\geqslant\bullet}_{w+1;0})$. Define

$$\begin{aligned}&Y_{\upsilon;w}\in T_{\mathrm{ev}_0(\upsilon)}X &&\text{by}\quad\exp_{\mathrm{ev}_0(\upsilon)}Y_{\upsilon;w}=\widetilde{u}^{\geqslant\bullet}_{\upsilon;w+1}\big(x^+_{e^w}(\upsilon^{\geqslant\bullet}_{w+1})\big)\\ &\xi^s_{\upsilon;w}\in\Gamma\big(\widetilde{u}^*_{(w);\upsilon;1}TX\big)&&\text{by}\quad\exp_{\widetilde{u}_{(w);\upsilon;1}}\xi^s_{\upsilon;w}=\widetilde{u}_{(w);Y_{\upsilon;w};\upsilon;1}\end{aligned}\qquad\forall\,\upsilon\in\Delta|_{\mathbf{u}'},\ \mathbf{u}'\in\widetilde{U}.$$

By the smoothness of (3.63) and the assumptions on (3.64),

$$\|\xi^s_{\upsilon;w}\|_{\upsilon_1,p,1},\ \|\xi^s_{\upsilon;w}\|_{C^1(\Sigma_{(w);\upsilon_1;0}(8\delta_{\bar\partial}))}\leqslant C|Y_{\upsilon;w}|=C\big|\xi^{\geqslant\bullet}_{\upsilon;w+1}\big(x^+_{e^w}(\upsilon^{\geqslant\bullet}_{w+1})\big)\big|\leqslant C'\sum_{\substack{w'\in\{P^+,P^-\}\sqcup\mathrm{Ver}_{\mathcal{C}}\\ w'>_\bullet w}}\varepsilon_{\upsilon;w'}\,.\tag{3.65}$$

For each $\upsilon\in\Delta|_{\mathbf{u}'}$, let $u^{\geqslant\bullet;0}_{\upsilon;w}$ be the map obtained by identifying $\widetilde{u}^{\geqslant\bullet}_{\upsilon;w+1}$ at the marked point $x^+_{e^w}(\upsilon^{\geqslant\bullet}_{w+1})$ with $\widetilde{u}_{(w);Y_{\upsilon;w};\upsilon;1}$ at the marked point $x^-_{e^w}(\upsilon_1)$. Define

$$\breve{u}^{\geqslant\bullet}_{\upsilon;w}=u^{\geqslant\bullet;0}_{\upsilon;w}\circ q_{\upsilon(E_w^{\geqslant\bullet}-E_{w;0});\upsilon(E_w^{\geqslant\bullet}-\chi(\mathcal{T}_{w;0}))}\circ\widetilde{q}_{\upsilon_w^{\geqslant\bullet};\chi(\mathcal{T}_{w;0})}\colon\ \Sigma^{\geqslant\bullet}_{\upsilon;w}\longrightarrow X.$$

We apply the proof of Proposition 3.1 with $w_0=w$ if $w\in\mathrm{Ver}_{\mathcal{C}}$ or Corollary 3.2 with $P=w$ if $w=P^{\pm}$ to this family of maps. It provides a family of maps

$$\widetilde{u}_{\upsilon;w}^{\geqslant\bullet}\equiv\exp_{u_{\upsilon;w}^{\geqslant\bullet}}\xi_{\upsilon;w}^{\geqslant\bullet}=\exp_{\breve{u}_{\upsilon;w}^{\geqslant\bullet}}\xi_{\upsilon;(w)}\colon\Sigma_{\upsilon;w}^{\geqslant\bullet}\longrightarrow X$$

satisfying the conclusions of Proposition 3.1 for $(\breve{u}_{\upsilon;w}^{\geqslant\bullet},\xi_{\upsilon;(w)})$ if $w\in\mathrm{Ver}_{\mathcal{C}}$ or the conclusions of Corollary 3.2 for $(\breve{u}_{\upsilon;w}^{\geqslant\bullet},\xi_{\upsilon;(w)})$ if $w=P^{\pm}$. Along the way, we obtain

$$\mathcal{D}_{\upsilon;w'}^{(w)}\in T_{\widetilde{u}_{(w);Y_{\upsilon;w};\upsilon;1}(x_{e^w}^-(\upsilon_1))}X\qquad\forall\,w'\in\widehat{V}_{w;0}$$

as in (3.32). Define

$$\mathcal{D}_{\upsilon;w'}=\Big(\Pi_{Y_{\upsilon;w}}^{-1}\mathcal{D}_{\upsilon;w'}^{(w)}\Big)\in T_{\mathrm{ev}_0(\upsilon)}X\quad\forall\,w'\in\widehat{V}_{w;0},\qquad\mathcal{D}_{\upsilon;e_w}=\Pi_{Y_{\upsilon;w}}^{-1}\mathrm{d}_{x_{e_w}^+(\upsilon_w^{\geqslant\bullet})}\widetilde{u}_{\upsilon;w}^{\geqslant\bullet}.$$

Since

$$\begin{aligned}&u_{\upsilon;w}^{\geqslant\bullet;0}\Big(q_{\upsilon(E_w^{\geqslant\bullet}-E_{w;0});\upsilon(E_w^{\geqslant\bullet}-\chi(\mathcal{T}_{w;0}))}\big(\widetilde{q}_{\upsilon_w^{\geqslant\bullet};\chi(\mathcal{T}_{w;0})}(x)\big)\Big)\\&=\exp_{u_{\upsilon;w}^{\geqslant\bullet}(x)}\Big(\xi_{\upsilon;w}^s\Big(q_{\upsilon(E_w^{\geqslant\bullet}-E_{w;0});\upsilon(E_w^{\geqslant\bullet}-\chi(\mathcal{T}_{w;0}))}\big(\widetilde{q}_{\upsilon_w^{\geqslant\bullet};\chi(\mathcal{T}_{w;0})}(x)\big)\Big)\Big)\qquad\forall\,x\in q_{\upsilon_w^{\geqslant\bullet}}^{-1}\Big(\bigcup_{w'\in V_w}\Sigma_{\mathbf{u}';w'}\Big),\end{aligned}$$

Lemma A.3 and (3.65) imply that $\xi_{\upsilon;w}^{\geqslant\bullet}$ satisfies the requirement of Corollary 3.4 for $(\mathscr{T}_w^{\geqslant\bullet},\mathcal{T}_{w;0}^{\geqslant\bullet})$. We patch $\xi_{\upsilon;w}^s$ with $\xi_{\upsilon;w+1}^s$ over $\Sigma_{\mathbf{u}';e^w}^+(4\delta)$, which does not affect any bounds over $\Sigma_{\upsilon_w^{\geqslant\bullet};e^w}^+(\delta)$. □

Proof of Corollary 3.5. We apply the proof of Corollary 3.4 to the tree framings $(\mathscr{T}_w,\mathcal{T}_{w;0})$ inductively over $w\in\mathrm{Ver}_{\mathcal{C}_+}$, with respect to the total order $<_\bullet$ on $\mathrm{Ver}_{\mathcal{C}_+}$. By the choice of $\widetilde{\Gamma}_-^{0,1}(\mathbf{u};\mathcal{C}_+)$ as in the sentence containing (3.27), we can assume $Y_{\upsilon;w}=0$ and $\xi_{\upsilon;w}^s$ for every $w\in\mathcal{C}_+$. Thus, the proof is much simpler in this case. We denote by $\widetilde{u}_{\upsilon;\mathcal{C}_+}$ the resulting map. In particular, the values of $\widetilde{u}_{\upsilon;\mathcal{C}_+}$ at the nodes corresponding to e_1^+,e_2^+ are equal to $\mathrm{ev}_0(\upsilon)$.

When the above induction terminates, the resulting bubble type does not form a pseudo-tree framing. On the other hand, the maximal chain of spheres $\mathcal{C}_+$ attaches either directly to $\mathcal{T}_{P_-}$ or to a single irreducible component of $\mathcal{C}_-$. Thus, the proof of Corollary 3.2 for $P=P_-$ still applies. □

3.4 Technical lemmas

Lemma 3.6. *There exists a constant $C\in\mathbb{R}$ such that*

$$\big\|\widehat{\zeta}_{\upsilon;w}\big\|_{\upsilon_w^>,p,1}\leqslant C\sum_{w'\in V_w^+}\Big(|\mathcal{D}_{\upsilon;w'}||\upsilon_{e_{w'}}|+\varepsilon_{\upsilon;w'}|\upsilon_{e_{w'}}|^2\Big)$$

for all $\upsilon\in\Delta$ and $w\in\mathrm{Ver}_0$.

Proof. Consider the quadratic expansion

$$\Pi_{\widehat{\zeta}_{\upsilon;w}}^{-1}\big(\bar{\partial}_J\exp_{\widehat{u}_{\upsilon;w}}\widehat{\zeta}_{\upsilon;w}\big)=\bar{\partial}_J\widehat{u}_{\upsilon;w}+D_{B;\upsilon;w}\widehat{\zeta}_{\upsilon;w}+N_{B;\upsilon;w}\big(\widehat{\zeta}_{\upsilon;w}\big),\tag{3.66}$$

where $N_{B;\upsilon;w}$ is a quadratic term. Analogous to [18, (4.27)], there exists $C\in\mathbb{R}$ such that

$$N_{B;\upsilon;w}(0)=0,\qquad\big\|N_{B;\upsilon;w}(\zeta)-N_{B;\upsilon;w}(\zeta')\big\|_{\upsilon_w^>,p}\leqslant C\big(\|\zeta\|_{\upsilon_w^>,p,1}+\|\zeta'\|_{\upsilon_w^>,p,1}\big)\|\zeta-\zeta'\|_{\upsilon_w^>,p,1}\tag{3.67}$$

for all $\zeta,\,\zeta'\in\Gamma_B\big(\widehat{u}_{\upsilon;w}\big)$ sufficiently small. By (3.49) with ζ replaced by $\widehat{\zeta}_{\upsilon;w}$ and (3.66),

$$\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\widehat{\zeta}_{\upsilon;w}}\Big(\bar\partial_J\widehat{u}_{\upsilon;w}+D_{B;\upsilon;w}\widehat{\zeta}_{\upsilon;w}+N_{B;\upsilon;w}\big(\widehat{\zeta}_{\upsilon;w}\big)\Big)=0\,. \tag{3.68}$$

Therefore, there exist constants $C,C'\in\mathbb{R}$ such that

$$\begin{aligned}\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon_w^>,p,1}\leqslant C\|D_{B;\upsilon;w}\widehat{\zeta}_{\upsilon;w}\|_{\upsilon_w^>,p}&=C\|\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\widehat{\zeta}_{\upsilon;w}}\big(D_{B;\upsilon;w}\widehat{\zeta}_{\upsilon;w}\big)\|_{\upsilon_w^>,p}\\&\leqslant C'\Big(\|\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\widehat{\zeta}_{\upsilon;w}}\bar\partial_J\widehat{u}_{\upsilon_w^>}\|_{\upsilon_w^>,p}+\|\widehat{\zeta}_{\upsilon;w}\|^2_{\upsilon_w^>,p,1}\Big).\end{aligned}$$

The first inequality above follows from (3.46), the equality there is due to the sentence after (3.48), and the last inequality above follows from (3.68), (3.48), and (3.67). Since $\widehat{\zeta}_{\upsilon;w}$ is small, we conclude that

$$\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon_w^>,p,1}\leqslant C\,\|\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\widehat{\zeta}_{\upsilon;w}}\bar\partial_J\widehat{u}_{\upsilon_w^>}\|_{\upsilon_w^>,p} \tag{3.69}$$

for some $C\in\mathbb{R}$.

By (3.38), (q2) in Section 2.2 with $(E_1,E_2)=(E_w^>-E_w^+,E_w^>)$, (3.37), and the assumption below (2.28),

$$\mathrm{supp}\big(\bar\partial_J\widehat{u}_{\upsilon;w}\big)\subset\bigg(\bigsqcup_{e\in E_w^+}\widetilde{A}^-_{e,\upsilon_w^>}(\delta_{\bar\partial})\bigg)\sqcup\big(\Sigma_{\upsilon_w^>}-\Sigma_{\upsilon_w^>;0}(8\delta_{\bar\partial})\big); \tag{3.70}$$

see (2.37) and (2.29) for the notation. The restrictions of $\bar\partial_J\widehat{u}_{\upsilon;w}$ to each $\widetilde{A}^-_{e,\upsilon_w^>}(\delta_{\bar\partial})$ and to

$$\Sigma^c_{\upsilon_w^>;0}(8\delta_{\bar\partial})\equiv\Sigma_{\upsilon_w^>}-\Sigma_{\upsilon_w^>;0}(8\delta_{\bar\partial})$$

extend to $\Sigma_{\upsilon_w^>}$ by zero.

By (3.38), (2.41), and (q2) in Section 2.2 with $(E_1,E_2)=(E_w^>-E_w^+,E_w^>)$,

$$\bar\partial_J\widehat{u}_{\upsilon;w}\big|_{\Sigma^c_{\upsilon_w^>;0}(8\delta_{\bar\partial})}=\widetilde{q}^*_{\upsilon;w}\Big(\bar\partial_J\widetilde{u}^+_{\upsilon;w}\big|_{\widetilde{q}_{\upsilon;w}\big(\Sigma^c_{\upsilon_w^>;0}(8\delta_{\bar\partial})\big)}\Big)=\widetilde{q}^*_{\upsilon;w}\Big(\bar\partial_J\widetilde{u}^+_{\upsilon;w}\Big). \tag{3.71}$$

Along with (3.37) and (3.43), this implies that

$$\bar\partial_J\widehat{u}_{\upsilon;w}\big|_{\Sigma^c_{\upsilon_w^>;0}(8\delta_{\bar\partial})}\in\widetilde{q}^*_{\upsilon;w}\Big(\big\{q_{\upsilon_w^{+,>}}\times\widetilde{u}^+_{\upsilon;w}\big\}^*\widetilde{\Gamma}_-(\mathbf{u}')\Big)=\Gamma^{0,1}_{B,-}\big(\widehat{u}_{\upsilon;w};0\big).$$

By the definition of the operator $\pi^{0,1;+}_{B;\upsilon;w;0}$, this means that

$$\pi^{0,1;+}_{B;\upsilon;w;0}\Big(\bar\partial_J\widehat{u}_{\upsilon;w}\big|_{\Sigma^c_{\upsilon_w^>;0}(8\delta_{\bar\partial})}\Big)=0. \tag{3.72}$$

By the boundedness of the operator $f^+_{B;\upsilon;w;\widehat{\zeta}_{\upsilon;w}}$ in (3.47), (3.45), and (3.72), there exists a constant $C\in\mathbb{R}$ such that

$$\Big\|\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\widehat{\zeta}_{\upsilon;w}}\Big(\bar\partial_J\widehat{u}_{\upsilon;w}\big|_{\Sigma^c_{\upsilon_w^>;0}(8\delta_{\bar\partial})}\Big)\Big\|_{\upsilon_w^>,p}\leqslant C\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon_w^>,p,1}\Big\|\bar\partial_J\widehat{u}_{\upsilon;w}\big|_{\Sigma^c_{\upsilon_w^>;0}(8\delta_{\bar\partial})}\Big\|_{\upsilon_{w'}^>,p}. \tag{3.73}$$

Combining (3.73), (3.71), and (3.35), we obtain

$$\Big\|\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\widehat{\zeta}_{\upsilon;w}}\Big(\bar{\partial}_J\widehat{u}_{\upsilon;w}\big|_{\Sigma^c_{\upsilon^>_w;0}(8\delta_{\bar\partial})}\Big)\Big\|_{\upsilon^>_w,p} \leqslant C\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon^>_w,p,1}\sum_{w'\in V^+_w}\|\bar{\partial}_J\widetilde{u}_{\upsilon;w'}\|_{\upsilon^>_{w'},p}. \tag{3.74}$$

Since $\bar{\partial}_J u=0$, there exists a continuous function $\epsilon_w\colon(\Delta,\mathbf{u})\longrightarrow(\mathbb{R},0)$ such that

$$\|\bar{\partial}_J\widetilde{u}_{\upsilon;w'}\|_{\upsilon^>_{w'},p} \leqslant \epsilon_w(\upsilon) \qquad \forall\ w'\in V^+_w. \tag{3.75}$$

By (3.70), (3.74), (3.75), and (3.48),

$$\begin{aligned}\|\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\widehat{\zeta}_{\upsilon;w}}\bar{\partial}_J\widehat{u}_{\upsilon^>_w}\|_{\upsilon^>_w,p} &\leqslant \sum_{e\in E^+_w}\Big\|\widetilde{\pi}^{0,1;+}_{B;\upsilon;w;\widehat{\zeta}_{\upsilon;w}}\Big(\bar{\partial}_J\widehat{u}_{\upsilon;w}\big|_{\widetilde{A}^-_{e,\upsilon^>_w}(\delta_{\bar\partial})}\Big)\Big\|_{\upsilon^>_w,p}+C\,\epsilon_w(\upsilon)\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon^>_w,p,1}\\ &\leqslant C'\sum_{e\in E^+_w}\Big\|\bar{\partial}_J\widehat{u}_{\upsilon;w}\big|_{\widetilde{A}^-_{e,\upsilon^>_w}(\delta_{\bar\partial})}\Big\|_{\upsilon^>_w,p}+C\,\epsilon_w(\upsilon)\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon^>_w,p,1}.\end{aligned} \tag{3.76}$$

After shrinking $\widetilde{U}$ and Δ if necessary, from (3.69) and (3.76) we obtain

$$\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon^>_w,p,1} \leqslant C\sum_{e\in E^+_w}\Big\|\bar{\partial}_J\widehat{u}_{\upsilon;w}\big|_{\widetilde{A}^-_{e,\upsilon^>_w}(\delta_{\bar\partial})}\Big\|_{\upsilon^>_w,p} \tag{3.77}$$

for some $C\in\mathbb{R}$.

By (3.38), (3.37), (3.35), and Lemma 2.2, there exists a constant $C\in\mathbb{R}$ such that

$$\Big\|\bar{\partial}_J\widehat{u}_{\upsilon;w}\big|_{\widetilde{A}^-_{e,\upsilon^>_w}(\delta_{\bar\partial})}\Big\|_{\upsilon^>_w,p} \leqslant C\big\|\mathrm{d}\widetilde{u}_{\upsilon;w^+_e}\big|_{\Sigma^+_{\upsilon^>_{w^+_e};e}(2|\upsilon_e|/\delta_{\bar\partial})}\big\|_{\upsilon^>_{w^+_e},p}|\upsilon_e|^{\frac{p-2}{p}} \qquad \forall\, e\in E^+_w. \tag{3.78}$$

By the first equation in (3.32) with w replaced by w^+_e, (2.44), and Lemma 3.8 for the pair (w^+_e,e), there exists a constant $C\in\mathbb{R}$ such that

$$\big\|\mathrm{d}\widetilde{u}_{\upsilon;w^+_e}\big|_{\Sigma^+_{\upsilon^>_{w^+_e};e}(2|\upsilon_e|/\delta_{\bar\partial})}\big\|_{\upsilon^>_{w^+_e},p} \leqslant C\big(|\mathcal{D}_{\upsilon;w^+_e}||\upsilon_e|^{\frac{2}{p}}+\varepsilon_{\upsilon;w^+_e}|\upsilon_e|^{\frac{p+2}{p}}\big) \qquad \forall\, e\in E^+_w\cap\mathrm{Edg}_0. \tag{3.79}$$

By (2.33), Corollary A.9, the second equation in (3.32) with w replaced by w^+_e, and (3.34), there exists a constant $C\in\mathbb{R}$ such that

$$\big\|\mathrm{d}\widetilde{u}_{\upsilon;w^+_e}\big|_{\Sigma^+_{\upsilon^>_{w^+_e};e}(2|\upsilon_e|/\delta_{\bar\partial})}\big\|_{\upsilon^>_{w^+_e},p} \leqslant C\big(|\mathcal{D}_{\upsilon;w^+_e}||\upsilon_e|^{\frac{2}{p}}+\varepsilon_{\upsilon;w^+_e}|\upsilon_e|^{\frac{p+2}{p}}\big) \qquad \forall\, e\in E^+_w\cap\chi(\mathcal{T}_0). \tag{3.80}$$

Lemma 3.6 follows from (3.77)-(3.80). □

Lemma 3.7. *There exists a constant $C\in\mathbb{R}$ such that*

$$\|\xi_{\upsilon;w}|_{\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial})}\|_{\upsilon^>_w,p,1}\leqslant C\sum_{w\preceq w'\prec w^+_e}\varepsilon_{\upsilon;w'}, \qquad \|\xi_{\upsilon;w}|_{\Sigma^{\mathrm{mn}}_{\upsilon^>_w;w_1}}\|_{\upsilon^>_w,p,1}\leqslant C\sum_{w\preceq w'\preceq w_1}\varepsilon_{\upsilon;w'}$$

for all $\upsilon\in\Delta$, $w\in\widehat{\mathrm{Ver}}_0$, $e\in\chi(\mathcal{T}_0)$, and $w_1\in\mathrm{Ver}_0$.

Proof. By (3.31) and (3.28), it suffices to prove Lemma 3.8 for $e\in\chi_w$ and $w_1\succeq w$. Let

$$\widehat{\xi}^+_{\upsilon;w}=\xi^+_{\upsilon;w}\circ\widetilde{q}_{\upsilon;w}\in\Gamma_B\big(\upsilon^>_w\big). \tag{3.81}$$

By (3.81), (3.40), and (3.51),

$$\widehat{u}_{\upsilon;w}=\exp_{u_{\upsilon^>_w}}\widehat{\xi}^+_{\upsilon;w}, \qquad \exp_{u_{\upsilon^>_w}}\xi_{\upsilon;w}=\exp_{\exp_{u_{\upsilon^>_w}}\widehat{\xi}^+_{\upsilon;w}}\widehat{\zeta}_{\upsilon;w}. \tag{3.82}$$

By (3.35), (3.29), and (3.34),

$$\begin{aligned} \xi^+_{\upsilon;w}(x)&=0 \qquad \forall\ x\in\widetilde{q}_{\upsilon;w}\big(\Sigma^{\mathrm{mn}}_{\upsilon^>_w;w}\big)\subset\Sigma_{\upsilon^{+,>}_w;w},\\ \xi^+_{\upsilon;w}(x)&=0 \qquad \forall\ x\in\widetilde{q}_{\upsilon;w}\big(\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial})\big),\ \ e\in E^+_w\cap\chi(\mathcal{T}_0). \end{aligned} \tag{3.83}$$

By (2.41) and (q3) in Section 2.2 with $(E_1,E_2)=(E^>_w-E^+_w,E^>_w)$, the restriction of $\widetilde{q}_{\upsilon;w}$ to

$$\Big(\Sigma_{\upsilon^>_w;0}(8\delta_{\bar\partial})-\bigsqcup_{e\in\mathrm{Edg}_0}\Sigma_{\upsilon^>_w;e}(\delta_{\bar\partial})\Big)-\Big(\widetilde{q}_{\upsilon;w}\big(\Sigma^{\mathrm{mn}}_{\upsilon^>_w;w}\big)\cup\bigsqcup_{e\in E^+_w\cap\chi(\mathcal{T}_0)}\widetilde{q}_{\upsilon;w}\big(\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial})\big)\Big)\subset\Sigma_{\upsilon^>_w}-\bigcup_{e\in E^+_w}\Sigma_{\upsilon^>_w;e}(2\delta_{\bar\partial}) \tag{3.84}$$

is an isometry onto its image.

By the second equality in (3.82), (3.81), (3.36), and Lemma A.3, there exists a constant $C\in\mathbb{R}$ such that

$$\begin{aligned} &\big|\xi_{\upsilon;w}(x)-\xi^+_{\upsilon;w}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|\leqslant C\,\big|\widehat{\zeta}_{\upsilon;w}(x)\big|,\\ &\big|\nabla^{\upsilon^>_w}\xi_{\upsilon;w}-\nabla^{\upsilon^>_w}\big(\xi^+_{\upsilon;w}\circ\widetilde{q}_{\upsilon;w}\big)\big|_x\\ &\qquad\leqslant C\Big(\big|\nabla^{\upsilon;w}\widehat{\zeta}_{\upsilon;w}\big|_x+\Big(|\mathrm{d}_xu_{\upsilon^>_w}|+\big|\nabla^{\upsilon^>_w}\big(\xi^+_{\upsilon;w}\circ\widetilde{q}_{\upsilon;w}\big)\big|_x\Big)\Big(|\xi^+_{\upsilon;w}|_{\widetilde{q}_{\upsilon;w}(x)}+|\widehat{\zeta}_{\upsilon;w}|_x\Big)\Big) \end{aligned} \tag{3.85}$$

By (3.85) and (3.36),

$$\begin{aligned} |\xi_{\upsilon;w}|_x&\leqslant C\Big(\big|\xi^+_{\upsilon;w}\big|_{\widetilde{q}_{\upsilon;w}(x)}+\big|\widehat{\zeta}_{\upsilon;w}\big|_x\Big),\\ |\nabla^{\upsilon^>_w}\xi_{\upsilon;w}|_x&\leqslant C\Big(\big|\nabla^{\upsilon^>_w}\big(\xi^+_{\upsilon;w}\circ\widetilde{q}_{\upsilon;w}\big)\big|_x+\big|\nabla^{\upsilon;w}\widehat{\zeta}_{\upsilon;w}\big|_x+|\mathrm{d}_xu_{\upsilon^>_w}|\big(|\xi^+_{\upsilon;w}|_{\widetilde{q}_{\upsilon;w}(x)}+|\widehat{\zeta}_{\upsilon;w}|_x\big)\Big) \end{aligned} \tag{3.86}$$

for all $x\in\Sigma_{\upsilon^>_w}$. By (3.86), (3.83), the sentence containing (3.84), (2.43), and (2.22),

$$\begin{gathered} \big\|\xi_{\upsilon;w}|_{\Sigma'}\big\|_{\upsilon^>_w,p,1}\leqslant C\Big(\big\|\xi^+_{\upsilon;w}|_{\widetilde{q}_{\upsilon;w}(\Sigma')}\big\|_{\upsilon^{+,>}_w,p,1}+\big\|\widehat{\zeta}_{\upsilon;w}\big\|_{\upsilon^>_w,p,1}\Big)\\ \forall\ \text{open}\ \ \Sigma'\subset\Sigma_{\upsilon^>_w}-\bigsqcup_{e\in\mathrm{Edg}_0}\Sigma_{\upsilon^>_w;e}(\delta_{\bar\partial}). \end{gathered} \tag{3.87}$$

By (3.87) with $\Sigma'=\Sigma^{\mathrm{mn}}_{\upsilon^>_w;w}$, the first line of (3.83), and the first equation in (3.51),

$$\big\|\xi_{\upsilon;w}|_{\Sigma^{\mathrm{mn}}_{\upsilon^>_w;w}}\big\|_{\upsilon^>_w,p,1}\leqslant C\big\|\widehat{\zeta}_{\upsilon;w}\big\|_{\upsilon^>_w,p,1}=C\,\varepsilon_{\upsilon;w}. \tag{3.88}$$

This establishes the second bound in Lemma 3.7 for $w_1=w$.

For $w_1\in\mathrm{Ver}_0$ with $w_1>w$, let $w'\in V_w^+$ be such that $w'\le w_1$. By (3.87) with $\Sigma'=\Sigma^{\mathrm{mn}}_{\upsilon^>_w;w_1}$ and the fact that $\widetilde{q}_{\upsilon;w}(\Sigma^{\mathrm{mn}}_{\upsilon^>_w;w_1})=\Sigma^{\mathrm{mn}}_{\upsilon^>_{w'};w_1}$,

$$\big\|\xi_{\upsilon;w}|_{\Sigma^{\mathrm{mn}}_{\upsilon^>_w;w_1}}\big\|_{\upsilon^>_w,p,1}\leqslant C\Big(\big\|\xi_{\upsilon;w'}|_{\Sigma^{\mathrm{mn}}_{\upsilon^>_{w'};w_1}}\big\|_{\upsilon^>_{w'},p,1}+\big\|\widehat{\zeta}_{\upsilon;w}\big\|_{\upsilon^>_w,p,1}\Big). \tag{3.89}$$

The second bound in Lemma 3.7 follows from (3.89), the second bound in Lemma 3.7 with w replaced by w', and the first equation in (3.51).

For $e\in\chi_w$, let $w'\in V_w^+$ be such that $w'\le w_e^+$. From (3.87) with $\Sigma'=\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial})$ and

$$\widetilde{q}_{\upsilon;w}\big(\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial})\big)=\Sigma_{\upsilon^>_{w'};e}(8\delta_{\bar\partial}),$$

it follows that

$$\big\|\xi_{\upsilon;w}|_{\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial})}\big\|_{\upsilon^>_w,p,1}\leqslant C\Big(\big\|\xi_{\upsilon;w'}|_{\Sigma_{\upsilon^>_{w'};e}(8\delta_{\bar\partial})}\big\|_{\upsilon^>_{w'},p,1}+\big\|\widehat{\zeta}_{\upsilon;w}\big\|_{\upsilon^>_w,p,1}\Big). \tag{3.90}$$

The first bound in Lemma 3.7 follows from (3.90), the first bound in Lemma 3.7 with w replaced by w', and the first equation in (3.51). □

Lemma 3.8. *There exists a constant $C\in\mathbb{R}$ such that*

$$\begin{aligned}
|\xi_{\upsilon;w}|_x &\leqslant C\Big(|\mathcal{D}_{\upsilon;w_e^+}||z^+_{e;\upsilon^>_w}(x)|+\varepsilon_{\upsilon;w_e^+}|z^+_{e;\upsilon^>_w}(x)|^2+\sum_{w\le w'<w_e^+}\varepsilon_{\upsilon;w'}\Big),\\
\big|\mathrm{d}_x\xi_{\upsilon;w}\big| &\leqslant C\Big(\big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}|z^+_{e;\upsilon^>_w}(x)|\big)|\mathrm{d}_x\widetilde{q}_{\upsilon^>_w;e}|+\sum_{w\le w'<w_e^+}\big|\nabla^{\upsilon;w'}\widehat{\zeta}_{\upsilon;w'}\big|_x\Big),
\end{aligned}$$

and

$$\begin{aligned}
\Big|T_{\nabla^J}\big(\xi_{\upsilon;w}(x),\mathrm{d}_x\xi_{\upsilon;w}\big)\Big| \leqslant C\Bigg(&\Big(\big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}|z^+_{e;\upsilon^>_w}(x)|\big)|\mathrm{d}_x\widetilde{q}_{\upsilon^>_w;e}|+\sum_{w\le w'<w_e^+}\big|\nabla^{\upsilon;w'}\widehat{\zeta}_{\upsilon;w'}\big|_x\Big)\sum_{w\le w'<w_e^+}\varepsilon_{\upsilon;w'}\\
&+|\mathcal{D}_{\upsilon;w_e^+}|\Big(|\mathcal{D}_{\upsilon;w_e^+}|^2+\varepsilon_{\upsilon;w_e^+}|\mathrm{d}_x\widetilde{q}_{\upsilon^>_w;e}|\Big)|z^+_{e;\upsilon^>_w}(x)|^2+\varepsilon^2_{\upsilon;w_e^+}|z^+_{e;\upsilon^>_w}(x)|^3\\
&+\Big(|\mathcal{D}_{\upsilon;w_e^+}||z^+_{e;\upsilon^>_w}(x)|+\varepsilon_{\upsilon;w_e^+}|z^+_{e;\upsilon^>_w}(x)|^2\Big)\sum_{w\le w'<w_e^+}\big|\nabla^{\upsilon;w'}\widehat{\zeta}_{\upsilon;w'}\big|_x\Bigg)
\end{aligned}$$

for all $\upsilon\in\Delta$, $w\in\mathrm{Ver}_0$, $e\in\mathrm{Edg}_0\sqcup\{e_{w_0}\}$, and $x\in\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial})$.

Proof. By (3.31) and (3.28), it suffices to prove Lemma 3.8 for every e with $w_e^+\ge w$. Let $J_{0;\upsilon}$ be the almost complex structure on a neighborhood V of $0\in T_{\mathrm{ev}_0(\upsilon)}X$ given by

$$J_{0;\upsilon}(y)(Y)=\big\{\mathrm{d}_y\exp_{\mathrm{ev}_0(\upsilon)}\big\}^{-1}\Big(J\big(\exp_{\mathrm{ev}_0(\upsilon)}y\big)\Big(\big\{\mathrm{d}_y\exp_{\mathrm{ev}_0(\upsilon)}\big\}(Y)\Big)\Big)\qquad\forall\,y\in V,\ Y\in T_{\mathrm{ev}_0(\upsilon)}X.$$

By the second equation in (3.51) and (3.39),

$$\xi_{\upsilon;w}(x)=\widehat{\zeta}_{\upsilon;w}(x)\in T_{\mathrm{ev}_0(\upsilon)}X\qquad\forall\ x\in\Sigma^+_{\upsilon^>_w;e_w}(32\delta_{\bar\partial})\subset q^{-1}_{\upsilon^>_w}\Big(\Sigma_{\mathbf{u}';w}-\bigsqcup_{e\in E_w^+}\Sigma_{\mathbf{u}';e}(\delta_{\bar\partial})\Big). \tag{3.91}$$

By (3.91), (3.31), and the sentence containing (2.28), the map

$$\widehat{\zeta}_{\upsilon;w}\colon\ \Sigma^+_{\upsilon^>_w;e_w}(32\delta_{\bar\partial})\longrightarrow T_{\mathrm{ev}_0(\upsilon)}X$$

is thus $J_{0;\upsilon}$-holomorphic and smooth. Shrinking $\widetilde{U}$ and Δ if necessary, we conclude from Lemma 3.6 that $\widehat{\zeta}_{\upsilon;w}$ is small. By Lemma A.8 and $\xi_{\upsilon;w}|_{x^+_{e_w}(\upsilon^>_w)}=0$, there thus exists $C\in\mathbb{R}$ such that

$$\begin{split}\Big|\mathrm{d}_x\widehat{\zeta}_{\upsilon;w}-\mathrm{d}_{x^+_{e_w}(\upsilon^>_w)}\widehat{\zeta}_{\upsilon;w}\Big|&\leqslant C\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon^>_w,p,1}\big|z^+_{e_w;\upsilon^>_w}(x)\big|,\\ \bigg|\widehat{\zeta}_{\upsilon;w}(x)-\frac{\partial\widehat{\zeta}_{\upsilon;w}}{\partial z^+_{e_w;\upsilon^>_w}}\Big|_{x^+_{e_w}(\upsilon^>_w)}z^+_{e_w;\upsilon^>_w}(x)\bigg|&=\big|\widehat{\zeta}_{\upsilon;w}(x)-\mathcal{D}_{\upsilon;w}z^+_{e_w;\upsilon^>_w}(x)\big|\leqslant C\|\widehat{\zeta}_{\upsilon;w}\|_{\upsilon^>_w,p,1}\big|z^+_{e_w;\upsilon^>_w}(x)\big|^2\end{split}\tag{3.92}$$

for all $x\in\Sigma^+_{\upsilon^>_w;e_w}(8\delta_{\bar\partial})$. Since $|\mathrm{d}\widetilde{q}_{\upsilon^>_w;e_w}|=1$ on $\Sigma^+_{\upsilon^>_w;e_w}(8\delta_{\bar\partial})$, the three bounds in Lemma 3.8 for $e=e_w$ and $x\in\Sigma^+_{\upsilon^>_w;e_w}(8\delta_{\bar\partial})$ follow from (3.91), (3.92), (3.32), and the first equation in (3.51).

