E_∞-Comodules and Topological Manifolds

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Abstract of the Dissertation

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The first story begins with a question of Steenrod. He asked if the product in the cohomology of a triangulated space, which is associative and graded commutative, can be induced from a cochain level product satisfying the same two properties. He answered it in the negative after identifying homological obstructions among a collection of chain maps he constructed. Using later language, his construction could be said to endow the simplicial chains with an E_∞-coalgebra structure.

The second story also begins with a question: when is a space homotopy equivalent to a topological manifold? For dimensions greater than 4, an answer was provided by the work of Browder, Novikov, Sullivan and Wall in surgery theory, which in a later development was algebraically expressed by Ranicki as a single chain level invariant: the total surgery obstruction.

After presenting the necessary parts of these stories, the goal of this work will be to express the total surgery obstruction associated to a triangulated space in terms of comodules over the E_∞-coalgebra structure build by Steenrod on its chains.

a los árboles en que estas ideas se escribieron

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Introduction

The goal of this dissertation is to relate the theory of algebraic surgery developed predominantly by Andrew Ranicki, with that of E_∞-structures on chain complexes.

Steenrod's construction of higher chain approximations to the diagonal inclusion has been encoded, by several authors, as a functor from simplicial sets to their normalized chains enriched with the structure of a coalgebra over an E_∞-operad. The first section of Chapter 1 presents the definition of an algebraic operad as well as the less common notions of coalgebra over an operad and comodule over one such coalgebra. The second section presents the specific E_∞-operad $\mathcal{S}$ related to Steenrod's construction following the work of McClure-Smith [25], Berger-Fresse [4] and others. In the last section of Chapter 1, the first of the two main technical results of this dissertation is presented as Theorem 1.3.5. It has as a corollary that the category of based ordered simplicial complexes embeds as a full subcategory into the category of $\mathcal{S}$-coalgebras. Similar results have been obtained at the level of the homotopy category by Mandell [20], Smirnov [38], Smith [39] and others.

The first section of Chapter 2 revisits the theory of sheaves and cosheaves over posets, see [8], [37] or [14] for other sources. It uses the connection between posets and Alexandrov topological spaces, extended in Lemma 2.1.5 to a duality preserving equivalence, to emphasize the symmetry between sheaves and cosheaves over posets. This section closes with some homological algebra of such sheaves and cosheaves with values in an abelian category. In the second section of Chapter 2, the sheaf theory developed in the previous section is specialized to posets associated to ordered simplicial complexes. The notion of tensor product of functor is used to define the Ranicki duality functors of complexes of sheaves and cosheaves, whose geometry is made apparent by the pair subdivision sheaf and cosheaf. The pair subdivision sheaf is also used to define the visible symmetric complex of a regular pseudomanifold, see Construction 2.2.18, which plays a central role in the application of the theory to manifold existence and uniqueness problems. The third section of Chapter 2 contains, as Theorem 2.3.4, the second main technical result of this work. It states that the category of complexes of sheaves over an ordered simplicial complex X with values in Ab embeds, as a full differential graded subcategory, into the category of comodules over the $\mathcal{S}$-coalgebra $\mathrm{C}_\bullet(X)$. This theorem is used to relate the algebraic surgery theory of Ranicki with comodules over E_∞-coalgebras. In particular, Theorem 2.3.13 and Theorem 2.3.15 provide existence and uniqueness statements for ANR homology manifold structures and topological manifold structures on the homotopy type of a Poincaré duality regular pseudomanifold, in terms of comodules on its $\mathcal{S}$-coalgebra of chains.

Chapter 1

Simplicial sets and $\mathcal{S}$-coalgebras

Convention. The term chain complex will be reserved for a homologically graded differential graded abelian group. The category of chain complexes, denoted by $\mathrm{Ab}_\bullet$, is enriched over itself, i.e. $\mathrm{Hom}_{\mathrm{Ab}_\bullet}(C, C') \in \mathrm{Ab}_\bullet$ for every pair C, $C' \in \mathrm{Ab}_\bullet$. In terms of this enrichment, chain maps correspond to 0-degree cycles, while chain homotopy equivalent morphisms correspond to homologous chains.

1.1 Operads, coalgebras and comodules

In this section, the definition of an algebraic operad is presented as well as the less common notions of coalgebra over an operad and comodule over one such coalgebra.

Definition 1.1.1. (Operad [22]) An (algebraic) **operad** consists of a collection of chain complexes $\mathcal{O}(n), n \geq 0$, a collection of chain maps

$$\gamma : \mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \to \mathcal{O}(j_1 + \cdots + j_k),$$

a chain map $\eta : R \to \mathcal{O}(1)$ and an action of the symmetric group Σ_k on $\mathcal{O}(k)$ satisfying the following conditions.

$O1$: (Associativity) The following diagram commutes, where $\sum_{s=1}^{k} j_s = j$, $\sum_{r=1}^{j} i_r = i$, $g_s = j_1 + \cdots + j_s$ and $h_s = i_{g_{s-1+1}} + .. + i_{g_s}$ for $1 \leq s \leq k$:

$$
\begin{tikzcd}
\mathcal{O}(k)\otimes\big(\bigotimes_{s=1}^{k}\mathcal{O}(j_s)\big)\otimes\big(\bigotimes_{r=1}^{j}\mathcal{O}(i_r)\big) \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[d, "\text{shuffle}"'] & \mathcal{O}(j)\otimes\big(\bigotimes_{r=1}^{j}\mathcal{O}(i_r)\big) \arrow[d, "\gamma"] \\
 & \mathcal{O}(i) \\
\mathcal{O}(k)\otimes\Big(\bigotimes_{s=1}^{k}\big(\mathcal{O}(j_s)\otimes\bigotimes_{q=1}^{j_s}\mathcal{O}(i_{g_s-1+q})\big)\Big) \arrow[r, "\mathrm{id}\otimes(\otimes_s\gamma)"'] & \mathcal{O}(k)\otimes\big(\bigotimes_{s=1}^{k}\mathcal{O}(h_s)\big). \arrow[u, "\gamma"']
\end{tikzcd}
$$

$O2$: (Unit) The following diagrams commute:

$$
\begin{tikzcd}
\mathcal{O}(k)\otimes R^k \arrow[r, "\cong"] \arrow[d, "\mathrm{id}\otimes\eta^k"'] & \mathcal{O}(k) \\
\mathcal{O}(k)\otimes\mathcal{O}(1)^k, \arrow[ur, "\gamma"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
R\otimes\mathcal{O}(j) \arrow[r, "\cong"] \arrow[d, "\eta\otimes\mathrm{id}"'] & \mathcal{O}(j) \\
\mathcal{O}(1)\otimes\mathcal{O}(j). \arrow[ur, "\gamma"'] &
\end{tikzcd}
$$

$O3$: (Equivariance) The following diagrams commute, where $\sigma \in \Sigma_k, \tau_s \in \Sigma_{j_s}$, the permutation $\sigma(j_1,\ldots,j_k) \in \Sigma_j$ permutes k blocks of letters as σ permutes k letters, and $\tau_1 \oplus \cdots \oplus \tau_k \in \Sigma_j$ is the block sum:

$$
\begin{tikzcd}
\mathcal{O}(k)\otimes\mathcal{O}(j_1)\otimes\cdots\otimes\mathcal{O}(j_k) \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\gamma"'] & \mathcal{O}(k)\otimes\mathcal{O}(j_{\sigma(1)})\otimes\cdots\otimes\mathcal{O}(j_{\sigma(k)}) \arrow[d, "\gamma"] \\
\mathcal{O}(j) \arrow[r, "\sigma(j_{\sigma(1)},\ldots,j_{\sigma(k)})"'] & \mathcal{O}(j),
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathcal{O}(k)\otimes\mathcal{O}(j_1)\otimes\cdots\otimes\mathcal{O}(j_k) \arrow[r, "\sigma\otimes\sigma^{-1}"] \arrow[d, "\gamma"'] & \mathcal{O}(k)\otimes\mathcal{O}(j_{\sigma(1)})\otimes\cdots\otimes\mathcal{O}(j_{\sigma(k)}) \arrow[d, "\gamma"] \\
\mathcal{O}(j) \arrow[r, "\sigma(j_{\sigma(1)},\ldots,j_{\sigma(k)})"'] & \mathcal{O}(j).
\end{tikzcd}
$$

Definition 1.1.2. (Coalgebra) Let $\mathcal{O}$ be an operad. An **$\mathcal{O}$-coalgebra** is a chain complex C together with chain maps

$$\theta : \mathcal{O}(j)\otimes C \to C^j$$

satisfying the following conditions.

$cA1$: (Associativity) Let $\sum_{s=1}^{k} j_s = j$, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(k) \otimes \mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \otimes C \arrow[r, "\gamma\otimes\mathrm{id}"] \arrow[dd, "\mathrm{id}\otimes\theta"'] & \mathcal{O}(j) \otimes C \arrow[d, "\theta"] \\
 & C^j \\
\mathcal{O}(j_1) \otimes \cdots \otimes \mathcal{O}(j_k) \otimes C^k \arrow[r, "\mathrm{shuffle}"'] & \mathcal{O}(j_1) \otimes C \otimes \cdots \otimes \mathcal{O}(j_k) \otimes C. \arrow[u, "\theta^k"']
\end{tikzcd}$$

$cA2$: (Unit) The following diagram commutes:

$$\begin{tikzcd}
R \otimes C \arrow[r, "\cong"] \arrow[d, "\gamma\otimes\mathrm{id}"'] & C \\
\mathcal{O}(1) \otimes C. \arrow[ur, "\theta"'] &
\end{tikzcd}$$

$cA3$: (Equivariance) Let $\sigma \in \Sigma_j$, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(j) \otimes C \arrow[r, "\sigma\otimes\mathrm{id}"] \arrow[d, "\theta"'] & \mathcal{O}(j) \otimes C \arrow[d, "\theta"] \\
C^j \arrow[r, "\sigma"'] & C^j.
\end{tikzcd}$$

A **morphisms of $\mathcal{O}$-coalgebras** is a chain map commuting strictly with all the above structure. The category of $\mathcal{O}$-coalgebras will be denoted by $\mathrm{coAlg}_{\mathcal{O}}$.

Definition 1.1.3. (Comodule) Let $\mathcal{O}$ be an operad and C an $\mathcal{O}$-coalgebra. A C**-comodule** is a chain complex D together with chains maps

$$\lambda : \mathcal{O}(j) \otimes D \to D \otimes C^{j-1}$$

satisfying the following conditions.

$cM1$: (Associativity) Let $\sum_{s=1}^{k} j_s = j$, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(k)\otimes\mathcal{O}(j_1)\otimes\cdots\otimes\mathcal{O}(j_k)\otimes D \arrow[rr, "\gamma\otimes\mathrm{id}"] \arrow[dd, "\mathrm{id}\otimes\lambda"'] & & \mathcal{O}(j)\otimes M \arrow[d, "\theta"] \\
& & D\otimes C^{j-1} \\
\mathcal{O}(j_1)\otimes\cdots\otimes\mathcal{O}(j_k)\otimes D\otimes C^{k-1} \arrow[r, "\text{shuffle}"'] & \mathcal{O}(j_1)\otimes D\otimes\cdots\otimes\mathcal{O}(j_k)\otimes C. \arrow[r, phantom] & {} \arrow[u, "\lambda\otimes\theta^{k-1}"']
\end{tikzcd}$$

$cM2$: (Unit) The following diagram commutes:

$$\begin{tikzcd}
R\otimes D \arrow[r, "\cong"] \arrow[d, "\gamma\otimes\mathrm{id}"'] & D \\
\mathcal{O}(1)\otimes D. \arrow[ur, "\theta"'] &
\end{tikzcd}$$

$cM3$: (Equivariance) Let $\sigma\in\Sigma_{j-1}\subset\Sigma_j$, then the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(j)\otimes D \arrow[r, "\sigma\otimes\mathrm{id}"] \arrow[d, "\theta"'] & \mathcal{O}(j)\otimes D \arrow[d, "\theta"] \\
D\otimes C^{j-1} \arrow[r, "\mathrm{id}\otimes\sigma"'] & D\otimes C^{j-1}.
\end{tikzcd}$$

A **morphisms of C-comodules** is a homeomorphism of abelian groups commuting strictly with all the above structure. The category of $\mathcal{O}$-comodules is enriched over $\mathrm{Ab}_\bullet$ and will be denoted by $\mathrm{coMod}_C^{\mathcal{O}}$.

Example 1.1.4. The operad $\mathcal{A}$ has $\mathcal{A}(j)=\mathbb{Z}[\Sigma_j]$ with unit map equal to the identity and product maps dictated by the equivariance formulas. An $\mathcal{A}$-coalgebra C is the same thing as a coassociative coalgebra. The operad product encodes all of the iterates and permutations of the coproduct of the coalgebra. A C-comodule D in the operadic sense is an C-bicomodule in the classical sense.

Definition 1.1.5. (E_∞-operad [13]) An operad $\mathcal{O}$ is said to be an E_∞ operad if it satisfies:

$E1$: (Unital) $\mathcal{O}(0)=\mathbb{Z}$.

$E2$: (Σ-free) Each Σ_j acts freely on $\mathcal{O}(j)$.

$E3$: (Contractible) Each $\mathcal{O}(j)$ has the homology of a point.

1.2 The operad $\mathcal{S}$

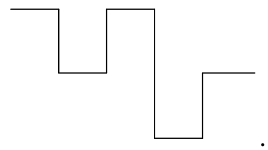
This section collects results related to an E_∞-operad studied by several researchers, whose combinatorial nature and explicit coaction on normalized chains makes it suitable for the applications of this work.

The coaction goes back to Steenrod construction in [40] of a chain approximation to the diagonal inclusion of triangulated spaces. The definition of an operad for which this coaction give rise to a natural coalgebra structure on the normalized chains of simplicial sets, appears in the proof of Deligne's conjecture by McClure-Smith [24] and is treated under the name "sequence operad" by the same authors in [25], where they present a filtration of it by E_n-suboperads. Work by Berger-Fresse in [4] uses the same operad with the name "surjection operad". Jonathan Potts' thesis [28] describes this operad with the name "step operad" and Jones-Adamaszek relate it to join operations of augmented symmetric simplicial sets in [1].

The definition of this E_∞-operad, which will be denoted $\mathcal{S}$ to stand for Steenrod, sequence, surjection or step, will be presented below together with its filtration by E_n-suboperads and its natural coaction on the normalized chain complex of simplicial sets.

Chain complex of $\mathcal{S}$. Let $\mathcal{S}(r)_d$ be the free abelian group generated by all functions from $\{1, \dots, r+d\}$ to $\{1, \dots, r\}$ quotiented by the submodule generated by all non-surjective functions and the surjections u for which there exists $i \in \{1, \dots, r+d-1\}$ so that $u(i) = u(i+1)$. Set $\mathcal{S}(0)_0 = \mathbb{Z}$ and $\mathcal{S}(0)_d = 0$ for $d > 0$. The module $\mathcal{S}(r)_d$ is free and any of the generators $u : \{1, \dots, r+d\} \to \{1, \dots, r\}$ will be identified with its ordered image $(u(1), \dots, u(r+d))$, which will be referred to as the **coordinates** of u.

To study examples a diagrammatical representation of the surjections proves useful, as an illustration that generalizes one has that $(1, 2, 1, 3, 2)$ is represented by



Define $\partial : \mathcal{S}(r)_d \to \mathcal{S}(r)_{d-1}$ by

$$u \mapsto \sum_{i=1}^{r+d} \varepsilon_i^u \cdot (u(1), \ldots, \widehat{u(i)}, \ldots, u(r+d))$$

with ε_i^u a sign to be specified. In order to determine ε_i^u, separate the coordinates of u into disjoint sets each characterized by one of the following properties:

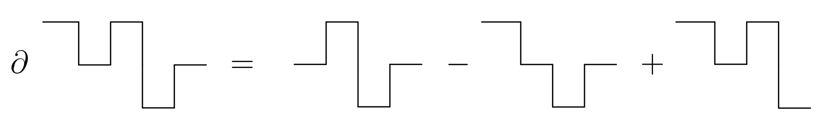
a) The value of the coordinate equals the value of a coordinate to its right i.e. $u(i) = u(i+j)$ for some positive j.

b) The value of the coordinate is different from all coordinates to its right but equal to one on its left i.e. $u(i) \neq u(i+j)$ and $u(i) = u(i-k)$ for all j and some k positives.

c) The value of the coordinate is different from all other coordinates i.e. $u(i) \neq u(j)$ for any i.

Consider the set $\{u(i_1), u(i_2), \ldots\}$ of coordinates satisfying a) indexed so that $j < j'$ implies $i_j < i_{j'}$. Define for them $\varepsilon_{i_j}^u = (-1)^{j-1}$. For coordinates $u(i)$ satisfying b) define $\varepsilon_i^u = -\varepsilon_{i_j}^u$ with $u(i_j) = u(i)$ satisfying a) and $u(i_{j'}) \neq u(i)$ for all $j' > j$. Coordinates $u(i)$ satisfying c) need no definition of ε_i^u since $(u(1), \ldots, \widehat{u(i)}, \ldots, u(r+d)) = 0$.

For example if $u = (1,2,1,3,2)$ then $u(1)$ and $u(2)$ satisfy a), $u(3)$ and $u(5)$ satisfy b) and $u(4)$ satisfy c) so



Operadic composition of $\mathcal{S}$. The maps

$$\circ_k : \mathcal{S}(r)_d \otimes \mathcal{S}(s)_e \to \mathcal{S}(r+s-1)_{d+e}$$

are defined as follows. Let $u \otimes v \in \mathcal{S}(r)_d \otimes \mathcal{S}(s)_e$ and let $\{i_1, \ldots, i_n\} = u^{-1}(k)$. For any splitting π of the coordinates of v into n subsequences

$$\big(v(l_0), \ldots, v(l_1)\big)\ \big(v(l_1), \ldots, v(l_2)\big)\ \cdots\ \big(v(l_{n-1}), \ldots, v(l_n)\big) \tag{1.1}$$

set the coordinates of a new surjection $u \circ_k^\pi v$ to be obtained from those of u by first replacing $u(i_j)$ by the j-th subsequence of v, adding $k-1$ to those coordinates and then adding $s-1$ to the coordinates $u(i)$ which are greater than n. Define

$$u \circ_k v = \sum_\pi \varepsilon_k^\pi \cdot (u \circ_k^\pi v)$$

with ε_k^π a sign to be specified. In order to do so, notice that $u^{-1}(k) = \{i_1, \ldots, i_n\}$ induces a collection of subsequences

$$\cdots \big(u(i_{j-1}+1), \ldots, u(i_j - 1)\big)\big(u(i_j)\big)\big(u(i_j+1), \ldots, u(i_{j+1}-1)\big)\big(u(i_{j+1})\big) \cdots \tag{1.2}$$

and that in order to do the replacements in the definition of $u \circ_k^\pi v$ we can think of passing the subsequences from (1.1) across those from (1.2). The sign ε_k^π will then be computed following the Koszul sign rule after defining the notion of degree for subsequences of coordinates.

Let w be an arbitrary surjection and

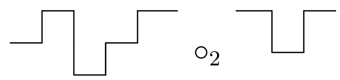
$$\cdots \big(w(m_{t-1}+1), \ldots, w(m_t)\big)\big(w(m_t - 1), \ldots, w(m_{t+1})\big) \cdots$$

be the partition of w into (consecutive and disjoint) subsequences determined by the coordinates $w(m_t)$ satisfying a). The degree of a general subsequence is then defined to be one less than the number of these subsequences that it overlaps with.

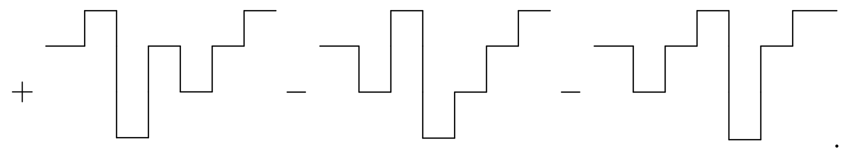
For example, to compute $u \circ_2 v = (2,1,3,2,1) \circ_2 (1,2,1)$ we first compute the degrees of the relevant subsequences, which requires the counting of overlaps with $(2)(1)(3,2,1)$ for subsequences of u and with $(1)(2,1)$ for those of v. Placing the degree as a subindex one obtains

$$(2)_0(1,3)_1(2)_0(1)_0\ ; \qquad \begin{array}{c} (1)_0(1,2,1)_1 \\ (1,2)_1(2,1)_0 \\ (1,2,1)_1(1)_0. \end{array}$$

Therefore,



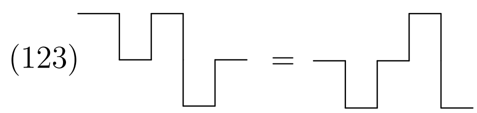
equals



Symmetric action on $\mathcal{S}$. Define an action of Σ_r on $\mathcal{S}(r)_d$ by

$$\sigma \cdot u = \big(\sigma(u(1)), \ldots, \sigma(u(r+d))\big).$$

For example,



Lemma 1.2.1. *The structure defined above makes $\mathcal{S} = \{\mathcal{S}(r)_\bullet\}$ into an E_∞-operad (see Definition 1.1.5).*

Proof. The action of Σ_r on $\mathcal{S}(r)_\bullet$ is free since it is free on the first coordinate of any surjection. The proof that $\mathcal{S}(r)_\bullet$ has the homology of a point is part (c) of Theorem 2.15 in [25]. □

Filtration of $\mathcal{S}$ by E_n-operads Consider a surjection u in $\mathcal{S}(r)_\bullet$ and a pair $i < j \leq r$. Define u_{ij} to be the sequence obtained from the sequence of coordinates of u by removing all elements different from i and j. For example, if $u = (2,1,3,1,2)$ then $u_{12} = (2,1,1,2)$, $u_{13} = (1,3,1)$ and $u_{23} = (2,3,2)$. To each such sequence assign the number of pairs of distinct consecutive coordinates and name it the **change number**. Using the previous example one sees that the change number of all u_{ij} is 2. For any u define its **filtration weight** to be the largest change number among all possible u_{ij} and define $\mathcal{S}^n$ to be the suboperad generated by all surjections whose filtration weight is less than or equal to n.

