Gluing Techniques in Calibrated Geometry

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Abstract of the Dissertation

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This thesis is concerned with the question: given a submanifold (perhaps with singularities), when is it possible to change the metric in some specific way so that the submanifold becomes homologically mass-minimizing?

We studied this question for "horizontal" change of metrics and conformal change of metrics for both compact and non-compact submanifolds. We also explored cases with singularities or boundaries. The main idea is to use the theory of calibrated geometry and gluing techniques.

As a special case, we confirm that any given oriented compact

connected submanifold which is not $\mathbb{R}$-homologous to zero in a Riemannian manifold can be calibrated after a highly controlled conformal change of the given metric. The statement remains true for any non-connected submanifold as well provided the convex hull of $\mathbb{R}$-homology classes represented by its oriented connected components does not contain zero.

To my beloved parents.

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Chapter 1

Introduction

Plateau's problem was raised by Joseph-Louis Lagrange in 1760 and later named after Joseph Plateau who did many interesting experiments using soap films. Over the past 90 years, much work has been devoted to the mathematical study of the Plateau problem (and its higher dimensional generalizations) by many people including Douglas, Rado, de Giorgi, Federer, Fleming and many more. As part of this development, in 1982 Reese Harvey and Blaine Lawson introduced the theory of calibrated geometries [8], which grew to be important because of its many applications, for example, to gauge theory and mirror symmetry, i.e., the compactication of Moduli space of Yang-Mills connections, and the SYZ conjecture.

One area where calibrated geometry applies is the theory of foliations. The story started with an interesting question raised by Gluck in [5]. It was asking: given a one-dimensional foliation on a compact manifold, when can one find a Riemannian metric so that every leaf of the foliation is geodesic, i.e., minimal? It was Sullivan who first discovered the complete answer in [23]: an oriented foliation by curves on a compact manifold is geometric taut if and only if it is homologically taut. Later he generalized his results to foliations of arbitrary dimension in [24]. Around the same time, Harvey and Lawson observed that the theory of calibration is naturally adapted to foliations, and gave an intrinsic characterization of those foliations which can be calibrated in [7]: an oriented foliation on a compact manifold is geometric tight if and only if it is homologically tight (and in fact, both cases here are equivalent to the foliation's being calibrated). They meanwhile analyzed relations between tautness and tightness, and several important results were given for constructing metrics, e.g. Lemma 2.12 and 2.14 (cited below in the proof of *Theorem 5.0.3*).

However, still, one piece of the entire story was missing. That is the question when a submanifold can be realized minimal or calibrated with respect to some metric. Since not all submanfolds can be realized as a leaf of a foliation, for example any smooth perturbation of $\mathbb{C}P^1$ in $\mathbb{C}P^2$ (because no C^∞ two-dimensional plane fields exist on $\mathbb{C}P^2$), generally it has few relations to the foliation case. (It may also be interesting to ask when a submanifold can be realized as a leaf of some foliation and, if it can, when the foliation can be chosen taut or tight.) To be minimal is not hard. (See *Corollary 4.1.8* and more generally also §*4.2*.) In order to be calibrated, for simplicity, an oriented compact connected submanifold has to be non-zero in the $\mathbb{R}$-coefficient homology of the ambient space. In 1991, by methods developed in [22], [23] and [8], Tasaki [25] proved the existence theorem of metrics for its inverse direction – $(\star)$

"Let M be a compact oriented submanifold embedded in a manifold X. If the real homology class represented by M in X is not equal to 0, then there exists a Riemannian metric g on X such that M is mass minimizing in its real homology class with respect to g."

In fact, his proof proves that M can be calibrated in (X, g). However, the argument depends much on the connectness of M (to have a potential calibration form, see the first line of p. 83) and no other properties can be said about the result metric.

In this paper, we concentrate as well on the reverse question and, by different methods (to construct "better", in the sense of local behaviors near the given submanifolds, potential calibration forms directly by algebraic topology methods instead of functional analysis ones), obtain more and nicer results in mainly three directions and this paper is organized correspondingly as follows.

We first introduce preliminary ingredients for this paper in §2. In the following §3, we provide full details in the **first direction**: *horizontal change of metrics*. It shows that, when the ambient space is compact, sufficiently many new metric satisfying ($\star$) can be constructed by horizontal change of any a priori given metric. This can be extended to non-connnect case of (suitable) several connect components possibly of different dimensions.

In §4, the **second direction** of *conformal change of metrics* is studied. We discover the fact that, in any conformal class of metrics (no matter whether the ambient space is compact or not), there are alway many metrics to fulfill ($\star$) and the same conclusion still holds for non-connnect case of (suitable) several connect components possibly of different dimensions. (Compare theorems 3.5.1 and 3.5.5 with 4.1.5 and 4.1.6.) In the second part of §4, we consider non-compact submanifolds. Particularly in §4.3, *Global Plateau Property* can be obtained in *Theorem 4.3.5*, due to the exactness of the calibration form.

In the **third direction** §5, *cases with singularities* are explored. *Theorem 5.0.3* confirms the existence of metrics in ($\star$) for submanifolds with mild singularities (not necessarily of dimension zero). In the section §6, we strengthen and generalize "equivariant" results of Tasaki [25].

One may also think that it seems easy to construct a new metric, by conformal change of metric, to force a priori given connected submanifold (of non-zero $\mathbb{R}$-homology class of ambient space) homologically minimal. However, even if this can be achieved, according to the author's knowledge, it is still unknown whether such submanifolds can be calibrated with respect to the new metric. The question here is essentially asking whether there are obstructions for a submanifold (or a rectifiable current) to be calibratable other than being mass minimizing in its current homology class (for a fixed metric). In fact, Federer considered similar questions and in [3] he established a duality in general context between homologically Ψ minimizing real flat chains and locally flat cocycles.

In the Acta Mathematica paper [8], for simplicity, Harvey and Lawson did not include boundary case or relative calibrated geometry in their original paper. However due to subsequent developments in pseudo-holomorphic curves in symplectic geometry and manifolds with special holonomy, these two cases become more and more important. Although it may be well known among some experts, it never appears on paper according to the author's knowledge. In section §7, we theoretically introduce **the last direction**, calibrations with boundary and relative calibrations. Thereafter, we successfully extend the previous results on conformal change of metrics to this setting.

Chapter 2

Preliminaries

2.1 From Calibrated Geometry

Let us briefly review some fundamental concepts and results that we will need from calibrated geometry. For a further understanding on calibrated geometry, readers are referred to its birth [8].

Definition 2.1.1 *Let ϕ be a smooth m-form on a Riemannian manifold (X, g). Then at each $x \in X$, we define the* **comass** *of ϕ_x to be*

$$\|\phi\|^*_{x,g} = \sup \{< \phi_x, \overrightarrow{V}_x >_g : \ \overrightarrow{V}_x \textit{ is a unit simple } m\textit{-vector at } x\},$$

where simple means there exists an orthonormal basis $\{e_i\}$ of T_xX such that $\overrightarrow{V}_x$ is a multiple of $e_1 \wedge e_2 \cdots \wedge e_m$. Furthermore, if A is a subset of X, the comass of ϕ on A is defined as

$$\|\phi\|^*_{A,g} = \sup_{x \in A} \|\phi\|^*_{x,g}.$$

When A is the entire space, the first subscript will be omitted conventionally.

Remark 2.1.2 *For any continuous form* ϕ *and a smooth metric* g, $\|\phi\|_g^*$ *can be considered as a pointwisely continuous function. Note that even when* ϕ *is smooth, generally* $\|\phi\|_g^*$ *is only a continuous function. Moreover, by definition, at any point* x,

$$\begin{aligned}\|\phi\|_{x,g}^* &= \max\{\phi(\overrightarrow{V}_x) : \overrightarrow{V}_x \textit{ is a simple } m\textit{-vector at } x \textit{ with } \|\overrightarrow{V}_x\|_g = 1\} \\ &= \max\{1/\|\overrightarrow{V}_x\|_g : \overrightarrow{V}_x \textit{ is a simple } m\textit{-vector at } x \textit{ with } \phi(\overrightarrow{V}_x) = 1\} \\ &= 1/\min\{\|\overrightarrow{V}_x\|_g : \overrightarrow{V}_x \textit{ is a simple } m\textit{-vector at } x \textit{ with } \phi(\overrightarrow{V}_x) = 1\}.\end{aligned}$$

Definition 2.1.3 *A smooth* m*-form* ϕ *on a Riemannian manifold* (X, g) *is said to be a* **calibration** *if* ϕ *is of comass one on* X *and* $d\phi = 0$. *A Riemannian manifold together with a calibration is called a* **calibrated manifold***.*

Definition 2.1.4 *In a calibrated manifold* (X, g, ϕ)*, if the restriction of* ϕ *to an oriented submanifold* Y *equals the induced volume form, we say that* ϕ **calibrates** Y *in* (X, g)*.*

Suppose ϕ calibrates a compact submanifold M in (X, g, ϕ). Let M' be another compact submaniold in the same $\mathbb{R}$-homology class of X. Then

$$\mathrm{Vol}_g M = \int_M i_M^*\phi = \int_{M'} {i_{M'}}^*\phi \leq \ \mathrm{Vol}_g M',$$

namely M is volume-minimizing in its "smooth" homology class. Actually this can be naturally generalized to the topological dual space of smooth forms, whose elements are called *de Rham* **currents** (cf. [2]).

Definition 2.1.5 *Let* T *be an arbitrary de Rham* m*-current with compact support on* (X, g)*. The* ***mass*** *of* T *is defined to be*

$$\mathbf{M}(T) = \sup\{T(\psi) : \|\psi\|_{X,g}^* < 1\}.$$

When $\mathbf{M}(T) < \infty$, T determines a *Radon* measure $\|T\|$ characterized by

$$\int_X f \cdot d\|T\| = \sup\{T(\psi) : \|\psi\|^*_{x,g} \leq f(x)\},$$

for any non-negative function f on X. Therefore $\mathbf{M}(T)$ turns out to be the total measure of X for $\|T\|$. Moreover, the *Radon-Nikodym* Theorem asserts the existence of a $\|T\|$ measurable tangent m-vector field $\overrightarrow{T}$ a.e. with vectors $\overrightarrow{T}_x \in \Lambda^p T_x X$ of unit length in the dual norm of the comass norm in *Definition 2.1.1*, satisfying

$$T(\psi) = \int_X \psi_x(\overrightarrow{T_x})\ d\|T\|(x), \tag{2.1.1}$$

or briefly,

$$T = \overrightarrow{T} \cdot \|T\|\ a.e.\ \|T\|.$$

If T stands for integration over an oriented m-dimensional submanifold S of X, then $\mathbf{M}(T) = \mathrm{vol}(S)$. Note that in order to have the *Radon* measure and decomposition (2.1.1), one sufficient condition is that T has local finite mass.

Definition 2.1.6 *In a calibrated manifold (X, g, ϕ), define*

$$\mathscr{G}(\phi) = \{\overrightarrow{V}_x : <\phi_x, \overrightarrow{V}_x>_g = 1,\ \text{where } \overrightarrow{V}_x \text{ is a unit simple } m\text{-vector at x}\}.$$

A current T of local finite comass is called a positive ϕ-current, if $\overrightarrow{T_x} \in \mathscr{G}(\phi)$ a.a. $x \in X$ for $\|T\|$.

Lemma 2.1.7 (Harvey and Lawson) *In a Riemannian manifold (X, g), suppose that ϕ is a smooth m-form with comass one and that T is an arbitrary m-current with compact support. Then*

$$T(\phi) \leq \mathbf{M}(T)$$

with equality if and only if T is a positive ϕ-current.

In particular, if S is a compact oriented m-dimensional submanifold (with possible boundary) in X, then

$$\int_S \phi \leq \mathrm{vol}(S)$$

with equality if and only if S is a ϕ-submanifold.

Proof. Without loss of generality, we assume $\mathbf{M}(T) < \infty$. Then $T = \overrightarrow{T} \cdot \|T\|$ a.e. for $\|T\|$. Thus $T(\phi) = \int \phi(\overrightarrow{T}) \cdot d\|T\| \leq \int d\|T\| = \mathbf{M}(T)$ and equality holds if and only if $\phi(\overrightarrow{T}) = 1$ a.e. for $\|T\|$, i.e., if and only if T is a positive ϕ-crruent.

If T stands for an oriented submanifold S, then $\|T\|$ is Hausdorff m-measure restricted to S and $\overrightarrow{T}$ is the field of oriented unite tangent m-vectors to S. It is easy to see that the second statement follows as a special case of the first. ∎

It was also pointed out in [8], that, by the nature isomorphism between homology of the complex of *de Rham* currents with compact support and $H_*(X;\mathbb{R})$, we have the following fundamental lemma in calibrated geometry.

Theorem 2.1.8 (Harvey and Lawson) *Suppose that X is a calibrated manifold with calibration ϕ, and that T is a positive ϕ-current with compact support. Let T' be any compactly supported current homologous to T (i.e., $T - T'$ is a boundary and in particular $dT = dT'$). Then*

$$\mathbf{M}(T) \leq \mathbf{M}(T')$$

with equality if and only if T' is a positive ϕ-current.

Proof. Since $T - T' = dS$, for some compactly supported $(m+1)$-current S, we have $T(\phi) - T'(\phi) = (T - T')(\phi) = dS(\phi) = S(d\phi) = 0$. Combining *Lemma 2.1.7*, we have

$$\mathbf{M}(T) = T(\phi) = T'(\phi) \leq \mathbf{M}(T')$$

with equality if and only if T' is also a positive ϕ-current. ∎

The above theorem shows that a calibrated submanifold (or current) is in fact mass minimizing in its current homology class. Moreover, in the same current homology class, other mass minimizing elements must be calibrated as well.

In the rest part of this paper, we will repeatedly use certain properties of comass. Among which, the elementary *Lemma 2.1.11 and 2.1.12* below are crucial to our methods.

Lemma 2.1.9 *For any metric g, an m-form ϕ and a positive function f on X. We have*

$$\|\phi\|^*_{f\cdot g} = f^{-\frac{m}{2}} \cdot \|\phi\|^*_g,$$

as pointwise functions.

Proof. By the formula in *Remark 2.1.2*. ∎

Lemma 2.1.10 *For any m-form ϕ. For metrics, if $g' \geq g$ on X, then*

$$\|\phi\|^*_{g'} \leq \|\phi\|^*_g,$$

as pointwise functions.

Proof. By the definition of comass or the formula in *Remark 2.1.2*. ∎

Lemma 2.1.11 (Comass Control for Gluing Procedure) *For any smooth m-form ϕ, positive smooth functions a and b, and smooth metrics g_1 and g_2, the following control inequality holds pointwisely,*

$$\|\phi\|^*_{ag_1+bg_2} \leq \frac{1}{\sqrt{a^m \cdot \frac{1}{\|\phi\|^{*2}_{g_1}} + b^m \cdot \frac{1}{\|\phi\|^{*2}_{g_2}}}}, \tag{2.1.2}$$

where $\frac{1}{0}$ and $\frac{1}{+\infty}$ are identified with $+\infty$ and 0 separately.

Proof. Let us fix a point x. Then for the subspace spanned by any simple m-vector $\overrightarrow{V}_x$, there exists an orthonormal basis $(e_1, e_2, ..., e_m)$ under g_1, to which g_2 is diagonalized as $diag(\lambda_1, ..., \lambda_m)$ where $\lambda_i > 0$ for $i = 1, 2, \cdots, m$. Suppose $\overrightarrow{V}_x = te_1 \wedge ... \wedge e_m$, then

$$\begin{aligned} \|\overrightarrow{V}_x\|^2_{ag_1+bg_2} &= t^2(a + b\lambda_1)...(a + b\lambda_m) \\ &= t^2[a^m + ... + b^m \Pi\lambda_i] \\ &\geq \quad t^2 a^m + t^2 b^m \Pi\lambda_i \\ &= a^m \|\overrightarrow{V}_x\|^2_{g_1} + b^m \|\overrightarrow{V}_x\|^2_{g_2}. \end{aligned} \tag{2.1.3}$$

Together with *Remark 2.1.2*, it is not hard to see that (2.1.3) implies conclusion (2.1.2). ∎

Lemma 2.1.12 *Suppose that (E, π) is a smooth disc (or other) bundle over M and that g is a Riemannian metric defined on E, then each fiber is perpendicular to M (considered as the zero section) if and only if $\pi^* vol_{g|_M}$ has comass one pointwisely along M, where $vol_{g|_M}$ stands for the volume form of M induced by the restriction metric $g|_M$.*

Remark 2.1.13 *Note that if the orthogonal condition is unsatisfied, the pull-back of the volume form will have strictly larger comass at dissatisfactory points. One simple example is to consider the slope-one diagonal line L in Euclidean xy-plane with fibers of parallel lines to x-axis. Then the volume form ω of L is $\frac{1}{\sqrt{2}}(dx+dy)|_L$ but $\pi^*\omega$ is $\sqrt{2}dy$ of comass $\sqrt{2}$.*

Proof. Fix a point x on M. Take $e_1,\ e_2,\ \cdots,\ e_m$ an oriented orthonormal basis of T_xM. Then we have decomposition

$$e_i = \sin\theta_i \cdot a_i + \cos\theta_i \cdot b_i,$$

where a_i is a unit vector perpendicular to the fiber through x, b_i is a unit fiber vector, and θ_i is the angle between e_i and the fiber through x for $i = 1, 2, \cdots, m$.

Denote $vol_{g|_M}$ by ω. By the choice of $\{e_i\}$, it follows that

$$\begin{aligned} 1 &= \omega(e_1 \wedge e_2 \cdots \wedge e_m) \\ &= \pi^*\omega(e_1 \wedge e_2 \cdots \wedge e_m) \\ &= \pi^*\omega(\sin\theta_1 \cdot a_1 \wedge \sin\theta_2 \cdot a_2 \cdots \wedge \sin\theta_m \cdot a_m) \\ &= \Pi \sin\theta_i \cdot \pi^*\omega(a_1 \wedge a_2 \cdots \wedge a_m). \end{aligned} \tag{2.1.4}$$

The third equality is from the fact that fiber directions are annihilators of $\pi^*\omega$.

