On some computational and analytic aspects of Chern-Weil forms

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Abstract of the Dissertation

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This thesis is divided into two parts. In the computational part, we calculate the characteristic forms of a trivial bundle equipped with a special metric. We apply this to prove a holomorphic Venice lemma. The same techniques are applied to give shorter proofs of Liu's calculation of Bando-Futaki invariants of projective hypersurfaces and Mourougane's calculation of the Bott-Chern forms of the Euler sequence of $\mathbb{P}(E)$. In the analytic part, we introduce a generalised Monge-Ampère equation and treat it using the method of continuity. We also prove a local existence result for a real version of this equation.

This work is dedicated to Swarna atta and B.C.N. Rao tatagaru.
R.I.P.

Contents

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Chapter 1

The Beginning...

1.1 Introduction

The study of vector bundles and characteristic classes is an important pursuit of modern geometry with applications to physics, differential and algebraic topology, algebraic geometry, and number theory (via Arakelov geometry). In the early twentieth century, the notion of continuously varying vector fields was needed to state and prove results like the Hairy Ball theorem. Thus tangent and normal bundle spaces were studied. The definition of vector bundles was the culmination of an effort to come up with new examples of manifolds via the fibre spaces of Seifert and the sphere bundles of Whitney [McC00]. Almost simultaneously these objects arose very naturally in algebraic geometry. Indeed, the classical concept of divisors was formulated in the language of line bundles and generalised to "matrix divisors" or holomorphic vector bundles by Weil [Wei38]. Lastly, concepts resembling line bundles popped up in physics through the Aharanov-Bohm effect and the Dirac string [Var]. Besides, the notion of "equivalence of vectors under change of coordinates" was necessary to formulate General relativity.

Tullio Levi-Civita introduced the notion of parallel transport in order to develop the differential calculus of vectors on curved surfaces. This notion was later generalised by Weyl, Ehressman and Cartan [Var] to connections on bundles. The basic idea is to define a directional derivative operator which tells us how to differentiate "vector-valued" functions i.e. sections. This opened up the field of differential geometry of vector bundles, i.e. finding "good" connections on vector bundles and relating the topology of vector bundles to the connections that can be put on them.

We recall that a manifold is a Hausdorff, connected, paracompact topological space that is locally isomorphic to $\mathbb{R}^n$. It is smooth if the transition functions are smooth, and is holomorphic if it is locally isomorphic to $\mathbb{C}^n$ with the transition functions being holomorphic.

In this thesis we shan't differentiate between a vector bundle and a trivialised vector bundle i.e. an open cover U_α along with transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow GL(k, \mathbb{C})$ where k is the rank of the complex vector bundle; the bundle itself is $\dfrac{\cup_\alpha U_\alpha \times \mathbb{C}^k}{(p, v_\alpha) - (p, g_{\alpha\beta} v_\beta)}$ and is smooth (holomorphic) if $g_{\alpha\beta}$ are smooth (holomorphic). As is usual in mathematics, the main problem of vector bundles is their classification upto isomorphism, which in our language is a map $\lambda_\alpha : U_\alpha \rightarrow GL(k, \mathbb{C})$ such that $\lambda_\alpha g_{\alpha\beta} \lambda_\beta^{-1} = g_{\alpha\beta}$. In this language, a section of a vector bundle is a collection of vector-valued functions f_α such that $f_\alpha = g_{\alpha\beta} f_\beta$. In what follows, we shall use the Einstein summation convention often.

Along with the classification problem, the problem of classification of bundles-with-connections is also an important one. In our language, a connection is a differential operator $d + A_\alpha$ acting on sections, where A_α is a collection of $k \times k$ matrix-valued 1-forms which transform as $A_\alpha = g_{\alpha\beta} A_\beta g_{\alpha\beta}^{-1} - dg_{\alpha\beta} g_{\alpha\beta}^{-1}$. Given an isomorphism of smooth vector bundles on a manifold, $\lambda : V \rightarrow W$, and a connection $d + A$ on W, one may define a connection on V by means of a pull-back, i.e $\tilde{A}_\alpha = \lambda_\alpha^{-1} A_\alpha \lambda_\alpha + \lambda_\alpha^{-1} d\lambda_\alpha$. As mentioned earlier, for many applications, one is interested in asking questions like "What kind of connections does a given vector bundle admit? How can one come up with new connections having good properties? Can one classify bundles-with-connections? Can studying connections help in classifying vector bundles ?"

Very often, we shall consider vector bundles that possess a metric i.e. positive-definite hermitian or symmetric matrices h_α such that $h_\alpha = g_{\alpha\beta} h_\beta g_{\alpha\beta}^{-1}$. The foremost example of a bundle-with-metric is that of the tangent bundle of a Riemannian manifold. Sometimes, bundles possessing metrics have naturally defined connections. Indeed, Levi-Civita realised that the correct way of measuring how fast a particle accelerates on a surface is to find its acceleration in space and project it on to the surface. This is a special case of the Levi-Civita connection on a Riemannian manifold. In general, we are interested in "metric-compatible" connections on vector bundles, i.e. in a trivialisation wherein the metric is the identity and the transition functions are unitary, the connection matrices A_α are skew-hermitian. Just like the Levi-Civita connection on a Riemannian manifold, a hermitian, holomorphic vector bundle on a complex manifold also admits a canonical connection. This connection is the so called "Chern connection" satisfying $A_\alpha = h_\alpha^{-1} \partial h_\alpha$ in a holomorphic trivialisation.

On a complex manifold whose holomorphic tangent bundle is equipped with a hermitian metric h which in local coordinates is $h = h_{i\bar{j}} dz^i \otimes d\bar{z}^j$, one may define a Riemannian metric g on its real tangent bundle (i.e. treating z as $x+\imath y$.) Indeed, if $\alpha_i + \imath\beta_i$ is a unitary frame for h, define $g = \sum_i \alpha_i \otimes \alpha_i + \beta_i \otimes \beta_i$. Then, $h - g = -\imath\omega$, where $\omega = \sum_i \alpha_i \wedge \beta_i = h_{i\bar{j}} dz^i \wedge d\bar{z}^j$. The metric h induces the canonical Chern connection whereas g gives rise to the Levi-Civita connection. It is clearly of interest to know when these two coincide. Such special manifolds are called Kähler manifolds. It may be proven that this happens when $d\omega = 0$. As a consequence of Hodge theory, on compact Kähler manifolds any $(1,1)$ form ω is locally $i\partial\bar{\partial}\phi$ for some smooth function ϕ where $\partial = \partial_x - \imath\partial_y$. Indeed, Kähler wrote down hermitian metrics of this form in an ad hoc manner, and found that they had good properties.

1.2 Curvature and Characteristic classes

Riemann realised that the measure of curvature of a surface is the failure of parallel transport to bring a vector to itself when transported around a loop. This maybe paraphrased as "the commutator of covariant derivatives is a measure of how curved the object is." Indeed, $[\nabla, \nabla] = [d + A, d + A] = dA + [A, A] = dA + A \wedge A$. Thus we define the curvature two form as $\Theta_\alpha = dA_\alpha + A_\alpha \wedge A_\alpha$. It transforms as

$$\Theta_\alpha = g_{\alpha\beta} \Theta_\beta g_{\alpha\beta}^{-1} \tag{1.1}$$

Chern-Weil theory associates to any polynomial Φ on $\mathrm{GL}(r, \mathbb{C})$, invariant under conjugation, a collection $\{\Phi(\Theta_\alpha\}_{\alpha\in A}$ which, according to 1.1, defines a global differential form $\Phi(\Theta)$ on X. The total Chern form $c(E, h)$ and the Chern character form $\mathrm{ch}(E, h)$ of a holomorphic Hermitian vector bundle (E, h) are special cases of this construction and are defined, respectively, by

$$\begin{aligned} c(E,h) &= \det\left(I + \frac{\sqrt{-1}}{2\pi}\Theta\right) = \sum_{k=0}^{r} c_k(E,h) \\ \mathrm{ch}(E,h) &= \mathrm{Tr}(\exp\left(\frac{\sqrt{-1}}{2\pi}\Theta\right)) = \sum_{k=0}^{n} \mathrm{ch}_k(E,h). \end{aligned} \tag{1.2}$$

In fact, it maybe proven [GH94] that such forms are closed and that their De Rham cohomology class is independent of the connection chosen[1]. In other

[1]For holomorphic vector bundles on complex manifolds, these forms are (k, k) forms and

words, such cohomology classes are invariants of vector bundles. These carry very important information about vector bundles. Computing them is thus paramount[2]. However, very rarely does one compute these by choosing a connection and calculating the above expressions. One uses other techniques such as the splitting principle to accomplish such a task. That being said, computing the forms themselves (as opposed to the classes) conveys important information about the connection. This is the basis for the theory of secondary characteristic classes. We shall look into this later in great detail.

1.3 Summary of main results

This thesis is divided into two parts. One is largely computational and the other has to do with the study of a PDE motivated by Chern-Weil theory. In the former part, we (My adviser Leon Takhtajan and I[PT11]) computed the characteristic and secondary characteristic forms for the Chern connection of a special non-diagonal metric on a trivial bundle. On a different note, we also gave a slightly different proof of the main theorem of [SS08] without using the existence of universal connections. We used our calculation of the characteristic forms to study the image of the Chern character map[3]. This result is as follows

Theorem 1.1. *For every $\bar{\partial}\partial$-exact form $\omega \in \mathcal{A}(X,\mathbb{C}) \cap \mathcal{A}^{\mathrm{even}}(X,\mathbb{R})$ there is a trivial vector bundle E over X with two smooth Hermitian metrics h_1 and h_2 such that*

$$\mathrm{ch}(E,h_1) - \mathrm{ch}(E,h_2) = \omega.$$

Here X is a complex manifold, ch is the Chern character, $\mathcal{A}(X,\mathbb{C})$ is the graded commutative algebra of smooth complex differential forms on X, and $\mathcal{A}^{\mathrm{even}}(X,\mathbb{R})$ consists of the real and even forms.

The result 1.1 was used to provide some insight into the Gillet-Soulé differential K-theory. The proof of theorem 1.1 required a simple linear algebra lemma :

Lemma 1.2. *If A is a matrix consisting of commutative entries, then,*

$$\frac{d\ln(\det(I+\lambda A))}{d\lambda} = tr(A(I+\lambda A)^{-1})$$

are ∂ and $\bar{\partial}$ closed

[2]Actually the Chern characters carry the same information as the Chern classes by the Newton identities

[3]The complex analogue of the so-called Venice lemma in [SS08]

If $A^2 = aA$, then,

$$(I+A)^{-1} = I - \frac{A}{1+a}$$
$$\det(I+A) = \exp\left(\frac{\mathrm{tr}(A)}{a}\ln(1+a)\right)$$

This lemma proved to be very versatile for computational purposes. Using this we gave much shorter proofs of the main calculations of [Liu10] and [Mou04].

In the analytic part of this work, I introduced and studied a PDE motivated by Chern-Weil theory. The results I obtained are also in [Pin12]. In what follows, (X, ω) shall denote a compact, Kähler n-dimensional manifold. The PDE is

$$\alpha_0(\omega + dd^c\phi)^n + \alpha_1 \wedge (\omega + dd^c\phi)^{n-1} + \ldots + \alpha_{n-1} \wedge (\omega + dd^c\phi) = \eta \qquad (1.3)$$

where $d\alpha_i = 0$, and $\int_X \eta = \int_X (\alpha_0\omega^n + \alpha_1 \wedge \omega^{n-1} + \ldots)$. Actually, a special case of this equation arose in a very different context (for bounding the Mabuchi energy) as a conjecture of X.X. Chen [Che00].

Equation 1.3, being a mixed Monge-Ampère type PDE, is known to present substantial difficulties. In fact, even the "pure" case (i.e. the hessian equations) was solved only recently [DK12, HMW10]. For $n = 2$, I reduced it (under some positivity assumptions) to the usual Monge-Ampère equation, introduced by Calabi, [Cal54] and solved by Aubin [Aub70], and Yau [Yau78]. For $n = 3$ one may reduce it (once again, under some assumptions) to

$$\alpha_0(\omega + dd^c\phi)^3 + \alpha_1 \wedge (\omega + dd^c\phi) = \eta$$

where $\omega > 0$, and $d\alpha_i = 0$. A local, real version of this equation is

$$\det(D^2u) + \mathrm{tr}(B^{-1}(x)D^2u) = f(x)$$

A variant of this local equation was studied by Krylov [Kry95], and a very general existence result was proven. However, it was done by reducing it to a Bellman equation and using the standard theory of Bellman equations. My purpose is to use the method of continuity directly so that the estimates thus obtained might give insight into getting estimates on manifolds (which is the ultimate goal). In this context, I proved that

Theorem 1.3. *The equation*

$$\det(D^2u) - \Delta u = f \text{ in } B$$
$$u = 0 \text{ on } \partial B$$

has a unique, smooth, convex solution if f is smooth and strictly positive on $\bar{B}$ where, B is the unit ball in $\mathbb{R}^3$.

If one changes the sign of the laplacian, one gets an equation of the type $\det(D^2u)+\Delta u = f$. When the method of continuity is applied to such an equation, as the "time" t varies along the continuity path, the equation becomes non-elliptic for large t. So, the best I can do currently is

Theorem 1.4. *The following Dirichlet problem on the ball B of radius 1 centred at the origin*

$$\begin{aligned} \det(D^2u) + \Delta u &= tf + (1-t)36 \\ u|_{\partial B} &= 0 \\ f &> 36 \end{aligned}$$

has a unique smooth solution at $t = T$ if f is smooth and for all $t \in [0, T)$, smooth solutions u_t exist and satisfy $D^2u_t > 3$.

A similar result on complex 3-dimensional manifolds was also proven.