The following appears as Theorem 3.5 in [25].

Lemma 1.2.2. *The constructions above defines a filtration by suboperads*

$$\mathcal{S}^1 \leq \mathcal{S}^2 \leq \cdots \leq \mathcal{S}^\infty = \mathcal{S}$$

with $\mathcal{S}^n$ an E_n-operad.

$\mathcal{S}$-coalgebra structure on normalized chains. Let Δ be the simplicial category as described in Definition A.17 and recall from Example A.22 the functor $\mathrm{C}_\bullet : \Delta \to \mathrm{Ab}_\bullet$ whose left Kan extension along the Yoneda embedding defines the normalized chains of simplicial sets. The purpose of this section is to construct a compatible collection of maps

$$\mathcal{S}(k) \otimes \mathrm{C}_\bullet[0, \dots, n] \to \mathrm{C}_\bullet[0, \dots, n]^{\otimes k},$$

indexed by $k, n \geq 0$, determining a functor represented as a dotted arrow in the following commutative diagram

$$\begin{tikzcd} & \mathrm{coAlg}_{\mathcal{S}} \arrow[d, "\text{forget}"] \\ \mathrm{sSet} \arrow[ur, dashed] \arrow[r, "\mathrm{C}_\bullet"'] & \mathrm{Ab}_\bullet . \end{tikzcd}$$

Let $[i_1, \dots, i_l]$ be the image of an order preserving function $[0, \dots, l] \to [0, \dots, n]$. This function induces a chain map $\mathrm{C}_\bullet[0, \dots, l] \to \mathrm{C}_\bullet[0, \dots, n]$ and the image the top dimensional generator of $\mathrm{C}_\bullet[0, \dots, l]$ will be identified with the generator $[i_1, \dots, i_l]$ in $\mathrm{C}_\bullet[0, \dots, n]$ if $i_j \neq i_{j+1}$ for all j, being 0 otherwise.

Let $u \in \mathcal{S}(r)_d$ be a surjection and $[0, \dots, n] \in \Delta$. Let π stand for a choice of $(r + d - 1)$ elements of $\{0, \dots, n\}$ satisfying

$$0 = n_0 \leq n_1 \leq \cdots \leq n_{r+d-1} \leq n_{r+d} = n$$

and associate to this choice π a collection of generators of $\mathrm{C}_\bullet[0, \dots, n]$

$$\big\{[n_{i-1}, \dots, n_i]\big\}_{i=0}^{r+d}$$

with consecutive vertices. Such generators will be referred to as **intervals**.

For $k \leq r$ define $L_\pi(k)$ to be 0 in case there exists a pair of intervals $[n_{i-1}, \dots, n_i]$ and $[n_{j-1}, \dots, n_j]$ with $u(i) = u(j) = k$ and a common vertex, or define $L_\pi(k)$ to be the generator in $\mathrm{C}_\bullet[0, \dots, n]$ whose set of vertices is the union of the vertices of all intervals $[n_{i-1}, \dots, n_i]$ with $u(i) = k$.

For every choice of π define an element in $\mathrm{C}_\bullet[0,\ldots,n]^{\otimes r}$ by

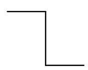
$$u_\pi[0,\ldots,n] = L_\pi(1)\otimes L_\pi(2)\otimes\cdots\otimes L_\pi(r)$$

and set

$$u[0,\ldots,n] = \sum_\pi \varepsilon_{u,\pi}\cdot u_\pi[0,\ldots,n]$$

with $\varepsilon_{u,\pi}$ a sign to be specified.

Example 1.2.3. If $u=(1,2)$, the value of $u[0,1,..,n]$ is equal to



$$[0,1,\ldots,n] = \textstyle\sum_i [0,\ldots,i]\otimes[i,\ldots,n]$$

$$= \begin{array}{cccc} 0 & & & \\ 0 & 1 & \ldots & n \end{array} + \begin{array}{cccc} 0 & 1 & & \\ & 1 & \ldots & n \end{array} + \cdots + \begin{array}{cccc} 0 & 1 & \ldots & n \\ & & & n, \end{array}$$

with signs computed to be all positive in Example 1.2.4.

Signs of the $\mathcal{S}$-coalgebra structure In order to specify the sign $\varepsilon_{u,\pi}$ one distinguishes between two types of intervals.

a) **Internal** intervals $[n_{i-1},\ldots,n_i]$ satisfy $u(i)=u(i+j)$ for some positive j.

b) **Final** intervals $[n_{i-1},\ldots,n_i]$ satisfy $u(i)\neq u(i+j)$ for all positive j.

Define the **degree** of an interval $[n_{i-1},\ldots,n_i]$ to be $n_i-n_{i-1}+1$ if it is internal or n_i-n_{i-1} if it is final.

Consider the permutation taking

$$(u(1),u(2),\ldots,u(r+d))\mapsto(1,\ldots,1,2,\ldots,2,\ldots,r,\ldots,r)$$

and obtain, by Koszul's rule, a sign $\varepsilon^{per}_{u,\pi}$ from the induced permutation of the graded intervals

$$\big([0,\ldots,n_1],[n_1,\ldots,n_2],\ldots,[n_{r+d-1},\ldots,n_{r+d}]\big).$$

Consider all internal intervals $[n_{i-1},\ldots,n_i]$ to be the indexing set of the sum $\sum n_i$. Let this sum be the exponent of a sign $\varepsilon^{pos}_{u,\pi}$ and set

$$\varepsilon_{u,\pi}=\varepsilon^{per}_{u,\pi}\cdot\varepsilon^{pos}_{u,\pi}.$$

Example 1.2.4. Let $u = (1,2)$ as in Example 1.2.3. For any $[0,\ldots,n]$ and any π, all signs $\varepsilon_{u,\pi}$ are positive since every $[n_{i-1},\ldots,n_i]$ is final, so $\varepsilon^{pos}_{u,\pi} = 1$, and $\varepsilon^{per}_{u,\pi} = 1$ because $(1,2)$ is already in the correct order.

Example 1.2.5. Let $u = (\ldots,2,1,2)$ be one of the two generators of $\mathcal{S}(2)_d$ and consider $[0,\ldots,d] \in \Delta$ of the same dimension as the degree of u.

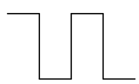
In order to compute the coaction of u on $[0,\ldots,d]$, thought of as one of the generators of $\mathrm{C}_\bullet[0,\ldots,d]$, one notices that the only choice for π

$$0 = n_0 \leq n_1 \leq \cdots \leq n_{r+d-1} \leq n_{r+d} = d$$

leading to a non-zero $u_\pi[0,\ldots,d]$ satisfies $n_i \neq n_{i+1}$ for all $i = 1,\ldots,d$. Because of the relation between the degree of the surjection and the dimension of the simplex, this choice is unique and given by $n_i = i-1$ for all $i = 1,\ldots,d$, in other words

$$0 \leq 0 < 1 < 2 < \cdots < (d-1) < d \leq d,$$

so up to a sign ε_d one has

$\cdots$  $\Big([0,1,\ldots,d]\Big)$

$$= \varepsilon_d \cdot \begin{matrix} 0 & 1 & \ldots & d \\ 0 & 1 & \ldots & d \end{matrix}$$

$$= \varepsilon_d \cdot \ [0,1,\ldots,d] \otimes [0,1,\ldots,d].$$

In order to determine the sign ε_d notice that only the last two intervals $[d-1,d]$ and $[d,d]$ are final with degrees 1 and 0 respectively. All other intervals $[i-1,i]$ are internal and have degree 2 except for $[0,0]$ which has degree 1. Therefore, a permutation contributes with a negative sign if and only if it exchanges $[0,0]$ and $[d-1,d]$. Consequently, the permutation

$$(\ldots,2,1,2) \mapsto (1,1,\ldots,2,2)$$

induces the permutation sign $\varepsilon^{per}_d = (-1)^d$. The position sign ε^{pos}_d, determined by the internal intervals, is equal in this case to $\sum_{i=0}^{d-1} i$ so

$$\varepsilon_d = \varepsilon^{per}_d \cdot \varepsilon^{pos}_d = (-1)^d \cdot \sum_{i=0}^{d-1} i = (-1)^{d(d+1)/2}.$$

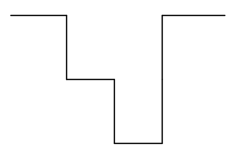
Example 1.2.6. Let $u = (1,2,3,1)$ and $[0,1,2] \in \Delta$, this example will compute $u[0,1,2]$. A choice of

$$\pi : 0 = n_0 \leq n_1 \leq n_2 \leq n_3 \leq n_4 = 2$$

leads to a non-zero term u_π if and only if $n_1 \neq n_3$. The following table summarizes for such possible choices the degrees of the associated internal and final intervals, as well as the resulting permutation and position signs.

n_1	n_2	n_3	$\lvert[n_0,\dots,n_1]\rvert$	$\lvert[n_1,\dots,n_2]\rvert$	$\lvert[n_2,\dots,n_3]\rvert$	$\lvert[n_3,\dots,n_4]\rvert$	ε^{per}	ε^{pos}
0	0	1	1 (i)	0 (f)	1 (f)	1 (f)	-1	1
0	0	2	1 (i)	0 (f)	2 (f)	0 (f)	1	1
0	1	1	1 (i)	1 (f)	0 (f)	1 (f)	-1	1
0	1	2	1 (i)	1 (f)	1 (f)	0 (f)	1	1
0	2	2	1 (i)	2 (f)	0 (f)	0 (f)	1	1
1	1	2	2 (i)	0 (f)	1 (f)	0 (f)	1	-1
1	2	2	2 (i)	1 (f)	0 (f)	0 (f)	1	-1

Therefore,



[0, 1, 2]

equals

$$-\begin{array}{ccc} 0 & 1 & 2 \\ 0 & & \\ 0 & 1 & \end{array} \quad + \begin{array}{ccc} 0 & & 2 \\ 0 & & \\ 0 & 1 & 2 \end{array} \quad - \begin{array}{ccc} 0 & 1 & 2 \\ 0 & 1 & \\ & 1 & \end{array}$$

$$+\begin{array}{ccc} 0 & & 2 \\ 0 & 1 & \\ & 1 & 2 \end{array} \quad + \begin{array}{ccc} 0 & & 2 \\ 0 & 1 & 2 \\ & & 2 \end{array} \quad - \begin{array}{ccc} 0 & 1 & 2 \\ & 1 & \\ & 1 & 2 \end{array} \quad - \begin{array}{ccc} 0 & 1 & 2 \\ & 1 & 2 \\ 0 & & 2. \end{array}$$

Lemma 1.2.7. *The maps define above*

$$\mathcal{S}(k) \otimes \mathrm{C}_\bullet[0,\dots,n] \to \mathrm{C}_\bullet[0,\dots,n]^{\otimes k}$$

determine a functor which is represented by the dotted arrow in the following commutative diagram

$$\begin{tikzcd} \Delta \arrow[r, dashed] \arrow[dr, "\mathrm{C}_\bullet"'] & \mathrm{coAlg}_{\mathcal{S}} \arrow[d, "forget"] \\ & \mathrm{Ab}_\bullet . \end{tikzcd}$$

Proof. This follows from part (b) of Theorem 2.15 in [25], see also part (b) of Remark 2.16 in the same reference. □

Definition 1.2.8. As in Definition A.19, a functor can be constructed by taking the left Kan extension along the Yoneda embedding of the functor of Lemma 1.2.7. Such functor is represented by the dotted arrow in the following commutative diagram

$$\begin{tikzcd} & \mathrm{coAlg}_{\mathcal{S}} \arrow[d, "\text{forget}"] \\ \mathrm{sSet} \arrow[ur, dashed] \arrow[r, "\mathrm{C}_\bullet"'] & \mathrm{Ab}_\bullet \end{tikzcd}$$

and the image of any simplicial set X by this functor will be referred to as the **$\mathcal{S}$-coalgebra structure on $\mathrm{C}_\bullet(X)$.**

For any $1 \leq k \leq \infty$, let $\mathcal{S}^k(2)$ denote the suboperad generated by the arity 2 part of the k-level of the filtration described in Lemma 1.2.2. Composing the above functor with the forgetful functor one has

$$\begin{tikzcd} & \mathrm{coAlg}_{\mathcal{S}} \arrow[d, "\text{forget}"] \\ \mathrm{sSet} \arrow[ur, "\mathrm{C}_\bullet"] \arrow[r, dashed, "\mathrm{C}_\bullet"'] & \mathrm{coAlg}_{\mathcal{S}^k(2)}, \end{tikzcd}$$

and the image of any simplicial set X by this functor will be referred to as the **$\mathcal{S}^k(2)$-coalgebra structure on $\mathrm{C}_\bullet(X)$.**

Notation 1.2.9. Let X be a simplicial set and consider the $\mathcal{S}$-coalgebra structure on $\mathrm{C}_\bullet(X)$. For every surjection $u \in S(k)_\bullet$ one has an abelian group homomorphism

$$\mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(X)^{\otimes k}.$$

For $u = (\dots, 2, 1, 2)$ one of the generators of $\mathcal{S}(2)_d$, the associated map will be denoted

$$\Delta_d : \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(X) \otimes \mathrm{C}_\bullet(X).$$

1.3 Simplicial sets and $\mathcal{S}$-coalgebras

In this section, the first of the two main technical results of this work is presented as Theorem 1.3.5. It follows from it that the category of based ordered simplicial complexes embeds into the category of coalgebras over the E_∞-operad $\mathcal{S}$, see Corollary 1.3.8. It also implies that the category of pointed small categories fully embeds into the category of coalgebras over the E_2-operad $\mathcal{S}^2$, see Corollary 1.3.10.

Similar results at the level of the homotopy category of simplicial sets have been obtained by Mandell [20], Smirnov [38], Smith [39] and others.

Lemma 1.3.1. *Let $X, Y \in \mathrm{sSet}$ and $\sigma \in X_n$. If $f : \mathrm{C}_\bullet(X) \to \mathrm{C}_\bullet(Y)$ is a homomorphism satisfying $(f \otimes f)\, \Delta_n\, \sigma = \Delta_n\, f\sigma$, then either*

$$f\sigma = 0 \quad \textit{or} \quad f\sigma \in Y_n.$$

Proof. Identifying non-degenerate simplices with their corresponding chains, Example 1.2.5 shows that for any n-dimensional simplex ρ one has $\Delta_n\, \rho = (-1)^{\varepsilon_n} \rho \otimes \rho$. In particular, the condition on f implies that $f\sigma = \sum_i a_i\, \tau_i$ for some $\tau_i \in Y_n$. Therefore,

$$(f \otimes f)\, \Delta_n\, \sigma = \Delta_n\, f\sigma$$

implies

$$\sum_i a_i\, \tau_i \otimes \sum_i a_i\, \tau_i = \sum_i a_i\, \tau_i \otimes \tau_i.$$

From which it follows that

$$a_i\, a_j = 0 \text{ if } i \neq j \text{ and } a_i^2 = a_i.$$

The only way these equations are satisfied is if all but possibly one of the coefficients a_i are zero, with the possible exception being equal to 1. It follows that $f\sigma = 0$ or $f\sigma = \tau_i$ for some $\tau_i \in Y_n$. □

The next definition comes from [23].

Definition 1.3.2. The following are three properties that a simplicial set X might have.

(A) X has property A if every face of a non-degenerate simplex of X is non-degenerate.

(B) X has property B if the $n+1$ vertices of any non-degenerate n-simplex of X are distinct.

(C) X has property C if for any set of $n+1$ distinct vertices of X, there is at most one non-degenerate n-simplex of X whose vertices are the elements of that set.

Definition 1.3.3. (Based simplicial sets) A simplicial set is said to **based** if it comes with a chosen vertex $*$. A **based simplicial map** between based simplicial sets is a simplicial map of the underlying simplicial sets preserving the base point. Denote the category of based simplicial sets by sSet_* and let $(-)_+ : \mathrm{sSet} \to \mathrm{sSet}_*$ be the functor adding a disjoint base point. Notice that $\mathrm{C}_\bullet(X_+, *)$ is isomorphic to $\mathrm{C}_\bullet(X)$ as $\mathcal{S}$-coalgebras.

Terminology 1.3.4. A functor $F : \mathcal{C} \to \mathcal{C}'$ is said to be **faithful**, respectively **full**, if for every $a, b \in \mathcal{C}$ the function

$$\mathrm{Hom}_{\mathcal{C}}(a, b) \to \mathrm{Hom}_{\mathcal{C}'}(Fa, Fb)$$

is injective, respectively surjective.

Theorem 1.3.5. *Let* $\mathrm{sSet}_*^{(n)}$ *denote the full subcategory of n-dimensional based simplicial sets as described in Definition A.17. Let* $\mathcal{S}^k$ *be the* E_n*-suboperad of* $\mathcal{S}$ *which is the k-th level of the filtration described in Lemma 1.2.2, see also Definition 1.2.8.*

1. *The functor* $\mathrm{C}_\bullet(-, *) : \mathrm{sSet}_* \to \mathrm{Ab}_\bullet$ *if faithful.*
2. *The functor* $\mathrm{C}_\bullet(-, *) : \mathrm{sSet}_*^{(1)} \to \mathrm{coAlg}_{\mathcal{S}^2(2)}$ *if full.*
3. *The functor* $\mathrm{C}_\bullet(-, *) : \mathrm{sSet}_*^{(2)} \to \mathrm{coAlg}_{\mathcal{S}^3(2)}$ *if full when restricted to simplicial sets satisfying property A.*
4. *The functor* $\mathrm{C}_\bullet(-, *) : \mathrm{sSet}_*^{(3)} \to \mathrm{coAlg}_{\mathcal{S}^4(2)}$ *if full when restricted to simplicial sets satisfying property B.*
5. *The functor* $\mathrm{C}_\bullet(-, *) : \mathrm{sSet}_*^{(n)} \to \mathrm{coAlg}_{\mathcal{S}^{n+1}(2)}$ *if full when restricted to simplicial sets satisfying properties B and C.*

To improve the readability of this work, the proof of Theorem 1.3.5 will be postponed until after a few corollaries are drawn from it.

Definition 1.3.6. (Ordered simplicial complexes) An **ordered simplicial complex** $X = (V, S)$ is a pair consisting of a partially ordered set V and a collection S of nonempty subsets of V with each $s \in S$ inheriting a total order, such that

$$\forall v \in V,\ [v] \in S \quad \text{and} \quad s' \subset s \in S \Rightarrow s' \in S.$$

A **map of ordered simplicial complexes** $(V, S) \to (V', S')$ is an order preserving map $F : V \to V'$ such that

$$[v_1, ..., v_k] \in S \Rightarrow [Fv_1, ..., Fv_k] \in S'.$$

This category is denoted SC and its based version is denoted SC_*.

Remark 1.3.7. The functor sending an ordered simplicial complex to the simplicial set whose non-degenerate n-simplices correspond to subsets in S of cardinality $n-1$, and whose degenerate simplices are freely generated; is a full and faithful functor whose essential image is the full subcategory of simplicial sets satisfying properties B and C. See [25] for more on this. The categories SC and SC_* will be identified with their essential image.

Corollary 1.3.8. *The functor*

$$\mathrm{C}_\bullet(-, *) : \mathrm{SC}_* \to \mathrm{coAlg}_{\mathcal{S}(2)}$$

is full and faithful.

Proof. Notice that an $\mathcal{S}(2)$-coalgebra map is in particular an $\mathcal{S}^k(2)$-coalgebra map for every $k > 0$. By part *1* of Theorem 1.3.5, $\mathrm{C}_\bullet(-, *) : \mathrm{SC}_* \to \mathrm{coAlg}_{\mathcal{S}(2)}$ is faithful and by part *5* it is full. □

Remark 1.3.9. A consequences of this corollary is that the category of based simplicial complexes fully embeds into that of coalgebras over an E_∞-operad.

Corollary 1.3.10. *Let* Cat_* *be the category of pointed small categories,* N *the nerve functor as described in Example A.23, and* $\pi^{(1)}$ *the functor projecting to the* 1*-dimensional skeleton. The composition*

$$\mathrm{Cat}_* \xrightarrow{\mathrm{N}} \mathrm{sSet}_* \xrightarrow{\pi^{(1)}} \mathrm{sSet}_*^{(1)} \xrightarrow{\mathrm{C}_\bullet} \mathrm{coAlg}_{\mathcal{S}^2(2)}$$

is a full and faithful functor.

Proof. The composition $\pi^{(1)} \circ \mathrm{N}$ is a full and faithful functor, while the functor $\mathrm{C}_\bullet : \mathrm{sSet}_*^{(1)} \to \mathrm{coAlg}_{\mathcal{S}^2(2)}$ is faithful and full by parts *1* and *2* of Theorem 1.3.5. $\square$

Remark 1.3.11. A consequences of this corollary is that the category of pointed small categories fully embeds into that of coalgebras over an E_2 operad.

Proof of Theorem 1.3.5 The proof of the five parts of Theorem 1.3.5 will be parcelized into three independent groups, beginning with part *5* followed by part *1* and finished with the remaining three parts. This choice is made to increase the readability of this work by presenting the less computational proof first, followed by the increasingly tedious case-by-case analysis involved in the other proofs.

Proof of 5. Let $X^{(n)} = (V, S)$ and $Y^{(n)} = (V', S')$ be n-dimensional based ordered simplicial complexes and consider an $\mathcal{S}^{n+1}(2)$-coalgebra map

$$f : \mathrm{C}_\bullet(X^{(n)}, *) \to \mathrm{C}_\bullet(Y^{(n)}, *).$$

Identifying simplices with their corresponding chains, Lemma 1.3.1 implies for any $\sigma \in S$ that $f\sigma = 0$ or $f\sigma \in S'$. In particular, for vertices one has that $f[v] = 0$ or $f[v]$ is a vertex of $Y^{(n)}$. Define a $F : V \to V'$ by

$$Fv = \begin{cases} fv & \text{if } fv \neq 0, \\ * & \text{if } fv = 0 \text{ or } v = *. \end{cases}$$

It needs to be shown that this is a morphism of based ordered simplicial complexes inducing f. This directly follows from establishing the next claims:

<u>Claim 1</u>. If $f[v_1, ..., v_k] \neq 0$ then $f[v_1, ..., v_k] = [Fv_1, ..., Fv_k]$ satisfying that $Fv_i < Fv_{i+1}$ for all i.

