On the Uniqueness of singular Kähler-Einstein metrics

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Abstract of the Dissertation

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In this dissertation we provide a new proof of the Bando-Mabuchi-Berndtsson uniqueness theorem for Kähler Einstein metrics with singularity along a divisor on Fano manifolds. In 1987, Bando and Mabuchi proved the uniqueness of smooth Kähler-Einstein metrics on a Fano manifold up to a holomorphic automorphism, and this automorphism is induced from a holomorphic vector field on the manifold. It has been noticed that the geodesic connecting two Kähler-Einstein metrics agrees with the path generated by the vector field. Hence it is natural to ask if we can use certain properties of geodesics to prove the uniqueness result.

However, the main difficulty comes from the lack of regularities on the geodesic. According Chen's results, only $\mathcal{C}^{1,\bar{1}}$ regularity can be guaranteed for the potentials on the geodesic. We develop a new technique to solve this problem, based on the convexity of *Ding*-functional on $\mathcal{C}^{1,\bar{1}}$ geodesics and Futaki's calculation on the spectrums of weighted Laplacian operators.

In addition, this method could be generalized to prove the uniqueness of conical Kähler-Einstein metrics on a Fano manifold, under the condition that certain energy functional is proper. The idea is to use twisted Kähler-Einstein metrics to approximate the singular one, and the converging process will preserve the uniqueness. In

the end, the energy condition provides the existence of such twisted Kähler-Einstein metrics.

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1 Introduction

The study of Kähler Einstein metrics on Fano manifolds is an old but lasting subject in complex geometry: on geometrical point of view, it characterizes the manifold with constant Ricci curvature, i.e. the Kähler metric satisfies

$$Ric(\omega) = \omega;$$

on analytical point of view, the complex Monge-Ampère equations arise from the study of this curvature equation, i.e. the Kähler potential $\varphi \in \mathcal{H}$ is the solution of the following equation

$$(\omega_0 + i\partial\bar{\partial}\varphi)^n = e^{h-\varphi}\omega_0^n$$

where $\mathcal{H} := \{\omega = \omega_0 + i\partial\bar{\partial}\varphi > 0\}$. Now as a PDE problem on manifolds, it's natural to ask two questions - existence and uniqueness. After Yau's celebrated work[20] on solving the Calabi Conjecture, Tian's α invariant[19] gives a sufficient condition to solve Monge Ampère equation on Fano manifolds in 1980's. Then many people contribute to this problem during these years. And quite recently, Chen-Donalson-Sun's work([9], [10], [11]) proves the existence of Kähler Einstein metrics on Fano manfolds is equivalent to K-stability condition.This settles down a long standing stability conjecture on Kähler Einstein metrics which goes back to Yau.Their work based on an investigation on some conical Kähler metrics in a special Hölder space $\mathcal{C}^{2,\gamma,\beta}$ for $\gamma < \frac{1}{\beta} - 1$, which was introduced by Donaldson[12].

The problem of uniqueness of Kähler Einstein metrics on Fano manifolds also keeps attractive during these years. It is first proved by Bando and Mabuchi[1] in 1987, and we will give an alternative proof in this paper. The statement is as follows

Theorem 1 *Let X be a compact complex manifold with $-K_X > 0$. Suppose ω_1 and ω_2 are two Kähler Einstein metrics on X, then there is a holomorphic automorphism F, such that*

$$F^*(\omega_2) = \omega_1$$

where this F is generated by a holomorphic vector field $\mathcal{V}$ on X.

They solve this problem by considering a special energy(Mabuchi energy) decreasing along certain continuity path. Then the existence of weak $\mathcal{C}^{1,\bar{1}}$ geodesic between any two smooth Kähler potentials is proved by X.X.Chen[8] in 2000, and this idea turns out to be an important tool in proving uniqueness theorems. For instance, Berman[5] gives a new proof of Bando-Mabuchi's theorem by arguing the geodesic connecting two Kähler Einstein metrics is actually smooth. And Berndtsson[7] proves the uniqueness of possible singular Kähler Einstein metrics along $\mathcal{C}^0$ geodesics. He observes the *Ding*-functional is convex along these geodesics from his curvature formula on the Bergman kernel[6]. Moreover, this curvature formula plays a major role to create a holomorphic vector fields when the functional is affine. This method is used by Berman again to prove the uniqueness of Donaldson's equation[4], and generalized to the $klt-pairs$ in [2].

The idea of this thesis is also initiated from the convexity of *Ding*-functional along geodesics from a different perspective. However, instead of using Berndtsson's curvature formula, we are going to use the Futaki's formula(refer to Chapter 2) of weighted Laplacian operator to derive the holomorphic vector fields. Unlike the former case, here the main difficulty arises from the change of metrics during the convergence of Laplacian operators. Fortunately, we have control on the mixed derivatives $\partial_\alpha\partial_{\bar{\beta}}\phi$ on the product manifold, i.e. Chen's existence theorem of weak geodesic[8] guarantees a uniform bound of mixed second derivatives of the potential in both space and time directions on the geodesic. Moreover, we can perturb the weak geodesic to a sequence of nearby smooth metrics $\{g_\epsilon\}$ with mixed second derivatives under control[8].

Next goal is to prove the uniqueness of $\mathcal{C}^{2,\gamma,\beta}$ conical Kähler-Einstein metrics, based on the new technique developed above. As mentioned before, the main ingredients of this technique consist of Chen's $\mathcal{C}^{1,\bar{1}}$ geodesics and a generalization of Futaki's formula, then these will be extended to prove the uniqueness of the so-called twisted Kähler-Einstein metrics, i.e. a smooth Kähler metric ω satisfies

$$Ric(\omega) = \omega + \theta$$

where θ is some non-negative closed (1,1) form on X. In fact, assuming the correct cohomology condition(see Chapter 11), we have

Theorem 2 *Suppose ω_0 and ω_1 are two solutions of twisted Kähler-Einstein equation with the same weight θ, then there exists a holomorphic automorphism F on X, such that $F^*(\omega_1) = \omega_0$ and $F^*(\theta) = \theta$, and this automorphism is induced from a holomorphic vector field $\mathcal{V}$. Moreover, if there is a point $p \in X$ such that the twister θ is strictly positive, then ω_0 is actually fixed, i.e.*

$$\omega_1 = \omega_0$$

on X.

Before considering the singular metrics, we shall investigate the perturbed conical Kähler-Einstein equation first, i.e. we put

$$\theta = (1-\beta)dd^c\chi_\epsilon$$

where the twister $\chi_\epsilon = \log(|s|^2 + \epsilon e^\psi)$ and ψ is some smooth positively curved metric on the line bundle associated with the divisor D. Then we have the uniqueness of the solution of the following equations

$$Ric(\omega_\epsilon) = \omega_\epsilon + dd^c\chi_\epsilon.$$

If we can take the limit when $\epsilon \to 0$, then in principle, it will bring us the uniqueness of the conical Kähler-Einstein metrics ω_β, i.e. ω_β satisfies

$$Ric(\omega_\beta) = \omega_\beta + (1-\beta)\delta_D$$

where δ_D is the integration current of the divisor D. However, this is not true in general. In some situations, we can't not even find solutions of the perturbed Kähler-Einstein equations even if the conical Kähler-Einstein metric exists. Hence we need another condition to guarantee the existence of twisted Kähler-Einstein metrics, i.e. the properness of twisted *Ding* functional.

Theorem 3 *Suppose the twisted Ding-functional $\mathcal{D}_\beta$ is proper, then there is only one $\mathcal{C}^{2,\alpha,\beta}$ solution ω_{φ_β} for the conical Kähler-Einstein equation with angle β along the divisor D on X.*

One direct consequence of above theorem is the uniqueness of Donaldson's equation, i.e.

$$Ric(\omega_\beta) = \beta\omega_\beta + (1-\beta)\delta_D.$$

for $0 < \beta < 1$. Here the Käher class is proportional to the anti-canonical class, and the twisted *Ding* functional is automatically proper in this case[12].

2 Futaki's formula and Hessian of $Ding$-functional

The manifolds X in our consideration is Fano, then we can assume the Kähler class $[\omega] = c_1(X)$, i.e. for each Kähler metric ω_g, there exists a smooth function F_g such that

$$Ric(\omega_g) - \omega_g = i\partial\bar{\partial}F_g,$$

hence we can define a weighted volume form as $e^F \det g$(we will write F_g as F when there is no confusion), and a pairing for any $u, v \in \mathcal{C}^\infty(X)$

$$(u, v)_g = \int_X u\bar{v}e^F \det g,$$

then Futaki[13] considers a weighted Laplacian operator

$$\Delta_F u = \Delta_g u - \nabla^j u \nabla_j F.$$

the reason to do this is because the new Laplacian operator is easy to do integration by parts under the weighted volume form

$$\int_X (\Delta_F u)\bar{u}e^F \det g = -\int_X (\nabla_j \nabla^j u + \nabla^j u \nabla_j F)\bar{u}e^F \det g$$

$$= \int_X \nabla^j u \nabla_j \bar{u} e^F \det g$$

$$= \int_X |\bar{\partial}u|^2 e^F \det g$$

where the norm of the 1-form is take with respect to the metric g. Hence it's an elliptic operator, and its spectral is discrete as $0 < \lambda_1 < \lambda_2 < \cdots$. Then for each eigenfunction $\Delta_F u = \lambda u$, Futaki[14] writes the following formula

$$\lambda \int_X |\bar{\partial}u|^2 e^F \det g = \int_X |\bar{\partial}u|^2 e^F \det g + \int_X |L_g u|^2 e^F \det g$$

where L_g is a second order differential operator defined as

$$L_g u = \nabla_{\bar{j}} \nabla^i u \frac{\partial}{\partial z^i} \otimes d\bar{z}^j.$$

Now observe the RHS of Futaki's formula is in fact $\int_X |\Delta_{F_g} u|^2 e^F \det g$, we can generalize it to all smooth function as

Lemma 4 *For any smooth function u on X, we have*

$$\int_X |\Delta_F u|^2 e^F \det g = \int_X |\bar\partial u|^2 e^F \det g + \int_X |L_g u|^2 e^F \det g.$$

Proof 1 *we can decompose $u = \Sigma_0^\infty a_i(u) e_i$ into the eigenspace of the operator Δ_{F_g}, and notice that the eigenfunction e_i is orthogonal with respect to each other under the weighted volume form and metric g. Then the first two terms in above equation will preserve this orthogonality, i.e. choose eigenfunctions u and w of Δ_F which are orthogonal to each other, then*

$$\int_X |\bar\partial u + \bar\partial w|^2 e^F \det g = \int_X |\bar\partial u|^2 e^F \det g + \int_X |\bar\partial w|^2 e^F \det g$$

and

$$\int_X |\Delta_F u + \Delta_F w|^2 e^F \det g = \int_X |\Delta_F u|^2 e^F \det g + \int_X |\Delta_F w|^2 e^F \det g$$

Moreover, the differential operator L_g keeps this orthogonality of eigenfunctions, but first notice

$$F_{,\alpha\bar\beta} = R_{\alpha\bar\beta} - g_{\alpha\bar\beta}$$

from the definition of F, then we compute as follows

$$\int_X \langle L_g u, L_g w\rangle_g e^F \det g = \int_X g^{\alpha\bar\lambda} g^{\mu\bar\beta} u_{,\bar\lambda\bar\beta} \bar w_{,\mu\alpha} e^F \det g$$

$$= -\int_X g^{\alpha\bar\lambda} g^{\mu\bar\beta} u_{,\bar\lambda\bar\beta\alpha} \bar w_{,\mu} e^F \det g - \int_X g^{\alpha\bar\lambda} g^{\mu\bar\beta} u_{,\bar\lambda\bar\beta} \bar w_{,\mu} F_{,\alpha} e^F \det g$$

$$= -\int_X g^{\alpha\bar\lambda} g^{\mu\bar\beta} u_{,\bar\lambda\alpha\bar\beta} \bar w_{,\mu} e^F \det g - \int_X g^{\mu\bar\beta} R^{\bar\gamma}_{\bar\beta} u_{,\bar\gamma} \bar w_{,\mu} e^F \det g$$

$$+ \int_X g^{\alpha\bar\lambda} g^{\mu\bar\beta} u_{,\bar\lambda} \bar w_{,\mu\bar\beta} F_{,\alpha} e^F \det g + \int_X g^{\mu\bar\beta} u_{,\bar\lambda} \bar w_{,\mu} F^{,\bar\lambda}_{\bar\beta} e^F \det g + \int_X g^{\alpha\bar\lambda} g^{\mu\bar\beta} u_{,\bar\lambda} \bar w_{,\mu} F_{,\alpha} F_{,\bar\beta} e^F \det g$$

$$= \int_X g^{\alpha\bar\lambda} g^{\mu\bar\beta} u_{,\bar\lambda\alpha} \bar w_{,\mu\bar\beta} e^F \det g + \int_X g^{\alpha\bar\lambda} g^{\mu\bar\beta} u_{\bar\lambda\alpha} \bar w_{,\mu} F_{,\bar\beta} e^F \det g$$

$$+ \int_X g^{\alpha\bar\lambda} g^{\mu\bar\beta} u_{,\bar\lambda} \bar w_{,\mu\bar\beta} F_{,\alpha} e^F \det g + \int_X (g^{\alpha\bar\lambda} u_{,\bar\lambda} F_{,\alpha})(g^{\mu\bar\beta} \bar w_{,\mu} F_{,\bar\beta}) e^F \det g - \int_X g^{\mu\bar\beta} u_{,\bar\beta} \bar w_\mu e^F \det g$$

$$= \int_X (g^{\alpha\bar\lambda} u_{,\alpha\bar\lambda} + g^{\alpha\bar\lambda} u_{,\bar\lambda} F_{,\alpha})(g^{\mu\bar\beta} \bar w_{\mu\bar\beta} + g^{\mu\bar\beta} \bar w_\mu F_{,\bar\beta}) e^F \det g$$

$$= \int_X (\Delta_F u, \Delta_F w)_g e^F \det g = 0.$$

Next let's consider an easy case: according to He[15], the second derivative of *Ding*-functional on a smooth geodesic equals

$$\frac{\partial^2 \mathcal{D}}{\partial t^2} = (\int_X e^{F_g} \det g)^{-1} \{ \int_X (|\bar{\partial}\varphi'|_g^2 - (\pi_\perp \varphi')^2) e^{F_g} \det g \}$$

where the metric g is induced by the Kähler form ω_φ, and the projection operator is defined as $\pi_\perp u = u - \int_X u e^{F_g} \det g / \int_X e^{F_g} \det g$. This implies *Ding*-functional is convex along smooth geodesics. Now suppose there is a smooth geodesic connecting two Kähler Einstein metrics, the *Ding*-functional must keep to be a constant along it. Hence we get

$$\int_X |\bar{\partial}\varphi'|_g^2 e^{F_g} \det g = \int_X (\pi_\perp \varphi')^2 e^{F_g} \det g,$$

then we see the first eigenvalue λ_1 of the weighted Laplacian operator Δ_{F_g} is 1, and $\pi_\perp \varphi'$ belong to the first eigenspace, i.e.

$$\Delta_{F_g}(\pi_\perp \varphi') = \pi_\perp \varphi'.$$

Now by Futaki's formula, we see

$$L_g(\pi_\perp \varphi') = 0,$$

then the induced vector field $V_t = \nabla^i \varphi' \frac{\partial}{\partial z^i}$ is holomorphic on X. Moreover, let's differentiate this vector field with respect to t on the geodesic

$$(g^{j\bar{k}} \varphi'_{\bar{k}})' = g^{j\bar{k}} \varphi''_{\bar{k}} - g^{j\bar{q}} \varphi'_{p\bar{q}} g^{p\bar{k}} \varphi'_{\bar{k}}$$

$$= g^{j\bar{k}} (g^{\alpha\bar{\beta}} \varphi'_\alpha \varphi'_{\bar{\beta}})_{,\bar{k}} - g^{j\bar{q}} \varphi'_{p\bar{q}} g^{p\bar{k}} \varphi'_{\bar{k}}$$

$$= g^{j\bar{k}} g^{\alpha\bar{\beta}} \varphi'_\alpha \varphi'_{,\bar{\beta}\bar{k}} = 0$$

by the holomorphicity of V_t. Finally, this gives us a holomorphic vector field $\mathcal{V} = V_t - \partial/\partial t$ on $X \times S$, and its induced automorphism will give the uniqueness of the two Kähler Einsteim metrics.