Suppose $e\in\mathrm{Edg}_0$ with $w^+_e>w$. Let $w'\in V^+_w$ be such that $w'\preceq w^+_e$. By (3.85), the first equation in (3.35), and (2.44),

$$\begin{split}&\big|\xi_{\upsilon;w}(x)-\xi_{\upsilon;w'}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|\leqslant C\,\big|\widehat{\zeta}_{\upsilon;w}(x)\big|,\\ &\big|\mathrm{d}_x\xi_{\upsilon;w}-\mathrm{d}_x(\xi_{\upsilon;w'}\circ\widetilde{q}_{\upsilon;w})\big|\leqslant C\Big(\big|\nabla^{\upsilon;w}\widehat{\zeta}_{\upsilon;w}\big|_x+\big|\mathrm{d}_x(\xi_{\upsilon;w'}\circ\widetilde{q}_{\upsilon;w})\big|\Big(|\xi_{\upsilon;w'}|_{\widetilde{q}_{\upsilon;w}(x)}+|\widehat{\zeta}_{\upsilon;w}|_x\Big)\Big)\end{split}$$

for all $x\in\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial})$. By the first bounds in (3.86) and in Lemma 3.8 for the pair (w',e), (2.22), and the first equation in (3.51),

$$\begin{split}|\xi_{\upsilon;w}|_x&\leqslant C'\Big(|\xi_{\upsilon;w'}|_{\widetilde{q}_{\upsilon;w}(x)}+|\widehat{\zeta}_{\upsilon;w}|_x\Big)\\ &\leqslant C\Big(|\mathcal{D}_{\upsilon;w^+_e}|\big|z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|+\varepsilon_{\upsilon;w^+_e}\big|z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|^2+\sum_{w\preceq w''\prec w^+_e}\varepsilon_{\upsilon;w''}\Big).\end{split}\tag{3.93}$$

By the second bounds in (3.86) and in Lemma 3.8 for the pair (w',e),

$$\begin{split}\big|\mathrm{d}_x\xi_{\upsilon;w}\big|&\leqslant C'\Big(\big|\mathrm{d}_x(\xi_{\upsilon;w'}\circ\widetilde{q}_{\upsilon;w})\big|+\big|\nabla^{\upsilon;w}\widehat{\zeta}_{\upsilon;w}\big|_x\Big)\\ &\leqslant C\bigg(\Big(|\mathcal{D}_{\upsilon;w^+_e}|+\varepsilon_{\upsilon;w^+_e}\big|z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|\Big)\big|\mathrm{d}_{\widetilde{q}_{\upsilon;w}(x)}\widetilde{q}_{\upsilon^>_{w'};e}\big||\mathrm{d}_x\widetilde{q}_{\upsilon;w}|+\sum_{w\preceq w''\prec w^+_e}\big|\nabla^{\upsilon;w''}\widehat{\zeta}_{\upsilon;w''}\big|_x\bigg).\end{split}\tag{3.94}$$

For every $x\in\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial})$, let

$$\begin{split}B_{\upsilon;w}(x)\equiv&\bigg(\Big(|\mathcal{D}_{\upsilon;w^+_e}|+\varepsilon_{\upsilon;w^+_e}\big|z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|\Big)\big|\mathrm{d}_{\widetilde{q}_{\upsilon;w}(x)}\widetilde{q}_{\upsilon^>_{w'};e}\big||\mathrm{d}_x\widetilde{q}_{\upsilon;w}|+\sum_{w\preceq w''\prec w^+_e}\big|\nabla^{\upsilon;w''}\widehat{\zeta}_{\upsilon;w''}\big|_x\bigg)\sum_{w\preceq w''\prec w^+_e}\varepsilon_{\upsilon;w''}\\ &+|\mathcal{D}_{\upsilon;w^+_e}|\Big(|\mathcal{D}_{\upsilon;w^+_e}|^2+\varepsilon_{\upsilon;w^+_e}\big|\mathrm{d}_{\widetilde{q}_{\upsilon;w}(x)}\widetilde{q}_{\upsilon^>_{w'};e}\big||\mathrm{d}_x\widetilde{q}_{\upsilon;w}|\Big)\big|z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|^2+\varepsilon^2_{\upsilon;w^+_e}\big|z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|^3\\ &+\Big(|\mathcal{D}_{\upsilon;w^+_e}|\big|z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|+\varepsilon_{\upsilon;w^+_e}\big|z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big|^2\Big)\sum_{w\preceq w''\prec w^+_e}\big|\nabla^{\upsilon;w''}\widehat{\zeta}_{\upsilon;w''}\big|_x\in\mathbb{R}^{\geqslant0}.\end{split}$$

By the last four inequalities and the last bound in Lemma 3.8 for the pair (w',e),

$$\big|T_{\nabla^J}(\xi_{\upsilon;w}(x),\mathrm{d}_x\xi_{\upsilon;w})\big|\leqslant B_{\upsilon;w}(x)\qquad\forall\,x\in\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial}).\tag{3.95}$$

For every $x\in\Sigma_{\upsilon_w^>;e}(8\delta_{\bar\partial})$,

$$\big|z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big)\big| \leqslant |z^+_{e;\upsilon^>_w}(x)| \quad \text{if} \quad w'=w_e^+, \qquad z^+_{e;\upsilon^>_{w'}}\big(\widetilde{q}_{\upsilon;w}(x)\big) = z^+_{e;\upsilon^>_w}(x) \quad \text{if} \quad w'\prec w_e^+. \tag{3.96}$$

The inequality above follows from (2.36) and (2.34), while the equality above follows from (2.41), (2.25), and (q2) in Section 2.2 with $(E_1,E_2)=(E_w^>-E_w^+,E_w^>)$. Since

$$\begin{aligned}
&\big|\mathrm{d}_{\widetilde{q}_{\upsilon;w}(x)}\widetilde{q}_{\upsilon^>_{w'};e}\big|=1, && |\mathrm{d}_x\widetilde{q}_{\upsilon;w}|=|\mathrm{d}_x\widetilde{q}_{\upsilon^>_w;e}| && \forall\, x\in\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial}),\ e\in E_w^+,\\
&\big|\mathrm{d}_{\widetilde{q}_{\upsilon;w}(x)}\widetilde{q}_{\upsilon^>_{w'};e}\big|=|\mathrm{d}_x\widetilde{q}_{\upsilon^>_w;e}|, && |\mathrm{d}_x\widetilde{q}_{\upsilon;w}|=1 && \forall\, x\in\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial}),\ e\in \mathrm{Edg}_0-E_w^+,
\end{aligned}$$

we conclude that

$$\big|\mathrm{d}_{\widetilde{q}_{\upsilon;w}(x)}\widetilde{q}_{\upsilon^>_{w'};e}\big||\mathrm{d}_x\widetilde{q}_{\upsilon;w}| = |\mathrm{d}_x\widetilde{q}_{\upsilon^>_w;e}| \qquad \forall\, x\in\Sigma_{\upsilon^>_w;e}(8\delta_{\bar\partial}). \tag{3.97}$$

Combining (3.93)-(3.95) with (3.96) and (3.97), we obtain all bounds in Lemma 3.8. □

4 Extensions of meromorphic differentials

We continue with the notation and setup of Sections 2 and 3. In Section 4.1, we describe extensions of meromorphic differentials under smoothings of the nodes of Riemann surfaces. Sections 4.2 and 4.3 present properties of these extensions that are used in Sections 5 and 6. The main statements are proved in Section 4.4.

4.1 Main statements

Let $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ be a graph framing and $\mathbf{u}\in\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be a J-holomorphic map. By the definition of $\mathcal{T}_{0;P}$, $g(\Sigma_{\mathbf{u}';w})=0$ for all $\mathbf{u}'\in\widetilde{U}$ and $w\in\mathrm{Ver}^c_{0;P}$. By (q5) in Section 2.2, the functions

$$\breve{z}_{w;\mathbf{u}'} \equiv \frac{1}{z^+_{e_w;\mathbf{u}'}}\colon\ \Sigma_{\mathbf{u}';w}-\big\{x^+_{e_w}(\mathbf{u}')\big\}\longrightarrow\mathbb{C}, \qquad w\in\mathrm{Ver}^c_{0;P}, \tag{4.1}$$

are thus holomorphic coordinates. Let

$$\breve{x}_{e;\mathbf{u}'} \equiv \breve{z}_{w;\mathbf{u}'}\big(x_e^-(\mathbf{u}')\big)\in\mathbb{C}, \qquad \forall\ w\in\mathrm{Ver}^c_{0;P},\ e\in E_w^+. \tag{4.2}$$

By (4.1), (q4) in Section 2.2, and the sentence after (2.27),

$$|\breve{z}_{w;\mathbf{u}'}(x)|,\, |\breve{x}_{e;\mathbf{u}'}|\leqslant \delta_{\bar\partial}^{-1} \qquad \forall\ w\in\mathrm{Ver}^c_{0;P},\ e\in E_w^+,\ x\in\Sigma_{\mathbf{u}';w}-\Sigma_{\mathbf{u}';e_w}(\delta_{\bar\partial}). \tag{4.3}$$

Analogous to (3.4), we can assume

$$z^-_{e;\mathbf{u}'}(x) = \breve{z}_{w;\mathbf{u}'}(x)-\breve{x}_{e;\mathbf{u}'} \qquad \forall\ w\in\mathrm{Ver}^c_{0;P},\ e\in E_w^+,\ x\in\Sigma^-_{\mathbf{u}';e}(8\delta_{\bar\partial}). \tag{4.4}$$

We denote by $\upsilon_{0;P}$, υ_0, and $\upsilon_{0\chi}$ the images of υ in

$$\Delta_{0;P}\equiv\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}\big(\mathrm{Edg}_{0;P}\big), \qquad \Delta_0\equiv\Delta\cap\widetilde{\mathcal{F}_0\mathscr{T}}, \qquad \text{and} \qquad \Delta_{0\chi}\equiv\Delta\cap\widetilde{\mathcal{F}\mathscr{T}}\big(\mathrm{Edg}_0\sqcup\chi(\mathcal{T}_0)\big),$$

respectively. In particular,

$$\rho_w(\upsilon)=\rho_w(\upsilon_0) \qquad \forall\, w\in \mathrm{Ver}_0. \tag{4.5}$$

For $\upsilon\in\Delta$, denote by

$$\mathcal{H}_S^{1,0}\big(\Sigma_{\upsilon;0}(8\delta_{\bar\partial})\big) \qquad \text{and} \qquad \mathcal{H}_S^{1,0}(\Sigma_{\upsilon;0})$$

the (infinite-dimensional) vector spaces of meromorphic differentials on $\Sigma_{\upsilon;0}(8\delta_{\bar\partial})$ and $\Sigma_{\upsilon;0}$, respectively, having at worst the following two types of poles:

- simple poles at the nodes of $\Sigma_{\upsilon;0}$ with the residues adding up to 0 at each node;
- poles at the marked points $x_s(\upsilon)$ with $s\in S$.

Let $\mathbb{E}_{0;S}\longrightarrow\Delta_0$ be the vector bundle with fibers

$$\mathbb{E}_{0;S}|_{\upsilon_0}=\mathcal{H}_S^{1,0}(\Sigma_{\upsilon_0;0}) \qquad \forall\,\upsilon_0\in\Delta_0.$$

If $S=\varnothing$, $\mathbb{E}_{0;S}$ is the vector bundle induced by the relative dualizing sheaf of the family

$$\mathfrak{U}_0\subset\mathfrak{U}|_{\Delta_0}\longrightarrow\Delta_0$$

of deformations of the contracted curves $\Sigma_{\mathbf{u}';0}$ of the elements of $\widetilde{U}$. Every $\mathbb{E}_{0;S}|_{\upsilon_0}$ determines a subspace of $\mathcal{H}_S^{1,0}\big(\Sigma_{\upsilon_0;0}(8\delta_{\bar\partial})\big)$, which is still denoted by $\mathbb{E}_{0;S}|_{\upsilon_0}$. Since $g(\Sigma_{\mathbf{u}';w})=0$ for all $w\in\mathrm{Ver}_{0;P}^c$ and $\mathfrak{m}(S)\subset\mathrm{Ver}_{0;P}$,

$$\operatorname{supp}\eta\subset q_{\upsilon_{0;P}}^{-1}\big(\Sigma_{\mathbf{u}';0;P}\big) \qquad \forall\,\eta\in\mathbb{E}_{0;S}|_{\upsilon_{0;P}}. \tag{4.6}$$

Let β_υ be as in (2.35). For every element η_υ of $\mathcal{H}_S^{1,0}\big(\Sigma_{\upsilon;0}(8\delta_{\bar\partial})\big)$, we view $\beta_\upsilon\eta_\upsilon$ as a smooth differential on $\Sigma_\upsilon-\{x_s(\upsilon)\}_{s\in S}$ that satisfies the same properties as the elements of $\mathcal{H}_S^{1,0}\big(\Sigma_{\upsilon;0}(8\delta_{\bar\partial})\big)$.

Proposition 4.1. *Let $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ be a graph framing and $\mathbf{u}\in\widetilde{\mathfrak{X}}_{\mathscr{T}}(X)$ be a J-holomorphic map. If Δ is a sufficiently small neighborhood of $\mathbf{u}$ in $\widetilde{\mathcal{F}\mathscr{T}}$, there exists a continuous family of injective homomorphisms*

$$R_\upsilon\colon\, \mathbb{E}_S|_{\pi_{\widetilde{\mathcal{F}\mathscr{T}}}(\upsilon)}\longrightarrow\mathcal{H}_S^{1,0}\big(\Sigma_{\upsilon;0}(8\delta_{\bar\partial})\big), \qquad \upsilon\in\Delta,$$

such that

(E1) for all $\mathbf{u}'\in\widetilde{U}$, $\upsilon\in\Delta|_{\mathbf{u}'}$, and $\psi\in\mathbb{E}_S|_{\mathbf{u}'}$,

$$R_{\mathbf{u}'}\psi=\psi.$$

Moreover, if $\mathbf{u}'\in\widetilde{U}$, $\upsilon\in\Delta|_{\mathbf{u}'}$, $\psi\in\mathbb{E}_S|_{\mathbf{u}'}$, $e_\bullet\in\chi(\mathcal{T}_{0;P})$, and

$$\big\{R_{\upsilon_{0;P}}\psi\big\}_x=\sum_{k=0}^{\infty}c_{e_\bullet;\upsilon}^{(k)}(\psi)\Big(z_{e_\bullet;\upsilon_{0;P}}^-(x)\Big)^k\mathrm{d}z_{e_\bullet;\upsilon_{0;P}}^- \qquad \forall\,x\in\Sigma_{\upsilon_{0;P};e_\bullet}^-(8\delta_{\bar\partial}) \tag{4.7}$$

for some $c_{e_\bullet;\upsilon}^{(k)}(\psi)\in\mathbb{C}$ with $k\in\mathbb{Z}^{\geqslant0}$, then

(E2) for all $e\in\widehat{\mathrm{Edg}}^{c}_{0;P}$ *with* $\langle e\rangle=e_\bullet$ *and* $x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial})$,

$$\big\{R_\upsilon\psi\big\}_x=\rho_{w_e^-}(\upsilon)\sum_{k=0}^{\infty}c^{(k)}_{e_\bullet;\upsilon}(\psi)\bigg(z^-_{e;\upsilon}(x)\rho_{w_e^-}(\upsilon)+\sum_{e_\bullet<e'\le e}\breve{x}_{e';\mathbf{u}'}\rho_{w^-_{e'}}(\upsilon)\bigg)^k\mathrm{d}z^-_{e;\upsilon}\,;$$

(E3) for all $w\in\mathrm{Ver}^c_{0;P}$ *with* $\langle e_w\rangle=e_\bullet$ *and* $x\in\Sigma^{\mathrm{mn}}_{\upsilon;w}$,

$$\big\{R_\upsilon\psi\big\}_x=\rho_w(\upsilon)\sum_{k=0}^{\infty}c^{(k)}_{e_\bullet;\upsilon}(\psi)\bigg(\breve{z}_{w;\mathbf{u}'}\big(q_\upsilon(x)\big)\rho_w(\upsilon)+\sum_{e_\bullet<e'\le e_w}\breve{x}_{e';\mathbf{u}'}\rho_{w^-_{e'}}(\upsilon)\bigg)^k\mathrm{d}\big(\breve{z}_{w;\mathbf{u}'}\circ q_\upsilon\big).$$

Corollary 4.2. *Let* β_υ *be as in (2.35) and* R_υ *be as in Proposition 4.1. There exists a constant* $C\in\mathbb{R}$ *such that for all* $\mathbf{u}'\in\widetilde{U}$, $\upsilon\in\Delta|_{\mathbf{u}'}$, *and* $\psi\in\mathbb{E}_S|_{\mathbf{u}'}$,

$$\operatorname{supp}\big(\beta_\upsilon R_\upsilon\psi\big)\subset\Sigma_{\upsilon;0}(4\delta_{\bar\partial}),\qquad \operatorname{supp}\big(\bar\partial(\beta_\upsilon R_\upsilon\psi)\big)\subset\bigcup_{e\in\chi(\mathcal{T}_0)}\big(\Sigma^+_{\upsilon;e}(4\delta_{\bar\partial})-\Sigma^+_{\upsilon;e}(2\delta_{\bar\partial})\big),$$
$$\big\|\big(\bar\partial(\beta_\upsilon R_\upsilon\psi)\big)|_{\Sigma^+_{\upsilon;e}(4\delta_{\bar\partial})-\Sigma^+_{\upsilon;e}(2\delta_{\bar\partial})}\big\|_{C^0}\leqslant C\big\|(R_\upsilon\psi)|_{\Sigma^+_{\upsilon;e}(4\delta_{\bar\partial})-\Sigma^+_{\upsilon;e}(2\delta_{\bar\partial})}\big\|_{C^0}\qquad\forall\,e\in\chi(\mathcal{T}_0).$$

Proof. By (2.35),

$$\operatorname{supp}(\beta_\upsilon)\subset\Sigma_{\upsilon;0}(4\delta_{\bar\partial}),\quad \operatorname{supp}(\bar\partial\beta_\upsilon)\subset\bigcup_{e\in\chi(\mathcal{T}_0)}\big(\Sigma^+_{\upsilon;e}(4\delta_{\bar\partial})-\Sigma^+_{\upsilon;e}(2\delta_{\bar\partial})\big),\quad \|\bar\partial\beta_\upsilon\|_{C^0}\leqslant C \tag{4.8}$$

for some $C\in\mathbb{R}$. The first statement in Corollary 4.2 follows from the first statement in (4.8). Since $R_\upsilon\psi\in\mathcal{H}^{1,0}_S\big(\Sigma_{\upsilon;0}(8\delta_{\bar\partial})\big)$, the last two statements in Corollary 4.2 then follow from the last two statements in (4.8), respectively. □

4.2 Further implications

For all $k\in\mathbb{Z}^{\geqslant0}$, $\mathbf{u}'\in\widetilde{U}$, $\upsilon\in\Delta|_{\mathbf{u}'}$, $\psi\in\mathbb{E}_S|_{\mathbf{u}'}$, and $e\in\chi(\mathcal{T}_0)$, define $c^{(k)}_{e;\upsilon}(\psi)\in\mathbb{C}$ by

$$\big\{R_{\upsilon_0}\psi\big\}_x=\sum_{k=0}^{\infty}c^{(k)}_{e;\upsilon}(\psi)\Big(z^-_{e;\upsilon_0}(x)\Big)^k\mathrm{d}z^-_{e;\upsilon_0}\qquad\forall\,x\in\Sigma^-_{\upsilon_0;e}(8\delta_{\bar\partial}). \tag{4.9}$$

By Cauchy's Integral Formula and the assumption that the nodes, there exists $C_k=C_k(\mathbf{u})$ such that

$$\big|c^{(k)}_{e;\upsilon}(\psi)\big|\leqslant C_k\big\|R_{\upsilon_0}\psi\big\|_{C^0(\Sigma^-_{\upsilon_0;e}(8\delta_{\bar\partial}))}\qquad\forall\,k\in\mathbb{Z}^{\geqslant0},\ e\in\chi(\mathcal{T}_0),\ \upsilon\in\Delta,\ \psi\in\mathbb{E}_0|_{\pi_{\widetilde{\mathcal{F}\mathcal{T}}}(\upsilon)}. \tag{4.10}$$

For $e\in\chi(\mathcal{T}_0)\cap\chi(\mathcal{T}_{0;P})$, (4.9) agrees with (4.7).

Corollary 4.3. *Suppose* $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$, $\mathbf{u}$, Δ, *and* R_υ *are as in Proposition 4.1 and* ψ *is as in (4.9). If* $\upsilon\in\Delta$ *and* $e^\bullet\in\chi(\mathcal{T}_0)$ *satisfy*

$$\upsilon_e\neq0\quad\forall\ e\in\widetilde{\mathrm{Edg}}^{c}_{0;P}\ \text{ with }\ e\succeq\langle e^\bullet\rangle, \tag{4.11}$$

then

(E′2) for all $e\in\widehat{\mathrm{Edg}}^c_{0;P}$ *with* $\langle e\rangle=\langle e^\bullet\rangle$ *and* $x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial})$,

$$\{R_\upsilon\psi\}_x=\rho_{w_e^-}(\upsilon)\sum_{k=0}^{\infty}\frac{c^{(k)}_{e^\bullet;\upsilon}(\psi)}{\rho^{k+1}_{w^-_{e^\bullet}}(\upsilon)}\bigg(z^-_{e;\upsilon}(x)\rho_{w_e^-}(\upsilon)+\sum_{\langle e^\bullet,e\rangle<e'\le e}\breve{x}_{e';\mathbf{u}'}\rho_{w^-_{e'}}(\upsilon)-\sum_{\langle e^\bullet,e\rangle<e'\le e^\bullet}\breve{x}_{e';\mathbf{u}'}\rho_{w^-_{e'}}(\upsilon)\bigg)^k\mathrm{d}z^-_{e;\upsilon}\,;$$

(E′3) for all $w\in\mathrm{Ver}^c_{0;P}$ *with* $\langle e_w\rangle=\langle e^\bullet\rangle$ *and* $x\in\Sigma^{\mathrm{mn}}_{\upsilon;w}$,

$$\begin{aligned}&\{R_\upsilon\psi\}_x\\&=\rho_w(\upsilon)\sum_{k=0}^{\infty}\frac{c^{(k)}_{e^\bullet;\upsilon}(\psi)}{\rho^{k+1}_{w^-_{e^\bullet}}(\upsilon)}\bigg(\breve{z}_{w;\mathbf{u}'}\big(q_\upsilon(x)\big)\rho_w(\upsilon)+\sum_{\langle e^\bullet,e_w\rangle<e'\le e_w}\breve{x}_{e';\mathbf{u}'}\rho_{w^-_{e'}}(\upsilon)-\sum_{\langle e^\bullet,e_w\rangle<e'\le e^\bullet}\breve{x}_{e';\mathbf{u}'}\rho_{w^-_{e'}}(\upsilon)\bigg)^k\mathrm{d}\big(\breve{z}_{w;\mathbf{u}'}\circ q_\upsilon\big).\end{aligned}$$

Proof. By (4.9) and (E2) in Proposition 4.1,

$$c^{(k)}_{e^\bullet;\upsilon}(\psi)=\rho^{k+1}_{w^-_{e^\bullet}}(\upsilon)\sum_{\ell=0}^{\infty}\binom{k+\ell}{k}c^{(k+\ell)}_{\langle e^\bullet\rangle;\upsilon}(\psi)\bigg(\sum_{\langle e^\bullet\rangle<e\le e^\bullet}\breve{x}_{e;\mathbf{u}'}\rho_{w_e^-}(\upsilon)\bigg)^\ell\qquad\forall\,k\in\mathbb{Z}^{\geqslant0}.\tag{4.12}$$

This implies that

$$\left.\begin{aligned}\frac{|c^{(k)}_{e^\bullet;\upsilon}(\psi)|}{|\rho^{k+1}_{w^-_{e^\bullet}}(\upsilon)|}&\leqslant C_k\big\|R_{\upsilon_{0;P}}\psi\big\|_{C^0(\Sigma^-_{\upsilon_{0;P};\langle e^\bullet\rangle}(8\delta_{\bar\partial}))},\\\bigg|\frac{c^{(k)}_{e^\bullet;\upsilon}(\psi)}{\rho^{k+1}_{w^-_{e^\bullet}}(\upsilon)}-c^{(k)}_{\langle e^\bullet\rangle;\upsilon}(\psi)\bigg|&\leqslant C_k|\upsilon|\big\|R_{\upsilon_{0;P}}\psi\big\|_{C^0(\Sigma^-_{\upsilon_{0;P};\langle e^\bullet\rangle}(8\delta_{\bar\partial}))}\end{aligned}\right\}\quad\forall\,k\in\mathbb{Z}^{\geqslant0}.\tag{4.13}$$

The sums in (E′2) and (E′3) thus converge. Substituting (4.12) into the expressions in (E′2) and (E′3), we obtain the expressions in (E2) and (E3) in Proposition 4.1, respectively. □

Fix $e^\bullet\in\chi(\mathcal{T}_0)$ and $w^\star\in\widehat{\mathrm{Ver}}^c_{0;P}$ with $w^+_{e^\bullet}\ge w^\star$. For $w\in\{P\}\cup\widehat{\mathrm{Ver}}^c_{0;P}$, $\mathbf{u}'\in\widetilde{U}$, $\upsilon\in\Delta|_{\mathbf{u}'}$ satisfying (4.11), and $\psi\in\mathbb{E}_S|_{\mathbf{u}'}$, define

$$\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w)=\frac{|c^{(1)}_{e^\bullet;\upsilon}(\psi)|}{|\rho^2_{w^-_{e^\bullet}}(\upsilon)|}+\|\psi\||\rho_{\langle w^\star,w\rangle}(\upsilon)|\ \in\mathbb{R}^{\geqslant0}.\tag{4.14}$$

In particular,

$$\begin{aligned}\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w')&\leqslant\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w)&&\text{if }\langle w^\star,w'\rangle\ge\langle w^\star,w\rangle;\\\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w')&\leqslant\frac{|\rho_{\langle w^\star,w'\rangle}(\upsilon)|}{|\rho_{\langle w^\star,w\rangle}(\upsilon)|}\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w)&&\text{if }\langle w^\star,w'\rangle\le\langle w^\star,w\rangle\end{aligned}\tag{4.15}$$

for all $w',w\in\widehat{\mathrm{Ver}}^c_{0;P}$. By the $k=1$ case of the first bound in (4.13), there exists $C\in\mathbb{R}$ such that

$$\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w)\leqslant C\|\psi\|\qquad\forall\,\upsilon\in\Delta,\ w\in\{P\}\cup\widehat{\mathrm{Ver}}^c_{0;P}.\tag{4.16}$$

Without loss of generality, we hereafter assume

$$\big|\breve{x}_{e;\mathbf{u}'}\big|\geqslant16\delta_{\bar\partial}\qquad\forall\,e\in\mathrm{Edg}^c_{0;P}\cup\big(\chi(\mathcal{T}_0)-\chi(\mathcal{T}_{0;P})\big),\ \mathbf{u}'\in\widetilde{U}\,.\tag{4.17}$$

Corollary 4.4. *Suppose* $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$, $\mathbf{u}'$, Δ, *and* R_υ *are as in Proposition 4.1,* $\psi\in\mathbb{E}_S|_{\mathbf{u}'}$ *is as in (4.9), and* $\upsilon\in\Delta$ *and* $e^\bullet,e'\in\chi(\mathcal{T}_0)$ *satisfy (4.11). Then, there exists* $C\in\mathbb{R}$ *such that*

(E″2) for all $e\in\widehat{\mathrm{Edg}}^c_{0;P}$ *with* $e\preceq e'$ *and* $x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial})$,

$$\left|\{R_\upsilon\psi\}_x\Big(\frac{\partial}{\partial z^-_{e;\upsilon}}\Big)-\frac{\rho_{w_e^-}(\upsilon)}{\rho_{w_{e'}^-}(\upsilon)}\,c^{(0)}_{e';\upsilon}(\psi)\right|\leqslant C\bigg(\big|\rho^2_{w_e^-}(\upsilon)\big|\Big(|\upsilon_e|+\big|z^-_{e;\upsilon}(x)\big|\Big)\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)$$
$$+\big|\rho^3_{w_e^-}(\upsilon)\big|\Big(|\upsilon_e|+\big|z^-_{e;\upsilon}(x)\big|\Big)^2\|\psi\|\bigg);$$

(E″3) for all $w\in\{P\}\cup\mathrm{Ver}^c_{0;P}$ *with* $w\preceq w^-_{e'}$ *and* $x\in\Sigma^{\mathrm{mn}}_{\upsilon;w}$,

$$\left|\{R_\upsilon\psi\}_x\Big(\frac{\partial}{\partial(\check{z}_{w;\mathbf{u}'}\circ q_\upsilon)}\Big)-\frac{\rho_w(\upsilon)}{\rho_{w_{e'}^-}(\upsilon)}\,c^{(0)}_{e';\upsilon}(\psi)\right|\leqslant C\big|\rho^2_w(\upsilon)\big|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w).$$

Proof. Suppose first that $\langle e'\rangle=\langle e^\bullet\rangle$. The bound in (E″2) with $\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)$ replaced by

$$\widetilde{C}^\bullet_{w^+_{e^\bullet};\upsilon}(\psi;w_e^+)\equiv\frac{|c^{(1)}_{e^\bullet;\upsilon}(\psi)|}{|\rho^2_{w^+_{e^\bullet}}(\upsilon)|}+\|\psi\||\rho_{\langle w^+_{e^\bullet},w\rangle}(\upsilon)| \tag{4.18}$$

then follows from the values of $R_\upsilon\psi$ and $R_{\upsilon_0}\psi$ in (E′2) in Corollary 4.3 at x and $x^-_{e'}(\upsilon_0)$, respectively, and (4.13). Since $w^+_{e^\bullet}\succeq w^\star$,

$$\widetilde{C}^\bullet_{w^+_{e^\bullet};\upsilon}(\psi;w_e^+)\preceq\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)$$

and the bound in (E″2) follows from (4.18). The bound in (E″3) follows from the value of $R_\upsilon\psi$ in (E′3) in Corollary 4.3 at x similarly.

If $\langle e'\rangle\neq\langle e^\bullet\rangle$, then

$$\rho_{\langle w^\star,w_e^+\rangle}(\upsilon),\ \rho_{\langle w^\star,w\rangle}(\upsilon)=\rho_P(\upsilon)=1\qquad\forall\ e\preceq e',\ w\preceq w^-_{e'}.$$

The claims (E″2) and (E″3) thus follow from the corresponding bounds for $(e^\bullet,w^\star)=(e',w^+_{e'})$ and (4.13) for $(k,e^\bullet)=(1,e')$. □

For each $w\in\{P\}\cup\widehat{\mathrm{Ver}}^c_{0;P}$, fix $e(w)\in\chi(\mathcal{T}_0)$ with $w^+_{e(w)}\succeq w$. For each $e\in\widehat{\mathrm{Edg}}^c_{0;P}$, let $\hat{e}=e(w_e^+)$.