<u>Claim 2</u>. If $f[v_1, ..., v_k] = 0$ then $Fv_i = Fv_{i+1}$ for some i or $fv_i = *$ for all i.

Proof of Claim 1. For vertices it holds trivially. Assume it holds for simplices of dimension $(k-1)$ and let $f[v_1, ..., v_k] = [w_1, \dots, w_k]$. Since f is a chain map one has

$$\sum (-1)^i [Fv_0, \dots, \widehat{Fv_i}, \dots, Fv_k] = \sum (-1)^i [w_0, \dots, \widehat{w_i}, \dots, w_k]$$

and the induction hypothesis proves the claim.

Proof of Claim 2. For vertices it holds trivially. Assume it holds for simplices of dimension $(k-1)$. Since f is a chain map one has

$$\sum(-1)^i f[v_0, \ldots, \widehat{v_i}, \ldots, v_k] = 0.$$

If $f[v_0, \ldots, \widehat{v_i}, \ldots, v_k] = 0$ for all i, then the induction hypothesis finishes the proof. If not, there must exist a pair $i < j$ so that $f[v_0, \ldots, \widehat{v_i}, \ldots, v_k] = f[v_0, \ldots, \widehat{v_j}, \ldots, v_k] \neq 0$. By Claim 1. that implies

$$[Fv_0, \ldots, \widehat{Fv_i}, \ldots, Fv_k] = [Fv_0, \ldots, \widehat{Fv_j}, \ldots, Fv_k]$$

so $Fv_i = Fv_{i+1}$. □

The next two groups of proofs rely on the standard identities, listed below, satisfied by the degeneracy and face maps of simplicial sets. These identities will be used without comment throughout the proofs.

i) $d_i d_j = d_{j-1} d_i$ if $i < j$,

ii) $d_i s_j = s_{j-1} d_i$ if $i < j$,

iii) $d_i s_j = id$ if $i = j$ or $i = j+1$,

iv) $d_i s_j = s_j d_{i-1}$ if $i > j+1$,

v) $s_i s_j = s_{j+1} s_i$ if $i = j$.

Proof of 1. Let $F, F' : X \to Y$ be based simplicial maps inducing the same chain map f. Identifying non-degenerate simplices with their corresponding chains, for any simplex σ, if $f\sigma \neq 0$ then $F\sigma = f\sigma = F'\sigma$. Since there are no degenerate 0-dimensional simplices, F and F' agree on X_0. Assume for an induction argument that F and F' agree up to a certain skeleton X_{k-1} and let $\sigma \in X_k$. If $\sigma = s_i\rho$ is degenerate then, using the induction hypothesis,

$$F\sigma = Fs_i\rho = s_iF\rho = s_iF\rho = F's_i\rho = F'\sigma.$$

The case $f\sigma \neq 0$ was already treated so assume σ is non-degenerate with $f\sigma = 0$. There must exist i, j and ρ, ρ' such that $F\sigma = s_i\rho$ and $F'\sigma = s_j\rho'$ with this data satisfying one of the following possibilities:

a) If $j = i$ then $\rho = \rho'$ since $\rho = d_i F\sigma = Fd_i\sigma = F'd_i\sigma = d_i F'\sigma = \rho'$. It follows that $F\sigma = s_i\rho = s_i\rho' = F'\sigma$.

b) If $j = i + 1$ then $\rho = s_i d_i \rho'$ and $\rho = \rho'$ since $\rho = d_i F\sigma = Fd_i\sigma = F'd_i\sigma = d_i F'\sigma = d_i s_{i+1}\rho' = s_i d_i \rho'$ and $\rho = d_{i+1}F\sigma = Fd_{i+1}\sigma = F'd_{i+1}\sigma = d_{i+1}F'\sigma = \rho'$. It follows that $F\sigma = s_i\rho = s_i s_i d_i \rho' = s_{i+1}s_i d_i\rho' = s_{i+1}\rho' = F'\sigma$.

c) If $j = i + k$ with $k > 1$ then $\rho = s_{i+k-1}d_{i+1}\rho'$ and $\rho' = s_i d_{i+k-1}\rho$ since $\rho = d_{i+1}F\sigma = Fd_{i+1}\sigma = F'd_{i+1}\sigma = d_{i+1}F'\sigma = d_{i+1}s_{i+k}\rho' = s_{i+k-1}d_{i+1}\rho'$ and $\rho' = d_{i+k}F'\sigma = F'd_{i+k}\sigma = Fd_{i+k}\sigma = d_{i+k}F\sigma = d_{i+k}s_i\rho = s_i d_{i+k-1}\rho'$. Applying d_{i+1} to the $\rho' = s_i d_{i+k-1}\rho$ gives $d_{i+1}\rho' = d_{i+k-1}\rho$. It follows that $F\sigma = s_i\rho = s_i s_{i+k-1}d_{i+1}\rho' = s_{i+k}s_i d_{i+1}\rho' = s_{i+k}s_i d_{i+k-1}\rho = s_{i+k}\rho' = F'\sigma$.

□

Proof of 2, 3 and 4. Lemma 1.3.1 will be use throughout this proof without mention, as will be the identification of non-degenerate simplices with their corresponding chains. Given a morphism f between the appropriate coalgebras, the following case-by-case procedure constructs a based simplicial map F with $\mathrm{C}_\bullet(F, *) = f$:

$\sigma \in X_0$:

a) $\underline{f\sigma \neq 0}$: set

$$F\sigma = f\sigma.$$

b) $\underline{f\sigma = 0 \text{ or } \sigma = *}$: set

$$F\sigma = *.$$

$\sigma \in X_1$:

a) $\underline{f\sigma \neq 0}$: set

$$F\sigma = f\sigma.$$

Since $(f \otimes f)\,\Delta_0\,\sigma = \Delta_0\,f\sigma$ and $\Delta_0\,\sigma = d_1\sigma \otimes \sigma + \sigma \otimes d_0\sigma$ one has

$$fd_j\sigma = d_j f\sigma \text{ for } j = 0, 1.$$

·) If $fd_j\sigma \neq 0$ then $Fd_j\sigma = fd_j\sigma = d_j f\sigma = d_j F\sigma$ for $j = 0, 1$.

·) If $fd_j\sigma = 0$ then $d_j f\sigma = 0$, therefore $d_j F\sigma = * = Fd_j\sigma$ for $j = 0, 1$.

b) $\underline{f\sigma = 0}$: set
$$F\sigma = s_0 F d_0 \sigma \bigl(= s_0 F d_1 \sigma \bigr).$$

Since $\partial f\sigma = f\partial\sigma$ one has $fd_0\sigma = fd_1\sigma$.

$\cdot$) $d_0 F\sigma = d_0 s_0 F d_0 \sigma = F d_0 \sigma$.

$\cdot$) $d_1 F\sigma = d_1 s_0 F d_0 \sigma = F d_0 \sigma = F d_1 \sigma$.

$\sigma \in X_2$: (assuming Y has Property A)

a) $\underline{f\sigma \neq 0}$: set
$$F\sigma = f\sigma.$$

By Property A, $d_j F\sigma$ is non-degenerate for all possible j. It follows that
$$d_j f\sigma \neq 0 \text{ for } j = 0, 1, 2.$$

Since $(f \otimes f)\, \Delta_0\, \sigma = \Delta_0\, f\sigma$ and $\Delta_0\, \sigma = d_1 d_2 \sigma \otimes \sigma + d_2 \sigma \otimes d_0 \sigma + \sigma \otimes d_0 d_0 \sigma$, one has in particular that $fd_2\sigma \otimes fd_0\sigma = d_2 f\sigma \otimes d_0 f\sigma$. It follows that
$$fd_2\sigma = d_2 f\sigma \text{ and } fd_0\sigma = d_0 f\sigma.$$

Since $(f \otimes f)\, \Delta_1\, \sigma = \Delta_1\, f\sigma$ and $\Delta_1\, \sigma = d_1\sigma \otimes \sigma - \sigma \otimes d_0\sigma - \sigma \otimes d_2\sigma$, one has in particular that
$$fd_1\sigma = d_1 f\sigma.$$

Therefore, $d_j F\sigma = d_j f\sigma = fd_j\sigma = Fd_j\sigma$ for $j = 1, 2, 3$.

b) $\underline{f\sigma = 0}$:

Since $f\partial\sigma = \partial f\sigma$ implies $fd_0\sigma - fd_1\sigma + fd_2\sigma = 0$ one has the following possibilities:

i) $\underline{fd_0\sigma = fd_1\sigma \neq 0 \;\&\; fd_2\sigma = 0}$: set
$$F\sigma = s_0 F d_0 \sigma.$$

$\cdot$) $d_0 F\sigma = d_0 s_0 F d_0 \sigma = F d_0 \sigma$.
$\cdot$) $d_1 F\sigma = d_1 s_0 F d_0 \sigma = F d_0 \sigma = F d_1 \sigma$.
$\cdot$) $d_2 F\sigma = d_2 s_0 F d_0 \sigma = s_0 d_1 F d_0 \sigma = s_0 F d_1 d_0 \sigma = s_0 F d_1 d_0 \sigma = s_0 F d_0 d_2 \sigma = F d_2 \sigma$.

ii) $\underline{fd_0\sigma = 0 \ \&\ fd_1\sigma = fd_2\sigma \neq 0}$: set

$$F\sigma = s_1 F d_1 \sigma$$

·) $d_0F\sigma = d_0s_1Fd_1\sigma = s_0d_0Fd_1\sigma = s_0Fd_0d_1\sigma = s_0Fd_0d_0\sigma = Fd_0\sigma$.
·) $d_1F\sigma = d_1s_1Fd_1\sigma = Fd_1\sigma$.
·) $d_2F\sigma = d_2s_1Fd_1\sigma = Fd_1\sigma = Fd_2\sigma$.

iii) $\underline{fd_0\sigma = fd_1\sigma = fd_2\sigma = 0}$: set

$$F\sigma = s_0 F d_0 \sigma.$$

·) $d_0F\sigma = d_0s_0Fd_0\sigma = Fd_0\sigma$.
·) $d_1F\sigma = d_1s_0Fd_0\sigma = Fd_0\sigma = s_0Fd_0d_0\sigma = s_0Fd_0d_1\sigma = Fd_1\sigma$.
·) $d_2F\sigma = d_2s_0Fd_0\sigma = d_2s_0s_0Fd_1d_0\sigma = d_2s_1s_0Fd_0d_2\sigma = s_0Fd_0d_2\sigma = Fd_2\sigma$.

$\sigma \in X_3$: (assuming Y has Property B)

a) $\underline{f\sigma \neq 0}$:

By Property A, $d_jF\sigma$ and $d_jd_iF\sigma$ are non-degenerate for all possible i, j. It follows that

$$d_jf\sigma \neq 0 \text{ and } d_jd_if\sigma \neq 0 \text{ for all possible } i, j.$$

Since $(f \otimes f)\,\Delta_0\,\sigma = \Delta_0\,f\sigma$ and $\Delta_0\,\sigma = d_1d_2d_3\sigma \otimes \sigma + d_2d_3\sigma \otimes d_0\sigma + d_3\sigma \otimes d_0d_0\sigma + \sigma \otimes d_0d_0d_0\sigma$, one has in particular that

$$fd_0\sigma = d_0f\sigma \text{ and } fd_3\sigma = d_3f\sigma.$$

Since $(f \otimes f)\,\Delta_2\,\sigma = \Delta_2\,f\sigma$ and $\Delta_2\,\sigma = -d_1\sigma \otimes \sigma - d_3\sigma \otimes \sigma - \sigma \otimes d_0\sigma - \sigma \otimes d_2\sigma$, one has in particular that $fd_1\sigma + fd_3\sigma = d_1f\sigma + d_3f\sigma$ and $fd_0\sigma + fd_2\sigma = d_0f\sigma + d_2f\sigma$. It follows that

$$fd_1\sigma = d_1f\sigma \text{ and } fd_2\sigma = d_2f\sigma.$$

Therefore, $d_jF\sigma = d_jf\sigma = fd_j\sigma = Fd_j\sigma$ for $j = 0, 1, 2, 3$.

b) $\underline{f\sigma = 0}$:

The following observation will restrict the cases to be analyzed. For any of the three specific pairs $(i, j) = (0, 2)$, $(0, 3)$ or $(1, 3)$ one has

$$fd_i\sigma \neq 0 \text{ or } fd_j\sigma \neq 0 \text{ imply } fd_i\sigma \neq fd_j\sigma.$$

·) Assume $fd_0\sigma = fd_2\sigma \neq 0$. By property B $d_1d_1Fd_0\sigma \neq d_1d_0Fd_0\sigma$. But $d_1d_1Fd_0\sigma = Fd_1d_1d_0\sigma = Fd_1d_0d_2\sigma = d_1d_0Fd_2\sigma = d_1d_0Fd_0\sigma$. A contradiction.

·) Assume $fd_0\sigma = fd_3\sigma \neq 0$. By property B $d_1d_1Fd_0\sigma \neq d_1d_0Fd_0\sigma$. But $d_1d_1Fd_0\sigma = Fd_1d_1d_0\sigma = Fd_1d_0d_3\sigma = d_1d_0Fd_3\sigma = d_1d_0Fd_0\sigma$. A contradiction.

·) Assume $fd_1\sigma = fd_3\sigma \neq 0$. By property B $d_1d_0Fd_0\sigma \neq d_0d_0Fd_0\sigma$. But $d_1d_0Fd_0\sigma = Fd_1d_0d_0\sigma = Fd_0d_0d_3\sigma = d_0d_0Fd_3\sigma = d_0d_0Fd_0\sigma$. A contradiction.

Since $f\partial\sigma = \partial f\sigma$ implies $fd_0\sigma - fd_1\sigma + fd_2\sigma - fd_3\sigma = 0$ one has the following possibilities:

i) $\underline{fd_0\sigma = fd_1\sigma \neq 0 \;\&\; fd_2\sigma = fd_3\sigma = 0}$: set

$$F\sigma = s_0Fd_0\sigma.$$

The faces of $Fd_0\sigma = Fd_1\sigma$ are non-degenerate by property A. In particular $0 \neq fd_0d_1 = fd_0d_2$ and $0 \neq fd_2d_0 = fd_0d_3$ implying that

$$Fd_2\sigma = s_0Fd_0d_2\sigma \text{ and } Fd_3 = s_0Fd_0d_3\sigma.$$

·) $d_0F\sigma = d_0s_0Fd_0\sigma = Fd_0\sigma$.
·) $d_1F\sigma = d_1s_0Fd_0\sigma = Fd_0\sigma = Fd_1\sigma$.
·) $d_2F\sigma = d_2s_0Fd_0\sigma = s_0d_1Fd_0\sigma = s_0Fd_1d_0\sigma = s_0Fd_0d_2\sigma = Fd_2\sigma$.
·) $d_3F\sigma = d_3s_0Fd_0\sigma = s_0d_2Fd_0\sigma = s_0Fd_2d_0\sigma = s_0Fd_0d_3\sigma = Fd_3\sigma$.

ii) $\underline{fd_0\sigma = 0 \;\&\; fd_1\sigma = fd_2\sigma \neq 0 \;\&\; fd_3\sigma = 0}$: set

$$F\sigma = s_1Fd_1\sigma.$$

The faces of $Fd_1\sigma = Fd_2\sigma$ are non-degenerate by property A. In particular $0 \neq fd_0d_1 = fd_0d_0$ and $0 \neq fd_2d_2 = fd_2d_3$ implying that

$$Fd_0\sigma = s_0Fd_0d_0\sigma \text{ and } Fd_3 = s_1Fd_1d_3\sigma.$$

$\cdot$) $d_0F\sigma = d_0s_1Fd_1\sigma = s_0d_0Fd_1\sigma = s_0Fd_0d_1\sigma = s_1Fd_0d_0\sigma = Fd_0\sigma$.

$\cdot$) $d_1F\sigma = d_1s_1Fd_1\sigma = Fd_1\sigma$.

$\cdot$) $d_2F\sigma = d_2s_1Fd_1\sigma = Fd_1\sigma = Fd_2\sigma$.

$\cdot$) $d_3F\sigma = d_3s_1Fd_1\sigma = s_1d_2Fd_1\sigma = s_1Fd_2d_1\sigma = s_1Fd_1d_3\sigma = Fd_3\sigma$.

iii) $\underline{fd_0\sigma = fd_1\sigma = 0 \ \& \ fd_2\sigma = fd_3\sigma \neq 0}$: set

$$F\sigma = s_2Fd_2\sigma.$$

The faces of $Fd_2\sigma = Fd_3\sigma$ are non-degenerate by property A. In particular $0 \neq fd_0d_3 = fd_2d_0$ and $0 \neq fd_1d_3 = fd_2d_1$ implying that

$$Fd_0\sigma = s_1Fd_1d_0\sigma \text{ and } Fd_1 = s_1Fd_1d_1\sigma.$$

$\cdot$) $d_0F\sigma = d_0s_2Fd_2\sigma = s_1d_0Fd_2\sigma = s_1Fd_0d_2\sigma = s_1Fd_1d_0\sigma = Fd_0\sigma$.

$\cdot$) $d_1F\sigma = d_1s_2Fd_0\sigma = s_1d_1Fd_2\sigma = s_1Fd_1d_2\sigma = s_1Fd_1d_1\sigma = Fd_1\sigma$.

$\cdot$) $d_2F\sigma = d_2s_2Fd_2\sigma = Fd_2\sigma$.

$\cdot$) $d_3F\sigma = d_3s_2Fd_2\sigma = Fd_2\sigma = Fd_3\sigma$.

iv) $\underline{fd_0\sigma = fd_1\sigma = fd_2\sigma = fd_3\sigma = 0}$.

If $fd_1d_1\sigma = fd_1d_2\sigma \neq 0$ then in order to determine the values of $Fd_1\sigma$ and $Fd_2\sigma$ one needs to know if $fd_0d_1\sigma = 0$ or not and if $fd_0d_2\sigma = 0$ or not. Since $fd_0d_1\sigma = fd_2d_0\sigma$ and $fd_0d_2\sigma = fd_1d_0\sigma$ this choice also determines $Fd_0\sigma$ and shows one of the combinations is impossible, namely $fd_0d_1\sigma \neq 0$ and $fd_0d_2\sigma = 0$. Since $fd_2d_1\sigma = fd_1d_3\sigma$ and $fd_2d_2\sigma = fd_2d_3\sigma$ this choice also determines $Fd_3\sigma$.

Similarly, if $fd_1d_1\sigma = fd_1d_2\sigma = 0$ then all $fd_id_j\sigma = 0$. Therefore, one has the following possibilities:

iv-a. $\underline{fd_0d_1\sigma = fd_1d_1\sigma = fd_0d_2\sigma \neq 0 \ \& \ fd_1d_3 = 0}$: set

$$F\sigma = s_1s_0Fd_1d_1\sigma.$$

$\cdot$) $Fd_0\sigma = s_0Fd_0d_0\sigma = s_0Fd_0d_1\sigma = s_0Fd_1d_1\sigma = s_0d_0s_0Fd_1d_1\sigma = d_0s_1s_0Fd_1d_1\sigma = d_0F\sigma$.

$\cdot$) $Fd_1\sigma = s_0Fd_0d_1\sigma = s_0Fd_1d_1\sigma = d_1s_1s_0Fd_1d_1\sigma = d_1Fd_\sigma$.

$\cdot$) $Fd_2\sigma = s_0Fd_0d_2\sigma = s_0Fd_1d_1\sigma = d_2s_1s_0Fd_1d_1\sigma = d_2F\sigma$.

$\cdot$) $Fd_3\sigma = s_0Fd_0d_3\sigma = s_0s_0Fd_0d_0d_3\sigma = s_1s_0Fd_0d_0d_3\sigma = s_1s_0Fd_1d_0d_1\sigma = s_1s_0Fd_1d_1d_1\sigma = d_3s_1s_0Fd_1d_1\sigma = d_3F\sigma$.

iv-b. $\underline{fd_1d_1\sigma = fd_2d_1\sigma = fd_0d_2\sigma = fd_0d_3\sigma = fd_2d_0\sigma \neq 0}$: set

$$F\sigma = s_0s_1Fd_1d_1\sigma.$$

·) $Fd_0\sigma = s_1Fd_1d_0\sigma = s_1Fd_0d_2\sigma = s_1Fd_1d_1\sigma = d_0s_0s_1Fd_1d_1\sigma = d_0F\sigma$.

·) $Fd_1\sigma = s_1Fd_1d_1\sigma = d_1s_1s_1Fd_1d_1\sigma = d_1F\sigma$.

·) $Fd_2\sigma = s_0Fd_0d_2\sigma = s_0Fd_1d_1\sigma = d_2s_2s_0Fd_1d_1\sigma = d_2s_0s_1Fd_1d_1\sigma = d_2F\sigma$.

·) $Fd_3\sigma = s_0Fd_0d_3\sigma = s_0Fd_1d_1\sigma = d_3s_2s_0Fd_1d_1\sigma = d_3s_0s_1Fd_1d_1\sigma = d_3F\sigma$.

iv-c. $\underline{fd_1d_1\sigma = fd_2d_1\sigma = fd_2d_2\sigma = fd_2d_3\sigma \neq 0 \ \&\ fd_1d_0 = 0}$: set

$$F\sigma = s_1s_1Fd_1d_1\sigma.$$

·) $fd_0\sigma = s_0Fd_0d_0\sigma = s_0s_0Fd_0d_0d_0\sigma = s_0s_0Fd_0d_1d_1\sigma = d_0s_1s_1Fd_1d_1 = d_0F\sigma$.

·) $Fd_1\sigma = s_1Fd_1d_1\sigma = d_1s_1s_1Fd_1d_1\sigma = d_1F\sigma$.

·) $Fd_2\sigma = s_1Fd_1d_2\sigma = s_1Fd_1d_1\sigma = d_2s_1s_1Fd_1d_1 = d_2F\sigma$.