3 Some L^2 theorems

In this section, we are going to use L^2 theorem to investigate the weighted Laplacian operator Δ_{F_g} and its spectrum, then we shall project our target to the front eigenspace in the proof of uniqueness theorem. First notice that we always have $\lambda_1 \geqslant 1$ by Futaki's formula. Then we are going to introduce some notations.

From now on, we shall assume the manifold X admits non-trivial holomorphic vector fields, and $H^{0,1}(X) = 0$. Then fix one t and restrict our attention to this fiber $X \times \{t\}$. Since $-K_X = [\omega]$, we can write

$$\omega_g = i\partial\bar{\partial}\phi_g$$

where ϕ_g is a plurisubharmonic metric on the line bundle $-K_X$. We claim the measure

$$e^{F_g} \det g = e^{-\phi_g},$$

and this is because locally $F_g = -\log \det g - \phi_g$. Then naturally the pairing between functions on X with this weight can be written as

$$(u, v)_g = \int_X u\bar{v}e^{-\phi_g}.$$

Here is the L^2 theorem coming to play with. Let's consider the space of all L^2 bounded $-K_X$ valued $(n, 0)$ forms under the metric ϕ_g, i.e. it consists of every function u on X such that

$$\int_X |u|^2 e^{-\phi_g} < +\infty,$$

we denote this space as $L^2_{(n,0)}(-K_X, \phi_g)$, and similarly we can consider all L^2 bounded $-K_X$ valued $(n, 1)$ forms under the weighted norm

$$\int_X g^{\alpha\bar{\beta}} v_\alpha \overline{v_\beta} e^{-\phi_g} < +\infty,$$

and we denote this space as $L^2_{(n,1)}(-K_X, \phi_g)$, then we can define an unbounded operator $\bar{\partial}$ between them

$$\bar{\partial} : L^2_{(n,0)}(-K_X, \phi_g) \dashrightarrow L^2_{(n,1)}(-K_X, \phi_g).$$

Notice that the domains of these two operator are not the whole L^2 spaces. In fact, we can define

$$dom(\bar{\partial}) := \{u \in L^2_{(n,0)}(-K_X, \phi_g);\ \bar{\partial}u \in L^2_{(n,1)}(-K_X, \phi_g)\},$$

but it is not densely defined in L^2 space when g_ϕ is a $\mathcal{C}^{1,\bar{1}}$ solution of geodesic equation on a fiber $X \times \{t\}$. Hence we should consider the Hilbert space $\mathcal{H}_1$ to be the closure of $dom(\bar{\partial})$ in $L^2_{(n,0)}(-K_X, \phi_g)$. We claim that $\mathcal{H}_1$ is not empty.

First notice that for any non-trivial holomorphic vector field $v \in L^2_{(n-1,0)}(-K_X)$, we can solve the following equation

$$\bar{\partial} u = \omega_{g_\phi} \wedge v,$$

since $\bar{\partial}(\omega_{g_\phi} \wedge v) = 0$ in the sense of distributions, but $\ker(\bar{\partial}) = Range(\bar{\partial})$ from $H^{0,1}(X) = 0$. Next, consider the subspace $\mathcal{W}$ containing all such u, i.e. define

$$\mathcal{W} := \{u \in L^2_{(n,0)}(-K_X, \phi_g);\ \bar{\partial} u = \omega_{g_\phi} \wedge v,\ \forall v \in L^2_{(n-1,0)}(-K_X)\},$$

then it is a non-empty subspace in $L^2_{(n,0)}(-K_X, \phi_g)$, and it's easy to check

$$\mathcal{W} \subset dom(\bar{\partial}),$$

hence we proved the claim. Now $\bar{\partial}$ is a densely defined, closed operator on the Hilbert space $\mathcal{H}_1$ - it's closed from the continuity property of differential operators in the distribution sense. We can discuss its Hilbert adjoint operator $\bar{\partial}^*_\phi$, which is a densely defined, closed operator on $L^2_{(n,1)}(-K_X, \phi_g)$. Moreover, they have closed ranges

Lemma 5 *$\bar{\partial}$ and $\bar{\partial}^*_{\phi_g}$ are densely defined, closed operators with closed ranges.*

Proof 2 *We need to estimate the L^2 norm of $\bar{\partial} u$. Take h to be a fixed smooth metric with positive Ricci curvature on X, and $u \in dom(\bar{\partial}) \cap \ker(\bar{\partial})^\perp$, we have*

$$\begin{aligned}\int_X |\bar{\partial} u|^2_g e^{-\phi_g} &\geqslant \int_X |\bar{\partial} u|^2_h \det h \\ &\geqslant c \int_X |u|^2 \det h \\ &\geqslant c' \int_X |u|^2_g e^{-\phi_g}.\end{aligned}$$

this estimate implies $\bar{\partial}$ has closed range, and hence its adjoint $\bar{\partial}^*_{\phi_g}$ by functional analysis reason.

Then we can define the Laplacian operator as $\Box_{\phi_g} = \bar{\partial}^*_{\phi_g}\bar{\partial}$, where also as an unbounded closed operator, i.e.

$$\Box_{\phi_g} : L^2_{(n,0)}(-K_X, \phi_g) \dashrightarrow L^2_{(n,0)}(-K_X, \phi_g)$$

and its domain of definition is

$$dom(\Box_{\phi_g}) := \{u \in L^2_{(n,0)}(-K_X, \phi_g);\ u \in dom(\bar{\partial}) \ and \ \bar{\partial}u \in dom(\bar{\partial}^*_{\phi_g})\}.$$

we claim this operator also has closed range. and

Proposition 6 *we have*

$$\ker \Box_{\phi_g} = coker \Box_{\phi_g},$$

hence they are both finite dimensional.

Proof 3 *First note* $\ker \Box_{\phi_g} = \ker \bar{\partial}$ *is the 1 dimensional space of constant functions on* X. *In order to prove* $coker \Box_{\phi_g}$ *also has finite rank, it's enough to prove the weighted Laplacian operator has closed range, since it's self-adjoint*

$$coker \Box_{\phi_g} = R(\Box_{\phi_g})^{\perp} = \ker \Box_{\phi_g}.$$

Now we are going to prove the closed range property, but this follows from the following estimate for $u \in dom(\Box_{\phi_g}) \cap \ker(\bar{\partial})^{\perp}$

$$||u||^2_g \leqslant C||\bar{\partial}u||^2_g$$

$$\leqslant C(\Box_{\phi_g}u, u)_g$$

$$\leqslant 2C||\Box_{\phi_g}u||^2_g + \frac{1}{2}||u||^2_g$$

and hence

$$||u||^2_g \leqslant C'||\Box_{\phi_g}u||^2_g,$$

which implies the claim.

Notice that this is not enough to guarantee the existence of discrete spectral, but we have a further estimate,

Lemma 7 *For all* $u \in dom(\Box_{\phi_g}) \cap \ker(\bar{\partial})^{\perp}$, *there is an uniform constant* C, *such that*

$$||u||_{W^{1,2}} \leqslant C||\Box_{\phi_g}u||^2_g.$$

Proof 4 *we still compare it with some fixed smooth weight(metric) h,*

$$
\begin{aligned}
||\bar{\partial} u||_h^2 &\leqslant C||\bar{\partial} u||_g^2 \\
&= C(\Box_{\phi_g} u, u)_g \\
&\leqslant C||\Box_{\phi_g} u||_g ||u||_g \\
&\leqslant C'||\Box_{\phi_g} u||_g ||u||_h \\
&\leqslant C''||\Box_{\phi_g} u||_g ||\bar{\partial} u||_h,
\end{aligned}
$$

then

$$||\bar{\partial} u||_h^2 \leqslant C''||\Box_{\phi_g} u||_g.$$

finally, an integration by part gives the desired estimate since

$$
\begin{aligned}
\int_X h^{\alpha\bar{\beta}} u_{,\alpha} \overline{u_{,\beta}} \det h &= -\int_X h^{\alpha\bar{\beta}} u_{,\alpha\bar{\beta}} \bar{u} \det h \\
&= -\int_X h^{\alpha\bar{\beta}} u_{,\bar{\beta}\alpha} \bar{u} \det h \\
&= \int_X h^{\alpha\bar{\beta}} u_{,\bar{\beta}} \overline{u_{,\bar{\alpha}}} \det h
\end{aligned}
$$

Then we can discuss the spectral of $\Box_{\phi_g}$, when g_ϕ is the $\mathcal{C}^{1,\bar{1}}$ function. Suppose λ is an eigenvalue of $\Box_{\phi_g}$, and let Λ be the corresponding eigenspace, we claim

Proposition 8 $\dim \Lambda < +\infty$

Proof 5 *Let $v_i \in \Lambda$ be a sequence of eigenfunctions with bound L^2 norm, i.e. $||v_i||_g^2 = 1$, then since*

$$
\begin{aligned}
||v_i||_{W^{1,2}} &\leqslant C||\Box_{\phi_g} v_i||_g \\
&= C\lambda,
\end{aligned}
$$

hence there exists a $W^{1,2}$ function v_∞ such that $v_i \to v_\infty$ in strong L^2 norm, by compact embedding theorem. And since $\Lambda = \ker(\Box_{\phi_g} - \lambda I)$ is a closed subspace of L^2

$$v_\infty \in \Lambda.$$

This implies every bounded sequence in Λ has a convergent subsequence, i.e. the unit ball in Λ is compact, hence $\dim \Lambda$ is finite.

Next we are going to discuss some computations when the weight ϕ_g is at least $\mathcal{C}^2$. First notice that formally

$$< \Box_{\phi_g} u, v >_g = < \bar{\partial} u, \bar{\partial} v >_g$$

for any pairing u, v. It's easy to see

$$\Box_{\phi_g} u = \Delta_{\phi_g} u$$

for all smooth functions u, when the metric ϕ_g is smooth. If we look closer at these operators, there is a more computable way to express them. For this purpose, let's assume ϕ_g is a $\mathcal{C}^2$ metric, then for any $(n, 1)$ form α with value in $-K_X$,

$$\bar{\partial}^*_{\phi_g} \alpha = \partial^{\phi_g} (\omega_g \lrcorner \alpha)$$

where $\partial^{\phi} v = e^{\phi} \partial (e^{-\phi} v) = \partial v - \partial \phi \wedge v$ for any $(n-1, 0)$ form with value in $-K_X$(that is a vector field on X). Hence if we define

$$v = \omega_g \lrcorner \alpha,$$

we will have

$$\bar{\partial}^*_{\phi_g} \alpha = \partial^{\phi_g} v$$

and the weighted Laplacian operator could be computed as

$$\Box_{\phi_g} u = \partial^{\phi_g} (\omega_g \lrcorner \bar{\partial} u)$$

for $u \in dom \Box_{\phi_g} \cap L^2_{(n,0)}(-K_X, \phi_g)$. Notice that there is commutation relation between the new defined operator ∂^{ϕ} and $\bar{\partial}$, that is

$$\partial^{\phi} \bar{\partial} + \bar{\partial} \partial^{\phi} = i \partial \bar{\partial} \phi \wedge \cdot \tag{1}$$

Now if u is any eigenfunction of the weighted Laplacian operator with eigenvalue λ, i.e. $\Box_{\phi_g} u = \lambda u$, we can decompose it into two equations

$$\omega_g \lrcorner \bar{\partial} u = v \quad \partial^{\phi_g} v = \lambda u.$$

here we can write $v = X \lrcorner 1$, where the constant function 1 is read as an $(n, 0)$ form with value in $-K_X$, and $X = X^{\alpha} \frac{\partial}{\partial z^{\alpha}}$ is a vector field in $(1, 0)$ direction on the manifolds. Next we are going to prove Futaki's formula by the commutation equality.

Lemma 9 *(Futaki's formula) Let u be a eigenfunction of weighted Laplacian with eigenvalue λ, i.e. $\Box_{\phi_g} u = \lambda u$, then*

$$\lambda \int_X |\bar{\partial} u|_g^2 e^{-\phi_g} = \int_X (|L_g u|^2 + |\bar{\partial} u|_g^2) e^{-\phi_g}.$$

Proof 6 *First notice u is pure real or imaginary. Hence here we will give the proof when u is real valued - the case when u is pure imaginary is similar. Now by the commutation relation of ∂^{ϕ_g}, we compute $\bar{\partial}(\lambda u)$*

$$-\partial^{\phi_g} \bar{\partial} v + i \partial \bar{\partial} \phi_g \wedge v = \lambda \bar{\partial} u,$$

notice that $i\partial\bar{\partial}\phi_g = \omega_g$, hence

$$-\partial^{\phi_g} \bar{\partial} v = (\lambda - 1) \bar{\partial} u,$$

pair it with $\bar{\partial} u$,

$$(\lambda - 1) \int_X |\bar{\partial} u|_g^2 e^{-\phi_g} = - \int_X \langle \partial^{\phi_g} \bar{\partial} v, \bar{\partial} u \rangle_g e^{-\phi_g}$$

$$= \int_X -g^{\lambda\bar{\mu}} \partial_\alpha (e^{-\phi_g} \partial_{\bar{\mu}} X^\alpha) \overline{\partial_{\bar{\lambda}} u}$$

$$= \int_X \partial_{\bar{\mu}} X^\alpha \overline{\partial_{\bar{\alpha}} X^\mu} e^{-\phi_g}.$$

Now notice that $X^\alpha = g^{\alpha\bar{\beta}} u_{,\bar{\beta}}$, under the normal coordinate when $g_{i\bar{j}} = \delta_{ij} \Lambda_i$,

$$\partial_{\bar{\mu}} X^\alpha \partial_\alpha X^{\bar{\mu}} = g^{\alpha\bar{\beta}} u_{,\bar{\beta}\bar{\mu}} g^{\lambda\bar{\mu}} u_{,\lambda\alpha}$$

$$= \frac{1}{\Lambda_\alpha \Lambda_\lambda} u_{,\bar{\alpha}\bar{\lambda}} u_{,\lambda\alpha}$$

$$= \frac{1}{\Lambda_\alpha \Lambda_\lambda} u_{,\bar{\alpha}\bar{\lambda}} u_{,\alpha\lambda}$$

$$= g_{\alpha\bar{\beta}} g^{\lambda\bar{\mu}} \partial_{\bar{\mu}} X^\alpha \overline{\partial_{\bar{\lambda}} X^\beta},$$

hence we proved the Futaki's formula

$$(\lambda - 1) \int_X |\bar{\partial} u|_g^2 e^{-\phi_g} = \int_X |\bar{\partial} X|_g^2 e^{-\phi_g}.$$