Since $\{a_i\}_1^m$ are of unit length, when $\|\pi^*\omega\|_x^*$ equals one, we have $\theta_i = \pi/2$ for $i = 1, 2, \cdots, m$, i.e., the fiber of x is perpendicular to T_xM.

Conversely, suppose fibers are perpendicular to M. We know that the only simple m-vector which ealizes the minimal in *Remark 2.1.2* is tangent to M, since all fiber-direction-involving part will contribute zero when paired with $\pi^*\omega$. Therefore the comass of $\pi^*\omega$ is one due to the fact that ω is the volume form and the first two equalities of (2.1.4). ∎

2.2 A Basic Fact about Forms

Lemma 2.2.1 *Suppose that X is an orientable or non-orientable, compact or non-compact, of finite type or not, manifold without boundary. If M is an m-dimensional oriented connected compact submanifold representing a non-zero homology class $[M]$ in $H_m(X;\mathbb{R})$, then for any positive number s, there exists a closed m-form ϕ on X such that*

$$\int_M i_M^* \phi = s.$$

Remark 2.2.2 *When X is compact and orientable, it follows directly from Poincaré duality. For general case, it can be derived from de Rham Theorem.*

Proof. In singular homology theory, we have *Kronecker* product $< \cdot, \cdot >$ between cochains and chains, which induces a homomorphism

$$\kappa \ : H^q(X;G) \to \ \mathrm{Hom}_{\mathbb{Z}}(H_q(X;\mathbb{Z}),\ G), \tag{2.2.1}$$

$$\kappa \ ([z^q])([z_q]) \triangleq < [z^q], [z_q] >,$$

where G is any Abellian group.

Let us recall a classical result.

Lemma 2.2.3 *κ is surjective. Moreover, there exists a homomorphism*

$$\iota : Hom_{\mathbb{Z}}(H_q(X;\mathbb{Z}), G) \to H^q(X;G),$$

such that $\kappa \circ \iota = id$, i.e., $Hom(H_q(X;\mathbb{Z}), G)$ is a splitting term of $H^q(X;G)$.

We just take the advantage of surjectivity of κ. When G is $\mathbb{R}$, by the *de Rham* Theorem, κ maps $H^q_{dR}(X)$ to $\mathrm{Hom}_{\mathbb{Z}}(H_q(X;\mathbb{Z}), \mathbb{R})$, which can be identified with $\mathrm{Hom}_{\mathbb{R}}(H_q(X;\mathbb{R}), \mathbb{R})$. Since $[M]$ is non-zero in $H_m(X;\mathbb{R})$ by our assumption, by *Lemma 2.2.3* and *de Rham* Theorem, there exists a homomorphism $F \in \mathrm{Hom}_{\mathbb{R}}(H_q(X;\mathbb{R}), \mathbb{R})$ sending $[M]$ to any a priori given positive number s and a chosen complement of their span in $H_q(X;\mathbb{R})$ to zero. Therefore, by the surjectivity of κ, we have at least one closed m-form defined on X with integral s on M (or a unique form for a fixed choice of ι). ∎

Proof of the surjectivity in Lemma 2.2.3. Denote the $\mathbb{Z}$-coefficient singular q-chain group, closed q-chain group, boundary q-chain group and the q-th homology group by S_q, Z_q, B_q and H_q respectively.

For any $\phi \in \mathrm{Hom}(H_q, G)$, it produces a homomorphism

$$\phi' : Z_q \to G,$$

$$\text{by } < \phi', z_q > = < \phi, [z_q] > .$$

Since we have the short exact sequence

$$0 \longrightarrow Z_q \longrightarrow S_q \xrightarrow{\partial} B_{q-1} \longrightarrow 0$$

and B_{q-1} is Abelian, the sequence splits. Therefore, ϕ' can extend to a homomorphism $\bar{\phi}$ from S_q to G.

(1) $\bar{\phi}$ is a closed cochain. For any chain $c_{q+1} \in S_{q+1}$,

$$< \delta\bar{\phi}, c_{q+1} >=< \bar{\phi}, \partial c_{q+1} >=< \phi', \partial c_{q+1} >=< \phi, [0] >= 0,$$

so it determines a class $[\bar{\phi}] \in H^q(X; G)$.

(2) $\kappa([\bar{\phi}]) = \phi \in \mathrm{Hom}(H_q, G)$. For any closed $z_q \in Z_q$,

$$\kappa([\bar{\phi}])([z_q]) =< [\bar{\phi}], [z_q] >=< \bar{\phi}, z_q >=< \phi', z_q >=< \phi, [z_q] >= \phi([z_q]).$$

∎

Note that if we are given mutually disjoint m-dimensional oriented connected compact submanifolds $M_1, \cdots, M_r$ with their homology **convex hull**

$$\mathcal{C} \triangleq \{\sum_{i=1}^{r} t_i[M_i] : \sum_{i=1}^{r} t_i = 1 \text{ and } t_i \geq 0\} \text{ in } H_m(X; \mathbb{R})$$

not containing the zero class. Then we can assign positive numbers s_j to each $[M_i]$ such that combinatorial relations between $\{[M_i]\}$ are faithfully inherited by $\{s_j\}$ as follows. Start from $[M_1]$ with arbitrary positive s_1. Suppose we are done for first k terms. If $[M_{k+1}]$ can not be spanned by first k terms, then assign arbitrary positive number s_{k+1} to it. Otherwise, $[M_{k+1}]$ can be written as $t_1[M_1] + \cdots + t_k[M_k]$, where $\{t_i\}$ are some non-negative numbers with sum positive. Then let $s_{k+1} = t_1 \cdot s_1 + \cdots + t_k \cdot s_k$. Similar as in the proof of *Lemma 2.2.1*, there exists a homomorphism $\digamma \in \mathrm{Hom}_{\mathbb{R}}(H_q(X; \mathbb{R}), \mathbb{R})$ sending $[M_i]$ to these s_i and a chosen complement of their span to zero. Correspondingly, there is at least one closed m-form with integral s_i on M_i. Thus we obtain the following general result.

Lemma 2.2.4 *Suppose that X a manifold without boundary, and that M_i are disjoint m-dimensional oriented connected compact submanifolds with their convex hull not containing the zero class in $H_m(X;\mathbb{R})$. Then there exists a closed m-form ϕ on X such that*

$$\int_{M_i} i_{M_i}^* \phi > 0.$$

Remark 2.2.5 *Since our analysis above is also valid for the case of countable components (to occur in §3.5 and §4), the same statement holds with no essential differences for that case as well.*

2.3 Bundle Structure around Submanifolds

Given a compact submanifold M in (X, g), consider its ϵ-neighborhood, $U_\epsilon \triangleq \{x \in X : \mathrm{dist}_g(x, M) \leq \epsilon\}$. When ϵ is small enough, there are no focal points in U_ϵ and the metric induces a disc-fibered bundle structure of U_ϵ, whose fiber is given by the exponential map restricted to normal directions along M. It is clear that the fibers are foliated by the distance function from M. We call the orthogonal complement (in TU_ϵ) to fiber directions **horizontal directions** and **horizontal change** of a metric means smoothly varying modifications on metrics along horizontal directions.

Since M is a retract of U_ϵ,

$$H^m(U_\epsilon;\mathbb{R}) \cong H^m(M;\mathbb{R}).$$

This implies that, for $[\phi_1], [\phi_2] \in H^m(U_\epsilon; \mathbb{R})$,

$$[\phi_1] = [\phi_2] \ \Leftrightarrow \int_M i_M^* \phi_1 = \int_M i_M^* \phi_2. \tag{2.3.1}$$

In future, we will use ϵ with $U_{3\epsilon}$ enjoying the above bundle structure such that, for the case of several components, their ϵ-neighborhoods are mutually disjoint.

2.4 Gluing of Forms

Let us focus on simple case first. For an oriented compact connected m-dimensional submanifold M in (X, g). If $[M] \neq 0$ in $H_m(X; \mathbb{R})$, then by §2.2, there exists a closed m-form ϕ defined on X with $\int_M i_M^* \phi = s > 0$. From §2.3, we have a bundle structure:

$$\begin{array}{ccc} D_\epsilon^{n-m} & \xrightarrow{i} & U_\epsilon \\ & & \downarrow{\scriptstyle \pi_g} \\ & & M \end{array}$$

Define $\tilde{\omega} \triangleq \pi_g^* \omega_M$. It is a closed m-form defined on U_ϵ with

$$\int_M i_M^* \tilde{\omega} = \int_M \omega_M = V (\triangleq \mathrm{Vol}_g M).$$

Therefore,

$$\int_M i_M^* \frac{s\tilde{\omega}}{V} = s = \int_M i_M^* \phi.$$

Set $\omega^* \triangleq s\frac{\tilde{\omega}}{V}$. By (2.3.1), $[\omega^*]$ equals $[\phi]$ in $H^m(U_\epsilon; \mathbb{R})$, which indicates

$$\phi = \omega^* + d\psi$$

for some smooth $(m-1)$-form ψ on U_ϵ. So we can glue forms by taking

$$\Phi = \omega^* + d((1-\rho(\mathbf{d}))\psi),$$

where ρ is a function of $\mathbf{d}$ (the distance function to M with respect to g) and it looks like:

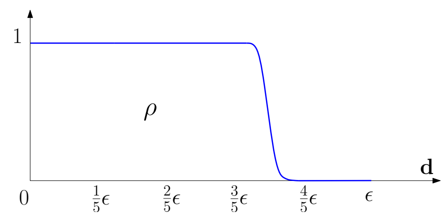

Figure 2.1: Graph of ρ.

Hence

$$\Phi = \begin{cases} \omega^* & 0 \leq \mathbf{d} \leq \frac{3}{5}\epsilon \\ \omega^* + d(1-\rho(\mathbf{d}))\psi & \frac{3}{5}\epsilon < \mathbf{d} \leq \frac{4}{5}\epsilon \\ \phi & \frac{4}{5}\epsilon < \mathbf{d} \end{cases} \quad (2.4.1)$$

Remark 2.4.1 $\mathbf{d}$ *is not smooth along* M *but the composition* $\rho(\mathbf{d})$ *is smooth on the entire* U_ϵ.

Following *Lemma 2.1.12*, we have

- **Property 1**: Pointwisely, $\|\Phi\|_g^* = \|\omega^*\|_g^* = \frac{s}{V}$ on M.

Chapter 3

Horizontal Change of Metric

3.1 Gluing of Metrics

Now let us consider the gluing of metrics.

According to the bundle structure described in §2.3, define

$$\bar{g} = (\frac{\mathcal{S}}{V})^{\frac{2}{m}} \pi_g^*(g|_M^h) \oplus g_F \tag{3.1.1}$$

in U_ϵ, where g_F is an arbitrary smooth metric for fiber (vertical) directions and $\pi_g^*(g|_M^h)(v_q, v_q') \triangleq g|_M^h(\pi_*^g(v_q), \pi_*^g(v_q'))$, where π_*^g is the push-forwarding map of π_g. Note that $\bar{g}$ preserves horizontal directions of g and the following is a nature extension of *Lemma 2.1.12*.

- **Property 2**: Pointwisely, $\|\omega^*\|_{\bar{g}}^* = 1$ on U_ϵ.

Proof. For any point q on U_ϵ. By the formula in *Remark 2.1.2*, we only need to prove that, the denominator,

$$\min\{\|W\|_{\bar{g}} : W \text{ is a simple } m\text{-vector at } q \text{ with } \omega^*(W) = 1\} \tag{3.1.2}$$

equals one. Suppose $\tilde{W} \in T_q(U_\epsilon)$ with $\omega^*(\tilde{W}) = 1$ but $\tilde{W}$ not a purely horizontal m-vector. Say $\tilde{W} = \tilde{W}^h + \tilde{W}^\nu$ is the decomposition to purely horizontal part and fiber-involving part. By the definition of $\pi_g^*(g|_M^h)$, we have $\omega^*(\tilde{W}^\nu) = 0$, so $\omega^*(\tilde{W}^h) = \omega^*(\tilde{W}) = 1$ with $\|\tilde{W}^h\|_{\bar{g}} < \|\tilde{W}\|_{\bar{g}}$. Therefore simple m-vectors realizing (3.1.2) must be purely horizontal. Furthermore it is unique by the reason of dimension. Name it $\bar{W}$. Then

$$1 = \omega^*(\bar{W}) = \frac{s\tilde{\omega}(\bar{W})}{V} = \frac{s\omega(\pi_*^g \bar{W})}{V} = \frac{s\|\pi_*^g \bar{W}\|_{g|_M^h}}{V} = \|\pi_*^g \bar{W}\|_{(\frac{s}{V})^{\frac{2}{m}} g|_M^h} = \|\bar{W}\|_{\bar{g}}.$$

Hence the claim follows. ∎

Remark 3.1.1 *From the proof, we notice that, it is sufficient to consider the unique horizontal m-vector $\bar{W}$, when one computes the comass of a multiple of the pull-back volume form of M via π_g^*. This is actually valid for any metric, which preserves horizontal directions of g. In particular, it works for the conformal change of metrics in §4.*

Let us take $\tilde{g} = \tilde{g}^h \oplus \tilde{g}^\nu$ by gluing horizontal and vertical metrics respectively:

$$\tilde{g}^h = \sigma^{\frac{1}{m}}((\frac{s}{V})^{\frac{2}{m}} + \mathbf{d}^2)\pi_g^*(g|_M^h) + (1-\sigma)^{\frac{1}{m}}\alpha g^h, \tag{3.1.3}$$

$$\tilde{g}^v = \sigma g_F + (1-\sigma)\alpha g^v, \tag{3.1.4}$$

where $\mathbf{d}$ is the distance function to M with respect to g, where α is a positive function (to be determined) defined on X, and where $\sigma = \sigma(\mathbf{d})$ is given as below: If we set g_F to be αg^v, then

$$\tilde{g}^v = \alpha g^v. \tag{3.1.5}$$

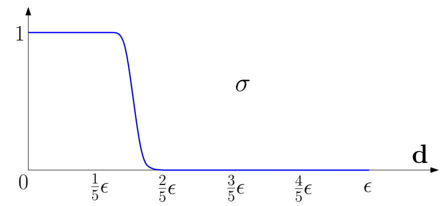

Figure 3.1: Graph of σ.

Choose appropriate α with following properties pointwisely:

$$\|\Phi\|^*_{\alpha g} < 1 \text{ on } X, \text{ and} \tag{3.1.6}$$

$$\alpha^m \|\bar{W}\|^2_g > 1 \text{ on } U_\epsilon. \tag{3.1.7}$$

$$\|\Phi\|^*_{\tilde{g}} = \begin{cases} 1 & 0 = \mathbf{d} \\ \|\omega^*\|^*_{\tilde{g}} = \frac{1}{\|\bar{W}\|_{\tilde{g}}} = \frac{1}{\|\bar{W}\|_{((\frac{s}{V})^{\frac{2}{m}} + \mathbf{d}^2)\pi^*_{\tilde{g}}(g|^h_M)}} < 1 & 0 < \mathbf{d} \leq \frac{1}{5}\epsilon \\ \begin{aligned} \|\omega^*\|^*_{\tilde{g}} &= \|\omega^*\|^*_{[\sigma^{\frac{1}{m}}((\frac{s}{V})^{\frac{2}{m}} + \mathbf{d}^2)\pi^*_g(g|^h_M) + \alpha(1-\sigma)^{\frac{1}{m}} g^h] \oplus \tilde{g}^v} \\ &\leq \frac{1}{\sqrt{\sigma \|\bar{W}\|^2_{((\frac{s}{V})^{\frac{2}{m}} + \mathbf{d}^2)\pi^*_{\tilde{g}}(g|^h_M)} + \alpha^m (1-\sigma)\|\bar{W}\|^2_{g^h}}} \\ &< 1 \end{aligned} & \frac{1}{5}\epsilon \leq \mathbf{d} \leq \frac{2}{5}\epsilon \\ \|\Phi\|^*_{\alpha g} < 1 & \frac{2}{5}\epsilon \leq \mathbf{d} \end{cases}$$

The second inequality is due to *Lemma 2.1.11*, the third by *Property 2* and (3.1.7), and the last by (3.1.6). In summary, we construct a closed m-form Φ and a metric $\tilde{g}$ satisfying

1. $\|\Phi\|^*_{\tilde{g}} \leq 1$, and

2. as a pointwise function, $\|\Phi\|_{\tilde{g}}^{*} = 1$ exactly on M.

Remark 3.1.2 *If X is compact, then α can be chosen as a sufficiently large constant which guarantees (3.1.6) and (3.1.7). It is clear that the same constant α still works if one shrinks the gluing neighborhood U_ϵ.*

3.2 X Compact and M Connected

Definition 3.2.1 *We say that a submanifold M is* **tamed** *in a calibrated manifold (X, ϕ, g), if each connected component of M is calibrated by either ϕ or $-\phi$ and as a pointwise function $\|\phi\|_g^*$ equals one exactly on M.*

Remark 3.2.2 *Note that mass has nothing to do with orientations of submanifolds. By theorem 2.1.8, each component of a tamed submanifold is mass minimizing in its current homology class. In particular, M is minimal in the sense of Riemannian Geometry.*

Theorem 3.2.3 *Suppose that (X, g) is a* `compact` *Riemannian manifold and that M is an oriented compact connected m-dimensional submanifold with $[M]$ non-zero in $H_m(X;\mathbb{R})$. Then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a horizontal change of g in U together with a closed m-form $\hat{\Phi}$ on X, such that M is tamed by $(X, \hat{\Phi}, \hat{g})$.*

Proof. By the compactness of M, there exists a small positive number ϵ which guarantees both the disc bundle structure of U_ϵ stated in §2.3 and $U_\epsilon \subset U$. According to §2.2 and §2.4, there exists a closed m-form Φ in (2.4.1). By §3.1 and *Remark* 3.1.2, a new metric $\tilde{g}$ can be constructed by the horizontal change (3.1.3) and (3.1.4) of g on U_ϵ with $\tilde{g} = \alpha g$ on $X - U_\epsilon$, where α is a large constant satisfying (3.1.6) and (3.1.7). Define $\hat{g} \triangleq \alpha^{-1}\tilde{g}$ and $\hat{\Phi} \triangleq \alpha^{-\frac{m}{2}}\Phi$. *Lemma 2.1.9*, gluing of metrics (3.1.3) and (3.1.4) imply that M is strongly calibrated by $(\hat{\Phi}, \hat{g})$ and that $\hat{g}$ equals g on $X - U_\epsilon$. ∎

Remark 3.2.4 *It is clear that $\hat{g}$ is far away from being unique. By the construction of (3.1.1), (3.1.3) and (3.1.4), the volume of M with respect to $\tilde{g}$ (no matter which initial metric we use) is always $\int_M \Phi$.*

With respect to each metric produced by our method, locally M is stronger than being minimal.