·) $Fd_3\sigma = s_1Fd_1d_3\sigma = s_1Fd_1d_1\sigma = s_1d_2s_1Fd_1d_1 = d_3s_1s_1Fd_1d_1 = d_3F\sigma$.

iv-d. $\underline{fd_jd_i\sigma = 0}$: set

$$F\sigma = s_0s_0s_0Fd_0d_0d_0\sigma.$$

$$Fd_i\sigma = s_0s_0Fd_0d_0d_i\sigma = s_0s_0Fd_0d_0d_0\sigma = d_is_0s_0s_0Fd_0d_0d_i\sigma = d_iF\sigma.$$

□

Chapter 2

Abelian sheaves and $\mathcal{S}$-comodules

2.1 Sheaf theory of posets

This section is divided into three parts. The first part builds on a know equivalence of categories between partially ordered sets and T_0-Alexandrov spaces. Each of these categories is equipped with a duality, opposite poset and topology of closed sets respectively, and Lemma 2.1.5 shows that the categorical equivalence is equivariant with respect to these dualities.

The second part uses the connection between posets and Alexandrov spaces to relate the categorical definition of sheaves and cosheaves with the topological one, presenting the close connection that arises between sheaves and cosheaves over posets.

The third part specializes to sheaves and cosheaves over posets with values in an abelian category and characterizes their projective objects, showing the existence of enough projectives under suitable conditions.

Alexandrov spaces and partially ordered sets

Definition 2.1.1. (Alexandrov Spaces) A topological space (X, τ) is said to be an **Alexandrov space** if an arbitrary intersection of open sets is an open set. This extra condition allows for the definition of a new topology on any Alexandrov space given by all closed sets of the original topology. This topology will be called the **dual topology** an denoted τ^{c}.

The full subcategory of topological spaces given by Alexandrov spaces satisfying the T_0 separation axiom (i.e. for any pair of points there is an open set containing one of them, but not both) will be denoted AT_0. Notice that $(X,\tau) \in \mathrm{AT}_0$ if and only if $(X,\tau^{\mathsf{c}}) \in \mathrm{AT}_0$.

Definition 2.1.2. Let $(P,\leq)$ be a poset. The set P can be made into a topological space in two ways. Define the topology $\tau_{\geq}$ in P to be generated by the subsets $b_{\geq} \stackrel{\text{def}}{=} \{a : b \geq a\}$ for all $b \in P$, and the topology $\tau_{\leq}$ to be generated by the subsets $b_{\leq} \stackrel{\text{def}}{=} \{c : b \leq c\}$ for all $b \in P$.

Let (X,τ) be a T_0-Alexandrov space. The set X can be made into a poset in two ways. For any $x \in X$ let $U_x \stackrel{\text{def}}{=} \bigcap_{x \in U \in \tau} U$. Define $X_{\subset} = (X,\leq)$ with $x \leq y$ if and only if $U_x \subset U_y$, and define $X_{\supset} = (X,\leq)$ with $x \leq y$ if and only if $U_x \supset U_y$.

Lemma 2.1.3. *The four assignments described above are functorial. Moreover, they define pairs of inverse functors*

$$(-)_{\geq} : \mathrm{Poset} \leftrightarrows \mathrm{AT}_0 : (-)_{\subset}$$

and

$$(-)_{\leq} : \mathrm{Poset} \leftrightarrows \mathrm{AT}_0 : (-)_{\supset}.$$

Proof. Given an order preserving function $f : P \to P'$ one needs to prove that f is continuous with respect to both topologies. Let $c'_{\geq}$ be a basis element of $\tau'_{\geq}$ and consider $f^{-1}(c'_{\geq})$. This set is open since it is straightforward to check that

$$f^{-1}(c'_{\geq}) = \bigcup_{\{b :\, c' \geq f(b)\}} b_{\geq}.$$

Analogously, for a basis element $a'_{\leq}$ of $\tau'_{\leq}$ one has

$$f^{-1}(a'_{\leq}) = \bigcup_{\{b :\, a' \leq f(b)\}} b_{\leq}.$$

Given a continuous function $f : X \to X'$ one needs to prove that if $U_x \subset U_y$ then $U_{f(x)} \subset U_{f(y)}$. To do so, notice that $f^{-1}(U_{f(y)})$ is an open set containing y and, since U_y is the smallest open set with that property, $U_y \subset f^{-1}(U_{f(y)})$. The assumption $U_x \subset U_y$ together with the previous observation imply that

$$f(x) \in f(U_x) \subset f(U_y) \subset U_{f(y)},$$

which, since $U_{f(x)}$ is the smallest open set containing $f(x)$, give the desired $U_{f(x)} \subset U_{f(y)}$.

Verifying these pairs of functors are inverse of each other follows directly from noticing that in $\tau_{\geq}$ one has $U_b = b_{\geq}$, while in $\tau_{\leq}$ one has $U_b = b_{\leq}$. $\square$

Definition 2.1.4. The (covariant) functor associating to a T_0-Alexandrov space the T_0-Alexandrov space with the dual topology is denoted by

$$(-)^{\mathsf{c}} : \mathrm{AT}_0 \to \mathrm{AT}_0 \,.$$

The (covariant) functor associating to a poset the poset with the opposite order is denoted by

$$(-)^{\mathrm{op}} : \mathrm{Poset} \to \mathrm{Poset} \,.$$

Lemma 2.1.5. *The functors defined in this section are related by the following identities:*

1a. $(-)_{\geq} = (-)_{\leq} \circ (-)^{\mathrm{op}}$. *1b.* $(-)_{\leq} = (-)_{\geq} \circ (-)^{\mathrm{op}}$.

2a. $(-)_{\geq} = (-)^{\mathsf{c}} \circ (-)_{\leq}$. *2b.* $(-)_{\leq} = (-)^{\mathsf{c}} \circ (-)_{\geq}$.

3a. $(-)_{\subset} = (-)^{\mathrm{op}} \circ (-)_{\supset}$. *3b.* $(-)_{\supset} = (-)^{\mathrm{op}} \circ (-)_{\subset}$.

4a. $(-)_{\subset} = (-)_{\supset} \circ (-)^{\mathsf{c}}$. *4b.* $(-)_{\supset} = (-)_{\subset} \circ (-)^{\mathsf{c}}$.

Proof. Pairs of corresponding identities are equivalent since

$$(-)^{\mathrm{op}} \circ (-)^{\mathrm{op}} = \mathrm{id} \ \text{ and } \ (-)^{\mathsf{c}} \circ (-)^{\mathsf{c}} = \mathrm{id}\,.$$

The third pair of identities follows from the first pair since

$$(-)_{\geq} \circ (-)_{\subset} = \mathrm{id} \ \text{ and } \ (-)_{\supset} \circ (-)_{\leq} = \mathrm{id}\,.$$

The fourth pair of identities follows from the second pair since

$$(-)_{\subset} \circ (-)_{\geq} = \mathrm{id} \ \text{ and } \ (-)_{\leq} \circ (-)_{\supset} = \mathrm{id}\,.$$

Proof of 1a. For any poset $(P, \leq)$ the topology $\tau_{\geq}$ is generated by sets $\{a : b \geq a\}$, while the topology $\tau_{\leq^{\mathrm{op}}}$ is generated by sets $\{a : b \leq^{\mathrm{op}} a\}$. These bases are equal so $\tau_{\leq} = \tau_{\geq^{\mathrm{op}}}$.

Proof of 2a. Consider an arbitrary poset $(P, \leq)$. It will be shown first that $\tau_{\geq} \subset \tau_{\leq}^{\mathsf{c}}$. To do so, consider a generator $\{a : b \geq a\}$ of $\tau_{\geq}$ and notice that

$$\{a : b \geq a\} \subset \left(\bigcup_{\{x \,:\, b \not\geq x\}} \{y : x \leq y\} \right)^{\mathsf{c}} \in \tau_{\leq}^{\mathsf{c}}$$

since $b \geq a$ and $a \geq x$ implies $b \geq x$. Also, one has

$$\{a : b \geq a\}^{\mathsf{c}} \;\subset\; \bigcup_{\{x\,:\,b \not\geq x\}} \{y : x \leq y\}$$

since $x \in \{a : b \geq a\}^{\mathsf{c}}$ implies $x \in \{y : x \leq y\}$ with $b \not\geq x$.

In order to show that $\tau_{\geq} \supset \tau_{\leq}^{\mathsf{c}}$ consider an arbitrary open set of $\tau_{\leq}^{\mathsf{c}}$ say $\left(\bigcup_{x \in I}\{y : x \leq y\}\right)^{\mathsf{c}} = \bigcap_{x \in I}\{y : x \leq y\}^{\mathsf{c}}$. Since $\tau_{\geq}$ is closed under arbitrary intersections, it suffices to show that $\{y : x \leq y\}^{\mathsf{c}}$ is open in $\tau_{\geq}$. This follows from a computation similar to the one above showing that

$$\{y : x \leq y\}^{\mathsf{c}} \;=\; \bigcup_{\{b\,:\,x \not\leq b\}} \{a : b \geq a\} \in \tau_{\geq},$$

and concludes the proof. □

Sheaves, cosheaves and their relationship over posets

Definition 2.1.6. (Presheaves and precosheaves) Let $\mathcal{C}$ and $\mathcal{V}$ be categories. A **presheaf**, respectively **precosheaf**, on $\mathcal{C}$ with values on $\mathcal{V}$ is a contravariant, respectively covariant, functor from $\mathcal{C}$ to $\mathcal{V}$. A **presheaf morphism**, respectively **precosheaf morphism**, is a natural transformation of such functors.

Denote these categories respectively by $\mathrm{PSh}(\mathcal{C}, \mathcal{V})$ and $\mathrm{PcoSh}(\mathcal{C}, \mathcal{V})$, or if $\mathcal{V}$ is understood from the context, simply by $\mathrm{PSh}(\mathcal{C})$ and $\mathrm{PcoSh}(\mathcal{C})$.

Definition 2.1.7. (Sites) A **site** is given by a (small) category $\mathcal{C}$ and a set $\mathrm{Cov}(\mathcal{C})$ of families of morphisms with fixed target $\{U_i \to U\}_{i \in I}$, called **coverings** of C, satisfying the following axioms:

S1: If $V \to U$ is an isomorphism then $\{V \to U\} \in \mathrm{Cov}(\mathcal{C})$.

S2: If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and for each $i \in I$ one has that $\{V_{ij} \to U_i\}_{j \in J_i} \in \mathrm{Cov}(\mathcal{C})$, then $\{V_{ij} \to U\}_{i \in I, j \in J_i} \in \mathrm{Cov}(\mathcal{C})$.

S3: If $\{U_i \to U\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$ and $V \to U$ is a morphism in $\mathcal{C}$ then the pullback $U_i \times_U V$ exists for each $i \in I$ and $\{U_i \times_U V \to V\}_{i \in I} \in \mathrm{Cov}(\mathcal{C})$.

Example 2.1.8. Let (X, τ) be a topological space. Think of τ as a category with set of objects τ and

$$\mathrm{Hom}_{\tau}(V, U) = \begin{cases} \{V \to U\} & \text{if } V \subset U, \\ \emptyset & \text{if } V \not\subset U, \end{cases}$$

and notice that this assignment Top $\to$ Cat is functorial. Define the set of coverings of τ by

$$\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\tau) \text{ if and only if } \bigcup_{i\in I} U_i = U.$$

The conditions for τ with this coverings to define a site are easily verified.

Remark 2.1.9. The functors $(-)_{\geq}$ and $(-)_{\leq}$ from Lemma 2.1.3 can be composed with the functor described in the previous example. Abusing notation, these resulting functors are denoted

$$\begin{aligned} (-)_{\geq} :\ & \text{Poset} \to \text{Cat} \\ & (P, \leq) \mapsto \tau_{\geq}. \end{aligned} \qquad \& \qquad \begin{aligned} (-)_{\geq} :\ & \text{Poset} \to \text{Cat} \\ & (P, \leq) \mapsto \tau_{\geq}. \end{aligned}$$

Example 2.1.10. Given any (small) category, define a site by declaring the coverings to be the identity morphisms only. In particular, since the assignment that takes any poset $(P, \leq)$ to a category with set of objects P and morphisms

$$\mathrm{Hom}_P(x, y) = \begin{cases} \{x \to y\} & \text{if } x \leq y \\ \emptyset & \text{if } x \not\leq y \end{cases}$$

is a full and faithful functor, any poset P can be thought of a site with trivial coverings.

The following definitions use some of the examples of limits and colimits described in Appendix A.

Definition 2.1.11. (Sheaves and cosheaves) Let $\mathcal{C}$ be a site, $\mathcal{D} \in \mathrm{PSh}(\mathcal{C})$ and $\mathcal{E} \in \mathrm{PcoSh}(\mathcal{C})$. The presheaf $\mathcal{D}$ is said to be a **sheaf** if for all $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ the first arrow in the following diagram represents the equalizer of the next two

$$\mathcal{D}(U) \to \prod_{i\in I} \mathcal{D}(U_i) \rightrightarrows \prod_{i,j\in I} \mathcal{D}(U_i \times_U U_j).$$

The full subcategory of sheaves in $\mathrm{PSh}(\mathcal{C})$ will be denoted by $\mathrm{Sh}(\mathcal{C})$.

The precosheaf $\mathcal{E}$ is said to be a **cosheaf** if for all $\{U_i \to U\}_{i\in I} \in \mathrm{Cov}(\mathcal{C})$ the last arrow in the following diagram represents the coequalizer of the first two

$$\coprod_{i,j\in I} \mathcal{E}(U_i \times_U U_j) \rightrightarrows \coprod_{i\in I} \mathcal{E}(U_i) \to \mathcal{E}(U).$$

The full subcategory of cosheaves in $\mathrm{PcoSh}(\mathcal{C})$ will be denoted by $\mathrm{coSh}(\mathcal{C})$.

Example 2.1.12. Let (X, τ) be a topological space. Sheaves and cosheaves on the site τ, defined in Example 2.1.8, agree with the usual topological sheaves and cosheaves on X.

Example 2.1.13. If a site has coverings given by the identity morphisms only, then the categories of sheaves and presheaves on such site agree; as also do the categories of cosheaves and precosheaves on it. In particular, following Example 2.1.10, for any poset $(P, \leq)$ one has $\mathrm{Sh}(P) = \mathrm{PSh}(P)$ and $\mathrm{coSh}(P) = \mathrm{PcoSh}(P)$ over this indiscrete site.

Definition 2.1.14. Let $(P, \leq)$ be a poset and $\mathcal{V}$ a complete and cocomplete category. From Remark 2.1.9 one obtains a covariant functor $(-)_{\geq} : P \to \tau_{\geq}$ when P is regarded as a category. Define the functor

$$\mathrm{Lan} : \mathrm{Sh}(P, \mathcal{V}) \to \mathrm{PSh}(\tau_{\leq}, \mathcal{V}),$$

which assigns to any $\mathcal{D} \in \mathrm{Sh}(P, \mathcal{V})$ the left Kan extension of $\mathcal{D}$ along $(-)^{\mathrm{op}}_{\geq}$, diagrammatically

$$\begin{tikzcd} P^{\mathrm{op}} \arrow[r, "\mathcal{D}"] \arrow[d, "(-)^{\mathrm{op}}_{\geq}"'] & \mathcal{V} \\ (\tau_{\geq})^{\mathrm{op}}. \arrow[ur, dashed, "\mathrm{Lan}\,\mathcal{D}"'] & \end{tikzcd}$$

The functor $\mathrm{Ran} : \mathrm{coSh}(P^{\mathrm{op}}, \mathcal{V}) \to \mathrm{PcoSh}(\tau_{\leq}, \mathcal{V})$ is defined using the contravariant functor $(-)_{\leq} : P \to \tau_{\leq}$ in a similar manner, diagrammatically

$$\begin{tikzcd} P^{\mathrm{op}} \arrow[r, "\mathcal{E}"] \arrow[d, "(-)_{\leq}"'] & \mathcal{V} \\ \tau_{\leq}. \arrow[ur, dashed, "\mathrm{Ran}\,\mathcal{E}"'] & \end{tikzcd}$$

Lemma 2.1.15. *Let $(P, \leq)$ be a poset and $\mathcal{V}$ a complete and cocomplete category. For any $\mathcal{D} \in \mathrm{Sh}(P, \mathcal{V})$ the presheaf $\mathrm{Lan}\,\mathcal{D}$ is a sheaf and the functor*

$$\mathrm{Lan} : \mathrm{Sh}(P, \mathcal{V}) \to \mathrm{Sh}(\tau_{\geq}, \mathcal{V})$$

is an equivalence of categories. Similarly, for any $\mathcal{E} \in \mathrm{coSh}(P^{\mathrm{op}}, \mathcal{V})$ the precosheaf $\mathrm{Ran}\,\mathcal{E}$ is a cosheaf and the functor

$$\mathrm{Ran} : \mathrm{coSh}(P^{\mathrm{op}}, \mathcal{V}) \to \mathrm{coSh}(\tau_{\leq}, \mathcal{V})$$

is an equivalence of categories.

Proof. One needs to verify that the presheaf $\operatorname{Lan}\mathcal{D}$ satisfies the sheaf condition. For any $U \in \tau_{\geq}$, Lemma A.16 and Lemma A.7 provide a formula for the left Kan extension

$$\operatorname{Lan}\mathcal{D}(U) = \operatorname{eq}\big(\prod_{y_{\geq}\subset U}\mathcal{D}(y) \rightrightarrows \prod_{x_{\geq}\subset y_{\geq}}\mathcal{D}(x)\big),$$

which is exactly the sheaf condition for the finest cover of U. (Recall that the collection $\{y_{\geq} : y \in P\}$ forms a basis of $\tau_{\geq}$). The inverse functor of Lan is given by restricting a sheaf on $\tau_{\geq}$ to the basis $\{y_{\geq} : y \in P\}$.

The proof for cosheaves is analogous using Lemma A.16 and Lemma A.12. □

Definition 2.1.16. Let $(P, \leq)$ be a poset. Consider the contravariant functor $(-)^{\mathsf{c}} : \tau_{\geq}^{\mathsf{c}} \to (\tau_{\geq})$ taking U^{c} to U and recall from Lemma 2.1.5 that $\tau_{\geq}^{\mathsf{c}} = \tau_{\leq}$. Abusing notation, define the functor

$$(-)^{\mathsf{c}} : \operatorname{Sh}(\tau_{\geq}) \to \operatorname{PcoSh}(\tau_{\leq})$$

which assigns to every $\mathcal{D} \in \operatorname{Sh}(\tau_{\geq})$ the precosheaf $\mathcal{D}^{\mathsf{c}} \in \operatorname{PcoSh}(\tau_{\leq})$ defined by the following commutative diagram

$$\begin{tikzcd}
(\tau_{\geq})^{\mathrm{op}} \arrow[r, "\mathcal{D}"] & \mathcal{V} \\
\tau_{\geq}^{\mathsf{c}} \arrow[u, "(-)^{\mathsf{c}}"] & \tau_{\leq} \arrow[l, "="'] \arrow[u, "\mathcal{D}^{\mathsf{c}}"'] .
\end{tikzcd}$$

Lemma 2.1.17. *Let $(P, \leq)$ be a poset and $\mathcal{V}$ a complete and cocomplete category where the sheaves and cosheaves under consideration take their values. For any $\mathcal{D} \in \operatorname{Sh}(\tau_{\geq})$ the precosheaf $\mathcal{D}^{\mathsf{c}}$ is a cosheaf and the functor*

$$(-)^{\mathsf{c}} : \operatorname{Sh}(\tau_{\geq}) \to \operatorname{coSh}(\tau_{\leq})$$

is an equivalence of categories making the following diagram commute

$$\begin{tikzcd}
\operatorname{Sh}(P) \arrow[r, "="] \arrow[d, "\operatorname{Lan}"'] & \operatorname{coSh}(P^{\mathrm{op}}) \arrow[d, "\operatorname{Ran}"] \\
\operatorname{Sh}(\tau_{\geq}) \arrow[r, "(-)^{\mathsf{c}}"'] & \operatorname{coSh}(\tau_{\leq})
\end{tikzcd}$$

Proof. It suffices to establish the commutativity of the diagram since Lan and Ran are equivalence of categories. To do so, consider the following diagram associated to any $\mathcal{D} \in \mathrm{Sh}(P)$

$$\begin{tikzcd}
 & & P^{\mathrm{op}} \arrow[dll, bend right, "(-)^{\mathrm{op}}_{\geq}"'] \arrow[dl, "\mathcal{D}"] \arrow[ddl, bend left, "(-)_{\leq}"] \\
(\tau_{\geq})^{\mathrm{op}} \arrow[r, "\mathrm{Lan}\,\mathcal{D}"] \arrow[d, leftrightarrow, "(-)^{\mathsf{c}}"'] & \mathcal{V} & \\
\tau_{\geq}^{\mathsf{c}} \arrow[r, leftrightarrow, "="'] & \tau_{\leq}. \arrow[u, "\mathrm{Ran}\,\mathcal{D}"'] \arrow[u, dashed, bend left, "(\mathrm{Lan}\,\mathcal{D})^{\mathsf{c}}"] &
\end{tikzcd}$$

Since $(\mathrm{Lan}\,\mathcal{D})^{\mathsf{c}} \circ (-)_{\geq} = \mathrm{Lan}\,\mathcal{D} \circ (-)^{\mathsf{c}} \circ (-)_{\leq} = \mathrm{Lan}\,\mathcal{D} \circ (-)_{\geq} = \mathcal{D}$, the universal property of right Kan extensions ensures the existence of a natural transformation $(\mathrm{Lan}\,\mathcal{D})^{\mathsf{c}} \to \mathrm{Ran}\,\mathcal{D}$. The inverse natural transformation is obtained similarly using the universal property of left Kan extensions and the functor $(-)^{\mathsf{c}} : \mathrm{Sh}(\tau_{\geq}) \to \mathrm{coSh}(\tau_{\leq})$. $\square$

Abelian sheaves over posets

Definition 2.1.18. (Projective objects) Let $\mathcal{A}$ be an abelian category. An object $\mathbb{P} \in \mathcal{A}$ is called **projective** if it satisfies any of the following equivalent conditions:

1. For any surjection $f : C \to B$ and any map $q : \mathbb{P} \to B$ there exists $g : \mathbb{P} \to C$ such that $q \circ g = f$, diagrammatically

$$\begin{tikzcd}
 & C \arrow[d, "q"] \\
\mathbb{P} \arrow[ur, dashed, "g"] \arrow[r, "f"'] & B \arrow[d] \\
 & 0.
\end{tikzcd}$$

2. Any exact sequence

$$0 \to A \to B \to \mathbb{P} \to 0$$

splits, i.e. it is isomorphic to

$$0 \to A \to A \oplus \mathbb{P} \to \mathbb{P} \to 0$$

with inclusion and projection maps.