4 Existence of the $\mathcal{C}^{1,\bar{1}}$ geodesic

Let X be a n dimensional compact complex Kähler manifold with Kähler form ω, then we can write locally

$$\omega = \Sigma_{i=1}^n g_{j\bar{k}} dz^j \wedge d\bar{z}^k.$$

Consider all the Kähler forms in the same cohomology class with ω, they can be identified with the space of all smooth strictly ω-plurisubharmonic functions on X, i.e.

$$\mathcal{H} := \{\varphi \in C^\infty(X);\ \omega_\varphi = \omega + dd^c\varphi > 0\},$$

and this is also called the space of Kähler potentials. It's easy to see that the tagent space of $\mathcal{H}$ consists of all smooth functions ψ on X, and we can introduce an natural L^2 metric on $T_\varphi\mathcal{H}$ as

$$< \psi_1, \psi_2 >_\varphi = \int_X \psi_1\psi_2\omega_\varphi^n$$

for two tagent vector $\psi_1, \psi_2 \in T_\varphi\mathcal{H}$. Then we can ask what geodesics are on this Riemannian manifold, and the geodesic equation turns out to be

$$\varphi''(t) - |\bar{\partial}\varphi'(t)|^2_{g_t} = 0$$

where g_t corresponds to the metric induced from the Kähler form ω_{φ_t}. In order to figure out the solution of this equation, we need to slightly change our consideration into a different setting. Think about the product space $(z, t) \in X \times [0, 1]$, we can add an extra dimension to complete it into a $n + 1$ dimensional complex space as follows

$$(z, t, e^{is}) \in X \times [0, 1] \times S^1$$

where $z = (z_1, \cdots, z_n)$ and $w = z_{n+1} = t + is$ is the last complex variable. If we denote the metrics defined by ω, ω_φ as g and g', then the top volume forms satisfy $\omega^n/n! = \det g$ and $\omega_\varphi^n/n! = \det g'$, and the geodesic equation is equivalent to

$$(\varphi'' - |\bar{\partial}\varphi'|^2_{g'}) \det g' = 0, \tag{2}$$

when $\det g' \neq 0$. Let $S = [0, 1] \times S^1$, then we can construct a Kähler form on $X \times S$ from the pull back $\pi^*\omega_\varphi$ as

$$\Omega = \Sigma_{j,k=1}^n g'_{j\bar{k}} dz^j \wedge d\bar{z}^k + (\bar{\partial}_k\varphi')dz^{n+1} \wedge d\bar{z}^k + (\partial_k\varphi')dz^k \wedge d\bar{z}^{n+1} + (\varphi'')dz^{n+1} \wedge d\bar{z}^{n+1}.$$

Notice that sometimes we will write $g'_{\alpha\bar{\beta}}$ with $\alpha, \beta = 1, \cdots, n + 1$ standing for the local coefficients of Ω, and then linear algebra tells us that equation(2) is equivalent to the following homogeneous Monge-Amprére equation

$$\Omega^{n+1} = 0. \tag{3}$$

In fact, it is possible to solve this equation with the following weak regularities

Theorem 10 *(Chen) Let $\varphi_0, \varphi_1 \in \mathcal{H}$, then there exists a unique $C^{1,\bar{1}}$ geodesic connecting them, i.e. the following homogenous Monge-Ampère equation has a unique weak solution $\varphi \in \overline{\mathcal{H}}$(the closure is taken under the $C^{1,\bar{1}}$ topology) on $X \times S$*

$$\det(g_{i\bar{j}} + \partial_i \partial_{\bar{j}} \varphi)_{(n+1)(n+1)} = 0$$

where $i, j = 1, \cdots, n+1$, and on the boundary $\partial(X \times S)$

$$\varphi(0, s, z) = \varphi_0(z),\ \varphi(1, s, z) = \varphi_1(z)$$

with the following estimate

$$||\varphi||_{C^1(X\times S)} + \max\{|\partial_i \partial_{\bar{j}} \varphi|\} < C$$

where C is a uniform constant only depending on φ_0 and φ_1.

In order to solve this equation, we refer to the famous continuity method, but first we need to establish its beginning point. Let ω be the background Kähler metric, there is another trivial way to construct $(n+1, n+1)$ Kähler form on the total space $X \times S$ from ω by

$$\tilde{\omega} = \pi^* \omega + dz^{n+1} \wedge d\bar{z}^{n+1}$$

with its potential $\tilde{\varphi} = \varphi - |z^{n+1}|^2$, and we shall also write $(g_{\alpha\bar{\beta}})$, φ instead of $(\tilde{g}_{\alpha\bar{\beta}})$, $\tilde{\varphi}$ when there is no confusion. Then let's begin with finding a strictly ω-plurisubharmonic function on the total space with prescribed boundary values φ_0 and φ_1

$$\phi(z, t) = t\varphi_0 + (1-t)\varphi_1 - Ct(1-t)$$

where C is a large positive constant such that $\omega + i\partial\bar{\partial}\phi > 0$ on $X \times S$. Notice that $\phi|_{X\times\{0\}} = \varphi_0$, $\phi|_{X\times\{1\}} = \varphi_1$ and $\phi|_{X\times\{t\}}$ is a strict ω-plurisubharmonic function for each $t \in [0, 1]$. Now we can consider a one parameter family of Monge-Amprére equations as follows

$$(*_\epsilon) \qquad \det(g_{\alpha\bar{\beta}} + \partial_\alpha \bar{\partial}_\beta \varphi_\epsilon) = \epsilon \det(g_{\alpha\bar{\beta}} + \partial_\alpha \bar{\partial}_\beta \phi), \tag{4}$$

for $\epsilon \in [0, 1]$. Now define the set

$$S := \{\epsilon \in [0, 1]; \ \ (*_\epsilon) \ \textit{is solvable}\}.$$

The goal is to prove $0 \in S$. First notice that $1 \in S$, since $\varphi_1(z, w) = \phi(z, w)$, and S is open thanks to the ellipticity of Monge-Amprére equations. We shall prove S is also closed from *a prior estimate* as follows

4.1 C^0 estimates and interior Laplacian estimates

Consider the following Dirichlet problem with boundary condition

$$\Delta_g h = -n-1, \quad h|_{\partial(X\times S)} = \phi|_{\partial(X\times S)}.$$

Then maximal principal implies $\max_{X\times S}(\varphi_\epsilon - h) \leqslant 0$, and hence $\varphi \leqslant h$. On the other hand, the domination principal for complex Monge-Amprére measure implies $\mu\{\varphi_\epsilon < \phi\} = 0$, where μ is the standard Lebesgue measure on X. Then

Proposition 11 *Let φ_ϵ be a sequence of smooth solution of equation $(*_\epsilon)$, then*

$$\phi \leqslant \varphi_\epsilon \leqslant h$$

Now let's invoke Yau's C^2 estimate with respect to φ_ϵ on the total space $X \times S$, then it gives

$$\Delta' F \geqslant R_1 - C(n+1)F + R_2 e^{c\varphi/n} F^{1+\frac{1}{n}} (\epsilon)^{-1},$$

where $F = e^{-c\varphi}(n+1+\Delta\varphi)$ and R_1, R_2, C, c are some uniform constants such that $R_2 > 0$. Since φ is uniformly bounded from above, we conclude the maximum of F is either uniformly bounded or achieved on the boundary. Hence

Proposition 12 *There is a uniform constant C such that*

$$\max_{X\times S}(\Delta\varphi_\epsilon + n + 1) \leqslant C(1 + \max_{\partial(X\times S)}(\Delta\varphi_\epsilon + n + 1)).$$

4.2 Boundary estimate

Suppose p is a point on the boundary $\partial(X \times S)$, there is a local parametrization of the boundary in a smaller open neighborhood of p, i.e. $\{z^{n+1} = 0\} = \partial(X \times S) \cap U$, where p corresponds to the origin $\{z = 0\}$ in U. Now since ω is a Kähler form, we can assume

$$g_{\alpha\bar{\beta}}(p) = \delta_{\alpha\beta}, \quad \alpha, \beta = 1, \cdots, n+1$$

and

$$\frac{1}{2}\delta_{\alpha\beta} \leqslant g_{\alpha\bar{\beta}} \leqslant 2\delta_{\alpha\beta}$$

for $\forall q \in U$. Furthermore, we can also assume from the positivity of ω_ϕ

$$g_{\alpha\bar{\beta}} + \partial_\alpha \partial_{\bar{\beta}}\phi > 2\kappa g_{\alpha\bar{\beta}}$$

for $\kappa > 0$ on $X \times S$. Hence near the origin p it implies

$$g_{\alpha\bar{\beta}} + \partial_\alpha \partial_{\bar{\beta}}\phi > \kappa\delta_{\alpha\bar{\beta}}.$$

for any $q \in U$. Next notice that on the tangential direction of $\partial(X \times S)$, the derivatives of difference vanish as

$$\frac{\partial(\varphi - \phi)}{\partial z^k} = 0, \quad \frac{\partial^2(\varphi - \phi)}{\partial z^j \partial \bar{z}^k} = 0$$

where $j, k = 1, \cdots, n$ and $\forall q \in U \cap \partial(X \times S)$. Then we claim

Proposition 13 *For any point $p \in \partial(X \times S)$, there is a uniform constant C, such that*

$$|\partial_{n+1}\bar{\partial}_{n+1}\varphi_\epsilon| \leqslant C(\max_{\partial(X\times S)} |\nabla\varphi_\epsilon|_g^2 + 1).$$

Proof 7 *Look at the point p, we can assume $g_\phi(p)$ is a diagonal matrix with eigenvalues $\lambda_1, \cdots, \lambda_{n+1}$, then the equality on the continuity path implies*

$$\det(\delta_{\alpha\bar{\beta}}\lambda_\alpha + \partial_\alpha\bar{\partial}_\beta(\varphi - \phi)) = \epsilon f$$

where $f = \det(g_\phi)$. By using the vanishing of the derivatives on the boundary, we have

$$\lambda_{n+1} + (\varphi - \phi)'' = \epsilon f + \Sigma_{j=1}^n \lambda_\alpha^{-1}\partial_\alpha(\varphi - \phi)'\bar{\partial}_\alpha(\varphi - \phi)',$$

which gives the deserved bound on φ''.

Next we are going to estimate the mixed derivatives terms in the complex Hessian of φ as follows

Proposition 14 *There is a uniform constant C such that*

$$\max_{X\times S}(n + 1 + \Delta\varphi_\epsilon) \leqslant C \max_{X\times S}(1 + |\nabla\varphi_\epsilon|_g^2)$$

Define the following differential operator as

$$\mathcal{L}u = \Sigma_{\alpha,\beta=1}^{n+1} g'^{\alpha\bar{\beta}} u_{,\alpha\bar{\beta}},$$

and let D denote a first order differential operator standing for $\frac{\partial}{\partial x}$ or $\frac{\partial}{\partial y}$, where $z^j = x + \sqrt{-1}y$ with $j = 1, \cdots, n$. Take

$$D(\log\det g') = D(\log tf)$$

and

$$g'^{\alpha\bar{\beta}} D\varphi_{\alpha\bar{\beta}} = D\log f - g'^{\alpha\bar{\beta}} Dg_{\alpha\bar{\beta}}.$$

$$g'^{\alpha\bar{\beta}}D(\varphi-\phi)_{\alpha\bar{\beta}} = D\log f - g'^{\alpha\bar{\beta}}Dg_{\phi,\alpha\bar{\beta}}.$$

then

$$\mathcal{L}D(\varphi-\phi) = g'^{\alpha\bar{\beta}}D(\varphi-\phi) \leqslant C(1+\Sigma_{\alpha=1}^{n+1}g'^{\alpha\bar{\alpha}}).$$

this is because for a positive Hermitian matrix $(h_{j\bar{k}})$, we have $|h_{j\bar{k}}| \leqslant h_{\alpha\bar{\alpha}} + h_{\beta\bar{\beta}}$.

Next we need to construct a barrier function v for the maximal principle. Let s, N be some undetermined positive number, and we define

$$v = (\varphi-\phi) + s(h-\phi) - Nt^2$$

on the neighborhood $\Omega_\delta = (X\times S)\cap B_\delta(0)$, where $B_\delta(0)$ is the coordinate ball centered at p with radius δ.

Lemma 15 *For any small $\kappa > 0$, there is a $\delta > 0$ such that*

$$\mathcal{L}v \leqslant -\kappa(1+\Sigma_{\alpha=1}^{n+1}g'^{\alpha\bar{\alpha}})$$

on Ω_δ and $v \geqslant 0$ on $\partial\Omega_\delta$.

Proof 8 *First notice that $v = 0$ on $\partial(X\times S)\cap B_\delta(0)$, and recall that $g_\phi > 4\kappa(\delta_{\alpha\beta})$ in $B_\delta(0)$, then*

$$\begin{aligned}\mathcal{L}(\varphi-\phi) &= g'^{\alpha\bar{\beta}}(g'_{\alpha\bar{\beta}} - g_{\phi,\alpha\bar{\beta}})\\ &= n+1 - g'^{\alpha\bar{\beta}}g_{\phi,\alpha\bar{\beta}}\\ &\leqslant n+1-4\kappa\Sigma_{\alpha=1}^{n=1}g'^{\alpha\bar{\alpha}}.\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}(h-\phi) &= g'^{\alpha\bar{\beta}}(h_{\alpha\bar{\beta}} - \phi_{\alpha\bar{\beta}})\\ &\leqslant C_1(1+\Sigma_{\alpha=1}^{n+1}g'^{\alpha\bar{\alpha}})\end{aligned}$$

for some uniform constant C_1. But the last term has $\mathcal{L}t^2 = 2g'^{(n+1)\overline{(n+1)}}$, which gives

$$\mathcal{L}v \leqslant (n+1) - 4\kappa\Sigma_{\alpha=1}^{n+1}g'^{\alpha\bar{\alpha}} + sC_1(1+\Sigma_{\alpha=1}^{n+1}g'^{\alpha\bar{\alpha}}) - 2Ng'^{(n+1)\overline{(n+1)}}$$

Then suppose $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \lambda_{n+1}$ are the eigenvalues of the matrix g', we have

$$\Sigma_{\alpha=1}^{n+1}\lambda_\alpha = \Sigma_{\alpha=1}^{n+1}g'^{\alpha\bar{\alpha}}, \qquad \lambda_{n+1}^{-1} \geqslant g'^{(n+1)\overline{(n+1)}},$$

and $\lambda_1\cdots\lambda_{n+1} = \epsilon f$. Hence

$$\kappa\Sigma_{\alpha=1}^{n+1}\lambda_{n+1}^{-1} + N\lambda_{n+1}^{-1} = \kappa\Sigma_{\alpha=1}^{n}\lambda_\alpha^{-1} + (\kappa+N)\lambda_{n+1}^{-1}$$

$$\geqslant (n+1)\kappa^{\frac{n}{n+1}} N^{\frac{1}{n+1}} (\epsilon f)^{-\frac{1}{n+1}}$$
$$\geqslant C_2 N^{\frac{1}{n+1}}.$$

Then for N large enough(C_2 could be large when ϵ is small), we can make

$$-C_2 N^{\frac{1}{n+1}} + (n+1) + sC_1 < -\kappa.$$

and $sC_1 < \kappa$ for s small enough, hence

$$\mathcal{L}v < -\kappa(1 + \Sigma_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}})$$

on Ω_δ. On the other hand, recall $\Delta_g(h-\phi) = -tr_g g_\phi < -4\kappa$ on Ω_δ, and this convexity gives the growth control near the boundary

$$h - \phi \geqslant C_0 t,$$

hence if we further choose δ small enough such that $(sC_0 - N\delta)t \geqslant 0$, the inequality $s(h-\phi) - Nt^2 \geqslant 0$ holds.