Proposition 3.2.5 *M is totally geodesic under the constructed $\hat{g}$ in the proof of Theorem 3.2.3.*

Proof. The only thing that we need to check is for any sufficiently close pair of points (p, q) on M, the shortest geodesic connecting them in $(X, \hat{g})$ stays in M. Here we assume that p and q are close enough to each other such that:

1. p lies in the strongly convex neighborhood of q, and
2. $\text{dist}_{\hat{g}}(p, q) < \frac{2}{5}\epsilon$ (the same ϵ in previous constructions).

Then there exists a unique shortest geodesic segment γ between them contained in $U_{\frac{1}{5}\epsilon}(M)$ (under either g or $\hat{g}$ because their ϵ-neighborhoods coincide by *Appedix .3*).

Our strategy is to prove it by contradiction. Suppose γ is not entirely in M as illustrated in the picture, then we can use projection π_g mapping γ down to a curve $\bar{\gamma}$ in M. For any point x on γ, we have a tangent vector w of γ at x with decomposition $w = w^h + w^\nu$. Since in $U_{\frac{1}{5}\epsilon}(M)$, $\hat{g}^h = \alpha^{-1}((\frac{s}{V})^{\frac{2}{m}} + \mathbf{d}^2)\pi_g^*(g|_M^h)$,

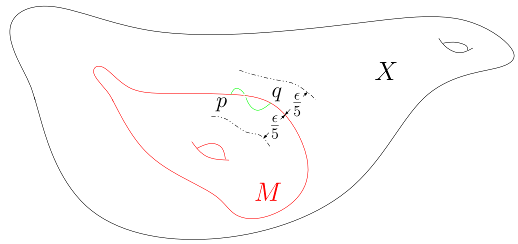

Figure 3.2: Graph for being totally geodesic.

we have

$$|w|_{\hat{g}} \geq |w^h|_{\hat{g}^h} \geq |w^h|_{\alpha^{-1}(\frac{s}{V})^{\frac{2}{m}}\pi_g^*(g|_M^h)} = |\pi_*^g(w)|_{\hat{g}|_M^h}.$$

So π_*^g is a length shrinking projection and $\bar{\gamma}$ will not be longer than γ. This is a contradiction with the fact that γ is the unique shortest curve connecting p and q. Hence we are done. ∎

Remark 3.2.6 *Since being minimal (or totally geodesic) in the sense of Riemannian Geometry is a local problem, from our argument, it is clear that any oriented compact submanifold can be realized minimal (or totally geodesic) with respect to some metric on the ambient manifold. In other words, there are no topological obstructions for an oriented compact submanifold to be minimal (or totally geodesic), if the ambient metric is allowed to vary. More generally, it also holds for a neat collection of countably many disjoint submanifolds (defined in §3.5), which means that all components of a neat countable collection will be minimal (or totally geodesic) simultaneously with respect to a common metric. Similar results for conformal change will be obtained in §4 for minimality.*

3.3 X Compact and M Non-Connected

Definition 3.3.1 *A family $\mathfrak{M}$ of disjoint connected oriented compact submanifolds of a manifold X (not necessarily compact) is called a* **collection** *and an element of $\mathfrak{M}$ is a* **component**. *The subset $\mathfrak{M}_k$ of all components of dimension k is its* **k-level**. *When it consists of finitely (or countably) many components, we call $\mathfrak{M}$ a* **finite** *(or* **countable***)* **collection**. *If $\mathfrak{M}$ has only one level in dimension m, then it is called an* **m-collection**. *We will let M denote the union of the components of $\mathfrak{M}$.*

From now on, we assume that X is compact in this section.

Theorem 3.3.2 *Suppose that M is a finite m-collection in (X, g). If the convex hull $\mathcal{C} = \{\sum_{i=1}^{s} t_i[M_i] : \sum_{i=1}^{s} t_i = 1 \text{ and } t_i \geq 0\}$ in $H_m(X;\mathbb{R})$ does not contain zero, then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a horizontal change of g in U together with a smooth closed m-form $\hat{\Phi}$ on X, such that every current of the form $T = \sum_{i=1}^{s} t_i M_i$, where t_i is non-negative with $\sum_{i=1}^{s} t_i > 0$, is calibrated in $(X, \hat{\Phi}, \hat{g})$ and consequently is mass minimizing in $[T]$ with $\mathbf{M}(T) = \sum_{i=1}^{s} t_i \cdot Vol_{\hat{g}}(M_i)$.*

If we drop the convex hull condition but still require that each component class is non-zero in the $\mathbb{R}$-homology of X , we can choose a hyperplane $\mathcal{P}_m$ through zero in $H_m(X;\mathbb{R})$, which avoids all the component classes $\{[M_i]\}_1^s$. $\mathcal{P}_m$ gives us two open chambers in $H_m(X;\mathbb{R})$. If we reverse orientations for components of $\mathfrak{M}$ in one chamber, then they are flipped into the other chamber through zero. Now all components of $\mathfrak{M}$, probably some of which are equipped with different orientation, satisfy the convex hull condition.

Corollary 3.3.3 *Let $\mathfrak{M}$ be a finite collection in a compact Riemannian manifold (X, g) with each component class non-zero in the $\mathbb{R}$-homology of X. Then* *for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a horizontal change of g in U, such that M is tamable in $(X, \hat{g})$.*

Proof of Theorem 3.3.2. By §2.2, the convex hull condition implies that there exists a smooth closed m-form ϕ on X such that $\int_{M_i} \phi > 0$ for every M_i. Since $\mathfrak{M}$ is a finite collection, we can choose a small positive ϵ (as described in §2.3) such that $\{U_\epsilon(M_i)\}$ are disjoint and contained in U. Now we use (2.4.1), the previous gluing procedure for forms in $U_\epsilon \subset U$, to obtain Φ.

For the metric gluing, we need pay attention to those constants $\alpha_i \triangleq \alpha(M_i)$ (see *Remark* 3.1.2) for each component. Although generally $\{\alpha_i\}$ are different, we can simply take α to be the maximal one to construct a new metric g_α by the method in §3.1., i.e., to glue local metrics of with αg. Easy to see that g_α makes Φ a calibration form. Let $\hat{g} \triangleq \alpha^{-1} g_\alpha$ and $\hat{\Phi} \triangleq \alpha^{-\frac{m}{2}} \Phi$, then each M_i is calibrated by $\hat{\Phi}$ in $(X, \hat{g})$.

Since $\{M_i\}$ are submanifolds, the measure $\|M_i\|$ induced by M_i is $\mathcal{H}|_{M_i}$, where $\mathcal{H}$ is the *Hausdorff* m-measure of $(X, \hat{g})$. Suppose $T = \sum_{i=1}^s t_i M_i$ with $t_i \geq 0$, then $\|T\| = \sum_i^s t_i \mathcal{H}|_{M_i}$ and

$$\begin{aligned}\mathbf{M}(T) &= \int |\overrightarrow{T}|_{\hat{g}} \; d\|T\| = \sum t_i \cdot \int_{M_i} |\overrightarrow{M}_i|_{\hat{g}} \; d\|M_i\| \\ &= \sum t_i \cdot \int_{M_i} d \; vol_{\hat{g}} = \sum t_i \cdot \mathrm{Vol}_{\hat{g}}(M_i).\end{aligned}$$

∎

Remark 3.3.4 *By Remark 3.2.4, the volume of M_i with respect to g_α (of any* *initial metric) is always* $\int_M \Phi$. *If* $[M_i] = [M_j]$, *it follows automatically that* $Vol_{g_\alpha}(M_i) = Vol_{g_\alpha}(M_j)$. *Therefore,* $Vol_{\hat{g}}(M_i) = Vol_{\hat{g}}(M_j)$.

Proposition 3.3.5 *For the metric* $\hat{g}$ *constructed in the proof of Theorem (3.3.2) or Corollary (3.3.3), we have* $\mathrm{dist}_{\hat{g}}(M_i, M_j) = \mathrm{dist}_g(M_i, M_j)$.

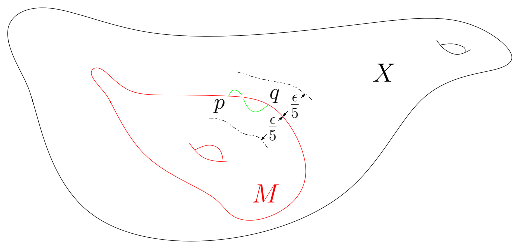

Figure 3.3: Graph for same distance.

Proof. Suppose that we do the horizontal change of g in U_ϵ and that γ (in the above picture) is one geodesic segment from p to q which realizes the distance between M_i and M_j with respect to g. According to §2.3, we know that $\gamma \cap U_\epsilon(M_i)$ is contained in a single fiber and it is $\{\exp_p(tv) : v = \gamma'_p/\|\gamma'_p\| \text{ and } t \in [0, \epsilon)\}$. The same phenomenon also happens around M_j. Since the fiber (vertical) part of g is unchanged,

$$\mathrm{dist}_{\hat{g}}(M_i, M_j) \leq l_{\hat{g}}(\gamma) = l_g(\gamma) = \mathrm{dist}_g(M_i, M_j). \tag{3.3.1}$$

From *Appendix .3*, we understand that the bundle structure will not change, namely the disc fiber with respect to g is exactly the disc fiber with respect to $\hat{g}$. By the same argument, we can prove the opposite inequality of (3.3.1) and the proof is complete. ∎

3.4 X Compact and M of Different Dimensions

In this section, $\mathfrak{M}$ will be assumed to be a finite collection (not necessarily of a single level) in a compact Remannian manifold (X, g). Then similar conclusions as in *Theorem 3.3.2* and *Corollary 3.3.3* can be proved.

Theorem 3.4.1 *Suppose that $\mathfrak{M}$ is a finite collection in a compact Riemannian manifold (X, g). For each k, let $\mathcal{C}_k \subset H_k(X; \mathbb{R})$ denote the convex hull of homology classes represented by components in k-level. Suppose $\mathcal{C}_k$ does not contain zero for each k. Then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a horizontal change of g supported in U, such that there exist a family of calibration forms $\{\hat{\Phi}_k\}$ in $(X, \hat{g})$, and every current of the form $T = \sum_{i=1}^{s_k} t_i M_i$, with $M_i \in \mathfrak{M}_k, t_i \geq 0$ and $\sum_{i=1}^{s_k} t_i > 0$, is calibrated by $\hat{\Phi}_k$. Consequently, each T is mass minimizing in its current homology class $[T]$ with mass $\mathbf{M}(T) = \sum_{i=1}^{s} t_i \mathrm{Vol}_{\hat{g}}(M_i)$.*

The same reason for *Corollary 3.3.3* leads to the following analog.

Corollary 3.4.2 *Let $\mathfrak{M}$ be a finite collection in a compact Riemannian manifold (X, g) with each component class non-zero in the $\mathbb{R}$-homology of X. Then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a horizontal change of g in U, such that each level $\mathfrak{M}_k$ is tamable in $(X, \hat{g})$.*

Proof of Theorem 3.4.1. Without loss of generality, consider the case of two levels with a single component in each. Denote these two components by A^a and B^b, where $a > b$. Then there exist an a-form ϕ and a b-form ψ with $\int_A \phi > 0$ and $\int_B \psi > 0$. Hence we can construct Φ and Ψ as in §2.4 for A and B respectively by gluing method in $U_\epsilon(A)$ and $U_\epsilon(B)$ (for sufficiently small

ϵ with the bundle structure described in §2.3 such that $U_\epsilon(A) \cap U_\epsilon(B) = \boldsymbol{\phi}$, $U_\epsilon(A \cup B) \subset U$). By the difference of dimensions, $\Phi = d\theta$ in $U_\epsilon(B)$. Take $\tilde{\Phi}$ to be $\Phi - d(\tilde{\rho}(\mathbf{d})\theta)$, where $\tilde{\rho}$ is given as:

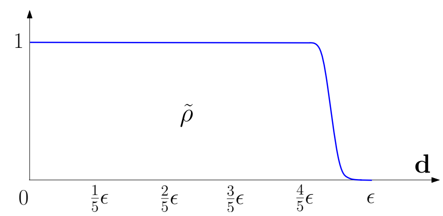

Figure 3.4: Graph for $\tilde{\rho}$.

Then $\tilde{\Phi}$ is zero on $U_{\frac{4}{5}\epsilon}(B)$. Based on the pair $(\tilde{\Phi}, g)$, we can get a new metric by gluing method in §3.1 such that A is calibrated by $(\tilde{\Phi}, \tilde{g})$. The same procedure leads us from $(\Psi, \tilde{g})$ to (Ψ, g') and the latter calibrates B.

Furthermore, we can obtain a triple $(\hat{\Phi}, \hat{\Psi}, \hat{g})$ as follows. By our construction, there exist constants λ and μ with $\tilde{g} = \lambda g$ away from $U_{\frac{2}{5}\epsilon}(A)$ and $g' = \mu\tilde{g}$ away from $U_{\frac{2}{5}\epsilon}(B)$ (see §3.1). Let $\hat{\Phi} = \lambda^{-\frac{a}{2}}\tilde{\Phi}$, $\hat{\Psi} = (\lambda\mu)^{-\frac{b}{2}}\Psi$ and $\hat{g} = (\lambda\mu)^{-1}g'$. Therefore, $\hat{g} = g$ away from $U_\epsilon(A \cup B)$. The punch line here $\|\hat{\Phi}\|^*_{\hat{g}} = \|\tilde{\Phi}\|^*_{\mu^{-1}g'} = \|\tilde{\Phi}\|^*_{\tilde{g}} < 1$ in $(U_{\frac{2}{5}\epsilon}(B))^c$ and meanwhile $\|\hat{\Phi}\|^*_{\hat{g}} = \|0\|^*_{\hat{g}} = 0$ in $U_{\frac{4}{5}\epsilon}(B)$. So $(\hat{\Phi}, \hat{g})$ and $(\hat{\Psi}, \hat{g})$ calibrate A and B respectively. ∎

Remark 3.4.3 *Another proof is to use the difference in dimensions and choose a proper constant factor α for each level without changing potential calibration forms. However it is a little lengthy and tricky but invalid for the conformal change in §4. So we omit it and prefer the above proof.*

Proposition 3.4.4 *For the metric $\hat{g}$ constructed in the proof of Theorem 3.4.1 or Corollary 3.4.2, any two components have the same distance as that with* *respect to g.*

Proof. Exactly the same argument as in Proposition 3.3.5. ∎

3.5 X Non-Compact with No Boundaries

We are able to extend our gluing methods to the case of non-compact ambient manifolds. However, due to the lack of compactness, some necessary modification has to be made correspondingly. In contrast to *Remark 3.1.2* for compact case, generally speaking, a constant function α can not meet the need of (3.1.6). For example, consider the surface X, obtained by rotating the graph of $y = e^x$ around the x-axis, with the induced metric from Euclidean $\mathbb{R}^3$. Take γ as the circle corresponding to $x = 0$. Then it is obvious that, without a global change of metric, there is no way to make γ homologically minimal.

This dissimilarity forces us to search a globally defined function α satisfying (3.1.6) and (3.1.7) for each component. After this necessary modification, similar results can be proved for the case of X non-compact and M a finite collection. The proof of *Theorem 3.5.1* given below also works for *Theorem 3.4.1*.

Theorem 3.5.1 *Suppose that $\mathfrak{M}$ is a finite collection in (X, g) with each component class non-zero in the $\mathbb{R}$-homology of X. Then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a horizontal change of g in U and a conformal change in $X - U$, such that each $\mathfrak{M}_k$ is tamable in $(X, \hat{g})$.*

Proof. Similar to the previous, for a fixed level k of $\mathfrak{M}$, we can choose a hyperplane $\mathcal{P}_k$ in $H_k(X; \mathbb{R})$ avoiding the corresponding classes of k-level, and then change orientations for components in one chamber. Then there exists a closed smooth k-form ϕ_k with positive integral over each k-dimensional component.

Without loss of generality, consider the case of two levels, each of which contains only one component. Let $M = M_1^{k_1} \coprod M_2^{k_2}$ with $k_1 > k_2$. There exist $\{\epsilon_i\}$ such that each $U_{\epsilon_i}(M_i^{k_i})$ has the bundle structure mentioned in §2.3 and that the closures of $U_{\epsilon_i}(M_i^{k_i})$ are disjoint compact sets. Then for level k_i, we can obtain forms Φ_{k_i} by gluing ϕ_{k_i} with local forms as in §2.4 on $U_{\epsilon_i}(M_i^{k_i})$ respectively.

Eliminate Φ_{k_1} on $U_{\frac{4}{5}\epsilon_2}(M_2^{k_2})$ to $\tilde{\Phi}_{k_1}$ as in the proof of *Theorem 3.4.1*. There exists a function α_{k_1} equal to one on $U_{\frac{4}{5}\epsilon_2}(M_2^{k_2})$ and satisfying (3.1.6) and (3.1.7) for g and $\tilde{\Phi}_{k_1}$. Using α_{k_1}, apply the horizontal change for g and $\tilde{\Phi}_{k_1}$ on level k_1 and denote the result metric $\tilde{g}$. Again, there exists a function α_{k_2} satisfying (3.1.6) and (3.1.7) for $\tilde{g}$ and Φ_{k_2}. Set

$$C = \max\{\alpha_{k_2} \text{ on the closure of } U_{\epsilon_1}(M_1^{k_1})\}$$

and define $\tilde{\Phi}_{k_2} \triangleq C^{-\frac{k_2}{2}} \cdot \Phi_{k_2}$. Since

$$\|\Phi_{k_2}\|^*_{\alpha_{k_2}\tilde{g}}| < 1 \text{ on } U_{\epsilon_1}(M_1^{k_1}) \subset X,$$

by *Lemma 2.1.9 and 2.1.10*, we have

$$\|\tilde{\Phi}_{k_2}\|^*_{\tilde{g}} < 1 \text{ pointwisely on } U_{\epsilon_1}(M_1^{k_1}).$$

This means that, there exists a positive function $\tilde{\alpha}(\geq 1)$, which equals to one on $U_{\epsilon_1}(M_1^{k_1})$, and satisfies (3.1.6) and (3.1.7) for $\tilde{g}$ and $\tilde{\Phi}_{k_2}$.