Theorem 1.5. *If $\alpha > 0$, $\omega > 0$ are smooth Kähler forms on a compact Kähler 3-manifold (X, ω_0), then, there exists a constant $C > 0$ depending only on α and ω_0 such that, the equation*

$$(\omega + dd^c u_t)^3 + \alpha^2(\omega + dd^c u_t) = \frac{e^{tf} \int (\omega^3 + \alpha^2\omega)}{\int e^{tf}(\omega^3 + \alpha^2\omega)}(\omega^3 + \alpha^2\omega) \quad (1.4)$$

has a unique smooth solution at $t = T$, if for all $t \in [0, T)$, smooth solutions exist and satisfy $\omega + dd^c u > C\omega_0$.

Chapter 2

The computational aspects

2.1 Smooth complex vector bundles

2.1.1 Chern-Simons secondary forms

Let X be a smooth n-dimensional manifold X, let

$$\mathcal{A}(X) = \bigoplus_{k=1}^{n} \mathcal{A}^k(X, \mathbb{C}) = \mathcal{A}^{\mathrm{even}}(X) \oplus \mathcal{A}^{\mathrm{odd}}(X)$$

be the graded commutative algebra of smooth complex differential forms on X, and let V be a C^∞-complex vector bundle over X with a connection ∇. Recall that the Chern character form $\mathrm{ch}(V, \nabla)$ for the pair (V, ∇) is defined by

$$\mathrm{ch}(V, \nabla) = \mathrm{tr} \exp\left(\frac{\sqrt{-1}}{2\pi}\nabla^2\right) \in \mathcal{A}^{\mathrm{even}}(X).$$

Here ∇^2 is the curvature of the connection ∇ — an $\mathrm{End}\, V$-valued 2-form on X — and tr is the trace in the endomorphism bundle $\mathrm{End}\, V$. The Chern character form is closed, $\mathrm{d}\,\mathrm{ch}(V, \nabla) = 0$, and its cohomology class in $H^*(X, \mathbb{C})$ does not depend on the choice of ∇.

Let ∇^0 and ∇^1 be two connections on V. In [CS74], S.S. Chern and J. Simons introduced secondary characteristic forms — the Chern-Simons forms. Namely, the Chern-Simons form $\mathrm{cs}(\nabla^1, \nabla^0) \in \mathcal{A}^{\mathrm{odd}}(X)$ defined modulo $\mathcal{A}^{\mathrm{even}}(X)$, satisfies the equation

$$d\,\mathrm{cs}(\nabla^1, \nabla^0) = \mathrm{ch}(V, \nabla^1) - \mathrm{ch}(V, \nabla^0), \tag{2.1}$$

and enjoys a functoriality property under the pullbacks with smooth maps.

Here we present a construction of the Chern-Simons form $\mathrm{cs}(\nabla^1, \nabla^0)$, which is similar to the construction of Bott-Chern forms for holomorphic vector bundles, given by H. Gillet and C. Soulé in [GS86]. Namely, for a given V put $\tilde{V} = \pi^*(V)$, where $\pi : X \times S^1 \mapsto X$ is a projection, and $S^1 = \{e^{i\theta}, 0 \leq \theta < 2\pi\}$. Explicitly, $\tilde{V}$ is a bundle over $X \times S^1$ whose fibre at every point $(x, \theta) \in X \times S^1$ is $V_x \otimes \mathbb{C} \simeq V_x$. For every θ define the map $i_\theta : X \mapsto X \times S^1$ by $i_\theta(x) = (x, e^{i\theta})$, and let $\tilde{\nabla}$ be a connection on $\tilde{V}$ such that

$$i_0^*(\tilde{\nabla}) = \nabla^0, \quad i_\pi^*(\tilde{\nabla}) = \nabla^1.$$

Denote by g a function defined by

$$g(\theta) = \begin{cases} 0 & \text{if} \quad 0 \leq \theta < \pi, \\ 1 & \text{if} \quad \pi \leq \theta < 2\pi \end{cases}$$

and extended 2π-periodically to $\mathbb{R}$. It defines a function $g : S^1 \mapsto \mathbb{R}$, which is discontinuous at 0 and π.

Definition. The Chern-Simons form is defined as

$$\mathrm{cs}(\nabla^1, \nabla^0) = \pi_*(g(\theta)\mathrm{ch}(\tilde{V}, \tilde{\nabla})) = \int_{S^1} g(\theta)\mathrm{ch}(\tilde{V}, \tilde{\nabla}) \in \mathcal{A}^{\mathrm{odd}}(X) \tag{2.2}$$

— direct image of $g(\theta)\mathrm{ch}(\tilde{V}, \tilde{\nabla})$ under the projection $\pi : X \times S^1 \mapsto X$ (integration over the fibres of π).

Remark 2.1.1. Connection $\tilde{\nabla}$ is trivial to construct. If in local coordinates $\nabla^i = d_x + A^i(x)$, where d_x is deRham differential on X and $i = 0, 1$, then

$$\tilde{\nabla} = d_x + d_\theta + A(x, \theta),$$

where $A(x, \theta)$ is 2π-periodic and $A(x, 0) = A^0(x)$, $A(x, \pi) = A^1(x)$.

Lemma 2.1. *The Chern-Simons form* $\mathrm{cs}(\nabla^1, \nabla^0)$ *satisfies the equation* (2.1), *and modulo* $d\mathcal{A}^{\mathrm{even}}(X)$ *it does not depend on the choice of connection* $\tilde{\nabla}$.

Proof. Using

$$(d_x + d_\theta)\mathrm{ch}(\tilde{\nabla}) = 0 \quad \text{and} \quad d_\theta g = (\delta_\pi - \delta_0)d\theta,$$

we obtain

$$\begin{aligned} d\,\mathrm{cs}(\nabla^1, \nabla^0) &= \int_{S^1} \left((d_x + d_\theta) - d_\theta\right)(\mathrm{ch}(\tilde{\nabla}))g(\theta) = -\int_{S^1} d_\theta(\mathrm{ch}(\tilde{\nabla}))g(\theta) \\ &= \int_{S^1} \mathrm{ch}(\tilde{\nabla})\, d_\theta g = \mathrm{ch}(\nabla^1) - \mathrm{ch}(\nabla^0). \end{aligned}$$

It is also easy to see that modulo exact forms $\mathrm{cs}(\nabla^1, \nabla^0)$ does not depend on the choice of $\tilde{\nabla}$. Namely, let $\tilde{\nabla} = d_x + d_\theta + A(x,\theta)$, $\tilde{\nabla}' = d_x + d_\theta + A'(x,\theta)$ be two such connections. Define a connection $\widehat{\nabla}$ in the bundle $\widehat{V}$ over $X \times S^1 \times S^1$ by

$$\widehat{\nabla} = d_x + d_{\theta_1} + d_{\theta_2} + \hat{A}(x, \theta_1, \theta_2),$$

where

$$\hat{A}(x, \theta_1, 0) = A(x, \theta_1), \quad \hat{A}(x, \theta_1, \pi) = A'(x, \theta_1) \quad \text{for all} \quad \theta_1 \in [0, 2\pi],$$

and $\hat{A}(x, 0, \theta_2) = A_0(x)$, $\hat{A}(x, \pi, \theta_2) = A_1(x)$ for all $\theta_2 \in [0, 2\pi]$. Then

$$\begin{aligned}
\int_{S^1} (\mathrm{ch}(\tilde{\nabla}) - \mathrm{ch}(\tilde{\nabla}'))g(\theta_1) &= \int_{S^1} (d_x + d_{\theta_1})\mathrm{cs}(\tilde{\nabla}, \tilde{\nabla}')g(\theta_1) \\
&= d_x \int_{S^1} \mathrm{cs}(\tilde{\nabla}, \tilde{\nabla}')g(\theta_1) - \mathrm{cs}(\tilde{\nabla}, \tilde{\nabla}')\Big|_{\theta_1=0}^{\theta_1=\pi} \\
&= d_x \int_{S^1} \mathrm{cs}(\tilde{\nabla}, \tilde{\nabla}')g(\theta_1) - \int_{S^1} \mathrm{ch}(\widehat{\nabla})g(\theta_2)\Big|_{\theta_1=0}^{\theta_1=\pi} \\
&= d_x \int_{S^1} \mathrm{cs}(\tilde{\nabla}, \tilde{\nabla}')g(\theta_1).
\end{aligned}$$

Here we have used that at $\theta_1 = 0$ and $\theta_1 = \pi$ the restriction of the form $\mathrm{ch}(\widehat{\nabla}) \in \mathcal{A}^{\mathrm{even}}(X \times S^1 \times S^1)$ to $X \times S^1$ has no components along S^1 and its integral over S^1 is zero. □

Remark 2.1.2. Formula (2.2) is similar to formula (1.5) in [SS08] but has different applications than definition (1.2) in [SS08].

Definition. Put

$$\mathrm{CS}(\nabla^1, \nabla^0) = \mathrm{cs}(\nabla^1, \nabla^0) \bmod d\mathcal{A}^{\mathrm{even}}(X),$$

which, according to Lemma 2.1, is a well-defined element in $\widetilde{\mathcal{A}}^{\mathrm{odd}}(X) = \mathcal{A}^{\mathrm{odd}}(X)/d\mathcal{A}^{\mathrm{even}}(X)$.

Remark 2.1.3. Formula (2.1) can be written as

$$\mathrm{cs}(\nabla^1, \nabla^0) = \int_{\pi}^{2\pi} \mathrm{ch}(\tilde{\nabla}),$$

and the choice of points π and 2π on the unit circle is immaterial. Using the change of variables, for every $\alpha < \beta$ on S^1 we get

$$\mathrm{cs}(\nabla^1, \nabla^0) = \int_\alpha^\beta \mathrm{ch}(\tilde{\nabla}), \tag{2.3}$$

where now $i_\alpha^*(\tilde{\nabla}) = \nabla^0$, $i_\beta^*(\tilde{\nabla}) = \nabla^1$.

Using (2.3) and Lemma 2.1, we immediately get

Corollary 2.2.

$$\mathrm{CS}(\nabla^2, \nabla^0) = \mathrm{CS}(\nabla^2, \nabla^1) + \mathrm{CS}(\nabla^1, \nabla^0).$$

2.1.2 K-theory

Let $K_0(X)$ be the Grothendieck group of X — a quotient of the free abelian group generated by the isomorphism classes $[V]$ of complex vector bundles V over X by the relations $[V] + [W] = [V \oplus W]$.

In [SS08], the authors defined a version of differential K-theory using the notion of a *structured bundle*. Namely, connections ∇^0 and ∇^1 on a complex vector bundle V over X are called equivalent, if $\mathrm{CS}(\nabla^1, \nabla^0) = 0$. It follows from Corollary 2.1 that it is an equivalence relation.

Definition. A pair $\mathcal{V} = (V, \{\nabla\})$, where $\{\nabla\}$ is an equivalence class of connections on V, is called a structured bundle.

Denote by $\mathrm{Struct}(X)$ the set of all equivalence classes of structured bundles over X. It is shown in [SS08] that it is a commutative semi-ring with respect to the direct sum $\oplus$ and tensor product $\otimes$ operations, and we denote by $\hat{K}_0(X)$ the corresponding Grothendieck ring.

We have two natural ring homomorphisms: the "forgetful map"

$$\delta : \hat{K}_0(X) \to K_0(X),$$

given by $[\mathcal{V}] \mapsto [V]$ for $\mathcal{V} = (V, \{\nabla\})$, and

$$\mathrm{ch} : \hat{K}_0(X) \to \mathcal{A}^{\mathrm{even}}(X),$$

given by the Chern character map $[\mathcal{V}] \mapsto \mathrm{ch}(V, \nabla)$. For a trivial bundle V with trivial connection $\nabla = d$, $\mathrm{ch}(V, d) = \mathrm{rk}(V)$ — the rank of V.

The mapping δ is surjective and for compact X its kernel consists of all differences $[\mathcal{V}] - [\mathcal{F}]$ such that $\mathcal{V} = (V, \{\nabla\})$, where V is stably trivial: $V \oplus M = N$ for some trivial bundles M and N, and $\mathcal{F} = (F, \{\nabla^F\})$, where F is trivial and $\mathrm{rk}(F) = \mathrm{rk}(V)$. Indeed, by definition,

$$\ker\delta = \{[\mathcal{U}] - [\mathcal{W}] \,|\, U \oplus M = W \oplus M \text{ for some trivial bundle } M\}.$$

Now let W' be a vector bundle satisfying $W \oplus W' = K$, where K is a trivial bundle[1]. Choose arbitrary connections $\nabla^{W'}$ and ∇^{K} on the bundles W' and K, and introduce the bundles $V = U \oplus W'$ and $F = K$ with the connections $\nabla = \nabla^{U} \oplus \nabla^{W'}$ and $\nabla^{F} = \nabla^{W} \oplus \nabla^{W'}$. The bundle V is stably trivial by construction: $V \oplus M = N$, where $N = F \oplus M$, and

$$[\mathcal{U}] - [\mathcal{W}] = [\mathcal{U} \oplus \mathcal{W}'] - [\mathcal{W} \oplus \mathcal{W}'] = [\mathcal{V}] - [\mathcal{F}].$$

It is an outstanding problem to describe the image of the Chern character map. The following simple result is crucial for our approach to the differential K-theory of Simons-Sullivan [SS08].

Proposition 2.1.1. *The image of the Chern character map*

$$\mathrm{ch} : \hat{K}_0(X) \to \mathcal{A}^{\mathrm{even}}(X)$$

contains all exact forms. Specifically, for every exact even form ω there is a trivial vector bundle $V = X \times \mathbb{C}^r$ with a connection $\nabla = d + A$ such that

$$\mathrm{ch}(V, \nabla) - \mathrm{ch}(V, d) = \omega.$$

The proof is based on the following useful result (see [DR84] and [Con85]); for the convenience of the reader, we prove it here as well.