Remark 2.1.19. The dual notion of an injective object will be omitted since it is not used in this work.

Definition 2.1.20. (Elementary projective sheaves and cosheaves) Let $\mathbb{P} \in \mathcal{A}$ be a projective object, $(P, \leq)$ a poset and y an element in P.

The **elementary projective sheaf** $\mathbb{P}_{\leq y} \in \mathrm{Sh}(P, \mathrm{Ab})$ with value $\mathbb{P}$ over y is defined by

$$\mathbb{P}_{\leq y}[x] = \begin{cases} \mathbb{P} & \text{if } x \leq y \\ 0 & \text{if } x \not\leq y, \end{cases}$$

with all non-zero morphisms equal to the identity.

The **elementary projective cosheaf** $\mathbb{P}_{y\leq} \in \mathrm{coSh}(P, \mathrm{Ab})$ with value $\mathbb{P}$ over y is defined by

$$\mathbb{P}_{y\leq}[z] = \begin{cases} \mathbb{P} & \text{if } y \leq z \\ 0 & \text{if } y \not\leq z, \end{cases}$$

with all non-zero morphisms equal to the identity.

Terminology 2.1.21. A poset $(P, \leq)$ is said to be **locally finite** if for all pairs $x, z \in P$ the set $\{y \in P : x \leq y \leq z\}$ is finite.

Lemma 2.1.22. *Let $\mathcal{A}$ be a cocomplete abelian category and $(P, \leq)$ a poset. A sheaf or a cosheaf over P with values in $\mathcal{A}$ is projective if, and if P is locally finite, only if, it is isomorphic to a direct sum of elementary projective ones.*

Proof. Only the proof for sheaves will be presented since small variations adapt it for cosheaves. Notice that by the universal property of coproducts a direct sum of projective objects is projective. Explicitly, a morphism $\bigoplus P_i \to B$ defines a collection of morphism $P_i \to B$ by precomposing with the respective inclusion. given a surjection $A \to B$ one gets a collection of lifts $P_i \to A$ and therefore a lift $\bigoplus P_i \to A$.

A sheaf $\mathbb{P}_{\leq y}$ or cosheaf $\mathbb{P}_{y\leq}$ is projective since it is straightforward to check that

$$\mathrm{Hom}_{\mathrm{Sh}}(\mathbb{P}_{\leq y}, \mathcal{D}) = \mathrm{Hom}(\mathbb{P}_{\leq y}[y], \mathcal{D}[y])$$

and

$$\mathrm{Hom}_{\mathrm{coSh}}(\mathbb{P}_{y\leq}, \mathcal{E}) = \mathrm{Hom}(\mathbb{P}_{y\leq}[y], \mathcal{E}[y]).$$

Let $\mathcal{P}$ be a projective sheaf and y an element of the poset. Notice that by considering sheaves with support only on y one concludes that $\mathcal{P}[y]$ is a projective object in $\mathcal{A}$. By the local finiteness of the poset under consideration every set $\{z : z > y\}$ as a smallest element. Define the **near star** of y, denoted $\mathrm{nst}(y)$, as the sub-poset containing all such minimal elements. Notice that all pairs of elements in this sub-poset are not comparable.

Define the sheaf $\mathcal{Q}_y$ by

$$\mathcal{Q}_y[z] = \begin{cases} \mathcal{P}[z] & \text{if } z \in \mathrm{nst}(y) \\ 0 & \text{if } z \notin \mathrm{nst}(y). \end{cases}$$

with all induced morphisms the zero map. The sheaf $\tilde{\mathcal{Q}}_y$ is defined by

$$\tilde{\mathcal{Q}}_y[z] = \begin{cases} \mathcal{P}_y[z] & \text{if } z \in \mathrm{nst}(y) \\ 0 & \text{if } z \notin \mathrm{nst}(y) \text{ and } z \neq y \\ \bigoplus\limits_{z' \in \mathrm{nst}(y)} \mathcal{P}[z'] & \text{if } z = y \end{cases}$$

with non zero induced morphisms given by the inclusions

$$\mathcal{P}[z] \to \bigoplus_{z' \in \mathrm{nst}(y)} \mathcal{P}[z'].$$

There are obvious surjections represented by solid arrows below

$$\begin{tikzcd} & \tilde{\mathcal{Q}}_y \arrow[d] \\ \mathcal{P} \arrow[ur, dashed, "g"] \arrow[r] & \mathcal{Q}_y \arrow[d] \\ & 0 \end{tikzcd}$$

and the morphism $g[z]$ is the identity for all $z \in \mathrm{nst}(y)$ and it is zero for all $z \notin \mathrm{nst}(y)$ such that $z \neq y$. The collection of maps $\mathcal{P}[z] \to \mathcal{P}[y]$ for $z \in \mathrm{nst}(y)$ gives a map $\bigoplus_{z \in \mathrm{nst}(y)} \mathcal{P}[z] \to \mathcal{P}[y]$ fitting into the following sequence

$$0 \longrightarrow \bigoplus_{z \in \mathrm{nst}(y)} \mathcal{P}[z] \longrightarrow \mathcal{P}[y] \xrightarrow{g[y]} \bigoplus_{z \in \mathrm{nst}(y)} \mathcal{P}[z] \longrightarrow 0$$

which is exact since, denoting the inclusion $\mathcal{P}[z] \to \bigoplus_{z \in \mathrm{nst}(y)} \mathcal{P}[z]$ by ι_z, for every $z > y$ one has

$$g[y] \circ \mathcal{P}[y < z] = \iota_z \circ g[z] = \iota_z.$$

The above sequence splits by the projectivity of the objects involved, see Definition 2.1.18, so there exists $B(y)$ such that

$$\mathcal{P}[y] \cong B(y) \oplus \big(\bigoplus_{z \in \mathrm{nst}(y)} \mathcal{P}[z] \big)$$

iterating the argument one gets

$$\mathcal{P}[y] \cong \bigoplus_{z \geq y} B(z)$$

for every y and therefore

$$\mathcal{P} = \bigoplus_{y \in P} B(y)_{\leq y}$$

concluding the proof. □

Remark 2.1.23. The condition in Lemma 2.1.22 that $\mathcal{A}$ be cocomplete is in practice too restrictive. For example, the category Ab_{f} of finitely generated abelian groups is not cocomplete. The conclusion of Lemma 2.1.22 remains true if one restricts to appropriate subcategories of sheaves or cosheaves on a poset where the relevant coproducts exist. Examples of such subcategories are the following.

Definition 2.1.24. (Sheaves and cosheaves with compact support) Let P be a poset and $\mathcal{A}$ an abelian category. Denote by $\mathrm{Sh}_{\mathsf{c}}(X, \mathcal{A})$ the full subcategory of $\mathrm{Sh}(X, \mathcal{A})$ whose objects satisfy $\mathcal{D}[x] \neq 0$ for at most finitely many $x \in P$. Define $\mathrm{coSh}_{\mathsf{c}}(X, \mathcal{A})$ similarly.

Definition 2.1.25. (Enough projectives) An abelian category is said to have **enough projectives** if for any object $B \in \mathcal{A}$ there exists a surjection

$$A \to B \to 0$$

with $A \in \mathcal{A}$ projective.

Lemma 2.1.26. *Let P be a poset and $\mathcal{A}$ an abelian category. If $\mathcal{A}$ has enough projectives then $\mathrm{Sh}_{\mathsf{c}}(P, \mathcal{A})$ and $\mathrm{coSh}_{\mathsf{c}}(P, \mathcal{A})$ do so as well. If in addition $\mathcal{A}$ is cocomplete, then $\mathrm{Sh}(P, \mathcal{A})$ and $\mathrm{coSh}(P, \mathcal{A})$ also have enough projectives.*

Proof. Assume $\mathcal{A}$ is cocomplete and let $\mathcal{D} \in \mathrm{Sh}(X, \mathcal{A})$. The first step is to construct a surjection of sheaves $\mathcal{Q} \to \mathcal{D} \to 0$ with $\mathcal{Q}[x] \in \mathcal{A}$ projective for every $x \in P$. Choose for each $x \in P$ a surjection $f[x] : \mathcal{Q}[x] \to \mathcal{D}[x] \to 0$ with $\mathcal{Q}[x]$ projective. For every pair $x \leq y$ there is a morphism $\mathcal{D}[y] \to \mathcal{D}[x]$ whose precomposition with $f[y] : \mathcal{Q}[y] \to \mathcal{D}[y]$ is represented by the horizontal solid arrow in the following diagram

$$\begin{tikzcd} & \mathcal{Q}[x] \arrow[d, "{f[x]}"] \\ \mathcal{Q}[y] \arrow[r] \arrow[ur, dashed] \arrow[d, "{f[y]}"'] & \mathcal{D}[x] \arrow[d] \\ \mathcal{D}[y] \arrow[ur] & 0. \end{tikzcd}$$

The choice of a morphism realizing the dotted arrow for any pair $x \leq y$ makes $\mathcal{Q}$ into a sheaf and f into a surjective morphism of sheaves. Define $\mathcal{P} \in \mathrm{Sh}(X)$ by

$$\mathcal{P}[x] = \bigoplus_{x \leq y} \mathcal{Q}[y]_{\leq y}.$$

This sheaf maps surjectively onto X, therefore onto $\mathcal{D}$, and it is projective by Lemma 2.1.22.

All other statements are proven in the same manner. $\square$

Notation 2.1.27. Let $\mathcal{A}$ be an abelian category. Denote by $\mathcal{A}_\bullet^+$ the **category of bounded below complexes** whose objects are homologically graded complexes which are zero below some degree. Notice that $\mathcal{A}_\bullet^+$ is enriched over $\mathrm{Ab}_\bullet$, and as usual one says that two morphisms are **chain homotopy equivalent** if they are homologous.

The following are standard result in homological algebra, see for example [45] section 5.7 for their proofs.

Lemma 2.1.28. *Let $\mathcal{A}$ be an abelian category with enough projectives.*

1. *For any $A_\bullet \in \mathcal{A}_\bullet^+$ there exists a complex of projective objects $\mathcal{P}_\bullet \in \mathcal{A}_\bullet^+$ and a morphism $P_\bullet \to A_\bullet$ inducing an isomorphism in homology.*

2. *If $\mathcal{P}_\bullet$ and $\mathcal{P}'_\bullet$ are projective and $f : \mathcal{P}_\bullet \to \mathcal{P}'_\bullet$ induces an isomorphism in homology, then there exists $g \in \mathrm{Hom}_{\mathcal{A}_\bullet^+}(\mathcal{P}_\bullet, \mathcal{P}'_\bullet) \in \mathrm{Ab}_\bullet$ such that $f \circ g$ and $g \circ f$ are chain homotopy equivalent to the respective identities.*

2.2 Simplicial complexes and Ranicki duality

In this section, the theory developed in the previous one is specialized in two ways. Only posets associated with simplicial complexes are considered and the sheaves and cosheaves studied have values either in the category of abelian groups Ab or its full subcategory Ab_{f} of finitely generated abelian groups.

The notion of tensor product of functors is used to define the tensor product of a complex of sheaves and a complex of cosheaves. In conjunction with linear duality and a couple of special complexes, this tensor product is used to define the Ranicki duality functors, whose geometric content is made apparent by the pair subdivision sheaf and cosheaf.

The section closes with the construction, using the pair subdivision sheaf, of the visible symmetric complex of a regular pseudomanifold.

Definition 2.2.1. Consider the category SC of ordered simplicial complexes as presented in Definition 1.3.6. Define a functor

$$\mathrm{SC} \to \mathrm{Poset}$$

sending an ordered simplicial complex $X = (V, S)$ to the poset $(S, \leq)$ with $\sigma \leq \tau$ if an only if $\sigma \subset \tau$, i.e. if σ is a face of τ.

For any $X \in \mathrm{SC}$, define the categories of **sheaves and cosheaves on** X, denoted $\mathrm{Sh}(X)$ and $\mathrm{coSh}(X)$, to be the corresponding sheaves and cosheaves categories on its associated poset.

The **barycentric subdivision functor** $\mathrm{SC} \to \mathrm{SC}$ is defined as the composition of the functor defined above and the functor $\mathrm{Poset} \to \mathrm{SC}$ sending a poset $(P, \leq)$ to the simplicial complex with P as set of vertices and simplices given by strictly ascending sets $[\sigma_0 < \cdots < \sigma_n]$ of elements in P.

Definition 2.2.2. (Open star and closure) Let X be an ordered simplicial complex and σ a simplex in X. The **open star** of σ, denote by $\mathrm{st}\,\sigma$, is defined as the subset of X formed by all simplices containing σ. The **closure** of σ, denote by $\mathrm{cl}\,\sigma$, is defined as the subcomplex of X formed by all simplices contained in σ. Notice that if $\sigma \leq \tau$ then $\mathrm{st}\,\sigma \supset \mathrm{st}\,\tau$ and $\mathrm{cl}\,\sigma \subset \mathrm{cl}\,\tau$.

Definition 2.2.3. The complex of sheaves $\mathrm{C}^\bullet$ with values in Ab is defined to assign to each simplex the chain complex of cochains on its open star, i.e.

$$\mathrm{C}^\bullet[\sigma] = \big(\mathrm{C}^\bullet(\mathrm{st}\,\sigma), \delta\big),$$

and to have induced morphisms given by inclusions.

The complex of cosheaves $\mathrm{C}_\bullet$ with values in Ab is defined to assign to each simplex the chain complex of cochains on its closure, i.e.

$$\mathrm{C}_\bullet[\sigma] = \big(\mathrm{C}_\bullet(\mathrm{cl}\,\sigma), \partial\big),$$

and to have induced morphisms given by inclusions.

The following definition is presented in level of generality suitable for the purposes of this work. For a more general discussion see for example [35].

Definition 2.2.4. (Tensor products of functors) Consider a pair of functors $F : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ and $G : \mathcal{C} \to \mathrm{Ab}$. The **tensor product of F and G over $\mathcal{C}$** is defined by

$$F \underset{\mathcal{C}}{\otimes} G = \mathrm{coeq}\Big(\bigoplus_{f\colon c_1 \to c_2} F(c_2) \otimes G(c_1) \rightrightarrows \bigoplus_{c} F(c) \otimes G(c) \Big),$$

and the **tensor product of G and F over $\mathcal{C}$** is defined by

$$G \underset{\mathcal{C}}{\otimes} F = \mathrm{coeq}\Big(\bigoplus_{f\colon c_1 \to c_2} G(c_1) \otimes F(c_2) \rightrightarrows \bigoplus_{c} F(c) \otimes G(c) \Big).$$

Example 2.2.5. Let R be a ring thought of as a category enriched over Ab with a single object. A right R-module corresponds to an Ab-enriched functor $A : R^{\mathrm{op}} \to \mathrm{Ab}$, while a left R-module corresponds to an Ab-enriched functor $B : R \to \mathrm{Ab}$. The functor tensor product $A \otimes_R B$ agrees with the usual tensor product of a left and a right R-module.

Definition 2.2.6. Given $\mathcal{D} \in \mathrm{Sh}(X)$ and $\mathcal{E} \in \mathrm{coSh}(X)$ define the sheaf $\mathcal{E} \underset{\mathrm{st}}{\boxtimes} \mathcal{D}$ by

$$(\mathcal{E} \underset{\mathrm{st}}{\boxtimes} \mathcal{D})[\sigma] = \mathcal{E}\,|_{\mathrm{st}\,\sigma} \underset{\mathrm{st}\,\sigma}{\otimes} \mathcal{D}\,|_{\mathrm{st}\,\sigma} = \bigoplus_{\sigma \leq \tau} \mathcal{E}[\tau] \otimes \mathcal{D}[\tau] / \sim$$

with $\big(\mathcal{D}[\iota](e) \otimes d\big) \sim \big(e \otimes \mathcal{E}[\iota](d)\big)$ for any $\iota : \sigma \to \tau$, and morphisms being induced by inclusions.

Analogously, define the cosheaf $\mathcal{E} \underset{\mathrm{cl}}{\boxtimes} \mathcal{D}$

$$(\mathcal{E} \underset{\mathrm{cl}}{\boxtimes} \mathcal{D})[\sigma] = \mathcal{E}\,|_{\mathrm{cl}\,\sigma} \underset{\mathrm{cl}\,\sigma}{\otimes} \mathcal{D}\,|_{\mathrm{cl}\,\sigma} = \bigoplus_{\rho \leq \sigma} \mathcal{E}[\rho] \otimes \mathcal{D}[\rho] / \sim$$

with $\big(\mathcal{E}[\iota](e)\otimes d\big)\sim\big(e\otimes\mathcal{D}[\iota](d)\big)$ for any $\iota:\rho\to\sigma$, and morphisms being induced by inclusions.

These assignments are functorial in both variables. The notation $\boxtimes_{\mathrm{st}}$ and $\boxtimes_{\mathrm{cl}}$ will also be used for the extension of these bifunctors to complexes.

Remark 2.2.7. Fix an ordered simplicial complex X. Let $\underline{\mathbb{Z}}$ be the constant cosheaf on X with value $\mathbb{Z}$. For any sheaf $\mathcal{D}$ the collection of maps

$$\big(\underline{\mathbb{Z}}\underset{\mathrm{st}}{\boxtimes}\mathcal{D}\big)[\sigma]=\Big(\bigoplus_{\sigma\leq\tau}\mathbb{Z}_\tau\otimes\mathcal{D}[\tau]/\sim\Big)\to\mathcal{D}[\sigma]$$

sending $1_\tau\otimes d$ to $\mathcal{D}[\underline{\iota}](d)$ with $\iota:\sigma\to\tau$ defines an isomorphism of sheaves.

Analogously, let $\overline{\mathbb{Z}}$ be the constant sheaf. For any cosheaf $\mathcal{E}$ the collection of maps

$$\big(\mathcal{E}\underset{\mathrm{cl}}{\boxtimes}\overline{\mathbb{Z}}\big)[\sigma]=\Big(\bigoplus_{\rho\leq\sigma}\mathcal{E}[\rho]\otimes\mathbb{Z}_\rho/\sim\Big)\to\mathcal{E}[\sigma]$$

sending $e\otimes 1_\rho$ to $\mathcal{E}[\iota](e)$ with $\iota:\rho\to\sigma$ defines an isomorphism of cosheaves.

Notation 2.2.8. Let $(-)^\vee:\mathrm{Sh}(X,\mathrm{Ab})\to\mathrm{coSh}(X,\mathrm{Ab})$ be the functor induced from linear duality. Explicitly, for any $\mathcal{D}\in\mathrm{Sh}(X,\mathrm{Ab})$ one has

$$\mathcal{D}^\vee[\sigma]=(\mathcal{D}[\sigma])^\vee\ \text{and}\ \mathcal{D}^\vee[\iota]=(\mathcal{D}[\iota])^\vee\ \text{for all}\ \iota:\sigma\to\tau.$$

Since the context will be clear enough to avoid confusions, the analogous functor $\mathrm{coSh}(X,\mathrm{Ab})\to\mathrm{Sh}(X,\mathrm{Ab})$ will be denoted by the same symbol $(-)^\vee$.

The same notation $(-)^\vee$ will be used for the extension of these functors to complexes.

Definition 2.2.9. (Ranicki duality functors) Let X be an ordered simplicial complex. Abusing notation, define respectively the **Ranicki duality functors** $\mathrm{T}:\mathrm{Sh}(X,\mathrm{Ab})_\bullet\to\mathrm{Sh}(X,\mathrm{Ab})_\bullet$ and $\mathrm{T}:\mathrm{coSh}(X,\mathrm{Ab})_\bullet\to\mathrm{coSh}(X,\mathrm{Ab})_\bullet$ as the following compositions

$$\mathrm{T}(-)=(-)^\vee\underset{\mathrm{st}}{\boxtimes}\mathrm{C}^\bullet\quad\text{and}\quad\mathrm{T}(-)=\mathrm{C}_\bullet\underset{\mathrm{cl}}{\boxtimes}(-)^\vee.$$

Example 2.2.10. Notice that $\overline{\mathbb{Z}}^\vee\cong\underline{\mathbb{Z}}$ and $\underline{\mathbb{Z}}^\vee\cong\overline{\mathbb{Z}}$, so by Remark 2.2.7 one has $\mathrm{T}\overline{\mathbb{Z}}\cong\mathrm{C}^\bullet$ and $\mathrm{T}\underline{\mathbb{Z}}\cong\mathrm{C}_\bullet$.