Now apply the horizontal change for $\tilde{g}$ and $\tilde{\Phi}_{k_2}$ on level k_2 and denote the result metric by $\hat{g}$. It is not hard to see that $M_1^{k_1}$ and $M_2^{k_2}$ are calibrated by

$\tilde{\Phi}_{k_1}$ and $\tilde{\Phi}_{k_2}$ respectively in $(X, \hat{g})$. By our construction, it is clear that the metric change is horizontal on U and conformal on its complement. ∎

Actually we can get the same conclusion (see *Theorem 3.5.5* below), when M is a suitable countable m-collection. In order to use gluing techniques, we introduce the following definitions.

Definition 3.5.2 *Let $\mathfrak{M} = \{M_i\}_{i=1,2,\cdots}$ be a countable collection in X. If for any positive integer j, the set $\cup_{i \neq j} M_i$ is closed, then we call $\mathfrak{M}$ a* **neat collection**.

Remark 3.5.3 *The neatness implies that*

$$\overline{M} = M, \text{ and}$$

$$\overline{M - M_j} \cap M_j = \emptyset.$$

These can guarantee that the modified metrics via horizontal change (or conformal change in §4) will be smooth.

Remark 3.5.4 *Since each M_i is compact, the neatness implies that $d_i \triangleq \mathrm{dist}_g(M_i, M - M_i) > 0$. Let $\epsilon_i = \frac{1}{3} d_i$, then $\{U_{\epsilon_i}(M_i)\}$ are disjoint and of positive distance from each other.*

For instance, on $X = \mathbb{R}^2 - \mathbb{Z}^2$ where $\mathbb{Z}^2$ is the lattice of points with integer coordinates. Let $M = \{(x, y) : \ \exists\, k, l \in \mathbb{Z} \text{ with } (x-k)^2 + (y-l)^2 = 0.1\}$.

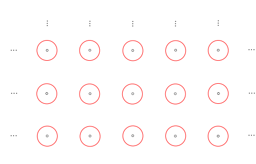

Figure 3.5: A neat collection.

Theorem 3.5.5 *Suppose that* $\mathfrak{M}$ *is a neat collection in* (X, g) *with each component class non-zero in the* $\mathbb{R}$*-homology of* X*. In addition, assume that every level of* $\mathfrak{M}$ *consists of finite components except the lowest level. Then for any open neighborhood* U *of* M*, a new metric* $\hat{g}$ *can be constructed by a horizontal change of* g *in* U *and a conformal change in* $X - U$*, such that each* $\mathfrak{M}_k$ *is tamable in* $(X, \hat{g})$*.*

Proof. It follows by the proof of *Theorem 3.5.1*. Note that the neatness guarantees the smoothness of the result metric. Also note that our method fails to descend from a level of infinitely many components. ∎

Chapter 4

Conformal Change of Metrics

4.1 Parallel Results to Chapter 3

Let us move to, in some sense, more interesting and clean case − change metrics in a fixed conformal class. In order to understand how our previous gluing techniques work for this kind of change, it is worth a glance at the simplest case first.

Theorem 4.1.1 *Suppose that* (X, g) *is a* `compact` *Riemannian manifold and* M *is an oriented compact connected* m*-dimensional submanifold with* $[M]$ *non-zero in* $H_m(X;\mathbb{R})$*. Then for any open neighborhood* U *of* M*, a new metric* $\hat{g}$ *can be constructed by a conformal change of* g *supported in* U*, such that there exists a closed smooth* m*-form* $\hat{\Phi}$ *defined on* X*, and* M *is tamed in* $(X, \hat{\Phi}, \hat{g})$*.*

Proof. By discussions in §2.2, we can find some closed m-form ϕ of X with positive integral over M. Based on ϕ, we can construct an m-form Φ described in §2.4 with an appropriate choice of ϵ. A key point here is that Φ is a multiple of $\pi_g^*\omega$ in $U_{\frac{3}{5}\epsilon}(M)$, namely a simple m-form. Therefore, its comass is smooth as a pointwise function in $U_{\frac{3}{5}\epsilon}(M)$. In particular, the comass is realized by the reciprocal of the norm of the unique oriented horizontal simple m-vector (see *Remark 3.1.1*). Take

$$g' = (\|\Phi\|_g^*)^{\frac{2}{m}} g,$$

then

$$\|\Phi\|_{g'}^* = 1 \text{ in } U_{\frac{3}{5}\epsilon}(M).$$

Now we can play similar gluing tricks. Let

$$\tilde{g} = \sigma^{\frac{1}{m}}(1 + \mathbf{d}^2) g' + \alpha (1-\sigma)^{\frac{1}{m}} g,$$

where $\sigma, \mathbf{d}$ and α are the same as given in §3.1. Then choose a constant α such that (3.1.6) and (3.1.7) hold. Set $\hat{g} \triangleq \alpha^{-1}\tilde{g}$ and $\hat{\Phi} \triangleq \alpha^{-\frac{m}{2}}\Phi$. It is easy to see by *Lemma 2.1.9* that M is tamed by $(\hat{\Phi}, \hat{g})$ and that $\hat{g}$ is in the conformal class of g with $\hat{g} = g$ in $X - U_\epsilon(M)$. (In fact, $\hat{g}$ and g are the same on $X - U_{\frac{2}{5}\epsilon}(M)$.) ∎

Remark 4.1.2 *For a curve, to be minimal and to be totally geodesic are equivalent. However this is not true for higher dimensional cases. In contrary to Proposition 3.3.5, one can not always make a submanifold totally geodesic by a conformal change of metric. Due to Appendix .1, M can be realized totally geodesic by a conformal change if and only if M is totally umbilical.*

The same local gluing ideas and elimination tricks for calibrations lead to the following results.

Theorem 4.1.3 *Suppose that $\mathfrak{M}$ is a finite collection in a* `compact` *Riemannian manifold (X, g). For each k, let $\mathcal{C}_k \subset H_k(X;\mathbb{R})$ denote the convex hull of all the components in k-level. If $[0] \notin \mathcal{C}_k$ for all k, then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a conformal change* *of g supported in U, such that there exists a family of smooth closed forms $\{\hat{\Phi}_k\}_{j=1}^s$, and each $\mathfrak{M}_k$ is strongly calibrated in $(X, \hat{\Phi}_k, \hat{g})$. Moreover, for each k-level, every current of the form $T = \sum_{i=1}^{r_k} t_i M_i$, where M_i belongs to k-level and each t_i is non-negative with $\sum_{i=1}^{r_k} t_i > 0$, is calibrated by $(\hat{\Phi}_k, \hat{g})$ and consequently is mass minimizing in the current homology class $[T]$ with $\mathbf{M}(T) = \sum_{i=1}^{s} t_i \cdot Vol_{\hat{g}}(M_i)$.*

Corollary 4.1.4 *Suppose that $\mathfrak{M}$ is a finite collection in a compact Riemannian manifold (X, g) with each component class non-zero in the $\mathbb{R}$-homology class of X. Then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a conformal change of g supported in U, such that each level is tamable in $(X, \hat{g})$.*

When X is non-compact (without boundary), we can prove the following nicer results compared with *Theorem 3.5.1* and *Theorem 3.5.5*.

Theorem 4.1.5 *Suppose that $\mathfrak{M}$ is a finite collection in (X, g) with each component class non-zero in the $\mathbb{R}$-homology of X. Then a new metric $\hat{g}$ can be constructed by a conformal change of g, such that each level is tamable in $(X, \hat{g})$.*

Theorem 4.1.6 *Suppose that $\mathfrak{M}$ is a neat collection in (X, g) with each component class non-zero in the $\mathbb{R}$-homology of X. In addition, assume that each level of M consists of finite components except the lowest level. Then a new metric $\hat{g}$ can be constructed by a conformal change of g, such that each level is tamable in $(X, \hat{g})$.*

Remark 4.1.7 *Suppose that (X, g) is hermitian with J as initial setups. Then the metric $\hat{g}$ in the above theorems and corollary is hermitian with respect to J as well. However due to the gluing process, generally speaking, no better* *regularity beyond C^∞ can be guaranteed. If we start with an non-analytic connected submanifold M in an analytic Kähler manifold (X, J, g), then M can not be calibrated in X with respect to an analytic $\hat{g}$. Because a regularity result of Morrey [17] says that if M is a C^2-differentiable minimal submanifold in a C^r ($r \geq$ or ω) Riemannian manifold (X, g), then M must has C^r-regularity.*

Corollary 4.1.8 *Suppose that M is an oriented compact submanifold in a Riemannian manifold (X, g). Then there exists $\hat{g}$ in the conformal class of g, such that M is minimal in $(X, \hat{g})$.*

Proof. By local calibrations. ∎

4.2 On Mean Curvature Vector Field

Let us take a short digression, which, in some sense, is a generalization of *Remark 4.1.2* and *Remark 4.1.7* by considering the following question:

Logically, since conformal change of metric can eliminate any initial mean curvature vector field of an oriented submanifold, it should be true that certain conformal change of metric can generate any expected smooth mean curvature vector field along the submanifold.

Theorem 4.2.1 *For an oriented compact submanifold M in (X, g) and arbitrary smooth section ξ in the normal bundle over M, one can conformally change g to $\breve{g}$ such that the mean curvature vector field alomg M with respect to $\breve{g}$ is ξ.*

Proof. First of all, by *Corollary 4.1.8*, one can conformally change g to $\hat{g}$ such that M is minimal in $(X, \hat{g})$. Since the ϵ-neighborhood is identified with the normal ϵ-disc bundle of M via exponential map restricted to normal directions, we can define a smooth function on U_ϵ by constructing a smooth function on $T^\perp M$:

$$f_x(y) = 1 - \frac{2}{m} < \xi_x,\ y >,$$

where x is a point on M and y lies in ϵ-disc fiber of $M \subset T_x^\perp M$. Then $f_x(0) = 1$ and $grad_g^\perp(f_x)(0) = -\frac{2}{m}\xi_x$ (since $d\exp(\cdot)$ is identity along M). Take $\breve{g} = f \cdot \hat{g}$. By *Appendix .1*,

$$\begin{aligned} H_{\breve{g}} &= H_{\hat{g}} - \tfrac{m}{2} \cdot grad_g^\perp f \\ &= 0 + \tfrac{m}{2} \cdot \tfrac{2}{m} \cdot \xi_x \\ &= \xi_x. \end{aligned}$$

∎

Remark 4.2.2 *In fact, one can prove the theorem directly by Appendix .1 without bothering Corollary 4.1.8. The orientability and compactness can be removed and the theorem is also valid for a neat collection.*

4.3 M Non-Compact

It is natural, at this point, to ask questions like − How about the case when M is non-compact? Can we generalize our results to this case? One difficulty is that **Mass**(M) is usually infinity because of the non-compactness. Although we could, sometimes, rescale it to be finite in some sense, it seems not quite natural. We would like to start with the simple case of dimension one.

Definition 4.3.1 *An* **infinite line** *in a Riemannian manifold is a connected properly embedded curve with infinite length for both directions. (Equivalently,* *it is complete with respect to the induced metric.)*

Definition 4.3.2 *A* **geometric line** *in a Riemannian manifold is an infinite line such that the length of any connected segment on it realizes the distance between the ending points in the ambient manifold.*

An interesting question is that suppose we have an infinite line in an non-compact Riemannian manifold (X, g) without boundary (no matter whether X is complete or not), can we make this infinite line a geometric line by a conformal change of metric? The answer can be asserted affirmatively.

Theorem 4.3.3 *Suppose that L is an infinite line in (X, g). Then for any open neighborhood U of L, a new metric $\hat{g}$ can be constructed by a conformal change of g supported in U, such that L is a geometric line with respect to $\hat{g}$. Moreover, if the initially metric g is complete, so is $\hat{g}$.*

Remark 4.3.4 *By a theorem of Nomizu and Ozeki in [19], every metric is conformal to a complete metric, so it is not very important to require that g is complete.*

Proof. By *Appendix .2*, after a conformal change from g to g', we have a good region $U_1(L; g')$, the set of points within distance one to L with respect to g', contained in U. Note that there is a fibration structure induced by g. The set within distance one under g' to L along the fibers is in $U_1(L; g')$.

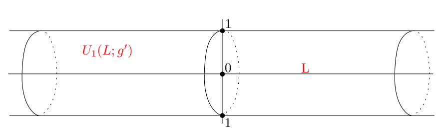

Figure 4.1: A good region.

Since L is not compact, its volume form induced by g is exact, i.e., df for some smooth function f defined on L. Set

$$\phi \triangleq d(\tau(\mathbf{d}^{g'})\pi_g^* f),$$

where $\mathbf{d}^{g'}$ is the distance function along fibers to L with respect to g' and τ looks like:

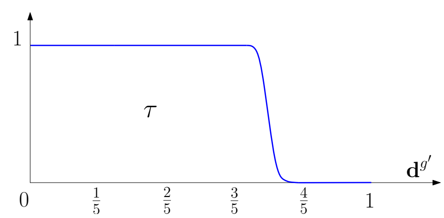

Figure 4.2: Graph of τ.

So ϕ is closed and it is the pull back of volume form of L by π_g^*, when $\mathbf{d}^{g'}$ is no bigger than $\frac{3}{5}$. Then $\|\phi\|_{g'}^*$ is smooth on $U_{\frac{3}{5}}(L;g')$. Take $\bar{g} = (\|\phi\|_{g'}^*)^2 g'$. We have $\|\phi\|_{\bar{g}}^* = 1$ on $U_{\frac{3}{5}}(L;g')$. Therefore we can continue our gluing procedures as before. Let

$$\tilde{g} = \bar{\sigma}^{\frac{1}{m}}(1 + (\mathbf{d}^{g'})^2)\bar{g} + \alpha(1-\bar{\sigma})^{\frac{1}{m}} g,$$

where α is a smooth function to be determined such that $\|\phi\|_{\tilde{g}}^* \leq 1$ on $U_1(L;g')$, and $\bar{\sigma}$ is chosen as: One construction of α is this. Set $h = \|\phi\|_{g'}^* + 2$ and

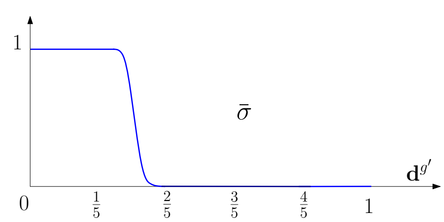

Figure 4.3: Graph of $\bar{\sigma}$.

smoothen it to $\tilde{h}$ with $\|h - \tilde{h}\|$ no bigger than one in the C^0-norm (see [12]).

Then $\tilde{h}$ is a smooth function bigger than $\|\phi\|^*_{g'}$ on $U_1(L;g')$. Now choose $\tilde{h}^2$ to be α. By *Theorem 2.1.2* for $m=1$, ϕ tames L in $(X,\tilde{g})$ with the following properties.

(1). $\|\phi\|^*_{\tilde{g}} \leq 1$ and the equality holds exactly on L,

(2). $\phi = 0$ in $X - U_{\frac{4}{5}}(L;g')$, and

(3). By *Stokes' Theorem*, $\int_{\tilde{\gamma}} \phi = 0$, where $\tilde{\gamma}$ is any curve with ending points on $\partial U_1(L;g')$.

For a pair of points p and q on L, we have a segment $[p,q]$ of L connecting them. Suppose that another different (not a reparametrized) curve γ also have the ending points p and q, then there are two possibilities.

The first is that the whole curve γ stays in $U_1(L;g')$, which implies $[p,q]$ and γ are homologous and by calibration ϕ,

$$l_{\tilde{g}}(\gamma) > |\int_\gamma \phi| = |\int_{[p,q]} \phi| = l_{\tilde{g}}([p,q]),$$

where $l_{\tilde{g}}$ means the length with respect to $\tilde{g}$.

The other is that γ leaves $U_1(L;g')$. Denote p' (or q') to be the nearest intersection point of γ with $\partial\overline{U_1(L;g')}$ from p (or q). Then we can replace $\gamma_{p'q'}$, the part between p' and q' of γ, by any curve c in $\overline{U_1(L;g')}$ connecting them (with a right choice of orientation). Call the new curve $\tilde{c}$, then it is obvious that $\tilde{c}$ is homologous to $[p,q]$. Because of properties listed above, we have

$$\begin{aligned} l_{\tilde{g}}(\gamma) &> |\int_{\gamma_{pp'}\cup\gamma_{p'q'}\cup\gamma_{q'q}}\phi| \\ &= |\int_{\gamma_{pp'}\cup\gamma_{q'q}}\phi| \\ &= |\int_{\gamma_{pp'}\cup\ c\ \cup\gamma_{q'q}}\phi| \\ &= |\int_{\tilde{c}}\phi| \\ &= |\int_{[p,q]}\phi| \\ &= l_{\tilde{g}}([p,q]) \end{aligned}$$

Hence we finish the proof. ∎

By the same idea, we can get results with strong Global Plateau Property for higher dimensional cases. Here (strong) **Global Plateau Property** of a submanifold M in a Riemannian manifold means that any oriented bounded domain Ω on M, with $\partial\Omega$ of finite mass, is (the unique) mass minimizer among all m-dimensional rectifiable current chains with the oriented boundary $\partial\Omega$. Here we exclude the cases of $\partial\Omega$ being the curve *Koch* snowflake and etc., but see *Remark 4.3.7*. If a current $K(\neq\Omega)$ shares the same boundary $\partial\Omega$ and $K-\Omega$ is a boundary of some rectifiable, compactly supported $(m+1)$-current, then this assertion follows whenever M can be (strongly) calibrated under the given metric. The difficulty here is the case when $[K-\Omega]$ is not zero in the $\mathbb{R}$-homology of the ambient manifold. However, the difficulty can be conquered by the exactness of calibration form.