Lemma 2.3. *Every $\eta \in \mathcal{A}^k(X)$ can be represented as a finite sum of the basic forms $f_1 df_2 \wedge \cdots \wedge df_{k+1}$, where $f_1, \ldots, f_{k+1}$ are smooth functions on X. If the form η is real, one can choose the basic forms such that all functions f_i are real-valued, and if η is zero on an open $U \Subset X$, there is a representation such that all functions f_i vanish on U.*

Proof. When X is a compact, one can choose a finite coordinate open cover $\{U_\alpha\}_{\alpha\in A}$ of X and a partition of unity $\{\rho_\alpha\}_{\alpha\in A}$ subordinated to it. Then $\eta = \sum_{\alpha\in A} \rho_\alpha\, \eta|_{U_\alpha}$, where in local coordinates $x^1, \ldots, x^n$ on U_α,

$$\eta|_{U_\alpha} = \sum_I f_{\alpha,I} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where $I = \{i_1, \ldots, i_k\}$ and $1 \le i_1 < \cdots < i_k \le n$. Let K_α be a compact set such that $\mathrm{supp}\,\rho_\alpha \subsetneq K_\alpha \subset U_\alpha$ and let b_α be a "bump function" — a smooth

[1]Such bundle W' exists since X is compact.

function on X which is 1 on $\operatorname{supp}\rho_\alpha$ and zero outside K_α. Then

$$\eta_\alpha = \sum_I \rho_\alpha f_{\alpha,I} d(x^{i_1} b_\alpha(x)) \wedge \cdots \wedge d(x^{i_k} b_\alpha(x)) \in \mathcal{A}^k(X)$$

is of required form and $\eta = \sum_{\alpha\in A} \eta_\alpha$. The second statement of the lemma is obvious from this construction.

In the general case one can use an embedding $f : X \to \mathbb{R}^N$ of the manifold X into Euclidean space (say the Whitney embedding into $\mathbb{R}^{2n}$). By considering the tubular neighbourhood of $f(X)$ in $\mathbb{R}^N$, it is easy to show that the pullback map $f^* : \mathcal{A}(\mathbb{R}^N) \to \mathcal{A}(X)$ is onto, which proves the result. □

Proof of Proposition 2.1. Induction by the degree in $d\mathcal{A}^{\mathrm{odd}}(X) \subset \mathcal{A}^{\mathrm{even}}(X)$. According to Lemma 2.3, it is sufficient to consider only basic forms in $\mathcal{A}^{\mathrm{odd}}(X)$.

For a basic 1-form $\alpha = f_1 df_2$ we have $\omega = d\alpha = df_1 \wedge df_2$, so that

$$\mathrm{ch}(L, \nabla) - \mathrm{ch}(L, d) = \mathrm{ch}(L, \nabla) - 1 = \omega,$$

where L is a trivial line bundle over X with $\nabla = d - 2\pi\sqrt{-1} f_1 df_2$.

Now suppose that all exact forms of degree $\leq 2k$ are in the image of ch. For a basic $(2k+1)$-form $\alpha = f_1 df_2 \wedge \cdots \wedge df_{2k+2}$ we have $\omega = d\alpha = df_1 \wedge df_2 \wedge \cdots \wedge df_{2k+2}$, which can be also written as

$$\omega = \frac{1}{(k+1)!}(df_1 \wedge df_2 + \cdots + df_{2k+1} \wedge df_{2k+2})^{k+1}.$$

Let V be a trivial line bundle over X with

$$\nabla = d - 2\pi\sqrt{-1}(f_1 df_2 + \cdots + f_{2k+1} df_{2k+2}),$$

so that

$$\nabla^2 = -2\pi\sqrt{-1}(df_1 \wedge df_2 + \cdots + df_{2k+1} \wedge df_{2k+2}).$$

Then $\mathrm{ch}(V, \nabla) - 1 - \omega$ is an exact form of degree $\leq 2k$ and, by induction, is in the image of ch. □

Remark 2.1.4. It the form ω is real then the connection ∇ in Proposition 2.1 is compatible with the metric on V given by the standard Hermitian metric on $\mathbb{C}^r$.

Remark 2.1.5. It immediately follows from the second statement of Lemma 2.2 and the proof of Proposition 2.1 that if form ω vanishes on open $U \subset X$, then connection $\nabla = d + A$ can be chosen such that $A = 0$ on U.

Corollary 2.4. *For every $\alpha \in \widetilde{\mathcal{A}}^{\mathrm{odd}}(X)$ there is a trivial vector bundle V with connection ∇ such that $\mathrm{CS}(\nabla, d) = \alpha$.*

Proof. For the given $\alpha \in \mathcal{A}^{\mathrm{odd}}(X)$ let $\Theta \in \mathcal{A}^{\mathrm{odd}}(X \times S^1)$ be such that under the inclusion map $i_\theta : X \to X \times S^1$ one has $i_\pi^*(\Theta) = \alpha$ and $i_\theta^*(\Theta) = 0$ for all θ in some neighborhood of 0. Applying Proposition 2.1 to the manifold $X \times S^1$ and the exact even form $-d\Theta$, we have

$$\mathrm{ch}(\tilde{V}, \tilde{\nabla}) - \mathrm{rk}(\tilde{V}) = -(d_x + d_\theta)\Theta.$$

Integrating over θ from π to 2π we get

$$\alpha = \mathrm{cs}(\nabla^1, \nabla^0) + d_x \int_\pi^{2\pi} \Theta,$$

for connections $\nabla^0 = i_0^*(\tilde{\nabla})$ and $\nabla^1 = i_\pi^*(\tilde{\nabla})$ on a trivial bundle V — a pullback of the trivial bundle $\tilde{V}$ to X. Finally, it follows from Remark 2.5 that one can choose connection $\tilde{\nabla}$ on $\tilde{V}$ such that $i_0^*(\tilde{\nabla}) = d$. Thus putting $\nabla = \nabla^1$ we obtain $\mathrm{CS}(\nabla, d) = \alpha \bmod d\mathcal{A}^{\mathrm{even}}(X)$. □

Remark 2.1.6. Corollary 2.2 gives somewhat stronger form of Proposition 2.6 in [SS08], the so-called "Venice lemma" of J. Simons[2]. It has been used in [SS08] to prove that one can remove the differential form from the definition of the differential K-theory given by M. Hopkins and I. Singer [HS02].

Remark 2.1.7. Here is a direct proof of Corollary 2.2 which is close to the original argument in [SS08]. Let η be a 1-form on X and let L be a trivial line bundle with the connection $\nabla = d - 2\pi\sqrt{-1}\eta$. It follows from the homotopy formula in [SS08] that

$$\mathrm{CS}(\nabla, d) = \int_0^1 \exp\{dt \wedge \eta + td\eta\} = \sum_{l \geq 1} \frac{1}{l!}\, \eta \wedge (d\eta)^{l-1}.$$

Thus for the basic 1-form $\alpha = f_1 df_2$ putting $\eta = \alpha$ we get $\alpha = \mathrm{CS}(\nabla, d)$. Now suppose that the result is valid for all odd forms of degree $\leq 2k - 1$, and let $\alpha = f_1 df_2 \wedge \cdots \wedge df_{2k+2}$ be a basic form of degree $2k + 1$. Putting $\eta = f_1 df_2 + \cdots + f_{2k+1} df_{2k+2}$ we obtain in $\widetilde{\mathcal{A}}^{\mathrm{odd}}(X)$,

$$\mathrm{CS}(\nabla, d) = \frac{1}{(k+1)!}\, \eta \wedge (d\eta)^k - \xi = \alpha - \xi,$$

where ξ is a sum of odd forms of degrees $\leq 2k-1$. By the induction hypothesis,

[2]D. Sullivan, private communication.

there is a trivial vector bundle V with the connection $\tilde{\nabla}$ such that $\mathrm{CS}(\tilde{\nabla}, d) = \xi$, so that $\alpha = \mathrm{CS}(\nabla \oplus \tilde{\nabla}, d)$.

Recall that a *flat connection* ∇ on a trivial vector bundle F is a connection with trivial holonomy around any closed path in X. Equivalently, $\nabla = d^g = d + g^{-1}dg$, where $g : X \to \mathrm{GL}(r, \mathbb{C})$, $r = \mathrm{rk}(F)$, is a global parallel frame. The corresponding structured bundle $\mathcal{F} = (F, \{\nabla\})$ is called flat. Since any two flat connections on a trivial bundle F are gauge equivalent, flat bundles of a fixed rank r correspond to a single point in $\mathrm{Struct}(X)$, which following [SS08], we denote by $[r]$. Also, denote by $\mathcal{T}(X)$ a subgroup in $\widetilde{\mathcal{A}}^{\mathrm{odd}}(X)$ consisting of $\mathrm{CS}(\nabla, \nabla')$ for all trivial bundles F and flat connections ∇, ∇' on F.

Remark 2.1.8. According to Lemma 2.3 in [SS08], the group $\mathcal{T}(X)$ has the following description. Let Θ be the bi-invariant closed odd form on the stable general linear group $\mathrm{GL}(\infty)$ such that the free abelian group generated by all distinct products of its components represent the entire cohomology ring of $\mathrm{GL}(\infty)$ over $\mathbb{Z}$. Then

$$\mathcal{T}(X) = \{g^*(\Theta) \,|\, \text{for all smooth } g : X \to \mathrm{GL}(\infty)\}/d\mathcal{A}^{\mathrm{even}}(X).$$

Now following [SS08], let

$$\mathrm{Struct}_{\mathrm{ST}}(X) = \{[\mathcal{V}] = [(V, \{\nabla\})] \in \mathrm{Struct}(X) \,|\, V \text{ is stably trivial}\}$$

be the stably trivial sub-semigroup of $\mathrm{Struct}(X)$, and for $\mathcal{V} \in \mathrm{Struct}_{\mathrm{ST}}(X)$ define

$$\widehat{\mathrm{CS}}(\mathcal{V}) = \mathrm{CS}(\nabla^N, \nabla \oplus \nabla^F) \in \mathcal{A}^{\mathrm{odd}}(X)/d\mathcal{A}^{\mathrm{even}}(X),$$

where $V \oplus F = N$ with trivial bundles F and N, and ∇^F, ∇^N are flat connections on these bundles. According to Proposition 2.4 in [SS08], for another choice of trivial bundles $\bar{F}$ and $\bar{N}$ with flat connections $\nabla^{\bar{F}}$, $\nabla^{\bar{N}}$ we have

$$\mathrm{CS}(\nabla^N, \nabla \oplus \nabla^F) - \mathrm{CS}(\nabla^{\bar{N}}, \nabla \oplus \nabla^{\bar{F}}) \in \mathcal{T}(X),$$

so that the mapping $\widehat{\mathrm{CS}} : \mathrm{Struct}_{\mathrm{ST}}(X) \to \widetilde{\mathcal{A}}^{\mathrm{odd}}(X)/\mathcal{T}(X)$ is a well-defined homomorphism of semigroups.

Remark 2.1.9. One can choose $F = X \times \mathbb{C}^r$ with $\nabla^F = d$ and $\nabla^N = d^g$, where g is the isomorphism between $V \oplus F$ and $N = X \times \mathbb{C}^k$, and put $\widehat{\mathrm{CS}} = \mathrm{CS}(\nabla \oplus d, d^g)$.

According to Corollary 2.2 the map $\widehat{\mathrm{CS}}$ is surjective, and according to Proposition 2.5 in [SS08], $\ker \widehat{\mathrm{CS}} = \mathrm{Struct}_{\mathrm{SF}}(X)$ — the subgroup of *stably flat* structured bundles. By definition, $\mathcal{V} \in \mathrm{Struct}_{\mathrm{ST}}(X)$ is stably flat, if

$$\mathcal{V} \oplus \mathcal{F} = \mathcal{N},$$

where $\mathcal{F} = (F, \{\nabla^F\})$ and $\mathcal{N} = (N, \{\nabla^N\})$ are trivial bundles with equivalence classes of flat connections. Since map $\widehat{\mathrm{CS}}$ is onto and $\widetilde{\mathcal{A}}^{\mathrm{odd}}(X)/\mathcal{T}(X)$ is a group, for every $\mathcal{V} \in \mathrm{Struct}_{\mathrm{ST}}(X)$ there is $\mathcal{W} \in \mathrm{Struct}_{\mathrm{ST}}(X)$ such that $\mathcal{V} \oplus \mathcal{W} \in \mathrm{Struct}_{\mathrm{SF}}(X)$. This introduces a group structure on the coset space $\mathrm{Struct}_{\mathrm{ST}}(X)/\mathrm{Struct}_{\mathrm{SF}}(X)$, and we arrive at the following statement.

Proposition 2.1.2. *The map* $\widehat{\mathrm{CS}}$ *induces a group isomorphism*

$$\widehat{\mathrm{CS}} : \mathrm{Struct}_{\mathrm{ST}}(X)/\mathrm{Struct}_{\mathrm{SF}}(X) \to \widetilde{\mathcal{A}}^{\mathrm{odd}}(X)/\mathcal{T}(X).$$

From this result we immediately obtain Theorem 1.15 in [SS08].

Corollary 2.5. *Every structured bundle over a compact manifold* X *has a structured inverse: for every* $\mathcal{V} = (V, \{\nabla\}) \in \mathrm{Struct}(X)$ *there exists* $\mathcal{W} = (W, \{\nabla^W\}) \in \mathrm{Struct}(X)$ *such that*

$$\mathcal{V} \oplus \mathcal{W} = \mathcal{N},$$

where $\mathcal{N} = (N, \{\nabla^N\})$ *is a trivial bundle with flat connection.*

Proof. For $\mathcal{V} = (V, \{\nabla\}) \in \mathrm{Struct}(X)$ let U be such that $V \oplus U = F$ — a trivial bundle over X. Then $\mathcal{F} = (F, \{\nabla \oplus \nabla^U\}) \in \mathrm{Struct}_{\mathrm{ST}}(X)$ for any choice of connection ∇^U on U. By Proposition 2.2, there exists $\mathcal{H} = (H, \{\nabla^H\}) \in \mathrm{Struct}_{\mathrm{ST}}(X)$ such that $\mathcal{F} \oplus \mathcal{H} \in \mathrm{Struct}_{\mathrm{SF}}(X)$, i.e., there are trivial bundles M and N with flat connections ∇^M and ∇^M such that $\mathcal{F} \oplus \mathcal{H} \oplus \mathcal{M} = \mathcal{N}$. Putting

$$\mathcal{W} = (U \oplus H \oplus M, \{\nabla^U \oplus \nabla^H \oplus \nabla^M\}),$$

we obtain $\mathcal{V} \oplus \mathcal{W} = \mathcal{N}$. □

From Corollary 2.3 we immediately obtain, as in [SS08], that

- The Grothendieck group $\hat{K}(X)$ consists of elements $[\mathcal{V}] - [r]$.
- The element $[\mathcal{V}] - [r] = 0$ if and only if $\mathcal{V} = (V\{\nabla\})$ is stably flat and $r = \mathrm{rk}(V)$.
- The mapping $\Gamma : \mathrm{Struct}_{\mathrm{ST}}(X)/\mathrm{Struct}_{\mathrm{SF}}(X) \to \hat{K}(X)$, defined by

  $$\Gamma([\mathcal{V}]) = [\mathcal{V}] - [\mathrm{rk}(V)],$$

  gives an isomorphism $\mathrm{Struct}_{\mathrm{ST}}(X)/\mathrm{Struct}_{\mathrm{SF}}(X) \simeq \ker \delta$.