Example 2.2.11. Let X be the interval an represent $\mathrm{C}^\bullet$ and $\mathrm{C}_\bullet$ by

$$\mathrm{C}^\bullet : \begin{tikzcd} \alpha \arrow[d, "-1"'] & & \gamma \arrow[d, "+1"] \\ \beta & \beta \arrow[l, hook'] \arrow[r, hook] & \beta \\ \bullet \arrow[rr, dash] & & \bullet \end{tikzcd} \qquad \mathrm{C}_\bullet : \begin{tikzcd} & & b \arrow[dl, "-1"'] \arrow[dr, "+1"] & & \\ a \arrow[r, hook] & a & & c & c \arrow[l, hook'] \\ \bullet \arrow[rrrr, dash] & & & & \bullet \end{tikzcd}.$$

Their linear duals are represented by

$$(\mathrm{C}^\bullet)^\vee : \begin{tikzcd} b \arrow[r, two heads] \arrow[d, "-1"'] & b & b \arrow[l, two heads] \arrow[d, "+1"] \\ a & & c \\ \bullet \arrow[rr, dash] & & \bullet \end{tikzcd} \qquad (\mathrm{C}_\bullet)^\vee : \begin{tikzcd} \alpha & \alpha \arrow[l, two heads] \arrow[dr, "-1"'] & & \gamma \arrow[r, two heads] \arrow[dl, "+1"] & \gamma \\ & & \beta & & \\ \bullet \arrow[rrrr, dash] & & & & \bullet \end{tikzcd}.$$

The Ranicki dual of $\mathrm{C}^\bullet$ defined as $(\mathrm{C}^\bullet)^\vee \underset{\mathrm{st}}{\boxtimes} \mathrm{C}^\bullet$ is represented by

$$\mathrm{T}\,\mathrm{C}^\bullet : \begin{tikzcd} & b\alpha \arrow[dl, "-1"'] \arrow[dr, "+1"] & & & b\gamma \arrow[dl, "-1"'] \arrow[dr, "+1"] & \\ a\alpha & & b\beta & b\beta \arrow[l, hook'] \arrow[r, hook] & b\beta & c\gamma \\ & \bullet \arrow[rrr, dash] & & & \bullet & \end{tikzcd}.$$

The Ranicki dual of $\mathrm{C}_\bullet$ defined as $\mathrm{C}_\bullet \underset{\mathrm{cl}}{\boxtimes} (\mathrm{C}_\bullet)^\vee$ is represented by

$$\mathrm{T}\,\mathrm{C}_\bullet : \begin{tikzcd} & & b\alpha \arrow[dl, "-1"'] \arrow[dr, "+1"] & & b\gamma \arrow[dl, "-1"'] \arrow[dr, "+1"] & & \\ a\alpha \arrow[r, hook] & a\alpha & & b\beta & & c\gamma & c\gamma \arrow[l, hook'] \\ \bullet \arrow[rrrrrr, dash] & & & & & & \bullet \end{tikzcd}.$$

Observe that in this example the collection of evaluation maps

$$(\mathrm{C}^\bullet[\sigma])^\vee \otimes \mathrm{C}^\bullet[\sigma] \to \mathbb{Z}$$

induces a morphism of complexes of sheaves

$$\varepsilon : \mathrm{T}\,\mathrm{C}^\bullet = \mathrm{T}^2\overline{\mathbb{Z}} \to \overline{\mathbb{Z}}$$

with $\varepsilon[\sigma]$ a homology isomorphism for every simplex σ.

Similarly defined, the morphism

$$\varepsilon : \mathrm{T}\,\mathrm{C}_\bullet = \mathrm{T}^2\underline{\mathbb{Z}} \to \underline{\mathbb{Z}}$$

is a homology isomorphism over each simplex.

Definition 2.2.12. (Pair subdivision) For any finite ordered simplicial complex define its **pair subdivision sheaf** $\mathrm{P}_\bullet^{Sh}$ as the Ranicki dual of the complex of sheaves $\mathrm{C}^\bullet$. Explicitly, one has chain isomorphisms

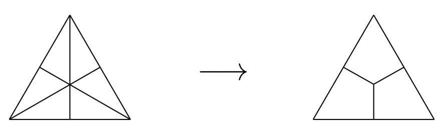
$$\mathrm{P}_\bullet^{Sh}[\rho] \cong \big(\bigoplus_{\rho\leq\sigma,\tau} \tau\otimes\sigma^*\big)/\big(\bigoplus_{\sigma\nleq\tau} \tau\otimes\sigma^*\big)$$

with maps $\mathrm{P}_\bullet^{Sh}[\rho\to\rho'] : \mathrm{P}_\bullet^{Sh}[\rho'] \to \mathrm{P}_\bullet^{Sh}[\rho]$ given by inclusions.

Define the **pair subdivision cosheaf** similarly by

$$\mathrm{P}_\bullet^{cSh} = \mathrm{T}\,\mathrm{C}_\bullet = \mathrm{C}_\bullet \underset{\mathrm{cl}}{\boxtimes} (\mathrm{C}_\bullet)^\vee.$$

Remark 2.2.13. The pair subdivision sheaf has a geometric interpretation justifying its name. Let X be an ordered simplicial complex and X' its barycentric subdivision as defined in 2.2.1. For any $\rho\in X$ define the **close dual cone** of ρ, denoted $\mathrm{dc}(\rho)$, as the subcomplex of X' containing all simplices of the form $[\rho_1<\rho_2<\cdots]$ with $\rho\leq\rho_1$. The chain complex $\mathrm{P}_\bullet^{Sh}[\rho]$ is chain isomorphic to the chain complex of a regular CW complex obtained by gluing along common faces certain simplices of $\mathrm{dc}(\rho)$. Two simplices in the closed dual cone are amalgamated along their common face if they have the same dimension and are represented by ascending subsets with the same initial and terminal simplices. For example, in the case of the (geometric realization of the) 2-dimensional simplex the amalgamation map looks as follows:



The subdivision map, which is the inverse of the amalgamation one defined above, induces a chain homotopy equivalence from $\mathrm{P}_\bullet^{Sh}[\rho]$ into the chains on the dual cone of ρ which are denoted $\mathrm{D}_\bullet[\rho]$. Notice that $\mathrm{D}_\bullet$ defines a complex of projective sheaves and, since $\mathrm{P}_\bullet^{Sh}$ is also projective, the complexes $\mathrm{D}_\bullet$ and $\mathrm{P}_\bullet^{Sh}$ are chain homotopy equivalent be Lemma 2.1.28. In particular, $\mathrm{P}_\bullet^{Sh}[\rho]$ is contractible for all $\rho\in X$. See [36] for more on the pair subdivision complex.

The Ranicki duality functors T are not in general involutions, not even up to homotopy. In order to have an involution-like property one needs to impose some finiteness conditions. This is accomplished by restricting T to $\mathrm{Sh}_{\mathsf{c}}(X, \mathrm{Ab}_{\mathsf{f}})_\bullet$ and $\mathrm{coSh}_{\mathsf{c}}(X, \mathrm{Ab}_{\mathsf{f}})_\bullet$, i.e. the categories of complexes of compactly supported sheaves, respectively cosheaves, with values in the category of finitely generated abelian groups.

Lemma 2.2.14. *There exist natural transformations*

$$\varepsilon : \mathrm{T}^2 \to \mathrm{id}_{\mathrm{Sh}_{\mathsf{c}}(X,\mathrm{Ab}_{\mathsf{f}})} \quad \textit{and} \quad \varepsilon : \mathrm{T}^2 \to \mathrm{id}_{\mathrm{coSh}_{\mathsf{c}}(X,\mathrm{Ab}_{\mathsf{f}})}$$

defined below, such that if $\mathcal{P}_\bullet$ stands for a complex of projective sheaves or cosheaves then

1. *The following pairs of complexes are chain isomorphic*

$$\mathrm{T}^2\mathcal{P}_\bullet \cong \mathcal{P}_\bullet \underset{\mathrm{st}}{\boxtimes} \mathrm{P}^{Sh}_\bullet \quad \textit{and} \quad \mathrm{T}^2\mathcal{P}_\bullet \cong \mathcal{P}_\bullet \underset{\mathrm{cl}}{\boxtimes} \mathrm{P}^{cSh}_\bullet$$

2. *The morphism $\varepsilon_{\mathcal{P}_\bullet} : (\mathrm{T}^2\mathcal{P}_\bullet) \to \mathcal{P}_\bullet$ is a chain homotopy equivalence.*

3. *The following diagram commutes*

$$\begin{tikzcd} \mathrm{T}\mathcal{P}_\bullet \arrow[r, "\mathrm{T}(\varepsilon_{\mathcal{P}_\bullet})"] \arrow[dr, "\mathrm{id}"'] & \mathrm{T}^3\mathcal{P}_\bullet \arrow[d, "\varepsilon_{\mathrm{T}\mathcal{P}_\bullet}"] \\ & \mathrm{T}\mathcal{P}_\bullet. \end{tikzcd}$$

Proof. Only the proof for sheaves will be presented since small variations adapt it for cosheaves. For any simplex $\sigma \in X$ denote its i-th face by $\partial_i\sigma$ and by $\delta^i\sigma$ any simplex such that $\partial_i\delta^i\sigma = \sigma$, notice that if $\delta^i\sigma$ exists then it is unique.

For any complex of sheaf $\mathcal{D}_\bullet$ one has $\mathrm{T}\,\mathcal{D}_\bullet[\rho] = \big(\bigoplus_{\rho\le\sigma} \mathcal{D}^\vee_\bullet[\sigma] \otimes \mathrm{C}^\bullet[\sigma]/\sim\big)$ which equals $\big(\bigoplus_{\rho\le\sigma\le\sigma'} \mathcal{D}^\vee_\bullet[\sigma] \otimes \mathrm{C}^\bullet[\sigma\le\sigma']\sigma'^*/\sim\big)$. Using that

$$d^*_\sigma \otimes \mathrm{C}^\bullet[\sigma \le \sigma'](\sigma'^*) \sim \mathcal{D}^\vee_\bullet[\sigma \le \sigma'](d^*_\sigma) \otimes \sigma'^*$$

one sees that as graded abelian groups

$$\mathrm{T}\,\mathcal{D}_\bullet[\rho] \cong \bigoplus_{\rho\le\sigma} \mathcal{D}^\vee_\bullet[\sigma] \otimes \sigma^*.$$

This isomorphism can be improved to a chain isomorphism by setting

$$\partial(d_\sigma \otimes \sigma^*) = \partial d_\sigma \otimes \sigma^* + (-1)^{|d_\sigma|} \sum_{\delta^i\sigma} (-1)^i \, \mathcal{D}_\bullet^\vee[\sigma \leq \delta^i\sigma](d_\sigma) \otimes (\delta^i\sigma)^*.$$

Applying the previous observation twice and using the finite dimensionality of the abelian groups involved one has that as graded abelian groups

$$\mathrm{T}^2\,\mathcal{D}_\bullet[\rho] \cong \bigoplus_{\rho\leq\sigma\leq\tau} \mathcal{D}_\bullet[\tau] \otimes \tau \otimes \sigma^*.$$

To describe the boundary making this into a chain isomorphism observe that $(\mathrm{T}\,\mathcal{D}_\bullet)^\vee[\sigma \leq \sigma'] : \bigoplus_{\sigma\leq\tau} \mathcal{D}_\bullet[\tau] \otimes \tau \to \bigoplus_{\sigma'\leq\tau} \mathcal{D}_\bullet[\tau] \otimes \tau$ is giving by projection. The boundary in $\bigoplus_{\rho\leq\sigma\leq\tau} \mathcal{D}_\bullet[\tau] \otimes \tau \otimes \sigma^*$ making it chain isomorphic to $\mathrm{T}^2\,\mathcal{D}_\bullet[\rho]$ is therefore

$$\begin{aligned}\partial(d_\tau \otimes \tau \otimes \sigma^*) = \; & \partial(d_\tau) \otimes \tau \otimes \sigma^* \\ & + (-1)^{|d_\tau|} \sum_{\partial^i\tau} (-1)^i \, \mathcal{D}_\bullet[\partial_i\tau \leq \tau](d_\tau) \otimes \partial_i\tau \otimes \sigma^* \\ & + (-1)^{|d_\tau|+|\tau|} \sum_{\delta^i\sigma\leq\tau} (-1)^i d_\tau \otimes \tau \otimes (\delta^i\sigma)^*.\end{aligned}$$

The above formula conceptually simplifies if $\mathcal{D}_\bullet$ is projective. In that case $\mathcal{D}[\partial_i\tau \leq \tau]$ is an inclusion map and the chain complex $\mathrm{T}^2[\rho]$ becomes chain isomorphic to $\mathcal{D}[\rho] \otimes \mathrm{P}_\bullet^{Sh}[\rho]$.

For any $\mathcal{D}_\bullet$ define $\varepsilon_{\mathcal{D}_\bullet}[\rho] : \mathrm{T}^2\,\mathcal{D}_\bullet[\rho] \to \mathcal{D}_\bullet[\rho]$ by sending $d_\tau \otimes \tau \otimes \sigma^*$ to $\mathcal{D}_\bullet[\rho \leq \tau](d_\tau) \cdot \langle \tau, \sigma^* \rangle$. Using the formula for the boundary it can be shown this defines a morphism of complexes of sheaves. In case $\mathcal{D}_\bullet$ is projective then $\varepsilon_{\mathcal{D}_\bullet}[\rho] : \mathcal{D}_\bullet[\rho] \otimes \mathrm{P}_\bullet^{Sh}[\rho] \to \mathcal{D}_\bullet[\rho]$ is the identity in the first factor and contracts the second. It is therefore a homology isomorphism and by Lemma 2.1.28 $\varepsilon_{\mathcal{D}_\bullet}$ is a chain homotopy equivalence.

The proof of the third part becomes a straightforward computation. Let $\mathcal{P}_\bullet$ be projective and consider any $\rho \in X$ and $(d_\sigma \otimes \sigma^*) \in \mathrm{T}\mathcal{P}_\bullet[\rho]$, then

$$\begin{tikzcd} (d_\sigma^* \otimes \sigma^*) \arrow[r, mapsto, "\mathrm{T}(\varepsilon_{\mathcal{P}_\bullet})[\rho]"] \arrow[dr, mapsto, "\mathrm{id}"'] & d_\sigma^* \otimes \left(\sum_{\sigma\leq\tau} \tau^* \otimes \tau\right) \otimes \sigma^* \arrow[d, mapsto, "\varepsilon_{\mathrm{T}\mathcal{P}_\bullet}"] \\ & d_\sigma^* \otimes \left(\sum_{\sigma\leq\tau} \tau^* \otimes \langle \tau, \sigma^* \rangle\right) \end{tikzcd}$$

□

Remark 2.2.15. For any projective complex $\mathcal{P}_\bullet$ of compactly supported sheaves or cosheaves with values on Ab_{f}, the lemma above can be used to endow the chain complex $\mathrm{Hom}(\mathrm{T}\mathcal{P}_\bullet, \mathcal{P}_\bullet)$ with an action of Σ_2 defined to send f to $(\varepsilon_{\mathcal{P}_\bullet} \circ \mathrm{T}f)$. This is an involution since

$$\varepsilon_{\mathcal{P}_\bullet} \circ \mathrm{T}(\varepsilon_{\mathcal{P}_\bullet} \circ \mathrm{T}f) = \varepsilon_{\mathcal{P}_\bullet} \circ \mathrm{T}^2 f \circ \mathrm{T}(\varepsilon_{\mathcal{P}_\bullet}) = f \circ \varepsilon_{\mathrm{T}\mathcal{P}_\bullet} \circ \mathrm{T}(\varepsilon_{\mathcal{P}_\bullet}) = f.$$

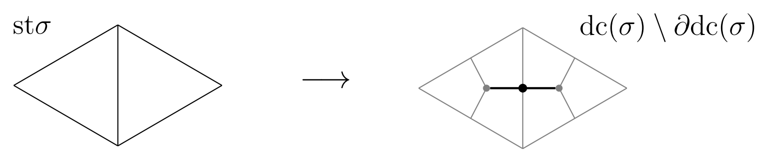
Notation 2.2.16. Recall from Definition 1.1.5 that the arity 2 part of the operad $\mathcal{S}$ carries a free action of Σ_2 and has the homology of a point. For any chain complex C with an action of Σ_2, the chain complex of Σ_2-equivariant maps from $\mathcal{S}(2)$ to C will be denoted C^{Σ_2}. Let $\varphi \in C^{\Sigma_2}$ and as in Example 1.2.5, let $(\dots, 2, 1, 2)$ be one of the degree d generators of $\mathcal{S}(2)_\bullet$. The image of this generator via φ will be denoted φ_d, and if φ is a cycle then one has

$$\partial \varphi_{d+1} = (1 - (-1)^d T)\varphi_d.$$

Cycles in C^{Σ_2} can be therefore thought of as homotopy fix points of the Σ_2-action on $\mathcal{C}$, compare with Example A.10.

Remark 2.2.17. Applying Lemma 2.2.14 to the pair subdivision sheaf one gets that $\varepsilon_{\mathrm{P}^{Sh}_\bullet} : \mathrm{T}\mathrm{P}^{Sh}_\bullet \cong \mathrm{T}^2\,\mathrm{C}^\bullet \to \mathrm{C}^\bullet$ is a chain homotopy equivalence.

Recall that by definition $\mathrm{C}^\bullet[\sigma]$ is isomorphic to $\bigoplus_{\sigma \leq \tau} \tau^*$, the complex of cochains on the open star of σ. The concept of dual cone can be used to give another geometric interpretation for $\mathrm{C}^\bullet$. Consider the amalgamation, described in Remark 2.2.13, of the close dual cone of a simplex σ into a subcomplex of the pair subdivision. The CW subcomplex corresponding to the open part of the dual cone of σ is parametrized by simplices τ satisfying $\tau \geq \sigma$, with a simplex of dimension k in the open star corresponds to a cell of dimension $k - |\sigma|$ in the amalgamated open dual cone. For example,



This geometric correspondence induces a chain homotopy equivalence between $\mathrm{C}^\bullet[\sigma]$ and $S^{-|\sigma|}\mathrm{C}^\bullet(\mathrm{dc}\,\sigma, \partial\,\mathrm{dc}\,\sigma)$; the complex, suspended by $|\sigma|$, of relative cochains on the closed dual cone modulo its boundary.

Construction 2.2.18. The goal of this construction is to naturally obtain a degree n cycle φ in $\mathrm{Hom}_{\mathrm{Sh}}(\mathrm{T}\mathrm{P}^{Sh}_\bullet, \mathrm{P}^{Sh}_\bullet)^{\Sigma_2}$ when the base X is a regular

n-dimensional pseudomanifold. This will be done taking the following steps. First, a morphism

$$\mathrm{D}_\bullet[-] \to \mathrm{Hom}_{\mathrm{Sh}}(\Sigma^{|-|}\mathrm{TP}^{Sh}_\bullet[-], \mathrm{P}^{Sh}_\bullet[-])^{\Sigma_2}$$

of complexes of sheaves will be constructed. Second, a collection of compatible homomorphism $\mathrm{C}_\bullet(X') \to \mathrm{D}_\bullet[\sigma]$ will be defined. third, the desired cycle will be obtained by evaluating the fundamental cycle of $\mathrm{C}_\bullet(X')$ in their composition.

Let X be an ordered simplicial complex and X' its barycentric subdivision. Let $\mathrm{D}_\bullet$, as in Remark 2.2.13, be the projective complex of sheaves assigning to each σ in X the complex of simplicial chains in $\mathrm{dc}(\sigma)$, the dual cone of σ; and to each pair $\sigma \leq \tau$ the inclusion $\mathrm{C}_\bullet(\mathrm{dc}(\tau)) \to \mathrm{C}_\bullet(\mathrm{dc}(\sigma))$. Each of this complexes is in a functorial manner an $\mathcal{S}$-coalgebra and in particular an $\mathcal{S}(2)$-coalgebra, so by the hom-tensor adjunction the $\mathcal{S}(2)$-structures define a morphism of complexes of sheaves

$$\mathrm{D}_\bullet[-] \to \mathrm{Hom}_{\Sigma_2}\left(\mathcal{S}(2), \mathrm{D}_\bullet[-] \otimes \mathrm{D}_\bullet[-]\right).$$

For every $\sigma \in X$ one has

$$\mathrm{D}_\bullet[\sigma] \otimes \mathrm{D}_\bullet[\sigma] \cong \mathrm{Hom}\Big(\mathrm{C}^\bullet\left(\mathrm{dc}(\sigma)\, , X' \setminus \mathrm{dc}(\sigma)\right), \mathrm{D}_\bullet[\sigma]\Big)$$

which is chain homotopy equivalent to

$$\mathrm{Hom}\Big(\mathrm{C}^\bullet\left(\,\mathrm{dc}(\sigma)\, , \partial\, \mathrm{dc}(\sigma)\right), \mathrm{D}_\bullet[\sigma]\Big).$$

This complex is by Remark 2.2.17 chain homotopy equivalent to

$$\mathrm{Hom}\big(\Sigma^{|\sigma|}\mathrm{TP}^{Sh}_\bullet[\sigma]\, , \mathrm{P}^{Sh}_\bullet[\sigma]\big),$$

so one gets a morphism of complexes of sheaves

$$\mathrm{D}_\bullet[-] \to \mathrm{Hom}_{\Sigma_2}\left(\mathcal{S}(2), \mathrm{Hom}_{\mathrm{Sh}}\left(\Sigma^{|-|}\mathrm{TP}^{Sh}_\bullet[-], \mathrm{P}^{Sh}_\bullet[-]\right)\right). \tag{2.1}$$

For any chain in the simplicial chain complex of X' one can construct an element in $\mathrm{D}_\bullet$ whose value in $\mathrm{D}_\bullet[\sigma]$ is computed by projecting the chain to the dual cone of the smallest vertex of σ, followed by taking its boundary and projecting it to the dual cone of the smallest edge of σ, and so on until reaching the projection to the dual cone of σ; in symbols if $\sigma = [v_0, \ldots, v_n]$ one has

$$c \longmapsto \pi_{[v_0,\ldots,v_n]} \circ \partial \circ \cdots \circ \pi_{[v_0,v_1]} \circ \partial \circ \pi_{[v_0]}(c)$$

with π_σ denoting the projection from the simplicial chain complex of X' onto $\mathrm{D}_\bullet[\sigma]$. Notice that this construction decreases degree by $|\sigma|$.

Let X be a simply connected regular pseudomanifold, i.e. an n-dimensional simplicial complex such that: each codimension 1 face is the boundary of exactly two distinct n-dimensional simplices, the boundary of the star of each simplex of codimension at least 2 is connected, and the sum of all n-dimensional simplices, denoted $[X]$, is a cycle. Passing to the barycentric subdivision, let $[X']$ denote the cycle corresponding to $[X]$. The image of $[X']$ in $\mathrm{D}_\bullet$ via the above construction is mapped by the morphism (2.1) to an n-dimensional cycle $\varphi \in \mathrm{Hom}_{\mathrm{Sh}}(\mathrm{TP}^{Sh}_\bullet, \mathrm{P}^{Sh}_\bullet)^{\Sigma_2}$. The pair $(\mathrm{P}^{Sh}_\bullet, \varphi)$ will be referred to as the **visible symmetric complex** of X.