Next we will complete the proof of proposition(14). Let $M = \max_{X\times S}(1 + |\nabla\varphi|_g)$, and for constants $A \gg B \gg C$ multiple of M, such that $B\delta^2 > |D(\varphi-\phi)|$, we define then $w \geqslant 0$ on $\partial\Omega_\delta$, and

$$w = Av + B|z|^2 + D(\varphi - \phi).$$

then $w \geqslant 0$ on $\partial\Omega_\delta$, and $w(0) = 0$. Hence compute

$$\mathcal{L}w \leqslant -A\kappa(1 + \Sigma_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}}) + 2B(\Sigma_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}}) + C(1 + \Sigma_{\alpha=1}^{n+1} g'^{\alpha\bar{\alpha}})$$
$$\leqslant (-A\kappa + 2B + C)(1 + \Sigma_\alpha g'^{\alpha\bar{\alpha}}).$$

Maximal principal implies $w \geqslant 0$ on Ω_δ and $\frac{\partial w}{\partial t}(p) \geqslant 0$, but

$$\frac{\partial w}{\partial t}(0) = A\frac{\partial v}{\partial t}(0) + \frac{\partial}{\partial t}(D\varphi - \phi)(0),$$

and

$$\frac{\partial v}{\partial t}(0) = \frac{\partial \varphi}{\partial t}(0) - \frac{\partial \phi}{\partial t}(0) + s\frac{\partial}{\partial t}(h - \phi).$$

Notice that $\phi'(0) \leqslant \varphi'(0) \leqslant h'(0)$, we have

$$\frac{\partial}{\partial t} D\varphi(0) \leqslant C_3 M$$

Finally, we repeat our argument to $-D$, the same estimate gives

$$-\frac{\partial}{\partial t} D\varphi(0) \leqslant C_3 M$$

This complete our proof.

4.3 C^1 estimates

The question is reduced to find out an upper bound for $|\nabla\varphi_\epsilon|_g$ along the continuity path, and here we will invoke the so called blowing up analysis. In the following, we will concentrate our attention on a small ball in X, so can assume the metric is just the Euclidean metric in $\mathbb{C}^{(n+1)}$. Now suppose on the contrary, there is a subsequence φ_{ϵ_i}(write as φ_i briefly) and points $x_i \in X$, such that

$$|\nabla\varphi_i|_g = \max_{X\times S}|\nabla\varphi_i| = 1/\kappa_i,$$

and by the above C^2 estimates, we have

$$\Delta\varphi_i \leqslant 1/\kappa_i^2,$$

and this growth estimate is important because it's invariant under rescaling. Let's define

$$\psi_i(x) = \varphi_i(x_i + \kappa_i x),$$

for $\forall x \in B_{\delta/\kappa_i}(0) \subset \mathbb{C}^{n+1}$, where δ is a fixed small number. Then after this rescaling, we see

$$\max_{B_{\delta/\kappa_i}}|\nabla\psi_i|_g = |\nabla\psi_i|_g(0) = 1, \qquad \Delta\psi_i \leqslant C$$

for some uniform constant C. In the same time, we can define

$$\phi_i(x) = \phi(x_i + \kappa_i x), \qquad h_i(x) = h(x_i + \kappa_i x),$$

and C^0 estimate follows as

$$C'^{-1} \leqslant \phi_i \leqslant \psi_i \leqslant h_i \leqslant C'.$$

The sequence of points x_i will converges to a point $p \in X \times S$. One case is that p is in the interior of $X \times S$, then by using a *subsequence of subsequence* argument, there exists a bounded limiting function ψ on $\mathbb{C}^{n+1}$, such that $\psi_i \to \psi$ in $C^{1,\eta}$ norm on any ball $B_R(0)$ in $\mathbb{C}^{n+1}$. Now notice

$$|\nabla\psi(0)|_g = 1 \tag{5}$$

and $\phi(p) \leqslant \psi(x) \leqslant h(p)$, for $\forall x \in \mathbb{C}^{n+1}$. We shall show ψ is in fact a bounded plurisubharmonic function on $\mathbb{C}^{n+1}$, then it must be a constant function, which contradicts to equation(9).

Notice that near the point p, we have the inequality

$$0 < (\delta_{\alpha\bar{\beta}} + \partial_\alpha \bar{\partial}_\beta \varphi_i) < \frac{C}{\kappa_i^2}(\delta_{\alpha\bar{\beta}}),$$

this implies

$$0 < \kappa_i^2(\delta_{\alpha\beta}) + (\partial_\alpha \bar{\partial}_\beta \psi_i) < C(\delta_{\alpha\beta})$$

after rescaling. Take the limits, we have

$$0 \leqslant i\partial\bar{\partial}\psi$$

on $\mathbb{C}^{n+1}$.

The other case is that $p \in \partial(X \times S)$, then ψ is a function on the half plane of $\mathbb{C}^{n+1}$. But it must be constant since it is squeezed by h and ϕ on the boundary. Hence it contradicts to equation(9) again.

5 *Ding*-functionals along the approximation geodesics

Chen's existence theorem has the following direct application to establish the smooth approximate of weak geodesics

Theorem 16 *(ϵ- approximation geodeiscs) Given $\varphi_0, \varphi_1 \in \mathcal{H}$, we can have a sequence of approximation geodesics $\varphi_\epsilon(t,z)$ as follows: for each small $\epsilon > 0$, there exists a unique solution of the equation*

$$(\varphi_{tt} - |\partial_X \varphi'|^2_{g_\varphi}) \det(g_\varphi) = \epsilon \det h$$

such that there exists a uniform constant C with

$$|\varphi_t'| + |\varphi_t''| + |\varphi|_{\mathcal{C}^1} + \max\{|\partial_\alpha \partial_{\bar\beta} \varphi|\} < C,$$

and φ_ϵ converges to the $\mathcal{C}^{1,\bar 1}$ geodesic φ in the weak $\mathcal{C}^{1,\bar 1}$ topology.

Notice that for any plurisubharmonic metric ϕ on $-K_X$, we can write its potential as $\varphi = \phi - \phi_0$, where ϕ and ϕ_0 are corresponding metrics on the line bundle $-K_X$. Now suppose ϕ_0, ϕ_1 are two smooth Kähler Einstein metrics on X, with their Kähler forms $\omega_i = i\partial\bar\partial\phi_i, i = 0, 1$ satisfying

$$\omega_i^n = \frac{e^{-\phi_i}}{\int_X e^{-\phi_i}}.$$

define the following functionals

$$\mathcal{F}(\phi) := -\log \int_X e^{-\phi}$$

and

$$\mathcal{E}(\phi) := \frac{1}{n+1} \Sigma_{j=0}^n \int_X \varphi \omega_0^j \wedge \omega_\phi^{n-j}$$

where $\omega_\phi = i\partial\bar\partial\phi$. Then the *Ding*-functional is defined as

$$\mathcal{D} = -\mathcal{E} + \mathcal{F} = -\frac{1}{n+1} \Sigma_{j=0}^n \int_X (\phi - \phi_0) \omega_0^j \wedge \omega_\phi^{n-j} - \log \int_X e^{-\phi}.$$

Notice the along a curve of metrics ϕ_t, the derivative of *Ding*-functional is

$$\frac{\partial \mathcal{D}}{\partial t} = \int_X \phi'(-\omega_\phi^n + \frac{e^{-\phi}}{\int_X e^{-\phi}}).$$

we see the critical point of this functional is the Kähler Einstein metric, and its second derivative is

$$\frac{\partial^2\mathcal{D}}{\partial t^2}=-\int_X(\phi''-|\partial\phi'|_g^2)\omega_\phi^n+(\int_X e^{-\phi})^{-1}\{\int_X(\phi''-|\partial\phi'|_g^2)e^{-\phi}+\int_X(|\partial\phi'|_g^2-(\pi_\perp\phi')^2)e^{-\phi}\}$$

where the metric $g=i\partial\bar{\partial}\phi_t$, and if we denote the term $f=\phi''-|\partial\phi'|_g^2$, $c_t=\int_X e^{-\phi}$ and $\delta_t=|\partial\phi'|_g^2-(\pi_\perp\phi')^2$, the equation reads

$$\frac{\partial^2\mathcal{D}}{\partial t^2}=-\int_X f\omega_\phi^n+\int_X(f+\delta_t)e^{-\phi}/c_t,$$

then we are going to consider the behavior of *Ding*-functional on the approximation geodesic. First from Chen's theorem, we can find a $\mathcal{C}^{1,\bar{1}}$ geodesic ϕ_t connecting the two Kähler Einstein metrics. Moreover for any small $\epsilon>0$, there is the smooth approximation geodesic $\phi_\epsilon(t,z)$ connecting the two end points ϕ_0,ϕ_1, which converges weakly to the $\mathcal{C}^{1,\bar{1}}$ geodesic. Now if we consider the *Ding*-functional on these approximation geodesics, we have estimates

$$\frac{\partial^2\mathcal{D}}{\partial t^2}\geqslant-\epsilon\int_X\det h$$

from $f=\epsilon\det h/\det g>0$ and $\int_X\delta_t e^{-\phi}>0$. Let $\epsilon\to0$, we see that *Ding*-functional keeps to be convex on $\mathcal{C}^{1,\bar{1}}$ geodesic. Now we can integrate it back along t

$$\frac{\partial\mathcal{D}}{\partial t}(1)-\frac{\partial\mathcal{D}}{\partial t}(0)=\int_{X\times I}-f\omega_\phi^n dt+\int_{X\times I}fe^{-\phi}/c_t dt+\int_{X\times I}\delta_t e^{-\phi}/c_t dt,$$

notice that at end points ϕ_0,ϕ_1 are both Kähler Einstein, hence the first derivative of *Ding*-functionals vanish. And on the approximation geodesic, we have the equation

$$f\det g=\epsilon\det h$$

and $f\leqslant\phi''<C$ uniformly independent of ϵ. Then the equation above reads

$$\begin{aligned}A\epsilon&=\int_{X\times I}fe^{-\phi}/c_t dt+\int_{X\times I}\delta_t e^{-\phi}/c_t dt\\&\geqslant\int_{X\times I}fe^{-\phi}dt+\int_{X\times I}\delta_t e^{-\phi}dt,\end{aligned}$$

because we have uniform $\mathcal{C}^0$ estimate on ϕ_ϵ. Now since we want to discuss the eigenfunctions on each fiber, we need to a lemma to pull back the estimate to fibers.

Lemma 17 *Suppose $F_\epsilon(t)$ is a sequence of non-negative function on $[0,1]$, with integration estimate*

$$\int_0^1 F_\epsilon dt < A\epsilon,$$

then for almost everywhere $t \in [0,1]$, we can find a subsequence(depending on t) F_{ϵ_j}, such that

$$F_{\epsilon_j} < C_t \epsilon_j$$

where C_t is a constant independent of ϵ.

Proof 9 *Let $\tilde{F}_\epsilon = F_\epsilon/\epsilon$, then by Fatou's lemma*

$$\int_0^1 \liminf_\epsilon \tilde{F}_\epsilon dt \leqslant \liminf_\epsilon \int_0^1 \tilde{F}_\epsilon dt \leqslant A,$$

hence the function $\liminf_\epsilon \tilde{F}_\epsilon \in L^1$, i.e. for almost everywhere t, there is a subsequence $\tilde{F}_{\epsilon_j}$ and a constant C_t such that

$$\tilde{F}_{\epsilon_j} < C_t,$$

hence

$$F_{\epsilon_j} < C_t \epsilon_j.$$

Now put $F_\epsilon = \int_X f_\epsilon e^{-\phi_\epsilon} + \int_X \delta_\epsilon e^{-\phi_\epsilon}$ and notice the two terms on RHS are both non-negative, we have proved

Proposition 18 *Consider the approximation geodesic ϕ_ϵ connection two Kähler Einstein metrics. For almost everywhere t, there is a constant C_t, such that for each such t, there exists a subsequence ϵ_j, such that the following estimates*

$$\int_X f e^{-\phi}(\epsilon_j) < C_t \epsilon_j$$

and

$$\int_X (|\partial \phi'|_g^2 - (\pi_\perp \phi')^2) e^{-\phi}(\epsilon_j) < C_t \epsilon_j$$

hold simutaneouly.

6 Convergence in the first eigenspace

In this section, we shall focus our attention to the one fiber $X \times \{t\}$, and picked up a subsequence ϕ_{ϵ_j} from above section. Then we can consider the sequence of weighted Laplacian operator $\Box_{\phi_\epsilon}$(we shall omit the subindex j here). For each ϵ, we can arrange its eigenvalues as $0 < \lambda_1^\epsilon \leqslant \lambda_2^\epsilon \leqslant \cdots$, corresponding with one eigenfunction $e_i(\epsilon)$, i.e.

$$\Box_{\phi_\epsilon} e_i(\epsilon) = \lambda_i^\epsilon e_i(\epsilon).$$

Then let $u_\epsilon(z)$ be a sequence of smooth functions on X, such that $u_\epsilon \perp \ker \bar{\partial}$. Then it decomposes into the eigenspace of weighted Laplacian operator $\Box_{\phi_\epsilon}$, i.e.

$$u_\epsilon = \Sigma_{i=1}^{N_\epsilon} a_i(\epsilon) e_i(\epsilon)$$

where $e_i \in \Lambda_i$, and in prior, N_ϵ could equal to $+\infty$ in the above notation. Then we can consider the action by the weighted Laplacian operator on this sequence of functions, i.e. we can write $\Box_{\phi_\epsilon} u_\epsilon$ as

$$v_\epsilon = \omega_{g_\epsilon} \lrcorner \bar{\partial} u$$

and

$$\partial^{\phi_\epsilon} v_\epsilon = \Sigma_{i=1}^{N_\epsilon} \lambda_i^\epsilon a_i(\epsilon) e_i(\epsilon).$$

Under certain constraint, we claim these vector fields v_ϵ will converge to a holomorphic one with the same equation satisfied,

Proposition 19 *Let u_ϵ be a sequence of functions as above. Suppose it satisfies the following conditions:*

1) $\Sigma_{i=1}^{N_\epsilon} |a_i(\epsilon)|^2 < A$ for an uniform constant A, and the sums does not converge to zero.

2) there exists a uniform constant K, such that $\lambda_{N_\epsilon}^\epsilon < K$ for each ϵ

3) the following estimate holds

$$\int_X (|\bar{\partial} u_\epsilon|_{g_\epsilon}^2 - (\pi_\perp u_\epsilon)^2) e^{-\phi_\epsilon} < C\epsilon. \tag{6}$$

then by passing to a subsequence, we have

$$u_\epsilon \to u_\infty$$

in strong L^2 sense, where $u_\infty \in W^{1,2}$ is nontrivial. Moreover there exists a nontrivial holomorphic $(n-1,0)$ form v_∞ with value in $-K_X$, such that

$$v_\epsilon \to v_\infty$$

in strong L^2 sense, and the equation

$$\omega_g \wedge v_\infty = \bar{\partial} u_\infty$$

holds in the sense of L^2 functions, where g is the metric found on the $\mathcal{C}^{1,\bar{1}}$ geodesic.

before proving the proposition, we need a lemma

Lemma 20 *Let f_j, g_j be two sequence of L^2 functions with $||f_j g_j||_{L^p} < C$ for some $p \geqslant 1$. Suppose that $\int_X |f_j|^2 d\mu < C'$ and $g_j \to g \in L^2$ in L^2 norm, then there exists an L^2 function f such that*

$$f_j g_j \to fg \in L^p$$

in the sense of distributions.