Denoting by $i = \Gamma \circ \widehat{\mathrm{CS}}^{-1} : \widetilde{\mathcal{A}}^{\mathrm{odd}}(X)/\mathcal{T}(X) \to \ker \delta$, we obtain the main part of the result in [SS08], the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \widetilde{\mathcal{A}}^{\mathrm{odd}}(X)/\mathcal{T}(X) & \xrightarrow{i} & \hat{K}(X) & \xrightarrow{\delta} & K(X) & \longrightarrow & 0 \\
& & \Big\downarrow d & & \Big\downarrow \mathrm{ch} & & \Big\downarrow \mathrm{ch} & & \\
0 & \longrightarrow & d\mathcal{A}^{\mathrm{odd}}(X) & \xrightarrow{j} & \mathcal{Z}^{\mathrm{even}}(X) & \longrightarrow & H^{\mathrm{even}}_{\mathrm{dR}}(X) & \longrightarrow & 0,
\end{array}$$

where $\mathcal{Z}^{\mathrm{even}}(X)$ is a subspace of closed forms in $\mathcal{A}^{\mathrm{even}}(X)$, and the map j is an inclusion.

2.2 Holomorphic vector bundles

2.2.1 Bott-Chern secondary forms

Let h_1 and h_2 be two Hermitian metrics on a holomorphic vector bundle E over a complex manifold X. In the classic paper [BC65], Bott and Chern have shown that there exist secondary characteristic forms, the so-called Bott-Chern forms — even differential forms $\mathrm{bc}(E; h_1, h_2) \in \tilde{\mathcal{A}}(X, \mathbb{C}) = \mathcal{A}(X, \mathbb{C})/(\mathrm{Im}\, \partial + \mathrm{Im}\, \bar{\partial})$, satisfying

$$\mathrm{ch}(E, h_2) - \mathrm{ch}(E, h_1) = \frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial\, \mathrm{bc}(E; h_1, h_2) \tag{2.4}$$

and the natural functorial property

$$\mathrm{bc}(f^*(E), f^*(h_1), f^*(h_2)) = f^*(\mathrm{bc}(E; h_1, h_2)) \tag{2.5}$$

for holomorphic maps $f : Y \to X$ of complex manifolds. The proof in [BC65, Proposition 3.15] uses the analogue of the homotopy formula in Chern-Weil theory.

Remark 2.2.1. In the smooth manifold case, using linear homotopy of connections ∇_t, it possible to integrate over t in the homotopy formula in a closed form and to obtain explicit formulas for the Chern-Simons forms (see, e.g., [SS08]). However in the complex manifold case for any homotopy h_t of Hermitian metrics, due to the presence of inverses h_t^{-1} in Θ_t, it is not possible to integrate over t in the homotopy formula in a closed form and to obtain explicit formulas for the Bott-Chern forms in terms of the Hermitian metrics h_1 and h_2 only.

In [GS86], Gillet and Soulé gave another definition of the Bott-Chern secondary classes which is also well-suited for short exact sequences of holomorphic vector bundles over X, which are used for defining the K-theory of X. Namely, let E be a holomorphic vector bundle over X with Hermitian metrics h_1 and h_2, let $\mathcal{O}(1)$ be the standard holomorphic line bundle of degree 1 over the complex projective line $\mathbb{P}^1$, and let $\tilde{E} = E \otimes \mathcal{O}(1)$ be the corresponding vector bundle over $X \times \mathbb{P}^1$. If $i_p : X \to X \times \mathbb{C}P^1$ is the natural inclusion map $i_p(x) = (x, p)$ then $i_p^*(\tilde{E}) \simeq E$ for all $p \in \mathbb{P}^1$. Let $\tilde{h}$ be a Hermitian metric on $\tilde{E}$ such that $i_0^*(\tilde{h}) = h_1$ and $i_\infty^*(\tilde{h}) = h_2$ (such a metric is constructed using partition of unity).

Definition. The Bott-Chern secondary form is defined as

$$\mathrm{bc}(E; h_1, h_2) = \int_{\mathbb{P}^1} \mathrm{ch}(\tilde{E}, \tilde{h}) \log |z|^2 \tag{2.6}$$

— direct image of $\log |z|^2 \mathrm{ch}(\tilde{E}, \tilde{h})$ under the projection $\pi : X \times \mathbb{P}^1 \to X$ (integration over the fibres of π). The integral is convergent since $\log |z|^2 \omega(z)$, where ω is any smooth $(1,1)$-form on $\mathbb{P}^1$, is integrable.

Lemma 2.6 (H. Gillet and C. Soulé)**.** *The Bott-Chern form* $\mathrm{bc}(E; h_1, h_2)$ *satisfies equations* (2.4) *and* (2.5), *and modulo* $\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}$ *does not depend on the choice of Hermitian metric* $\tilde{h}$*.*

The proof of (2.4) uses Poincaré-Lelong formula:

$$\frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial \log |z|^2 = \delta_\infty - \delta_0$$

(see [GS86, BGS88b]), and Lemma 2.1 uses a simplified version of this argument. As in the previous section, we put

$$\mathrm{BC}(E; h_1, h_2) = \mathrm{bc}(E; h_1, h_2) \bmod (\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial}).$$

Remark 2.2.2. Note that formula (2.6) the for Bott-Chern forms uses the Green function $\log |z|^2$ of the Laplace operator on $\mathbb{P}^1$, whereas formula (2.2) for the Chern-Simons form uses the Green's function $g(\theta)$ of the operator $\dfrac{d}{d\theta}$ on S^1.

Remark 2.2.3. In fact, Gillet and Soulé in [GS86] (and with J.-M. Bismut in [BGS88b]) defined Bott-Chern forms for short exact sequences of holomorphic vector bundles over X. Namely, let $\mathscr{E}$

$$0 \longrightarrow F \xrightarrow{\ i\ } E \longrightarrow H \longrightarrow 0$$

be such an exact sequence, where holomorphic bundles F, E and H are equipped with Hermitian metrics h_F, h_E and h_H. Put[3] $F(1) = F \otimes \mathcal{O}(1)$ and consider the map $\mathrm{id} \otimes \sigma : F \to F(1)$, where σ is a holomorphic section of the bundle $\mathcal{O}(1)$ over $\mathbb{P}^1$ with a single zero at ∞. Let

$$\tilde{E} = (F(1) \oplus E)/F$$

be the quotient bundle over $X \times \mathbb{P}^1$, where F is mapped diagonally into $F(1) \oplus E$ by $(\mathrm{id} \otimes \sigma) \oplus i$. Then under the embedding $i_p : X \to X \times \mathbb{P}^1$ we will have[4] $i_0^*(\tilde{E}) = E$ and $i_\infty^*(\tilde{E}) = F \oplus H$ since $E/F \simeq H$. There exists a Hermitian metric $\tilde{h}$ on $\tilde{E}$ such that $i_0^*(\tilde{h}) = h_E$ and $i_\infty^*(\tilde{h}) = h_F \oplus h_H$, and the Bott-Chern secondary form for the exact sequence $\mathscr{E}$ and Hermitian metrics h_F, h_E, h_H is defined by Gillet and Soulé [GS86] by the formula

$$\mathrm{bc}(\mathscr{E}; h_E, h_F, h_H) = \int_{\mathbb{P}^1} \mathrm{ch}(\tilde{E}, \tilde{h}) \log |z|^2.$$

Similar to (2.4), the Bott-Chern forms satisfy the equation

$$\mathrm{ch}(F \oplus H, h_F \oplus h_H) - \mathrm{ch}(E, h_E) = \frac{\sqrt{-1}}{2\pi} \bar{\partial}\partial \, \mathrm{bc}(\mathscr{E}; h_E, h_F, h_H),$$

are functorial, and vanish when the exact sequence $\mathscr{E}$ holomorphically splits and $h_E = h_F \oplus h_H$ (see [GS86, BGS88b]).

2.2.2 Chern forms of trivial bundles

We start with the following simple linear algebra result.

Lemma 2.7. *Let α_i, β_i, $i = 1, \ldots, k$, be odd elements in some graded commutative algebra $\mathcal{A}$ over $\mathbb{C}$ (e.g., the algebra of complex differential forms on X), and let A be a $k \times k$ matrix with even elements $A_{ij} = \alpha_i \beta_j$, and put $a = \mathrm{tr}\, A = \sum_{i=1}^k \alpha_i \beta_i$. Then for every $\lambda \in \mathbb{C}$,*

$$(I - \lambda A)^{-1} = I + \frac{\lambda}{1 + \lambda a} A = I + (\lambda - \lambda^2 a + \cdots + (-1)^k \lambda^{k+1} a^k) A,$$

where I is $k \times k$ identity matrix, and also

$$\det(I - \lambda A) = \frac{1}{1 + \lambda a} = 1 - \lambda a + \cdots + (-1)^k \lambda^k a^k.$$

[3]Here and in what follows we use the same notation for bundles over X and their pullbacks under the projection $\pi : X \times \mathbb{P}^1 \to X$.

[4]It is for this construction that it is necessary to twist the bundle F by $\mathcal{O}(1)$.

Proof. Consider the following identity

$$\frac{d}{d\lambda}\log\det(I-\lambda A) = -\operatorname{tr} A(I-\lambda A)^{-1},$$

whose validity for matrices over $\mathbb{C}$ and small λ follows from Jordan canonical form, and for matrices with even nilpotent entries — from the definition of the determinant[5]. Since $A^2 = -aA$, we obtain

$$(I-\lambda A)^{-1} = I + \frac{\lambda}{1+\lambda a}A,$$

so that

$$\frac{d}{d\lambda}\log\det(I-\lambda A) = -\frac{a}{1+\lambda a} = -\frac{d}{d\lambda}\log(1+\lambda a).$$

Integrating and using $\det I = 1$, we get the result. □

The next result is an explicit computation of the total Chern form of a trivial vector bundle with a special non-diagonal Hermitian metric.

Lemma 2.8. *Let $E_r = X \times \mathbb{C}^r$ be a trivial rank r vector bundle over X with a Hermitian metric h given by*

$$h = h(\sigma, f_1, \dots, f_{r-1}) = g^* g, \quad \textit{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 & \bar{f}_1 \\ 0 & 1 & 0 & \dots & 0 & \bar{f}_2 \\ 0 & 0 & 1 & \dots & 0 & \bar{f}_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & \bar{f}_{r-1} \\ 0 & 0 & 0 & \dots & 0 & e^{\sigma/2} \end{pmatrix},$$

and $f_1, \dots, f_{r-1} \in C^\infty(X, \mathbb{C})$, $\sigma \in C^\infty(X, \mathbb{R})$. Then

$$c(E_r, h) = c(E_1, e^\sigma) + \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial \log\left(1 - \frac{\sqrt{-1}}{2\pi}U\right),$$

where $U = e^{-\sigma}\sum_{i=1}^{r-1}\partial f_i \wedge \bar{\partial}\bar{f}_i$, $E_1 = \det E_r$ is a trivial line bundle over X, and for a nilpotent element a of order r, $\log(1-a) = -(a + \frac{a^2}{2} + \cdots + \frac{a^{r-1}}{r-1})$. Equivalently,

$$c_1(E_r, h) = \frac{\sqrt{-1}}{2\pi}\bar{\partial}\partial\sigma, \quad c_k(E_r, h) = -\frac{1}{k-1}\left(\frac{\sqrt{-1}}{2\pi}\right)^k \bar{\partial}\partial\, U^{k-1}, \quad k = 2, \dots, r.$$

---

[5] In the latter case log and inverse are given, correspondingly by the finite sums — truncated power series.