2.3 Topological manifolds and $\mathcal{S}$-comodules

In this section, as Theorem 2.3.4, the second of the two main technical results of this work is presented. It states that the category of complexes of sheaves over an ordered simplicial complex X with values in Ab embeds as a differential graded full subcategory of the category of comodules over $\mathrm{C}_\bullet(X)$ as an $\mathcal{S}$-coalgebra.

This theorem is used to relate the algebraic surgery theory of Ranicki with comodules over E_∞-coalgebras. In particular, Theorem 2.3.13 and Theorem 2.3.15 provide existence and uniqueness statements for homology manifold structures and topological manifold structures on the homotopy type of a Poincaré duality regular pseudomanifold, using comodules on its $\mathcal{S}$-coalgebra of chains.

Notation 2.3.1. The tensor product over X with the complex of cosheaf $\mathrm{C}_\bullet$ defines a functor from $\mathrm{Sh}(X, \mathrm{Ab})_\bullet$ to $\mathrm{Ab}_\bullet$, see Definition 2.2.4 and Definition 2.2.3 for unfamiliar terminology. For $\mathcal{D}_\bullet$ a complex of sheaves, $\mathcal{D}_\bullet \otimes_X \mathrm{C}_\bullet$ is given by

$$\bigoplus_{\sigma \in X} \mathcal{D}_\bullet[\sigma] \otimes \mathrm{C}_\bullet[\sigma] / \sim$$

with $d_\tau \otimes \mathrm{C}_\bullet[\sigma \leq \tau](c_\sigma) \sim \mathcal{D}_\bullet[\sigma \leq \tau](d_\tau) \otimes c_\sigma$ and differential graded structure induce from the tensor product of chain complexes. It is isomorphic as abelian group to

$$\bigoplus_{\sigma\in X} \mathcal{D}_\bullet[\sigma]\otimes\sigma,$$

and elements of the form $d\otimes\sigma$ for some $\sigma\in X$ will be referred to as **canonical representatives** of elements in $\mathcal{D}_\bullet\otimes_X \mathrm{C}_\bullet$. Compare with the proof of Lemma 2.2.14.

Notice that given $F:\mathcal{D}_\bullet\to\mathcal{D}'_\bullet$ a morphism of complexes of sheaves, the induced morphism is given in terms of canonical representatives by

$$f(d\otimes\sigma) = F[\sigma](d)\otimes\sigma.$$

Lemma 2.3.2. *The functor $-\otimes_X \mathrm{C}_\bullet$ lifts along the forgetful functor to the category of comodules over the $\mathcal{S}$-coalgebra $\mathrm{C}_\bullet(X)$. Diagrammatically,*

$$\begin{tikzcd} & \mathrm{coMod}^{\mathcal{S}}_{\mathrm{C}_\bullet(X)} \arrow[d, "forget"] \\ \mathrm{Sh}(X,\mathrm{Ab})_\bullet \arrow[ur, dashed] \arrow[r, "-\underset{X}{\otimes}\mathrm{C}_\bullet"'] & \mathrm{Ab}_\bullet . \end{tikzcd}$$

Proof. For every $\sigma\in X$ the complex $\mathrm{C}_\bullet[\sigma]=\mathrm{C}_\bullet(\mathrm{cl}\,\sigma,\partial)$ is an $\mathcal{S}$-coalgebra naturally, so the functor $\mathrm{C}_\bullet : X\to\mathrm{Ab}_\bullet$ can be lifted along the forgetful functor to $\mathrm{C}_\bullet : X\to\mathrm{coMod}^{\mathcal{S}}_{\mathrm{C}_\bullet(X)}$ with structure maps

$$\mathcal{S}(k)\otimes\mathrm{C}_\bullet[\sigma]\to\mathrm{C}_\bullet[\sigma]^{\otimes k}\to\mathrm{C}_\bullet[\sigma]\otimes\mathrm{C}_\bullet(X)^{\otimes(k-1)}.$$

The functor tensor product $\mathcal{D}_\bullet\otimes_X\mathrm{C}_\bullet$ inherits a $\mathcal{S}$-comodule structure over $\mathrm{C}_\bullet(X)$ from its second factor. □

Remark 2.3.3. By forgetting structure, $\mathcal{D}_\bullet\otimes_X\mathrm{C}_\bullet$ is also a comodule over $\mathrm{C}_\bullet(X)$ thought of as an $\mathcal{S}(2)$ coalgebra, i.e. a coalgebra over the operad generated by the arity 2 part of the operad $\mathcal{S}$. The coaction of the generator $(\dots,1,2,1)$ of $\mathcal{S}(2)$ of degree k will be denoted by ∇_k and, according to Lemma 2.3.2, it is defined for any class $d\otimes c\in\mathcal{D}_\bullet\otimes_X\mathrm{C}_\bullet$ by

$$\nabla_n(d\otimes c)=d\otimes\Delta_n(c),$$

with the notation Δ_n introduced in 1.2.9.

Theorem 2.3.4. *The differential graded functor*

$$- \underset{X}{\otimes} \mathrm{C}_\bullet : \mathrm{Sh}(X, \mathrm{Ab})_\bullet \longrightarrow \mathrm{coMod}^{\mathcal{S}(2)}_{\mathrm{C}_\bullet(X)}$$

is full and faithful.

Proof. Let $F : \mathcal{D}_\bullet \to \mathcal{D}'_\bullet$ be a morphisms of complexes of sheaves and assume the morphism induced by the functor $- \otimes_X \mathrm{C}_\bullet$ is 0. Using canonical representatives, this implies that the abelian group homomorphism

$$\bigoplus_{\sigma \in X} (F[\sigma] \otimes \mathrm{id}_\sigma) : \bigoplus_{\sigma \in X} \mathcal{D}_\bullet[\sigma] \otimes \sigma \longrightarrow \bigoplus_{\sigma \in X} \mathcal{D}'_\bullet[\sigma] \otimes \sigma$$

is 0, hence $F[\sigma] = 0$ for each $\sigma \in X$, i.e. $F = 0$.

Given an $\mathcal{S}$-$\mathrm{C}_\bullet(X)$-comodule map $f : \mathcal{D}_\bullet \otimes_X \mathrm{C}_\bullet \to \mathcal{D}'_\bullet \otimes_X \mathrm{C}_\bullet$ one needs to construct a sheaf map inducing it. Let $d_\sigma \otimes \sigma$ be a canonical representative and write its image in terms of canonical representatives $f(d_\sigma \otimes \sigma) = \sum_{\tau \in X} d'_\tau \otimes \tau$. Since $\nabla_n \circ f = (f \otimes \mathrm{id}) \circ \nabla_n$ one has for all $n \geq 0$ that

$$\begin{tikzcd}
d_\sigma \otimes \sigma \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\nabla_n"'] & \sum_{\tau \in X} d'_\tau \otimes \tau \arrow[d, mapsto, "\nabla_n"] \\
d_\sigma \otimes \Delta_n \sigma \arrow[r, mapsto, "f \otimes \mathrm{id}"'] & (?)_n \sim \sum_{\tau \in X} d'_\tau \otimes \Delta_n \tau.
\end{tikzcd}$$

The above equation will be used to show that $d'_\tau = 0$ for all $\tau \neq \sigma$. Let n be the largest $|\tau|$ so $d'_\tau \neq 0$ and assume $n > |\sigma|$, then

$$(?)_n = 0 \sim \sum_{|\tau| = n} d'_\tau \otimes \tau \otimes \tau$$

so $d'_\tau = 0$ for all τ of dimension n. Iterating this argument one has that $d'_\tau = 0$ for all τ of dimension greater than $|\sigma|$. For $n = |\sigma|$ one has

$$(?)_n = \sum_{\tau \in X} d'_\tau \otimes \tau \otimes \sigma \sim \sum_{|\tau| = |\sigma|} d'_\tau \otimes \tau \otimes \tau$$

so $d'_\tau = 0$ for all $\tau \neq 0$. For each $\sigma \in X$ define the chain map

$$\begin{aligned} f_\sigma : \quad \mathcal{D}_\bullet[\sigma] &\to \mathcal{D}_\bullet[\sigma] \\ e_\sigma &\mapsto e'_\sigma. \end{aligned}$$

If the above collection of chain maps defines a morphism from $\mathcal{D}_\bullet$ to $\mathcal{D}'_\bullet$ then it induces f, so it needs to be shown that for every $\iota : \rho \to \sigma$ one has $\mathcal{D}'_\bullet[\iota] \circ f_\sigma = f_\rho \circ \mathcal{D}_\bullet[\iota]$. Assume for an induction argument that this holds for all morphisms $\iota : \rho \to \sigma$ with $|\sigma| < n$. For any simplex $\sigma = [v_0, \ldots, v_n]$ denote its i-th face by $\sigma_i = [v_0, \ldots, \hat{v}_i, \ldots, v_n]$ and $\iota^i : \sigma_i \to \sigma$. The induction assumption and the functoriality of sheaves imply that it suffices to show that

$$\mathcal{D}'_\bullet[\iota^i] \circ f_\sigma = f_{\sigma_i} \circ \mathcal{D}_\bullet[\iota^i] \tag{2.2}$$

for all simplices σ of dimension n and $i = 0, \ldots, n$.

For any such $\sigma \in X$ and $d \in \mathcal{D}_\bullet[\sigma]$ consider the following diagram

$$\begin{tikzcd} d \otimes \sigma \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\nabla_0"'] & f_\sigma(d) \otimes \sigma \arrow[d, mapsto, "\nabla_0"] \\ d \otimes \Delta_0\sigma \arrow[r, mapsto, "f\otimes \mathrm{id}"'] & (?) \sim f_\sigma(d) \otimes \Delta_0\tau. \end{tikzcd} \tag{2.3}$$

Recall from Example 1.2.3 that $\Delta_0[0, \ldots, n] = \sum_i [0, \ldots, i] \otimes [i, \ldots, n]$ so projecting $\mathcal{D}'_\bullet \otimes_X \mathrm{C}_\bullet \otimes \mathrm{C}_\bullet(X)$ onto $\mathcal{D}'_\bullet \otimes_X \mathrm{C}_\bullet \otimes [v_{n-1}, v_n]$ makes the equation in (2.3) be

$$f\big(d \otimes \iota^n_\bullet \sigma_n\big) \sim f_\sigma(d) \otimes \iota^n_\bullet \sigma_n$$

which implies

$$\big(f_{\sigma_n} \circ \mathcal{D}_\bullet[\iota^n]\big)(d) = \big(\mathcal{D}'_\bullet[\iota^n] \circ f_\sigma\big)(d).$$

A completely analogous argument using $T\nabla_0$ verifies equation (2.2) for $i = 0$. For all $0 < i < n$ one consider the following diagram associated to ∇_1

$$\begin{tikzcd} d \otimes \sigma \arrow[r, mapsto, "f"] \arrow[d, mapsto, "\nabla_1"'] & f_\sigma(d) \otimes \sigma \arrow[d, mapsto, "\nabla_1"] \\ d \otimes \Delta_1\sigma \arrow[r, mapsto, "f\otimes \mathrm{id}"'] & (?) \sim f_\sigma(d) \otimes \Delta_1\tau. \end{tikzcd} \tag{2.4}$$

Recall that $\Delta_1[0, \ldots, n] = \sum_{i<j} \pm [0, \ldots, i, j, \ldots, n] \otimes [i, \ldots, j]$ so projecting $\mathcal{D}'_\bullet \underset{X}{\otimes} \mathrm{C}_\bullet \otimes \mathrm{C}_\bullet(X)$ onto $\mathcal{D}'_\bullet \underset{X}{\otimes} \mathrm{C}_\bullet \otimes [v_{i-1}, v_{i+1}]$ makes the equation in (2.4) be

$$f\big(d \otimes \iota^i_\bullet \sigma_i\big) \sim f_\sigma(d) \otimes \iota^i_\bullet \sigma_i$$

which implies

$$\big(f_{\sigma_i} \circ \mathcal{D}_\bullet[\iota^i]\big)(d) = \big(\mathcal{D}'_\bullet[\iota^i] \circ f_\sigma\big)(d)$$

and completes the verification of equation (2.2). □

Remark 2.3.5. One can consider the functor tensor product over X of the cochain functor $\mathrm{C}^\bullet$ and any $\mathcal{E}_\bullet \in \mathrm{coSh}(X, \mathcal{A})_\bullet$. The analogue of the results above can be proven by similar arguments, but they will not be used in this work. In particular, the category of complexes of cosheaves over X can be thought of as a full subcategory of the category of modules on the $\mathcal{S}$-algebra of cochains on X.

L-theory for sheaf-like $\mathcal{S}$-comodules

Definition 2.3.6. (Sheaf-like comodules and duality) An $\mathcal{S}$-comodules D over $\mathrm{C}_\bullet(X)$ is said to be **sheaf-like** if it is isomorphic to one of the form $\mathcal{D}_\bullet \otimes_{\mathrm{X}} \mathrm{C}_\bullet$ with $\mathcal{D}_\bullet$ a projective complex of sheaves with values on the category Ab_{f} of finitely generated abelian groups. The Ranicki duality functor, Definition 2.2.9, induces by Lemma 2.3.4 a contravariant functor on the subcategory of sheaf-like comodules. Explicitly, let $D \cong \mathcal{D}_\bullet \otimes_{\mathrm{X}} \mathrm{C}_\bullet$ and $D' \cong \mathcal{D}'_\bullet \otimes_{\mathrm{X}} \mathrm{C}_\bullet$ be sheaf-like and $f \in \mathrm{Hom}_{\mathrm{coMod}}(D, D')$. By Lemma 2.3.4 there exists $F \in \mathrm{Hom}_{\mathrm{Sh}}(\mathcal{D}, \mathcal{D}')$ so that $F \otimes_{\mathrm{X}} \mathrm{C}_\bullet = f$. Define, abusing notation, $\mathrm{T}f : \mathrm{T}D' \to \mathrm{T}D$ to be

$$(\mathrm{T}F) \otimes_{\mathrm{X}} \mathrm{C}_\bullet : (\mathrm{T}\,\mathcal{D}') \otimes_{\mathrm{X}} \mathrm{C}_\bullet \to (\mathrm{T}\,\mathcal{D}) \otimes_{\mathrm{X}} \mathrm{C}_\bullet\,.$$

Remark 2.3.7. The natural transformation $\varepsilon : T^2 \to \mathrm{id}_{\mathrm{Sh}}$ of Lemma 2.2.14 induces an analogous natural transformation for the duality of sheaf-like comodules. In particular, the chain complex $\mathrm{Hom}_{\mathrm{coMod}}(\mathrm{T}D, D)$ has an action of Σ_2 given by $f \mapsto \varepsilon_D \circ Tf$. For any sheaf-like comodule $D = \mathcal{D} \otimes_{\mathrm{X}} \mathrm{C}_\bullet$, Theorem 2.3.4 gives a chain isomorphism

$$\mathrm{Hom}_{\mathrm{Sh}}(\mathrm{T}\,\mathcal{D}, \mathcal{D})^{\Sigma_2} \cong \mathrm{Hom}_{\mathrm{coMod}}(\mathrm{T}D, D)^{\Sigma_2}.$$

See 2.2.16 for the definition of these complexes.

Definition 2.3.8. (Connective sheaf-like and Poincaré comodules) A sheaf-like comodule is said to be **connective** if it is chain homotopy equivalent, as $\mathcal{S}$-comodule, to one which equals 0 in negative degrees.

An n-**dimensional weak Poincaré comodule** is a pair (D, φ) with D a connective sheaf-like comodule and φ a cycle of degree n in $\mathrm{Hom}_{\mathrm{coMod}}(\mathrm{T}D, D)^{\Sigma_2}$ such that φ_0 is a homology isomorphism. (See Remark 2.2.16 for unfamiliar notation.)

An n-**dimensional strong Poincaré comodule** is an n-dimensional weak Poincaré comodule (D, φ) so that φ_0 is a chain homotopy equivalence of $\mathcal{S}$-comodules.

Notation 2.3.9. Let $f : C \to C'$ be a chain map between chain complexes. Denote the **mapping cone** of f by $\mathrm{Cone}(f)$, i.e. the chain complex $\Sigma C \oplus C'$ with boundary defined by $(\Sigma(c) + c') \mapsto \Sigma(\partial(c)) + f(c) + \partial(c')$, where $\Sigma(-)$ stands for suspension.

Definition 2.3.10. (Cobordism) A **weak cobordism** between n-dimensional weak Poincaré comodules (D, φ) and (D', φ') consists of a connective sheaf-like comodule E with a degree $(n+1)$-chain $\phi \in \mathrm{Hom}_{\mathrm{coMod}}(\mathrm{T}E, E)^{\Sigma_2}$ and a couple of maps $f : D \to E$ and $f : D' \to E$ satisfying:

1. $\partial\phi = (f \circ \varphi \circ \mathrm{T}f) - (f' \circ \varphi' \circ \mathrm{T}f')$.
2. $(\phi/\varphi)_0 \overset{\text{def}}{=} \big(\phi_0 + (\varphi_0 \circ \mathrm{T}f) + (\varphi'_0 \circ \mathrm{T}f')\big) : \mathrm{T}E \to \mathrm{Cone}(f \oplus (-f'))$ is a homology isomorphism.

A **strong cobordism** between n-dimensional strong Poincaré comodules (D, φ) and (D', φ') is a weak cobordism between them such that $(\phi/\varphi)_0$ is a chain homotopy equivalence of $\mathcal{S}$-comodules.

Example 2.3.11. The central example of a weak Poincaré comodule for the applications of this work comes from Construction 2.2.18. Let X be a simply-connected regular pseudomanifold which is an n-dimensional Poincaré duality space. By Theorem 2.3.4 and Remark 2.3.7 the image by $-\otimes_{\mathrm{X}}\mathrm{C}_\bullet$ of the visible symmetric complex, denoted $(\mathrm{P}, \varphi_{\mathrm{P}})$, would be an n-dimensional weak Poincaré comodule if $(\varphi_{\mathrm{P}})_0$ induces a homology isomorphism.

To see this is the case, recall that by definition one has for any complex of sheaves $\mathcal{D}_\bullet$ that $\mathcal{D}_\bullet \otimes_{\mathrm{X}} \mathrm{C}_\bullet = \bigoplus_{\sigma \in X} \mathcal{D}_\bullet[\sigma] \otimes \mathrm{C}_\bullet[\sigma]/\!\sim$ and $\mathrm{C}_\bullet[\sigma] = \mathrm{C}_\bullet(\mathrm{cl}\,\sigma)$. Contracting each second factor one gets a chain map inducing an isomorphism in homology

$$\mathcal{D}_\bullet \otimes_{\mathrm{X}} \mathrm{C}_\bullet \to \bigoplus_{v \in X^{(0)}} \mathcal{D}_\bullet[v]/\hat{\sim} \tag{2.5}$$

with $\mathcal{D}_\bullet[v \le \sigma](d_v) \,\hat{\sim}\, \mathcal{D}_\bullet[v' \le \sigma](d_{v'})$ for every $\sigma \in X$. This assignment is functorial with a morphism of complexes of sheaf $F : \mathcal{D} \to \mathcal{D}'$ inducing the chain map $\bigoplus_{v \in X^{(0)}} F[v]$.

Recall from Remark 2.2.13 that $\mathrm{P}^{Sh}_\bullet[\sigma]$ is chain homotopy equivalent to the simplicial chains on the closed dual cone of σ, in symbols

$$\mathrm{P}^{Sh}_\bullet[\sigma] \overset{che}{\to} \mathrm{D}_\bullet[\sigma] = \mathrm{C}_\bullet(\mathrm{dc}(\sigma));$$

and from Remark 2.2.17 that $\mathrm{TP}^{Sh}_\bullet[\sigma]$ is chain homotopy equivalent to the relative simplicial cochains suspended by $|\sigma|$ of the closed dual cone of σ modulo its boundary, in symbols

$$\mathrm{TP}^{Sh}_\bullet[\sigma] \overset{che}{\to} \Sigma^{|\sigma|}\mathrm{C}^\bullet(\mathrm{dc}(\sigma), \partial\,\mathrm{dc}(\sigma)).$$

By the previous two observations, the morphism (2.5) and the definition of φ_{P} one has a commutative diagram

$$\begin{tikzcd}
\mathrm{TP}^{Sh}_\bullet \underset{X}{\otimes} \mathrm{C}_\bullet \arrow[r, "(\varphi_{\mathrm{P}})_0"] \arrow[d] & \mathrm{P}^{Sh}_\bullet \underset{X}{\otimes} \mathrm{C}_\bullet \arrow[d] \\
\mathrm{C}^\bullet(X') \arrow[r, "-\cap[X']"'] & \mathrm{C}_\bullet(X')
\end{tikzcd}$$

with vertical arrows representing homology isomorphisms. Since X' is a Poincaré duality space, the morphism $- \cap [X']$ induces an isomorphism in homology with a degree shift of n and therefore, the morphism $(\varphi_{\mathrm{P}})_0$ does as well.

Definition 2.3.12. An ANR homology n-manifold is a finite dimensional absolute neighborhood retract X satisfying for every $x \in X$

$$\mathrm{H}_i(X, X \setminus \{x\}) = \begin{cases} \mathbb{Z} & \text{if } i = n \\ 0 & \text{if } i \neq n. \end{cases}$$

Theorem 2.3.13. *Let X be a simply-connected regular pseudomanifold which is an n-dimensional Poincaré duality space with $n > 4$.*

For any homotopy equivalence between an ANR homology n-manifold and X, there exists a weak cobordism between a strong n-dimensional Poincaré comodule and $(\mathrm{P}, \varphi_{\mathrm{P}})$. Conversely, for every weak cobordism between a strong n-dimensional Poincaré comodule and $(\mathrm{P}, \varphi_{\mathrm{P}})$, there exists a homotopy equivalence between X and an ANR homology n-manifold.

Two such homotopy equivalences are related by an h-cobordism relative to boundary if and only if their corresponding strong Poincaré comodules are related by a strong cobordism.

In order to obtain existence and uniqueness statements for topological manifold structures on X one needs to “tame the fundamental group” of the weak cobordisms involved.