Theorem 4.3.5 *Suppose that M is an embedded connected submanifold with a complete metric g_M induced from (X,g). If M is unbounded with respect to g_M, then for any open neighborhood U of M, a new metric $\hat{g}$ can be constructed by a conformal change of g supported in U, such that M can be tamed with* *strong Global Plateau Property in* $(X, \hat{g})$.

Proof. Similarly, by the completeness of (M, g_M), we can obtain g' and good region $U_1(M; g') \subset U$ from *Appendix .2*. Construct a calibration form $\phi = d(\tau(\mathbf{d}^{g'})\pi_g^*(\psi))$, where $d\psi$ is the volume form on M. Then construct $\hat{g}$, following the gluing method, in the same conformal class of g such that ϕ tames M in $(X, \hat{g})$.

For the Global Plateau Property of M in $(X, \hat{g})$. Take any oriented bounded domain Ω on M, with $\partial\Omega$ of finite mass. Suppose that a current $K(\neq \Omega)$ shares the same boundary $\partial\Omega$ and $K - \Omega$ is a compactly supported rectifiable current. Then

$$
\begin{aligned}
\mathbf{M}(\Omega) = \mathrm{Vol}(\Omega) = \int_\Omega i_M^*\phi = \int_{\partial\Omega}\psi \\
&= (dK)(\tau(\mathbf{d}^{g'})\pi_g^*(\psi)) \\
&= K(\phi) \\
&\leq \mathbf{M}(K).
\end{aligned}
$$

Since $K \neq \Omega$ and ϕ strongly calibrates M, the inequality is strict by *Theorem 2.1.8*. ∎

Remark 4.3.6 *As showed in the proof, any current calibrated by an exact calibration form will automatically have Global Plateau Property.*

Remark 4.3.7 *Although, in the definition of strong Global Plateau Property, we require the* $\partial\Omega$ *has finite mass, it turns out that dropping this restriction does not make any difference.*

4.4 Global Plateau Property for M Compact

After discussing non-compact cases, one may ask whether we construct metrics with Global Plateau Property for M compact case. Here we have to modify the definition to be either (both) the domain Ω or (and) its complement $\Omega - M$ is (are) the unique (two) mass minimizer(s) with boundary $\partial\Omega$.

Theorem 4.4.1 *Suppose that (X, g) is a Riemannian manifold and M is an oriented compact connected m-dimensional submanifold with $[M]$ non-zero in $H_m(X;\mathbb{R})$. Then a new metric $\hat{g}$ can be constructed by a conformal change of g, such that there exists a closed smooth m-form $\hat{\Phi}$ defined on X, and M is strongly calibrated in $(X, \hat{\Phi}, \hat{g})$ with strong Global Plateau Property.*

Proof. Suppose we take global form ϕ' to glue forms and use α satisfying (3.1.6) and (3.1.7) for the metric gluing. Denote the new metric by $\breve{g}$.

For a domain Ω on M (suppose $\mathbf{M}(\Omega) \leq \mathrm{Vol}(M)/2$) and a rectifiable competitor K with the same boundary, if the connected part of K to $\partial\Omega$, denoted by K', is supported in $\overline{U_\epsilon}$, then $[\Omega - K']$ is homologous to a positive integer multiple of $[M]$. Therefore, the mass of Ω is smaller than K's. If K' is not supported in $\overline{U_\epsilon}$, by Slicing theorem, since $\mathbf{d}$ is Lipschitz, there exists very small $\delta < \frac{\epsilon}{30}$ such that $S \triangleq K' \cap \partial\overline{U_{\epsilon-\delta}}$ is rectifiable with finite mass. Then take a sequence K_i with $\partial K_i = \partial\Omega - S$ to approach the minimal mass among rectifiable currents with the same boundary. By the compactness theorem [4], we have a subsequence with a minimal rectifiable limit $\tilde{K}$. If it has no points on $U_{\epsilon-\delta} - \overline{U_{\epsilon-2\delta}}$, then by the above argument, $\mathbf{M}(\Omega) < \mathbf{M}(K)$. Suppose $\tilde{K}$ has a point $\tilde{a} \in U_{\epsilon-\delta} - \overline{U_{\epsilon-2\delta}}$.

Claim: In $(X, \breve{g})$, suppose $\tilde{K}$ is a minimal rectifiable current $\bar{K} \triangleq \tilde{K}|_{U_{\epsilon-\delta}-\overline{U_{\frac{4\epsilon}{5}}}}$, then it has a positive lower bound mass β depending on $\breve{g}$ only.

Now we can change ϕ' to $\phi \triangleq \frac{\phi'}{N}$ for sufficient large N such that

$$\int_M \phi < \beta.$$

Then we can use ϕ and α' which is same as α on $X - U_{\frac{4\epsilon}{5}}$ to do gluing. Clearly, the new metric $\hat{g}$ is $\breve{g}$ on $X - U_{\frac{4\epsilon}{5}}$ makes M strongly calibrated with strong Global Plateau Property in $(X, \hat{\Phi}, \hat{g})$.

∎

Proof of Claim. By Nash's embedding theorem [18], we can isometrically embed $(X, \breve{g})$ in to an Euclidean space $(\mathbb{R}^s, g_E)$ for a large s. Note that $\bar{K}$ induces a varifold. Since the norm of $\delta\bar{K}$ in $\mathbb{R}^s$ is bounded, we can apply Corollary (3) on page 446 in Allard [1].

Theorem 4.4.2 (Allard) *Suppose* $0 \leq A < \infty$, $a \in$ *support of* $\|V\|$, $V \in \mathbf{V}_k(U)$, *where* U *is an open region of* $\mathbb{R}^s$. *If* $0 < R <$ *distance*$(a, \mathbb{R}^s - U)$ *and*

$$\|\delta V\|\mathbf{B}(a,r) \leq A\|V\|\mathbf{B}(a,r) \qquad \textit{whenever } 0 < r < R,$$

then $r^{-k}\|V\|\mathbf{B}(a,r)\exp Ar$ *is nondecreasing in* r, $0 < r < R$, *in particular,*

$$\Theta^k(\|V\|, a) \in \mathbb{R}.$$

Since the density of a rectifiable current is a.e. at least one, by taking a good point a on the current near a' with $|a - a'| < \frac{\epsilon}{60}$ and seting $r = \frac{\epsilon}{60}$, we have a lower bound as claimed. ∎

Chapter 5

Case with Singularities

Beyond the case of smooth submanifolds, singular situation plays an attractive role in current research and we explore this direction in this chapter. For simplicity, we first consider the case of singularity of a single point.

Theorem 5.0.3 *Suppose that S is an oriented compact submanifold with one singular point o in (X, g). Assume $[S]$ is non-zero in the $\mathbb{R}$-homology of X. If a local part $S_\epsilon \triangleq B_\epsilon(o; g) \cap S$ for some $\epsilon > 0$ can be calibrated in $(B_\epsilon(o; g), g)$, then there exists a metric $\hat{g}$ coinciding with g in $B_{\frac{\epsilon}{2}}(o; g)$ such that S can be calibrated in $(X, \hat{g})$.*

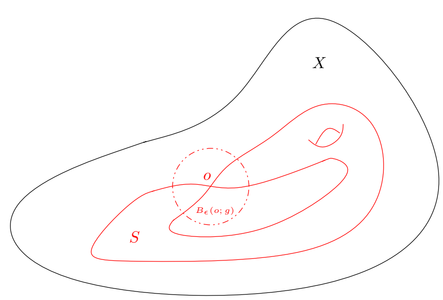

Figure 5.1: A picture for the case with a singular point.

Remark 5.0.4 *In the theorem,* $\frac{\epsilon}{2}$ *can be replaced by* $\varkappa\epsilon$ *for any* $0 < \varkappa < 1$.

Proof. Suppose that ϵ in the assumption is sufficiently small so that the open disc $D = B_\epsilon(o; g)$ corresponds to an open disc in some local chart. Let ϕ be the local calibration form. Then $\phi = d\psi$ where ψ is some smooth $(m-1)$-form defined on D. Suppose the following compact region $\Gamma_1 \cup \Omega \cup \Gamma_2$ (constructed by the fiber structure over the set $(\Gamma_1 \cup \Omega \cup \Gamma_2) \cap S$ when h is small) is included in $D - B_{\frac{2\epsilon}{3}}(o; g)$.

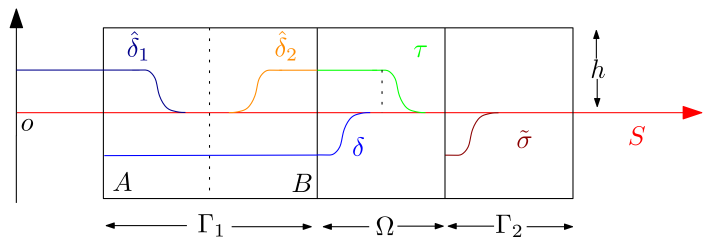

Figure 5.2: The gluing region.

Define $\pi_g^*\omega \triangleq d(\pi_g^*(\psi|_S))$ in $\Gamma_1 \cup \Omega \cup \Gamma_2$, where ω is the volume form on $S \cap (\Gamma_1 \cup \Omega \cup \Gamma_2)$. Set

$$\Phi = d(\tau\psi + (1-\tau)\pi_g^*(\psi|_S)),$$

where τ is a gluing function in Ω illustrated in the picture above with value one near Γ_1 and zero near Γ_2. (The picture here is just an illustration, since the region "hight" h is generally smaller than one to guarantee no overlapping.) Since $\Phi(\overrightarrow{T_x S_g}) = 1$, where $x \in S \cap (\Gamma_1 \cup \Omega \cup \Gamma_2)$ and $\overrightarrow{T_x S_g}$ is the unique oriented unit horizontal m-vector through x on S, it can be achieved by shrinking h (with respect to g) that the smooth function

$$\Phi(\overrightarrow{T_y S_g}) > \frac{1}{2}$$

in $\Gamma_1 \cup \Omega \cup \Gamma_2$ for all y in $\Gamma_1 \cup \Omega \cup \Gamma_2$, where $\overrightarrow{T_y S_g}$ is the unique oriented unit horizontal m-vector through y with respect to g. Let

$$\bar{g} = f \cdot g,$$

where

$$f = \delta + (1 - \delta)(\Phi(\overrightarrow{T_y S_g}))^{\frac{2}{m}}$$

defined in $\Gamma_1 \cup \Omega$. On the support of δ, $\Phi = \phi$. Since (ϕ, g) is a local calibration pair given in the assumption, we know $f \geq (\Phi(\overrightarrow{T_y S_g}))^{\frac{2}{m}}$ on $\Gamma_1 \cup \Omega$ and $f \equiv 1$ on Γ_1. Then $\bar{g}$ can be extended to Υ, where Υ is the region embraced by the (blue) curves in the picture below (which is an "h-disc bundle" containing $\Gamma_1 \cup \Omega \cup \Gamma_2$), with properties:

(a). Φ calibrates $S \cap (\Upsilon - \Omega)$ on $(\Upsilon - \Omega, \bar{g})$,

(b). $\bar{g} = g$ in Γ_1, and

(c). $\Phi(\overrightarrow{T_y S_{\bar{g}}}) \leq 1$ with equality on $\Upsilon - \Gamma_1 - \Omega$, where $\overrightarrow{T_y S_{\bar{g}}}$ is defined similar as $\overrightarrow{T_y S_g}$ but with respect to $\bar{g}$ instead.

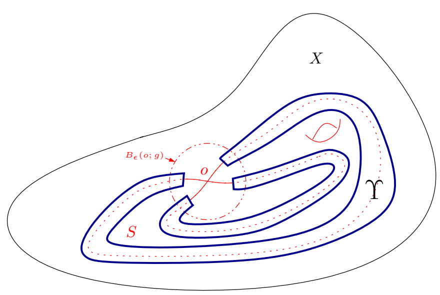

Figure 5.3: Region Υ.

In order to glue $\bar{g}$ and g together and meanwhile to guarantee Φ a calibration, two powerful lemmas from [7] are needed.

Lemma 5.0.5 (Harvey and Lawson) *Let $\xi \in \Lambda^p\mathbb{R}^n$ be a simple p-vector with $V = span\{\xi\}$. Suppose $\phi \in \Lambda^p\mathbb{R}^n$ satisfies $\phi(\xi) = 1$. Then there exists a unique oriented complementary subspace W to V with the following property. For any basis $v_1, \cdots, v_n$ of $\mathbb{R}^n$ such that $\xi = v_1 \wedge ... \wedge v_p$ and $v - p + 1, \cdots, v_n$ is basis for W, one has that*

$$\phi = v_1^* \wedge \cdots \wedge v_p^* + \sum a_I v_I^*, \tag{5.0.1}$$

where $a_I = 0$ whenever $i_{p-1} \leq p$.

Lemma 5.0.6 (Harvey and Lawson) *Let ϕ, $V = span\{\xi\}$, and W be as in Lemma 5.0.5, and consider an inner product $< \cdot, \cdot >$ on $\mathbb{R}^n$ such that $V \perp W$ and $\|\xi\| = 1$. Choose any constant $C^2 > \binom{n}{p}\|\phi\|^*$ and define a new inner product on $\mathbb{R}^n = V \oplus W$ by setting $< \cdot, \cdot >' = < \cdot, \cdot >_V + C^2 < \cdot, \cdot >_W$. Then in this new metric we have*

$$\|\phi\|^* = 1 \text{ and } \phi(\xi) = \|\xi\| = 1.$$

Remark 5.0.7 *According to the oringinal proofs, if $\phi(\xi) = \vartheta$ (positive) instead of one, one can apply Lemma 5.0.5 to $\vartheta^{-1}\phi$ to get a similar conclusion that $\|\phi\|^* = \vartheta$ with $\|\xi\| = 1$ and $\phi(\xi) = \vartheta$ by choosing $C^2 > \vartheta^{-1}\binom{n}{p}\|\phi\|^*$.*

By *Lemma 5.0.5* for Φ, $\overrightarrow{T_yS_{\bar{g}}}$ and $\bar{g}$ on Υ, there exists a smoothly varying $(n-m)$-dimensional plane field $\mathscr{W}$ transverse to the horizontal directions on Υ. Following *Lemma 5.0.6* and *Remark 5.0.7*, for any metric g_W along $\mathscr{W}$, there is a sufficiently large constant $\bar{\alpha}$ (by the compactness of Υ) such that if we set $\tilde{g} = \bar{g}^h \oplus \bar{\alpha} g_W$, then $\|\Phi\|^*_{\tilde{g}} = \Phi(\overrightarrow{T_yS_{\bar{g}}})$ which is no larger than 1 on Υ by property (c).

Now construct a smooth metric $\breve{g}$ on Ξ as follows based on property (b).

$$\breve{g} = \begin{cases} g & \text{near } o \\ g + (1-\hat{\delta}_1)((0 \cdot \bar{g}^h) \oplus \bar{\alpha} g_W) & \text{on } A \\ (1-\hat{\delta}_2)((0 \cdot g^h) \oplus g^\nu) + \tilde{g} & \text{on } B \\ \tilde{g} & \text{on } \Omega \\ \tilde{\sigma}\tilde{g} + (1-\tilde{\sigma})\bar{g} & \text{on } \Gamma_2 \\ \bar{g} & \text{far away from } o \end{cases} \tag{5.0.2}$$

In (5.0.2), $\oplus$ means the orthogonal splitting of a (pseudo-)metric and $+$ is the addition between two (pseudo-)metrics.

Note that on Γ_2, W is actually the vertical distribution (fiber directions) and Φ is $\pi_g^*(\omega)$ – a simple horizontal m-form. It is not hard to check that Φ is a calibration form on $(\Xi, \breve{g})$ by *Lemma 2.1.9*, *2.1.10* and *2.1.11*. Note that Ξ can be retracted to S and thereafter we can play previous gluing tricks for form and metric respectively in smaller regions of S to get a global calibration pair $(\hat{\Phi}, \hat{g})$ for S.

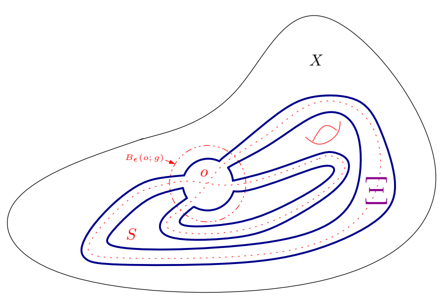

Figure 5.4: Region Ξ.

∎

In fact, the last step of the proof, i.e., gluing forms does not require that S is a retract of some open neighborhood of S. As long as there exists a global defined form which stands for $[\Phi]$ on the intersection of Ξ and some open neighborhood of S, then we can proceed. Therefore, the above ideas work well for more general cases.