Proof. Let $\Theta = \bar\partial(h^{-1}\partial h)$ be the curvature form associated with the Hermitian metric h. We need to prove that for every $\lambda \in \mathbb{C}$,

$$\begin{aligned}\det(I+\lambda\Theta) &= 1+\lambda\bar\partial\partial\sigma + \lambda\bar\partial\partial\log(1-\lambda U)\\ &= 1+\lambda\bar\partial\partial\sigma - \lambda^2\frac{\bar\partial\partial U}{1-\lambda U} - \lambda^3\frac{\bar\partial U\wedge\partial U}{(1-\lambda U)^2},\end{aligned}$$

where

$$\frac{1}{1-\lambda U} = \sum_{k=0}^{r-1}\lambda^k U^k \quad \text{and} \quad \frac{1}{(1-\lambda U)^2} = \sum_{k=0}^{r-1}(k+1)\lambda^k U^k.$$

It is convenient to represent the matrix $I+\lambda\Theta$ in the following block form

$$I+\lambda\Theta = \begin{pmatrix} I+\lambda\Theta_{11} & \lambda\Theta_{12}\\ \lambda\Theta_{21} & 1+\lambda\Theta_{22}\end{pmatrix},$$

where $(r-1)\times(r-1)$ matrix Θ_{11}, $(r-1)$-vectors $\Theta_{12}, \Theta_{21}^t$, and the scalar Θ_{22} are given by

$$\Theta_{11} = \left\{-\bar\partial(\bar f_i e^{-\sigma}\partial f_j)\right\}_{i,j=1}^{r-1}, \quad \Theta_{12} = \left\{\bar\partial\partial\bar f_i - \bar\partial(\bar f_i F) - \bar\partial(\bar f_i\partial\sigma)\right\}_{i=1}^{r-1},$$
$$\Theta_{21}^t = \left\{\bar\partial(e^{-\sigma}\partial f_i)\right\}_{i=1}^{r-1}, \quad \Theta_{22} = \bar\partial\partial\sigma + \bar\partial F,$$

and $F = e^{-\sigma}\sum_{l=1}^{r-1}\bar f_l\partial f_l$. The row operations $R_i \mapsto R_i + \bar f_i R_r$ transform the matrix $I+\lambda\Theta$ to the form

$$\begin{pmatrix} I-\lambda A & b\\ c & d\end{pmatrix},$$

where

$$A = \left\{e^{-\sigma}\bar\partial\bar f_i\wedge\partial f_j\right\}_{i,j=1}^{r-1}, \quad b = \left\{\bar f_i + \lambda(\bar\partial\partial\bar f_i - \bar\partial\bar f_i\wedge F - \bar\partial\bar f_i\wedge\partial\sigma)\right\}_{i=1}^{r-1},$$

and we put $c = \lambda\Theta_{21}$, $d = 1+\lambda\Theta_{22}$.

Now it follows from the representation

$$\begin{pmatrix} I-\lambda A & b\\ c & d\end{pmatrix} = \begin{pmatrix} I & b\\ c(I-\lambda A)^{-1} & d\end{pmatrix}\begin{pmatrix} I-\lambda A & 0\\ 0 & 1\end{pmatrix}$$

that

$$\det(I+\lambda\Theta) = \det(I-\lambda A)\left(d - c(I-\lambda A)^{-1}b\right),$$

which we compute explicitly using Lemma 2.7. Namely,

$$\det(I+\lambda\Theta)=\frac{1}{1-\lambda U}\Bigg(1+\lambda(\bar\partial\partial\sigma+\bar\partial F)$$

$$-\sum_{i,j=1}^{r-1}\lambda\bar\partial(e^{-\sigma}\partial f_i)\wedge\left(\delta_{ij}+\frac{\lambda e^{-\sigma}\bar\partial\bar f_i\wedge\partial f_j}{1-\lambda U}\right)\wedge(\bar f_j+\lambda(\bar\partial\partial\bar f_j-\bar\partial\bar f_j\wedge(F+\partial\sigma))\Bigg).$$

Using equations

$$\bar\partial F=-U-\bar\partial\sigma\wedge F+e^{-\sigma}\sum_{i=1}^{r-1}\bar f_i\bar\partial\partial f_i,$$

and

$$\partial U=-\partial\sigma\wedge U+\Psi_+,\quad \bar\partial U=-\bar\partial\sigma\wedge U+\Psi_-,$$

$$\bar\partial\partial U=-\bar\partial\partial\sigma\wedge U+\bar\partial\sigma\wedge\partial\sigma\wedge U+\partial\sigma\wedge\Psi_--\bar\partial\sigma\wedge\Psi_++\Phi,$$

where

$$\Psi_+=e^{-\sigma}\sum_{i=1}^{r-1}\partial f_i\wedge\bar\partial\partial\bar f_i,\quad \Psi_-=e^{-\sigma}\sum_{i=1}^{r-1}\bar\partial\partial f_i\wedge\bar\partial\bar f_i,\quad \Phi=e^{-\sigma}\sum_{i=1}^{r-1}\bar\partial\partial f_i\wedge\bar\partial\partial\bar f_i,$$

and simplifying, we obtain

$$\begin{aligned}\det(I+\lambda\Theta)&=1+\frac{\lambda}{1-\lambda U}\Bigg(\bar\partial\partial\sigma-\lambda\Phi+\lambda\bar\partial\sigma\wedge\Psi_+-\lambda\bar\partial\sigma\wedge U\wedge F+\lambda\Psi_-\wedge F\\&-\lambda\bar\partial\sigma\wedge\partial\sigma\wedge U+\lambda\Psi_-\wedge\partial\sigma-\frac{\lambda}{1-\lambda U}\Big(-\bar\partial\sigma\wedge U\wedge F+\Psi_-\wedge F\\&+\lambda\bar\partial\sigma\wedge\partial\sigma\wedge U\wedge U+\lambda\Psi_-\wedge\Psi_+-\lambda\Psi_-\wedge U\wedge F-\lambda\Psi_-\wedge U\wedge\partial\sigma\\&-\lambda\bar\partial\sigma\wedge U\wedge\Psi_++\lambda\bar\partial\sigma\wedge U\wedge U\wedge F\Big)\Bigg)\\&=1+\frac{\lambda}{1-\lambda U}\Big(\bar\partial\partial\sigma+\lambda(-\Phi+\bar\partial\sigma\wedge\Psi_+-\bar\partial\sigma\wedge\partial\sigma\wedge U+\Psi_-\wedge\partial\sigma)\Big)\\&-\lambda^3\frac{\bar\partial U\wedge\partial U}{(1-\lambda U)^2}=1+\lambda\bar\partial\partial\sigma-\lambda^2\frac{\bar\partial\partial U}{1-\lambda U}-\lambda^3\frac{\bar\partial U\wedge\partial U}{(1-\lambda U)^2}.\qquad\square\end{aligned}$$

Corollary 2.9. *The following identities hold*

$$\sum_{l=1}^{r}(-1)^l \sum_{1\le i_1<\ldots<i_{l-1}\le r-1} \mathrm{ch}_k(E_l, h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}}))$$
$$= \frac{\delta_{kr}}{r-1}\left(\frac{\sqrt{-1}}{2\pi}\right)^r \bar{\partial}\partial\left(\frac{U^{r-1}}{(r-1)!}\right)$$
$$= \frac{\delta_{kr}}{r-1}\left(\frac{\sqrt{-1}}{2\pi}\right)^r \bar{\partial}\partial\left(e^{(r-1)\sigma}\partial f_1\wedge\bar{\partial}\bar{f}_1\wedge\cdots\wedge\partial f_{r-1}\wedge\bar{\partial}\bar{f}_{r-1}\right),$$

for $k = 1, \ldots, r$, *and*

$$\sum_{l=1}^{r}(-1)^l \sum_{1\le i_1<\ldots<i_{l-1}\le r-1} \mathrm{ch}_{r+1}(E_l, h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}}))$$
$$= \frac{1}{(r+1)!}\left(\sum_{m=1}^{r} m c_{r+1-m}(E_r, h)c_m(E_r, h)\right)$$

where it is understood that $h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}}) = e^\sigma$ *for* $l = 1$.

Proof. Replacing the functions f_i in the definition of the Hermitian metric $h(\sigma; f_1, \ldots, f_{r-1})$ by $t_i f_i$ with real t_i, we can consider the Chern forms $c_k(E_r, h)$ as polynomials in $t_1, \ldots, t_{r-1}$ with coefficients in the commutative ring $\mathcal{A}(X,\mathbb{C}) \cap \mathcal{A}^{\mathrm{even}}(X,\mathbb{R})$. In follows from the explicit formulas in Lemma 2.8, that these are polynomials in variables $\alpha_i = t_i^2$, which can be considered as nilpotent elements of order 2 since the forms $\partial f_i \wedge \bar{\partial}\bar{f}_i$ have the same property, and that the Chern forms $c_k(E_l, h(\sigma; t_{i_1} f_{1_1}, \ldots, t_{i_{l-1}} f_{i_{l-1}})$ can be obtained from the Chern form $c_k(E_r, h(\sigma; t_1 f_1, \ldots, t_{r-1} f_{r-1})$ by setting $t_j = 0$ for $j \in J$ — a complementary subset to $I = \{i_1, \ldots, i_{l-1}\}$ in the set $\{1, \ldots, r-1\}$. The Chern character forms are related to the Chern forms by Newton's identities

$$(-1)^k k!\mathrm{ch}_k = -kc_k + c_{k-1}\mathrm{ch}_1 - 2!c_{k-2}\mathrm{ch}_2 + \cdots + (-1)^k(k-1)!c_1\mathrm{ch}_{k-1},$$

so that the same relation holds between Chern character forms for the vector bundles E_l and E_r. Each of the terms in the above expression, i.e. $-kc_k$, $c_{k-1}\mathrm{ch}_1$, etc satisfy the property that their value for $(E_l, t_{i_1} f_{1_1}, \ldots, t_{i_{l-1}} f_{i_{l-1}}))$ can be obtained from E_r by replacing t_J with 0 for an appropriate multi-index. Now, any polynomial $p_k(\alpha_1, \ldots, \alpha_{r-1})$ of degree $k < r$ in order two nilpotents

α_i satisfies the property

$$\sum_{l=0}^{r-1}\sum_{|I|=l}(-1)^l p_k(0,0,\dots,\alpha_{i_1},\dots,\alpha_{i_l},\dots)=0$$

This property maybe proven easily for monomials and hence for polynomials. When $k=r$, $(-1)^{r-1}\sum_{l=0}^{r}\sum_{|I|=l}(-1)^l p_r(0,0,\dots,\alpha_{i_1},\dots)$ is the $\alpha_1\alpha_2\dots\alpha_{r-1}$ term of the polynomial p_r.

Hence the theorem is proven for $k<r$. It is easy to see (inductively) that ch_k is a polynomial of degree at most $k-1$. For the case $k=r$ the term $-c_r(E_r,h)/(k-1)!$ is the only one that survives, and when $k=r+1$ the "top degree" term actually corresponds to the top degree one in

$$\frac{1}{(r+1)!}\left(c_r(E_r)\mathrm{ch}_1(E_r)-2!c_{r-1}(E_r)\mathrm{ch}_2(E_r)+\dots\right)$$

The top term in ch_k corresponds to $\frac{(-1)^{k-1}}{(k-1)!}c_k$. Hence for $k=r+1$ the right-hand side of equation 2.9 is

$$\frac{1}{(r+1)!}\left(\sum_{m=1}^{r} mc_{r+1-m}(E_r)c_m(E_r)\right)$$

□

Remark 2.2.4. Setting $\bar\theta=h^{-1}\bar\partial h$, it is easy to obtain

$$\mathrm{tr}(\theta\wedge\bar\theta)=e^{-\sigma}\partial f\wedge\bar\partial\bar f+e^{-\sigma}\partial\bar f\wedge\bar\partial f.$$

To get rid of the second term and to write down the simplest nontrivial Bott-Chern form $\mathrm{bc}_1(h,I)$, where I is a trivial Hermitian metric on E_2, we need to add the "Wess-Zumino term" (rather its $(1,1)$-component) to the "kinetic term" $\mathrm{tr}(\theta\wedge\bar\theta)$. Such formula was first obtained by A. Alekseev and S. Shatashvili in [AS89], where for the case of Minkowski signature the decomposition

$$\begin{pmatrix}1 & \bar f\\ f & |f|^2+e^{\sigma}\end{pmatrix}=\begin{pmatrix}1 & 0\\ f & 1\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & e^{\sigma}\end{pmatrix}\begin{pmatrix}1 & \bar f\\ 0 & 1\end{pmatrix}$$

is replaced by the Gauss decomposition for $\mathrm{SL}(2,\mathbb{C})$.

Remark 2.2.5. The Bott-Chern forms (or rather their exponents) also appear quite naturally in supersymmetric quantum field theories as ratios of non-chiral partition functions for the higher dimensional analogues of the *bc*-systems [LMNS97].

2.2.3 The main result

The first result is an analogue of Lemma 2.3 for complex manifolds.

Lemma 2.10. *Let X be a complex manifold. Every $\omega \in \mathcal{A}^{k,k}(X,\mathbb{C}) \cap \mathcal{A}^{2k}(X,\mathbb{R})$ can be written as a finite linear combination of wedge products of real $(1,1)$-forms of the type $\sqrt{-1}e^{\sigma}\partial f \wedge \bar\partial \bar f$, where $\sigma \in C^\infty(X,\mathbb{R})$ and $f \in C^\infty(X,\mathbb{C})$. Moreover, if ω is zero on open $U \subset X$, than one can choose these forms such that all functions σ and f vanish on U.*

Proof. Let ω be a real form of type (k,k). According to Lemma 2.2., it is a finite sum of the terms

$$f_0 df_1 \wedge df_2 \wedge \cdots \wedge df_{2k} = f_0(\partial + \bar\partial) f_1 \wedge (\partial + \bar\partial) f_2 \wedge \cdots \wedge (\partial + \bar\partial) f_{2k}.$$

Since ω is of type (k,k), it is a finite sum of (k,k)-components of the forms above. Every such component is a function times the wedge product of the following factors (where f and g are some of the f_i's)

$$\partial f \wedge \bar\partial g + \bar\partial f \wedge \partial g = i\partial(f+ig) \wedge \bar\partial(f-ig) - i\partial f \wedge \bar\partial f - i\partial g \wedge \bar\partial g. \quad \square$$

Remark 2.2.6. Actually the above proof also proves lemma 1.2. We shall use this lemma in the section on applications.