Corollary 4.5. *Suppose* $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$, $\mathbf{u}'$, Δ, R_υ, ψ, $e^\bullet$, *and* $w^\star$ *are as in Corollary 4.4. Then, there exists* $C\in\mathbb{R}$ *such that*

(R1) for all $e\in\widehat{\mathrm{Edg}}^c_{0;P}$ *and* $x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial})$,

$$|R_\upsilon\psi|_x\leqslant C\bigg(\frac{|\rho_{w_e^-}(\upsilon)|}{|\rho_{w_{\hat e}^-}(\upsilon)|}\big|c^{(0)}_{\hat e;\upsilon}(\psi)\big|+|\rho^2_{w_e^-}(\upsilon)|\Big(|\upsilon_e|+|z^-_{e;\upsilon}(x)|\Big)\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)$$
$$+|\rho^3_{w_e^-}(\upsilon)|\Big(|\upsilon_e|+|z^-_{e;\upsilon}(x)|\Big)^2\|\psi\|\bigg)|\mathrm{d}_xz^-_{e;\upsilon}|;$$

(R2) for every $e\in\chi(\mathcal{T}_0)$,

$$\|R_\upsilon\psi\|_{C^0(\Sigma_{\upsilon;e}(8\delta_{\bar\partial}))} \leqslant \big|c^{(0)}_{e;\upsilon}(\psi)\big| + C|\rho^2_{w_e^-}(\upsilon)|\widetilde{C}^{\bullet}_{w^\star;\upsilon}(\psi;w_e^-);$$

(R3) for every $w\in\{P\}\cup\mathrm{Ver}^c_{0;P}$,

$$\|R_\upsilon\psi\|_{C^0(\Sigma^{\mathrm{mn}}_{\upsilon;w})} \leqslant \frac{|\rho_w(\upsilon)|}{|\rho_{w^-_{e(w)}}(\upsilon)|}\big|c^{(0)}_{e(w);\upsilon}(\psi)\big| + C|\rho^2_w(\upsilon)|\widetilde{C}^{\bullet}_{w^\star;\upsilon}(\psi;w).$$

Proof. The bound in (R1) follows from (E″2) in Corollary 4.4 with $e'=\widehat{e}$ and (4.15). The bound in (R2) is a consequence of (R1). The bound in (R3) follows from (E″3) in Corollary 4.4 with $e'=e(w)$ and (4.15). □

Corollary 4.6. *Suppose* $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$, $\mathbf{u}'$, Δ, R_υ, ψ *are as in Corollary 4.4 and* $\upsilon\in\Delta$ *and* $e^\bullet\in\chi(\mathcal{T}_0)$ *satisfy (4.11). Then, there exists* $C\in\mathbb{R}$ *such that*

$$\left|\frac{1}{\rho_{w_e^-}(\upsilon)}\,c^{(0)}_{e;\upsilon}(\psi) - \frac{1}{\rho_{w^-_{e(w)}}(\upsilon)}\,c^{(0)}_{e(w);\upsilon}(\psi)\right| \leqslant C\,|\rho_{\langle w,w_e^+\rangle}(\upsilon)|\,\widetilde{C}^{\bullet}_{w^\star;\upsilon}\big(\psi;\langle w,w_e^+\rangle\big)$$

for all $e\in\chi(\mathcal{T}_0)$ *and* $w\in\{P\}\cup\mathrm{Ver}^c_{0;P}$.

Proof. Choose $x\in\Sigma^{\mathrm{mn}}_{\upsilon;\langle w,w_e^+\rangle}$. By (E″3) in Corollary 4.4 with w replaced by $\langle w,w_e^+\rangle$ and e' replaced by $e,e(w)$,

$$\left|\frac{1}{\rho_{\langle w,w_e^+\rangle}(\upsilon)}\,\{R_\upsilon\psi\}_x\Big(\frac{\partial}{\partial(\mathring{z}_{\langle w,w_e^+\rangle;\mathbf{u}'}\circ q_\upsilon)}\Big) - \frac{1}{\rho_{w_e^-}(\upsilon)}\,c^{(0)}_{e;\upsilon}(\psi)\right| \leqslant C\big|\rho_{\langle w,w_e^+\rangle}(\upsilon)\big|\widetilde{C}^{\bullet}_{w^\star;\upsilon}\big(\psi;\langle w,w_e^+\rangle\big),$$

$$\left|\frac{1}{\rho_{\langle w,w_e^+\rangle}(\upsilon)}\,\{R_\upsilon\psi\}_x\Big(\frac{\partial}{\partial(\mathring{z}_{\langle w,w_e^+\rangle;\mathbf{u}'}\circ q_\upsilon)}\Big) - \frac{1}{\rho_{w^-_{e(w)}}(\upsilon)}\,c^{(0)}_{e(w);\upsilon}(\psi)\right| \leqslant C\big|\rho_{\langle w,w_e^+\rangle}(\upsilon)\big|\widetilde{C}^{\bullet}_{w^\star;\upsilon}\big(\psi;\langle w,w_e^+\rangle\big).$$

The claim follows from these inequalities. □

By Corollary 4.6 with (w,e) replaced by $(w_e^-,\widehat{e})$ and (4.15),

$$\frac{|\rho_{w_e^-}(\upsilon)|}{|\rho_{w^-_{\widehat{e}}}(\upsilon)|}\big|c^{(0)}_{\widehat{e};\upsilon}(\psi)\big| \leqslant \frac{|\rho_{w_e^-}(\upsilon)|}{|\rho_{w^-_{e(w_e^-)}}(\upsilon)|}\big|c^{(0)}_{e(w_e^-);\upsilon}(\psi)\big| + C|\rho^2_{w_e^-}(\upsilon)|\widetilde{C}^{\bullet}_{w^\star;\upsilon}(\psi;w_e^-) \qquad \forall\,e\in\widetilde{\mathrm{Edg}}^c_{0;P}. \tag{4.19}$$

4.3 Some applications

We continue with the notation of Subsection 4.2. This subsection provides estimates that are applied to the computation of the obstructions in Section 5.

Corollary 4.7. *Suppose* $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$, $\mathbf{u}$, *and* Δ *are as in Proposition 4.1 and* $e^\bullet,e^\circ\in\chi(\mathcal{T}_0)$ *satisfy* $e^\circ\neq e^\bullet$ *and* $\langle e^\circ\rangle=\langle e^\bullet\rangle$. *Let* $\psi=\psi_\upsilon\in\mathbb{E}_0|_{\mathbf{u}'}-\{0\}$ *for* $\upsilon\in\Delta|_{\mathbf{u}'}$, $\mathbf{u}'\in\widetilde{U}$, *be a continuous family such that*

$$c^{(0)}_{e^\bullet;\upsilon}(\psi)=0 \quad \forall\,\upsilon\in\Delta, \qquad c^{(1)}_{\langle e^\bullet\rangle;\mathbf{u}}(\psi)\neq 0. \tag{4.20}$$

Then, there exists $C\in\mathbb{R}$ such that

$$\|\psi\|,\, \widetilde{C}^{\bullet}_{w^\star;\upsilon}(\psi;w_e^+)\leqslant C|\upsilon|\frac{\big|\upsilon_{e^\circ}c^{(0)}_{e^\circ;\upsilon}(\psi)\big|}{|\rho^2_{e^\circ}(\upsilon)|}$$

for all $e\in\chi(\mathcal{T}_0)$ and $\upsilon\in\Delta$ satisfying (4.11).

Proof. By the inequality in (4.20) and the $k=1$ case of the second bound in (4.13), there exist $C,C'\in\mathbb{R}$ such that

$$\|\psi\|\leqslant C\big|c^{(1)}_{\langle e^\bullet\rangle;\upsilon}(\psi)\big|\leqslant C'\,\frac{|c^{(1)}_{e^\bullet;\upsilon}(\psi)|}{|\rho^2_{w^-_{e^\bullet}}(\upsilon)|}\qquad\forall\,\upsilon\in\Delta.$$

By (E$'$2) in Corollary 4.3 with $(\upsilon,x)=\big(\upsilon_0,x^-_{e^\circ}(\upsilon_0)\big)$, the equality in (4.20), and the first bound in (4.13),

$$|\rho_{e^\circ}(\upsilon)|\big|\rho_{\langle e^\bullet,e^\circ\rangle}(\upsilon)\big|\frac{|c^{(1)}_{e^\bullet;\upsilon}(\psi)|}{|\rho^2_{w^-_{e^\bullet}}(\upsilon)|}\leqslant C\Big(\big|\upsilon_{e^\circ}c^{(0)}_{e^\circ;\upsilon}(\psi)\big|+|\upsilon||\rho_{e^\circ}(\upsilon)|\big|\rho_{\langle e^\bullet,e^\circ\rangle}(\upsilon)\big|\|\psi\|\Big)\qquad\forall\,\upsilon\in\Delta$$

for some $C\in\mathbb{R}$. The claim for $\|\psi\|$ follows from the above two inequalities; combined with (4.16), this in turn implies the claim for $\widetilde{C}^{\bullet}_{w^\star;\upsilon}(\psi;w_e^+)$. $\square$

Corollary 4.8. *Suppose $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$, $\mathbf{u}$, and Δ are as in Proposition 4.1 and $e_1,e_2\in\chi(\mathcal{T}_0)$ and $w^\star\in\mathrm{Ver}^c_{0;P}$ satisfy*

$$\big|\{e^\bullet,e_1,e_2\}\big|=3,\qquad\langle e^\bullet\rangle=\langle e_1\rangle=\langle e_2\rangle,\qquad\langle e_1,e_2\rangle\succeq\langle e^\bullet,e_1\rangle=\langle e^\bullet,e_2\rangle=e_{w^\star}.\tag{4.21}$$

Let $\psi=\psi_\upsilon\in\mathbb{E}_0|_{\mathbf{u}'}-\{0\}$ for $\upsilon\in\Delta|_{\mathbf{u}'}$, $\mathbf{u}'\in\widetilde{U}$, be a continuous family such that

$$c^{(0)}_{e^\bullet;\upsilon}(\psi)=0\quad\forall\,\upsilon\in\Delta,\qquad\qquad c^{(2)}_{\langle e^\bullet\rangle;\mathbf{u}}(\psi)\neq0.\tag{4.22}$$

Then, there exists $C\in\mathbb{R}$ such that

$$\widetilde{C}^{\bullet}_{w^\star;\upsilon}(\psi;w_e^+)\leqslant C|\upsilon|\left(\frac{\big|\upsilon_e c^{(0)}_{e;\upsilon}(\psi)\big|}{|\rho^2_e(\upsilon)|}+\frac{\big|\upsilon_{e_1}c^{(0)}_{e_1;\upsilon}(\psi)\big|}{|\rho^2_{e_1}(\upsilon)|}+\frac{\big|\upsilon_{e_2}c^{(0)}_{e_2;\upsilon}(\psi)\big|}{|\rho^2_{e_2}(\upsilon)|}\right)$$

for all $e\in\chi(\mathcal{T}_0)$ with $\langle e\rangle=\langle e^\bullet\rangle$ and $\upsilon\in\Delta$ satisfying (4.11).

Proof. Given distinct $e,e'\in\chi(\mathcal{T}_0)$ with $\langle e\rangle=\langle e'\rangle$, define $\breve{x}_{e,e'}(\upsilon)$, $\breve{x}^{e,e'}_{\mathbf{u}'}\in\mathbb{C}-\{0\}$ by

$$\breve{x}_{e,e'}(\upsilon)=\sum_{\langle e,e'\rangle\prec e''\preceq e}\breve{x}_{e'';\mathbf{u}'}\rho_{w^-_{e''}}(\upsilon)\;-\sum_{\langle e,e'\rangle\prec e''\preceq e'}\breve{x}_{e'';\mathbf{u}'}\rho_{w^-_{e''}}(\upsilon),$$

$$\breve{x}^{e,e'}_{\mathbf{u}'}=\breve{x}_{e^+;\mathbf{u}'}-\breve{x}_{e'^+;\mathbf{u}'}\qquad\text{with}\qquad e^+,e'^+\in V^+_{\langle w^+_e,w^+_{e'}\rangle},\quad e^+\preceq e,\quad e'^+\preceq e'.$$

In particular,

$$\big|\breve{x}_{e,e'}(\upsilon)-\breve{x}^{e,e'}_{\mathbf{u}'}\rho_{\langle e,e'\rangle}(\upsilon)\big|\leqslant C|\upsilon||\rho_{\langle e,e'\rangle}(\upsilon)|.\tag{4.23}$$

By (E$'$2) in Corollary 4.3 for $\upsilon=\upsilon_0$ and $x=x_e^-(\upsilon_0)$, the equality in (4.22), and the first bound in (4.13), there exists $C\in\mathbb{R}$ such that

$$\left|\frac{\upsilon_e c^{(0)}_{e;\upsilon}(\psi)}{\rho_e(\upsilon)\breve{x}_{e,e^\bullet}(\upsilon)}-\frac{c^{(1)}_{e^\bullet;\upsilon}(\psi)}{\rho^2_{w^-_{e^\bullet}}(\upsilon)}-\breve{x}^{e,e^\bullet}_{\mathbf{u}'}\frac{c^{(2)}_{e^\bullet;\upsilon}(\psi)}{\rho^3_{w^-_{e^\bullet}}(\upsilon)}\rho_{\langle e^\bullet,e\rangle}(\upsilon)\right|\leqslant C\|\psi\||\upsilon|\big|\rho_{\langle e^\bullet,e\rangle}(\upsilon)\big| \tag{4.24}$$

for all $e\in\chi(\mathcal{T}_0)-\{e^\bullet\}$ with $\langle e\rangle=\langle e^\bullet\rangle$. By (E$'$2) in Corollary 4.3 for $\upsilon=\upsilon_0$, $e=e_1,e_2$, and $x=x^-_{e_1}(\upsilon_0),x^-_{e_2}(\upsilon_0)$, there similarly exists $C\in\mathbb{R}$ such that

$$\left|\frac{\upsilon_{e_1}c^{(0)}_{e_1;\upsilon}(\psi)}{\rho_{e_1}(\upsilon)\breve{x}_{e_1,e^\bullet}(\upsilon)}-\frac{\upsilon_{e_2}c^{(0)}_{e_2;\upsilon}(\psi)}{\rho_{e_2}(\upsilon)\breve{x}_{e_2,e^\bullet}(\upsilon)}-\breve{x}^{e_1,e_2}_{\mathbf{u}'}\frac{c^{(2)}_{e^\bullet;\upsilon}(\psi)}{\rho^3_{w^-_{e^\bullet}}(\upsilon)}\rho_{\langle e_1,e_2\rangle}(\upsilon)\right|\leqslant C\|\psi\||\upsilon|\big|\rho_{\langle e_1,e_2\rangle}(\upsilon)\big| \tag{4.25}$$

for e_1,e_2 satisfying the assumptions of the corollary.

By (4.13) and the last two assumptions in (4.22),

$$\|\psi\|\leqslant C\left|c^{(2)}_{\langle e^\bullet\rangle;\upsilon}(\psi)\right|\leqslant C'\left|\frac{c^{(2)}_{e^\bullet;\upsilon}(\psi)}{\rho^3_{w^-_{e^\bullet}}(\upsilon)}\right|\leqslant C''\|\psi\| \qquad \forall\,\upsilon\in\Delta \tag{4.26}$$

for some $C,C',C''\in\mathbb{R}$. By (4.23) and the last property in (4.21),

$$C|\rho_{w^\star}(\upsilon)|\leqslant|\breve{x}_{e_1,e^\bullet}(\upsilon)|,\ |\breve{x}_{e_2,e^\bullet}(\upsilon)|\leqslant C'|\rho_{w^\star}(\upsilon)|$$

for some $C,C'\in\mathbb{R}^+$. By (4.25) and (4.26), there thus exist $C,C'\in\mathbb{R}^+$ such that

$$\begin{aligned}\left|\frac{c^{(2)}_{e^\bullet;\upsilon}(\psi)}{\rho^3_{w^-_{e^\bullet}}(\upsilon)}\right|,\ \|\psi\|&\leqslant\frac{C}{|\rho_{\langle e_1,e_2\rangle}(\upsilon)|}\left(\frac{|\upsilon_{e_1}c^{(0)}_{e_1;\upsilon}(\psi)|}{|\rho_{e_1}(\upsilon)\breve{x}_{e_1,e^\bullet}(\upsilon)|}+\frac{|\upsilon_{e_2}c^{(0)}_{e_2;\upsilon}(\psi)|}{|\rho_{e_2}(\upsilon)\breve{x}_{e_2,e^\bullet}(\upsilon)|}\right)\\&\leqslant\frac{C'|\upsilon|}{|\rho_{w^\star}(\upsilon)|}\left(\frac{|\upsilon_{e_1}c^{(0)}_{e_1;\upsilon}(\psi)|}{|\rho^2_{e_1}(\upsilon)|}+\frac{|\upsilon_{e_2}c^{(0)}_{e_2;\upsilon}(\psi)|}{|\rho^2_{e_2}(\upsilon)|}\right)\end{aligned} \tag{4.27}$$

Combined with (4.24) for $e=e_1$, (4.23) for $(e,e')=(e_1,e^\bullet)$, and the last property in (4.21), this gives

$$\left|\frac{c^{(1)}_{e^\bullet;\upsilon}(\psi)}{\rho^2_{w^-_{e^\bullet}}(\upsilon)}\right|\leqslant C\left(\left|\frac{\upsilon_{e_1}c^{(0)}_{e_1;\upsilon}(\psi)}{\rho_{e_1}(\upsilon)\rho_{\langle e^\bullet,e_1\rangle}(\upsilon)}\right|+\|\psi\||\rho_{\langle e^\bullet,e_1\rangle}(\upsilon)|\right)\leqslant C'\left(\frac{|\upsilon_{e_1}c^{(0)}_{e_1;\upsilon}(\psi)|}{|\rho_{e_1}{}^2(\upsilon)|}+\frac{|\upsilon_{e_2}c^{(0)}_{e_2;\upsilon}(\psi)|}{|\rho^2_{e_2}(\upsilon)|}\right) \tag{4.28}$$

for some $C,C'\in\mathbb{R}$. Combining (4.27) with (4.28), we obtain the claim of the corollary for all $e\in\chi(\mathcal{T}_0)$ with $w_e^+\succeq w^\star$.

Suppose $e\in\chi(\mathcal{T}_0)$ with $\langle e\rangle=\langle e^\bullet\rangle$ and $w_e^+\not\succeq w^\star$. By (4.26), (4.24), and (4.28),

$$\begin{aligned}\|\psi\||\rho_{\langle w^\star,w_e^+\rangle}(\upsilon)|\leqslant C\left|\frac{c^{(2)}_{e^\bullet;\upsilon}(\psi)}{\rho^3_{w^-_{e^\bullet}}(\upsilon)}\right||\rho_{\langle e^\bullet,e\rangle}(\upsilon)|&\leqslant C'\left(\left|\frac{\upsilon_e c^{(0)}_{e;\upsilon}(\psi)}{\rho_e(\upsilon)\rho_{\langle e^\bullet,e\rangle}(\upsilon)}\right|+\left|\frac{c^{(1)}_{e^\bullet;\upsilon}(\psi)}{\rho^2_{w^-_{e^\bullet}}(\upsilon)}\right|\right)\\&\leqslant C''\left(\frac{|\upsilon_e c^{(0)}_{e;\upsilon}(\psi)|}{|\rho_e{}^2(\upsilon)|}+\frac{|\upsilon_{e_1}c^{(0)}_{e_1;\upsilon}(\psi)|}{|\rho_{e_1}{}^2(\upsilon)|}+\frac{|\upsilon_{e_2}c^{(0)}_{e_2;\upsilon}(\psi)|}{|\rho^2_{e_2}(\upsilon)|}\right).\end{aligned}$$

Along with (4.28) again, this completes the proof of the corollary. $\square$

Corollary 4.9. *Suppose* $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ *is a pseudo-tree framing and* $\mathbf{u}'$, Δ, R_υ, ψ, $e^\bullet$, *and* $w^\star$ *are as in Corollary 4.6. Let* $w\in\{P\}\cup\mathrm{Ver}_{0;P}^c$ *and* $e\in\chi(\mathcal{T}_0)$. *There exists* $C\in\mathbb{R}$ *such that*

- *if* $\langle w_e^+,w\rangle\neq\langle w^\star,w\rangle$, *then*

$$\frac{|\rho_e^2(\upsilon)|}{|\rho^2_{\langle w,w_e^+\rangle}(\upsilon)|}\bigg(\frac{|\rho_w(\upsilon)|}{|\rho_{w^-_{e(w)}}(\upsilon)|}\big|c^{(0)}_{e(w);\upsilon}(\psi)\big|+|\rho_w^2(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w)\bigg) \leqslant C\Big(|\upsilon_e|^2\big|c^{(0)}_{e;\upsilon}(\psi)\big|+|\rho_e^2(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)\Big);$$

- *if* $\langle w^\star,w_e^+\rangle\preceq\langle w^\star,w\rangle$, *then*

$$\frac{|\rho_e^2(\upsilon)|}{|\rho^2_{\langle w^\star,w\rangle}(\upsilon)|}\bigg(\frac{|\rho_w(\upsilon)|}{|\rho_{w^-_{e(w)}}(\upsilon)|}\big|c^{(0)}_{e(w);\upsilon}(\psi)\big|+|\rho_w^2(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w)\bigg) \leqslant C\Big(\frac{|\rho_e^2(\upsilon)|}{|\rho^2_{e^\bullet}(\upsilon)|}|\upsilon_{e^\bullet}|^2\big|c^{(0)}_{e^\bullet;\upsilon}(\psi)\big|+|\rho_e^2(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)\Big);$$

- *if* $w_e^+\succeq\langle w^\star,w\rangle$, *then*

$$\frac{|\rho^2_{\langle w^\star,w_e^+\rangle}(\upsilon)||\rho_e^2(\upsilon)|}{|\rho^4_{\langle w^\star,w\rangle}(\upsilon)|}\bigg(\frac{|\rho_w(\upsilon)|}{|\rho_{w^-_{e(w)}}(\upsilon)|}\big|c^{(0)}_{e(w);\upsilon}(\psi)\big|+|\rho_w^2(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w)\bigg) \leqslant C\Big(|\upsilon_e|^2\big|c^{(0)}_{e;\upsilon}(\psi)\big|+|\rho_e^2(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)\Big).$$

Proof. If $\langle w_e^+,w\rangle\neq\langle w^\star,w\rangle$, then

$$w\succeq\langle w^\star,w_e^+\rangle,\qquad \langle w^\star,w\rangle,\ \big\langle w^\star,\langle w_e^+,w\rangle\big\rangle\succeq\langle w^\star,w_e^+\rangle.$$

The first inequality in the corollary follows from Corollary 4.6 and the first inequality in (4.15). Since

$$\langle w^\star,\langle w^+_{e^\bullet},w_e^+\rangle\rangle=\langle w^\star,w_e^+\rangle,$$

the second inequality in the corollary follows from Corollary 4.6 with $e=e^\bullet$ and the first inequality in (4.15).

If $w_e^+\succeq\langle w^\star,w\rangle$, then

$$\langle w,w_e^+\rangle,\ \langle w^\star,w_e^+\rangle\succeq\langle w^\star,w\rangle=\big\langle w^\star,\langle w,w_e^+\rangle\big\rangle.$$

Along with the second inequality in (4.15), this implies that

$$\frac{|\rho_{\langle w^\star,w_e^+\rangle}(\upsilon)|}{|\rho_{\langle w^\star,w\rangle}(\upsilon)|}\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w)\leqslant\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+).$$

Combined with Corollary 4.6 and the first inequality in (4.15), this establishes the last inequality in the corollary. □

Corollary 4.10. *Suppose* $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$, $\mathbf{u}'$, Δ, R_υ, ψ, $e^\bullet$, *and* $w^\star$ *are as in Corollary 4.9. Let* $e\in\widetilde{\mathrm{Edg}}^c_{0;P}$. *There exists* $C\in\mathbb{R}$ *such that*

- *if* $w_e^+ \not\preceq w^\star$, *then*

$$\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ e'>e}} \frac{|\rho_{e'}^2(\upsilon)|}{|\rho_e^2(\upsilon)|}\bigg(\frac{|\rho_e(\upsilon)|}{|\rho_{w_{\widehat{e}}^-}(\upsilon)|}\big|c_{\widehat{e};\upsilon}^{(0)}(\psi)\big| + |\rho_e^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^\bullet(\psi;w_e^+)\bigg) \\ \leqslant C\sum_{e'\in\chi(\mathcal{T}_0)}\Big(|\upsilon_{e'}|^2\big|c_{e';\upsilon}^{(0)}(\psi)\big|+|\rho_{e'}^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^\bullet(\psi;w_{e'}^+)\Big);$$

- *if* $w_e^+ \preceq w^\star$, *then*

$$\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ e'\not>e}} \frac{|\rho_{e'}^2(\upsilon)|}{|\rho_e^2(\upsilon)|}\bigg(\frac{|\rho_e(\upsilon)|}{|\rho_{w_{\widehat{e}}^-}(\upsilon)|}\big|c_{\widehat{e};\upsilon}^{(0)}(\psi)\big| + |\rho_e^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^\bullet(\psi;w_e^-)\bigg) \\ \leqslant C\sum_{e'\in\chi(\mathcal{T}_0)}\bigg(\frac{|\rho_{e'}^2(\upsilon)|}{|\rho_{e^\bullet}^2(\upsilon)|}|\upsilon_{e^\bullet}|^2\big|c_{e^\bullet;\upsilon}^{(0)}(\psi)\big|+|\rho_{e'}^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^\bullet(\psi;w_{e'}^+)\bigg);$$

  and

$$\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ e'>e}} \frac{|\rho_{\langle w^\star,w_{e'}^+\rangle}^2(\upsilon)||\rho_{e'}^2(\upsilon)|}{|\rho_e^4(\upsilon)|}\bigg(\frac{|\rho_e(\upsilon)|}{|\rho_{w_{\widehat{e}}^-}(\upsilon)|}\big|c_{\widehat{e};\upsilon}^{(0)}(\psi)\big| + |\rho_e^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^\bullet(\psi;w_e^+)\bigg) \\ \leqslant C\sum_{e'\in\chi(\mathcal{T}_0)}\Big(|\upsilon_{e'}|^2\big|c_{e';\upsilon}^{(0)}(\psi)\big|+|\rho_{e'}^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^\bullet(\psi;w_{e'}^+)\Big).$$

Proof. If $w_e^+ \not\preceq w^\star$ and $e'\in\chi(\mathcal{T}_0)$ with $e'>e$, then

$$\langle w_{e'}^+, w_e^+\rangle = w_e^+ \succ \langle w^\star, w_e^+\rangle.$$

If $w_e^+ \preceq w^\star$ and $e'\in\chi(\mathcal{T}_0)$ with $e'>e$, then

$$w_{e'}^+ \succ w_e^+ = \langle w^\star, w_e^+\rangle.$$

The first and the third inequalities in the corollary thus follow from the corresponding bounds in Corollary 4.9 with (w,e) replaced by (w_e^+,e'). If $w_e^+\preceq w^\star$ and $e'\in\chi(\mathcal{T}_0)$ with $e'\not>e$, then

$$\langle w^\star, w_{e'}^+\rangle \preceq w_e^- = \langle w^\star, w_e^-\rangle, \qquad \widetilde{C}_{w^\star;\upsilon}^\bullet(\psi;w_e^-) \leqslant \widetilde{C}_{w^\star;\upsilon}^\bullet(\psi;w_{e'}^+).$$

Combined with the second bound in Corollary 4.9 with (w,e) replaced by (w_e^+,e'), this implies the second inequality in the corollary. □

4.4 Proof of Proposition 4.1

The Taylor series expansion (4.7) implies that $\sum_{k\geqslant 0}c_k z^k$ converges if $|z|<8\delta_{\bar\partial}$. By (q4) in Section 2.2,

$$|z_{e;\upsilon}^-(x)| \leqslant 8\delta_{\bar\partial} \qquad \forall\, e\in\widehat{\mathrm{Edg}}_{0;P}^{\,c},\ x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial}). \tag{4.29}$$

By (4.29) and (3.11),

$$\big|z_{e;\upsilon}^-(x)\rho_{w_e^-}(\upsilon)\big| + \Big|\sum_{e_0\prec e'\preceq e}\breve{x}_{e';\mathbf{u}'}\rho_{w_{e'}^-}(\upsilon)\Big| \leqslant 8\delta_{\bar\partial}+0 = 8\delta_{\bar\partial} \tag{4.30}$$

for all $e\in\chi(\mathcal{T}_{0;P})$ and $x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial})$. By (4.29), (3.11), the sentence containing (2.28), and (4.3),

$$\big|z_{e;\upsilon}^-(x)\rho_{w_e^-}(\upsilon)\big| + \Big|\sum_{e_0\prec e'\preceq e}\breve{x}_{e';\mathbf{u}'}\rho_{w_{e'}^-}(\upsilon)\Big| \leqslant 8\delta_{\bar\partial}|\upsilon|+\frac{1}{\delta_{\bar\partial}}\frac{|\upsilon|}{1-|\upsilon|} < 4\delta_{\bar\partial}+\frac{\delta_{\bar\partial}}{8} < 8\delta_{\bar\partial} \tag{4.31}$$

for all $e\in\widehat{\mathrm{Edg}}_{0;P}^c-\chi(\mathcal{T}_{0;P})$ and $x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial})$. The series in (E2) thus converges. By (4.3), (3.11), and the sentence containing (2.28),

$$\big|\breve{z}_{w;\mathbf{u}'}\big(q_\upsilon(x)\big)\rho_w(\upsilon)\big| + \Big|\sum_{e_0\prec e'\preceq e_w}\breve{x}_{e';\mathbf{u}'}\rho_{w_{e'}^-}(\upsilon)\Big| \leqslant \frac{1}{\delta_{\bar\partial}}|\upsilon|+\frac{1}{\delta_{\bar\partial}}\frac{|\upsilon|}{1-|\upsilon|} < \frac{\delta_{\bar\partial}}{16}+\frac{\delta_{\bar\partial}}{8} < 8\delta_{\bar\partial} \tag{4.32}$$

for all $w\in\mathrm{Ver}_{0;P}^c$ and $x\in q_\upsilon^{-1}\big(\Sigma_{\mathbf{u}';w}\big)-\Sigma_{\upsilon;e_w}(\delta_{\bar\partial})$. Since $\Sigma_{\upsilon;w}^{\mathrm{mn}}\subset q_\upsilon^{-1}\big(\Sigma_{\mathbf{u}';w}\big)-\Sigma_{\upsilon;e_w}(\delta_{\bar\partial})$, the series in (E3) also converges.

Let $<$ be a (strict) total order on $\widehat{\mathrm{Edg}}_{0;P}^c$ extending $\prec$. For every $e\in\widehat{\mathrm{Edg}}_{0;P}^c$, let

$$E_e^< = \mathrm{Edg}_{0;P}\cup\big\{e'\in\widehat{\mathrm{Edg}}_{0;P}^c\colon e'<e\big\}, \qquad E_e^\leqslant = E_e^<\cup\{e\}, \qquad \upsilon_e^< = \upsilon\big(E_e^<\big), \qquad \upsilon_e^\leqslant = \upsilon\big(E_e^\leqslant\big).$$

By (3.11),

$$\begin{aligned} &\rho_w(\upsilon_e^\leqslant)=\rho_w(\upsilon_e^<) && \forall\ e\in\widehat{\mathrm{Edg}}_{0;P}^c,\ w\in\mathrm{Ver}_{0;P}^c-\{w_e^+\},\\ &\rho_{w_e^+}(\upsilon_e^\leqslant)=v_e\,\rho_{w_e^-}(\upsilon_e^<) && \forall\ e\in\widetilde{\mathrm{Edg}}_{0;P}^c. \end{aligned} \tag{4.33}$$

For $e\in\widehat{\mathrm{Edg}}_{0;P}^c$, we will inductively construct injective homorphisms

$$R_{\upsilon_e^\leqslant}\colon \mathbb{E}_S|_{\mathbf{u}'} \longrightarrow \mathcal{H}_S^{1,0}\big(\Sigma_{\upsilon_e^\leqslant;0}(8\delta_{\bar\partial})\big), \qquad \mathbf{u}'\in\widetilde{U},\ \upsilon\in\Delta|_{\mathbf{u}'}, \tag{4.34}$$

so that (E1), (E2), and (E3) with $\upsilon=\upsilon_e^\leqslant$ are satisfied.

Fix a continuous family of isomorphisms

$$R_{\upsilon_{0;P}}\colon \mathbb{E}_S|_{\mathbf{u}'} \longrightarrow \mathbb{E}_S|_{\upsilon_{0;P}}\subset \mathcal{H}_S^{1,0}\big(\Sigma_{\upsilon_{0;P};0}(8\delta_{\bar\partial})\big), \qquad \mathbf{u}'\in\widetilde{U},\ \upsilon_{0;P}\in\Delta_{0;P}|_{\mathbf{u}'},$$

so that (E1) holds for $\upsilon=\upsilon_{0;P}$. By the first equation in (3.11),

$$\rho_w(\upsilon_{0;P})=0 \qquad \forall\ \upsilon_{0;P}\in\Delta_{0;P},\ w\in\mathrm{Ver}_{0;P}^c.$$

Combined with the second equation in (3.11) and (4.6), this gives (E2) and (E3) with $\upsilon=\upsilon_{0;P}$.