Definition 2.3.14. (Admissible cobordisms) Using the notation of Definition 2.3.10, a weak cobordism is said to be **admissible** if the the mapping cone of $(\phi/\varphi)_0$ is chain homotopy equivalent as $\mathcal{S}$-comodule to a sheaf-like comodule which equals 0 in degrees less than or equal to 1.

Theorem 2.3.15. *Let X be a simply-connected regular pseudomanifold which is an n-dimensional Poincaré duality space with $n > 4$.*

For any homotopy equivalence between a topological n-manifold and X, there exists an admissible weak cobordism between a strong n-dimensional Poincaré comodule and $(\mathrm{P}, \varphi_{\mathrm{P}})$. *Conversely, for every admissible weak cobordism between a strong n-dimensional Poincaré comodule and* $(\mathrm{P}, \varphi_{\mathrm{P}})$*, there exists a homotopy equivalence between X and a topological n-manifold.*

Two such homotopy equivalences are related by an h-cobordism relative to boundary if and only if their corresponding strong Poincaré comodules are related by a strong cobordism.

Remark 2.3.16. Starting with a homotopy equivalence in either of the theorems above, the weak cobordism obtained has the further property that the morphism from each of its boundary components induces an isomorphism in homology.

Also, in both theorems above, given a strong Poincaré complex weak cobordant to $(\mathrm{P}, \varphi_{\mathrm{P}})$, there exist another strong Poincaré complex, which is strong cobordant to the original one, and a weak cobordism between it and $(\mathrm{P}, \varphi_{\mathrm{P}})$ such that the morphism from each of its boundary components induces and isomorphism in homology.

Proof of Theorem 2.3.13 and Theorem 2.3.15. Both of the proofs are obtain by relating to the algebraic surgery theory as developed by Ranicki in [30].

The differential graded category of connective sheaf-like comodules over $\mathrm{C}_\bullet(X)$ with duality T is, by Theorem 2.3.4, equivalent to the category of connective complexes of projective sheaves over X with Ranicki duality. This category is equivalent to the category of connective complexes of X-based modules defined in [30, p.63], see [33, p.169] for a proof, with the duality defined in [30, p.75].

The weak Poincaré comodule $(\mathrm{P}, \varphi_{\mathrm{P}})$ represents, under the above identification, the "(1/2)-visible symmetric signature" of Ranicki, see Remark 16.8 [30, p.181]. Also under this identification, weak cobordisms, admissible weak cobordism and strong cobordisms correspond to symmetric cobordisms in the algebraic bordism categories $\Lambda\langle 0\rangle(\mathbb{Z}, X)$, $\Lambda\langle 1/2\rangle(\mathbb{Z}, X)$ and $\Lambda\langle 0\rangle(\mathbb{Z})_*(X)$ respectively, as defined in pages 157, 164 and 158 of [30].

The statements now follow from Theorem 17.4, Theorem 18.5, Proposition 25.7 and Remark 25.13 of [30].

$\square$

Appendix A

Categorical Background

In this appendix the notions of limits, colimits and Kan extensions are collected, emphasizing their use in constructions associated to simplicial sets.

Definition A.1. (Diagrams) A **diagram** in $\mathcal{C}$ indexed by $\mathcal{I}$ is a functor $\mathcal{I} \to \mathcal{C}$ with $\mathcal{I}$ a small category.

Example A.2. (Constant diagrams) For any small category $\mathcal{I}$ and object c in a category $\mathcal{C}$, define the **constant diagram** indexed by $\mathcal{I}$ with value c by declaring the image of any objet in $\mathcal{I}$ to be c and of any morphism to be id_c.

Example A.3. Let G be a group and $\mathcal{G}$ be the category with one object $*$ and $\mathrm{Hom}_{\mathcal{G}}(*, *) \cong G$. A diagram $\mathcal{G} \to \mathcal{C}$ is the same data as and object in $\mathcal{C}$ with a G action.

Limits

Definition A.4. (Limits) Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram. The **limit** of D consists of an object $\lim_{\mathcal{I}} D$ in $\mathcal{C}$ and a natural transformation φ from the constant diagram indexed by $\mathcal{I}$ with value $\lim_{\mathcal{I}} D$ to D, satisfying the following universal property. For any constant diagram indexed by $\mathcal{I}$ provided with a natural transformation ϕ to D, there exists a unique morphism f from its constant value *cone* to $\lim_{\mathcal{I}} D$ such that for every $i \in \mathcal{I}$ one has

$\varphi(i) \circ f = \phi(i)$. Diagrammatically,

$$\begin{tikzcd}
D(i) \arrow[rr, "D(i \to j)"] & & D(j) \\
& \lim_{\mathcal{I}} D \arrow[ul, "\varphi(i)"'] \arrow[ur, "\varphi(j)"] & \\
& cone \arrow[u, "f"] \arrow[uul, bend left, "\phi(i)"] \arrow[uur, bend right, "\phi(j)"'] &
\end{tikzcd}$$

commutes for every $(i \to j) \in \mathrm{Hom}_{\mathcal{I}}(i, j)$.

Example A.5. (Orbits) Let $\mathcal{C}$ be a small category. Recall from Example A.3 that an object in $\mathcal{C}$ with an action of a group G can be thought of as a diagram $D : \mathcal{G} \to \mathcal{C}$. Denote $D(*) = X \in \mathcal{C}$ and notice that for any $g \in G$ one has the commutative diagram

$$\begin{tikzcd}
X \arrow[rr, "g"] & & X \; . \\
& \lim_{\mathcal{G}} D \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

The universal property of limits implies that the limit of D equals the set of **orbits** of the action, i.e. $\mathrm{colim}_{\mathcal{G}}\, D = X_G$.

Example A.6. (Initial objects, products and equalizers) Consider a diagram in $\mathcal{C}$ with index category one of the following:

a) $\emptyset$ b) $\overbrace{\bullet \quad \bullet \; ... \quad \bullet}^{\Lambda}$ c) $\bullet \rightrightarrows \bullet$.

If the limit of the diagram exists it is called respectively

a) The **initial object**. For example, the empty topological space or the zero abelian group.

b) The **product**, which is denoted $\prod_\Lambda$ or $\times ... \times$. For example, cartesian product of topological spaces or tensor product of abelian groups.

c) The **equalizer**, which is denoted $\mathrm{eq}(-)$. For example, for spaces one has $\mathrm{coeq}(X \overset{f}{\underset{g}{\rightrightarrows}} Y) = \{x \in X : f(x) = g(x)\}$. For abelian groups, the equalizer of a pair of maps where one of them is the zero map equals the kernel of the other map.

The following statement, whose proof can be found in [18, p.112], shows that a category where all products and equalizers exist is such that the limit of any diagram exists. Such categories are called **complete**.

Lemma A.7. Let $D : \mathcal{I} \to \mathcal{C}$ be an diagram in a complete category $\mathcal{C}$. The limit of D is given by

$$\mathrm{eq}\left(\prod_i D(i) \rightrightarrows \prod_{i \to j} D(j)\right)$$

where one of the maps comes from projecting to the source of each morphism and then applying the corresponding morphism induced by D, while the other is induced from directly projecting to the target of each morphism.

Example A.8. (Pullbacks) The limit of a diagram in a cocomplete small category of the form

$$\begin{tikzcd} & \bullet \arrow[d] \\ \bullet \arrow[r] & \bullet \end{tikzcd} \quad \longrightarrow \quad \begin{tikzcd} & C \arrow[d, "g"'] \\ A \arrow[r, "f"] & B \end{tikzcd}$$

is according to Lemma A.7 equal to

$$\mathrm{eq}\left(A \times C \times B \underset{f \circ p_1 \times g \circ p_2}{\overset{p_3 \times p_3}{\rightrightarrows}} B \times B\right) = \{(x, y, b) : f(x) = g(y) = b\}.$$

This limit will be denoted by $A \times_B C \cong \{(x, y) : f(x) = g(y)\}$ and referred to as the **pullback** of $A \xrightarrow{f} B \xleftarrow{g} C$.

Colimits

Definition A.9. (Colimits) Let $D : \mathcal{I} \to \mathcal{C}$ be a diagram. The **colimit** of D consists of an object $\mathrm{colim}_{\mathcal{I}} D$ in $\mathcal{C}$ and a natural transformation φ from D to the constant diagram indexed by $\mathcal{I}$ with value $\mathrm{colim}_{\mathcal{I}} D$, satisfying the following universal property. For any constant diagram indexed by $\mathcal{I}$ provided with a natural transformation ϕ to D, there exists a unique morphism f from $\mathrm{colim}_{\mathcal{I}} D$ to its constant value *cocone* such that for every $i \in \mathcal{I}$ one has $\phi(i) = f \circ \varphi(i)$. Diagrammatically,

$$
\begin{tikzcd}
D(i) \arrow[rr, "D(i\to j)"] \arrow[dr, "\varphi(i)"] \arrow[ddr, bend right, "\phi(i)"'] & & D(j) \arrow[dl, "\varphi(j)"'] \arrow[ddl, bend left, "\phi(j)"] \\
& \operatorname{colim}_{\mathcal{I}} D \arrow[d, "f"'] & \\
& \mathit{cocone} &
\end{tikzcd}
$$

commutes for every $(i \to j) \in \mathrm{Hom}_{\mathcal{I}}(i, j)$.

Example A.10. (Fix points) Let $\mathcal{C}$ be a small category. Recall from Example A.3 that an object in $\mathcal{C}$ with an action of a group G can be thought of as a diagram $D : \mathcal{G} \to \mathcal{C}$. Denote $D(*) = X \in \mathcal{C}$ and notice that for any $g \in G$ one has the commutative diagram

$$
\begin{tikzcd}
X \arrow[rr, "g"] \arrow[dr] & & X \arrow[dl] \\
& \operatorname{colim}_{\mathcal{G}} D &
\end{tikzcd} .
$$

The universal property of colimits implies that the colimit of D equals the **fix point** set of the action, i.e. $\operatorname{colim}_{\mathcal{G}} D = X^G$.

Example A.11. (Terminal objects, coproducts and coequalizers) Consider a diagram in $\mathcal{C}$ with index category one of the following:

a) $\emptyset$ b) $\overbrace{\bullet \quad \bullet \; \ldots \quad \bullet}^{\Lambda}$ c) $\bullet \rightrightarrows \bullet$.

If the colimit of the diagram exists it is called respectively

a) The **terminal object**. For example, the topological space with one element or the zero abelian group.

b) The **coproduct**, which is denoted $\coprod_{\Lambda}$ or $\sqcup \ldots \sqcup$. For example, disjoint union of topological spaces or direct sum of abelian groups.

c) The **coequalizer**, which is denoted $\mathrm{coeq}(-)$. For example, for spaces one has $\mathrm{coeq}(X \overset{f}{\underset{g}{\rightrightarrows}} Y) = Y \big/ f(x) \sim g(x)$. For abelian groups, the coequalizer of a pair of maps where one of them is the zero map equals the cokernel of the other map.

The following lemma shows that a category where all coproducts and coequalizers exist is such that the colimit of any diagram exists. Such categories are called **cocomplete**.

Lemma A.12. Let $D : \mathcal{I} \to \mathcal{C}$ be an diagram in a cocomplete category $\mathcal{C}$. The colimit of D is given by

$$\operatorname{coeq}\Big(\coprod_{i\to j} D(i) \rightrightarrows \coprod_{i} D(i)\Big)$$

where one of the maps comes from the identity $D(i) \to D(i)$, while the other comes from the morphism $D(i) \to D(j)$ induced by D from the morphisms $i \to j$.

Proof. This is a variation of the proof of the analogue statement for limits in Lemma A.7. □

Example A.13. (Pushouts) The colimit of a diagram in a cocomplete small category of the form

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd} \longrightarrow \begin{tikzcd} B \arrow[r, "f"] \arrow[d, "g"'] & C \\ A & \end{tikzcd}$$

is according to Lemma A.12 equal to

$$\operatorname{coeq}\Big(B \sqcup B \overset{\mathrm{id}\sqcup\mathrm{id}}{\underset{f\sqcup g}{\rightrightarrows}} A \sqcup B \sqcup C\Big) = A \sqcup C \big/ f(b) \sim g(b).$$

This colimit will be denoted by $A \sqcup_B C$ and referred to as the **pushout** of $A \overset{g}{\leftarrow} B \overset{f}{\to} C$.

Kan extensions

Definition A.14. (Kan extensions) Let $F : \mathcal{C} \to \mathcal{A}$ and $E : \mathcal{C} \to \mathcal{B}$ be a pair of functors. The **right Kan extension** of F along E is a functor $\operatorname{Ran}_E F : \mathcal{B} \to \mathcal{A}$ and a natural transformation $\phi : F \to \operatorname{Ran}_E F \circ E$ satisfying the following universal property. For any pair $R : \mathcal{B} \to \mathcal{A}$ and $\psi : F \to \mathcal{R} \circ E$

there exists a unique $\theta : R \to \operatorname{Ran}_E F$ such that $\psi = \phi \circ \theta_F$ with $\theta_F(c) = \theta(F(c))$ for all $c \in \mathcal{C}$. Diagrammatically,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "E"'] \arrow[dr, phantom, "\phi\,\Nearrow", pos=0.3] & \mathcal{A} \\
\mathcal{B} \arrow[ur, dashed, "\operatorname{Ran}_E F"'] \arrow[ur, bend right=60, "R"'] \arrow[r, phantom, "\Nwarrow\,\theta"] & {}
\end{tikzcd}$$

The **left Kan extension** of F along E is a functor $\operatorname{Lan}_E F : \mathcal{B} \to \mathcal{A}$ and a natural transformation $\phi : \operatorname{Lan}_E F \circ E \to F$ satisfying the following universal property. For any pair $R : \mathcal{B} \to \mathcal{A}$ and $\psi : \mathcal{R} \circ E \to F$ there exists a unique $\theta : \operatorname{Lan}_E F \to R$ such that $\psi = \theta_F \circ \phi$ with $\theta_F(c) = \theta(F(c))$ for all $c \in \mathcal{C}$. Diagrammatically,

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "E"'] \arrow[dr, phantom, "\phi\,\Swarrow", pos=0.3] & \mathcal{A} \\
\mathcal{B} \arrow[ur, dashed, "\operatorname{Lan}_E F"'] \arrow[ur, bend right=60, "R"'] \arrow[r, phantom, "\Searrow\,\theta"] & {}
\end{tikzcd}$$

Kan extensions need not exist. But if $\mathcal{A}$ is cocomplete then one can prove the existence of the right Kan extension by exhibiting a formula. Correspondingly, if $\mathcal{A}$ is complete then the left Kan extension exists and it is also given by a formula, both of which are presented in Lemma A.16. One begins by defining the following category.

Definition A.15. (Comma category) Let $\mathcal{A} \overset{S}{\to} \mathcal{C} \overset{T}{\leftarrow} \mathcal{B}$ be a diagram of categories. The **comma category** $(S \downarrow T)$ has objects all triples (h, a, b) with $S(a) \overset{h}{\mapsto} T(b)$ and morphisms $(f, g) : (h, a, b) \to (h', a', b')$ all pairs satisfying

$$\begin{tikzcd}
S(a) \arrow[r, "h"] \arrow[d, "S(f)"'] & T(b) \arrow[d, "T(g)"] \\
S(a') \arrow[r, "h'"] & T(b').
\end{tikzcd}$$

When $\mathcal{A}$ is the category with one object $*$ and one morphism and $S(*) = c$, the category $(S \downarrow T)$ will be denoted simply by $(c \downarrow T)$. The category $(S \downarrow c)$ is similarly defined.

The following statement and a proof can be found in [18, p.237] and [18, p.244].

Lemma A.16. Let $F : \mathcal{C} \to \mathcal{A}$ and $E : \mathcal{C} \to \mathcal{B}$ be a pair of functors.

1. If $\mathcal{A}$ is cocomplete then for any $b \in B$

$$\operatorname{Ran}_E F(b) = \operatorname*{colim}_{(E \downarrow b)} F.$$

2. If $\mathcal{A}$ is complete then for any $b \in B$

$$\operatorname{Lan}_E F(b) = \lim_{(b \downarrow E)} F.$$

Definition A.17. (Simplicial category and simplicial sets) Let Δ denote the **simplicial category** whose objects are the finite non-empty totally ordered sets, commonly denoted $[0, 1, \dots, n]$, and whose morphisms are the order preserving functions. Any such function can be obtained as a composition of basic ones, called **cofaces** and **codegeneracies**, which insert or delete a single element. The cofaces and codegeneracies will be respectively denoted $d_i : [n-1] \to [n]$ and $s_i : [n+1] \to [n]$ and they satisfy well know relations.

Define the category of **simplicial sets** to be

$$s\,\mathrm{Set} = \mathrm{Hom}_{\mathrm{Cat}}(\Delta^{op}, \mathrm{Set}),$$

the category of contravariant functors from the simplicial category to the category of sets.

For $X \in$ sSet, the image of $[0, \dots, n]$ will be denoted X_n and referred to as the set of n**-simplices** of X. Simplices which are the image of lower dimensional simplices are said to be **degenerate** and a simplicial set is said to be n**-dimensional** if there exists a non-degenerate n-simplex and for $m > n$ all m-simplices are degenerate.

Remark A.18. (Yoneda) There exists a full and faithful functor from the simplicial category Δ into the category of simplicial sets given on objects by

$$\begin{array}{ccc} \Delta & \longrightarrow & s\,\mathrm{Set} \\ [0, \dots, n] & \longmapsto & \mathrm{Hom}_\Delta([0, \dots, n], -). \end{array}$$

Such functor will be called the **Yoneda embedding** and the images of $[0, \dots, n] \in \Delta$ will be denoted by Δ^n. Notice that for any $X_\bullet \in s\,\mathrm{Set}$ one has

$$\mathrm{Hom}_{s\,\mathrm{Set}}(\Delta^n, X_\bullet) = X_n,$$

a fact referred to as **Yoneda lemma**.

Definition A.19. (Realization and nerve) Let $\mathcal{C}$ be a cocomplete category and $F : \Delta \to \mathcal{C}$ a functor. Denote the Yoneda embedding $\Delta \to s\,\mathrm{Set}$ by $\mathcal{Y}$. The **realization with respect to** F is the right Kan extension of F along $\mathcal{Y}$. The **nerve with respect to** F is the functor defined on objects by $c \longmapsto \big([0,\dots,n] \mapsto \mathrm{Hom}_{\mathcal{C}}(F[0,\dots,n],c)\big)$. Diagrammatically,

$$\begin{tikzcd} \Delta \arrow[r, "F"] \arrow[d, "\mathcal{Y}"'] & \mathcal{C} \arrow[dl, dashed, bend left] \\ s\,\mathrm{Set} \arrow[ur, dashed] & \end{tikzcd}$$

Remark A.20. It is a theorem of Daniel Kan [11] that the realization and nerve with respect to a functor form a universal adjoint pair.

Example A.21. (Geometric realization and singular complex) Consider the embedding $\Delta \to \mathrm{Top}$ of the simplicial category into the category of topological spaces sending $[0,\dots,n]$ to the standard topological n-simplex $|\Delta^n|$. The realization with respect to this functor of any $X_\bullet \in s\,\mathrm{Set}$ can be described by Lemma A.16 and Lemma A.12 as the coequalizer of

$$\coprod_{\begin{tikzcd}[sep=small] \Delta^n \arrow[dr, bend left] \arrow[dd, bend right] & \\ & X_\bullet \\ \Delta^k \arrow[ur, bend right] & \end{tikzcd}} |\Delta^n| \;\rightrightarrows\; \coprod_{\Delta^n \to X_\bullet} |\Delta^n|$$

or equivalently using the Yoneda lemma as

$$\mathrm{coeq}\Big(\coprod_{n\geq 0} X_n \times |\Delta^n| \rightrightarrows \coprod_{n\geq 0} X_n \times |\Delta^n|\Big),$$

which utilizing the cofaces and codegeneracies $d_i : [0,\dots,n-1] \to [0,\dots,n]$ and $s_i : [0,\dots,n+1] \to [0,\dots,n]$ in Δ can be expressed as

$$\coprod_{n\geq 0} X_n \times |\Delta^n| \Big/ \begin{array}{l} d_i^* x \times p \sim x \times d_{i*} p \\ s_i^* x \times p \sim x \times s_{i*} p. \end{array}$$

This topological space will be called the **geometric realization** of $X_\bullet$ and will be denoted by $|X_\bullet|$.

In this context, the nerve of a topological space X will be called the **singular simplicial complex** of X and it is the simplicial set $\mathrm{Sing}_\bullet(X)$ given by

$$[0,\dots,n] \mapsto \mathrm{Hom}_{\mathrm{Top}}(|\Delta^n|, X).$$

Example A.22. (Normalized chain complex) Consider the embedding $\Delta \to \mathrm{Ab}_\bullet$ of the simplicial category into the category of chains complexes sending $[0, \ldots, n]$ to the standard chain complex $\mathrm{C}_\bullet(\Delta^n)$. As in the previous example the realization with respect to this functor of any $X_\bullet \in s\,\mathrm{Set}$ can be described as

$$\bigoplus_{n \geq 0} X_n \otimes \mathrm{C}_\bullet(\Delta^n) \Big/ \begin{array}{l} d_i^* x \otimes c \sim x \otimes d_{i\,*} c \\ s_i^* x \otimes c \sim 0. \end{array}$$

This chain complex will be called the **normalized chain complex** of $X_\bullet$.

Example A.23. (Nerve of a category) Consider the embedding $\Delta \to \mathrm{Cat}$ of the simplicial category into the category of small categories sending $[0, \ldots, n]$ to the category $\underline{n}$ with one object for each $i \in \{0, 1, ..., n\}$ and one morphisms $i \to j$ whenever $i \leq j$. The **nerve of a category** $\mathcal{C} \in \mathrm{Cat}$ is the simplicial set $\mathrm{N}_\bullet(\mathcal{C})$ defined as the nerve with respect to this functor. Explicitly $\mathrm{N}_\bullet(\mathcal{C})$ is given by

$$[0, \ldots, n] \mapsto \mathrm{Hom}_{\mathrm{Cat}}(\underline{n}, \mathcal{C}).$$

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