Remark 2.2.7. For complex manifolds admitting a finite coordinate open cover and for submanifolds of $\mathbb{C}^n$, there is a different version of this lemma which would eventually lead to a simpler proof of the main theorem for these manifolds : Every $\omega \in \mathcal{A}^{k,k}(X,\mathbb{C}) \cap \mathcal{A}^{2k}(X,\mathbb{R})$ can be written as a finite linear combination of wedge products of real $(1,1)$-forms of the type $\sqrt{-1}g\bar\partial\partial\rho$ where ρ and g are smooth real functions on X. We will follow the proof of Lemma 2.3. Namely, let $\{U_\alpha\}_{\alpha\in A}$ be a finite coordinate open cover of X and $\{\rho_\alpha\}_{\alpha\in A}$ be a partition of unity subordinate to it, so that $\omega = \sum_{\alpha\in A} \rho_\alpha\, \omega|_{U_\alpha}$. Denoting by

$$z^1 = x^1 + \sqrt{-1}\, y^1, \ldots, z^n = x^n + \sqrt{-1}\, y^n$$

local complex coordinates in U_α, we can write

$$\omega|_{U_\alpha} = \sum_{I,J} f_{\alpha,IJ} dx^{i_1} \wedge \cdots \wedge dx^{i_l} \wedge dy^{j_1} \wedge \cdots \wedge dy^{j_m},$$

where $I = \{i_1, \ldots, i_l\}, J = \{j_1, \ldots, j_m\}$, $f_{\alpha,IJ} \in C^\infty(U_\alpha, \mathbb{R})$ and $1 \le i_1 < \cdots < i_l \le n$, $1 \le j_1 < \cdots < j_m \le n$, $l + m = 2k$. Since the form ω was supposed to be of (k,k) type, so are the forms $\omega|_{U_\alpha}$. On the other hand, the (k,k)-component of these forms can be obtained by rewriting them in complex coordinates using

$$dx^i = \frac{1}{2}(dz^i + d\bar{z}^i), \quad dy^i = \frac{1}{2\sqrt{-1}}(dz^i - d\bar{z}^i), \; i = 1, \dots, n,$$

and collecting terms of the type (k,k). If one of such terms has a factor $dz^i \wedge d\bar{z}^l$, $i,k \in I$, then it necessarily has a factor

$$(dz^i \wedge d\bar{z}^l + d\bar{z}^i \wedge dz^l),$$

if it comes from $dx^i \wedge dx^l$. Similarly, one has factors $(dz^j \wedge d\bar{z}^m + d\bar{z}^j \wedge dz^m)$, $j,m \in J$, coming from $dy^j \wedge dy^m$, and $\sqrt{-1}(dz^i \wedge d\bar{z}^j - d\bar{z}^i \wedge dz^j)$, coming from $dx^i \wedge dy^j$, $i \in I$ and $j \in J$.

In the first two cases corresponding factors can be written as

$$2\sqrt{-1}\partial\bar{\partial}(\mathrm{Im}(z^i\bar{z}^l)) \quad \text{and} \quad 2\sqrt{-1}\partial\bar{\partial}(\mathrm{Im}(z^j\bar{z}^m)),$$

whereas in the third case it takes the form $2\sqrt{-1}\partial\bar{\partial}(\mathrm{Re}(z^i\bar{z}^j))$. As in the proof of Lemma 2.3, let K_α be a compact set such that $\operatorname{supp}\rho_\alpha \subsetneq K_\alpha \subset U_\alpha$ and let b_α be the corresponding bump function. Then we see that all the terms will take the form $2\sqrt{-1}\partial\bar{\partial}\rho$, where $\rho(z) = \mathrm{Im}(b_\alpha(z)z^i\bar{z}^j)$ or $\rho(z) = \mathrm{Re}(b_\alpha(z)z^i\bar{z}^j)$. This proves the first part of the statement. The second statement of the lemma is obvious from the construction. For submanifolds of $\mathbb{C}^n$, the local part of the argument above carries over globally using a tubular neighbourhood argument just as in Lemma 2.3.

Definition. A form $\omega \in \mathcal{A}^{k,k}(X,\mathbb{C}) \cap \mathcal{A}^{2k}(X,\mathbb{R})$ is called a *form of pure type* if it is a wedge product of the forms $\sqrt{-1}\, e^\sigma \partial f \wedge \bar{\partial}\bar{f}$; if one or more factors contain $\sqrt{-1}\,\bar{\partial}\partial\rho$, it is called a *composite form.*

As mentioned in chapter 1, we have the following complex manifold analogue of Proposition 2.1.1.

Theorem 2.11. *For every $\bar{\partial}\partial$-exact form $\omega \in \mathcal{A}(X,\mathbb{C}) \cap \mathcal{A}^{\mathrm{even}}(X,\mathbb{R})$ there is a trivial vector bundle E over X with two Hermitian metrics h_1 and h_2 such that*

$$\mathrm{ch}(E,h_1) - \mathrm{ch}(E,h_2) = \omega.$$

Proof. It is convenient to introduce a virtual Hermitian bundle $\mathcal{E} = E - E$ with corresponding Hermitian metrics h_1 and h_2, and to rewrite the above equation as $\mathrm{ch}\,\mathcal{E} = \omega$. We observe that the thus defined Chern character form for virtual

Hermitian bundles is multiplicative: if $\mathcal{W}_1 = W_1 - W_1$ with Hermitian metrics h_{11} and h_{12}, and $\mathcal{W}_2 = W_2 - W_2$ with Hermitian metrics h_{21} and h_{22}, then

$$\operatorname{ch}\mathcal{W}_1 \operatorname{ch}\mathcal{W}_2 = \operatorname{ch}\mathcal{W},$$

where $\mathcal{W} = W - W$ and $W = (W_1 \otimes W_2) \oplus (W_1 \otimes W_2)$ with corresponding Hermitian metrics

$$h_1 = (h_{11} \otimes h_{21}) \oplus (h_{12} \otimes h_{22}) \quad \text{and} \quad h_2 = (h_{11} \otimes h_{22}) \oplus (h_{12} \otimes h_{21}).$$

Slightly abusing notations, we will write $\mathcal{W} = \mathcal{W}_1 \otimes \mathcal{W}_2$.
Let ω be a pure form of degree (k, k), $k > 1$, given by

$$\omega = \frac{1}{k-1} \left(\frac{\sqrt{-1}}{2\pi} \right)^k \bar{\partial}\partial \left(e^{(k-1)\sigma} \partial f_1 \wedge \bar{\partial}\bar{f}_1 \wedge \cdots \wedge \partial f_{k-1} \wedge \bar{\partial}\bar{f}_{k-1} \right).$$

It follows from Corollary 2.9 that

$$\omega = \operatorname{ch}_k \mathcal{F}_k \quad \text{and} \quad \operatorname{ch}_i \mathcal{F}_k = 0, \quad i = 1, \ldots, k-1, \tag{2.7}$$

where $\mathcal{F}_k = F_k - F_k$ and

$$\begin{aligned} F_k &= \bigoplus_{l=1}^{k} \tfrac{1}{2}\big(1 + (-1)^l\big) \binom{k-1}{l-1} E_l \\ &= \bigoplus_{l=1}^{k} \tfrac{1}{2}\big(1 - (-1)^l\big) \binom{k-1}{l-1} E_l \end{aligned}$$

with Hermitian metrics

$$\begin{aligned} h_{1k} &= \bigoplus_{l=1}^{k} \tfrac{1}{2}\big(1 + (-1)^l\big) \bigoplus_{1 \le i_1 < \ldots < i_{l-1} \le k-1} h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}}) \\ h_{2k} &= \bigoplus_{l=1}^{k} \tfrac{1}{2}\big(1 - (-1)^l\big) \bigoplus_{1 \le i_1 < \ldots < i_{l-1} \le k-1} h(\sigma, f_{i_1}, \ldots, f_{i_{l-1}}) \end{aligned}$$

for $k > 2$, whereas $F_2 = E_2 = E_1 \oplus E_1$ and $h_{12} = h(\sigma, f), h_{22} = e^{\sigma} \oplus 1$. Now, we may use induction to finish the proof. The base case is for the top forms for which there exist virtual bundles of the type $\mathcal{F}_n$ so that $\omega = ch(\mathcal{F}_n)$. By the induction hypothesis, there exists an $\mathcal{E}$ so that $ch(\mathcal{F}_k) - \omega = ch(\mathcal{E})$. Thus, $\omega = ch(\mathcal{F}_k - \mathcal{E})$. $\square$

Remark 2.2.8. As mentioned earlier, we may give a simpler proof for a large class of manifolds using the remark after lemma 3.4. Firstly, the statement holds for $(1,1)$-forms. Namely, since every real $\bar\partial\partial$-exact $(1,1)$ form is given by $\omega = \frac{\sqrt{-1}}{2\pi}\bar\partial\partial\sigma$, where $\sigma \in C^\infty(X,\mathbb{R})$, consider the trivial holomorphic line bundle E_1 with the Hermitian metric $h = e^\sigma$, so that

$$\operatorname{ch}(E,h) = \exp\omega = 1 + \omega + \frac{1}{2}\omega^2 + \cdots + \frac{1}{n!}\omega^n.$$

To get rid of all terms in this expression except ω, consider Hermitian metrics $e^{\alpha_i\sigma}$, $i = 1,\ldots,n+1$, and choose pair-wise distinct α_i such that the following system of equations

$$\begin{pmatrix} 1 & 1 & 1 & \ldots & 1 \\ \alpha_1 & \alpha_2 & \alpha_3 & \ldots & \alpha_{n+1} \\ \alpha_1^2 & \alpha_2^2 & \alpha_3^2 & \ldots & \alpha_{n+1}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \alpha_1^n & \alpha_2^n & \alpha_3^n & \ldots & \alpha_{n+1}^n \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

has an integer solution $c_1,\ldots,c_{n+1}$. Namely, for any choice of $n+1$ different rational numbers α_i the numbers c_i are also rational. If their least common denominator is $N > 1$, then for the numbers $\beta_i = \alpha_i/N$ the corresponding solution is integral. Now putting

$$(E,h_1) = \bigoplus_{c_i>0} c_i(E_1, e^{\beta_i\sigma}) \quad \text{and} \quad (E,h_2) = \bigoplus_{c_i<0} (-c_i)(E_1, e^{\beta_i\sigma}),$$

where $n(L,h)$ stands for the direct sum of n copies of a line bundle L with the Hermitian metrtic h, we get

$$\operatorname{ch}(E,h_1) - \operatorname{ch}(E,h_2) = \omega.$$

Hence, a form of the type $\bar\partial\partial(g\bar\partial\partial\rho)$ can be written as $ch(\mathcal{E}_1 \otimes \mathcal{E}_2)$. Using this, the theorem follows easily.

Remark 2.2.9. It immediately follows from the second statement of Lemma 2.10 and the proof of Theorem 1.1, that if form ω vanishes on open $U \subset X$, then Hermitian metrics h_1 and h_2 can be chosen such that $h_1 = h_2 = I$ — identity matrix — on U.

Corollary 2.12. *For every $\omega \in \mathcal{A}(X,\mathbb{C}) \cap \mathcal{A}^{\text{even}}(X,\mathbb{R})$ of degree not greater than $2n-2$, there is a trivial vector bundle E over X with two Hermitian* *metrics h_1 and h_2 such that in $\tilde{\mathcal{A}}(X,\mathbb{C})$*

$$\mathrm{BC}(E;h_1,h_2)=\omega.$$

Proof. It is analogous to the proof of Corollary 2.4. Namely, let $\Omega\in\mathcal{A}(X\times\mathbb{P}^1,\mathbb{C})\cap\mathcal{A}^{\mathrm{even}}(X\times\mathbb{P}^1,\mathbb{R})$ be such that under the inclusion map $i_p: X\to X\times\mathbb{P}^1$ one has $i_\infty^*(\Omega)=-\omega$ and $i_0^*(\Omega)=0$ in some neighborhood of 0 in $\mathbb{P}^1$. It follows from Theorem 1.1 that there is a trivial vector bundle $\tilde{E}$ over $X\times\mathbb{P}^1$ with two Hermitian metrics $\tilde{h}_1$ and $\tilde{h}_2$ such that

$$\frac{\sqrt{-1}}{2\pi}\,\bar{\partial}\partial\Omega=\mathrm{ch}(\tilde{E},\tilde{h}_1)-\mathrm{ch}(\tilde{E},\tilde{h}_2),$$

where the metrics $\tilde{h}_1$ and $\tilde{h}_2$ can be chosen such that $i_0^*(\tilde{h}_1)=i_0^*(\tilde{h}_2)=I$. Denoting by E a trivial vector bundle over X — a pullback of $\tilde{E}$ — and putting $h_1=i_\infty^*(\tilde{h}_1), h_2=i_\infty^*(\tilde{h}_2)$, we obtain, modulo $\operatorname{Im}\partial+\operatorname{Im}\bar{\partial}$,

$$\begin{aligned}\mathrm{bc}(E;I,h_1)-\mathrm{bc}(E;I,h_2)&=\int_{\mathbb{P}^1}\frac{\sqrt{-1}}{2\pi}\,\bar{\partial}\partial\,\Omega\log|z|^2\\&=\int_{\mathbb{P}^1}\frac{\sqrt{-1}}{2\pi}\,\bar{\partial}_z\partial_z\Omega\log|z|^2\\&=\int_{\mathbb{P}^1}\frac{\sqrt{-1}}{2\pi}\,\Omega\,\bar{\partial}_z\partial_z\log|z|^2\\&=i_\infty^*(\Omega)-i_0^*(\Omega)\\&=-\omega.\end{aligned}$$

Therefore in $\tilde{\mathcal{A}}(X,\mathbb{C})$,

$$\omega=-\mathrm{BC}(E;I,h_1)+\mathrm{BC}(E;I,h_2)=\mathrm{BC}(E;h_1,h_2).\quad\square$$

2.3 Applications

2.3.1 Differential K-theory

According to the definition of differential K-theory in [GS86], the K-group $\hat{K}_0(X)$ for complex manifold X is defined as the free abelian group generated by the triples (E,h,η), where E is a holomorphic vector bundle over X with Hermitian metric h and $\eta\in\tilde{A}(X,\mathbb{C})$ with the following relations. For every exact sequence $\mathscr{E}$

$$0 \longrightarrow F \xrightarrow{i} E \longrightarrow H \longrightarrow 0$$

of holomorphic vector bundles over X, endowed with arbitrary Hermitian metrics h_F, h_E and h_H, impose

$$(F, h_F, \eta') + (H, h_H, \eta'') = (E, h_E, \eta' + \eta'' - \mathrm{BC}(\mathscr{E}, h_E, h_F, h_H)), \tag{2.8}$$

where $\mathrm{BC}(\mathscr{E}, h_E, h_F, h_H) = \mathrm{bc}(\mathscr{E}, h_E, h_F, h_H) \bmod (\mathrm{Im}\,\partial + \mathrm{Im}\,\bar{\partial})$ (see Remark 2.2.3). It follows from (2.8) that in $\hat{K}_0(X)$

$$(E, h_1, \eta_1) = (E, h_2, \eta_2 - \mathrm{BC}(E, h_1, h_2)). \tag{2.9}$$

Now following [SS08], we define two Hermitian metrics h_1 and h_2 on the holomorphic vector bundle E to be equivalent, if $\mathrm{BC}(E, h_1, h_2) = 0$, and define a *structured* holomorphic Hermitian vector bundle $\mathcal{E}$ as a pair $(E, \{h\})$, where $\{h\}$ is the equivalence class of a Hermitian metric h. Our goal is to impose a relations on a free abelian group generated by $\mathcal{E}$ such the resulting group $H\hat{K}_0(X)$ is isomorphic to the "reduced" differential K-theory group $\hat{K}_0^{\mathrm{rd}}(X)$, a subgroup of $\hat{K}_0(X)$ with forms η of degrees not greater than $2n - 2$.