Suppose $e\in\widehat{\mathrm{Edg}}_{0;P}^c$ and we have constructed a continuous family of isomorphisms

$$R_{\upsilon_e^<}\colon \mathbb{E}_S|_{\mathbf{u}'} \longrightarrow \mathcal{H}_S^{1,0}\big(\Sigma_{\upsilon_e^<;0}(8\delta_{\bar\partial})\big), \qquad \mathbf{u}'\in\widetilde{U},\ \upsilon\in\Delta|_{\mathbf{u}'},$$

so that (E1), (E2), and (E3) with $\upsilon=\upsilon_e^<$ are satisfied. Given $\psi\in\mathbb{E}_S|_{\mathbf{u}'}$ as in (4.7) with $e_0=\langle e\rangle$, let $c_k=c_{e_0;\upsilon}^{(k)}(\psi)$. By (E2) with $\upsilon=\upsilon_e^<$,

$$\big\{R_{\upsilon_e^<}\psi\big\}_x=\rho_{w_e^-}(\upsilon_e^<)\sum_{k=0}^{\infty}c_k\bigg(z_{e;\upsilon_e^<}^-(x)\rho_{w_e^-}(\upsilon_e^<)+\sum_{e_0<e'\le e}\breve{x}_{e';\mathbf{u}'}\rho_{w_{e'}^-}(\upsilon_e^<)\bigg)^k\mathrm{d}z_{e;\upsilon_e^<}^- \tag{4.35}$$

for all $x\in\Sigma_{\upsilon_e^<;e}(8\delta_{\bar\partial})$. We define a holomorphic 1-form $\mathfrak{R}_{\upsilon_e}\big(R_{\upsilon_e^<}\psi\big)$ on $\Sigma_{\upsilon_e^\leqslant;e}(8\delta_{\bar\partial})$ by

$$\big\{\mathfrak{R}_{\upsilon_e}\big(R_{\upsilon_e^<}\psi\big)\big\}_x=\rho_{w_e^-}(\upsilon_e^<)\sum_{k=0}^{\infty}c_k\bigg(z_{e;\upsilon_e^\leqslant}^-(x)\rho_{w_e^-}(\upsilon_e^<)+\sum_{e_0<e'\le e}\breve{x}_{e';\mathbf{u}'}\rho_{w_{e'}^-}(\upsilon_e^<)\bigg)^k\mathrm{d}z_{e;\upsilon_e^\leqslant}^- \tag{4.36}$$

for all $x\in\Sigma_{\upsilon_e^\leqslant;e}(8\delta_{\bar\partial})$. By (4.30) or (4.31), the infinite sum converges. By (2.26),

$$\big\{\mathfrak{R}_{\upsilon_e}\big(R_{\upsilon_e^<}\psi\big)\big\}_x=\big\{q_{\upsilon_e^<;\upsilon_e^\leqslant}^*\big(R_{\upsilon_e^<}\psi\big)\big\}_x\qquad\forall\,x\in\Sigma_{\upsilon_e^\leqslant;e}^-(8\delta_{\bar\partial})-\Sigma_{\upsilon_e^\leqslant;e}^-(\delta_{\bar\partial}). \tag{4.37}$$

If in addition $\upsilon_e=0$, i.e. $\upsilon_e^\leqslant=\upsilon_e^<$, then

$$\big\{\mathfrak{R}_{\upsilon_e}\big(R_{\upsilon_e^<}\psi\big)\big\}_x=\big\{R_{\upsilon_e^<}\psi\big\}_x\qquad\forall\,x\in\Sigma_{\upsilon_e^\leqslant;e}(8\delta_{\bar\partial}). \tag{4.38}$$

Suppose $e\in\chi(\mathcal{T}_0)$. Define $R_{\upsilon_e^\leqslant}$ in (4.34) by

$$\big\{R_{\upsilon_e^\leqslant}\psi\big\}_x=\begin{cases}\big\{q_{\upsilon_e^<;\upsilon_e^\leqslant}^*\big(R_{\upsilon_e^<}\psi\big)\big\}_x, & \text{if } x\in\Sigma_{\upsilon_e^\leqslant;0}(8\delta_{\bar\partial})-\big(\Sigma_{\upsilon_e^\leqslant;e}^-(\delta_{\bar\partial})\cup\Sigma_{\upsilon_e^\leqslant;e}^+(8\delta_{\bar\partial})\big);\\ \big\{\mathfrak{R}_{\upsilon_e}\big(R_{\upsilon_e^<}\psi\big)\big\}_x, & \text{if } x\in\Sigma_{\upsilon_e^\leqslant;e}(8\delta_{\bar\partial}).\end{cases} \tag{4.39}$$

By (4.37), the two expressions above agree on the overlap. By the target space of $R_{\upsilon_e^<}\psi$, (q2) in Section 2.2, and $\mathfrak{R}_{\upsilon_e}\big(R_{\upsilon_e^<}\psi\big)$ being holomorphic,

$$R_{\upsilon_e^\leqslant}\psi\in\mathcal{H}_S^{1,0}\big(\Sigma_{\upsilon_e^\leqslant;0}(8\delta_{\bar\partial})\big). \tag{4.40}$$

By (4.38), the right-hand side of (4.39) reduces to $R_{\upsilon_e^<}\psi$ if $\upsilon_e=0$. Thus, (E1) with $\upsilon=\upsilon_e^\leqslant$ follows from the same equation with $\upsilon=\upsilon_e^<$.

By the first case of (4.39), (E2) with $\upsilon=\upsilon_e^<$, (2.26), and (4.33), (E2) with $\upsilon=\upsilon_e^\leqslant$ holds for all $e'\in\widehat{\mathrm{Edg}}_{0;P}^c$ with $e'\neq e$. By the second case of (4.39), (4.36), and (4.33), the property (E2) with $\upsilon=\upsilon_e^\leqslant$ holds for the edge e. By the first case of (4.39), (E3) with $\upsilon=\upsilon_e^<$, (2.23), and (4.33), (E3) with $\upsilon=\upsilon_e^\leqslant$ holds for all $w\in\mathrm{Ver}_{0;P}^c$.

Suppose $e\in\widetilde{\mathrm{Edg}}_{0;P}^c$. By (q4) in Section 2.2 and (4.1),

$$z_{e;\upsilon_e^\leqslant}^-(x)=\breve{z}_{w_e^+;\mathbf{u}'}\big(q_{\upsilon_e^\leqslant}(x)\big)\,\upsilon_e\qquad\forall\,x\in\Sigma_{\upsilon_e^\leqslant;e}^+(8\delta_{\bar\partial})-\Sigma_{\upsilon_e^\leqslant;e}^+(\delta_{\bar\partial}).$$

Along with (4.36) and (4.33), this implies that

$$\begin{aligned}&\big\{\mathfrak{R}_{\upsilon_e}\big(R_{\upsilon_e^<}\psi\big)\big\}_x\\&=\upsilon_e\rho_{w_e^-}(\upsilon_e^<)\sum_{k=0}^{\infty}c_k\bigg(\breve{z}_{w_e^+;\mathbf{u}'}\big(q_{\upsilon_e^\leqslant}(x)\big)\upsilon_e\rho_{w_e^-}(\upsilon_e^<)+\sum_{e_0<e'\le e}\breve{x}_{e';\mathbf{u}'}\rho_{w_{e'}^-}(\upsilon_e^<)\bigg)^k\mathrm{d}\big(\breve{z}_{w_e^+;\mathbf{u}'}\circ q_{\upsilon_e^\leqslant}\big)\\&=\rho_{w_e^+}(\upsilon_e^\leqslant)\sum_{k=0}^{\infty}c_k\bigg(\breve{z}_{w_e^+;\mathbf{u}'}\big(q_{\upsilon_e^\leqslant}(x)\big)\rho_{w_e^+}(\upsilon_e^\leqslant)+\sum_{e_0<e'\le e}\breve{x}_{e';\mathbf{u}'}\rho_{w_{e'}^-}(\upsilon_e^\leqslant)\bigg)^k\mathrm{d}\big(\breve{z}_{w_e^+;\mathbf{u}'}\circ q_{\upsilon_e^\leqslant}\big)\end{aligned} \tag{4.41}$$

for all $x\in\Sigma^+_{\upsilon_e^{\leqslant};e}(8\delta_{\bar\partial})-\Sigma^+_{\upsilon_e^{\leqslant};e}(\delta_{\bar\partial})$. By (4.32), the infinite sum above converges on

$$\Sigma^0_{\upsilon_e^{\leqslant};w_e^+}\equiv q^{-1}_{\upsilon_e^{\leqslant}}\big(\Sigma_{\mathbf{u}';w_e^+}\big)-\Sigma_{\upsilon_e^{\leqslant};e}(\delta_{\bar\partial}).$$

Thus, $\mathfrak{R}_{\upsilon_e}\big(R_{\upsilon_e^<}\psi\big)$ extends to a holomorphic differential on $\Sigma^0_{\upsilon_e^{\leqslant};w_e^+}$.

If $e\in\widetilde{\mathrm{Edg}}^c_{0;P}$, we define $R_{\upsilon_e^{\leqslant}}$ in (4.34) by

$$\big\{R_{\upsilon_e^{\leqslant}}\psi\big\}_x=\begin{cases}\big\{q^*_{\upsilon_e^<;\upsilon_e^{\leqslant}}\big(R_{\upsilon_e^<}\psi\big)\big\}_x, & \text{if } x\in\Sigma_{\upsilon_e^{\leqslant};0}(8\delta_{\bar\partial})-\big(q^{-1}_{\upsilon_e^{\leqslant}}\big(\Sigma_{\mathbf{u}';w_e^+}\big)\cup\Sigma_{\upsilon_e^{\leqslant};e}(\delta_{\bar\partial})\big);\\ \big\{\mathfrak{R}_{\upsilon_e}\big(R_{\upsilon_e^<}\psi\big)\big\}_x, & \text{if } x\in q^{-1}_{\upsilon_e^{\leqslant}}\big(\Sigma_{\mathbf{u}';w_e^+}\big)\cup\Sigma_{\upsilon_e^{\leqslant};e}(8\delta_{\bar\partial}).\end{cases}\tag{4.42}$$

By (4.37), the two expressions above agree on the overlap. By the reasoning below (4.39), (4.42) satisfies (4.40), (E1) with $\upsilon=\upsilon_e^{\leqslant}$, (E2) with (υ,e) replaced by $(\upsilon_e^{\leqslant},e')$ and $e'\notin E^+_{w_e^+}$, and (E3) with $\upsilon=\upsilon_e^{\leqslant}$ and $w\neq w_e^+$. By the second case of (4.42) and (4.41), (E3) with $\upsilon=\upsilon_e^{\leqslant}$ holds for $w=w_e^+$.

For every $e^*\in E^+_{w_e^+}$,

$$w_e^+=w_{e^*}^-,\qquad (\upsilon_e^<)_{e^*},\,(\upsilon_e^{\leqslant})_{e^*}=0.\tag{4.43}$$

By the second case of (4.42), (4.41), (4.4), and the first equation in (4.43),

$$\begin{split}\big\{R_{\upsilon_e^{\leqslant}}\psi\big\}_x&=\rho_{w_e^+}(\upsilon_e^{\leqslant})\sum_{k=0}^{\infty}c_k\bigg(z^-_{e^*;\upsilon_e^{\leqslant}}(x)\rho_{w_e^+}(\upsilon_e^{\leqslant})+\breve{x}_{e^*;\mathbf{u}'}\rho_{w_e^+}(\upsilon_e^{\leqslant})+\sum_{e_0<e'\le e}\breve{x}_{e';\mathbf{u}'}\rho_{w^-_{e'}}(\upsilon_e^{\leqslant})\bigg)^k\mathrm{d}z^-_{e^*;\upsilon_e^{\leqslant}}\\&=\rho_{w^-_{e^*}}(\upsilon_e^{\leqslant})\sum_{k=0}^{\infty}c_k\bigg(z^-_{e^*;\upsilon_e^{\leqslant}}(x)\rho_{w^-_{e^*}}(\upsilon_e^{\leqslant})+\sum_{e_0<e'\le e^*}\breve{x}_{e';\mathbf{u}'}\rho_{w^-_{e'}}(\upsilon_e^{\leqslant})\bigg)^k\mathrm{d}z^-_{e^*;\upsilon_e^{\leqslant}}\end{split}$$

for every $x\in\Sigma^-_{\upsilon_e^{\leqslant};e^*}(8\delta_{\bar\partial})$. This establishes (E2) with (υ,e) replaced by $(\upsilon_e^{\leqslant},e^*)$ and $x\in\Sigma^-_{\upsilon_e^{\leqslant};e^*}(8\delta_{\bar\partial})$. By the second equations in (4.43) and (q4) in Section 2.2,

$$\mathrm{d}z^-_{e^*;\upsilon_e^<}\big|_{\Sigma^+_{\upsilon_e^<;e^*}(8\delta_{\bar\partial})}=0,\qquad \mathrm{d}z^-_{e^*;\upsilon_e^{\leqslant}}\big|_{\Sigma^+_{\upsilon_e^{\leqslant};e^*}(8\delta_{\bar\partial})}=0.\tag{4.44}$$

Combined with the first case of (4.42) and (E2) with $\upsilon=\upsilon_e^<$, (4.44) implies

$$\big\{R_{\upsilon_e^{\leqslant}}\psi\big\}_x=0=\rho_{w^-_{e^*}}(\upsilon_e^{\leqslant})\sum_{k=0}^{\infty}c_k\bigg(z^-_{e^*;\upsilon_e^{\leqslant}}(x)\rho_{w^-_{e^*}}(\upsilon_e^{\leqslant})+\sum_{e_0<e'\le e^*}\breve{x}_{e';\mathbf{u}'}\rho_{w^-_{e'}}(\upsilon_e^{\leqslant})\bigg)^k\mathrm{d}z^-_{e^*;\upsilon_e^{\leqslant}}$$

for all $x\in\Sigma^+_{\upsilon_e^<;e^*}(8\delta_{\bar\partial})$.

If e^* is the maximal element of $\widehat{\mathrm{Edg}}^c_{0;P}$ with respect to $<$, then $\upsilon_{e^*}^{\leqslant}=\upsilon_{0\chi}$. When the above induction terminates, we thus obtain a continuous family of isomorphisms

$$R_{\upsilon_{0\chi}}\colon\,\mathbb{E}_S|_{\mathbf{u}'}\longrightarrow\mathcal{H}^{1,0}_S\big(\Sigma_{\upsilon_{0\chi};0}(8\delta_{\bar\partial})\big),\qquad \mathbf{u}'\in\widetilde{U},\ \upsilon_{0\chi}\in\Delta_{0\chi}|_{\mathbf{u}'},\tag{4.45}$$

so that (E1), (E2), and (E3) with $\upsilon=\upsilon_{0\chi}$ are satisfied. Define

$$R_\upsilon=q^*_{\upsilon_{0\chi};\upsilon}\circ R_{\upsilon_{0\chi}}\colon\,\mathbb{E}_S|_{\mathbf{u}'}\longrightarrow\mathcal{H}^{1,0}_S\big(\Sigma_{\upsilon;0}(8\delta_{\bar\partial})\big)\qquad\forall\,\mathbf{u}'\in\widetilde{U},\ \upsilon\in\Delta|_{\mathbf{u}'}.$$

(E1) then follows from the same equation with $\upsilon=\upsilon_{0\chi}$ and $q_{\mathbf{u}';\mathbf{u}'}=\mathrm{id}$. By (3.11),

$$\rho_w(\upsilon_{0\chi})=\rho_w(\upsilon)\qquad\forall\,w\in\mathrm{Ver}_0,\ \upsilon\in\Delta.$$

Therefore, (E2) follows from (E2) with $\upsilon=\upsilon_{0\chi}$ and (2.26), while (E3) follows from (E3) with $\upsilon=\upsilon_{0\chi}$ and (2.23).

5 Computation of obstructions

Let $(\mathscr{T}, \mathcal{T}_0, \mathcal{T}_{0;P})$ be a graph framing; see the beginning of Section 2.2. With $\mathbb{E}_0 \equiv \mathbb{E}_{0;\varnothing}$ as in Section 4.1 and ev_0 as in (2.29), let

$$\mathrm{Obs}_0 = \mathbb{E}_0^*|_{\widetilde{U}} \otimes_J \mathrm{ev}_0^* TX \longrightarrow \widetilde{U} \tag{5.1}$$

be the obstruction bundle of the two-step pregluing construction of Section 2.2 with a fixed norm $\|\cdot\|$. Let R_υ be as in Proposition 4.1. With β_υ as in (2.35) and $\widetilde{\xi}_{\upsilon;1}$ as in (2.32), we define a homomorphism

$$\begin{gathered}\Theta_\upsilon\colon\ \Gamma^{0,1}(\upsilon) \longrightarrow \mathrm{Obs}_0|_{\pi_{\widetilde{\mathcal{F}\mathscr{T}}}(\upsilon)},\\ \big\{\Theta_\upsilon(\eta)\big\}\psi = \frac{\mathfrak{i}}{2\pi}\int_{\Sigma_\upsilon}\big(\beta_\upsilon R_\upsilon\psi\big)\wedge\big(\Pi^{-1}_{\widetilde{\xi}_{\upsilon;1}\circ q_{\upsilon_1;\upsilon_{01}}\circ\widetilde{q}_\upsilon}\,\eta\big) \qquad \forall\,\psi\in\mathbb{E}_0|_{\pi_{\widetilde{\mathcal{F}\mathscr{T}}}(\upsilon)}.\end{gathered} \tag{5.2}$$

By (2.32) and the first statement in Corollary 4.2, $\widetilde{\xi}_{\upsilon;1}$ is defined on the support of $\beta_\upsilon R_\upsilon\psi$. Since $\beta_\upsilon R_\upsilon\psi$ has at worst simple poles at the nodes of Σ_υ, the integral in (5.2) is well-defined. By Hölder's Inequality and (E1) in Proposition 4.1,

$$\|\Theta_\upsilon(\eta)\| \leqslant C\|\eta\|_{\upsilon,p} \qquad \forall\,\eta\in\Gamma^{0,1}(\upsilon) \tag{5.3}$$

for some $C\in\mathbb{R}$.

In Section 5, we consider the quadratic expansion

$$\Pi_\xi^{-1}\big(\bar\partial_J \exp_{u_\upsilon}\xi\big) = \bar\partial_J u_\upsilon + D_\upsilon\xi + N_\upsilon(\xi) \qquad \forall\,\xi\in\Gamma(\upsilon) \tag{5.4}$$

of the $\bar\partial_J$-operator, where D_υ is the linear term as in (2.4) and N_υ is a quadratic term. Analyzing the image of each term of the right-hand side of (5.4) under the homomorphism Θ_υ and introducing a bootstrapping mechanism, we obtain estimates in Sections 5.1 and 5.2 that play crucial roles in Section 6.

5.1 Terms of the quadratic expansion

For $e\in\chi(\mathcal{T}_0)$, $\upsilon\in\Delta|_{\mathbf{u}'}$, $\psi\in\mathbb{E}_S|_{\mathbf{u}'}$, and $\mathbf{u}'\in\widetilde{U}$, let

$$\mathscr{D}^{(\ell)}_{\upsilon;e}\psi = \sum_{k=0}^{\ell-1} c^{(k)}_{e;\upsilon}(\psi)\frac{\upsilon_e^{k+1}}{(k+1)!}\frac{\partial^{k+1}\widetilde{\xi}_{\upsilon;1}}{\partial(z^+_{e;\upsilon_1})^{k+1}}\Big|_{x_e^+(\upsilon_1)} \quad \forall\,\ell\in\mathbb{Z}^+, \qquad \mathscr{D}_{\upsilon;e}\psi = \mathscr{D}^{(1)}_{\upsilon;e}\psi. \tag{5.5}$$

Lemma 5.1. *For every $\ell\in\mathbb{Z}^+$, there exists $C\in\mathbb{R}$ such that*

$$\begin{aligned}&\Big|\big\{\Theta_\upsilon(\bar\partial_J u_\upsilon)\big\}\psi - \sum_{e\in\chi(\mathcal{T}_0)}\mathscr{D}^{(\ell)}_{\upsilon;e}\psi\Big|\\ &\qquad\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\|R_{\upsilon_0}\psi\|_{C^0(\Sigma^-_{\upsilon_0;e}(8\delta_{\bar\partial}))}|\upsilon_e|^2\Big(\big|\mathrm{d}_{x_e^+(\upsilon_1)}\widetilde{u}_{\upsilon;1}\big|^2 + \|\widetilde{\xi}_{\upsilon;1}|_{\Sigma^+_{\upsilon_1;e}(8\delta_{\bar\partial})}\|_{\upsilon_1,p,1}\big(|\upsilon_e|^2+|\upsilon_e|^{\ell-1}\big)\Big)\end{aligned}$$

for all $\upsilon\in\Delta|_{\mathbf{u}'}$, $\psi\in\mathbb{E}_S|_{\mathbf{u}'}$, and $\mathbf{u}'\in\widetilde{U}$.

Proof. Similarly to the proof of Lemma 3.8, $\widetilde{\xi}_{\upsilon;1}$ is $J_{0;\upsilon}$-holomorphic. By the Mean Value Inequality [10, Lemma 4.3.1] and Hölder's Inequality, there exists $C\in\mathbb{R}$ such that

$$\big|\mathrm{d}_{x_e^+(\upsilon_1)}\widetilde{u}_{\upsilon;1}\big| \leqslant C\|\widetilde{\xi}_{\upsilon,1}|_{\Sigma^+_{\upsilon_1;e}(8\delta_{\bar\partial})}\|_{\upsilon_1,p,1} \qquad \forall\, e\in\chi(\mathcal{T}_0). \tag{5.6}$$

By Lemma A.8, there exists $C\in\mathbb{R}$ such that

$$\begin{aligned} \big|\mathrm{d}_x\widetilde{u}_{\upsilon;1}-\mathrm{d}_{x_e^+(\upsilon_1)}\widetilde{u}_{\upsilon;1}\big| &\leqslant C\|\widetilde{\xi}_{\upsilon,1}|_{\Sigma^+_{\upsilon_1;e}(8\delta_{\bar\partial})}\|_{\upsilon_1,p,1}|z^+_{e;\upsilon_1}(x)|\\ \Big|\widetilde{\xi}_{\upsilon,1}(x)-T^{(\ell)}_{\widetilde{\xi}_{\upsilon;1};0}\big(z^+_{e;\upsilon_1}(x)\big)\Big| &\leqslant C\|\widetilde{\xi}_{\upsilon,1}|_{\Sigma^+_{\upsilon_1;e}(8\delta_{\bar\partial})}\|_{\upsilon_1,p,1}|z^+_{e;\upsilon_1}(x)|^{\ell+1}, \quad \ell\in\mathbb{Z}^+ \end{aligned} \tag{5.7}$$

for all $x\in\Sigma^+_{\upsilon_1;e}(8\delta_{\bar\partial})$ and $e\in\chi(\mathcal{T}_0)$. Therefore,

$$\begin{aligned} \|\widetilde{\xi}_{\upsilon,1}|_{\Sigma^+_{\upsilon_1;e}(2|\upsilon_e|/\delta_{\bar\partial})}\|_{\upsilon_1,p,1} &\leqslant C|\upsilon_e|^{\frac{2}{p}}\Big(\big|\mathrm{d}_{x_e^+(\upsilon_1)}\widetilde{u}_{\upsilon;1}\big|+|\upsilon_e|\|\widetilde{\xi}_{\upsilon,1}|_{\Sigma^+_{\upsilon_1;e}(8\delta_{\bar\partial})}\|_{\upsilon_1,p,1}\Big),\\ \|\widetilde{\xi}_{\upsilon;1}\|_{C^0(\Sigma^+_{\upsilon_1;e}(2|\upsilon_e|/\delta_{\bar\partial}))} &\leqslant C|\upsilon_e|\Big(\big|\mathrm{d}_{x_e^+(\upsilon_1)}\widetilde{u}_{\upsilon;1}\big|+|\upsilon_e|\|\widetilde{\xi}_{\upsilon,1}|_{\Sigma^+_{\upsilon_1;e}(8\delta_{\bar\partial})}\|_{\upsilon_1,p,1}\Big) \end{aligned} \tag{5.8}$$

for all $e\in\chi(\mathcal{T}_0)$.

By the proof of [18, (4.23), (4.25)],

$$\begin{aligned} &\Big|\big\{\Theta_\upsilon(\bar\partial_J u_\upsilon)\big\}\psi-\frac{1}{2\pi\mathfrak{i}}\sum_{e\in\chi(\mathcal{T}_0)}\sum_{k=0}^{\infty}\oint_{\partial\Sigma^+_{\upsilon_1;e}(2|\upsilon_e|/\delta_{\bar\partial})}\widetilde{\xi}_{\upsilon;1}c^{(k)}_{e;\upsilon}(\psi)\frac{\upsilon_e^{k+1}}{(z^+_{e;\upsilon_1})^{k+2}}\mathrm{d}z^+_{e;\upsilon_1}\Big|\\ &\qquad\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\|R_{\upsilon_0}\psi\|_{C^0(\Sigma^-_{\upsilon_0;e}(8\delta_{\bar\partial}))}\|\widetilde{\xi}_{\upsilon;1}\|_{C^0(\Sigma^+_{\upsilon_1;e}(2|\upsilon_e|/\delta_{\bar\partial}))}|\upsilon_e|^{\frac{p-2}{p}}\big\|\mathrm{d}\widetilde{\xi}_{\upsilon;1}|_{\Sigma^+_{\upsilon_1;e}(2|\upsilon_e|/\delta_{\bar\partial})}\big\|_{\upsilon_1,p} \end{aligned}$$

for every $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$. By the second inequality in (5.7) and Cauchy's Integral Formula, there exists $C\in\mathbb{R}$ such that

$$\begin{aligned} &\Big|\frac{1}{2\pi\mathfrak{i}}\sum_{k=0}^{\infty}\oint_{\partial\Sigma^+_{\upsilon_1;e}(2|\upsilon_e|/\delta_{\bar\partial})}\widetilde{\xi}_{\upsilon;1}c^{(k)}_{e;\upsilon}(\psi)\frac{\upsilon_e^{k+1}}{(z^+_{e;\upsilon_1})^{k+2}}\mathrm{d}z^+_{e;\upsilon_1}-\mathscr{D}^{(\ell)}_{\upsilon;e}\psi\Big|\\ &\qquad\qquad\leqslant C\|R_{\upsilon_0}\psi\|_{C^0(\Sigma^-_{\upsilon_0;e}(8\delta_{\bar\partial}))}\|\widetilde{\xi}_{\upsilon,1}|_{\Sigma^+_{\upsilon_1;e}(8\delta_{\bar\partial})}\|_{\upsilon_1,p,1}|\upsilon_e|^{\ell+1} \end{aligned}$$

for every $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$. The claim follows from the last two inequalities along with (5.8) and (5.6). □

Lemma 5.2. *There exists $C\in\mathbb{R}$ such that*

$$\Big|\big\{\Theta_\upsilon(D_\upsilon\xi)\big\}\psi\Big|\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\big\|R_{\upsilon_0}\psi\big\|_{C^0(\Sigma^-_{\upsilon_0;e}(8\delta_{\bar\partial}))}\|\xi\|_{C^0(\Sigma_{\upsilon;e}(8\delta_{\bar\partial}))}|\upsilon_e|$$

for all $\upsilon\in\Delta|_{\mathbf{u}'}$, $\mathbf{u}'\in\widetilde{U}$, $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$, and $\xi\in\Gamma(\upsilon)$.

Proof. By (E2) in Proposition 4.1, (4.5), and the $q=1$ case of the second bound in (2.39),

$$\int_{\Sigma_{\upsilon;e}(8\delta_{\bar\partial})}|R_\upsilon\psi||z^+_{e;\upsilon}|\,,\ \int_{\Sigma^+_{\upsilon;e}(8\delta_{\bar\partial})-\Sigma^+_{\upsilon;e}(\delta_{\bar\partial})}|R_\upsilon\psi|\ \leqslant\ C|\upsilon_e|\big\|R_{\upsilon_0}\psi\big\|_{C^0(\Sigma^-_{\upsilon_0;e}(8\delta_{\bar\partial}))} \tag{5.9}$$

for all $e\in\widehat{\mathrm{Edg}}_{0;P}^{c}$, $\xi\in\Gamma(\upsilon)$, and $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$. By [21, Corollary 3.3] and Integration by Parts as in the proof of [18, Lemma 4.4(7)], there exists $C\in\mathbb{R}$ such that

$$\Big|\big\{\Theta_\upsilon(D_\upsilon\xi)\big\}\psi\Big| \leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\Big(\int_{\Sigma_{\upsilon;e}(8\delta_{\bar\partial})}|R_\upsilon\psi||z^+_{e;\upsilon}||\xi| + \int_{\Sigma^+_{\upsilon;e}(4\delta_{\bar\partial})-\Sigma^+_{\upsilon;e}(2\delta_{\bar\partial})}|R_\upsilon\psi||\xi|\Big) \tag{5.10}$$

for all $\xi\in\Gamma(\upsilon)$ and $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$. Combining (5.10) with (5.9), we obtain the claim. □

For all $\upsilon\in\Delta$ and $w\in\{P\}\sqcup\widehat{\mathrm{Ver}}_{0;P}^{c}$, let ξ_υ and $\varepsilon_{\upsilon;w}$ be as in (3.14) and Corollary 3.2, respectively.

Lemma 5.3. *There exists $C\in\mathbb{R}$ such that*

$$\big|\big\{\Theta_\upsilon\big(N_\upsilon(\xi_\upsilon)\big)\big\}\psi\big| \leqslant C\|\psi\||\upsilon|\sum_{e\in\chi(\mathcal{T}_0)}|\rho_e(\upsilon)|$$

for all $\upsilon\in\Delta|_{\mathbf{u}'}$, $\mathbf{u}'\in\widetilde{U}$, and $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$.

Proof. By [21, Proposition 3.13], the quadratic term N_υ satisfies

$$N_\upsilon(0)=0, \qquad \big\|N_\upsilon(\zeta)-N_\upsilon(\zeta')\big\|_{\upsilon,p} \leqslant C\big(\|\zeta\|_{\upsilon,p,1}+\|\zeta'\|_{\upsilon,p,1}\big)\|\zeta-\zeta'\|_{\upsilon,p,1} \tag{5.11}$$

for some $C\in\mathbb{R}$ and for all small $\zeta,\,\zeta'\in\Gamma(\upsilon)$. By Lemma A.2, there exists $C\in\mathbb{R}$ such that

$$|N_\upsilon(\xi)|_x \leqslant C\Big(\big|T_{\nabla^J}\big(\xi(x),\{\nabla^\upsilon\xi\}_x\big)\big| + \big(|\mathrm{d}_x u_\upsilon|+|\nabla^\upsilon\xi|_x\big)|\xi(x)|^2\Big) \qquad \forall\, x\in\Sigma_\upsilon,\ \upsilon\in\Delta, \tag{5.12}$$

where T_{∇^J} is the torsion tensor of ∇^J as in (2.1).

By the first property in Corollary 4.2, there exists $C\in\mathbb{R}$ such that

$$\big|\big\{\Theta_\upsilon\big(N_\upsilon(\xi)\big)\big\}\psi\big| \leqslant C\int_{\Sigma_{\upsilon;0}(8\delta_{\bar\partial})}\big|N_\upsilon(\xi)\big|\big|R_\upsilon\psi\big| \qquad \forall\,\xi\in\Gamma(\upsilon),\ \psi\in\mathbb{E}_0|_{\mathbf{u}'}. \tag{5.13}$$

By Hölder's Inequality, (5.11), and the last property in (M1) in Proposition 3.1, there exists $C\in\mathbb{R}$ such that

$$\sum_{e\in\chi(\mathcal{T}_0)}\int_{\Sigma_{\upsilon;e}(8\delta_{\bar\partial})}\big|N_\upsilon(\xi_\upsilon)\big|\big|R_\upsilon\psi\big| \leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\Big(\|R_{\upsilon_0}\psi\|_{C^0(\Sigma_{\upsilon;e}(8\delta_{\bar\partial}))}\sum_{w\leq w_e^-}\varepsilon^2_{\upsilon;w}\Big). \tag{5.14}$$

Combined with (R2) in Corollary 4.5, (4.13) for $e^\bullet=e$, (4.16), and (3.16), this implies that

$$\begin{aligned}\sum_{e\in\chi(\mathcal{T}_0)}\int_{\Sigma_{\upsilon;e}(8\delta_{\bar\partial})}\big|N_\upsilon(\xi_\upsilon)\big|\big|R_\upsilon\psi\big| &\leqslant C'\|\psi\|\sum_{e,e'\in\chi(\mathcal{T}_0)}\big|\rho_{w_e^-}(\upsilon)\big|\big|\rho_{\langle w_e^-,w^+_{e'}\rangle,w^+_{e'}}(\upsilon)\big|^2\\ &\leqslant C\|\psi\||\upsilon|\sum_{e\in\chi(\mathcal{T}_0)}|\rho_e(\upsilon)|.\end{aligned} \tag{5.15}$$

Similarly, the last property in (M2) in Proposition 3.1 and (R3) in Corollary 4.5 imply that

$$\begin{aligned}\sum_{w\in\{P\}\cup\mathrm{Ver}^c_{0;P}}\int_{\Sigma^{\mathrm{mn}}_{\upsilon;w}}\big|N_\upsilon(\xi_\upsilon)\big|\big|R_\upsilon\psi\big| &\leqslant C\sum_{w\in\{P\}\cup\mathrm{Ver}^c_{0;P}}\Big(\|R_{\upsilon_0}\psi\|_{C^0(\Sigma^{\mathrm{mn}}_{\upsilon;w})}\sum_{w'\leq w}\varepsilon^2_{\upsilon;w'}\Big)\\ &\leqslant C'\|\psi\|\sum_{\substack{w\in\{P\}\cup\mathrm{Ver}^c_{0;P}\\ e'\in\chi(\mathcal{T}_0)}}\big|\rho_w(\upsilon)\big|\big|\rho_{\langle w,w^+_{e'}\rangle,w^+_{e'}}(\upsilon)\big|^2 \leqslant C''\|\psi\||\upsilon|\sum_{e\in\chi(\mathcal{T}_0)}|\rho_e(\upsilon)|.\end{aligned} \tag{5.16}$$

By (5.12), (R1) in Corollary 4.5, (3.15), and the third bound in (M3) in Proposition 3.1,

$$\sum_{e\in\widetilde{\mathrm{Edg}}_{0;P}^c}\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}\big|N_\upsilon(\xi_\upsilon)\big|\big|R_\upsilon\psi\big| \leqslant C\|\psi\||\upsilon|\sum_{e\in\chi(\mathcal{T}_0)}|\rho_e(\upsilon)|.$$

Along with (5.13), (5.15), and (5.16), this establishes the claim. □

Corollary 5.4. *There exists a continuous function $\epsilon\colon(\mathbb{R},0)\longrightarrow(\mathbb{R},0)$ with the property that*

$$\Big|\sum_{e\in\chi(\mathcal{T}_0)}\upsilon_e\, c_{e;\upsilon}^{(0)}(\psi)\Big(\mathcal{D}_e\mathbf{u}'\big(\tfrac{\partial}{\partial z_{e;\mathbf{u}'}^+}\big)\Big)\Big| \leqslant \epsilon\big(|\upsilon|\big)\,\|\psi\|\sum_{e\in\chi(\mathcal{T}_0)}|\rho_e(\upsilon)|$$

for all $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$ and $\upsilon\in\Delta|_{\mathbf{u}'}$ with $\mathbf{u}'\in\widetilde{U}$ such that the map $\exp_{\upsilon_\upsilon}\xi_\upsilon$ as in (3.14) is J-holomorphic.