Proof 10 *First note there exists an L^2 function f such that $f_j \to f$ in weak L^2 topology. Then we check*

$$\int_X (fg - f_j g_j) d\mu = \int_X g(f - f_j) d\mu + \int_X f_j (g - g_j) d\mu,$$

the first term on the RHS of above equation converges to zero from the weak convergence of f_j, and the second term converges to zero too, since

$$|\int_X f(g - g_j) d\mu|^2 \leqslant (\int_X |f|^2 d\mu)(\int_X |g - g_j|^2 d\mu) \to 0.$$

hence $f_j g_j$ converges to fg in the sense of distributions. Moreover, from the L^p bound of $f_j g_j$, we have an L^p function k such that $f_j g_j \to k$ in weak L^p topology. Then

$$fg = k$$

as L^p functions.

Remark 1 *Suppose the sequence $|f_j|$ is uniformly bounded in lemma 13, then the limit f is an L^∞ function, then $fg \in L^2$ automatically.*

Proof 11 *(of proposition19.) First we can write equation (2) as*

$$\Sigma_{i=1}^{N_\epsilon} (\lambda_i^\epsilon - 1) |a_i(\epsilon)|^2 < C\epsilon$$

by Futaki's formula, we know

$$\int_X |L_{g_\epsilon} u_\epsilon|^2 e^{-\phi_\epsilon} = \Sigma_{i=1}^{N_\epsilon} \lambda_i^\epsilon (\lambda_i^\epsilon - 1) |a_i(\epsilon)|^2$$

$$\leqslant KC\epsilon$$

from condition (2) and (3). But if we write $v_\epsilon = X_\epsilon \lrcorner 1$ *for some vector field* $X_\epsilon = X_\epsilon^\alpha \frac{\partial}{\partial z^\alpha}$*, then*

$$(L_g u)_{\bar{j}}^{\ i} == g^{i\bar{k}} u_{,\bar{k}\bar{j}} = \frac{\partial X^i}{\partial \bar{z}^j},$$

hence the L^2 *norm is*

$$|L_g u|^2 = g_{\alpha\bar{\beta}} g^{\mu\bar{\lambda}} \frac{\partial X^\alpha}{\partial \bar{z}^\lambda} \overline{\frac{\partial X^\beta}{\partial \bar{z}^\mu}} = |\frac{\partial X}{\partial \bar{z}}|_g^2.$$

now we choose a fixed smooth background metric h *to estimate*

$$|\frac{\partial X}{\partial \bar{z}}|_h^2 = h_{\alpha\bar{\beta}} h^{\mu\bar{\lambda}} \frac{\partial X^\alpha}{\partial \bar{z}^\lambda} \overline{\frac{\partial X^\beta}{\partial \bar{z}^\mu}}$$

$$= h_{\alpha\bar{\beta}} h^{\mu\bar{\lambda}} g^{\alpha\bar{\eta}} u_{,\bar{\eta}\bar{\lambda}} g^{\gamma\bar{\beta}} u_{,\gamma\mu} = \frac{1}{\Lambda_\alpha^2} |u_{,\bar{\alpha}\bar{\lambda}}|^2$$

$$\leqslant \Sigma(\frac{\Lambda_\lambda}{\Lambda_\alpha}) \Sigma \frac{1}{\Lambda_\alpha \Lambda_\lambda} |u_{,\bar{\alpha}\bar{\lambda}}|^2$$

$$\leqslant C(tr_g h) |\frac{\partial X}{\partial \bar{z}}|_g^2$$

where we compute in some normal coordinate. And correspondingly, the L^2 *norm of* X *can be estimated by*

$$|X|_h^2 = h_{\alpha\bar{\beta}} g^{\alpha\bar{\lambda}} u_{,\bar{\lambda}} \overline{g^{\beta\bar{\eta}} u_{,\bar{\eta}}}$$

$$= \frac{1}{\Lambda_\alpha^2} |u_{,\bar{\alpha}}|^2$$

$$\leqslant \Sigma(\frac{1}{\Lambda_\alpha}) \Sigma \frac{1}{\Lambda_\alpha} |u_{,\bar{\alpha}}|^2$$

$$\leqslant (tr_g h) |\bar{\partial} u|_g^2.$$

Recall that $f = \phi'' - |\partial \phi'|_g^2$ *is bounded from above, then we can estimate the* L^2 *norm of* $\bar{\partial} v$ *as*

$$\int_X |\frac{\partial X}{\partial \bar{z}}|_h^2 \det h \leqslant C \int_X |\frac{\partial X}{\partial \bar{z}}|_h^2 \frac{1}{f} \det h$$

$$\leqslant \frac{C}{\epsilon} \int_X |\frac{\partial X}{\partial \bar{z}}|_g^2 (tr_g h) \det g$$

$$\leqslant \frac{C'}{\epsilon}\int_X |\frac{\partial X}{\partial \bar{z}}|_g^2 e^{-\phi_g} \leqslant C''.$$

note X is a vector in $(1,0)$ direction, which means locally its coefficients are functions. Hence its full gradient is uniformly bounded in L^2 norm, i.e.

$$\int_X |\nabla X_\epsilon|_h^2 \det h < C$$

for some constant independent of ϵ. We claim it's also L^1 bounded. Recall from our choice of ϵ, we have

$$\int_X f e^{-\phi_\epsilon} < C_1\epsilon,$$

then we can estimate

$$\int_X e^{F_\epsilon} \det h = \frac{1}{\epsilon}\int_X f e^{-\phi_\epsilon} < C_1,$$

hence

$$(\int_X |X|_h \det h)^2 \leqslant C(\int_X |X|_h^2 e^{F_g} \det g)^2$$

$$\leqslant C(\int_X |X|_h^2 (\det g)^2 e^{F_g})(\int_X e^{F_g})$$

$$\leqslant C'(\int_X |\bar{\partial} u|_g^2 e^{-\phi_g}) < C''.$$

Hence it's uniformly L^1 bounded, then by Poincáre inequality, we know $||X||_{L^2} < C$ for some uniform constant. These together imply the sequence of vector fields X_ϵ are uniformly $W^{1,2}$ bounded. Now by compact imbedding theorem, there exists a vector field $X = X^\alpha \frac{\partial}{\partial z^\alpha} \in W^{1,2}$ such that $X_\epsilon \to X$ in strong L^2 norm.

Moreover, observe that

$$(\int_X |\frac{\partial X}{\partial \bar{z}}|_h e^{-\phi_g})^2 = (\int_X |\frac{\partial X}{\partial \bar{z}}|_h e^{F_g} \det g)^2$$

$$\leqslant (\int_X |\frac{\partial X}{\partial \bar{z}}|_h^2 (\det g)^2 e^{F_g})(\int_X e^{F_g})$$

$$\leqslant (C\int_X |\frac{\partial X}{\partial \bar{z}}|_g^2 e^{-\phi_g})(\int_X e^{F_g})$$

$$\leqslant C'\epsilon \int_X e^{F_g} = C'\int_X f e^{-\phi_g}$$

$$< C''\epsilon \to 0$$

from our choice of sequence ϵ. Hence $\bar\partial X \to 0$ in weak L^1 sense, but this is enough to imply $\bar\partial X = 0$ in the sense of distributions. Then X is in fact a holomorphic $(1,0)$ vector field on the manifolds, and we can define $v_\infty = X \lrcorner 1$, which is a $-K_X$ valued holomorphic $(n-1,0)$ form.

On the other hand, for the function u_ϵ itself, we have

$$\int_X |\bar\partial u_\epsilon|_h^2 \det h \leqslant C \int_X |\bar\partial u_\epsilon|_{g_\epsilon}^2 e^{-\phi_\epsilon}$$

$$= C\Sigma_{i=1}^{N_\epsilon} \lambda_i^\epsilon |a_i(\epsilon)|^2 \leqslant C',$$

hence u_ϵ has a uniform $W^{1,2}$ bound, and it converges to a function $u_\infty \in W^{1,2}$ in strong L^2 norm. Then by condition (1), the L^2 norm of u_∞ is non-trivial. Moreover, we know the equation

$$g^\epsilon_{\alpha\bar\beta} X^\alpha_\epsilon = u(\epsilon)_{,\bar\beta}$$

holds for every ϵ. Now $g^\epsilon_{\alpha\bar\beta}$ is uniformly bounded from above, hence converges to $g_{\alpha\bar\beta}$ in weak L^∞, where $g_{\alpha\bar\beta}$ is the weak $\mathcal{C}^{1,\bar 1}$ solution of the geodesic equation. And $X_\epsilon \to X$ in strong L^2, hence by the Remark after lemma 13, we see that the equation

$$g_{\alpha\bar\beta} X^\alpha = \partial_{\bar\beta} u_\infty$$

holds in the sense of L^2 functions. In particular, they are equal almost everywhere.

Finally, observe that $u_\infty \perp \ker\bar\partial$, since

$$\int_X u_\infty e^{-\phi} = \lim_{\epsilon\to 0} \int_X u_\epsilon e^{-\phi_\epsilon} = 0.$$

Hence if v_∞ is trivial, then $\bar\partial u = 0$, i.e. $u \in \ker\bar\partial$, which implies $u = 0$, a contradiction. So v_∞ is non-trivial too.

Notice that before taking the limits, the vector field v_ϵ also satisfies another equation, i.e.

$$\partial^{\phi_\epsilon} v_\epsilon = \Sigma_{i=1}^{N_\epsilon} \lambda_i^\epsilon a_i(\epsilon) e_i(\epsilon).$$

the LHS converges weakly to $\partial^\phi v_\infty$, since for any smooth testing $(n,0)$ form W,

$$\int_X v_\epsilon \wedge \overline{\bar\partial W} e^{-\phi_\epsilon} \to \int_X v_\infty \wedge \overline{\bar\partial W} e^{-\phi}$$

and the RHS converges to u_∞ since condition (3). And the RHS

$$||\Sigma_{i=1}^{N_\epsilon}\lambda_i^\epsilon a_i(\epsilon)e_i(\epsilon) - u_\epsilon||^2 \leqslant K\Sigma_{i=1}^{N_\epsilon}(\lambda_i^\epsilon - 1)|a_i(\epsilon)|^2$$

converges to zero. We have equality

$$\partial^\phi v_\infty = u_\infty$$

holds in the weak sense. But since both sides of above equation are L^2 functions, the equation actually holds as L^2 functions. This reminds us that u_∞ might be the eigenfunction of the operator $\Box_\phi$ with eigenvalue 1. In fact, we have

Corollary 21 *Let u_ϵ be a sequence of functions satisfying condition (1) - (3) in proposition 7, then there exists a function $u_\infty \in W^{1,2}$ such that*

$$u_\epsilon \to u_\infty$$

in strong L^2 sense, and u_∞ is a nontrivial eigenfunction of the operator $\Box_{\phi_g}$ with eigenvalue 1.

Proof 12 *First notice $u_\infty \in dom(\Box_{\phi_g})$. This is because $\bar\partial u = \omega_g \wedge v_\infty$, hence $u \in \mathcal{W} \subset dom(\bar\partial)$, and $\bar\partial u \in dom(\bar\partial^*_{\phi_g})$ since v_∞ is holomorphic. Now for any smooth testing $(n,0)$ form W with value in $-K_X$, we compute*

$$\int_X \bar\partial^*_{\phi_g}\bar\partial u_\infty \wedge \overline{W}e^{-\phi_g} = (\bar\partial^*_{\phi_g}\bar\partial u_\infty, W)_g$$

$$= \langle \bar\partial u_\infty, \bar\partial W\rangle_g$$

$$= \langle \omega_g \wedge v_\infty, \bar\partial W\rangle_g$$

$$= \int_X v_\infty^\alpha \overline{\partial_{\bar\alpha} W} e^{-\phi_g}$$

$$= (\partial^{\phi_g} v_\infty, W)_g$$

$$= \int_X u_\infty \wedge \overline{W}e^{-\phi_g}.$$

hence $\Box_{\phi_g} u_\infty = u_\infty$ as L^2 functions.

7 the eigenspace decomposition of ϕ' (the easy case)

In this section, we shall construct a sequence of functions u_ϵ, which could satisfy the condition $(1)-(3)$ in proposition(19) from ϕ'_ϵ, then construct a holomorphic vector field from there. However, we need to discuss case by case this time, i.e. let

$$\pi_\perp \phi'_\epsilon = \Sigma_{i=1}^{+\infty} a_i(\epsilon) e_i(\epsilon),$$

then

$$\Box_{\phi_\epsilon}(\pi_\perp \phi'_\epsilon) = \Sigma_{i=1}^{+\infty} \lambda_i^\epsilon a_i(\epsilon) e_i(\epsilon).$$

Note the restriction from the vanishing of $Ding$-functional gives

$$\Sigma_{i=1}^{+\infty}(\lambda_i^\epsilon - 1)|a_i(\epsilon)|^2 < C\epsilon \tag{7}$$

by passing to the chosen subsequence ϵ_j. And notice that

$$\int_X |\bar\partial \phi'_\epsilon|_h^2 \leqslant C \int_X |\bar\partial \phi'_\epsilon|_{g_\epsilon}^2 e^{-\phi_\epsilon}$$
$$\leqslant C \int_X \phi''_\epsilon e^{-\phi_\epsilon} < C',$$

then there exists a function $\psi \in W^{1,2}$ such that $\phi'_\epsilon \to \psi$ in strong L^2 norm. Hence we can assume

$$\frac{1}{2} < \Sigma_{i=1}^{+\infty}|a_i(\epsilon)|^2 < 2 \tag{8}$$

for ϵ small enough.

Remark 2 *In fact ,we have $|\phi_\epsilon|_{\mathcal{C}^1} < C$, hence $||\phi_\epsilon||_{W^{1,p}} < C$ for any p large. Then by compact imbedding theorem, we can assume*

$$\phi_\epsilon \to \phi$$

in $\mathcal{C}^{0,\alpha}$ norm.

In fact, we are going to prove

Theorem 22 *There is a holomorphic vector field v on the manifolds, such that*

$$\omega_g \wedge v = \bar\partial \psi$$

where ψ is the L^2 limit of ϕ'_ϵ and g is the $\mathcal{C}^{1,\bar 1}$ solution of geodesic equation. Moreover, ψ is a eigenfunction of the operator $\Box_{\phi_g}$ with eigenvalue 1, i.e.

$$\Box_{\phi_g}\psi = \psi.$$

In order to prove this theorem, we shall discuss case by case. First there are two possibilities for the convergence of eigenvalue λ_i^ϵ:

Case 1, there exist a finite integer k such that the following two things hold

i) for each $1 \leqslant i \leqslant k$, $\lambda_i^\epsilon \to 1$ as $\epsilon \to 0$;

ii) λ_{k+1}^ϵ does not converges to 1.

Case 2, for each $1 \leqslant i < +\infty$, $\lambda_i^\epsilon \to 1$ as $\epsilon \to 0$.

Let's discuss *Case* 1 first in this section. In this case, we shall define

$$u_\epsilon := \Sigma_{i=1}^k a_i(\epsilon) e_i(\epsilon).$$

Notice that the divergence of λ_i^ϵ implies $\lambda_i^\epsilon > 1 + \delta$ for some small $\delta > 0$, by passing to a subsequence. Then since λ_i^ϵ is a non-decreasing sequence in i, we have for all $i > k$

$$\lambda_i^\epsilon > 1 + \delta$$

for the same subsequence. Now by equation (3), we see

$$C\epsilon > \Sigma_{i=k+1}^{+\infty} (\lambda_i^\epsilon - 1)|a_i(\epsilon)|^2$$

$$\geqslant \Sigma_{i=k+1}^{+\infty} \delta |a_i(\epsilon)|^2,$$

hence $\Sigma_{i=k+1}^{+\infty} |a_i(\epsilon)|^2 \to 0$ when $\epsilon \to 0$. This gives condition (1), i.e.

$$\Sigma_{i=1}^k |a_i(\epsilon)|^2 > 1/4.$$

condition (2) is satisfied because $\lambda_k^\epsilon \to 1$ by the assumption, and condition (3) is automatically satisfied by equation (3). Hence we can generate a holomorphic vector field v_∞ from proposition(19).