Theorem 5.0.8 *Suppose that S is an oriented connected compact submanifold with a compact singular set $\mathscr{S}$ in (X, g). Assume that $[S]$ is non-zero in the current $\mathbb{R}$-homology of X. Also assume that $H^m(X;\mathbb{R}) \to H^m(U;\mathbb{R})$ is onto for some open neighborhood U of S. If a local part $S_\epsilon \triangleq B_\epsilon(\mathscr{S}; g) \cap S$ for some $\epsilon > 0$ can be calibrated in $(B_\epsilon(\mathscr{S}; g), g)$, where $B_\epsilon(\mathscr{S}; g)$ stands for the ϵ-neighborhood of $\mathscr{S}$ with respect to g. Then there exists a metric $\hat{g}$ coinciding with g in $B_{\frac{\epsilon}{2}}(\mathscr{S}; g)$ such that S can be calibrated in $(X, \hat{g})$.*

Remark 5.0.9 *By the Almgren regularity theorem, $\mathscr{S}$ has codimension at least 2 and then $\partial S \subset \mathscr{S}$ implies $\partial S = \emptyset$.*

Besides standard local models of complex varieties of singularities in *Kähler* manifolds, one famous type worth mentioning is the *Simons* cones. For dimension 7, it is the cone generated by $S^3(\frac{1}{\sqrt{2}}) \times S^3(\frac{1}{\sqrt{2}})$ in $S^7(1)$ of Euclidean $\mathbb{R}^8$. Since Lawson [14] constructed smooth calibrations for cases $S^{r-1}(\sqrt{\frac{r}{r+s}}) \times S^{s-1}(\sqrt{\frac{s}{r+s}})$ with $r + s > 10$, or $r + s = 9$ and $|r - s| < 5$, or $r = s = 4$, we know that a local model of *Simons* cones meets the requirment of *Theorem 5.0.3*. Recently, Haskins [9], Haskins and Kapouleas [10] and [11] provide many local prototypes of special *Lagrangian* cone type. In fact, we need diffeomorphic types only. For example, in an open patch U in a 6-dimensional manifold, if the pair (U, the part of a 3-dimensional submanifold in U) is diffeomorphic to ($\mathbb{C}^3$, a special *Lagrangian* cone), then it satisfies the local condition of *Theorem 5.0.3* by considering the pulling back forms and metrics.

One can consider product cases as well, e.g., $S^1\times$ a *Simons* cone or a *Simons* cone $\times$ a special *Lagrangian* cone.

Our theorem here merely confirms the existence of a metric which makes S globally minimal. Under the same hypotheses, we may also ask: is it possible to *conformally* change g such that S can be calibrated for the new metric? Little progress has been made on this direction so far. However, it is not hard to prove, under the above assumptions but without the requirement that $[S]$ is non-zero in the $\mathbb{R}$-homology of X, that there exist metrics in the conformal class of g which force the mean curvature to be zero away from the above singular set. This includes all *Lawlor* cones (cf. [13]) which is minimal in standard Euclidean spaces.

Theorem 5.0.10 *Suppose that S is an oriented compact manifold with a singular point o in (X^n, g). If $\mathring{S}_\epsilon \triangleq B_\epsilon(o;g)\cap S-\{o\}$ is minimal on $(B_\epsilon(o;g),g)$, then for any open neighborhood U of S, a new metric $\hat{g}$ can be constructed by a conformal change g supported in $U-B_{\frac{\epsilon}{2}}(o;g)$, such that the mean curvature vector, with respect to $\hat{g}$, is zero on $S-o$.*

Proof. Suppose that ϵ in the assumption is sufficiently small so that $B_\epsilon(o;g)$ is an open disc in some local chart. We can use the same method to construct a local calibration pair $-$ the pull-back of the volume form $\tilde{\omega}$ on $U_{\epsilon'}(S-B_{\frac{7\epsilon}{12}}(o;g))$, which is ϵ' neighborhood of $S-B_{\frac{7\epsilon}{12}}(o;g)$ along fibers induced by g (choose ϵ' small enough to guarantee the bundle structure and its inclusion in $U-B_{\frac{7\epsilon}{12}}(o;g)$) and the metric $g'=(\|\tilde{\omega}\|_g^*)^{\frac{2}{m}}g$ as in *Theorem 4.1.1* .

Now glue g and g' by taking:

$$\hat{g}=\hat{\rho}(r)g+(1-\hat{\rho}(r))g',$$

where $r = r(\pi_g(\cdot))$ is the composition of projection and the g-distance to o along S, and $\hat{\rho}$ looks like

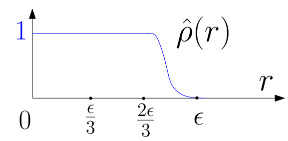

Figure 5.5: Graph for $\hat{\rho}$.

By the *Lemma 2.1.12*, the metric g' equals g along $S - B_{\frac{7\epsilon}{12}}(o; g)$. Hence $\hat{g} = g$ along S. Since $\mathring{S}_\epsilon \cap (B_\epsilon(o; g) - B_{\frac{7}{12}\epsilon}(o; g))$ is already minimal with respect to g, by *Appendix .1*, we have

$$\begin{aligned} H_{g'} &= H_g - \frac{m}{2} \cdot grad_g^{\perp}((\|\tilde{\omega}\|_g^*)^{\frac{2}{m}}), \quad \text{i.e.,} \\ 0 &= 0 - \frac{m}{2} \cdot grad_g^{\perp}((\|\tilde{\omega}\|_g^*)^{\frac{2}{m}}), \ \text{therefore} \\ 0 &= grad_g^{\perp}((\|\tilde{\omega}\|_g^*)^{\frac{2}{m}}). \end{aligned}$$

It follows that the mean curvature vector field on $\mathring{S}_\epsilon \cap (B_\epsilon(o; g) - B_{\frac{7}{12}\epsilon}(o; g))$ is still zero with respect to $\hat{g}$ by the computation:

$$\begin{aligned} H_{\hat{g}} &= H_g - \frac{m}{2} \cdot grad_g^{\perp}((\hat{\rho} + (1 - \hat{\rho})(\|\tilde{\omega}\|_g^*)^{\frac{2}{m}}) \\ &= 0 + \hat{\rho} \cdot \frac{m}{2} \cdot grad_g^{\perp}((\|\tilde{\omega}\|_g^*)^{\frac{2}{m}}) \\ &= 0. \end{aligned}$$

∎

More generally, we have:

Theorem 5.0.11 *Suppose that S is an oriented connected compact submanifold with compact singular set $\mathscr{S}$ as a submanifold in (X, g). If $\mathring{S}_\epsilon \triangleq B_\epsilon(\mathscr{S}; g) \cap$*

$S - \mathscr{S}$ is minimal in $(B_\epsilon(\mathscr{S}; g), g)$ for some $\epsilon > 0$, where $B_\epsilon(\mathscr{S}; g)$ stands for the ϵ-neighborhood of $\mathscr{S}$ with respect to g. Then for any open neighborhood U of S, a new metric $\hat{g}$ can be constructed by a conformal change g supported in $U - B_{\frac{\epsilon}{2}}(\mathscr{S}; g)$, such that the mean curvature vector, with respect to $\hat{g}$, is zero on $S - \mathscr{S}$.

Chapter 6

“Equivariant” Cases

We can naturally strengthen “equivariant” theorems in [25] by our methods. First, let us review the result from [25].

Theorem 6.0.12 (Tasaki) *Let K be a compact connected Lie transformation group of a manifold X and M be a compact oriented submanifold embedded in a manifold X. We assume that M is invariant under the action of K. If the real homology class represented by M in X is not equal to 0 , then there exists a K-invariant Riemannian metric g on X such that M is mass minimizing in its real homology class with respect to g.*

Notice that M in the above theorem is necessarily connected. Also pointed out by Prof. Tasaki, this result can be generalized to case when K is not connected but the K-action is orientation preserving.

Theorem 6.0.13 (Tasaki) *Let K be a compact Lie transformation group of a manifold X and M be a compact connected oriented submanifold embedded in a manifold X. We assume that M is invariant under the action of K and the action is orientation preserving. If the real homology class represented by M in X is not equal to 0 , then there exists a K-invariant Riemannian metric*

g on X such that M is mass minimizing in its real homology class with respect to g.

By our method, we can obtain:

Theorem 6.0.14 *Let K be a compact Lie transformation group of a manifold X and M be a compact connected oriented submanifold embedded in a manifold X. We assume that M is invariant under the action of K and the action is orientation preserving. Then for any K-invariant Riemannian metric g^K, there exists a K-invariant metric $\hat{g}^K$ in the same conformal class of g^K such that M can be calibrated in $(X, \hat{g}^K)$.*

Proof. It can be easily derived from the proof of *Theorem 6.0.15*. ∎

If we restrict ourselves to the K-connected case, another generalization is:

Theorem 6.0.15 *Let K be a compact connected Lie transformation group of a manifold X and M be a finite collection (or a countable neat collection with only lowest level consisting of infinte components) with each component class non-zero in the real homology of X. We assume that each component of M is invariant under the action of K. Then for any K-invariant Riemannian metric g^K, there exists a K-invariant metric $\hat{g}^K$ in the conformal class of g^K such that M can be strongly calibrated in $(X, \hat{g}^K)$.*

Proof. By *Theorem 3.5.5*, without loss of generality, we only need consider one level case.

First, the projection map of the local disc-fibered bundle structure induced by g^K around each component is K-invariant. Therefore, the projection π is

K-equivariant and by §2.4, we know the local forms are K-invariant.

Since K is connected, its action on each component of M is automatically orientation preserving.

By the discussions in §2.2, we can find some smooth closed m-form ϕ of X with positive integrals over all components of M (possibly with reversed orientation for some components). Since K is a compact Lie group, we have *Haar*-measure $d\mu$, with $\int_K d\mu = 1$, based on which one may average ϕ. Denote the result form by ϕ^K and then we have

$$\int_{M_i} \phi^K = \int_{M_i} \phi.$$

Now we are ready to construct an m-form Φ described in §2.4 with an appropriate choice of ϵ by locally gluing ω^*, a right mutilple of pull-back of volume form induced by g^K, and ϕ^K as follows. Suppose that $\phi^K = \omega^* + d\psi$, where ψ is a one dimension lower form. Then we get $\phi^K = \omega^* + d\psi^K$, where $\psi^K \triangleq \int_K \psi d\mu$. Set

$$\Phi = \omega^* + d((1-\rho)\cdot\psi^K).$$

As in *Theorem 4.1.1*, g'^K, $\mathbf{d}$ and σ are all K-invariant. Then choose α such that (3.1.6) and (3.1.7) hold. Take $\alpha^K = \int_K \alpha d\mu$. Do the gluing as in §3.1 around each component:

$$\hat{g}^K = \sigma^{\frac{1}{m}}(1+\mathbf{d}^2)\cdot g'^K + \alpha^K(1-\sigma)^{\frac{1}{m}}\cdot g^K,$$

where m is the dimension of the component. It is easy to see $(\Phi, \hat{g}^K)$ is a K-invariant calibration pair, which calibrates all components of M. ∎

- **Question:**

How about the case when K does not fix all components of M?

Chapter 7

Calibrations on Manifolds with Boundary and Relative Calibrations

Since we treat the boundary case and relative case essentially in the same way, this chapter is organized as follows. In §7.1, details of boundary case are given for the theory. Instead, in §7.2, several classical examples arising from the relative case are mentioned. In the last section, the natural generalized version of conformal change result on metric is proved.

7.1 Calibrations on Manifolds with Boundary

Parallel to Harvey and Lawson's original idea, we wish to adapt the concept of calibration forms such that each calibrated current is mass minimizing in its relative homology class. In order to realize the idea, any contribution from the boundary part should be zero. In fact, this leads to the right choice of calibration for this case.

Definition 7.1.1 *Suppose $(X, \partial X, g)$ is a smooth Riemannian manifold with boundary. Define*

$$\mathcal{E}^p(X, \partial X) \triangleq \{\phi \in \mathcal{E}^p(X) : \phi|_{\partial X} = 0\},$$

where $\mathcal{E}^p(X)$ is the set of all smooth p-forms on X and "$|_{\partial X}$" means pulling back map induced by the inclusion $: \partial X \to X$. Therefore, we have a cochain complex with usual operator d, which induces

$$H^p_{deR}(X, \partial X) \triangleq \{\phi \in \mathcal{E}^p(X, \partial X) : \ d\phi = 0\}/\{d\psi : \ \psi \in \mathcal{E}^{p-1}(X, \partial X)\}.$$

An important *de Rham* type lemma is the following.

Lemma 7.1.2 (de Rham) *Suppose $(X, \partial X, g)$ is a smooth finite type Riemannian manifold with boundary, then*

$$H^p_{deR}(\mathcal{E}^*(X, \partial X)) \cong H^p(\mathcal{C}^*(X, \partial X)) \cong H_p(X, \partial X)^*,$$

where both singular homology and cohomology are assumed with `real` *coefficient.*

Proof. By the short exact sequence

$$0 \to \mathcal{E}^*(X, \partial X) \to \mathcal{E}^*(X) \to \mathcal{E}^*(\partial X) \to 0,$$

and the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{E}^*(X, \partial X) & \longrightarrow & \mathcal{E}^*(X) & \longrightarrow & \mathcal{E}^*(\partial X) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathcal{C}^*(X, \partial X) & \longrightarrow & \mathcal{C}^*(X) & \longrightarrow & \mathcal{C}^*(\partial X) & \longrightarrow & 0 \end{array}$$

we have the following commutative diagram:

$$\begin{array}{ccccccccc}
H^{k-1}_{deR}(\mathcal{E}^*) & \longrightarrow & H^{k-1}_{deR}(\mathcal{E}^*(\partial)) & \longrightarrow & H^{k}_{deR}(\mathcal{E}^*(X,\partial)) & \longrightarrow & H^{k}_{deR}(\mathcal{E}^*) & \longrightarrow & H^{k}_{deR}(\mathcal{E}^*(\partial)) \\
\downarrow \cong & & \downarrow \cong & & \downarrow & & \downarrow \cong & & \downarrow \cong \\
H^{k-1}(\mathcal{C}^*) & \longrightarrow & H^{k-1}(\mathcal{C}^*(\partial)) & \longrightarrow & H^{k}(\mathcal{C}^*(X,\partial)) & \longrightarrow & H^{k}(\mathcal{C}^*) & \longrightarrow & H^{k}(\mathcal{C}^*(\partial))
\end{array}$$

where X is omitted and ∂ stands for ∂X.

The vertical isomorphisms except the middle column are due to *de Rham* Theorem. Therefore, by five lemma, the middle one is also an isomorphism. The last isomorphism in the statement follows from the universal coefficient theorem (another way is to use five lemma again for the natural pairing) for manifolds of finite type. ∎

Now we are ready to modify definitions in [8] for the case with boundary as follows.

Definition 7.1.3 *Suppose that $(X, \partial X, g)$ is a smooth Riemannian manifold of finite type with boundary. $\phi \in \mathcal{E}^m(X)$ is called a calibration (with boundary) if it satisfies*

(a) comass of ϕ is one,

(b) $d\phi = 0$, and

(c) $\phi|_{\partial X} = 0$.

The triple (X, g, ϕ) is called a calibrated geometry (with boundary).

Definition 7.1.4 *In a calibrated geometry (X, g, ϕ), an m-dimensional submanifold M which represents a non-zero relative homology class in $H_m(X, \partial X)$, is called calibrated if $\phi|_M$ is the induced volume form of M in (X, g).*

As in [8], let us define

Definition 7.1.5 *Suppose ϕ is a smooth p-form of comass 1 on $(X, \partial X, g)$. We define $\mathscr{G}(\phi)$ to be the union of the sets*

$$\mathscr{G}_x(\phi) \triangleq \{\xi_x \in G(p, T_xX) : < \phi_x, \xi_x >= 1\},$$

where

$$G(p, T_xX) = \{\zeta \in \Lambda^P T_xX : \ \zeta \text{ is a unit simple } p\text{-vector}\}.$$

Each locally rectifiable current $T \in \mathscr{R}_p^{loc}(X)$ determines an associated "volume" measure $\|T\|$, and for $\|T\|$-a.e. x there is an oriented "tangent" p-plane, $\vec{T}(x) \in G(p, T_xX)$. The current T is characterized by

$$T(\psi) = \int < \vec{T}, \psi > \ d\|T\|$$

for any form ψ.

Definition 7.1.6 *Let M be an m dimensional submanifold in $(X, \partial X, g)$, and suppose that ϕ is a smooth p-form of comass one. We call M is a ϕ-submanifold, if the unit oriented tangent of M at any point belongs to $\mathscr{G}(\phi)$.*

Similarly, we also have the notions ϕ-chains, ϕ-cycles, positive ϕ-chains, positive ϕ-cycles, positive ϕ-currents, closed positive ϕ-currents and the following statements.

Lemma 7.1.7 (Harvey and Lawson) *In a Riemannian manifold $(X, \partial X, g)$, suppose that ϕ is a smooth m-form with comass one and that T is an arbitrary m-current with compact support. Then*

$$T(\phi) \leq \mathbf{M}(T)$$

with equality if and only if T is a positive ϕ-current.

In particular, if S is a compact oriented m-dimensional submanifold (with possible boundary) in X, then

$$\int_S \phi \leq \mathrm{vol}(S)$$

with equality if and only if S is a ϕ-submanifold.

Next theorem is the most important and it is called "fundamental theorem in calibrated geometry".

Theorem 7.1.8 (Harvey and Lawson) *Suppose that $(X, \partial X, g)$ is a calibrated manifold with calibration ϕ, and that T is a positive ϕ-current with compact support. Let T' be any compactly supported current homologous to T (i.e., $T - T'$ is a boundary and in particular $dT = dT'$) in $H_*(X, \partial X)$. Then*

$$\mathbf{M}(T) \leq \mathbf{M}(T')$$

with equality if and only if T' is a positive ϕ-current.

In particular, if M is an m-dimensional ϕ-submanifold possibly with boundary and M' is a compact submanifold homologous to M in $H_m(X, \partial X)$, then

$$\mathrm{Vol}(M) \leq \mathrm{Vol}(M')$$

with equality if and only if M' is calibrated by ϕ as well.

Proof. Since $T - T' = dS + K$, for some compactly supported $(m+1)$-current S in X and some m-current K in ∂X, we have $T(\phi) - T'(\phi) = (T - T')(\phi) = dS + K(\phi) = S(d\phi) + K(\phi) = 0$. Combining *Lemma 7.1.7*, we have

$$\mathbf{M}(T) = T(\phi) = T'(\phi) \leq \mathbf{M}(T')$$

with equality if and only if T' is also a positive ϕ-current. ∎

It is apparent that if a calibrated submanifold does not touch the boundary of ambient manifold, then it basically falls into the realm of [8]. Therefore we will mainly focus on the cases of submanifolds with boundary in the boundary of ambient manifold for the rest of this note.

Suppose that M is a submanifold in (X, g) with boundary ∂M contained in ∂X. Pick one point x in ∂M, then T_xM and $T_x\partial M$ can be identified as a linear subspace of T_xX. Using the metric g_x, there exists a unique unit inner vector $\mathcal{V}_x$ perpendicular to $T_x\partial M$ such that span$(\{\mathcal{V}_x\} \cup T_x\partial M)$ is T_xM. When x varies in ∂M, we get a vector field and name it the **boundary vector field** of M in (X, g).