Proof. By the $\ell=1$ case of Lemma 5.1, Lemma 5.2, (2.22), and (M1) in Proposition 3.1, there exist $C,C'\in\mathbb{R}$ such that

$$\begin{aligned}\Big|\big\{\Theta_\upsilon(\bar\partial_J u_\upsilon+D_\upsilon\xi_\upsilon)\big\}\psi-\sum_{e\in\chi(\mathcal{T}_0)}\mathscr{D}_{\upsilon;e}\psi\Big| &\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\big\|R_{\upsilon_0}\psi\big\|_{C^0(\Sigma_{\upsilon_0;e}^-(8\delta_{\bar\partial}))}\big(|\upsilon_e|+\|\xi_\upsilon|_{\Sigma_{\upsilon;e}(8\delta_{\bar\partial})}\|_{\upsilon,p,1}\big)|\upsilon_e|\\ &\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\big\|R_{\upsilon_0}\psi\big\|_{C^0(\Sigma_{\upsilon_0;e}^-(8\delta_{\bar\partial}))}\big(|\upsilon_e|+\sum_{w\preceq w_e^-}\varepsilon_{\upsilon;w}\big)|\upsilon_e|\end{aligned}\tag{5.17}$$

for all $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$ and $\upsilon\in\Delta|_{\mathbf{u}'}$ with $\mathbf{u}'\in\widetilde{U}$. Along with (3.18), this gives

$$\Big|\big\{\Theta_\upsilon(\bar\partial_J u_\upsilon+D_\upsilon\xi_\upsilon)\big\}\psi-\sum_{e\in\chi(\mathcal{T}_0)}\mathscr{D}_{\upsilon;e}\psi\Big| \leqslant C\,|\upsilon|\sum_{e\in\chi(\mathcal{T}_0)}\big\|R_{\upsilon_0}\psi\big\|_{C^0(\Sigma_{\upsilon_0;e}^-(8\delta_{\bar\partial}))}|\upsilon_e|\tag{5.18}$$

for some $C\in\mathbb{R}$.

Since $\exp_{u_\upsilon}\xi_\upsilon$ is J-holomorphic,

$$\bar\partial_J u_\upsilon+D_\upsilon\xi_\upsilon+N_\upsilon(\xi_\upsilon)=0.\tag{5.19}$$

By (5.19), (5.18), and Lemma 5.3,

$$\Big|\sum_{e\in\chi(\mathcal{T}_0)}\mathscr{D}_{\upsilon;e}\psi\Big| \leqslant C|\upsilon|\sum_{e\in\chi(\mathcal{T}_0)}\Big(|\upsilon_e|\big\|R_{\upsilon_0}\psi\big\|_{C^0(\Sigma_{\upsilon_0;e}^-(8\delta_{\bar\partial}))}+|\rho_e(\upsilon)|\|\psi\|\Big).$$

By the $\ell=1$ case of (5.5), this implies that

$$\Big|\sum_{e\in\chi(\mathcal{T}_0)}\upsilon_e\, c_{e;\upsilon}^{(0)}(\psi)\Big(\mathrm{d}_{x_e^+(\upsilon_1)}\widetilde{u}_{\upsilon;1}\big(\tfrac{\partial}{\partial z_{e;\upsilon_1}^+}\big)\Big)\Big| \leqslant C|\upsilon|\sum_{e\in\chi(\mathcal{T}_0)}\Big(|\upsilon_e|\big\|R_{\upsilon_0}\psi\big\|_{C^0(\Sigma_{\upsilon_0;e}^-(8\delta_{\bar\partial}))}+|\rho_e(\upsilon)|\|\psi\|\Big).$$

By the continuity of the family $\widetilde{u}_{\upsilon;1}$, there exists a continuous function $\epsilon\colon(\mathbb{R},0)\longrightarrow(\mathbb{R},0)$ such that

$$\Big|\sum_{e\in\chi(\mathcal{T}_0)}\upsilon_e\, c_{e;\upsilon}^{(0)}(\psi)\Big(\mathcal{D}_e\mathbf{u}'\big(\tfrac{\partial}{\partial z_{e;\mathbf{u}'}^+}\big)\Big)\Big| \leqslant \epsilon\big(|\upsilon|\big)\sum_{e\in\chi(\mathcal{T}_0)}\Big(|\upsilon_e|\big\|R_{\upsilon_0}\psi\big\|_{C^0(\Sigma_{\upsilon_0;e}^-(8\delta_{\bar\partial}))}+|\rho_e(\upsilon)|\|\psi\|\Big).$$

Combined with (R2) in Corollary 4.5 and the $k=0,1$ cases in (4.13), this establishes the claim. □

5.2 A bootstrapping estimate and applications

Let $\widehat{u}_{\upsilon;2}$ and $\widehat{\zeta}_{\upsilon;2}$ be as in (3.59), ξ_υ be as in (3.14), and $\varepsilon_{\upsilon;w}$, $\mathcal{D}_{\upsilon;w}$, and $\gamma_{\upsilon;e}$ be as in Corollary 3.2. The following proposition is the main statement of this section.

Proposition 5.5. *Suppose $e_\bullet \in \chi(\mathcal{T}_{0;P})$ is such that $\psi|_{x^-_{e_\bullet}(\mathbf{u})} \neq 0$ for some $\psi \in \mathbb{E}_0|_{\mathbf{u}}$. Then there exists $C \in \mathbb{R}$ with the property that*

$$\varepsilon_{\upsilon;P} \equiv \big\|\widehat{\zeta}_{\upsilon;2}\big\|_{\upsilon,p,1} \leqslant C\Big(\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ \langle e\rangle\neq e_\bullet}} |\rho_e(\upsilon)| + \varepsilon_{\upsilon;w^+_{e_\bullet}} |\upsilon_{e_\bullet}|^2\Big) \tag{5.20}$$

for all $\upsilon \in \Delta$ such that the map $\exp_{\widehat{u}_{\upsilon;2}}\widehat{\zeta}_{\upsilon;2}$ is J-holomorphic. If in addition

$$w^\star \in \widehat{\mathrm{Ver}}^c_{0;P}, \qquad \langle e_{w^\star}\rangle = e_\bullet, \qquad \upsilon_e \neq 0 \quad \forall\, e \in \widehat{\mathrm{Edg}}^c_{0;P}, \tag{5.21}$$

then

$$\varepsilon_{\upsilon;w} \leqslant C\Bigg(\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ w^+_e \not\geq w}} \frac{|\rho_e(\upsilon)|}{|\rho_w(\upsilon)|} + \sum_{\substack{e\in\chi(\mathcal{T}_0)\\ w^+_e \geq w}} \frac{|\rho_{\langle w^\star, w^+_e\rangle}(\upsilon)||\rho_e(\upsilon)|}{|\rho^2_w(\upsilon)|}\Bigg) \tag{5.22}$$

for all $w \in \{P\} \cup \widehat{\mathrm{Ver}}^c_{0;P}$ with $w \preceq w^\star$ and

$$\big|\mathcal{D}_{\upsilon;w}\big| \leqslant C\Bigg(\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ w^+_e \not\geq w}} \frac{|\rho_e(\upsilon)|}{|\rho_w(\upsilon)|} + |\upsilon_{e_w}| \sum_{\substack{e\in\chi(\mathcal{T}_0)\\ w^+_e \geq w}} \frac{|\rho_{\langle w^\star, w^+_e\rangle}(\upsilon)||\rho_e(\upsilon)|}{|\rho^2_w(\upsilon)|}\Bigg) \tag{5.23}$$

for all $w \in \widehat{\mathrm{Ver}}^c_{0;P}$ with $w \preceq w^\star$.

Proposition 5.5 improves the bounds in (3.15) arising from Corollary 3.2. For example, suppose $[\mathbf{u}]$ is of the bubble type of the last diagram in Figure 1 and all assumptions in Proposition 5.5 are satisfied. Then

$$\chi(\mathcal{T}_{0;P}) = \chi(\mathcal{T}_0) = \mathrm{Edg} = \{e_\bullet\}, \qquad \widehat{\mathrm{Ver}}^c_{0;P} = \{w^+_{e_\bullet}\} = \{w^\star\}.$$

By (3.15), $\varepsilon_{\upsilon;P}$ is then bounded by $C|\upsilon|$ (even if $\exp_{\widehat{u}_{\upsilon;2}}\widehat{\zeta}_{\upsilon;2}$ is not J-holomorphic); there is no small a priori bound for $\mathcal{D}_{\upsilon;w^\star}$. On the other hand, Proposition 3.1 implies that

$$\varepsilon_{\upsilon;P} \leqslant C|\upsilon|^2 \qquad \text{and} \qquad \big|\mathcal{D}_{\upsilon;w^\star}\big| \leqslant C|\upsilon|.$$

Corollary 5.6. *Suppose $e_\bullet$ and $w^\star$ are as in Proposition 5.5 and $e^\bullet \in \chi(\mathcal{T}_0)$ satisfies $w^+_{e^\bullet} \geq w^\star$. Then there exists $C \in \mathbb{R}$ with the property that*

$$\Big|\sum_{e\in\chi(\mathcal{T}_0)} \upsilon_e\, c^{(0)}_{e;\upsilon}(\psi)\Big(\mathcal{D}_e\mathbf{u}'\big(\tfrac{\partial}{\partial z^+_{e;\mathbf{u}'}}\big)\Big)\Big| \leqslant C \sum_{e\in\chi(\mathcal{T}_0)} \Bigg(\Big(\sum_{e'\in\chi(\mathcal{T}_0)} \frac{|\rho^2_{e'}(\upsilon)|}{|\rho^2_{e^\bullet}(\upsilon)|}\Big)|\upsilon_e|^2\big|c^{(0)}_{e;\upsilon}(\psi)\big| + |\rho^2_e(\upsilon)|\big|\widetilde{C}^{\bullet}_{w^\star;\upsilon}(\psi; w^+_e)\Bigg)$$

for all $\psi \in \mathbb{E}_0|_{\mathbf{u}'}$ and $\upsilon \in \Delta|_{\mathbf{u}'}$ with $\mathbf{u}' \in \widetilde{U}$ such that (5.21) holds and the map $\exp_{u_\upsilon}\xi_\upsilon$ as in (3.14) is J-holomorphic.

Proof. Analogous to the proof of Corollary 5.4, the claim follows from the equation (5.19) and Lemmas 5.8 and 5.9 below. $\square$

Lemma 5.7. *There exists $C\in\mathbb{R}$ with the property that*

$$\Big|\sum_{e\in\chi(\mathcal{T}_{0;P})} c_{e;\upsilon}^{(0)}(\psi)\upsilon_e\mathcal{D}_{\upsilon;w_e^+}\Big| \leqslant C\|\psi\|\sum_{e\in\chi(\mathcal{T}_{0;P})}\Big(|\mathcal{D}_{\upsilon;w_e^+}||\upsilon||\upsilon_e|+\varepsilon_{\upsilon;w_e^+}|\upsilon_e|^2\Big)$$

for all $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$ and $\upsilon\in\Delta|_{\mathbf{u}'}$ with $\mathbf{u}'\in\widetilde{U}$ such that the map $\exp_{u_\upsilon}\xi_\upsilon$ as in (3.14) is J-holomorphic.

Proof. By (3.60) and (2.22),

$$\|\widehat{\zeta}_{\upsilon;2}\|_{\upsilon,p,1},\ \|\widehat{\zeta}_{\upsilon;2}\|_{C^0} \leqslant C\sum_{e\in\chi(\mathcal{T}_{0;P})}\Big(|\mathcal{D}_{\upsilon;w_e^+}||\upsilon_e|+\varepsilon_{\upsilon;w_e^+}|\upsilon_e|^2\Big). \tag{5.24}$$

Consider the quadratic expansion of the $\bar\partial_J$-operator

$$\Pi_{\widehat{\zeta}_{\upsilon;2}}^{-1}\big(\bar\partial_J\exp_{\widehat{u}_{\upsilon;2}}\widehat{\zeta}_{\upsilon;2}\big) = \bar\partial_J\widehat{u}_{\upsilon;2} + D_{\widehat{u}_{\upsilon;2}}\widehat{\zeta}_{\upsilon;2} + N_{\widehat{u}_{\upsilon;2}}\big(\widehat{\zeta}_{\upsilon;2}\big).$$

Let $\beta_{P;\upsilon}$ and $R_{P;\upsilon}$ be as in (2.35) and Proposition 4.1, respectively, but with the contracted subgraph $\mathcal{T}_0$ chosen to be $\mathcal{T}_{0;P}$. Define

$$\begin{aligned}&\Theta_{P;\upsilon}\colon\ \Gamma^{0,1}(\widehat{u}_{\upsilon;2}) \longrightarrow \mathrm{Obs}_0|_{\pi_{\widetilde{\mathcal{F}\mathcal{T}}}(\upsilon)},\\ &\big\{\Theta_{P;\upsilon}(\eta)\big\}\psi = \frac{\mathfrak{i}}{2\pi}\int_{\Sigma_\upsilon}\big(\beta_{P;\upsilon}R_{P;\upsilon}\psi\big)\wedge\big(\Pi^{-1}_{\widetilde{\xi}_{\upsilon;2}\circ q_{\upsilon_2;\upsilon_{02}}\circ\widetilde{q}_{\upsilon;\chi(\mathcal{T}_{0;P})}}\eta\big) \qquad \forall\,\psi\in\mathbb{E}_0|_{\pi_{\widetilde{\mathcal{F}\mathcal{T}}}(\upsilon)}\end{aligned} \tag{5.25}$$

with $\widetilde{\xi}_{\upsilon;2}$ as in (3.56) and υ_2 and $\upsilon_{0;2}$ as in the sentence containing (3.54). Along with (3.57) and (3.62), the proof of the $\ell=1$ case of Lemma 5.1 implies that there exist $C,C'\in\mathbb{R}$ such that

$$\begin{aligned}\Big|\big\{\Theta_{P;\upsilon}(\bar\partial_J\widehat{u}_{\upsilon;2})\big\}\psi - \sum_{e\in\chi(\mathcal{T}_{0;P})} c_{e;\upsilon}^{(0)}(\psi)\upsilon_e\mathcal{D}_{\upsilon;w_e^+}\Big| &\leqslant C\sum_{e\in\chi(\mathcal{T}_{0;P})}\|\psi\||\upsilon_e|^2\Big(\|\mathcal{D}_{\upsilon;w_e^+}|+\|\widetilde{\xi}_{\upsilon;2}|_{\Sigma^+_{\upsilon_2;e}(8\delta_{\bar\partial})}\|_{\upsilon_2,p,1}\Big)\\ &\leqslant C'\|\psi\|\sum_{e\in\chi(\mathcal{T}_{0;P})}|\upsilon_e|^2\big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}\big) \qquad \forall\,\psi\in\mathbb{E}_0|_{\mathbf{u}'}.\end{aligned} \tag{5.26}$$

By the proof of Lemma 5.2 and (5.24),

$$\begin{aligned}\Big|\big\{\Theta_{P;\upsilon}(D_{\widehat{u}_{\upsilon;2}}\widehat{\zeta}_{\upsilon;2})\big\}\psi\Big| &\leqslant C\sum_{e\in\chi(\mathcal{T}_{0;P})}\|\psi\|\|\widehat{\zeta}_{\upsilon;2}\|_{C^0}|\upsilon_e|\\ &\leqslant C'\|\psi\||\upsilon|\sum_{e\in\chi(\mathcal{T}_{0;P})}\Big(|\mathcal{D}_{\upsilon;w_e^+}||\upsilon_e|+\varepsilon_{\upsilon;w_e^+}|\upsilon_e|^2\Big) \qquad \forall\,\psi\in\mathbb{E}_0|_{\mathbf{u}'}.\end{aligned} \tag{5.27}$$

By (5.3) with Θ_υ replaced by $\Theta_{P;\upsilon}$, (5.11), and (5.24),

$$\Big|\big\{\Theta_{P;\upsilon}\big(N_{\widehat{u}_{\upsilon;2}}(\widehat{\zeta}_{\upsilon;2})\big)\big\}\psi\Big| \leqslant C\|\psi\|\sum_{e\in\chi(\mathcal{T}_{0;P})}\Big(|\mathcal{D}_{\upsilon;w_e^+}|^2|\upsilon_e|^2+\varepsilon^2_{\upsilon;w_e^+}|\upsilon_e|^4\Big) \quad \forall\,\psi\in\mathbb{E}_0|_{\mathbf{u}'}. \tag{5.28}$$

Since the map $\exp_{\widehat{u}_{\upsilon;2}}\widehat{\zeta}_{\upsilon;2}=\exp_{u_\upsilon}\xi_\upsilon$ is J-holomorphic,

$$\bar\partial_J\widehat{u}_{\upsilon;2} + D_{\widehat{u}_{\upsilon;2}}\widehat{\zeta}_{\upsilon;2} + N_{\widehat{u}_{\upsilon;2}}\big(\widehat{\zeta}_{\upsilon;2}\big) = 0. \tag{5.29}$$

Combined with (5.26)-(5.28), this establishes the claim. □

Proof of Proposition 5.5. By the first assumption of the proposition, there exist $C\in\mathbb{R}$ and a continuous family $\psi=\psi|_{\mathbf{u}'}\in\mathbb{E}_0|_{\mathbf{u}'}$ with $\mathbf{u}'\in\widetilde{U}$ such that

$$C\big|c^{(0)}_{e_\bullet;\upsilon}(\psi)\big| > \|\psi\| \qquad \forall\,\upsilon\in\Delta.$$

Combined with Lemma 5.7 and (3.15) for $w=w_e^+$ with $e\in\chi(\mathcal{T}_{0;P})-\{e_\bullet\}$, this implies that

$$\big|\mathcal{D}_{\upsilon;w_{e_\bullet}^+}\big||\upsilon_{e_\bullet}| \leqslant C\Big(\sum_{\substack{e\in\chi(\mathcal{T}_{0;P})\\ e\neq e_\bullet}}\big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}\big)|\upsilon_e|+\varepsilon_{\upsilon;w_{e_\bullet}^+}|\upsilon_{e_\bullet}|^2\Big) \leqslant C'\Big(\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ \langle e\rangle\neq e_\bullet}}|\rho_e(\upsilon)|+\varepsilon_{\upsilon;w_{e_\bullet}^+}|\upsilon_{e_\bullet}|^2\Big) \tag{5.30}$$

for some $C,C'\in\mathbb{R}$. The bound in (5.20) follows from (3.62), (5.24), (5.30), and the first bound in (3.15).

Suppose in addition (5.21) holds. We claim that there exists $C\in\mathbb{R}$ such that

$$\big|\mathcal{D}_{\upsilon;w}\big| \leqslant C\Big(\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ w_e^+\not\geq w}}\frac{|\rho_e(\upsilon)|}{|\rho_w(\upsilon)|}+\varepsilon_{\upsilon;w}|\upsilon_{e_w}|\Big) \qquad \forall\; w\in\widehat{\mathrm{Ver}}^c_{0;P} \text{ s.t. } w_{e_\bullet}^+\preceq w\preceq w^\star\,. \tag{5.31}$$

By (5.30), this is indeed the case for $w=w_{e_\bullet}^+$. Suppose $w_{e_\bullet}^+\prec w\preceq w^\star$ and (5.31) holds with w replaced by $w^-\equiv w_{e_w}^-$. Combining this with the first two bounds in (M2) in Proposition 3.1 for $w=w^-$ and (3.15) for $w'\in V_{w^-}^+-\{w\}$, we obtain (5.31) itself.

By (3.15), (5.22) holds for $w=w^\star$. Suppose $w_{e_\bullet}^+\preceq w\preceq w^\star$ and (5.22) holds for this w. Combining it with (5.31), we obtain (5.23) for the same w. Suppose $w\prec w^\star$ and (5.22) and (5.23) hold for $w^+\in V_w^+$ such that $w^+\preceq w^\star$. Combining them with the first inequality in (M2) in Proposition 3.1, (3.15) for $w'\in V_w^+-\{w^+\}$, and the fact that

$$\langle w^\star,w_e^+\rangle\prec w^+ \qquad \forall\; e\in\chi(\mathcal{T}_0)\;\text{ s.t. }\; w_e^+\succeq w,\; w_e^+\not\geq w^+,\; w^\star\succeq w^+, \tag{5.32}$$

we obtain (5.22) for this w. □

Lemma 5.8. *Suppose $e_\bullet,e^\bullet\in\widehat{\mathrm{Edg}}^c_{0;P}$ and $w^\star\in\widehat{\mathrm{Ver}}^c_{0;P}$ satisfy the assumptions in Corollary 5.6. Then there exists $C\in\mathbb{R}$ such that*

$$\begin{aligned}&\Big|\big\{\Theta_\upsilon(\bar\partial_J u_\upsilon+D_\upsilon\xi_\upsilon)\big\}\psi-\sum_{e\in\chi(\mathcal{T}_0)}\mathscr{D}_{\upsilon;e}\psi\Big|\\ &\qquad\qquad\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\Big(\Big(\sum_{e'\in\chi(\mathcal{T}_0)}\tfrac{|\rho^2_{e'}(\upsilon)|}{|\rho^2_{e^\bullet}(\upsilon)|}\Big)|\upsilon_e|^2\big|c^{(0)}_{e;\upsilon}(\psi)\big|+|\rho^2_e(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)\Big)\end{aligned}$$

for all $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$ and $\upsilon\in\Delta|_{\mathbf{u}'}$ with $\mathbf{u}'\in\widetilde{U}$ such that (5.21) holds.

Proof. By (5.22) and (5.32), there exists $C\in\mathbb{R}$ such that

$$\sum_{w'\preceq w}\varepsilon_{\upsilon;w'}\leqslant C\Big(\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ w_e^+\not\geq w}}\frac{|\rho_e(\upsilon)|}{|\rho_w(\upsilon)|}+\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ w_e^+\succeq w}}\frac{|\rho_{\langle w^\star,w_e^+\rangle}(\upsilon)||\rho_e(\upsilon)|}{|\rho^2_w(\upsilon)|}\Big)$$

for all $w\in\{P\}\cup\mathrm{Ver}^c_{0;P}$ with $w\preceq w^\star$. Combined with (3.15), this implies that

$$\begin{aligned}\sum_{w'\preceq w}\varepsilon_{\upsilon;w'}&=\sum_{\langle w^\star,w\rangle\prec w'\preceq w}\varepsilon_{\upsilon;w'}+\sum_{w'\preceq\langle w^\star,w\rangle}\varepsilon_{\upsilon;w'}\\&\leqslant C\Bigg(\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ \langle w,w_e^+\rangle\succ\langle w^\star,w\rangle}}\frac{|\rho_e(\upsilon)|}{|\rho_{\langle w,w_e^+\rangle}(\upsilon)|}+\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ \langle w^\star,w_e^+\rangle\preceq\langle w^\star,w\rangle}}\frac{|\rho_e(\upsilon)|}{|\rho_{\langle w^\star,w\rangle}(\upsilon)|}+\sum_{\substack{e\in\chi(\mathcal{T}_0)\\ w_e^+\succeq\langle w^\star,w\rangle}}\frac{|\rho_{\langle w^\star,w_e^+\rangle}(\upsilon)||\rho_e(\upsilon)|}{|\rho^2_{\langle w^\star,w\rangle}(\upsilon)|}\Bigg)\end{aligned}\tag{5.33}$$

for all $w\in\{P\}\cup\mathrm{Ver}^c_{0;P}$. By (5.33) and Corollary 4.9, there exists $C\in\mathbb{R}$ such that

$$\begin{aligned}\sum_{w\in\{P\}\cup\mathrm{Ver}^c_{0;P}}&\Bigg(\bigg(\frac{|\rho_w(\upsilon)|}{|\rho_{w^-_{e(w)}}(\upsilon)|}\big|c^{(0)}_{e(w);\upsilon}(\psi)\big|+|\rho^2_w(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w)\bigg)\sum_{w'\preceq w}\varepsilon^2_{\upsilon;w'}\Bigg)\\&\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\Bigg(\bigg(\sum_{e'\in\chi(\mathcal{T}_0)}\frac{|\rho^2_{e'}(\upsilon)|}{|\rho^2_{e^\bullet}(\upsilon)|}\bigg)|\upsilon_e|^2\big|c^{(0)}_{e;\upsilon}(\psi)\big|+|\rho^2_e(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)\Bigg).\end{aligned}\tag{5.34}$$

By (R2) and (R3) in Corollary 4.5 and by (5.34),

$$\begin{aligned}\sum_{e\in\chi(\mathcal{T}_0)}&\Big(\|R_{\upsilon_0}\psi\|_{C^0(\Sigma_{\upsilon;e}(8\delta_{\bar\partial}))}\sum_{w\preceq w_e^-}\varepsilon^2_{\upsilon;w}\Big)+\sum_{w\in\{P\}\cup\mathrm{Ver}^c_{0;P}}\Big(\|R_{\upsilon_0}\psi\|_{C^0(\Sigma^{\mathrm{mn}}_{\upsilon;w})}\sum_{w'\preceq w}\varepsilon^2_{\upsilon;w'}\Big)\\&\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\Bigg(\bigg(\sum_{e'\in\chi(\mathcal{T}_0)}\frac{|\rho^2_{e'}(\upsilon)|}{|\rho^2_{e^\bullet}(\upsilon)|}\bigg)|\upsilon_e|^2\big|c^{(0)}_{e;\upsilon}(\psi)\big|+|\rho^2_e(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)\Bigg).\end{aligned}\tag{5.35}$$

Along with the Cauchy-Schwarz Inequality and (R2) in Corollary 4.5, this implies that

$$\begin{aligned}\sum_{e\in\chi(\mathcal{T}_0)}&\Big(\|R_{\upsilon_0}\psi\|_{C^0(\Sigma_{\upsilon;e}(8\delta_{\bar\partial}))}\big(|\upsilon_e|+\sum_{w\preceq w_e^-}\varepsilon_{\upsilon;w}\big)|\upsilon_e|\Big)\leqslant2\sum_{e\in\chi(\mathcal{T}_0)}\Big(\|R_{\upsilon_0}\psi\|_{C^0(\Sigma_{\upsilon;e}(8\delta_{\bar\partial}))}\Big(|\upsilon_e|^2+\sum_{w\preceq w_e^-}\varepsilon^2_{\upsilon;w}\Big)\Big)\\&\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\Bigg(\bigg(\sum_{e'\in\chi(\mathcal{T}_0)}\frac{|\rho^2_{e'}(\upsilon)|}{|\rho^2_{e^\bullet}(\upsilon)|}\bigg)|\upsilon_e|^2\big|c^{(0)}_{e;\upsilon}(\psi)\big|+|\rho^2_e(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)\Bigg).\end{aligned}$$

Combined with (5.17), this establishes the claim. □

Lemma 5.9. *Suppose $e_\bullet,e^\bullet\in\widehat{\mathrm{Edg}}^c_{0;P}$ and $w^\star\in\widehat{\mathrm{Ver}}^c_{0;P}$ satisfy the assumptions in Corollary 5.6. Then there exists $C\in\mathbb{R}$ such that*

$$\big|\{\Theta_\upsilon\big(N_\upsilon(\xi_\upsilon)\big)\}\psi\,\big|\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\Bigg(\bigg(\sum_{e'\in\chi(\mathcal{T}_0)}\frac{|\rho^2_{e'}(\upsilon)|}{|\rho^2_{e^\bullet}(\upsilon)|}\bigg)|\upsilon_e|^2\big|c^{(0)}_{e;\upsilon}(\psi)\big|+|\rho^2_e(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)\Bigg)$$

for all $\psi\in\mathbb{E}_0|_{\mathbf{u}'}$ and $\upsilon\in\Delta|_{\mathbf{u}'}$ with $\mathbf{u}'\in\widetilde{U}$ such that (5.21) holds.

Proof. By (5.14), the first inequality in (5.16), and (5.35),

$$\begin{aligned}\sum_{e\in\chi(\mathcal{T}_0)}&\int_{\Sigma_{\upsilon;e}(8\delta_{\bar\partial})}\big|N_\upsilon(\xi_\upsilon)\big||R_\upsilon\psi|+\sum_{w\in\{P\}\cup\mathrm{Ver}^c_{0;P}}\int_{\Sigma^{\mathrm{mn}}_{\upsilon;w}}\big|N_\upsilon(\xi_\upsilon)\big||R_\upsilon\psi|\\&\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\Bigg(\bigg(\sum_{e'\in\chi(\mathcal{T}_0)}\frac{|\rho^2_{e'}(\upsilon)|}{|\rho^2_{e^\bullet}(\upsilon)|}\bigg)|\upsilon_e|^2\big|c^{(0)}_{e;\upsilon}(\psi)\big|+|\rho^2_e(\upsilon)|\widetilde{C}^\bullet_{w^\star;\upsilon}(\psi;w_e^+)\Bigg)\end{aligned}\tag{5.36}$$

for some $C\in\mathbb{R}$.

Suppose $e\in\widetilde{\mathrm{Edg}}^c_{0;P}$. By (2.40) and the Cauchy-Schwarz Inequality (CSI),

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|\mathrm{d}\widetilde{q}_{\upsilon;e}||\mathrm{d}z^-_{e;\upsilon}|\bigg)|\mathcal{D}_{\upsilon;w_e^+}|\sum_{w'\preceq w_e^-}\varepsilon_{\upsilon;w'}\leqslant C\Big(|\mathcal{D}_{\upsilon;w_e^+}|^2|\upsilon_e|+\sum_{w'\preceq w_e^-}\varepsilon^2_{\upsilon;w'}\Big).$$

By the first bound in (2.39) and CSI,

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|\mathrm{d}\widetilde{q}_{\upsilon;e}|\big(|\upsilon_e|+|z^-_{e;\upsilon}|\big)|\mathrm{d}z^-_{e;\upsilon}|\bigg)|\mathcal{D}_{\upsilon;w_e^+}|\sum_{w'\preceq w_e^-}\varepsilon_{\upsilon;w'}\leqslant C\Big(|\mathcal{D}_{\upsilon;w_e^+}|^2|\upsilon_e|^2+\sum_{w'\preceq w_e^-}\varepsilon^2_{\upsilon;w'}\Big).$$

By (2.38), the $q=1$ case of the second bound in (2.39), and CSI,

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|z^+_{e;\upsilon}||\mathrm{d}\widetilde{q}_{\upsilon;e}||\mathrm{d}z^-_{e;\upsilon}|\bigg)\varepsilon_{\upsilon;w_e^+}\sum_{w'\preceq w_e^-}\varepsilon_{\upsilon;w'}\leqslant C\Big(\varepsilon^2_{\upsilon;w_e^+}|\upsilon_e|^2+\sum_{w'\preceq w_e^-}\varepsilon^2_{\upsilon;w'}\Big).$$

By (q4) in Section 2.2, the $q=1$ case of the second bound in (2.39), (2.40), and CSI,

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|z^+_{e;\upsilon}||\mathrm{d}\widetilde{q}_{\upsilon;e}|\big(|\upsilon_e|+|z^-_{e;\upsilon}|\big)|\mathrm{d}z^-_{e;\upsilon}|\bigg)\varepsilon_{\upsilon;w_e^+}\sum_{w'\preceq w_e^-}\varepsilon_{\upsilon;w'}\leqslant C\Big(\varepsilon^2_{\upsilon;w_e^+}|\upsilon_e|^3+\sum_{w'\preceq w_e^-}\varepsilon^2_{\upsilon;w'}\Big).$$

By (q4) in Section 2.2 and the $q=1$ case of the second bound in (2.39),

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|z^+_{e;\upsilon}|^2\big(|\upsilon_e|+|z^-_{e;\upsilon}|\big)^\ell|\mathrm{d}z^-_{e;\upsilon}|\bigg)\big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}\big)|\mathcal{D}_{\upsilon;w_e^+}|\leqslant C\big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}\big)|\mathcal{D}_{\upsilon;w_e^+}||\upsilon_e|^{\ell+1}$$

with $\ell=0,1$. By (q4) in Section 2.2, the first bound in (2.39), and CSI,

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|z^+_{e;\upsilon}|^2|\mathrm{d}\widetilde{q}_{\upsilon;e}|\big(|\upsilon_e|+|z^-_{e;\upsilon}|\big)^2|\mathrm{d}z^-_{e;\upsilon}|\bigg)|\mathcal{D}_{\upsilon;w_e^+}|\varepsilon_{\upsilon;w_e^+}\leqslant C\Big(|\mathcal{D}_{\upsilon;w_e^+}|^2|\upsilon_e|^2+\varepsilon^2_{\upsilon;w_e^+}|\upsilon_e|^3\Big).$$

By (q4) in Section 2.2 and the $q=1$ case of the second bound in (2.39),

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|z^+_{e;\upsilon}|^3\big(|\upsilon_e|+|z^-_{e;\upsilon}|\big)^\ell|\mathrm{d}z^-_{e;\upsilon}|\bigg)\varepsilon^2_{\upsilon;w_e^+}\leqslant C\varepsilon^2_{\upsilon;w_e^+}|\upsilon_e|^{\ell+1},\qquad \ell=0,1,2.$$

By Hölder's Inequality and the third bound in (M3) in Proposition 3.1,

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|\gamma_{\upsilon;e}|\bigg)\sum_{w'\preceq w_e^-}\varepsilon_{\upsilon;w'}\leqslant C\sum_{w'\preceq w_e^-}\varepsilon^2_{\upsilon;w'}.$$

By Hölder's Inequality, the third bound in (M3) in Proposition 3.1, the second bound in (2.39), and CSI,

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|z^+_{e;\upsilon}|^2|\gamma_{\upsilon;e}||\mathrm{d}z^-_{e;\upsilon}|\bigg)\varepsilon_{\upsilon;w_e^+}\leqslant C\Big(\varepsilon^2_{\upsilon;w_e^+}|\upsilon_e|^2+\sum_{w'\preceq w_e^-}\varepsilon^2_{\upsilon;w'}\Big).$$

Along with (q4) in Section 2.2, the same reasoning gives

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|z_{e;\upsilon}^+|\big(|\upsilon_e|+|z_{e;\upsilon}^-|\big)^{\ell}|\gamma_{\upsilon;e}||\mathrm{d}z_{e;\upsilon}^-|\bigg)|\mathcal{D}_{\upsilon;w_e^+}|\leqslant C\Big(|\mathcal{D}_{\upsilon;w_e^+}|^2|\upsilon_e|^2+\sum_{w'\preceq w_e^-}\varepsilon_{\upsilon;w'}^2\Big),\qquad \ell\geqslant 0,$$

$$\bigg(\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}|z_{e;\upsilon}^+|^2\big(|\upsilon_e|+|z_{e;\upsilon}^-|\big)^{\ell}|\gamma_{\upsilon;e}||\mathrm{d}z_{e;\upsilon}^-|\bigg)\varepsilon_{\upsilon;w_e^+}\leqslant C\Big(\varepsilon_{\upsilon;w_e^+}^2|\upsilon_e|^4+\sum_{w'\preceq w_e^-}\varepsilon_{\upsilon;w'}^2\Big),\qquad \ell\geqslant 1.$$

In the above estimates, C refers to a sufficiently large constant.