Moreover, we could see $||\pi_\perp \phi'_\epsilon - u_\epsilon||_{L^2}$ converges to zero in above argument, hence we actually have

$$\psi = u_\infty$$

after taking the limit. And hence it's the eigenfunction of $\Box_{\phi_g}$ with eigenvalue 1, by corollary (14). Hence we proved theorem(22) in this case.

8 the hard case

Now we are going to deal with *Case* 2, i.e. we assume

$$\lambda_i^\epsilon \to 1$$

for each $1 \leqslant i < +\infty$. Here we still subdivide it into two subcases as follows:

subCase 1, for any $1 < k < \infty$, the partial sum $\Sigma_{i=1}^{k-1}|a_i(\epsilon)|^2 \to 0$, when $\epsilon \to 0$.

subCase 2, there exists a finite number K, such that $\Sigma_{i=1}^{K-1}|a_i(\epsilon)|^2$ does not converge to zero.

Before going to the subcases, we need a lemma first

Lemma 23 *Let $e_i(\epsilon)$ be the eigenfunction of the weighted Laplacian $\Box_{\phi_\epsilon}$ with eigenvalue λ_i^ϵ, i.e.*

$$\Box_{\phi_\epsilon} e_i(\epsilon) = \lambda_i^\epsilon e_i(\epsilon).$$

Suppose there exists an uniform constant C, such that $\lambda_i^\epsilon < 1 + C\epsilon$, then $e_i(\epsilon)$ converges to a non-trivial eigenfunction e_i of the operator $\Box_{\phi_g}$ with eigenvalue 1. *Moreover, suppose there is another $j \neq i$, such that λ_j satisfies the same condition, then e_i, e_j are mutually orthogonal to each other.*

Proof 13 *we define $u_\epsilon = e_i(\epsilon)$, then condition* (1) *and* (2) *hold automatically. And condition (3) is also satisfied because*

$$\int_X (|\bar\partial u_\epsilon|^2_{g_\epsilon} - (\pi_\perp u_\epsilon)^2) e^{-\phi_\epsilon} = (\lambda_i^\epsilon - 1) < C\epsilon,$$

hence by proposition(22), we get

$$e_i(\epsilon) \to e_i$$

in strong L^2 sense, where $e_i \in W^{1,2}$ is a eigenfunction of $\Box_{\phi_g}$ with eigenvalue 1. *Now for $j \neq i$, we have similar convergence and eigenfunction e_j, but*

$$\int_X e_i \bar{e}_j e^{-\phi_g} = \lim_{\epsilon \to 0} \int_X e_i(\epsilon) \overline{e_j(\epsilon)} e^{-\phi_\epsilon} = 0$$

by the strong L^2 convergence of $e_i(\epsilon)$, and L^∞ convergence of ϕ_ϵ.

Now let's begin to discuss the *subCase* 1. For any fixed k, by equation (8), we can find a large integer $N_{\epsilon,k}$ such that

$$\Sigma_{i=1}^{N_{\epsilon,k}}|a_i(\epsilon)|^2 \geqslant 1/4$$

by the assumption in this subcase, for ϵ small

$$\Sigma_{i=k}^{N_{\epsilon,k}}|a_i(\epsilon)|^2 \geqslant 1/8.$$

but then by equation (7),

$$\frac{1}{8}(\lambda_k^\epsilon - 1) \leqslant \Sigma_{i=k}^{N_{\epsilon,k}}(\lambda_i^\epsilon - 1)|a_i(\epsilon)|^2 < C\epsilon,$$

because the sequence λ_i^ϵ is non-decreasing. Hence we proved for each k,

$$\lambda_k^\epsilon < 1 + 8C\epsilon$$

for ϵ small enough. Now by lemma 16, we get an eigenfunction e_k for each $1 \leqslant i < \infty$, and they are orthogonal to each other. However, this is impossible since the eigenspace with eigenvalue 1 of an elliptic operator $\Box_{\phi_g}$ has only finite rank. Hence the *subCase* 1 actually never happens.

9 the final case

Let's discuss *subCase* 2. Under the assumption in this case, we can find K_1, a finite integer, to be the first number such that $\Sigma_{i=1}^{K_1-1}|a_i(\epsilon)|^2$ does not converge to zero. Then by passing to a subsequence, we can assume $\Sigma_{i=1}^{K_1-1}|a_i(\epsilon)|^2 > \delta_1$ for some fixed positive number δ_1. Now consider the truncated sequence

$$\Lambda_1(\phi') = \Sigma_{i=K_1}^{+\infty} a_i(\epsilon)e_i(\epsilon).$$

suppose there exists another integer $K_2 > K_1$, such that $\Sigma_{i=K_1}^{K_2-1}|a_i(\epsilon)|^2$ does not converge to zero, and then we can assume $\Sigma_{i=K_1}^{K_2-1}|a_i(\epsilon)|^2 > \delta_2$. We can repeat this argument, to find $0 < K_1 < K_2 < K_3 < \cdots$, but we claim this process will terminate in finite steps.

Lemma 24 *There exists an finite integer n, such that*

$$\Sigma_{i=K_n}^{+\infty}|a_i(\epsilon)|^2 \to 0.$$

Proof 14 *Let's define a sequence of sequence of functions $u_\epsilon^{(j)}$ as*

$$u_\epsilon^{(0)} := \Sigma_{i=1}^{K_1-1} a_i(\epsilon)e_i(\epsilon)$$
$$u_\epsilon^{(1)} := \Sigma_{i=K_1}^{K_2-1} a_i(\epsilon)e_i(\epsilon)$$
$$\cdots$$
$$u_\epsilon^{(j)} :== \Sigma_{i=K_j}^{K_{j+1}-1} a_i(\epsilon)e_i(\epsilon)$$

and so on. We now claim $u_\epsilon^{(j)}$ satisfying all the conditions (1) - (3) in proposition(19). Condition (1) is satisfied automatically by assumption, and condition (2) is satisfied since $\lambda_k^\epsilon \to 1$ for any fixed k. Condition (3) is satisfied too because of equation (3), i.e.

$$\Sigma_{i=K_j}^{K_{j+1}-1}(\lambda_i^\epsilon - 1)|a_i(\epsilon)|^2 < C\epsilon,$$

then by proposition(19) and corollary (14), we see there exists an non-trivial $W^{1,2}$ function $u^{(j)}$ such that

$$u_\epsilon^{(j)} \to u^{(j)}$$

in strong L^2 norm. And $u^{(j)}$ is a eigenfunction of operator $\Box_{\phi_g}$ with eigenvalue 1. However, notice that u_ϵ^j and $u_\epsilon^{(k)}$ are mutually orthogonal, and by the same argument used in lemma 16, this implies

$$u^{(j)} \perp u^{(k)}$$

for all different j and k. Now we can find finite many such $u^{(j)}$ since they are all in the eigenspace with eigenvalue 1 of the weighted Laplacian operator $\Box_{\phi_g}$, hence we proved the lemma.

Next we are going to complete the proof of theorem(22). Now let's define

$$u_\epsilon := \Sigma_{i=1}^{K_n-1} a_i(\epsilon) e_i(\epsilon)$$

where K_n is the number appearing in lemma(24). Now people can check the three conditions in proposition(19) are satisfied, and hence there exists a $W^{1,2}$ function u such that

$$u_\epsilon \to u$$

in L^2 sense, and u is a eigenfunction with eigenvalue 1 of operator $\Box_{\phi_g}$, and there is a holomorphic vector field v such that

$$\omega_g \wedge v = \bar{\partial} u.$$

Moreover, the difference of the L^2 norm is

$$||\pi_\perp \phi'_\epsilon - u_\epsilon||_{L^2} = \Sigma_{i=K_n}^{+\infty} |a_i(\epsilon)|^2 \to 0$$

by our choice of K_n, hence we have

$$\psi = u.$$

And we complete the proof.

Remark 3 *If there is no any non-trivial holomorphic vector field on X, then* proposition 12 *directly implies $\phi' = 0$ almost everywhere on $X \times I$ from above case by case discussion. Without using* corollary 14*, we don not need to invoke any eigenfunction of the first eigenspace of the weighted Laplacian operator in the limit. Hence we proved uniqueness in this case.*

10 Time direction

Up to now, we construct a holomorphic vector field v_t on a fiber $X \times t$ for almost everywhere $t \in [0,1]$. And this vector field can be computed as

$$v_t = \omega_g \lrcorner \bar{\partial}\psi$$

where $\phi'_\epsilon \to \psi$ in strong L^2 norm at time t. Notice that there are more information to use for the convergence of ϕ'_ϵ. In fact, we know $|\phi'|, |\phi_{t\bar{z}}|$ and $|\phi_{z\bar{t}}|$ are all uniformly bounded on $X \times I$, i.e.

$$|\phi'|_{\mathcal{C}^1} < C,$$

then we can assume $\phi'_\epsilon \to \phi' \in \mathcal{C}^1(X \times I)$, in $\mathcal{C}^{0,\alpha}$ norm. Hence the two limits actually agree with each other, i.e.

$$\psi = \phi'$$

as L^2 functions on X. Now the holomorphic vector field can be written as

$$v_t = \omega_g \lrcorner \bar{\partial}\phi'.$$

Then we can define the following subset of the unit interval

$$S := \{t \in I;\ there\ is\ a\ holomorphic\ vector\ field\ v_t\ on\ X \times \{t\}\ satisfying\ \omega_g \wedge v_t = \bar{\partial}\phi'\}$$

we know the set $I - S$ has measure zero. Next we are going to prove a stronger result

Proposition 25 *The subset S coincides with the whole unit interval, i.e.*

$$S = I.$$

Proof 15 *First recall that $\phi_\epsilon \to \phi$ in $\mathcal{C}^{0,\alpha}(X \times I)$ norm, by the uniform bound on $\mathcal{C}^1$ norm of ϕ. Then on each fiber $X \times \{t\}$, the convergence still holds, i.e.*

$$\phi_\epsilon \to \phi$$

in $\mathcal{C}^{0,\alpha}(X)$, and this implies

$$g_{\epsilon,\alpha\bar{\beta}} \to g_{\alpha\bar{\beta}}$$

in the sense of distribution on the fiber $X \times \{t\}$. Pick up a point $\underline{t} \in I - S$, and a sequence $t_i \in S$ such that $t_i \to \underline{t}$. Observe that the space of all holomorphic vector fields is finite dimensional, i.e. let

$$\Gamma(X) := H^0(TX),$$

then Γ is a finite dimensional vector space. Write $v_{t_i} = X_i \lrcorner 1$, where $v_{t_i} \in \Gamma$ is the vector field satisfying the equation in the definition of S. Observe that v_t is the unique solution to the following equation

$$\partial^{\phi_t} v_t = \square_{\phi_t} \phi' = \pi_{\perp} \phi'$$

under the condition $H^{0,1}(X) = 0$, then the standard L^2 estimate(Berndtsson[7]] gives us

$$||v_t||_h \leqslant C||\pi_{\perp}\phi'||_h$$

for some fixed metric h and uniform constant C independent of time t. Consider the sequence $\{X_i\} \in H^0(TX)$, the uniform bounds on the L^2 norm of X_i shows it must converges under the fixed metric h, i.e. there exists a vector field $X \in \Gamma$ such that

$$||X - X_i||_h^2 \to 0.$$

Let's write $g_{\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}(\underline{t})$ and $g_{i,\alpha\bar{\beta}} = g_{\alpha\bar{\beta}}(t_i)$, then

$$||X - X_i||_g^2 \leqslant C||X - X_i||_h^2,$$

hence converges to zero too. Now we claim the equation

$$\omega_g \wedge X = \bar{\partial}\phi'$$

holds in the sense of distribution. Put $\chi(z)$ be any smooth compact supported testing function on X(we can further assume χ is supported in some coordinate chart), we fix a pair of index α, β, and compute

$$\int_X (g_{\alpha\bar{\beta}} X^\alpha - g_{i,\alpha\bar{\beta}} X_i^\alpha)\chi(z) \det h$$

$$= \int_X \chi(g_{\alpha\bar{\beta}} - g_{i,\alpha\bar{\beta}}) X^\alpha \det h + \int_X \chi(X^\alpha - X_i^\alpha) g_{i,\alpha\bar{\beta}} \det h,$$

since $g_{i,\alpha\bar{\beta}}$ is uniformly bounded, the second term in above equation converges to zero in strong L^2 sense. And the first term, we can decompose it into

$$\int_X \chi(g_{\alpha\bar{\beta}} - g_{i,\alpha\bar{\beta}}) X^\alpha \det h$$

$$= \int_X \chi(g_{\alpha\bar{\beta}} - g^{\epsilon}_{\alpha\bar{\beta}}) X^\alpha \det h - \int_X \chi(g_{i,\alpha\bar{\beta}} - g^{\epsilon}_{i,\alpha\bar{\beta}}) X^\alpha \det h + \int_X \chi(g^{\epsilon}_{i,\alpha\bar{\beta}} - g^{\epsilon}_{\alpha\bar{\beta}}) X^\alpha \det h,$$

the first and second terms converge to zero as $\epsilon \to 0$, and for the third term, we integration by parts

$$\int_X \chi(g^{\epsilon}_{i,\alpha\bar{\beta}} - g^{\epsilon}_{\alpha\bar{\beta}})X^{\alpha} \det h = \int_X \chi_{,\bar{\beta}}(\phi^{\epsilon}_{i,\alpha} - \phi^{\epsilon}_{\alpha})X^{\alpha} \det h$$

$$= \int_X \chi_{,\bar{\beta}}(t_i - \underline{t})\phi'_{,\alpha}(t)X^{\alpha} \det h$$

$$\leqslant A|t_i - \underline{t}|$$

where A is a constant independent of ϵ. Hence

$$\int_X \chi(g_{\alpha\bar{\beta}} - g_{i,\alpha\bar{\beta}})X^{\alpha} \det h \to 0$$

as $t_i \to \underline{t}$, and we proved

$$g_{i,\alpha\bar{\beta}}X^{\alpha}_i \to g_{i,\alpha\bar{\beta}}X^{\alpha}_i$$

in the sense of distributions. But we know $\phi'_i \to \phi'$ in $\mathcal{C}^{0,\alpha}$ norm, hence $\bar{\partial}\phi'_i \to \bar{\partial}\phi'$ in the sense of distribution too. Finally, the limit equation

$$g_{\alpha\bar{\beta}}X^{\alpha} = \phi'_{,\bar{\beta}}$$

holds in distribution sense on $X \times \{\underline{t}\}$. Now since both sides in above equation are L^{∞} functions, we see the equation actually holds in the sense of L^2 functions by the same argument in Remark 1*.*

Now it makes sense to talk about the time derivative of vector fields v_t in distribution sense, i.e. on the $\mathcal{C}^{1,\bar{1}}$ geodesic, we compute in the sense of distributions

$$\phi''_{,\bar{\beta}} = (g_{\alpha\bar{\beta}}X^{\alpha})',$$

and computation implies

$$(g^{\alpha\bar{\lambda}}\phi'_{,\alpha}\phi'_{,\bar{\lambda}})_{,\bar{\beta}} = \phi'_{\alpha\bar{\beta}}X^{\alpha} + g_{\alpha\bar{\beta}}(X^{\alpha})'.$$

note the RHS is in fact equal to

$$\nabla_{\bar{\beta}}(\phi'_{,\alpha}X^{\alpha}) = \phi'_{,\alpha\bar{\beta}}X^{\alpha} + \phi'_{,\alpha}X^{\alpha}_{,\bar{\beta}} = \phi'_{,\alpha\bar{\beta}}X^{\alpha},$$

here Leibniz rule makes sense since X is holomorphic. Hence we get

$$g_{\alpha\bar{\beta}}(X^{\alpha})' = 0$$

which is equivalent to the vanishing of $\frac{\partial}{\partial t}v_t = 0$, i.e. we have an unchanged holomorphic vector field v on the geodesic.