First we observe that it follows from (2.9) that the mapping

$$\mathcal{E} = (E, \{h\}) \mapsto \varepsilon(\mathcal{E}) = (E, h, 0) \in \hat{K}_0^{\mathrm{rd}}(X) \tag{2.10}$$

is well-defined. Next we show that when extended to to the free abelian group generated by the structured holomorphic Hermitian bundles, this mapping is onto. Indeed, for every $\eta \in \tilde{A}(X, \mathbb{C}) \cap A^{\mathrm{even}}(X, \mathbb{R})$ of degree not greater than $2n - 2$, let F be the trivial vector bundle over X with two Hermitian metrics h_1 and h_2 such that, according to Corollary 2.12,

$$(F, h_1, \eta) = (F, h_2, 0).$$

in $\hat{K}_0(X)$. Since

$$\begin{aligned}(E \oplus F, h \oplus h_1, \eta) &= (E, h, 0) + (F, h_1, \eta) \\ &= (E, h, \eta) + (F, h_1, 0),\end{aligned}$$

we obtain

$$(E, h, \eta) = (E, h, 0) + (F, h_2, 0) - (F, h_1, 0).$$

Finally, we define the group $H\hat{K}_0(X)$ as the quotient of the free abelian group generated by $\mathcal{E}$ modulo the relations — pullbacks of the defining relations for

$\hat{K}_0^{\mathrm{rd}}(X)$ by the mapping ε. Explicitly, for every exact sequence $\mathscr{E}$ of holomorphic vector bundles over X with Hermitian metrics h_F, h_E and h_H satisfying $\mathrm{BC}(\mathscr{E}; h_E, h_F, h_H) = 0$, we impose

$$(F, \{h_F\}) + (H, \{h_H\}) = (E, \{h_E\}).$$

2.3.2 Bando-Futaki invariants

Let (M, ω) be a compact Kähler manifold. A natural question is: Is there a metric in the Kähler class of ω such that its k^{th} Chern form is harmonic? This is not a very abstruse question. Indeed, in the case of the first Chern form, harmonicity is equivalent to having constant scalar curvature. For this purpose, Bando [Ban06] and Futaki [Fut83] introduced the some invariants namely, the Bando-Futaki invariants associated to a given Kähler class ω (henceforth denoted as $\mathcal{F}_k(X, \omega)$ where X is a holomorphic vector field) as obstructions to the harmonicity of the Chern forms. They are defined as follows:

$$\begin{aligned} c_k - H(c_k) &= \frac{i}{2\pi}\partial\bar{\partial} f_k \\ \mathcal{F}_k(X, \omega) &= \int_M L_X f_k \wedge \omega^{n-k+1} \end{aligned}$$

In Liu's paper [Liu10], these invariants were computed for a smooth hypersurface M of $\mathbb{CP}^n$ for the Fubini-Study Kähler class. Liu speculated that an "abstraction" of the procedure used would be desirable (in order to compute the same for complete intersections). Here, we simplify some aspects of Liu's proof (whilst following the same basic strategy) thus providing a possible abstraction of that method. Our starting point is the expression for the curvature of the hypersurface defined by $F = 0$ where F is a homogeneous polynomial with non-zero gradient[6]. We shall follow the same notation as in [Liu10]. In particular, $f = F[1, \frac{Z_1}{Z_0}, \ldots]$ and if $\frac{\partial f}{\partial z_1} \neq 0$, by the implicit function theorem, z_1 is a holomorphic function of the other coordinates. Define $a_i = \frac{\partial z_1}{\partial z_i}$. Let $\widetilde{g}$ be the metric on M induced by the Fubini-study metric ω_{FS}. Let $F_k = \frac{\partial F}{\partial Z_k}$

[6]Note that, one may compute the inverse of the metric and hence the curvature in a simpler manner by using lemma 1.2

and $\rho = \frac{\sum_{k=0}^{k=n} |F_k|^2}{(1+|z|^2)|F_1|^2}$.

$$\widetilde{g_{\mu\nu}} = \frac{\delta_{\mu\nu} + a_\mu \bar{a}_\nu}{1+|z|^2} - \frac{(\bar{z}_\mu + \bar{z}_1 a_\mu)(z_\nu + z_1 \bar{a}_\nu)}{(1+|z|^2)^2}$$

$$\Theta_{\mu\nu} = \sum_{i,j} \tilde{g_{ij}} dz_i \wedge d\bar{z}_j \delta_{\mu\nu} - \sum_j \tilde{g_{\mu j}} d\bar{z}_j \wedge dz_\nu - \frac{1}{\rho}(\sum \frac{\partial a_\mu}{\partial z_i} dz_i \wedge \frac{\partial \bar{a}_s}{\partial \bar{z}_j} g^{\tilde{\nu s}} d\bar{z}_j)$$

Now, we shall state and prove lemma 2.3 of [Liu10]

Lemma 2.13. *The qth Chern form of the degree d hypersurface M is*

$$c_q(\Theta) = \sum_{k=0}^{q} \alpha_{qk} (\frac{i}{2\pi}\omega)^k \wedge (\frac{i}{2\pi}\partial\bar{\partial}\xi)^{q-k}$$

where,

$$\begin{aligned}
\alpha_{00} &= 1 \\
\alpha_{qq} &= C(n+1,q) - d\alpha_{(q-1)(q-1)} \\
\alpha_{q(q-k)} &= -[d\alpha_{(q-1)(q-k-1)} + \alpha_{(q-1)(q-k)}] \ for \ k = 1, \ldots, q-1 \\
\alpha_{q0} &= (-1)^q
\end{aligned}$$

where q ranges from 1 to $n-1$, and, $C(n,a)$ is the number of ways of choosing a things out of n things. where $\xi = \ln(\frac{\sum_{k=0}^{n} |F_k|^2}{(\sum_{k=0}^{k=n} |Z_k|^2)^{d-1}})$.

Proof. We shall use the bra-ket notation to make the following computations look more suggestive. We shall also use lemma 1.2 quite often during the course of the proof.

$$\Theta_{ij} = \omega\delta_{ij} + v_i \wedge w_j + \alpha_i \wedge \beta_j$$

$$\det(I + t\Theta) = \det(I(1+t\omega) + t(|v\rangle\langle w| + |\alpha\rangle\langle\beta|))$$

$$= (1+t\omega)^{n-1} \det(I + \frac{t}{1+t\omega}(|v\rangle\langle w| + |\alpha\rangle\langle\beta|))$$

$$= (1+t\omega)^{n-1} \det(I + \frac{t}{1+t\omega}|v\rangle\langle w|) \det(I + \frac{\frac{t}{1+t\omega}|\alpha\rangle\langle\beta|}{I + \frac{t}{1+t\omega}|v\rangle\langle w|})$$

$$= (1+t\omega)^{n-1} \det(I + \lambda|v\rangle\langle w|) \det(I + \lambda(I - \frac{t|v\rangle\langle w|}{(1+t\gamma) + t\langle w|v\rangle})|\alpha\rangle\langle\beta|)$$

$$= (1+t\omega)^{n-1} \det(I + \frac{t}{1+t\omega}|v\rangle\langle w|) \det(I + \frac{\frac{t}{1+t\omega}|\alpha\rangle\langle\beta|}{I + \frac{t}{1+t\omega}|v\rangle\langle w|})$$

$$= (1+t\omega)^{n-1} \det(I + \lambda|v\rangle\langle w|) \det(I + \lambda A)$$

where $\omega = \widetilde{g}_{\mu\nu} dz_\mu \wedge d\bar{z}_\nu$, $v_\mu = -\sum_j \widetilde{g}_{\mu j} d\bar{z}_j$, $w_\nu = dz_\nu$, $\alpha_\mu = -\frac{1}{\rho}\frac{\partial a_\mu}{\partial z_i} dz_i$, $\beta_\nu = \frac{\partial \bar{a}_s}{\partial \bar{z}_j}\tilde{g}^{\nu s} d\bar{z}_j$, $\lambda = \frac{t}{1+t\omega}$, $|u\rangle = \frac{t}{(1+t\omega)+t\sum w_i \wedge v_i}|v\rangle$, $|a\rangle\langle b| = a_i \wedge b_j$.
Now notice that $A^2 = (\langle\beta|\alpha\rangle - \langle\beta|u\rangle\langle w|\alpha\rangle)A = -tr(A)A$. Using lemma 1.2, we see that

$$\det(1+t\Theta) = (1+t\omega)^{n+1}\frac{1}{1+t\omega+\frac{t}{\rho}\frac{\partial a_\mu}{\partial z_i}dz_i \wedge \frac{\partial \bar{a}_s}{\partial \bar{z}_j}\tilde{g}^{\mu s} d\bar{z}_j}$$

From [Liu10] we see that $\frac{1}{\rho}\frac{\partial a_\mu}{\partial z_i}dz_i \wedge \frac{\partial \bar{a}_s}{\partial \bar{z}_j}\tilde{g}^{\mu s}d\bar{z}_j = (d-1)\omega + \partial\bar{\partial}\xi$. Hence, we see that the coefficient of t^k in the above expression is

$$\begin{aligned} c_a(\Theta) &= \sum_b \sum_l C(n+1,b) d^l \omega^{b+l} (-1)^{a-b} C(a-b,l)(\partial\bar{\partial}\xi)^{a-b-l} \\ &= \sum_{k=0}^{a}\sum_{b=0}^{k} C(n+1,b) d^{k-b}\omega^k (-1)^{a-b} C(a-b,k-b)(\partial\bar{\partial}\xi)^{a-k} \end{aligned}$$

From this, the lemma follows. □

Now, we may compute the Bando-Futaki invariants using a generating series version of Liu's approach. The theorem we wish to prove is theorem (1.1) of [Liu10] :

Theorem 2.14. *Let M be a hypersurface in $\mathbb{CP}^n$ defined by a homogeneous polynomial F of degree $d \le n$. Let X be a holomorphic vector field on $\mathbb{CP}^n$ such that $XF = \kappa F$ for a constant κ. Then, the q-th Bando-Futaki Invariant is*

$$\mathcal{F}(X,\omega_{FS}) = -(n+1-d)^{n-q}\frac{(d-1)(n+1)}{n}\sum_{j=0}^{q-1}(-d)^j (j+1) C(n, q-j-1)\kappa$$

To prove this, our basic strategy is the same as Liu's, in that, we shall not compute the invariant directly. Instead, $i_X(c(\Theta)) - i_X(H(c(\Theta))) - \bar{\partial}(i_X(\partial f)) = 0$ (where c and f are the Chern and the Futaki polynomials respectively). Then, we shall write the left hand side as $\bar{\partial}\eta = 0$ and after that we shall find the harmonic part of eta to finally compute the integral. We shall use generating polynomials throughout. In the course of the proof we shall have to use lemma 1.2 repeatedly.

Proof. First, we recall that $\det(I+t\Theta) = \frac{(1+t\omega)^{n+1}}{1+t(\omega d+\partial\bar{\partial}\xi)}$. The harmonic part of the same maybe obtained by putting $\xi = 0$. Hence,

$$\begin{aligned} c - Hc &= (1+t\omega)^{n+1}\left(\frac{1}{1+t(\omega d + \partial\bar{\partial}\xi)} - \frac{1}{1+t\omega d}\right) \\ &= -t\partial\bar{\partial}\left(\frac{\xi(1+t\omega)^{n+1}}{(1+t(\omega d+\partial\bar{\partial}\xi))(1+t\omega d)}\right) \\ &= \partial\bar{\partial} f \end{aligned}$$

In what follows, θ is the "Hamiltonian" function [Liu10] such that $i_X\omega = -\bar{\partial}\theta$. We shall use the fact that i_X is a derivation (and hence the quotient and the product rules for derivatives maybe used when interpreted suitably).

$$\begin{aligned} i_X(Hc) &= \frac{(n+1)(1+t\omega)^n t i_X(\omega)(1+t\omega d) - t i_X(\omega) d (1+t\omega)^{n+1}}{(1+t\omega d)^2} \\ &= \bar{\partial}\left(\frac{t(1+t\omega)^n\theta(d-(n+1)-nt\omega d)}{(1+t\omega d)^2}\right) \\ &= \bar{\partial}\alpha_2 \end{aligned}$$

$$(I+t\Theta)^{-1} = \frac{1}{1+t\omega}\left(I + \frac{t}{1+t\omega}(|v\rangle\langle w| + |\alpha\rangle\langle\beta|)\right)^{-1}$$

$$= \frac{1}{1+t\omega}\left(I+\frac{t}{1+t\omega}|v\rangle\langle w|\right)^{-1}\left(I+\frac{t}{1+t\omega}\left(I+\frac{t}{1+t\omega}|v\rangle\langle w|\right)^{-1}|\alpha\rangle\langle\beta|\right)^{-1}$$

$$= \frac{1}{1+t\omega}\left(I - \frac{\frac{t}{1+t\omega}|v\rangle\langle w|}{1+\frac{t}{1+t\omega}\langle w|v\rangle}\right)\left(I - \frac{\frac{t}{1+t\omega}(I+\frac{t}{1+t\omega}|v\rangle\langle w|)^{-1}|\alpha\rangle\langle\beta|}{1+\frac{t}{1+t\omega}\langle\beta|(I+\frac{t}{1+t\omega}|v\rangle\langle w|)^{-1}|\alpha\rangle}\right)$$