By (5.12), (2.44), and (M3) in Proposition 3.1, there exists $C\in\mathbb{R}$ such that

$$\begin{split}|N_\upsilon(\xi_\upsilon)|_x\leqslant C\bigg(&\varepsilon_{\upsilon;w_e^+}^2|z_{e;\upsilon}^+(x)|^3+\Big(|\mathcal{D}_{\upsilon;w_e^+}||z_{e;\upsilon}^+(x)|+\varepsilon_{\upsilon;w_e^+}|z_{e;\upsilon}^+(x)|^2+\sum_{w'\prec w_e^+}\varepsilon_{\upsilon;w'}\Big)\gamma_{\upsilon;e}\\&+\Big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}|z_{e;\upsilon}^+(x)|\Big)|\mathrm{d}_x\widetilde{q}_{\upsilon;e}|\sum_{w'\prec w_e^+}\varepsilon_{\upsilon;w'}+|\mathcal{D}_{\upsilon;w_e^+}|\Big(|\mathcal{D}_{\upsilon;w_e^+}|+\varepsilon_{\upsilon;w_e^+}|\mathrm{d}_x\widetilde{q}_{\upsilon;e}|\Big)|z_{e;\upsilon}^+(x)|^2\bigg)\end{split}\tag{5.37}$$

for all $\upsilon\in\Delta$, $x\in\Sigma_{\upsilon;e}(8\delta_{\bar\partial})$, and $e\in\widetilde{\mathrm{Edg}}_{0;P}^{c}$.

Combining (5.37), (R1) in Corollary 4.5, (4.19), and the bounds in the paragraph before (5.37), we find that

$$\begin{split}\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}\big|N_\upsilon(\xi_\upsilon)\big|\big|R_\upsilon\psi\big|\leqslant& C\big(|\mathcal{D}_{\upsilon;w_e^+}|^2+\varepsilon_{\upsilon;w_e^+}^2\big)\bigg(\frac{|\rho_e(\upsilon)|}{|\rho_{w_{\hat e}^-}(\upsilon)|}\big|c_{\hat e;\upsilon}^{(0)}(\psi)\big|+|\rho_e^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^{\bullet}(\psi;w_e^+)\bigg)\\&+C|\mathcal{D}_{\upsilon;w_e^+}|^2|\rho_e^2(\upsilon)||\rho_{w_e^-}(\upsilon)|\|\psi\|\\&+C\bigg(\frac{|\rho_{w_e^-}(\upsilon)|}{|\rho_{w_{e(w_e^-)}^-}(\upsilon)|}\big|c_{e(w_e^-);\upsilon}^{(0)}(\psi)\big|+|\rho_{w_e^-}^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^{\bullet}(\psi;w_e^-)\bigg)\sum_{w'\preceq w_e^-}\varepsilon_{\upsilon;w'}^2\end{split}\tag{5.38}$$

for all $e\in\widetilde{\mathrm{Edg}}_{0;P}^{c}$. Along with (3.15) and the fact that

$$|\rho_{w_e^-}(\upsilon)|\leqslant|\rho_{\langle w^\star,w_e^+\rangle}(\upsilon)|\qquad\forall\ e\in\widetilde{\mathrm{Edg}}_{0;P}^{c}\ \ \text{s.t.}\ \ w_e^+\not\preceq w^\star,$$

this gives

$$\begin{split}\int_{\Sigma_{\upsilon;e}(\delta_{\bar\partial})}\big|N_\upsilon(\xi_\upsilon)\big|\big|R_\upsilon\psi\big|\leqslant& C\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ e'\succ e}}\frac{|\rho_{e'}^2(\upsilon)|}{|\rho_e^2(\upsilon)|}\bigg(\frac{|\rho_e(\upsilon)|}{|\rho_{w_{\hat e}^-}(\upsilon)|}\big|c_{\hat e;\upsilon}^{(0)}(\psi)\big|+|\rho_e^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^{\bullet}(\psi;w_e^+)\bigg)\\&+C\bigg(\Big(\frac{|\rho_{w_e^-}(\upsilon)|}{|\rho_{w_{e(w_e^-)}^-}(\upsilon)|}\big|c_{e(w_e^-);\upsilon}^{(0)}(\psi)\big|+|\rho_{w_e^-}^2(\upsilon)|\widetilde{C}_{w^\star;\upsilon}^{\bullet}(\psi;w_e^-)\Big)\sum_{w'\preceq w_e^-}\varepsilon_{\upsilon;w'}^2\bigg)\end{split}\tag{5.39}$$

for all $e\in\widetilde{\mathrm{Edg}}^c_{0;P}$ with $w_e^+\not\le w^\star$. Along with (5.22) and (5.23), (5.38) gives

$$\begin{aligned}\int_{\Sigma_{v;e}(\delta_{\bar\partial})}\big|N_v(\xi_v)\big|\big|R_v\psi\big| &\leqslant C\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ e'\not> e}}\frac{|\rho^2_{e'}(v)|}{|\rho^2_e(v)|}\bigg(\frac{|\rho_e(v)|}{|\rho_{w^-_{\hat e}}(v)|}\big|c^{(0)}_{\hat e;v}(\psi)\big|+|\rho_e(v)|^2\widetilde{C}^\bullet_{w^\star;v}(\psi;w_e^-)\bigg)\\ &\quad+C\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ e'> e}}\frac{|\rho^2_{\langle w^\star,w^+_{e'}\rangle}(v)||\rho^2_{e'}(v)|}{|\rho^4_e(v)|}\bigg(\frac{|\rho_e(v)|}{|\rho_{w^-_{\hat e}}(v)|}\big|c^{(0)}_{\hat e;v}(\psi)\big|+|\rho_e(v)|^2\widetilde{C}^\bullet_{w^\star;v}(\psi;w_e^+)\bigg)\\ &\quad+C\bigg(\Big(\frac{|\rho_{w_e^-}(v)|}{|\rho_{w^-_{e(w_e^-)}}(v)|}\big|c^{(0)}_{e(w_e^-);v}(\psi)\big|+|\rho^2_{w_e^-}(v)|\widetilde{C}^\bullet_{w^\star;v}(\psi;w_e^-)\Big)\sum_{w'\preceq w_e^-}\varepsilon^2_{v;w'}\bigg)\end{aligned}\tag{5.40}$$

for all $e\in\widetilde{\mathrm{Edg}}^c_{0;P}$ with $w_e^+\preceq w^\star$. Combining (5.39) for $w_e^+\not\le w^\star$ and (5.40) for $w_e^+\preceq w^\star$ with Corollary 4.10 and (5.34), we obtain

$$\begin{aligned}&\sum_{e\in\widetilde{\mathrm{Edg}}^c_{0;P}}\int_{\Sigma_{v;e}(\delta_{\bar\partial})}\big|N_v(\xi_v)\big|\big|R_v\psi\big|\\ &\qquad\leqslant C\sum_{e\in\chi(\mathcal{T}_0)}\bigg(\Big(\sum_{e'\in\chi(\mathcal{T}_0)}\frac{|\rho^2_{e'}(v)|}{|\rho^2_{e^\bullet}(v)|}\Big)|v_e|^2\big|c^{(0)}_{e;v}(\psi)\big|+|\rho^2_e(v)|\widetilde{C}^\bullet_{w^\star;v}(\psi;w_e^+)\bigg)\end{aligned}\tag{5.41}$$

for some $C\in\mathbb{R}$. The claim follows from (5.13), (5.36), and (5.41). □

5.3 Proof of Proposition 1.3

For the purpose of establishing Propositions 1.3 and 1.4, it is sufficient to assume that $[\mathbf{u}]$ and $[\mathbf{u}_r]$ are so that

(A1) $[\mathbf{u}]$ is an element of $\overline{\mathfrak{M}}_g(X,A;J)$ of the bubble type $\mathscr{T}=(\mathcal{T},\mathfrak{d})$ of genus $g\in\mathbb{Z}^+$,

(A2) $\{[\mathbf{u}_r]\}_{r=1}^\infty$ is a sequence of elements of $\overline{\mathfrak{M}}_g(X,A;J)$ converging to $[\mathbf{u}]$, and

(A3) all $[\mathbf{u}_r]$'s are of the same bubble type $\mathscr{T}^\dagger=(\mathcal{T}^\dagger,\mathfrak{d}^\dagger)$ with

$$\mathcal{T}^\dagger=\mathcal{T}^\dagger_P\equiv\big(\mathrm{Ver}^\dagger,\mathrm{Edg}^\dagger,S^\dagger=\varnothing,\mathfrak{g}^\dagger,\mathfrak{m}^\dagger\big)\qquad\text{and}\qquad \mathrm{PC}(\mathscr{T}^\dagger)=\varnothing.$$

By (A3), every contracted subgraph $\mathcal{T}^\dagger_0$ of $\mathscr{T}^\dagger$ satisfies $g_a(\mathcal{T}^\dagger_0)=0$. An example of such $\mathscr{T}^\dagger$ is the last diagram on the first row in Figure 3 with one of the two irreducible components forming a maximal contracted subgraph. By the stability condition on $[\mathbf{u}_r]$'s, $\mathcal{T}^\dagger_0$ does not correspond to a contracted chain of spheres (with only 2 points shared with other components of the domain), due to the stability conditions on $[\mathbf{u}_r]$'s.

For every maximal contracted subgraph $\mathcal{T}_0$ of $\mathscr{T}$, let

$$\mathcal{D}_e\mathbf{u}'=\mathcal{D}_{x_e^+(\mathbf{u}')}u'=\mathrm{d}_{x_e^+(\mathbf{u}')}u',\quad \mathcal{D}^{(m)}_e\mathbf{u}'=\mathcal{D}^{(m)}_{x_e^+(\mathbf{u}')}u'\qquad\forall\ e\in\chi(\mathcal{T}_0),\ \mathbf{u}'\in\widetilde{U},\ m\in\mathbb{Z}^+;$$

see (1.4) for notation. Lemma 5.11 below is the main statement of this section. By (1.10), it implies Proposition 1.3.

Lemma 5.10. *Suppose $g=1$. If $[\mathbf{u}]$ and all $[\mathbf{u}_r]$'s satisfy (A1)-(A3), then*

$$\mathfrak{c}\Big(\big\{\mathcal{D}_e\mathbf{u}\colon e\in\chi(\mathcal{T}_0)\big\}\Big)=1 \qquad \forall\ \mathcal{T}_0\in\mathrm{PC}(\mathscr{T}).$$

Proof. By (A3), $\Sigma_{\mathbf{u}_r}$ either is a smooth genus 1 curve or consists of a circle of spheres. The reasoning after (A3) implies that the restriction of u_r to each irreducible component of $\Sigma_{\mathbf{u}_r}$ is non-constant. Lemma 5.10 then follows from [18, Propositions 5.3, 5.2]. □

Lemma 5.11. *Suppose $g=2$ and $\mathscr{T}$ is of Type 1, 1a, or 1b. If $[\mathbf{u}]$ and all $[\mathbf{u}_r]$'s satisfy (A1)-(A3), then*

$$\mathfrak{c}\Big(\big\{\mathcal{D}_e\mathbf{u}\colon e\in\chi(\mathcal{T}_0)\big\}\Big)=1 \qquad \forall\ \mathcal{T}_0\in\mathrm{PC}(\mathscr{T}). \tag{5.42}$$

Proof. Fix $\mathcal{T}_0\in\mathrm{PC}(\mathscr{T})$. Since $\mathscr{T}$ is of Type 1, $1a$, or $1b$, $g_a(\mathcal{T}_0)=1$.

Suppose $\mathrm{Edg}^\dagger$ contains a separating edge e, i.e. $\mathcal{T}^\dagger$ is represented by one of the diagrams in the second row in Figure 3. The limiting map $\mathbf{u}$ then has a distinguished separating node; see the second diagram in the first row in Figure 1. Since $g_a(\mathcal{T}^\dagger)=2$, the removal of e from $\mathcal{T}^\dagger$ gives two genus 1 curves consisting of irreducible components of $\Sigma_{\mathbf{u}_r}$. One of these two curves converges to a genus 1 curve containing all irreducible components of $\Sigma_{\mathbf{u};0}$. We denote the corresponding stable maps by $[\tilde{\mathbf{u}}_r]$ and $[\tilde{\mathbf{u}}]$. Since these maps satisfy the assumptions of Lemma 5.10, we obtain (5.42).

Suppose $\mathrm{Edg}^\dagger$ contains a non-separating edge e so that the node $x_e(\mathbf{u}_r)$ converges to some node $x_e(\mathbf{u})\in\Sigma_{\mathbf{u}}$ which is not a principal node of $\Sigma_{\mathbf{u};0}$, i.e. a node not intrinsic to the principal curve $\Sigma_{\mathbf{u};0;P}$ of $\Sigma_{\mathbf{u};0}$. In this case, $\mathcal{T}^\dagger$ corresponds to either one of the last three diagrams in the top row of Figure 3 or one of the last two diagrams in the bottom row. The removal of e from $\mathcal{T}^\dagger$ determines a sequence of genus 1 maps $[\tilde{\mathbf{u}}_r]$ converging to a genus 1 map $[\tilde{\mathbf{u}}]$ with a bubble type $(\widetilde{\mathcal{T}},\tilde{\mathfrak{d}})$. Since $x_e(\mathbf{u})$ is not a node of $\Sigma_{\mathbf{u};0;P}$, $\mathcal{T}_{0;P}$ is contained in a primary contracted graph $\widetilde{\mathcal{T}}_0$ of $\widetilde{\mathcal{T}}$. Thus, $\chi(\widetilde{\mathcal{T}}_0)$ is non-empty and can be identified with a subset of $\chi(\mathcal{T}_0)$. Since $[\tilde{\mathbf{u}}_r]$ and $[\tilde{\mathbf{u}}]$ satisfy the assumptions of Lemma 5.10, this establishes the statement of Lemma 5.11.

Suppose $\mathrm{Edg}^\dagger$ contains no edges of the first two kinds. We can then assume that

(A4) every $e\in\mathrm{Edg}^\dagger$ is a non-separating edge and the corresponding node $x_e(\mathbf{u}_r)$ converges to a principal node $x_e(\mathbf{u})$ of $\Sigma_{\mathbf{u};0}$.

In Section 2.2, $\mathcal{T}_0$ denotes an arbitrary contracted subgraph, not necessarily a *primary* contracted subgraph. We apply the setup of Section 2.2 with $\mathcal{T}_0$ chosen to be the *principal* subgraph $\mathcal{T}_{0;P}$ of $\mathcal{T}_0$ fixed above. The subbundles in (2.15) then become

$$\widetilde{\mathcal{FT}}_0=\widetilde{\mathcal{FT}}\big(\mathrm{Edg}_{0;P}\big),\quad \widetilde{\mathcal{FT}}_1=\widetilde{\mathcal{FT}}\Big(\mathrm{Edg}-\big(\mathrm{Edg}_{0;P}\cup\chi(\mathcal{T}_{0;P})\big)\Big),\quad \widetilde{\mathcal{FT}}_{01}=\widetilde{\mathcal{FT}}\big(\mathrm{Edg}-\chi(\mathcal{T}_{0;P})\big).$$

Following (3.54), let

$$\Delta_2=\Delta\cap\widetilde{\mathcal{FT}}_1, \qquad \Delta_2^{\varnothing}=\Delta^{\varnothing}\cap\widetilde{\mathcal{FT}}_1$$

and denote the projections of $\upsilon\in\widetilde{\mathcal{FT}}$ to $\widetilde{\mathcal{FT}}_1$ and $\widetilde{\mathcal{FT}}_{01}$ by υ_2 and υ_{02}, respectively. Section 2.2 provides a modified gluing map

$$\tilde{q}_\upsilon=\tilde{q}_{\upsilon;\chi(\mathcal{T}_{0;P})}\colon\Sigma_\upsilon\longrightarrow\Sigma_{\upsilon_{02}}\equiv\Sigma_{\upsilon_{01}} \qquad \forall\,\upsilon\in\Delta$$

and two continuous families of nearly J-holomorphic maps $\widetilde{u}_{\upsilon;2}\equiv\widetilde{u}_{\upsilon;1}$ and

$$u_\upsilon=\widetilde{u}_{\upsilon;2}\circ q_{\upsilon_2;\upsilon_{02}}\circ\widetilde{q}_\upsilon\colon\Sigma_\upsilon\longrightarrow X;$$

see (2.41), (2.30), and (2.42), respectively. Let $\widetilde{\xi}_{\upsilon;2}\equiv\widetilde{\xi}_{\upsilon;1}$ be as in (2.32) and $\widetilde{\rho}_e(\upsilon)$ and $\rho_e(\upsilon)$ be as in (3.10).

By the proof of the $r=1$ case of [18, Lemma 3.5(2a)], there exist continuous functions

$$\begin{gathered}\breve{\varepsilon}_{e'}\colon(\Delta,\widetilde{U})\longrightarrow(\mathrm{ev}_0^*TX,0),\qquad e'\in\chi(\mathcal{T}_0)\qquad\text{s.t.}\\ \mathrm{d}_{x_e^+(\upsilon_2)}\widetilde{u}_{\upsilon;2}\big(\tfrac{\partial}{\partial z_{e;\upsilon_2}^+}\big)=\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ \langle e'\rangle=e}}\Big(\mathrm{d}_{x_{e'}^+(\mathbf{u}')}u'\big(\tfrac{\partial}{\partial z_{e';\mathbf{u}'}^+}\big)+\breve{\varepsilon}_{e'}(\upsilon)\Big)\widetilde{\rho}_{e'}(\upsilon)\end{gathered}\tag{5.43}$$

for all $e\in\chi(\mathcal{T}_{0;P})$, $\mathbf{u}'\in\widetilde{U}$, and $\upsilon\in\Delta|_{\mathbf{u}'}$. By [18, Corollary 3.8(2b),(2c)], there exists $C\in\mathbb{R}$ such that

$$\big\|\mathrm{d}\widetilde{\xi}_{\upsilon;2}|_{\Sigma_{\upsilon_2;e}(8\delta_{\bar\partial})}\big\|_{\upsilon_2,p}\leqslant C\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ \langle e'\rangle=e}}|\widetilde{\rho}_{e'}(\upsilon)|,\quad\big|\widetilde{\xi}_{\upsilon;2}(x)\big|\leqslant C\big|z_{e;\upsilon_2}^+(x)\big|\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ \langle e'\rangle=e}}|\widetilde{\rho}_{e'}(\upsilon)|\tag{5.44}$$

for all $e\in\chi(\mathcal{T}_{0;P})$ and $x\in\Sigma_{\upsilon_2;e}^+(8\delta_{\bar\partial})$. By Lemma 2.2 with $(f,\widetilde{f})$ replaced by $(u_\upsilon,\widetilde{u}_{\upsilon;2})$, Lemma A.8, (5.43), and the first inequality in (5.44),

$$\big\|\bar\partial_J u_\upsilon|_{\widetilde{A}_{e,\upsilon}^-(\delta_{\bar\partial})}\big\|_{\upsilon,p}\leqslant C\sum_{\substack{e'\in\chi(\mathcal{T}_0)\\ \langle e'\rangle=e}}|\rho_{e'}(\upsilon)|\qquad\forall\,e\in\chi(\mathcal{T}_{0;P})\tag{5.45}$$

for some $C\in\mathbb{R}$.

With $\Gamma_-(\mathbf{u}')$ as in (2.21) and $\zeta_{\upsilon;1}$ as in (2.31), we define

$$\Gamma_-(\upsilon)=\Big\{\Big(\Pi_{\zeta_{\upsilon;1}}\big(\xi\circ q_{\upsilon_1}\big)\Big)\circ\widetilde{q}_\upsilon\colon\,\xi\in\Gamma_-(\mathbf{u}')\Big\}\subset\Gamma(\upsilon)\qquad\forall\,\mathbf{u}'\in\widetilde{U},\,\upsilon\in\Delta|_{\mathbf{u}'}.$$

Denote by $\Gamma_+(\upsilon)$ the L^2-complement of $\Gamma_-(\upsilon)$ in $\Gamma(\upsilon)$. Similar to [15, Lemma 3.16(1)],

$$\|\zeta\|_{\upsilon,p,1}\leqslant C\|D_\upsilon\zeta\|_{\upsilon,p}\qquad\forall\;\upsilon\in\Delta,\;\zeta\in\Gamma_+(\upsilon)\tag{5.46}$$

for some $C\in\mathbb{R}$.

Since $[\mathbf{u}_r]$ converges to $[\mathbf{u}]$, for sufficiently large r there exist $\upsilon_r\in\Delta$ with $\upsilon_r\to 0$ and small $\widehat{\zeta}_{\upsilon_r;2}\in\Gamma_+(\upsilon_r)$ such that $[\mathbf{u}_r]$ is represented by

$$u_r=\exp_{u_{\upsilon_r}}\widehat{\zeta}_{\upsilon_r;2}\,:\,\Sigma_{\upsilon_r}\longrightarrow X.$$

Since every u_r is J-holomorphic, (5.29) with $\widehat{u}_{\upsilon;2}=u_{\upsilon_r}$ holds. By the same reasoning as in the proof of (3.77), there exists a constant $C\in\mathbb{R}$ such that

$$\|\widehat{\zeta}_{\upsilon_r;2}\|_{\upsilon_r,p,1}\leqslant C\sum_{e\in\chi(\mathcal{T}_{0;P})}\big\|\bar\partial_J u_{\upsilon_r}|_{\widetilde{A}_{e,\upsilon_r}^-(\delta_{\bar\partial})}\big\|_{\upsilon,p}.\tag{5.47}$$

Combining (5.47) with (5.45), we obtain

$$\|\widehat{\zeta}_{\upsilon_r;2}\|_{\upsilon_r,p,1} \leqslant C \sum_{e\in\chi(\mathcal{T}_0)} |\rho_e(\upsilon)| \tag{5.48}$$

for some $C\in\mathbb{R}$.

By the $\ell=1$ case of Lemma 5.1 with Θ_υ replaced by $\Theta_{P;\upsilon}$, (5.43), and (5.44), there exist $C,C'\in\mathbb{R}$ such that

$$\begin{aligned}
&\Big|\big\{\Theta_{P;\upsilon_r}(\bar{\partial}_J u_{\upsilon_r})\big\}\psi - \sum_{e\in\chi(\mathcal{T}_{0;P})} \mathscr{D}_{\upsilon_r;e}\psi\Big| \\
&\leqslant C \sum_{e\in\chi(\mathcal{T}_{0;P})} \|\psi\| |\upsilon_{r;e}|^2 \Big(\Big|\mathrm{d}_{x_e^+(\upsilon_{r;2})} \widetilde{u}_{\upsilon_r;2}\big(\tfrac{\partial}{\partial z^+_{e;\upsilon_{r;2}}}\big)\Big| + \big\|\widetilde{\xi}_{\upsilon_r;2}|_{\Sigma_{\upsilon_{r;2};e}(8\delta_{\bar{\partial}})}\big\|_{\upsilon_{r;2},p,1}\Big) \leqslant C' \|\psi\| |\upsilon_r| \sum_{e'\in\chi(\mathcal{T}_0)} |\rho_{e'}(\upsilon_r)|
\end{aligned} \tag{5.49}$$

for all $\psi\in\mathbb{E}_0|_{\mathbf{u}'_r}$. By Lemma 5.2 with Θ_υ replaced by $\Theta_{P;\upsilon}$, (2.22), and (5.48),

$$\big\|\Theta_{P;\upsilon_r}(D_{\upsilon_r}\widehat{\zeta}_{\upsilon_r;2})\big\| \leqslant C \|\widehat{\zeta}_{\upsilon_r;2}\|_{\upsilon_r,p,1} |\upsilon_r| \leqslant C' |\upsilon_r| \sum_{e'\in\chi(\mathcal{T}_0)} |\rho_{e'}(\upsilon_r)|. \tag{5.50}$$

By (5.3) with Θ_υ replaced by $\Theta_{P;\upsilon}$, (5.11), and (5.48),

$$\big\|\Theta_{P;\upsilon_r}\big(N_{\upsilon_r}\big(\widehat{\zeta}_{\upsilon_r;2}\big)\big)\big\| \leqslant C \|\widehat{\zeta}_{\upsilon_r;2}\|^2_{\upsilon_r,p,1} \leqslant C' \sum_{e'\in\chi(\mathcal{T}_0)} |\rho_{e'}(\upsilon_r)|^2. \tag{5.51}$$

By (5.29) for $\widehat{u}_{\upsilon;2} = u_{\upsilon_r}$ and (5.49)-(5.51), there exists $C\in\mathbb{R}$ such that

$$\Big|\sum_{e\in\chi(\mathcal{T}_{0;P})} \mathscr{D}_{\upsilon_r;e}\psi\Big| \leqslant C \|\psi\| |\upsilon_r| \sum_{e'\in\chi(\mathcal{T}_0)} |\rho_{e'}(\upsilon_r)| \qquad \forall\, \psi\in\mathbb{E}_0|_{\mathbf{u}'_r}.$$

By the $\ell=1$ case of (5.5), this gives

$$\Big|\sum_{e\in\chi(\mathcal{T}_{0;P})} \upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\psi) \Big(\mathrm{d}_{x_e^+(\upsilon_{r;2})} \widetilde{u}_{\upsilon_r;2}\big(\tfrac{\partial}{\partial z^+_{e;\upsilon_{r;2}}}\big)\Big)\Big| \leqslant C \|\psi\| |\upsilon_r| \sum_{e'\in\chi(\mathcal{T}_0)} |\rho_{e'}(\upsilon_r)| \qquad \forall\, \psi\in\mathbb{E}_0|_{\mathbf{u}'_r}. \tag{5.52}$$

By (5.43) and (5.52), there exists a continuous function $\epsilon\colon (\Delta,\widetilde{U}) \longrightarrow (\mathbb{R},0)$ such that

$$\Big|\sum_{e'\in\chi(\mathcal{T}_0)} \rho_{e'}(\upsilon_r)\, c^{(0)}_{\langle e'\rangle;\upsilon_r}(\psi) \Big(\mathcal{D}_{e'}\mathbf{u}'_r\big(\tfrac{\partial}{\partial z^+_{e';\mathbf{u}'_r}}\big)\Big)\Big| \leqslant \|\psi\|\, \epsilon(\upsilon_r) \sum_{e'\in\chi(\mathcal{T}_0)} |\rho_{e'}(\upsilon_r)| \qquad \forall\, \psi\in\mathbb{E}_0|_{\mathbf{u}'_r}.$$

By the assumption (A4), $\rho_{e'}(\upsilon_r) \neq 0$ for all $e' \in \chi(\mathcal{T}_0)$. Since $g_a(\mathcal{T}_{0;P}) = 1$, there exists $\widetilde{C} \in \mathbb{R}^+$ such that

$$\|\psi\| \leqslant \widetilde{C} |c^{(0)}_{e;\upsilon}(\psi)| \qquad \forall\, \mathbf{u}'\in\widetilde{U},\ \psi\in\mathbb{E}_0|_{\mathbf{u}'},\ \upsilon\in\Delta|_{\mathbf{u}'},\ e\in\chi(\mathcal{T}_{0;P}).$$

Since $\mathrm{rk}(\mathbb{E}_0)\geqslant 0$, $|\upsilon_r|\to 0$, and $\mathbf{u}'_r\to\mathbf{u}$, the last two inequalities establish (5.42). □

6 On the proof of Proposition 1.4

We continue with the setup of Section 5.3 and the assumptions (A1)-(A3). We now also assume that $g=2$ and that $\mathscr{T}$ is of Type 2. Thus, $\mathrm{PC}(\mathscr{T})$ contains a unique element $\mathcal{T}_0$ with $g_a(\mathcal{T}_0)=2$. The triple $(\mathscr{T},\mathcal{T}_0,\mathcal{T}_{0;P})$ forms a pseudo-tree framing, and the pair $(\mathscr{T}_e,\mathcal{T}_{e;0})$ with $e\in\chi(\mathcal{T}_{0;P})$ forms a tree framing; see Section 3.1 for the notation and terminology.

Suppose $\mathrm{Edg}^\dagger$ contains a separating edge e. The removal of e from $\mathcal{T}^\dagger$ then determines two sequences $\tilde{\mathbf{u}}_r^{(1)}$ and $\tilde{\mathbf{u}}_r^{(2)}$ of genus 1 maps with one marked point each that converge to genus 1 maps $\tilde{\mathbf{u}}^{(1)}$ and $\tilde{\mathbf{u}}^{(2)}$ of bubble types $\mathscr{T}^{(1)}$ and $\mathscr{T}^{(2)}$. The latter maps are obtained by removing the node corresponding to e from $\Sigma_{\mathbf{u}}$. Thus, $\mathrm{PC}(\mathscr{T}^{(1)})$ and $\mathrm{PC}(\mathscr{T}^{(2)})$ consist of one element $\mathcal{T}_0^{(1)}$ and $\mathcal{T}_0^{(2)}$ each and

$$\chi(\mathcal{T}_0)=\chi\big(\mathcal{T}_0^{(1)}\big)\sqcup\chi\big(\mathcal{T}_0^{(2)}\big),\qquad \chi\big(\mathcal{T}_0^{(1)}\big),\,\chi\big(\mathcal{T}_0^{(2)}\big)\neq 0,\qquad \mathbb{E}|_{\mathbf{u}}=\mathbb{E}|_{\tilde{\mathbf{u}}^{(1)}}\oplus\mathbb{E}|_{\tilde{\mathbf{u}}^{(2)}}.$$

Applying Lemma 5.10 to $\tilde{\mathbf{u}}_r^{(1)}$ and $\tilde{\mathbf{u}}_r^{(2)}$ each, we obtain

$$\mathfrak{c}\Big(\big\{\mathcal{D}_e\mathbf{u}\colon\, e\in\chi\big(\mathcal{T}_0^{(i)}\big)\big\}\Big)=1\qquad i=1,2.$$

This implies that

$$[\mathbf{u}]\in\bigcup_{\ell\geqslant 2}\iota_\ell(\mathcal{Z}_{2;\ell})\subset\mathfrak{M}^0_{\tau 2};\tag{6.1}$$

see (1.8) for the notation.

In the remainder of this section, we assume that $\mathrm{Edg}^\dagger$ contains no separating edge and thus $[\mathbf{u}]$ and $[\mathbf{u}_r]$ satisfy the assumptions (A1)-(A4). Since $[\mathbf{u}_r]$ converges to $[\mathbf{u}]$, for sufficiently large r there exists a sequence $\{\upsilon_r\}$ of elements of Δ such that $\upsilon_r\to 0$ and every $[\mathbf{u}_r]$ can be represented by a J-holomorphic map

$$\mathbf{u}_r=\big(\Sigma_{\upsilon_r},\,\tilde{u}_{\upsilon_r}=\exp_{u_{\upsilon_r}}\xi_{\upsilon_r}:\Sigma_{\upsilon_r}\longrightarrow X\big)\tag{6.2}$$

as in (3.6). By (A3),

$$\upsilon_{r;e}\neq 0\qquad\forall\,e\in\mathrm{Edg}-\mathrm{Edg}_{0;P}.\tag{6.3}$$

Replacing υ_r by a subsequence if necessary, we can assume there exists a non-empty subset $\chi^\bullet$ of $\chi(\mathcal{T}_0)$ such that

$$\lim_{r\to\infty}\frac{\rho_{e'}(\upsilon_r)}{\rho_e(\upsilon_r)}\in\mathbb{C}^*,\quad \lim_{r\to\infty}\frac{\rho_{e''}(\upsilon_r)}{\rho_e(\upsilon_r)}=0\qquad\forall\,e,e'\in\chi^\bullet,\ e''\in\chi(\mathcal{T}_0)-\chi^\bullet.\tag{6.4}$$

The elements of $\chi^\bullet$ are called dominant directions. Let

$$\chi_\bullet=\big\{\langle e\rangle\colon\, e\in\chi^\bullet\big\}\subset\chi(\mathcal{T}_{0;P}).$$

Along with (1.11) and (6.1), the next four propositions imply Proposition 1.4. If $[\mathbf{u}]$ is as in Propositions 6.1 or 6.2, then

$$[\mathbf{u}]\in\iota_1\big(\overline{\mathcal{M}}_{2,1}\times\mathcal{Z}^{(2)}_{0;1;1}\cup\overline{\mathcal{W}}_{2,1}\times\mathcal{Z}^{(3)}_{0;1;1}\big)\qquad\text{if }\ |\chi(\mathcal{T}_0)|=1;$$
$$[\mathbf{u}]\in\iota_\ell\big(\overline{\mathcal{M}}_{2,\ell}\times\mathcal{Z}^{(2)}_{0;\ell;1}\cup\overline{\mathcal{W}}_{2,\ell}\times\mathcal{Z}^{(3)}_{0;\ell;1}\cup\mathcal{Z}_{2;\ell}\big)\qquad\text{if }\ \ell\equiv|\chi(\mathcal{T}_0)|\geqslant 2.$$

If $[\mathbf{u}]$ is as in Propositions 6.3 or 6.4, then

$$[\mathbf{u}]\in\iota_\ell\big(\mathcal{Z}_{2;\ell}\cup\overline{\mathcal{C}}_{2,\ell}\times\mathcal{Z}_{0;\ell;12}^{(2)}\big)$$

with $\ell=|\chi(\mathcal{T}_0)|$.