We finished the proof of uniqueness theorem by taking the holomorphic vector field

$$\mathcal{V} := \frac{\partial}{\partial t} - V,$$

then it's easy to check $\mathcal{L}_{\mathcal{V}}(i\partial\bar{\partial}\phi_t) = 0$ during the flow, hence the induced the automorphism F preserves the metric along the geodesic.

11 Twisted Kähler-Einstein metrics

Let X be a compact complex Kähler manifold with $-K_X > 0$, and S be a semi-positive $\mathbb{R}$-line bundle on X, i.e. there exists a smooth Hermitian metric ψ on the line bundle S such that

$$\theta = i\partial\bar{\partial}\psi \geqslant 0.$$

Notice here θ is a globally defined closed $(1,1)$ form, and then we are going to consider the following twisted Kähler-Einstein equation

$$Ric(\omega) = \omega + \theta \tag{9}$$

on X. Here we assume $-(K_X + S) > 0$, and a Kähler form ω can always be written as

$$\omega = i\partial\bar{\partial}\phi$$

where ϕ is a positively curved smooth metric on the $\mathbb{R}$-line bundle $-(K_X + S)$. And we will also use another notation when there is no confusion, i.e.

$$\omega_\varphi = \omega_0 + dd^c\varphi$$

where ω_0 is a fixed background Kähler metric in the same cohomology class, and φ is the Kähler potential. We shall consider all such metric with L^∞ potentials, i.e.

$$|\phi' - \phi| < +\infty$$

Let's denote these metrics as $PSH_\infty(-K_X - S)$, and we always assume $\int_X \omega^n = 1$ in the following. Now suppose there exists two Kähler metrics ϕ_0 and ϕ_1 satisfying equation (9), i.e.

$$\omega^n_{\phi_i} = \frac{e^{-\phi_i - \psi}}{\int_X e^{-\phi_i - \psi}} \tag{10}$$

for $i = 1, 2$. We then claim they are the same up to a holomorphic automorphism, i.e.

Theorem 26 *Suppose $\omega_1 = i\partial\bar{\partial}\phi_1$ and $\omega_2 = i\partial\bar{\partial}\phi_2$ are two solutions of twisted Kähler-Einstein equation with the same weight θ, then there exists a holomorphic automorphism F on X, such that $F^*(\omega_2) = \omega_1$ and $F^*(\theta) = \theta$. Moreover, this automorphism is induced from a holomorphic vector field $\mathcal{V}$.*

This is the twisted version of uniqueness theorem on Kähler-Einstein metrics on Fano manifolds. In order to investigate this equation from variational methods, we shaw introduce the twisted *Ding*-functional, whose critical point corresponds to the twisted Kähler-Einstein metric.

Definition 1 *The twisted Ding-functional $\mathcal{D}$ is a functional defined on the space of all plurisubharmonic metrics $\phi \in PSH_\infty(-K_X - S)$, such that*

$$\mathcal{D} := -\mathcal{E} + \mathcal{F}_\psi$$

where

$$\mathcal{E}(\phi) := \frac{1}{n+1}\int_X \Sigma_{j=0}^n (\phi - \phi_0)\omega_\phi^j \wedge \omega_{\phi_0}^{n-j}$$

and

$$\mathcal{F}_\psi(\phi) := -\log \int_X e^{-\phi-\psi}.$$

Remark 4 *Since ϕ is a metric on the line bundle $-(K_X + S)$ and ψ is a metric on S, we see $\tau = \phi + \psi$ is a metric on $-K_X$, and $e^{-\tau}$ is a volume form on X.*

Notice that the two functionals $\mathcal{E}$ and $\mathcal{F}$ will be changed in a new normalization, but *Ding*-functional is a normalization invariant. Now suppose there is a smooth curve ϕ_t in the space of metrics, and we can compute derivatives of twisted *Ding*-functional on it, i.e.

$$\frac{\partial \mathcal{D}}{\partial t} = \int_X \phi'(\omega_\phi^n - \frac{e^{-\phi-\psi}}{\int_X e^{-\phi-\psi}}).$$

Hence twisted Kähler-Einstein metric is its critical point, and

$$\frac{\partial^2 \mathcal{D}}{\partial t^2} = \int_X (\phi'' - |\partial\phi'|^2_{g_\phi})\omega_\phi^n + (\int_X e^{-\tau})^{-1}\{\int_X (\phi'' - (\pi_\perp^\tau \phi')^2)e^{-\tau}\}$$

where $\tau = \phi + \psi$, and the orthogonal projection is $\pi_\perp^\tau u = u - \int u e^{-\tau}/\int_X e^{-\tau}$. Now suppose $\phi_t, 0 \leqslant t \leqslant 1$ is a smooth geodesic connecting ϕ_1 and ϕ_2, we see

$$(\int_X e^{-\tau})\frac{\partial^2 \mathcal{D}}{\partial t^2} = \int_X (|\bar\partial\phi'|^2_{g_\tau} - (\pi_\perp^\tau\phi')^2)e^{-\tau} + \int_X (|\bar\partial\phi'|^2_{g_\phi} - |\bar\partial\phi'|^2_{g_\tau})e^{-\tau} \tag{11}$$

where g_τ is the metric corresponding to the Kähler form $\omega_\tau = i\partial\bar\partial\tau$, and hence

$$\omega_\tau \geqslant \omega_\phi.$$

This implies the second term on the RHS of equation(11) is non-negative, i.e. $g_\phi^{\alpha\bar\beta}\phi'_{,\bar\beta}\phi'_{,\alpha} \geqslant g_\tau^{\alpha\bar\beta}\phi'_{,\bar\beta}\phi'_{,\alpha}$, and the first term is non-negative from the *Futaki's formula* with respect to the weighted Laplacian operator [16]

$$\Box_\tau = \bar\partial_\tau^*\bar\partial.$$

Hence we proved the following

Proposition 27 *The twisted Ding-functional is convex along smooth geodesics.*

We can further observe that on the smooth geodesic ϕ_t, the twisted *Ding-functional* must keep to be a constant, i.e. $\partial^2\mathcal{D}/\partial t^2 \equiv 0$ on the geodesic, and then the following two equations

$$\delta_\tau(t) := \int_X (|\bar{\partial}\phi'|^2_{g_\tau} - (\pi^\tau_\perp \phi')^2)e^{-\tau} = 0 \tag{12}$$

$$k(t) := \int_X (|\bar{\partial}\phi'|^2_{g_\phi} - |\bar{\partial}\phi'|^2_{g_\tau})e^{-\tau} = 0 \tag{13}$$

hold simultaneously for each $0 \leqslant t \leqslant 1$. The equation (12) implies there exists a time independent holomorphic vector field $V = X^\alpha \frac{\partial}{\partial z^\alpha}$, where

$$X^\alpha = g_\tau^{\alpha\bar{\beta}}\phi'_{,\bar{\beta}}$$

such that the Lie derivative

$$\mathcal{L}_{\mathcal{V}}(\tau_t) = 0$$

where $\mathcal{V}_t = V - \frac{\partial}{\partial t}$, i.e. the induced automorphism F preserves the metric as

$$F^*(\omega_\tau(t)) = \omega_\tau(0)$$

for any $0 \leqslant t \leqslant 1$. Then from the twisted Kähler-Einstein equation we see

$$F^*(Ric(\omega_{\phi_1})) = F^*(\omega_{\phi_1} + \theta) = \omega_{\phi_0} + \theta = Ric(\omega_{\phi_0}),$$

hence $(F^*\omega_{\phi_1})^n = (\omega_{\phi_0})^n$ because they are in the same cohomology. By the uniqueness of Monge-Ampère equation, we get

$$F^*(\omega_{\phi_1}) = \omega_{\phi_0},$$

the equation

$$F^*(\theta) = \theta$$

follows directly. Up to here, we proved theorem 26 under the assumption of smooth geodesics. Moreover, the equation (13) implies the twisted Kähler-Einstein metric is really unique if θ is strictly positive. Next, we shall prove the theorem in the case when there is only $\mathcal{C}^{1,\bar{1}}$ geodesic connecting two twisted Kähler-Einstein metrics.

Proof 16 (of theorem 26) *Let's consider the ϵ-approximation geodesics connecting ϕ_1 and ϕ_2, i.e. the solution of the following equation*

$$(\phi'' - |\partial\phi'|^2_{g_\phi}) \det g_\phi = \epsilon \det h$$

with boundary values

$$\phi(0,z) = \phi_1(z); \quad \phi(1,z) = \phi_2(z).$$

Now if we define $f = \phi'' - |\partial\phi'|^2_{g_\phi} > 0$, then we can see

$$f \det g_\phi = \epsilon \det h$$

and the second time derivative of twisted Ding-functional is

$$\frac{\partial^2 \mathcal{D}}{\partial t^2} = -\int_X f \det g_\phi + \left(\int_X e^{-\tau}\right)^{-1}\left\{\int_X (f + \delta_\tau + k)e^{-\tau}\right\}.$$

Notice that $\int_X f_\tau e^{-\tau}$, $\int_X \delta_\tau e^{-\tau}$ and $\int_X ke^{-\tau}$ are all non-negative, hence

$$\int_{X\times I} (f + \delta_\tau + k)e^{-\tau} dt \leqslant \epsilon C$$

for some uniform constant C independent of ϵ. If put $f_\tau = \phi'' - |\partial\phi'|^2_{g_\tau} = f + k > 0$, we see that for almost everywhere $t \in [0,1]$, there exists a constant $C(t)$ and a subsequence $\epsilon_j(t)$ such that

$$\int_X f_\tau e^{-\tau}(\epsilon_j) < C(t)\epsilon_j; \qquad \int_X \delta_\tau e^{-\tau}(\epsilon_j) < C(t)\epsilon_j$$

from Fatou's lemma. Furthermore, all the uniform $\mathcal{C}^{1,\bar{1}}$ estimates hold for metrics $\tau(\epsilon)$ on the approximation geodesics, because

$$\tau_\epsilon = \phi_\epsilon + \psi$$

and ψ is a fixed smooth twister here. Hence by considering the weighted Laplacian operator $\Box_\tau$, the same argument in [16] implies there exists a time independent holomorphic vector field $V = X^\alpha \frac{\partial}{\partial z^\alpha}$, such that the induced automorphism F of X will preserve the metric, i.e. $F_t^*(\omega_\tau(t)) = \omega_\tau(0)$, then as we argued before in smooth case,

$$F^*(\omega_{\phi_1}) = \omega_{\phi_0}$$

and

$$F^*(\theta) = \theta$$

for all $t \in [0,1]$.

In prior, the holomorphic vector field $V = X^\alpha \frac{\partial}{\partial z^\alpha}$ obtained from above proof is computed with respect to the metric τ, hence

$$X^\alpha = g_\tau^{\alpha\bar{\beta}} \phi'_{,\bar{\beta}}.$$

However, we know the condition

$$F_t^*(\omega_{\phi_t}) = \omega_{\phi_0} + \theta - F_t^*\theta$$

for each $t \in [0,1]$ from the proof. Hence the geodesic g_{ϕ_t} is smooth both in space and time directions, and the twisted *Ding*-functional $\mathcal{D}$ keeps being a constant along the geodesic ϕ_t, i.e.

$$\frac{\partial \mathcal{D}}{\partial t} \equiv 0$$

for all $t \in [0,1]$. Hence ϕ_t keeps to be the local minimizer of the twisted $\mathcal{D}$-functional along the whole curve, i.e. it satisfies

$$Ric(\omega_{\phi_t}) = \omega_{\phi_t} + \theta.$$

From the same argument as above we have

$$F_t^*(\omega_{\phi_t}) = \omega_{\phi(0)}; \quad F_t^*(\theta) = \theta$$

for each $0 \leqslant t \leqslant 1$. Then the curve generated by the one parameter group of automorphisms F_t coincides with the geodesic ω_{ϕ_t}, and this is to say the $\mathcal{C}^{1,\bar{1}}$ geodesic is in fact smooth. Moreover, we can prove

Corollary 28 *Suppose there is one point $p \in X$, such that the closed (1,1) form θ is strictly positive, i.e. $\theta(p) > 0$, then the twisted Kähler-Einstein metric ω_1 is actually unique.*

Proof 17 *Suppose we have two different twisted Kähler-Einstein metrics ω_1 and ω_2, then we can assume there is a smooth geodesic connecting them by the argument before the corollary. And this curve is the one parameter group of Automorphism F_t generated by a nontrivial holomorphic vector field V. Now take the time derivative in the integral equation, we see*

$$F_t^*(\mathcal{L}_{\mathcal{V}}(\omega_{\phi_t})) = 0,$$

and hence

$$\partial(V \lrcorner\, \omega_{\phi_t}) = \omega'_{\phi_t} = \omega'_\tau$$

which implies

$$\partial(V \lrcorner\, \theta) = 0$$

Now notice the existence of twisted Kähler-Einstein metrics on the manifold X implies the first betti number $b_1 = 0$, i.e. there is no nontrivial harmonic (0,1) form, hence $V \lrcorner\, \theta = 0$, i.e.

$$X^\alpha \theta_{\alpha\bar{\beta}} = 0$$

and then we can write

$$X^\alpha = g_\phi^{\alpha\bar{\beta}} \phi'_{\bar{\beta}};$$

for any time $0 \leqslant t \leqslant 1$. Now in a neighborhood of the point $p \in U$, the (1,1) form θ keeps to be strictly positive, i.e.

$$\theta > \epsilon\omega$$

in U. But the equation $X^\alpha \theta_{\alpha\bar{\beta}} = 0$ implies the holomorphic vector field V is identically zero in an open set U, hence it must be identically zero on X, which is a contradiction.

Finally by combining the results from theorem 26 and corollary 28, the proof of theorem2 is finished.

12 Smooth perturbation of conical Kähler-Einstein metrics

We shall introduce conical Kähler-Einstein metrics first in this section, and consider the perturbation of these metrics. Let D be a smooth divisor on the manifold X, such that the associated line bundle S_D of this divisor is semi-positive. Suppose the $\mathbb{R}$-line bundle $-(K_X + (1-\beta)S_D)$ is strictly positive, where $0 < \beta < 1$ is any real number. Then each L^∞ strictly plurisubharmonic metric ϕ on this $\mathbb{R}$-line bundle associates with a Kähler form

$$\omega_\phi = i\partial\bar{\partial}\phi > 0$$

as before.