As what happens for dimension one (consider a shortest curve in its relative homology class), a calibrated submanifold which represents a relative homology class of any dimension must be "perpendicular" to the boundary of ambient manifold along its own boundary.

Theorem 7.1.9 *If M is calibrated in (X, g, ϕ), then its boundary vector field $\mathcal{V}$ is perpendicular to ∂X, i.e., $\mathcal{V}$ is the restriction of the unit inner normal vector field to M.*

Proof. Suppose $\mathcal{V}_x = \cos\theta_x \cdot T_x + \sin\theta_x \cdot N_x$, where N_x is the unit inner normal vector to ∂X at x, T_x is a unit vector tangential to ∂X at x, and θ_x is the angle between $\mathcal{V}_x$ and T_x. Let us fix the point x and omit the subscripts. Since the volume element of T_xM at a boundary point x is just the wedge product of $\mathcal{W}$ (the induced volume element of $T_x\partial M$) and $\mathcal{V}$, from M's being calibrated, we have

$$1 = \phi(\mathcal{V} \wedge \mathcal{W}) = \phi((\cos\theta \cdot T + \sin\theta \cdot N) \wedge \mathcal{W}) = \phi(\sin\theta \cdot N \wedge \mathcal{W}) \leq \sin\theta$$

Therefore, $\theta = \frac{\pi}{2}$ for any point $x \in \partial M$, i.e., $\mathcal{V}$ is perpendicular to ∂X along ∂M. ∎

7.2 Relative Calibrations

Definition 7.2.1 *Suppose (X, M, g) is a triple of a smooth manifold without boundary, a submanifold inside and a Riemannian metric defined on X.*

$$\mathcal{E}^p(X, M) \triangleq \{\phi \in \mathcal{E}^p(X) : \phi|_M = 0\},$$

where $\mathcal{E}^p(X)$ is the set of all smooth p-forms on X and "$|_M$" means pulling back map induced by the inclusion : $M \to X$. Therefore, we have a cochain complex with usual operator d, which induces

$$H^p_{deR}(X, M) \triangleq \{\phi \in \mathcal{E}^p(X, M) : \ d\phi = 0\}/\{d\psi : \ \psi \in \mathcal{E}^{p-1}(X, M)\}.$$

Definition 7.2.2 *Suppose that (X, M, g) is a smooth Riemannian manifold of finite type. $\phi \in \mathcal{E}^m(X)$ is called a calibration if it satisfies*

(a) comass of ϕ is one,

(b) $d\phi = 0$, and

(c) $\phi|_M = 0$.

The (X, M, g, ϕ) is called a relative calibrated geometry.

Apparently, results in the previous section also work for relative case.

Example [A] In [6], one beautiful existence theorem is the following.

Theorem 7.2.3 (Gromov) *Every closed quasi-Lagrangian submanifold* $W \subset \mathbb{C}^n$ *admits a (non-constant!) holomorphic disk* $(D^2, \partial D^2) \to (\mathbb{C}^n, W)$.

Any triangulation of D for the pseudoholomorphic disk will give us a sigular relative two cycle in $(\mathbb{C}^n, W)$. If we only focus on the image set, then it will be a relatively calibrated current in $(\mathbb{C}^n, W)$ with respect to standard metric and symplectic form. Since its (symplectic) area is not zero, it stands for a non-zero class in the relative homology class of $H_2(\mathbb{C}^n, W; \mathbb{R})$ and meanwihle mass minimizing. This argument reconfirms the homological minimality of pseudo-holomorphic disks with boundary in *Lagrangian* submanifolds in symplectic geometry.

From the inclusion maps

$$W \to \mathbb{C}^n \to (\mathbb{C}^n, W),$$

we have the exact sequence

$$\cdots \to H_2(\mathbb{C}^n) \to H_2(\mathbb{C}^n, W) \to H_1(W) \to H_1(\mathbb{C}^n) \to \cdots$$

Since the first and last homology are zero for $\mathbb{C}^n$, we see that the existence of pseudoholomorphic disk with boundary on a given compact *Lagrangian* submanifold implies that the submnanifold has non-trivial first homology class. Together with the previous result, we have

Theorem 7.2.4 (Gromov [6], Sikorav [22]) *Every closed quasi-Lagrangian submanifold* $W \subset \mathbb{C}^n$ *supports a non-zero first real homology class.*

In particular, there are no *Lagrangian* spheres in the standard complex vector spaces (except $S^1 \subset \mathbb{C}^1$).

Sikorav [22] gave a proof without bothering Gromov's result, where he also pointed out the following direct proof based on *Theorem 7.2.3*.

Proof. Since $\mathbb{C}^n$ has trivial first homology classes, the standard symplectic form $\omega = d\phi$. For any compact *Lagrangian* submanifold W, there is some non-constant pseudoholomorphic disk (identifying with its image) such that $\partial D \subset W$. If $H_1(W;\mathbb{R}) = 0$, then $\phi = df$ on M and

$$0 < \int_D \omega = \int_{\partial D} \phi|_W = \int_{\partial D} df = 0.$$

Contradiction! ∎

Another examples from symplectic geometry involve the notation "displaceable". To be displaceable of a *Lagrangian* submanifold implies the existence of a pseudoholomorphic disk with boundary on it. If we assume the ambient symplectic space is **exact**, i.e. the symplectic form $\omega = d\phi$, we have the following by the same argument.

Corollary 7.2.5 *In any exact symplectic manifold (X, ω), any displaceable Lagrangian submanifold has non-trivial first real homology class.*

An **exact Lagrangian submanifold** L in an exact symplectic manifold X means $\phi|_L = df$, where $d\phi$ equals the symplectic 2-form on X and $f \in C^\infty(L)$.

Corollary 7.2.6 *For any exact Lagrangian submanifold in an exact symplectic manifold, there are no relative symplectic currents.*

EXAMPLE $\boxed{\text{B}}$ Any associative with boundary in a coassociative in a G_2 manifold.

Leung, Wang and Zhu [16] prove the existence of thin associative submanifolds between two nearby disjoint conassociatives M_1, M_2 in a G_2 manifold (X, ϕ^3). It is obtained by suitably moving the three dimensional submanifold generated by a complex curve in one coassociative. This induces a large class of relative calibrated submanifolds (or currents) and therefore leads to a quite non-trivial and interesting relative homology of $H_3(X, M_1 \cup M_2)$.

Inspired by Salur [20] and Dr. Haskins' remark, more questions about G_2 may be asked (for example restrict ourselves to full G_2 holonomy case).

Q1: Is it possible to consider the G_2 cobordism theory between six dimensional manifolds or *Calabi-Yau* threefolds?

Q2: Is it possible to stretch G_2 manifolds between two six dimensional manifolds or *Calabi-Yau* threefolds aribitrarily to infinity? (maybe the realizable "length" is discrete or finite)

As Dr. Haskins pointed out, if two ends can be smoothly extended to infinity cylinders still with non-negative *Ric*, then, in the first question, it must be a cylinder connecting two essentially same six dimensional manifolds or *Calabi-Yau* threefolds with strictly small holonomy in G_2. However it seems not easy to keep the non-negativity.

Q3: Is it true that any totally geodesic six dimensional submanifold in a G_2 manifold is Calabi-Yau with full $SU(3)$ holonomy? Does it imply a cylinder splitting? **(The answer is a twisted cylinder proved by others.)**

Relevant to this question, the following is ture.

Proposition 7.2.7 *Suppose that (X, Φ, g) is a calibrated geometry (relative or not) with $\nabla\Phi = 0$, and that M (could be ∂X) is an oriented totally geodesic hypersurface. Denote its normal vector field by N (uniquely determined by orientation), then $(M, (N \lrcorner \Phi)|_M, g|_M)$ is a calibrated geometry (possibly relative depending on the choice of M).*

Proof. Denote the form $(N \lrcorner \Phi)|_M$ by ϕ. Since M is totally geodesic, it follows that ∇N vanishes along M. By choosing a geodesic normal basis e_i at any point p in M, we have

$$\begin{aligned} d\phi &= e_i \wedge \nabla^{\perp}_{e_i}(\Phi(N, \cdots)|_M) \\ &= e_i \wedge ((\nabla_{e_i}\Phi)(N, \cdots)|_M + \Phi(\nabla_{e_i}N, \cdots)|_M) \\ &= 0. \end{aligned}$$

Its comass one property clearly follows from the contraction construction. ∎

Remark 7.2.8 *If the submanifold is totally umbilic at each point instead of being totally geodesic, then the same computation leads us to*

$$d\phi = i_M^*\{\lambda(\Phi - N^* \wedge (N \lrcorner \Phi))\} = \lambda\Phi|_M$$

along M, where λ is the "totally umbilic function".

Easy to see that ϕ in the proof is parallel.

EXAMPLE $\boxed{\text{C}}$ Again, G_2 manifolds.

The calibration form on a G_2 manifold is from the standard Φ_s in $\mathbb{R}^7 \cong$ Imaginary part of $\mathbb{O}$ with

$$\Phi_s = \omega_{234} - \omega_{278} - \omega_{638} - \omega_{674} - \omega_{265} - \omega_{375} - \omega_{485}.$$

Here we think $\omega_{abc} = e_a^* \wedge e_b^* \wedge e_c^*$, where $e_1 = 1,\ e_2 = i,\ e_3 = j,\ e_4 = k,\ e_5 = e,\ e_6 = ie,\ e_7 = je,\ e_8 = ke$ with i, j, k, standard imaginary basis for $\mathbb{H}$, anti-commutative with a unit vector e in $\mathbb{R}^8 \cong \mathbb{H} \oplus (\mathbb{H} \cdot e)$. Since Φ and N are parallel along M, we can identify N with the vector e at a point p, then we have $\phi = \omega_{26} + \omega_{37} + \omega_{48}$ which is a non-degenerate 2-form.

Define $J_p : T_pM \to T_pM$ by

$$g|_M(J_pV, W) = \phi(V, W)$$

for $\forall\ V, W \in T_pM$, explicitly by sending e_2 to e_6, e_3 to e_7 and e_4 to e_8. Then, by parallel transporting J_p, we have a compatible integrable complex structure J, with respect to which the holonomy of M lies in $SU(3)$. Therefore (M, J, ϕ) is a *Calabi-Yau* threefold with a non-zero parallel three form $\Phi|_M$.

EXAMPLE $\boxed{\text{D}}$ Kähler manifolds.

If there exists an oriented totally geodesic hypersurface M in a Kähler manifold (X^{2n}, J, ω), then similarly, for

$$\phi_{2k-1} = N \lrcorner \omega^k / k!$$

or

$$\psi_{2k-2} = JN \lrcorner N \lrcorner \omega^k / k!,$$

M with the induce metric is a calibrated manifold.

In fact $\phi_{2k-1} = \phi_1 \wedge \psi_{2k-2}$ and ϕ_1 foliates M, namely by the intersections with complex curves perpendicular to it. This is one example filled with (oriented) geodesics related to the famous question in introduction.

7.3 Conformal Change of Metric

From *Theorem 7.1.9*, we see that the orthogonal condition along boundary of a submanifold is necessary for its being calibrated. If one allows conformal change of metric, then it turns out this is the unique requirement.

Theorem 7.3.1 *Suppose that M is an oriented connected k-dimensional submanifold in a Riemannian manifold (X, g) of finite type, with $\partial M \subset \partial X$, and that $[M]$ is non-zero in $H_k(X, \partial X; \mathbb{R})$. Then the following are equivalent:*

1. *the boundary vector field of M is perpendicular to ∂X, and*

2. *there exists a metric $\hat{g}$ conformal to g and a calibration ϕ with respect to $\hat{g}$ such that M is calibrated in $(X, \hat{g}, \phi)$.*

Remark 7.3.2 *The main difficulty for this case is the construction of fibration structure in a neighborhood of M in X with each fiber perpendicular to the submanifold (in order to make the pull back volume form of comass one) and any fiber through boundary point contained in ∂X (for the pull back volume form's being zero when restricted to ∂X), whereas, for the case with no boundary, one can simply take the fibration induced by distance function.*

Proof. 2) $\Rightarrow$ 1) is trivial by *Theorem 7.1.9*. Now consider 1) $\Rightarrow$ 2).

First, extend X, M and g smoothly beyond the boundary a little bit. Let the new submanifold be $\overline{M}$ Consider a sufficiently small neighborhood $\overline{U}$ of $\overline{M}$. Via the horizontal change, we know that, with respect to the result metric $\bar{g}$, the original M is now totally geodesic, which allows us to do the following construction for the required fibration structure.

Determined by the boundary vector field (note that $\bar{g}|_M = g|_M$), we have a geodesic foliation of M near its boundary by γ_p starting from any point $p \in \partial M$. Suppose the uniform existent length is T (with negative direction) without conjugate points along those γ_p.

There exists a natural $(n - m + 1)$-dimensional submanifold in $\overline{U}$ determined by γ_p and the exponential map of normal ϵ-disk bundle of $\overline{M}$ along $\bar{\gamma}_p$ (extension of γ_p) for small ϵ. We truncate the submanifold by ∂X and call it $\Gamma_p(\epsilon, T)$ showed in the pictures with color blue.

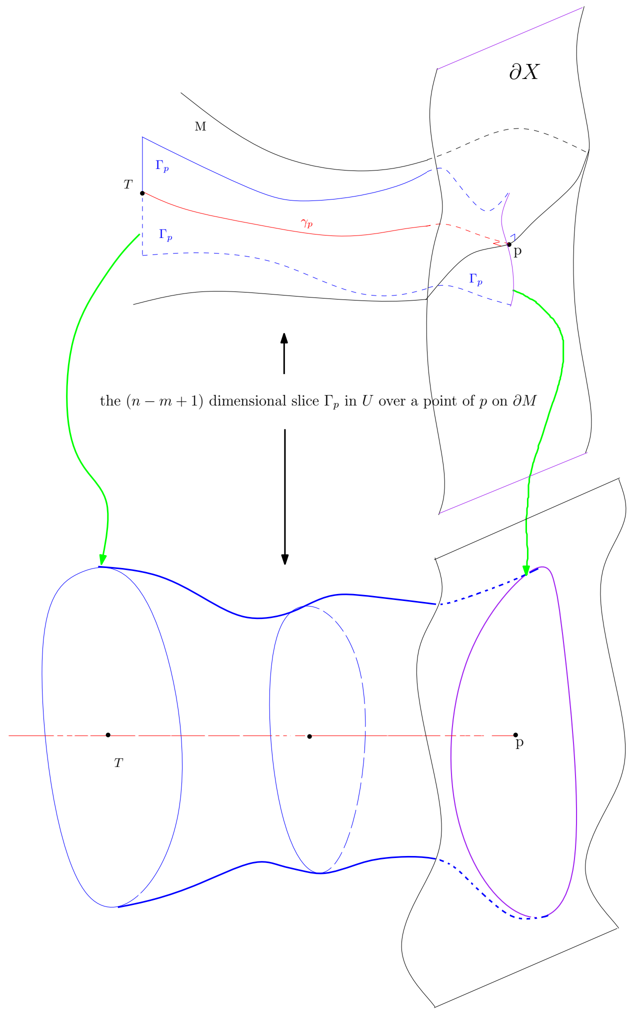

Figure 7.1: $\Gamma_p(\epsilon, T)$.

Take ϵ, t small enough such that $\exp_p^{-1}|_{\Gamma_p(\epsilon,t)}(\cdot)$ induces a diffeomorphism on $\Gamma_p(\epsilon, t)$ for any p on ∂M. When ∂M is compact, ϵ can be a constant. Otherwise take ϵ as a smooth function.

Consider the foliation $\exp_p^{-1}|_{\Gamma_p}(\Gamma_p)$ with respect to $\bar{g}|_{\Gamma_p}$ for the "left half part", i.e., $T \leq x \leq T/2$ defined below. Now we can reduce the question to construct a foliation on the region of $T_pX/T_p(\partial M)$ in the next picture (γ_p is still geodesic in $(\Gamma_p, \bar{g}|_{\Gamma_p})$).

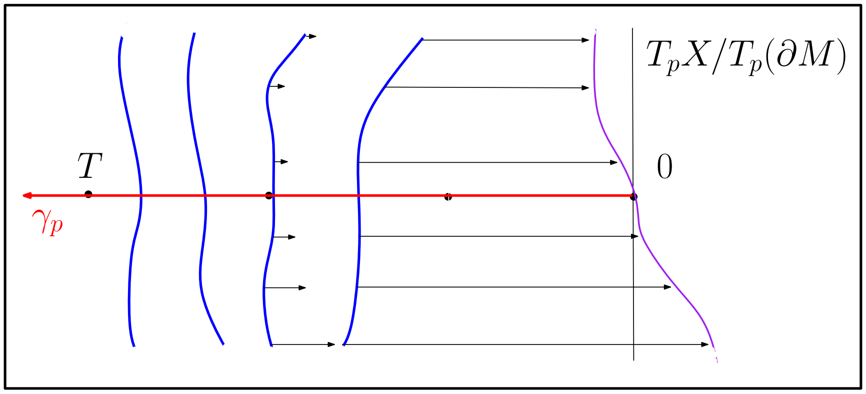

Figure 7.2: A construction of bundle.

Define

$$\overline{\xi}(x, \bar{y}) = \xi(x, \bar{y}) - \check{\sigma}(x)(\xi(x, \bar{y}) - \xi_0(\bar{y})),$$

where x is the intersection of the fiber ξ with line γ_p (same symbol for the preimage), $\bar{y}$ means the orthogonal coordinates components, $\xi_0(\bar{y})$ is the unique (if necessary, shrink ϵ appropriately for "the graph property") point on the preimage $\exp_p^{-1}|_{\Gamma_p}(\partial X \cap \Gamma_p)$, corresponding to $\bar{y}$, and $\check{\sigma}$ is increasing to hight one at $T/2$ like

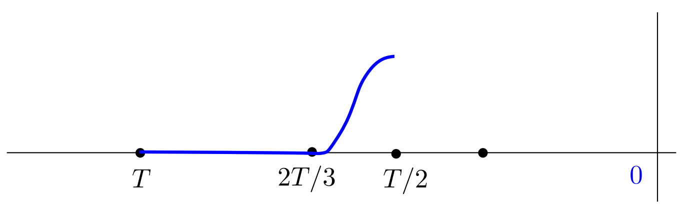

Figure 7.3: Graph of $\check{\sigma}$.