Proposition 6.1. *If $\chi^\bullet=\{e\}$ and $[\Sigma_{\mathbf{u};0};x_e(\mathbf{u})]$ is not in $\overline{\mathcal{W}}_{2,1}$, then either $[\mathbf{u}]$ satisfies (6.1) or*

$$\mathcal{D}_e\mathbf{u}=0,\qquad \mathfrak{c}\Big(\big\{\mathcal{D}_{e'}\mathbf{u}\colon\, e'\in\chi(\mathcal{T}_0)\big\}\cup\{\mathcal{D}_e^{(2)}\mathbf{u}\}\Big)=2.$$

Proposition 6.2. *If $\chi^\bullet=\{e\}$ and $[\Sigma_{\mathbf{u};0};x_e(\mathbf{u})]$ is in $\overline{\mathcal{W}}_{2,1}$, then*

$$\mathcal{D}_e\mathbf{u}=0,\qquad \mathfrak{c}\Big(\big\{\mathcal{D}_{e'}\mathbf{u}\colon\, e'\in\chi(\mathcal{T}_0)\big\}\cup\{\mathcal{D}_e^{(2)}\mathbf{u},\,\mathcal{D}_e^{(3)}\mathbf{u}\}\Big)=2.$$

Proposition 6.3. *If either $|\chi^\bullet|\geqslant 3$ or*

$$\chi^\bullet=\{e_1,e_2\}\qquad\text{and}\qquad\big[\Sigma_{\mathbf{u};0};x_{e_1}(\mathbf{u}),x_{e_2}(\mathbf{u})\big]\notin\overline{\mathcal{C}}_{2,2},$$

then $[\mathbf{u}]$ satisfies (6.1).

Proposition 6.4. *If $\chi^\bullet=\{e_1,e_2\}$ and $[\Sigma_{\mathbf{u};0};x_{e_1}(\mathbf{u}),x_{e_2}(\mathbf{u})]\in\overline{\mathcal{C}}_{2,2}$, then either $[\mathbf{u}]$ satisfies (6.1) or*

$$\mathfrak{c}\big(D_{e_1}\mathbf{u},D_{e_2}\mathbf{u}\big)=1,\qquad \mathfrak{c}\Big(\big\{\mathcal{D}_{e'}\mathbf{u}\colon\, e'\in\chi(\mathcal{T}_0)\big\}\cup\big\{\big(\mathcal{D}_{e_1}^{(2)}\mathbf{u}+\mathcal{D}_{e_2}^{(2)}\mathbf{u}\big)|_{(\ker\{\mathcal{D}_{e_1}\mathbf{u}+\mathcal{D}_{e_2}\mathbf{u}\})^{\otimes 2}}\big\}\Big)=2.$$

Denote by $\Sigma_{\mathbf{u};0;P}^\bullet$ the minimal connected union of the irreducible components of $\Sigma_{\mathbf{u};0;P}$ containing all $x_e(\mathbf{u})$ with $e\in\chi_\bullet$ and by $\Sigma_{\mathbf{u};0;P}^{\bullet;c}$ the complement of $\Sigma_{\mathbf{u};0;P}^\bullet$ in $\Sigma_{\mathbf{u};0;P}$. Let $g_a^c(\mathbf{u})$ be the sum of the arithmetic genera of the topological components of $\Sigma_{\mathbf{u};0;P}^{\bullet;c}$. There are four primary possibilities for $\Sigma_{\mathbf{u};0;P}^\bullet$ that are similar to the five types in Figure 1:

$$\begin{aligned}
&\text{Case }(0)\colon\; g_a^c(\mathbf{u})=0, &&\text{Case }(1a)\colon\; g_a^c(\mathbf{u})=1,\;\; \big|\Sigma_{\mathbf{u};0;P}^\bullet\cap\Sigma_{\mathbf{u};0;P}^{\bullet;c}\big|=2,\\
&\text{Case }(2)\colon\; g_a^c(\mathbf{u})=2, &&\text{Case }(1b)\colon\; g_a^c(\mathbf{u})=1,\;\; \big|\Sigma_{\mathbf{u};0;P}^\bullet\cap\Sigma_{\mathbf{u};0;P}^{\bullet;c}\big|=1.
\end{aligned}$$

A secondary characterization for $\Sigma_{\mathbf{u};0;P}^\bullet$ is used in the proof of Proposition 6.3; see page 72.

Suppose $\psi\equiv\psi_\upsilon\in\mathbb{E}_0|_{\mathbf{u}'}-\{0\}$ for $\upsilon\in\Delta|_{\mathbf{u}'}$, $\mathbf{u}'\in\widetilde{U}$, is a continuous family such that

$$\big(\upsilon_r\,c_{e;\upsilon_r}^{(0)}(\psi)\big)_{e\in\chi(\mathcal{T}_0)}\neq 0\in\mathbb{C}^{\chi(\mathcal{T}_0)} \tag{6.5}$$

for all $r\in\mathbb{Z}^+$ sufficiently large. Then,

$$\mathcal{L}_r(\psi)\equiv\big[\upsilon_r\,c_{e;\upsilon_r}^{(0)}(\psi)\big]_{e\in\chi(\mathcal{T}_0)}\in\mathbb{P}\big(\mathbb{C}^{\chi(\mathcal{T}_0)}\big)$$

is a well-defined element. After passing to a subsequence, we can assume that $\mathcal{L}_r(\psi)$ converges to some line $\mathcal{L}(\psi)\in\mathbb{P}\big(\mathbb{C}^{\chi(\mathcal{T}_0)}\big)$. If ψ and η are two families satisfying (6.5) and $\mathcal{L}(\psi)\neq\mathcal{L}(\eta)$, we denote by

$$\mathcal{P}(\psi,\eta)\in\mathrm{Gr}\big(2;\chi(\mathcal{T}_0)\big) \tag{6.6}$$

the 2-plane generated by ψ and η.

By the assumptions (A1)-(A3), there always exists a continuous family $\psi=\psi_\upsilon\in\mathbb{E}_0$ such that (6.5) holds. If in addition

$$|\chi(\mathcal{T}_0)|\geqslant 3 \qquad \text{or} \qquad \chi(\mathcal{T}_0)=\{e_1,e_2\} \;\text{ with }\; [\Sigma_{\mathbf{u};0};x_{e_1}(\mathbf{u}),x_{e_2}(\mathbf{u})]\notin\overline{\mathcal{C}}_{2,2},$$

then (6.5) holds for every family ψ_υ with $\psi_{\mathbf{u}}\neq 0$.

Lemma 6.5. *Suppose $\psi=\psi_\upsilon\in\mathbb{E}_0|_{\mathbf{u}'}$ for $\upsilon\in\Delta|_{\mathbf{u}'}$, $\mathbf{u}'\in\widetilde{U}$, is a continuous family such that $\psi_{\mathbf{u}}|_{x^-_{e_\bullet}(\mathbf{u})}\neq 0$ for some $e_\bullet\in\chi_\bullet$. Then,*

$$\begin{gathered} a_{e^\bullet}\neq 0 \qquad \forall\,(a_e)_{e\in\chi(\mathcal{T}_0)}\in\mathcal{L}(\psi)-\{0\},\; e^\bullet\in\chi(\mathcal{T}_0)\ \text{s.t.}\ \langle e^\bullet\rangle=e_\bullet,\\ \sum_{e\in\chi(\mathcal{T}_0)}a_e\,\mathcal{D}_e\mathbf{u}\big(\tfrac{\partial}{\partial z^+_{e;\mathbf{u}}}\big)=0 \qquad \forall\,(a_e)_{e\in\chi(\mathcal{T}_0)}\in\mathcal{L}(\psi). \end{gathered} \tag{6.7}$$

Proof. By the assumption on ψ, there exists $C\in\mathbb{R}$ such that

$$\|\psi\|\leqslant C\,\big|c^{(0)}_{e_\bullet;\upsilon}(\psi)\big| \qquad \forall\,\upsilon\in\Delta. \tag{6.8}$$

Let $e^\bullet\in\chi^\bullet$ be such that $\langle e^\bullet\rangle=e_\bullet$. By (6.8) and the $k=0$ case of the second inequality of (4.13), there exist $C',C\in\mathbb{R}$ such that

$$\|\psi\||\rho_{e^\bullet}(\upsilon)|\leqslant C'\big|c^{(0)}_{e_\bullet;\upsilon}(\psi)\big||\rho_{e^\bullet}(\upsilon)|\leqslant C\,\big|\upsilon_{e^\bullet}c^{(0)}_{e^\bullet;\upsilon}(\psi)\big| \qquad \forall\,\upsilon\in\Delta. \tag{6.9}$$

By the first inequality in (4.13), (6.4), and (6.9),

$$\begin{split} \sum_{e\in\chi(\mathcal{T}_0)}\big|\upsilon_{r;e}c^{(0)}_{e;\upsilon_r}(\psi)\big|\leqslant C\|\psi\|\sum_{e\in\chi(\mathcal{T}_0)}|\rho_e(\upsilon_r)|\leqslant C'\|\psi\||\rho_{e^\bullet}(\upsilon_r)|\\ \leqslant C''\big|\upsilon_{r;e^\bullet}c^{(0)}_{e^\bullet;\upsilon_r}(\psi)\big|\leqslant C''\sum_{e\in\chi(\mathcal{T}_0)}\big|\upsilon_{r;e}c^{(0)}_{e;\upsilon_r}(\psi)\big|. \end{split} \tag{6.10}$$

This establishes the first claim.

Since every map in (6.2) is J-holomorphic, Corollary 5.4 applies. By the continuity of $\mathcal{D}_e\mathbf{u}'$ in $\mathbf{u}'\in\widetilde{U}$, it implies that there exists a continuous function $\epsilon\colon(\Delta,\widetilde{U})\longrightarrow(\mathbb{R},0)$ such that

$$\Big|\sum_{e\in\chi(\mathcal{T}_0)}\upsilon_{r;e}\,c^{(0)}_{e;\upsilon_r}(\psi)\Big(\mathcal{D}_e\mathbf{u}\big(\tfrac{\partial}{\partial z^+_{e;\mathbf{u}}}\big)\Big)\Big|\leqslant\epsilon(\upsilon_r)\sum_{e\in\chi(\mathcal{T}_0)}\Big(\big|\upsilon_{r;e}\,c^{(0)}_{e;\upsilon_r}(\psi)\big|+\|\psi\||\rho_e(\upsilon_r)|\Big). \tag{6.11}$$

The second claim follows from (6.10) and (6.11). $\square$

Lemma 6.6. *Suppose $e^\bullet,e^\circ\in\chi^\bullet$ are such that $e^\bullet\neq e^\circ$ and $\langle e^\bullet\rangle=\langle e^\circ\rangle$. If $\eta=\eta_\upsilon\in\mathbb{E}_0|_{\mathbf{u}'}$ for $\upsilon\in\Delta|_{\mathbf{u}'}$, $\mathbf{u}'\in\widetilde{U}$, is a continuous family such that*

$$c^{(0)}_{e^\bullet;\upsilon}(\eta)=0 \quad \forall\,\upsilon\in\Delta, \qquad\qquad c^{(1)}_{\langle e^\bullet\rangle;\mathbf{u}}(\eta)\neq 0, \tag{6.12}$$

then $\upsilon_{r;e^\circ}c^{(0)}_{e^\circ;\upsilon_r}(\eta)\neq 0$ for all $r\in\mathbb{Z}^+$ sufficiently large and

$$\sum_{e\in\chi(\mathcal{T}_0)}a_e\,\mathcal{D}_e\mathbf{u}\big(\tfrac{\partial}{\partial z^+_{e;\mathbf{u}}}\big)=0 \qquad \forall\,(a_e)_{e\in\chi(\mathcal{T}_0)}\in\mathcal{L}(\eta). \tag{6.13}$$

Proof. Let $e_\bullet = \langle e^\bullet \rangle$ and $w^\star = w^+_{e^\bullet}$. By Corollary 4.7 for $\psi = \eta$ and (6.3),

$$\|\eta\|,\ \widetilde{C}^\bullet_{w^\star;\upsilon_r}(\eta; w^+_e) \leqslant C|\upsilon_r| \frac{|\upsilon_{r;e^\circ} c^{(0)}_{e^\circ;\upsilon_r}(\eta)|}{|\rho^2_{e^\circ}(\upsilon_r)|} \tag{6.14}$$

for all $e \in \chi(\mathcal{T}_0)$ and $r \in \mathbb{Z}^+$ sufficiently large. Along with (6.3), this establishes the first claim.

By (6.4), there exists $C \in \mathbb{R}$ such that

$$\sum_{e\in\chi(\mathcal{T}_0)} \frac{|\rho_e(\upsilon_r)|^2}{|\rho_{e^\bullet}(\upsilon_r)|^2},\ \sum_{e\in\chi(\mathcal{T}_0)} \frac{|\rho_e(\upsilon_r)|^2}{|\rho_{e^\circ}(\upsilon_r)|^2} \leqslant C. \tag{6.15}$$

By (6.14) and (6.15),

$$\sum_{e\in\chi(\mathcal{T}_0)} |\rho^2_e(\upsilon_r)| \widetilde{C}^\bullet_{w^\star;\upsilon_r}(\eta; w^+_e) \leqslant C'|\upsilon_r| \big|\upsilon_{r;e^\circ} c^{(0)}_{e^\circ;\upsilon_r}(\eta)\big| \leqslant C'|\upsilon_r| \sum_{e\in\chi(\mathcal{T}_0)} \big|\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)\big| \tag{6.16}$$

for some $C' \in \mathbb{R}$.

Since every map in (6.2) is J-holomorphic, Corollary 5.6 applies. Along with the continuity of $\mathcal{D}_e \mathbf{u}'$ in $\mathbf{u}' \in \widetilde{U}$, it implies that there exist $C \in \mathbb{R}$ and a continuous function $\epsilon\colon (\Delta, \widetilde{U}) \longrightarrow (\mathbb{R}, 0)$ such that

$$\begin{aligned} &\Big|\sum_{e\in\chi(\mathcal{T}_0)} \upsilon_{r;e}\, c^{(0)}_{e;\upsilon_r}(\eta) \Big(\mathcal{D}_e \mathbf{u}\big(\tfrac{\partial}{\partial z^+_{e;\mathbf{u}}}\big)\Big)\Big| \\ &\qquad \leqslant C \sum_{e\in\chi(\mathcal{T}_0)} \bigg(\Big(\epsilon(\upsilon_r) + |\upsilon_{r;e}| \sum_{e'\in\chi(\mathcal{T}_0)} \tfrac{|\rho^2_{e'}(\upsilon_r)|}{|\rho^2_{e^\bullet}(\upsilon_r)|}\Big) \big|\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)\big| + |\rho^2_e(\upsilon_r)| \widetilde{C}^\bullet_{w^\star;\upsilon_r}(\eta; w^+_e)\bigg). \end{aligned} \tag{6.17}$$

By (6.15) and (6.16), there thus exist $C \in \mathbb{R}$ and continuous functions $\epsilon, \epsilon_1 : (\Delta, \widetilde{U}) \longrightarrow (\mathbb{R}, 0)$ such that

$$\begin{aligned} \Big|\sum_{e\in\chi(\mathcal{T}_0)} \upsilon_{r;e}\, c^{(0)}_{e;\upsilon_r}(\eta) \Big(\mathcal{D}_e \mathbf{u}\big(\tfrac{\partial}{\partial z^+_{e;\mathbf{u}}}\big)\Big)\Big| &\leqslant C \sum_{e\in\chi(\mathcal{T}_0)} \Big(\epsilon(\upsilon_r) \big|\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)\big| + |\rho^2_e(\upsilon_r)| \widetilde{C}^\bullet_{w^\star;\upsilon_r}(\eta; w^+_e)\Big) \\ &\leqslant \epsilon_1(\upsilon_r) \sum_{e\in\chi(\mathcal{T}_0)} \big|\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)\big|\,. \end{aligned} \tag{6.18}$$

The second claim of the lemma follows from the first claim and (6.18). □

Lemma 6.7. *Suppose $e^\bullet, e_1, e_2 \in \chi^\bullet$ are such that*

$$\big|\{e^\bullet, e_1, e_2\}\big| = 3, \qquad \langle e^\bullet \rangle = \langle e_1 \rangle = \langle e_2 \rangle, \qquad \langle e_1, e_2 \rangle \succeq \langle e^\bullet, e_1 \rangle = \langle e^\bullet, e_2 \rangle. \tag{6.19}$$

If $\eta = \eta_\upsilon \in \mathbb{E}_0|_{\mathbf{u}'}$ for $\upsilon \in \Delta|_{\mathbf{u}'}$, $\mathbf{u}' \in \widetilde{U}$, is a continuous family such that

$$c^{(0)}_{e^\bullet;\upsilon}(\eta) = 0 \quad \forall\, \upsilon \in \Delta, \qquad c^{(1)}_{\langle e^\bullet \rangle;\mathbf{u}}(\eta) = 0, \qquad c^{(2)}_{\langle e^\bullet \rangle;\mathbf{u}}(\eta) \neq 0, \tag{6.20}$$

then $\big(\upsilon_{r;e_1} c^{(0)}_{e_1;\upsilon_r}(\eta),\ \upsilon_{r;e_2} c^{(0)}_{e_2;\upsilon_r}(\eta)\big) \neq (0,0)$ for all $r \in \mathbb{Z}^+$ sufficiently large and (6.13) holds.

Proof. Let $e_\bullet = \langle e^\bullet \rangle$ and $w^\star = w^+_{\langle e^\bullet, e_1\rangle}$. By Corollary 4.8 for $\psi = \eta$, (6.3), and (4.14),

$$\|\eta\| |\rho_{\langle w^\star, w_e^+\rangle}(\upsilon_r)|,\ \widetilde{C}^\bullet_{w^\star;\upsilon_r}(\eta; w_e^+) \leqslant C|\upsilon_r| \left(\frac{|\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)|}{|\rho_e^2(\upsilon_r)|} + \frac{|\upsilon_{r;e_1} c^{(0)}_{e_1;\upsilon_r}(\eta)|}{|\rho_{e_1}^2(\upsilon_r)|} + \frac{|\upsilon_{r;e_2} c^{(0)}_{e_2;\upsilon_r}(\eta)|}{|\rho_{e_2}^2(\upsilon_r)|} \right)$$

for all $e \in \chi(\mathcal{T}_0)$ such that $\langle e \rangle = e_\bullet$. Taking $e = e^\bullet$ above and using (6.3), we obtain the first claim.

Along with (6.15) with e° replaced by e_1 and e_2, the inequality above gives

$$|\rho_e^2(\upsilon_r)| \widetilde{C}^\bullet_{w^\star;\upsilon_r}(\eta; w_e^+) \leqslant C|\upsilon_r| \Big(|\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)| + |\upsilon_{r;e_1} c^{(0)}_{e_1;\upsilon_r}(\eta)| + |\upsilon_{r;e_2} c^{(0)}_{e_2;\upsilon_r}(\eta)| \Big) \tag{6.21}$$

for all $e \in \chi(\mathcal{T}_0)$ such that $\langle e \rangle = e_\bullet$. By the first two assumptions in (6.20), $\eta_{\mathbf{u}}|_{x^-_{\langle e\rangle}(\mathbf{u})} \neq 0$ for every $e \in \chi(\mathcal{T}_0)$ with $\langle e \rangle \neq e_\bullet$. By (4.16) and (6.9) with $(\psi, e^\bullet)$ replaced by (η, e), there thus exist $C, C' \in \mathbb{R}$ such that

$$|\rho_e^2(\upsilon_r)| \widetilde{C}^\bullet_{w^\star;\upsilon_r}(\eta; w_e^+) \leqslant C|\rho_e^2(\upsilon_r)| \|\eta\| \leqslant C'|\upsilon_r| |\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)| \tag{6.22}$$

for all $e \in \chi(\mathcal{T}_0)$ such that $\langle e \rangle \neq e_\bullet$. By (6.21) and (6.22),

$$\sum_{e \in \chi(\mathcal{T}_0)} |\rho_e^2(\upsilon_r)| \widetilde{C}^\bullet_{w^\star;\upsilon_r}(\eta; w_e^+) \leqslant C|\upsilon_r| \sum_{e \in \chi(\mathcal{T}_0)} |\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)| \tag{6.23}$$

for some $C \in \mathbb{R}$.

Since every map in (6.2) is J-holomorphic, Corollary 5.6 applies and so (6.17) holds. By (6.15) and (6.23), there thus exist $C \in \mathbb{R}$ and continuous functions $\epsilon, \epsilon_1 \colon (\Delta, \widetilde{U}) \longrightarrow (\mathbb{R}, 0)$ such that

$$\begin{aligned} \Big| \sum_{e \in \chi(\mathcal{T}_0)} \upsilon_{r;e}\, c^{(0)}_{e;\upsilon_r}(\eta) \Big(\mathcal{D}_e \mathbf{u}\big(\tfrac{\partial}{\partial z^+_{e;\mathbf{u}}} \big) \Big) \Big| &\leqslant C \sum_{e \in \chi(\mathcal{T}_0)} \Big(\epsilon(\upsilon_r) |\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)| + |\rho_e^2(\upsilon_r)| \widetilde{C}^\bullet_{w^\star;\upsilon_r}(\eta; w_e^+) \Big) \\ &\leqslant \epsilon_1(\upsilon_r) \sum_{e \in \chi(\mathcal{T}_0)} |\upsilon_{r;e} c^{(0)}_{e;\upsilon_r}(\eta)| \,. \end{aligned} \tag{6.24}$$

The second claim of the lemma follows from the first claim and (6.24). □

Proof of Proposition 6.3, Case (0). We split the proof based on a secondary characterization of $[\mathbf{u}]$ and $[\mathbf{u}_r]$:

(.2): $\chi_\bullet \ni e_1, e_2$ with $e_1 \neq e_2$ and $[\Sigma_{\mathbf{u};0;P}; x_{e_1}(\mathbf{u}), x_{e_2}(\mathbf{u})] \notin \overline{\mathcal{C}}_{2,2}$;

(.1g): $\chi_\bullet = \{e_1, e_2\}$ with $e_1 \neq e_2$ and $[\Sigma_{\mathbf{u};0;P}; x_{e_1}(\mathbf{u}), x_{e_2}(\mathbf{u})] \in \overline{\mathcal{C}}_{2,2}$ or
$\chi_\bullet = \{e_\bullet\}$ with $[\Sigma_{\mathbf{u};0;P}; x_{e_\bullet}(\mathbf{u})] \notin \overline{\mathcal{W}}_{2,1}$;

(.1w): $\chi_\bullet = \{e_\bullet\}$ with $[\Sigma_{\mathbf{u};0;P}; x_{e_\bullet}(\mathbf{u})] \in \overline{\mathcal{W}}_{2,1}$.

In Case (0), let $\chi'_\bullet$ consist of all elements of $\chi_\bullet$ for which the map (1.3) is well-defined and $S'_\bullet$ be the corresponding set of marked points. In the first sub-case above, the image of $S'_\bullet$ under this map contains at least 2 points. In the second sub-case, it consists of a single regular point. In the last sub-case, this image is a single branched point.

Suppose $\chi_\bullet$ is as in Case (0.2). There then exist continuous families $\psi_1=\psi_{1;\mathbf{u}'}, \psi_2=\psi_{2;\mathbf{u}'}\in\mathbb{E}|_{\mathbf{u}'}$ for $\mathbf{u}'\in\widetilde{U}$ such that

$$c^{(0)}_{\langle e_1\rangle;\mathbf{u}}(\psi_1),\, c^{(0)}_{\langle e_2\rangle;\mathbf{u}}(\psi_2)\neq 0, \qquad c^{(0)}_{\langle e_1\rangle;\mathbf{u}}(\psi_2),\, c^{(0)}_{\langle e_2\rangle;\mathbf{u}}(\psi_1)=0.$$

By the $k=0$ case of the second inequality in (4.13), the first assumption in (6.4), (6.8), and (6.9), there thus exist $C,C'\in\mathbb{R}$ and a continuous function $\epsilon\colon(\Delta,\widetilde{U})\longrightarrow(\mathbb{R},0)$ such that

$$\begin{aligned}\big|v_{e_{3-i}}c^{(0)}_{e_{3-i};v}(\psi_i)\big| &\leqslant C|\rho_{e_{3-i}}(v)|\Big(\big|c^{(0)}_{\langle e_{3-i}\rangle;v}(\psi_i)\big|+|v|\|\psi_i\|\Big)\\ &\leqslant \epsilon(v)|\rho_{e_i}(v)|\big|c^{(0)}_{\langle e_i\rangle;v}(\psi_i)\big| \leqslant C'\epsilon(v)\big|v_{e_i}c^{(0)}_{e_i;v}(\psi_i)\big| \qquad \forall\ v\in\Delta,\ i=1,2.\end{aligned}$$

Along with the first statement of Lemma 6.5, this implies that

$$\mathcal{L}(\psi_i)\in\big\{[a_e]_{e\in\chi(\mathcal{T}_0)}\colon\ a_{e_i}\neq 0,\ a_{e_{3-i}}=0\big\}, \qquad i=1,2.$$

These lines are thus distinct. By the second statement of Lemma 6.5 for $\psi=\psi_1,\psi_2$,

$$\sum_{e\in\chi(\mathcal{T}_0)}a_e\,\mathcal{D}_e\mathbf{u}\big(\tfrac{\partial}{\partial z^+_{e;\mathbf{u}}}\big)=0 \qquad \forall\,(a_e)_{e\in\chi(\mathcal{T}_0)}\in\mathcal{P}(\psi_1,\psi_2)\,.$$

This establishes (6.1).

Suppose $\chi_\bullet$ is as in Case $(0.1g)$. By the assumptions of Proposition 6.3, there exist distinct $e^\bullet,e^\circ\in\chi^\bullet$ such that $\langle e^\bullet\rangle=\langle e^\circ\rangle$ and $[\Sigma_{\mathbf{u};0;P};x_{e_\bullet}(\mathbf{u})]\notin\overline{\mathcal{W}}_{2,1}$. Thus, there exist continuous families $\psi=\psi_{\mathbf{u}'}\in\mathbb{E}_0|_{\mathbf{u}'}$ for $\mathbf{u}'\in\widetilde{U}$ and $\eta=\eta_v\in\mathbb{E}_0|_{\mathbf{u}'}$ for $v\in\Delta|_{\mathbf{u}'}$, $\mathbf{u}'\in\widetilde{U}$, satisfying the assumptions of Lemmas 6.5 and 6.6, respectively. By the first statement of Lemma 6.5 and the equality in (6.12),

$$\mathcal{L}(\psi)\in\big\{[a_e]_{e\in\chi(\mathcal{T}_0)}\colon\ a_{e^\bullet}\neq 0\big\} \qquad\text{and}\qquad \mathcal{L}(\eta)\in\big\{[a_e]_{e\in\chi(\mathcal{T}_0)}\colon\ a_{e^\bullet}=0\big\}. \tag{6.25}$$

These lines are thus distinct. By the second statements of Lemmas 6.5 and 6.6,

$$\sum_{e\in\chi(\mathcal{T}_0)}a_e\,\mathcal{D}_e\mathbf{u}\big(\tfrac{\partial}{\partial z^+_{e;\mathbf{u}}}\big)=0 \qquad \forall\,(a_e)_{e\in\chi(\mathcal{T}_0)}\in\mathcal{P}(\psi,\eta)\,. \tag{6.26}$$

This establishes (6.1).

Suppose $\chi_\bullet$ is as in Case $(0.1w)$. Then, $|\chi^\bullet|\geqslant 3$. Choose $e^\bullet,e_1,e_2\in\chi^\bullet$ satisfying (6.19). There then exist continuous families $\psi=\psi_{\mathbf{u}'}\in\mathbb{E}_0|_{\mathbf{u}'}$ for $\mathbf{u}'\in\widetilde{U}$ and $\eta=\eta_v\in\mathbb{E}_0|_{\mathbf{u}'}$ for $v\in\Delta|_{\mathbf{u}'}$, $\mathbf{u}'\in\widetilde{U}$, satisfying the assumptions of Lemmas 6.5 and 6.7, respectively. By the first statement of Lemma 6.5 and the first equality in (6.20),

$$\mathcal{L}(\psi)\in\big\{[a_e]_{e\in\chi(\mathcal{T}_0)}\colon\ a_{e^\bullet}\neq 0\big\} \qquad\text{and}\qquad \mathcal{L}(\eta)\in\big\{[a_e]_{e\in\chi(\mathcal{T}_0)}\colon\ a_{e^\bullet}=0\big\}.$$

These lines are thus distinct. The second statements of Lemmas 6.5 and 6.7 imply (6.26) and consequently (6.1). □

Suppose $[\mathbf{u}]$ and $[\mathbf{u}_r]$ are as in Case $(1a)$ or $(1b)$. In the first case, illustrated in Figure 6, the principal subgraph $\mathcal{T}_{0;P}$ of $[\mathbf{u}]$ satisfies the sentence containing (3.25). In the second case, illustrated in Figure 5, $\mathcal{T}_{0;P}$ satisfies the sentence containing (3.19). In either case, there exists a continuous family $\psi_{\mathbf{u}'}\in\mathbb{E}_0|_{\mathbf{u}'}$ for $\mathbf{u}'\in\widetilde{U}$ to which Corollary 5.6 and Lemma 6.5 apply. The latter provides a line $\mathcal{L}(\psi)\subset\mathbb{C}^{\chi(\mathcal{T}_0)}$ satisfying (6.7) and (6.25).

In order to obtain a second vanishing condition in Cases $(1a)$ and $(1b)$, fix $e^\bullet\in\chi^\bullet$. Let

$$\mathbb{E}_0^\bullet=\big\{\eta\in\mathbb{E}_0|_{\upsilon}\colon\,\eta|_{x^-_{\langle e^\bullet\rangle}(\upsilon)}=0,\ \upsilon\in\Delta_{0;P}\big\}\longrightarrow\Delta_{0;P}.$$

This line subbundle of $\mathbb{E}_0|_{\Delta_{0;P}}$ extends to a line subbundle $\mathbb{E}_0^\bullet$ of $\mathbb{E}_0|_{\Delta_0}$ such that $\eta|_{x^-_{e^\bullet}(\upsilon)}=0$ for all $\eta\in\mathbb{E}_0^\bullet|_\upsilon$ and $\upsilon\in\Delta$. A section η_υ of this subbundle such that $\eta|_{\mathbf{u}}\neq0$ gives rise to a second vanishing condition; the latter depends on the proposition in question (6.1, 6.2, 6.3, or 6.4).

In the case of Proposition 6.3, the above family η_υ determines a line $\mathcal{L}(\eta)$ satisfying (6.25). In order to show that $\mathcal{L}(\eta)$ also satisfies (6.13), the estimate of Corollary 5.6 with ψ_υ replaced by η_υ needs to be improved to incorporate the nodes of $\Sigma_{\mathbf{u};0;P}$ not in $\Sigma^\bullet_{\mathbf{u};0;P}$. This will be done in a separate paper by combining Corollaries 3.3 and 3.5 in Cases $(1b)$ and $(1a)$, respectively, with suitable estimates for η_υ on different regions of the domain Σ_υ with $\upsilon\in\Delta_0$. The required estimates are the analogues of those in Corollaries 4.5-4.10.

In Propositions 6.1 and 6.2, $\chi^\bullet$ consists of a single element $e^\bullet$. The settings of these propositions correspond to the secondary characterizations $(.1w)$ and $(.1g)$, respectively, while (.2) does not occur. The vanishing condition on $\mathcal{D}_{e^\bullet}\mathbf{u}$ in Cases (0), $(1a)$, and $(1b)$ follows directly from (5.23) with $w^\star=w^+_{e^\bullet}$; it is also the second conclusion of Lemma 6.5. In Case (0), the properties in (1.6) involving $\mathcal{D}^{(2)}_{e^\bullet}\mathbf{u}$ and $\mathcal{D}^{(3)}_{e^\bullet}\mathbf{u}$ follow from the $\ell=2,3$ cases of Lemma 5.1 and a refined version of Corollary 5.6. In Cases $(1a)$ and $(1b)$, estimates on the section η_υ as in the previous paragraph are needed as well.