Definition 2 *A singular Kähler metric ω is a conical Kähler metric with angle β along D if the following conditions are satisfied:*

(i) ω is a closed positive (1,1) current on X, and is smooth on $X \backslash D$;

(ii) for every point $p \in D$, there exists a constant C, such that in $D \cap U = \{z_1 = 0\}$, where U is a coordinate neighborhood of p, we have

$$C^{-1}\omega_\beta \leqslant \omega \leqslant C\omega_\beta$$

where ω_β is a local model conical metric on $D \cap U$, i.e.

$$\omega_\beta := \sqrt{-1}\left(\frac{dz^1 \wedge d\bar{z}^1}{|z^1|^{2-2\beta}} + \Sigma_{i=2}^n dz^i \wedge d\bar{z}^i\right)$$

From now on, we shall suppose ω_ϕ is always a conical Kähler metric, and then we can talk about the Kähler-Einstein equation in this setting: we call ω_ϕ is a *conical Kähler-Einstein metric* if the following equation is satisfied in current sense

$$Ric(\omega_\phi) = \omega_\phi + (1-\beta)\delta_D \tag{14}$$

where δ_D is the integration current of the divisor D, and it's equivalent to the following Monge-Ampère equation

$$\omega_\phi^n = \frac{e^{-\phi}/|s|^{2-2\beta}}{\int_X e^{-\phi}/|s|^{2-2\beta}}, \tag{15}$$

where $D = \{s = 0\}$. Notice that $e^{-\phi}/|s|^{2-2\beta}$ is a volume form on X by the cohomology condition, since $\log|s|^2$ corresponds to a plurisubharmonic metric on the line bundle S.

Before going to this singular case, it's natural to consider the perturbed version of this equation, i.e. we shall consider the following smooth approximation equation

$$\omega^n_{\phi_\epsilon} = \frac{e^{-\phi_\epsilon}}{\mu_\epsilon(|s|^2 + \epsilon e^{\psi})^{1-\beta}} \tag{16}$$

where ψ is any semi-positively curved smooth metric on the line bundle S, and

$$\mu_\epsilon = \int_X \frac{e^{-\phi_\epsilon}}{(|s|^2 + \epsilon e^{\psi})^{1-\beta}}$$

is the normalization constant. This is equivalent to the following geometric equality

$$Ric(\omega_{\phi_\epsilon}) = \omega_{\phi_\epsilon} + (1-\beta)\chi_\epsilon \tag{17}$$

where

$$\chi_\epsilon = dd^c \log\left(|s|^2 + \epsilon e^{\psi}\right)$$

Now we claim that χ_ϵ is an non-negative closed (1,1) form as

Lemma 29 *For any $\epsilon > 0$, we have*

$$dd^c \log\left(|s|^2 + \epsilon e^{\psi}\right) \geqslant 0.$$

Proof 18

$$\begin{aligned} &\partial\bar\partial \log\left(|s|^2 + \epsilon e^{\psi}\right) \\ &= \partial\left\{\frac{s d\bar s + \epsilon e^{\psi}\bar\partial\psi}{|s|^2 + \epsilon e^{\psi}}\right\} \\ &= \frac{\epsilon e^{\psi}}{|s|^2 + \epsilon e^{\psi}}\left\{\partial\bar\partial\psi + |ds - s\partial\psi|^2\right\}, \end{aligned}$$

and notice the above term is non-negative if the smooth metric ψ is.

Now suppose we have a smooth solution ω_{ϕ_ϵ} of equation(16), then theorem(2) tells us

Corollary 30 *The smooth solution of the perturbed conical Kähler-Einstein equation ω_{ϕ_ϵ} is actually unique.*

Proof 19 *It's enough to prove there exist one point* $p \in X$*, such that* $\chi_\epsilon(p) > 0$*. First notice that this is the case if the metric* $i\partial\bar{\partial}\psi$ *is not completely degenerate. Otherwise, we have*

$$\partial\bar{\partial}\psi \equiv 0$$

on X*. But meanwhile we should have*

$$\bar{\partial}\psi = \frac{d\bar{s}}{\bar{s}}$$

if χ_ϵ *vanishes. And then* $i\partial\bar{\partial}\psi = \delta_D$*, where* δ_D *is the integration current of* D*, which is a contradiction.*

13 Construction of the perturbed solution

In this section, we shall discuss the existence of the solution of the perturbed Kähler-Einstein equation. In general, this is unknown even if the solution of equation (15) exists. So, here we need an extra assumption: the twisted *Ding*-functional is proper.

In the following, we shall write the twisted *Ding*-functional as $\mathcal{D}_\epsilon$ with respect to the smooth twister χ_ϵ, and $\mathcal{D}_\beta$ with respect to the singular twister $(1-\beta)\delta_D$, i.e. we have

$$\mathcal{D}_\epsilon(\phi) = -\mathcal{E}(\phi) - \log \int_X \frac{e^{-\phi}}{(|s|^2 + \epsilon e^{\psi})^{1-\beta}}$$

and

$$\mathcal{D}_\beta(\phi) = -\mathcal{E}(\phi) - \log \int_X \frac{e^{-\phi}}{|s|^{2-2\beta}}.$$

Notice that the critical point of the twisted *Ding*-functional $\mathcal{D}_\epsilon$ (or $\mathcal{D}_\beta$) is the solution of the twisted Kähler-Einstein equation with twister χ_ϵ (or $(1-\beta)\delta_D$). Next, we shall introduce another important functional *Aubin's J* functional, i.e.

$$\mathcal{J}(\phi) := \int_X \varphi\omega_0^n - \mathcal{E}(\varphi)$$

where $\varphi = \phi - \phi_0$. This functional is a kind of $W^{1,2}$ norm of the potential φ, and we can compare it with *Ding*-functional

Definition 3 *The twisted Ding-functional $\mathcal{D}_\epsilon$(or $\mathcal{D}_\beta$) is called proper if there exists some constant $a > 0$ and b such that*

$$\mathcal{D}_\cdot(\phi) \geqslant a\mathcal{J}(\phi) + b$$

for all $\phi \in PSH(-K_X - S)$.

Remark 5 *Notice that the properness of the twisted Ding-functional is in fact independent of the twister as long as the twister is smooth. This is because the twisted Ding-functionals are comparable for two different smooth twisters, i.e. let ψ_1 and ψ_2 be the corresponding metrics associated to the twisters θ_1 and θ_2 , then there exists a constant C such that*

$$-C < \psi_1 - \psi_2 < C$$

then the twisted functionals satisfy

$$\mathcal{F}_{\psi_2}(\phi) = -\log \int_X e^{-\phi-\psi_2} = -\log \int_X e^{-\phi-\psi_1+(\psi_1-\psi_2)}$$

hence

$$|\mathcal{F}_{\psi_1} - \mathcal{F}_{\psi_2}| < C'$$

for some uniform constant C'. And since $\mathcal{E}$ and $\mathcal{J}$ functionals are independent of the twister, the assertion follows.

Remark 6 *For the special twister χ_ϵ, observe that the major term $\mathcal{F}$ in the twisted Ding-functional has the following relation*

$$\mathcal{F}_\epsilon \leqslant \mathcal{F}_{\epsilon'}$$

if $\epsilon \leqslant \epsilon'$, hence $\mathcal{D}_\epsilon$ is decreasing when ϵ becomes smaller. Moreover, if $\epsilon = 0$, it achieves its minimum , i.e.

$$\mathcal{D}_\beta \leqslant \mathcal{D}_\epsilon$$

for all $\epsilon > 0$ small. Hence in the practice, we can simply require $\mathcal{D}_\beta$ to be proper.

Generally speaking, the purpose for introducing properness of *Ding*-functional is to solve the following continuity path

$$Ric(\omega_{\phi_t}) = t\omega_{\phi_t} + (1-t)\omega_0 + (1-\beta)\chi_\epsilon.$$

where ω_0 is a fixed smooth Kähler metric in the same cohomology with ω_ϕ, and $\chi_\epsilon = i\partial\bar{\partial}\psi_\epsilon$ for $\psi_\epsilon = \log(|s|^2 + \epsilon e^\psi)$. This equation is solvable up to $t = 1$ if the twisted $K-energy$ [4] is proper, and this condition is equivalent to the properness of the twisted *Ding*-functional from the argument of Berman[4], so the existence of smooth perturbed solutions are guaranteed from here. However, this is not the end of the story since we want to make sure these perturbed solutions can approximate the conical one in some sense.

In order to do this, we need to consider a kind of conical Kähler metrics with better regularity. Donaldson[12] introduced a special Hölder space $\mathcal{C}^{2,\alpha,\beta}$ where $0 < \alpha < \frac{1}{\beta} - 1$ for real valued functions on X, and we can define the so called $\mathcal{C}^{2,\alpha,\beta}$ conical Kähler metric[12] by requiring that a local Kähler potential lies in $\mathcal{C}^{2,\alpha,\beta}$, i.e. in a local coordinate chart near the divisor D, we can always write

$$\omega = i\partial\bar{\partial}(\varphi + \psi)$$

where $\varphi \in \mathcal{C}^{2,\alpha,\beta}$, and ψ is some smooth function. Be aware that this condition is stronger than the condition of conical Kähler metrics. Simply speaking, for fixed angle β, a $\mathcal{C}^{2,\alpha,\beta}$ conical Kähler metric is a conical Kähler metric with uniform $\mathcal{C}^{2,\alpha,\beta}$ norm.

Now let's take ω_{φ_β} to be a $\mathcal{C}^{2,\alpha,\beta}$ conical Kähler-Einstein metrics on X and assume the twisted *Ding*-functional $\mathcal{D}_\beta$ is proper, then Chen, Donaldson and Sun's work[9] provided a way to construct the following a family of sequence of perturbed solutions

$$\omega_{\phi_\epsilon}(t,.) = \omega_0 + i\partial\bar{\partial}\phi_\epsilon,$$

and it satisfies the following equation

$$Ric(\omega_{\phi_\epsilon(t)}) = t\omega_{\phi_\epsilon(t)} + (1-t)\omega_{\varphi_\epsilon} + (1-\beta)\chi_\epsilon.$$

Notice that for $t=1$, $\omega_{\phi_\epsilon}(1,.) = \omega_{\phi_\epsilon}$ is exactly the solution of the twisted Kähler-Einstein equation with smooth twister $(1-\beta)\chi_\epsilon$, i.e.

$$Ric(\omega_{\phi_\epsilon}) = \omega_{\phi_\epsilon} + (1-\beta)\chi_\epsilon,$$

and when $t=0$, the metric $\omega_{\phi_\epsilon}(0,.) = \psi_\epsilon$ will approximate the original metric ω_β because we require

$$\omega^n_{\psi_\epsilon} = e^{-\varphi_\epsilon + h_{\omega_0}} \frac{1}{(|s|^2_h + \epsilon)^{1-\beta}} \omega_0^n$$

where $\omega_{\varphi_\epsilon}$ is a small perturbation of the original metric such that $\omega_{\varphi_\epsilon} \to \omega_\beta$ in $C^\gamma(X)$. Moreover, we know that when $\epsilon \to 0$, the sequence of metrics $\phi_\epsilon(t,.)$ converges to a metric $\phi_0(t,.)$ globally in C^γ and locally in $C^{3,\gamma}$ outside the divisor D, and the limiting metric $\phi_0(t,.)$ will satisfy the following equation outside D

$$\omega^n_{\phi_0} = e^{-t\phi_0 - (1-t)\varphi_\beta + h_{\omega_0}} \frac{1}{|s|_h^{2-2\beta}} \omega_0^n \tag{18}$$

with

$$\phi_0(0,.) = \varphi_\beta.$$

Observe that above equation is in fact equivalent to

$$\omega^n_{\phi_0} = e^{-t(\phi_0 - \varphi_\beta)} \omega^n_{\varphi_\beta} \tag{19}$$

with $t \in [0,1]$. Now we claim the curve of metrics $\phi_0(t,0)$ is in fact fixed. The argument of the claim is similar with the end of the paper [9], and we shall recall it here for convenience of the reader

Lemma 31 *The continuous family $\phi_0(t,.)$ is independent of the time t, and hence*

$$\phi_0(1,.) = \varphi_\beta.$$

Proof 20 *We shall argue like the end of the paper [9]. First notice that the weighted Laplacian operator Δ_{φ_β} is continuous and invertible as a map*

$$\Delta_{\varphi_\beta} : \mathcal{C}_0^{2,\gamma,\beta} \to \mathcal{C}^{,\gamma,\beta}$$

for some $\gamma < \frac{1}{\beta} - 1$, and $\mathcal{C}_0^{2,\gamma,\beta}$ is the space of functions in $\mathcal{C}^{2,\gamma,\beta}$ with zero average. In fact, by Donaldson's Hölder estimate of conical metrics in [12], we can prove the first eigenvalue of this operator is strictly positive, i.e.

$$\Delta_{\varphi_\beta} > \lambda$$

for some constant $\lambda > 0$, and any $u \in \mathcal{C}_0^{2,\gamma,\beta}$. Now since the term $e^{-t(\phi_0 - \varphi_\beta)}$ lies in $\mathcal{C}^{,\gamma,\beta}$, Implicit Function Theorem implies the existence of a continuous family of solution of the following equation

$$\omega_\psi^n = e^{-t(\phi_0 - \varphi_\beta)}$$

with $\psi(t,.) \in \mathcal{C}_0^{2,\gamma,\beta}$, for $t \in [0, \epsilon_0)$. Notice that $\psi(0,.) = \varphi_\beta$, hence $\psi(t,.)$ must coincide with $\phi_0(t,.)$ up to a constant in this short time. Then we can guarantee that $\phi_0(t,.) \in \mathcal{C}^{2,\gamma,\beta}$ for $t \in [0, \epsilon_0)$.

Next we consider a constant continuity path $\varphi_t \equiv \varphi_\beta$, which satisfies the equation

$$\omega_{\varphi_t}^n = e^{-t(\varphi_t - \varphi_\beta)} \omega_{\varphi_\beta}^n.$$

Now by Implicit Function Theorem again, we have a unique path of solution for $t < \min(\epsilon_0, \lambda/2)$, and hence

$$\phi_0(t,.) = \varphi_\beta, \quad \forall\ 0 \leqslant t < \min(\epsilon_0, \lambda/2).$$

Finally we can repeat this procedure again and again, it will reach $t = 1$, i.e.

$$\phi_0(1,.) = \varphi_\beta.$$

Recall from our previous construction, we see the sequence of twisted Kähler-Einstein metric ϕ_ϵ will converges to φ_β in C^γ, hence we have

Proposition 32 *Suppose ω_{φ_β} is a $\mathcal{C}^{2,\alpha,\beta}$ solution for the conical Kähler-Einstein equation along the divisor D on X, and the twisted Ding-functional is proper, we can construct a sequence of perturbed Kähler-Einstein metrics ϕ_ϵ, such that*

$$\phi_\epsilon \to \varphi_\beta$$

globally in C^γ, and locally in $C^{k,\gamma}$ for some $k > 3$ outside of D.

and this brings the proof our uniqueness.

Proof 21 (of theorem 3) *Suppose ω_{φ_β} and ω'_{φ_β} are two different $\mathcal{C}^{2,\alpha,\beta}$ conical Kähler-Einstein metrics on X with cone angle β along the divisor D. From above argument, we can construct a sequence of smooth twisted Kähler-Einstein metrics ϕ_{ϵ_j} to approximate φ_β in some Hölder space C^γ. Then repeat the construction again, we can find a subsequence $\phi_{\epsilon_{n(j)}}$ to approximate φ'_β by the actual uniqueness of ϕ_ϵ, and this implies $\phi_\beta = \phi'_\beta$ on X, which is a contradiction.*

In the end of the paper, a new point in the above proof should be mentioned. Unlike the end of the paper[9], the vanishing of tangential holomorphic vector fields is not necessary in the proof of lemma(31), i.e. the properness of twisted *Ding* functional has not been used here. So we can try to generalize this method by starting from the construction of some suitable approximation of conical Kähler-Einstein metrics without properness of *Ding* functional.

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