By a direct computation of

$$\begin{aligned}\frac{d}{dx}\overline{\xi}(x, \bar{y}) &= \frac{d}{dx}\xi(x, \bar{y}) - \frac{d}{dx}\check{\sigma}(x) \cdot (\xi(x, \bar{y}) - \xi_0(\bar{y})) - \check{\sigma}(x)\frac{d}{dx}\xi(x, \bar{y}),\\ &= (1 - \check{\sigma}(x))\frac{d}{dx}\xi(x, \bar{y}) - \frac{d}{dx}\check{\sigma}(x) \cdot \xi(x, \bar{y})\\ &> 0\end{aligned}$$

for any fixed $\bar{y}$ and $T \leq x < T/2$ (T is negative in our notation), it shows that the stretched one (along each Γ_p) is a foliation (no intersection).

Because the horizontal change preserves the orthogonal property along M and the linear stretch construction (note that $\xi_0 \perp \gamma_p$ at 0) guarantees that constructed foliation leaves, $\exp_p|_{\Gamma_p}(\bar{\xi})$ for $T \leq x \leq T/2$, are perpendicular to M by *Gauss Lemma*, we know that the extended foliation (actually fibration), by the exponential map of normal ϵ-disk bundle for the part with distance larger than $|T|$, meets our needs in *Remark 7.3.2*.

Denote the neighborhood by U_ϵ. Clearly $(M, \partial M)$ is a retract pair of $(U_\epsilon, \partial U_\epsilon)$. So

$$H^k(U_\epsilon, \partial U_\epsilon) \cong \mathbb{R} \cdot [\tilde{\omega}],$$

where $\tilde{\omega}$ is the pull back of volume form induced by g (NOT $\bar{g}$) via the above fibration structure. By *Lemma 2.12*, we have that $\|\tilde{\omega}\|_g^*$ equals one along M and at each point q it reaches one if and only if paired with the oriented tangent k-vector of M at p among all unit simple k-vector of T_qX.

Now by *de Rham lemma 7.1.2*, there is a closed k-form ϕ with $\phi|_{\partial X} = 0$ and $\int_M \phi = \mathrm{Vol}_g(M)$. Since

$$\int_M \phi = \mathrm{Vol}_g(M) = \int_M \tilde{\omega},$$

$[\phi] = [\tilde{\omega}]$ and therefore $\phi = \tilde{\omega} + d\psi$, where ψ is $(k-1)$-form on U_ϵ with zero restriction into ∂X.

Set

$$\Phi \triangleq \tilde{\omega} + d((1-\rho)\psi),$$

where $\rho(\mathbf{d})$ is a gluing function as below of variable $\mathbf{d}$, where $\mathbf{d}$ is the distance function (w.r.t g) along fibers.

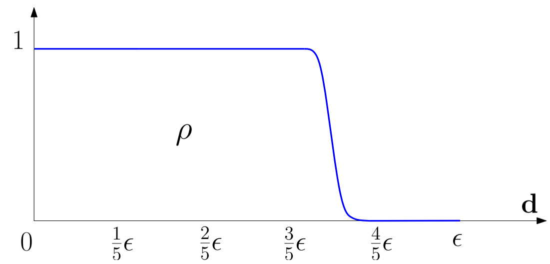

Figure 7.4: Graph of ρ for boundary case.

It is easy to see that Φ is closed and of comass one along M plus $\Phi|_{\partial X} = 0$. By the gluing method of metrics for no boundary case, we know after suitable conformal change on g, we have many new metric $\hat{g}$, with respect to which M is relatively calibrated by Φ. ∎

Remark 7.3.3 *Modified version for relative case can be obtained similarly. Results parallel to §4.1 can be achieved as well.*

Appendices

.1 A Formula on Mean Curvatures by a Conformal Change of Metric

Theorem .1.1 *Suppose M is an m-dimensional submanifold in (X, g) and a conformal metric change is given by $\tilde{g} = f \cdot g$ where f is a positive function. Then we have the formula*

$$f \cdot \tilde{H}_p = H_p - \frac{m}{2f} \cdot grad^{\perp}_{g,p}(f). \tag{.1.1}$$

Here H is the mean verctor field of M in (X, g), $\tilde{H}$ is the mean verctor field of M in $(X, \tilde{g})$, and $grad^{\perp}_{g,p}(\cdot)$ means the normal gradient at p to g, i.e., the normal part of $grad_{g,p}(\cdot)$.

Proof. Since both sides of (.1.1) are coordinates independent. We only need to verify it pointwisely. Let us take a coordinates $(x_1, \cdots, x_m, \cdots, x_n)$ of X centered at p, first m components of which contribute to a coordinates of M. We can meanwhile make $\{\partial/\partial x_i|_p\}_1^n$ an orthonormal frame with respect to g. For g,

$$\Gamma^k_{ij} = \frac{1}{2} g^{kl} \{ \frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \},$$

and for $\tilde{g}$,

$$\begin{aligned}
\tilde{\Gamma}^k_{ij} &= \frac{1}{2} \tilde{g}^{kl} \{ \frac{\partial \tilde{g}_{lj}}{\partial x_i} + \frac{\partial \tilde{g}_{il}}{\partial x_j} - \frac{\partial \tilde{g}_{ij}}{\partial x_l} \} \\
&= \frac{1}{2} g^{kl} \{ \frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ij}}{\partial x_l} \} + \frac{1}{2} \tilde{g}^{kl} \{ \frac{\partial f}{\partial x_i} g_{lj} + \frac{\partial f}{\partial x_j} g_{il} - \frac{\partial f}{\partial x_l} g_{ij} \}.
\end{aligned}$$

Set $i, j \in \{1, \cdots, m\}$ and $\kappa \in \{m+1, \cdots, n\}$. By the orthonormality at p,

$$\begin{aligned}\tilde{\Gamma}_{ij}^{\kappa} &= \Gamma_{ij}^{\kappa} + \frac{1}{2}\tilde{g}^{\kappa\kappa}\{\frac{\partial f}{\partial x_i}g_{\kappa j} + \frac{\partial f}{\partial x_j}g_{i\kappa} - \frac{\partial f}{\partial x_\kappa}g_{ij}\} \\ &= \Gamma_{ij}^{\kappa} - \frac{1}{2f}\frac{\partial f}{\partial x_\kappa}\delta_{ij}.\end{aligned}$$

Denote $\partial/\partial x_s|_p$ and $f^{-1/2} \cdot \partial/\partial x_s|_p$ by e_s and $\tilde{e}_s$ separately for $s \in \{1, \cdots, n\}$, then

$$\begin{aligned}\tilde{H} &= \sum_{i,\kappa} < \tilde{\nabla}_{\tilde{e}_i}\tilde{e}_i, \tilde{e}_\kappa >_{\tilde{g}} \tilde{e}_\kappa \\ &= \sum_{i,\kappa} < \tilde{\nabla}_{e_i}e_i, e_\kappa >_g \frac{e_\kappa}{f} \\ &= \sum_{i,\kappa} < \nabla_{e_i}e_i - \frac{1}{2f}\cdot grad_{g,p}^{\perp}(f), e_\kappa >_g \frac{e_\kappa}{f} \\ &= \frac{1}{f}\cdot(H - \frac{m}{2f}\cdot grad_{g,p}^{\perp}(f)).\end{aligned}$$

Hence the proof is complete. ∎

.2 A Concrete Construction of Metric and Region for §4.3

Since there is no essential difference for our construction in this section for higher dimensions, we focus on dimension one case only.

Lemma .2.1 *Suppose that L is an infinite line in a complete manifold (X, g) and that U is an open neighborhood containing L. Then there exists g' in g's conformal class such that $U_1(L; g')$ lies in U.*

Proof. Pick a point O on L. Without loss of generality, assume $U \neq X$. Since L is embedded, if necessary, we can shrink U to guarantee that, for every point in U, there is a unique nearest point on L. Define

$$r(d) \triangleq \sup\{r : \text{for any x with } d(x, O) = d,\ B(x, r) \in U\}.$$

Since the induced metric is complete, $r(d)$ is a well-defined continuous function with $\mathbb{R}$ values. For each d, $\{x | d(x, O) = d\}$ is compact due to the completeness of L, so $r(d)$ is always a finite number. Set $\tilde{r}(d) \triangleq inf_{s \leq d}\{r(s)\}$, then it is an everywhere positive, decreasing and continuous function.

Let

$$e(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}} & when\ |x| \leq 1 \\ 0 & when\ |x| > 1 \end{cases},$$

and define

$$h(d) \triangleq (\frac{1}{\tilde{r}} * e)(d).$$

If we take $\tilde{h}(d) \triangleq h(d + 1)$, then

1. $\tilde{h}$ is smooth and increasing, and

2. $\tilde{h}(d) = h(d+1) \geq \frac{1}{\tilde{r}(d)}$.

Set

$$\hat{r}(d) \triangleq \frac{1}{\tilde{h}(d)}.$$

By the above two properties, $\hat{r}$ is smooth and decreasing with $\hat{r}(d) \leq \tilde{r}(d)$. Now we have a smooth neighborhood $U_{\hat{r}}(L; g) \triangleq \{y : y$ is of distance at most

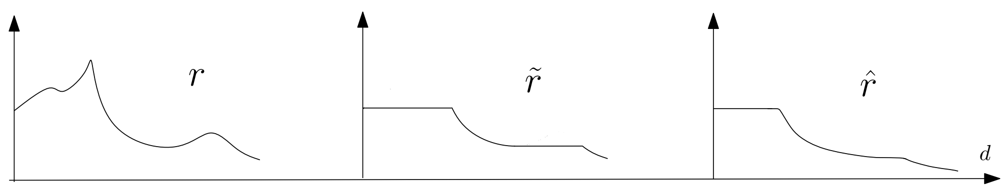

Figure 5: Modifications on function r.

$\hat{r}(d(x))$ from $x \in L\}$ contained in U. Since we want to rescale metric such that $U_{\hat{r}}(L)$ includes all points with distance at most one from L under the new metric, we only need to concentrate on the infinity part. Suppose $\hat{r}$ decreases below 1 from some d_0. (If there are no such d_0, we are done.) For the part of $d \geq d_0 + 1$, we define $\bar{g} \triangleq \hat{r}(d + 3 - \hat{r}(d))^{-2} g$. Then, it follows that

$$\begin{aligned} & d + 3 - \hat{r}(d) \text{ is increasing} \\ \Rightarrow \quad & \hat{r}(d + 3 - \hat{r}(d)) \text{ is decreasing} \\ \Rightarrow \quad & \hat{r}(d + 3 - \hat{r}(d))^{-2} \text{ is increasing.} \end{aligned}$$

Hence on the union of balls $\cup_{\{x | d(x) = d\}} B_{\hat{r}(d)}(x; g)$ for any $d \geq d_0 + 1$, we have

$$\frac{1}{\hat{r}(d + 3 - \hat{r}(d))^2} \geq \frac{1}{\hat{r}(d - \hat{r} + 3 - \hat{r}(d - \hat{r}))^2} \geq \frac{1}{\hat{r}(d+1)^2}$$

The last inequality is due to the fact that both $\hat{r}$ and $\hat{r}(d - \hat{r})$ are smaller than one. Choose arbitrary curve γ from x to $\partial B_{\hat{r}(d)}(x;g)$ entirely contained in $B_{\hat{r}(d)}(x;g)$. Then

$$\begin{aligned} l_{\bar{g}}(\gamma) &= \int_\gamma ds_{\bar{g}} \\ &\geq \int_\gamma \frac{1}{\hat{r}(d-\hat{r}(d)+3-\hat{r}(d-\hat{r}(d)))} ds_g \\ &= \frac{1}{\hat{r}(d-\hat{r}(d)+3-\hat{r}(d-\hat{r}(d)))} \int_\gamma ds_g \\ &\geq \frac{1}{\hat{r}(d+1)} \hat{r}(d) \\ &> 1. \end{aligned}$$

It is easy to extend $\bar{g}$ to a smooth metric g' conformal to g on $U_{\hat{r}}(L)$ with $U_1(L;g') \subset U_{\hat{r}}(L;g) \subset U$. ∎

.3 A Generalized Gauss Lemma

Since, for an one dimensional foliation, it is always locally orientable, one can define a local **length flow** with respect to metric according to a choice of orientation. By an observation of Sullivan [23], and a special case in Harvey and Lawson [7], we have the following theorems.

Theorem .3.1 (Sullivan) *A one dimensional flow is geodesible if and only if there is a transverse field of codimension one planes invariant under the local length flow.*

Theorem .3.2 (Harvey and Lawson) *A one dimensional foliation* Γ *in* (X, g) *is geodesic if and only if its perpendicular plane field* $\mathscr{P}$ *is invariant under the length flow.*

Proof. Locally, choose an orientation of Γ and denote the unit tangent vector field by V. For any local (nowhere zero) smooth section N of $\mathscr{P} \cong TX/T\Gamma$, we have

$$0 = N < V, V >= 2 < \nabla_N V, V >,$$

which implies

$$\begin{aligned} 0 &= V < V, N > \\ &=< \nabla_V V, N > + < V, \nabla_V N > \\ &=< \nabla_V V, N > + < V, \nabla_N V + \mathfrak{L}_V N > \\ &=< \nabla_V V, N > + < V, \mathfrak{L}_V N > . \end{aligned}$$

Hence, in the last line, the first term equals zero is equivalent to say the second term is zero, namely Γ is geodesic if and only if its perpendicular plane field is preserved by the local length flow. ∎

Since what we encounter in the proof of *Propersition 3.3.5* is the case that $\mathscr{P}$ is locally integrable, we will focus on this special case from now on.

Corollary .3.3 (Generalized Gauss Lemma) *Suppose that* Γ *is a one dimensional foliation for an open ball* B^n *in* $\mathbb{R}^n$ *with a metric* g. *If the perpendicular plane field of* Γ *is locally integrable, then* Γ *is geodesic if and only if (locally) integral pieces of the perpendicular plane field are preserved by the length flow along* Γ.

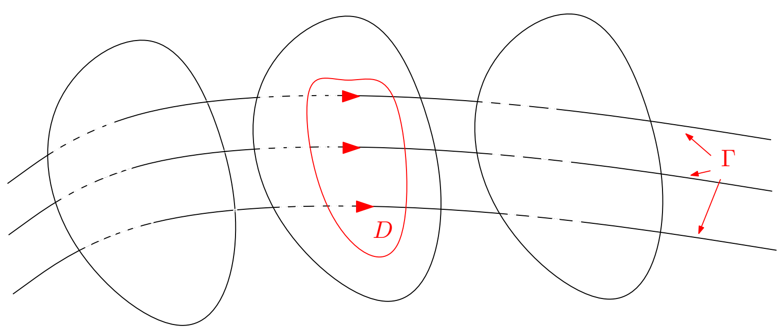

Figure 6: Foliation of dimension one.

Remark .3.4 *Suppose that the local perpendicular plane field* $\mathscr{P}$ *of a geodesic foliation* Γ *through* p *is integrable. Say* P_p *is an integral piece. Then, by the length flow, it follows that* $\mathscr{P}$ *is integrable near* p.

Remark .3.5 *For the integrable case, we have a different proof by constructing a distinguished coordinates of the foliation via local length flow and the classical* $Clairaut's$ *Theorem on geodesics of surfaces of revolution in* $\mathbb{R}^3$ *is a by-product from that argument.*

The reason why we assign the appellation "Generalized Gauss Lemma" to *Lemma .3.3* is the following. For a fixed p on (X, g). Since $d(\exp_p 0)(\cdot)$ is identity on T_pX, $\exp_p$ is a local diffeomorphism in a small neighborhood U of 0. Without loss of generality, assume that the unit sphere S in (T_pX, g_p) is contained in U. For any vector $v \in S$, take a connected open neighborhood $\tilde{W}$ of v in S and denote the sector (generated by $\tilde{W}$ to 0) by W (topologically $(0,1] \times \tilde{W}$). Then $\exp_p(W - 0)$ is foliated by $\mathscr{F} \triangleq \{\exp_p(\mathring{I}_w); w \in \tilde{W}\}$, where $\mathring{I}_w$ is the interval between 0 and w excluding 0 but including w. Obviously, $\mathscr{P}\mathscr{P} \triangleq \{\exp_p(t\tilde{W});\ 0 < t \leq 1\}$ (potentially perpendicular plane field of $\mathscr{F}$) are integrable and preserved by the length flow along $\mathscr{F}$. Although the foliation $\mathscr{P}\mathscr{P}$ seems smashed at p, let us zoom in and look at its limit behavior. For the geodesic $\exp_p(\mathring{I}_v)$, its limit tangent vector (outward) at p is v and the limit of $\mathscr{P}\mathscr{P}$ at p is $d(\exp_p 0)(T_vS)$. Again, due to the fact $d(\exp_p 0)(\cdot) = \mathrm{Id}$, the limits $\mathscr{P}\mathscr{P}_0$ and $\mathscr{F}_0$ at (p, v) are orthogonal. Therefore, it follows that $\mathscr{P}\mathscr{P}$ is the perpendicular plane field of $\mathscr{F}$.

Corollary .3.6 *Suppose that Γ is a one-dimensional geodesic foliation of an open ball (B^n, g) and that its perpendicular plane field is locally integrable. Let $g = g^\Gamma \oplus g^\perp$ be the metric decomposition of g along Γ and its orthogonal part. If $\hat{g} = g^\Gamma \oplus \hat{g}^\perp$ is a smooth metric by replacing $g^\perp$ by $\hat{g}^\perp$, then Γ is geodesic as well with respect to $\hat{g}$.*

Proof. Since g^Γ and Γ's perpendicular plane field are unchanged, Γ and $\hat{g}$ satisfy the conditions in *Theorem* .3.3. Thus the corollary follows. ∎

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