In Proposition 6.4, $\chi^\bullet$ consists of two elements e_1 and e_2. The set $\chi_\bullet$ may consist of one or two elements, corresponding to the secondary characterizations $(.1w)$ and $(.1g)$; (.2) again does not occur. The setting of Proposition 6.4 is the most delicate of the four propositions. In Cases (0), $(1a)$, and $(1b)$, the first degeneration condition in (1.7) follows from Corollary 5.4 by an argument similar to the proof of Corollary 5.6. However, the second vanishing condition in (1.7) requires a modification of the spaces $\Gamma_+(\upsilon)$ below (3.42) and (3.58) to distinguish the one-dimensional linear span of $\mathcal{D}_{e_1}\mathbf{u}$ and $\mathcal{D}_{e_2}\mathbf{u}$ (if both vanish, then the second condition in (1.7) automatically holds). This modification suffices to establish Proposition 6.4 in Case (0). In Cases $(1a)$ and $(1b)$, Proposition 6.4 is obtained by combining this modification with estimates on the section η_υ as above.

In Case 2, $\psi|_{x^-_e(\upsilon)}=0$ for all $\psi\in\mathbb{E}_0|_{\Delta_{0;P}}$ and $e\in\chi_\bullet$. There is then a natural splitting of $\mathbb{E}_0|_{\Delta_{0;P}}$ into two line bundles $\mathbb{E}_1$ and $\mathbb{E}_2$. It can be extended to a splitting of $\mathbb{E}_0|_{\Delta_0}$ so that $\eta|_{x^-_{\langle e^\bullet\rangle}(\upsilon)}=0$ for all $\eta\in\mathbb{E}_2$, $\upsilon\in\Delta_0$, and for some fixed $e^\bullet\in\chi^\bullet$. An analogue of Corollary 5.6 in this setting is then obtained using estimates on a nonzero section ψ_υ of $\mathbb{E}_1$ analogous to those of Corollaries 4.5-4.10. It yields one condition on the derivatives of $\mathbf{u}$. A second vanishing condition is obtained by using a nonzero section η_υ of $\mathbb{E}_2$.

A Appendix

In the remainder of this dissertation, we collect a few basic observations.

A.1 Exponential maps and differentiation

If $f\colon M\longrightarrow X$ is a smooth map between smooth manifolds and $E\longrightarrow X$ is a smooth vector bundle, let

$$\Gamma(f;E)\equiv\Gamma(M;f^*E),\qquad \Gamma^1(f;E)\equiv\Gamma(M;T^*M\otimes f^*E).$$

A connection ∇ induces a connection ∇^f on $f^*E\longrightarrow M$. If $\alpha\colon(a,b)\longrightarrow X$ is a smooth curve and $\xi\in\Gamma(\alpha;E)$, let

$$\frac{D}{\mathrm{d}t}\xi\equiv\nabla^{\alpha}_{\partial_t}\xi\in\Gamma(\alpha;E),\qquad \frac{D^\ell}{\mathrm{d}t^\ell}\xi\equiv\frac{D}{\mathrm{d}t}\bigg(\frac{D^{\ell-1}}{\mathrm{d}t^{\ell-1}}\xi\bigg)\ \ \forall\ \ell\geqslant2,$$

where ∂_t is the standard unit vector field on $\mathbb{R}$.

For a Riemmanian metric g_X on X and a g_X-compatible connection ∇, let exp be the exponential map, R_∇ the curvature tensor, and T_∇ the torsion tensor for ∇. For every $x\in X$ and $v\in T_xX$, denote by Π_v the parallel transport with respect to the connection ∇ along the geodesic

$$\gamma_v:[0,1]\longrightarrow X,\qquad t\mapsto\exp_x(tv).$$

Lemma A.1. *For every $x\in X$, $w_0\in T_xX$, and smooth map $\xi:(-\epsilon,\epsilon)\longrightarrow T_xX$,*

$$\frac{D^\ell}{\mathrm{d}t^\ell}\bigg|_{t=0}\Pi_{tw_0}\xi(t)=\frac{\mathrm{d}^\ell\xi}{\mathrm{d}t^\ell}(0)\qquad\forall\ \ell\in\mathbb{Z}^+.$$

Proof. Let $\{\zeta_i\}$ be a basis for T_xX and $\zeta_i(t)\equiv\Pi_{tw_0}\zeta_i$. Then

$$\frac{D}{\mathrm{d}t}\zeta_i(t)=0\qquad\forall\ t\in(-\epsilon,\epsilon).\tag{A.1}$$

Suppose $\xi(t)=\sum_i f_i(t)\zeta_i$. Then $\Pi_{tw_0}\xi(t)=\sum_i f_i(t)\zeta_i(t)$. By (A.1),

$$\frac{D^\ell}{\mathrm{d}t^\ell}\bigg|_{t=0}\Pi_{tw_0}\xi(t)=\frac{D^\ell}{\mathrm{d}t^\ell}\bigg|_{t=0}\Big(\sum_i f_i(t)\zeta_i(t)\Big)=\sum_i\frac{\mathrm{d}^\ell f_i}{\mathrm{d}t^\ell}(0)\zeta_i=\frac{\mathrm{d}^\ell\xi}{\mathrm{d}t^\ell}(0).\qquad\square$$

We next refine the estimate of [21, Lemma 3.9]. Given $x\in X$ and $v,w_0,w_1\in T_xX$, let $\alpha\colon(-\epsilon,\epsilon)\longrightarrow X$ be a smooth curve and $\xi\in\Gamma(\alpha;TX)$ such that

$$\alpha(0)=x,\qquad\alpha'(0)=v,\qquad\xi(0)=w_0,\qquad\tfrac{D}{\mathrm{d}s}\xi(s)\big|_{s=0}=w_1.\tag{A.2}$$

Put

$$\begin{aligned}\Phi_x(v;w_0,w_1)&\equiv\Pi^{-1}_{\xi(0)}\big(\tfrac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0}\exp(\xi(s))\big)\in T_xX,\\ \widetilde\Phi_x(v;w_0,w_1)&\equiv\Phi_x(v;w_0,w_1)-\big(v+w_1-T_\nabla(v,w_0)\big);\end{aligned}\tag{A.3}$$

these definitions are independent of the choices of α and ξ satisfying (A.2). Let

$$A(v_1,v_2,v_3)\equiv\tfrac12\Big(R_\nabla(v_1,v_2)v_3-(\nabla_{v_1}T_\nabla)(v_2,v_3)+T_\nabla\big(T_\nabla(v_1,v_2),v_3\big)\Big)\quad\forall\,v_1,v_2,v_3\in T_xX,\ x\in X.$$

Lemma A.2. *There exists $C\in C^\infty(TX;\mathbb{R})$ such that*

$$\big|\,\widetilde{\Phi}_x(v;w_0,w_1)+\tfrac{1}{2}T_\nabla(w_1,w_0)-A(w_0,v,w_0)\,\big|\leqslant C(w_0)\big(|v||w_0|^3+|w_0|^2|w_1|\big)$$

for all $x\in X$ and $v,w_0,w_1\in T_xX$.

Proof. Let α and ξ be as in (A.2). Put $u(s,t)\equiv\exp_{\alpha(s)}(t\xi(s))$,

$$\begin{aligned}F_{v,w_0,w_1}(t)&=u_s(0,t), \qquad \text{and}\\ H_{v,w_0,w_1}(t)&=\Pi_{tw_0}\Big(v+t\big(w_1-T_\nabla(v,w_0)\big)+t^2\big(\tfrac{1}{2}T_\nabla(w_0,w_1)+A(w_0,v,w_0)\big)\Big).\end{aligned}$$

Since

$$u_s(0,0)=v,\quad u_t(s,0)=\xi(s),\quad \tfrac{D}{\mathrm{d}t}u_t=0,\quad \tfrac{D}{\mathrm{d}s}\big|_{(0,0)}u_t=w_1,\quad \tfrac{D}{\mathrm{d}s}\big|_{(0,0)}u_s=w_1-T_\nabla(v,w_0), \tag{A.4}$$

we find that

$$\begin{aligned}&F_{v,w_0,w_1}(0)=v=H_{v,w_0,w_1}(0),\\ &\tfrac{D}{\mathrm{d}t}\big|_{t=0}F_{v,w_0,w_1}(t)=w_1-T_\nabla(v,w_0)=\tfrac{D}{\mathrm{d}t}\big|_{t=0}H_{v,w_0,w_1}(t).\end{aligned} \tag{A.5}$$

From (A.4), we also obtain

$$\begin{aligned}\tfrac{D^2}{\mathrm{d}t^2}\Big|_{t=0}F_{v,w_0,w_1}(t)&=\big(\tfrac{D}{\mathrm{d}t}\tfrac{D}{\mathrm{d}t}u_s\big)_{(0,0)}=\Big(\tfrac{D}{\mathrm{d}t}\tfrac{D}{\mathrm{d}s}u_t-\tfrac{D}{\mathrm{d}t}\big(T_\nabla(u_s,u_t)\big)\Big)_{(0,0)}\\ &=\Big(\tfrac{D}{\mathrm{d}s}\tfrac{D}{\mathrm{d}t}u_t+R_\nabla(u_t,u_s)u_t-(\nabla_{u_t}T_\nabla)(u_s,u_t)-T_\nabla(\tfrac{D}{\mathrm{d}t}u_s,u_t)-T_\nabla(u_s,\tfrac{D}{\mathrm{d}t}u_t)\Big)_{(0,0)}\\ &=T_\nabla(w_0,w_1)+2A(w_0,v,w_0).\end{aligned}$$

Along with Lemma A.1, this implies that

$$\tfrac{D^2}{\mathrm{d}t^2}\Big|_{t=0}F_{v,w_0,w_1}(t)=\tfrac{D^2}{\mathrm{d}t^2}\Big|_{t=0}H_{v,w_0,w_1}(t). \tag{A.6}$$

As in the proof of [21, Lemma 3.9],

$$F_{\cdot,w_0,\cdot}(t)-H_{\cdot,w_0,\cdot}(t)\in\mathrm{Hom}\big(T_xX\oplus T_xX;\,T_{\exp(tw_0)}X\big).$$

By (A.5) and (A.6),

$$\big|F_{v,w_0,w_1}(t)-H_{v,w_0,w_1}(t)\big|\leqslant C(w_0,t)t^3\big(|v|+|w_1|\big),$$

where C is a smooth function on $TX\times\mathbb{R}$. Since

$$F_{v,w_0,w_1}(t)-H_{v,w_0,w_1}(t)=F_{v,tw_0,tw_1}(1)-F_{v,tw_0,tw_1}(1),$$

there exists $C\in C^\infty(TX;\mathbb{R})$ such that

$$\begin{aligned}\big|F_{v,w_0,w_1}(1)-H_{v,w_0,w_1}(1)\big|&=\big|F_{v,w_0/|w_0|,w_1/|w_0|}(|w_0|)-H_{v,w_0/|w_0|,w_1/|w_0|}(|w_0|)\big|\\ &\leqslant C(w_0)\big(|v||w_0|^3+|w_0|^2|w_1|\big).\end{aligned}$$

□

Lemma A.3. *Suppose X is a compact Riemannian manifold, $\alpha\colon[-\epsilon,\epsilon]\longrightarrow X$ is a smooth curve, $\xi,\xi^+\in\Gamma(\alpha;TX)$, and $\zeta\in\Gamma\big(\exp_\alpha\xi^+;TX\big)$. If ξ^+ and ζ are small and satisfy*

$$\exp_\alpha\xi=\exp_{\exp_\alpha\xi^+}\zeta, \tag{A.7}$$

then there exists $C\in\mathbb{R}$ such that

$$\big|\,\xi-\xi^+-\Pi_{\xi^+}^{-1}\zeta\,\big|_t\leqslant C|\xi^+|_t|\zeta|_t, \tag{A.8}$$

$$\big|\,\tfrac{D}{\mathrm{d}t}\xi-\tfrac{D}{\mathrm{d}t}\xi^+-\Pi_{\xi^+}^{-1}\tfrac{D}{\mathrm{d}t}\zeta\,\big|_t\leqslant C\Big(\big|\tfrac{\mathrm{d}\alpha}{\mathrm{d}t}\big|_t+\big|\tfrac{D}{\mathrm{d}t}\xi^+\big|_t+\big|\tfrac{D}{\mathrm{d}t}\zeta\big|_t\Big)\Big(|\xi^+|_t+|\zeta|_t\Big) \tag{A.9}$$

for all $t\in(-\epsilon,\epsilon)$.

Proof. Since the difference in (A.8) is a smooth function of ξ^+ and ζ and vanishes if $\xi^+=0$ or $\zeta=0$, (A.8) holds. Differentiating both sides of (A.7) and taking the parallel transports, we obtain

$$\Pi_\xi^{-1}\mathrm{d}\exp_\alpha\xi=\Pi_\xi^{-1}\circ\Pi_\zeta\Big(\Pi_\zeta^{-1}\mathrm{d}\exp_{\exp_\alpha\xi^+}\zeta\Big). \tag{A.10}$$

By Lemma A.2,

$$\begin{aligned}&\big|\Pi_\xi^{-1}\mathrm{d}\exp_\alpha\xi-\mathrm{d}\alpha-\nabla^\alpha\xi\big|_t\leqslant C\big(|\mathrm{d}\alpha|_t+|\nabla^\alpha\xi|_t\big)|\xi|_t\,, &(A.11)\\ &\Big|\Pi_\zeta^{-1}\mathrm{d}\exp_{\exp_\alpha\xi^+}\zeta-\Pi_{\xi^+}\big(\mathrm{d}\alpha+\nabla^\alpha\xi^+\big)-\nabla^{\exp_\alpha\xi^+}\zeta\Big|_t\\ &\qquad\leqslant C\Big(|\mathrm{d}\alpha|_t+|\nabla^\alpha\xi^+|_t+\big|\nabla^{\exp_\alpha\xi^+}\zeta\big|_t\Big)\big(|\xi^+|_t+|\zeta|_t\big). &(A.12)\end{aligned}$$

By (A.7),

$$\begin{aligned}\Big|\big(\Pi_\xi^{-1}\circ\Pi_\zeta\circ\Pi_{\xi^+}-\mathrm{id}\big)\big(\mathrm{d}\alpha+\nabla^\alpha\xi^+\big)\Big|_t&\leqslant C\big(|\mathrm{d}\alpha|_t+|\nabla^\alpha\xi^+|_t\big)|\xi^+|_t|\zeta|_t,\\ \Big|\big(\Pi_\xi^{-1}\circ\Pi_\zeta-\Pi_{\xi^+}^{-1}\big)\big(\nabla^{\exp_\alpha\xi^+}\zeta\big)\Big|_t&\leqslant C\big|\nabla^{\exp_\alpha\xi^+}\zeta\big|_t|\xi^+|_t|\zeta|_t.\end{aligned} \tag{A.13}$$

The inequality (A.9) follows from (A.10)-(A.13) and (A.8). □

A.2 Derivatives of J-holomorphic maps

Let $E\longrightarrow X$ be a complex vector bundle over a smooth manifold and $\Delta\subset\mathbb{C}$ be a disk around the origin 0. If ∇,∇' are (complex) connections in E, there exists

$$\theta\in\Gamma\big(X;T^*X\otimes_{\mathbb{R}}\mathrm{End}_{\mathbb{C}}(E)\big)\quad\text{s.t.}\quad\nabla'_vs=\nabla_vs+\theta(v)s(x)\quad\forall\,v\in T_xX,\ x\in X,\ s\in\Gamma(X;E). \tag{A.14}$$

Any C^∞-map $u\colon\Delta\longrightarrow X$ induces connections ∇^u,∇'^u in $u^*E\longrightarrow\Delta$ and

$$\nabla'^u=\nabla^u+u^*\theta\equiv\nabla^u+\theta\circ\mathrm{d}u.$$

A (holomorphic) connection $\hat\nabla$ in $T\Delta\longrightarrow\Delta$ induces a (holomorphic) connection in $T^*\Delta\longrightarrow\Delta$, which we still denote by $\hat\nabla$. If $\hat\nabla'$ is another (holomorphic) connection in $T\Delta$, there exists a (holomorphic) one-form

$$\hat\theta\in\Gamma(\Delta;T^*\Delta)\qquad\text{s.t.}\quad\hat\nabla'_v\alpha=\hat\nabla_v\alpha+\hat\theta(v)\alpha(p)\quad\forall\,v\in T_p\Delta,\ p\in\Delta,\ \alpha\in\Gamma(\Delta;T^*\Delta). \tag{A.15}$$

The pairs $(\nabla,\hat\nabla)$ and $(\nabla',\hat\nabla')$ induce connections D^u and D'^u in the bundles

$$T^*\Delta^{\otimes_{\mathbb{R}}k}\otimes_{\mathbb{R}}u^*E\longrightarrow\Delta.$$

Lemma A.4. *If* $\xi\in\Gamma(\Delta;T^*\Delta\otimes_{\mathbb R}u^*E)$ *is such that* $\xi(0)=0$, *then* $(D^u\xi)_0\in T_0^*\Delta^{\otimes_{\mathbb R}2}\otimes_{\mathbb R}E_{u(0)}$ *is independent of the choices of* ∇ *and* $\hat\nabla$.

Proof. By (A.14) and (A.15), there exists

$$\begin{gathered}\vartheta\in\Gamma\big(\Delta;T^*\Delta\otimes_{\mathbb R}\mathrm{End}_{\mathbb R}(T^*\Delta\otimes_{\mathbb R}u^*E)\big)\qquad\text{s.t.}\\ D'^u_v\xi=D^u_v\xi+\vartheta(v)\xi(p)\quad\forall\,v\in T_p\Delta,\ p\in\Delta,\ \xi\in\Gamma(\Delta;T^*\Delta\otimes_{\mathbb R}u^*E);\end{gathered}$$

in fact, $\vartheta=\hat\theta\otimes_{\mathbb R}\mathrm{id}+\mathrm{id}\otimes_{\mathbb R}u^*\theta$. Since $\xi(0)=0$, the above identity implies the claim. □

Lemma A.5. *Suppose* $\mathrm{d}_0u=0$ *and* $v\in T_0\Delta$. *If* $\xi\in\Gamma(\Delta;T^*\Delta\otimes_{\mathbb R}u^*E)$ *is such that* $\xi(0)=0$ *and*

$$D^u\xi\colon T_0\Delta\otimes_{\mathbb R}T_0\Delta\longrightarrow E_{u(0)},\qquad(v_1,v_2)\longrightarrow\{D^u_{v_1}\xi\}(v_2),\tag{A.16}$$

is $\mathbb C$*-bilinear, then the space*

$$\mathrm{Span}_{\mathbb C}\Big\{\{D^u_v\xi\}(v),\{\{D^u_v(D^u\xi)\}(v)\}(v)\Big\}\tag{A.17}$$

is independent of the choices of ∇ *and* $\hat\nabla$.

Proof. By the proof of Lemma A.4,

$$\begin{aligned}D'^u(D'^u\xi)&=D'^u\big(D^u\xi+\{\hat\theta\otimes_{\mathbb R}\mathrm{id}+\mathrm{id}\otimes_{\mathbb R}u^*\theta\}\xi\big)\\&=D^u(D^u\xi)+\big\{\hat\theta\otimes_{\mathbb R}\mathrm{id}\otimes_{\mathbb R}\mathrm{id}+\mathrm{id}\otimes_{\mathbb R}\hat\theta\otimes_{\mathbb R}\mathrm{id}+\mathrm{id}\otimes_{\mathbb R}\mathrm{id}\otimes_{\mathbb R}u^*\theta\big\}(D^u\xi)+D'^u\big(\{\hat\theta\otimes_{\mathbb R}\mathrm{id}+\mathrm{id}\otimes_{\mathbb R}u^*\theta\}\xi\big).\end{aligned}$$

Thus, by the vanishing assumptions on d_0u and $\xi(0)$ and Lemma A.4,

$$\begin{aligned}\big\{\{D'^u_v(D'^u\xi)\}(v)\big\}(v)\big\}&=\big\{\{D^u_v(D^u\xi)\}(v)\big\}(v)\big\}\\&\quad+\big\{D^u_{\hat\theta(v)v}\xi\big\}(v)+\big\{D^u_v\xi\big\}\big(\hat\theta(v)v\big)+\big\{D^u_v\xi\big\}\big(\hat\theta(v)v\big).\end{aligned}\tag{A.18}$$

By the $\mathbb C$-bilinearity assumption,

$$\big\{D^u_{\hat\theta(v)v}\xi\big\}(v),\big\{D^u_v\xi\big\}\big(\hat\theta(v)v\big)=\hat\theta(v)\cdot\{D^u_v\xi\}(v).$$

Thus, the terms on the last line in (A.18) are in the $\mathbb C$-span of $\{D^u_v\xi\}(v)=\{D'^u_v\xi\}(v)$, which implies the claim. □

If in addition

$$D^uD^u\xi\colon T_0\Delta\otimes_{\mathbb R}T_0\Delta\otimes_{\mathbb R}T_0\Delta\longrightarrow E_{u(0)},\qquad(v_1,v_2,v_3)\longrightarrow\big\{\{D^u_{v_1}(D^u\xi)\}(v_2)\big\}(v_3),\tag{A.19}$$

is $\mathbb C$-trilinear (at least modulo the image of the map (A.16)) for some choice of $(\nabla,\hat\nabla)$, then the property (A.17) is independent of the choice of $v\in T^*\Delta-0$ (since any such two choices differ by the multiplication by an element of $\mathbb C^*$).

We apply this to $\xi=\mathrm{d}u$ for a J-holomorphic map u such that $\mathrm{d}_0u=0$. First we show the following lemma.

Lemma A.6. *The map* $D^u\mathrm{d}u$ *is* $\mathbb C$*-bilinear.*

Proof. Since the map is $\mathbb{R}$-bilinear, it suffices to show

$$\{D^u_{\mathrm{j}v_1}\mathrm{d}u\}(v_2), \{D^u_{v_1}\mathrm{d}u\}(\mathrm{j}v_2) = J\{D^u_{v_1}\mathrm{d}u\}(v_2) \quad \forall\, v_1,\, v_2 \in T_0\Delta. \tag{A.20}$$

Extend v_2 to a vector around 0 on Δ. Since u is J-holomorphic, $\mathrm{d}_0u = 0$ and ∇ (and hence ∇^u) is J-linear,

$$\begin{aligned}\{D^u_{v_1}\mathrm{d}u\}(\mathrm{j}v_2) &= \nabla^u_{v_1}\big\{\mathrm{d}u(\mathrm{j}v_2)\big\} - \mathrm{d}_0u(\hat{\nabla}_{v_1}\mathrm{j}v_2)\\ &= J\nabla^u_{v_1}\big\{\mathrm{d}u(v_2)\big\} = J\{D^u_{v_1}\mathrm{d}u\}(v_2).\end{aligned}$$

This establishes the second statement in (A.20) holds.

By the $\mathbb{R}$-linearity of D^u, the first statement in (A.20) holds if and only if

$$\nabla^u_{\partial_t}(\mathrm{d}u(\partial_s))|_0 = \nabla^u_{\partial_s}(\mathrm{d}u(\partial_t)\})|_0, \qquad \nabla^u_{\partial_s}(\mathrm{d}u(\partial_s))|_0 = -\nabla^u_{\partial_t}(\mathrm{d}u(\partial_t))|_0, \tag{A.21}$$

where ∂_s and ∂_t are the standard coordinate vector fields on Δ. Let $(\{x_i\}_{1\leqslant i\leqslant 2n})$ be a coordinate chart in X around $u(0)$. With $u_i = x_i \circ u$,

$$\mathrm{d}u = \sum_i \mathrm{d}u_i \otimes_{\mathbb{R}} u^*\frac{\partial}{\partial x^i}$$

on a neighborhood of 0 in Δ. Since $\mathrm{d}_0u = 0$,

$$\mathrm{d}_0u^i = 0 \qquad \forall\ i. \tag{A.22}$$

By Lemma A.4, the first and second identities in (A.21) are thus equivalent to

$$\frac{\partial^2 u_i}{\partial s\partial t}(0) = \frac{\partial^2 u_i}{\partial t\partial s}(0)\,, \qquad \frac{\partial^2 u_i}{\partial s\partial s}(0) = -\frac{\partial^2 u_i}{\partial t\partial t}(0), \tag{A.23}$$

respectively; this establishes the first identity in (A.21). Since u is J-holomorphic,

$$\mathrm{d}u(\partial_s) + J(u)\mathrm{d}u(\partial_t) = 0. \tag{A.24}$$

Thus,

$$\begin{aligned}&\nabla^u_{\partial_s}(\mathrm{d}u(\partial_s)) + (\nabla^u_{\partial_s}u^*J)\mathrm{d}u(\partial_t) + J(u)\nabla^u_{\partial_s}(\mathrm{d}u(\partial_t))\\ &\qquad = 0 = J(u)\big(\nabla^u_{\partial_t}(\mathrm{d}u(\partial_s)) + (\nabla^u_{\partial_t}u^*J)\mathrm{d}u(\partial_t) + J(u)\nabla^u_{\partial_t}(\mathrm{d}u(\partial_t)).\end{aligned} \tag{A.25}$$

Evaluating this at 0 and using $\mathrm{d}_0u = 0$ and the first identity in (A.21), we obtain the second identity in (A.21). □

Now we are ready to show the following result.

Lemma A.7. *The map $D^uD^u\mathrm{d}u$ is $\mathbb{C}$-trilinear.*

Proof. By Lemma A.5, it suffices to work with the Eucleadian connection $\hat{\nabla}$ on Δ. Denote s and t by s^1 and s^2, respectively. Notice that

$$\nabla^u_v\bigg(u^*\frac{\partial}{\partial x^i}\bigg)\Big|_0 = \nabla_{\mathrm{d}_0u(v)}\bigg(\frac{\partial}{\partial x^i}\bigg)\Big|_{u(0)} = 0, \qquad \nabla^u_v u^*J = 0,$$

for any $v\in T_0\Delta$ and $1\leqslant i\leqslant 2n$. Taking (A.22) into account, we see that

$$\{D^u(D^u\mathrm{d}u)\}(0)=\sum_i\sum_{j,k,l=1}^{2}\frac{\partial^3 u_i}{\partial s^j\partial s^k\partial s^l}(0)\,\mathrm{d}s^j\otimes_{\mathbb{R}}\mathrm{d}s^k\otimes_{\mathbb{R}}\mathrm{d}s^l\otimes_{\mathbb{R}}u^*\frac{\partial}{\partial x^i}. \tag{A.26}$$

By the bilinearity of the map (A.16), the $\mathbb{C}$-linearity of the map (A.19) in the third component is equivalent to

$$\begin{aligned}\sum_i\frac{\partial^3 u_i}{\partial s\partial s\partial s}(0)\frac{\partial}{\partial x^i}\Big|_{u(0)}+J(u(0))\sum_i\frac{\partial^3 u_i}{\partial s\partial s\partial t}(0)\frac{\partial}{\partial x^i}\Big|_{u(0)}&=0,\\ \sum_i\frac{\partial^3 u_i}{\partial s\partial t\partial t}(0)\frac{\partial}{\partial x^i}\Big|_{u(0)}+J(u(0))\sum_i\frac{\partial^3 u_i}{\partial t\partial t\partial t}(0)\frac{\partial}{\partial x^i}\Big|_{u(0)}&=0.\end{aligned} \tag{A.27}$$

By (A.25),

$$\begin{aligned}\nabla^u_{\partial_s}\big(\nabla^u_{\partial_s}(\mathrm{d}u(\partial_s))\big)+\big(\nabla^u_{\partial_s}(\nabla^u_{\partial_s}u^*J)\big)\mathrm{d}u(\partial_t)&\\ +2(\nabla^u_{\partial_s}u^*J)\nabla^u_{\partial_s}\big(\mathrm{d}u(\partial_t)\big)+J(u)\nabla^u_{\partial_s}\big(\nabla^u_{\partial_s}(\mathrm{d}u(\partial_t))\big)&=0.\end{aligned}$$

Evaluating this at 0 and using $\mathrm{d}_0u=0$ and $\nabla^u u^*J|_0=0$, we obtain the first identity in (A.27). The second holds by symmetry. □

A.3 Local properties of J-holomorphic maps

For $n\in\mathbb{Z}^+$ and $R\in\mathbb{R}^+$, denote by $B_R^n\subset\mathbb{C}^n$ the open ball of radius R centered at the origin. If $n=1$, we simply write $B_R\equiv B_R^1$. For $z\in\mathbb{C}$, let

$$\partial\equiv\frac{\partial}{\partial z}\qquad\text{and}\qquad\bar\partial\equiv\frac{\partial}{\partial\bar z}.$$

For each $\alpha=(m_1,m_2)\in\mathbb{Z}^{\geqslant 0}\oplus\mathbb{Z}^{\geqslant 0}$, define

$$\alpha!\equiv m_1!m_2!\,,\qquad |\alpha|\equiv m_1+m_2,\qquad D^\alpha\equiv\partial^{m_1}\bar\partial^{m_2}\colon C^\infty\longrightarrow C^\infty,\qquad z^\alpha\equiv z^{m_1}\bar z^{m_2}$$

with $D^{(0,0)}=\mathrm{id}$. If f is a function differentiable to order k with $k\in\mathbb{Z}^{\geqslant 0}$, denote by

$$T^{(k)}_{f;0}(z)\equiv\sum_{|\alpha|=0}^{k}\frac{D^\alpha_0 f}{\alpha!}z^\alpha$$

the k-th order Taylor polynomial at $z=0$ and let

$$|D^k f|\equiv\sum_{|\alpha|=k}|D^\alpha f|.$$

Lemma A.8. *Let δ, ϵ, $p\in\mathbb{R}^+$ with $p>2$, $\ell\in\mathbb{Z}^+$, and J be an almost complex structure on $\mathbb{C}^n$ such that $J(0)$ is the standard complex structure $J_{\mathbb{C}^n}$. There exists $C_\ell\in\mathbb{R}^+$ such that*

$$\big|\mathrm{d}_z u-\mathrm{d}_0u\big|\leqslant C_\ell\|u\|_{L^p_1}|z|,\qquad\big|u(z)-T^{(\ell)}_{u;0}(z)\big|\leqslant C_\ell\|u\|_{L^p_1}|z|^{\ell+1}\qquad\forall\ z\in B_\delta$$

for every J-holomorphic map $u\colon B_{2^{\ell+1}\delta}\longrightarrow\mathbb{C}^n$ with $u(0)=0$ and $\|\mathrm{d}u\|_{L^p}\leqslant\epsilon$. Furthermore, C_ℓ can be chosen to depend continuously on δ, ϵ, p, and J with respect to the $C^{\ell+1}$-topology.

Proof. By [21, Corollary 4.3], there exists $C_{\delta,p}\in\mathbb{R}^+$ depending on δ and p continuously such that

$$\|f\|_{C^k}\leqslant C_{\delta,p}\|f\|_{L^p_{k+1}}, \qquad k=0,1,\ldots,\ell,\ f\in C^\infty\big(B_{2^{\ell+1-k}\delta};\mathbb{C}^n\big). \tag{A.28}$$

By Taylor's Theorem, it is thus sufficient to show that there exists $C\in\mathbb{R}^+$ such that

$$\|u\|_{L^p_{\ell+2}(B_\delta)}\leqslant C\|u\|_{L^p_1} \tag{A.29}$$

for every J-holomorphic map $u\colon B_{2^{\ell+1}\delta}\longrightarrow\mathbb{C}^n$ with $u(0)=0$ and $\|\mathrm{d}u\|_{L^p}\leqslant\epsilon$.

By (A.28), there exists $C'_{\delta,\epsilon,p}\in\mathbb{R}^+$ such that

$$\|f\|_{C^0}\leqslant C'_{\delta,\epsilon,p} \qquad \forall\ f\in C^\infty(B_{2^{\ell+1}\delta};\mathbb{C}^n)\text{ with } f(0)=0,\ \|\mathrm{d}f\|_{L^p}\leqslant\epsilon. \tag{A.30}$$

By (A.28) and (A.30), there exists $C\in\mathbb{R}^+$ such that

$$\big\|J(f)\big\|_{L^p_k}\leqslant C\Big(1+\|f\|_{L^p_k}\Big) \qquad \forall\ k\in[\![\ell+1]\!],\ f\in C^\infty\big(B_{2^{\ell+2-k}\delta};\mathbb{C}^n\big)\text{ with } f(0)=0,\ \|\mathrm{d}f\|_{L^p}\leqslant\epsilon. \tag{A.31}$$

By [10, Proposition B.4.9] and (A.31), there exists $C\in\mathbb{R}^+$ such that

$$\|u\|_{L^p_{\ell+2}(B_\delta)}\leqslant C\|u\|_{L^p_{\ell+1}(B_{2\delta})}\leqslant\ldots\leqslant C^{\ell+1}\|u\|_{L^p_1(B_{2^{\ell+1}\delta})}=C^{\ell+1}\|u\|_{L^p_1} \tag{A.32}$$

for every J-holomorphic map $u\colon B_{2^{\ell+1}\delta}\longrightarrow\mathbb{C}^n$ with $u(0)=0$ and $\|\mathrm{d}u\|_{L^p}\leqslant\epsilon$. This establishes (A.29). □

Corollary A.9. *Let δ, ϵ, $p\in\mathbb{R}^+$ with $p>2$, $\ell\in\mathbb{Z}^+$, and J be an almost complex structure on a compact Riemannian manifold X. There exists $C_\ell\in\mathbb{R}^+$ such that*

$$\big|\mathrm{d}_z u-\mathrm{d}_0 u\big|\leqslant C_\ell\|u\|_{L^p_1}|z|, \qquad \big|u(z)-T^{(\ell)}_{u;0}(z)\big|\leqslant C_\ell\|u\|_{L^p_1}|z|^{\ell+1} \qquad \forall\ z\in B_\delta$$

for every J-holomorphic map $u\colon B_{4\delta}\longrightarrow X$ with $\|\mathrm{d}u\|_{L^p}\leqslant\epsilon$.

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