Now, noticing that $\langle w|v\rangle = -\omega$ and $\langle \beta|\alpha\rangle = (d-1)\omega + \partial\bar{\partial}\xi$, we see that,

$$(I+t\Theta)^{-1} = \frac{1}{1+t\omega}(I - t|v\rangle\langle w|)(I - \frac{t(I-t|v\rangle\langle w|)|\alpha\rangle\langle\beta|}{1+t\omega d + t\partial\bar{\partial}\xi - t^2\langle\beta|v\rangle\langle w|\alpha\rangle})$$

$$\begin{aligned}
i_X(c) &= \det(I+t\Theta)\mathrm{tr}(t i_X(\Theta)(I+t\Theta)^{-1}) \\
&= -t\bar{\partial}(\det(I+t\Theta)\mathrm{tr}(\nabla X(I+t\Theta)^{-1})) \\
&= -t\bar{\partial}\left(\det(I+t\Theta)\frac{1}{1+t\omega}\mathrm{tr}(\nabla X(I - t|v\rangle\langle w|\right) \\
&\quad -\nabla X\left(\frac{t(I-(\omega t^2+2t)|v\rangle\langle w|)|\alpha\rangle\langle\beta|}{1+t\omega d + t\partial\bar{\partial}\xi - t^2\langle\beta|v\rangle\langle w|\alpha\rangle})\right) \\
&= -t\bar{\partial}(\frac{\det(I+t\Theta)}{1+t\omega}(\mathrm{div}(X) + t\langle w|\nabla X|v\rangle \\
&\quad + \frac{t\langle\beta|\nabla X|\alpha\rangle - t\langle\beta|\nabla X|v\rangle\langle w|\alpha\rangle(\omega t^2+2t)}{1+t\omega d + t\partial\bar{\partial}\xi - t^2\langle\beta|v\rangle\langle w|\alpha\rangle}))
\end{aligned}$$

We use the following equations from [Liu10]

$$\begin{aligned}
(\nabla X)^l_k &= -\widetilde{g}^{lj}\partial_k\bar{\partial}_j\theta \\
\Phi &= -\frac{1}{\rho}X^l_{;k}\frac{\partial a_l}{\partial z^p}\frac{\partial \bar{a}_s}{\partial \bar{z}^q}\widetilde{g}^{k\bar{s}}dz^p\wedge d\bar{z}^q \\
&= \mathrm{div}(X)((d-1)\omega + \partial\bar{\partial}\xi) - \partial\bar{\partial}\theta + \partial\bar{\partial}\Delta\theta \\
&- (n+1)\theta((d-1)\omega + \partial\bar{\partial}\xi)
\end{aligned}$$

Hence, upon simplification (recall that since $a_i = \frac{\partial z_1}{\partial z_i}$, $\langle w|\alpha\rangle = 0$),

$$\begin{aligned}
i_X(c) &= -t\bar{\partial}\left(\frac{(1+t\omega)^n}{1+t\omega d + t\partial\bar{\partial}\xi}(\mathrm{div}(X) + t\bar{\partial}\partial\theta - \frac{t\Phi}{1+t\omega d + t\partial\bar{\partial}\xi})\right) \\
&= \bar{\partial}\alpha_1 \\
i_X(\partial f) &= t\frac{-X(\xi)(1+t\omega)^{n+1}}{(1+t(\omega d+\partial\bar{\partial}\xi))(1+t\omega d)} \\
&- \frac{t^2(1+t\omega)^{n+1}\partial\xi}{(1+t(\omega d+\partial\bar{\partial}\xi))^2(1+t\omega d)^2}\bar{\partial}[\theta((n+1-d+nt\omega d)(1+t(\omega d+\partial\bar{\partial}\xi)) \\
&-(1+t\omega)(1+t\omega d)d) + X(\xi)((1+t\omega)(1+t\omega d))]
\end{aligned}$$

It is easy to see that, for an appropriate form γ, we have,

$$\alpha_1-\alpha_2-i_X(\partial f)=\bar\partial\gamma-t\frac{(1+t\omega)^n}{1+t(\omega d+\partial\bar\partial\xi)^2}(\mathrm{div}(X)(1+t\omega)$$
$$+(n+1)t\theta((d-1)\omega+\partial\bar\partial\xi))+\frac{t(1+t\omega)^n}{(1+t\omega d)^2}\theta(nt\omega d+n+1-d)-i_X(\partial f)$$

We shall use this identity [Lu99],

$$\mathrm{div}(X)-X(\xi)-(n-d+1)\theta \;=\; -\kappa$$

Replacing $\mathrm{div}(X)$ by the above identity and simplifying we have,

$$\alpha_1-\alpha_2-i_X(\partial f)=\bar\partial\gamma+t^2\frac{\kappa(1+t\omega)^{n+1}}{(1+t(\omega d+\partial\bar\partial\xi))^2}$$
$$-\bar\partial(\frac{t^2(1+t\omega)^{n+1}\partial\xi}{(1+t(\omega d+\partial\bar\partial\xi))^2(1+t\omega d)}$$
$$\times[\frac{\theta((n+1-d+nt\omega d)(1+t(\omega d+\partial\bar\partial\xi))-(1+t\omega)(1+t\omega d)d)}{1+t\omega d}$$
$$+(1+t\omega)X(\xi)])$$

Thus, the harmonic part is $t^2\frac{\kappa(1+t\omega)^{n+1}}{(1+t(\omega d+\partial\bar\partial\xi))^2}$. Notice that (the integral of a non-top form is defined to be zero),

$$\int_M L_Xf\wedge\frac{1}{1-\omega}=\int_M(di_X+i_X\partial)f\wedge\frac{1}{1-\omega}$$
$$=\int_M\left(\alpha_1-\alpha_2-t^2\frac{\kappa(1+t\omega)^{n+1}}{(1+t(\omega d+\partial\bar\partial\xi))^2}\right)\wedge\frac{1}{1-\omega}$$
$$=\int_M\left(\alpha_1-\frac{t(1+t\omega)^n\theta(d-(n+1)-nt\omega d)}{(1+t\omega d)^2}-t^2\frac{\kappa(1+t\omega)^{n+1}}{(1+t\omega d)^2}\right)\wedge\frac{1}{1-\omega}$$

where Stokes' theorem was used to deduce that $\int_M di_Xf\wedge\frac{1}{1-\omega}=0$, to replace $1+t(\omega d+\partial\bar\partial\xi)$ by $1+t\omega d$ and, to ignore the integral of the anharmonic part of $\alpha_1-\alpha_2-i_X(\partial f)$.

From lemma 2.6 of [Liu10], it follows that $\int_M\frac{\alpha_1}{1-\omega}=0$. After replacing t by $\frac{i}{2\pi}$, one may easily compute the integral using the facts that $\int_M\theta\omega^{n-1}=\frac{\kappa}{n}$ and $\int\omega^{n-1}=d$. This completes the proof. □

2.3.3 Computation of Bott-Chern forms

Let X be a compact, complex manifold. Given an exact sequence of hermitian holomorphic bundles (the metrics are arbitrary and $\bar{V}$ indicates V has a metric) $\mathcal{F} = 0 \to \bar{S} \to \bar{E} \to \bar{Q} \to 0$, if $\phi(x)$ is an invariant polynomial, then, Chern-Weil theory associates a form $\phi(\frac{i\Theta}{2\pi})$ where Θ is the curvature form of a connection. The cohomology class of this form is well-defined and $[\phi(\bar{E}) - \phi(\bar{S} \oplus \bar{Q})] = [0]$ where, the bundles are endowed with Chern connections. In this situation, Gillet and Soule [BGS88b, BGS88a, BGS88c] axiomatically introduced the Bott-Chern classes $\tilde{\phi}(\mathcal{F})$ to satisfy the following equation (along with natural functoriality conditions)

$$\phi(\bar{E}) - \phi(\bar{S} \oplus \bar{Q}) = \frac{i}{2\pi}\partial\bar{\partial}\tilde{\phi}(\mathcal{F})$$

They proved that, modulo image of ∂ and $\bar{\partial}$, these classes are uniquely defined. It was also proved that, if ψ was another invariant polynomial,

$$\begin{aligned} \widetilde{\psi\phi}(\mathcal{F}) &= \tilde{\phi}(\mathcal{F})\psi(\bar{S} \oplus \bar{Q}) + \tilde{\psi}(\mathcal{F})\phi(\bar{E}) \\ \widetilde{\psi + \phi}(\mathcal{F}) &= \tilde{\psi}(\mathcal{F}) + \tilde{\phi}(\mathcal{F}) \end{aligned}$$

These secondary classes were then used to define a differential K-theory, arithmetic characteristic classes and to prove the arithmetic Riemann-Roch theorem. Computing these forms is necessary for Arakelov geometry. Mourougane [Mou04] computed these forms when ϕ is the Chern class and $\mathcal{F}$ is the metrised relative Euler sequence on the projectivisation $\mathbb{P}(E)$ of a hermitian holomorphic vector bundle E on X i.e. on $0 \to \mathcal{O}_E(-1) \to \pi^*(E^*) \to T\mathbb{P} \otimes \mathcal{O}_E(-1) \to 0$ endowed with the natural metric inherited from E. Our notation will be exactly the same as Mourougane's.

To begin with, we quickly review Bott-Chern's homotopy formula (in a special case) for computing these classes. If $0 \to S \xrightarrow{\iota} E \xrightarrow{p} Q \to 0$ is an exact sequence with the metrics on S and Q inherited from the one on E, then, consider the family of connections $\nabla_u = \nabla + (u-1)P_Q \nabla P_S$ where ∇ denotes the Chern connection, $P_Q = p^* p$, $P_S = \iota\iota^*$ are the orthogonal projections. In a frame wherein, E splits to second order as $S \oplus Q$ i.e. in a normal frame, it turns out that [Mou04]

$$c_k(\bar{E}) - c_k(\bar{S} \oplus \bar{Q}) = \frac{i}{2\pi}\partial\bar{\partial}\int_0^1 \frac{\Phi_k(u) - \Phi_k(0)}{u} du$$

where $\Phi_k(u) = \mathrm{coeff}_\lambda \det_k(\Theta(\nabla_u) + \lambda P_S)$ and $\det(I + tA) = \sum t^k \det_k(a)$. If S is a line bundle, then, $\Phi_{d+1}(u) = \det_d((1-u)\Theta_Q + up\Theta_E p^*)$.

Consider a point $(x_0, [a_0^*])$ on $\mathbb{P}(E)$. Choose coordinates $x_1, \dots x_n$ at x_0 and a frame $e_1^*, \dots, e_r^*$ for E normal at x_0 (i.e. $\langle e_i^*, e_j^* \rangle = \delta_{ij} - 2\pi \sum c_{\lambda\mu} j i x_\lambda \bar{x}_\mu + O(|x|^3)$) such that $a_0^* = e_1^*(x_0)$. If $z_j = \frac{a_j}{a_1}$ are coordinates on the fibres on the open set $a_1 \neq 0$, we see that [Mou04]

$$\begin{aligned}
\Phi_d(u) &= \det_{d-1}(c_{jk} + (1-u)\frac{i}{2\pi} dz_j \wedge d\bar{z}_k) \\
\Phi(u,t) &= \sum t^d \Phi_{d+1}(u) \\
&= \det(I + t(c_{jk} + (1-u)\frac{i}{2\pi} dz_j \wedge d\bar{z}_k))
\end{aligned}$$

Our goal is to give a simple proof of Theorem 1 of [Mou04] [7].

Theorem 2.15.

$$\widetilde{c_t}(h) = -H(-\Omega_t) c_t^{'}(E^*)$$

where,

$$\begin{aligned}
\mathcal{H}_s &= \sum_1^s \frac{1}{i} = \int_0^1 \frac{1-(1-u)^s}{u} \\
H(X) &= \sum \mathcal{H}_s X^s \\
\Omega_t &= \sum (-t)^d \Omega_{d,1} \\
\Omega_{p,1} &= \frac{i}{2\pi} \sum_{2 \leq i_1, \dots, t_p \leq r} c_{i_1 i_2} \dots c_{i_{p-1} i_p} dz_{i_p} \wedge d\bar{z}_{i_1} \\
c_k^{'}(E^*) &= \det_k(c_{ij}) \\
\widetilde{c_t} &= \sum t^d \widetilde{c_{d+1}} \\
c_t^{'}(E^*) &= \sum t^d c_d^{'}(E^*)
\end{aligned}$$

[7] Note that, one may, in a similar way also give a simpler proof of proposition 4 of [Mou04]

Proof. Let $[C]_{jk} = c_{jk}$ and $|v\rangle = dz_j$. So,

$$\begin{aligned}
\Phi(u,t) &= \det(I+tC)\det(I+t(I+tC)^{-1}(1-u)\frac{i}{2\pi}|v\rangle\langle v|) \\
&= \det(I+tC)\frac{1}{1+t(1-u)\frac{i}{2\pi}\langle v|(I+tC)^{-1}|v\rangle} \\
&= c_t^{'}(E^*)\frac{1}{1+t(1-u)\frac{i}{2\pi}\langle v|(I+tC)^{-1}|v\rangle}
\end{aligned}$$

where the second equality follows from lemma 1.2. Hence,

$$\begin{aligned}
\widetilde{c_t}(h) &= \int_0^1 \frac{c_t^{'}(E^*)t\frac{i}{2\pi}\langle v|(I+tC)^{-1}|v\rangle}{(1+(1-u)(t\frac{i}{2\pi}\langle v|(I+tC)^{-1}|v\rangle))(1+(t\frac{i}{2\pi}\langle v|(I+tC)^{-1}|v\rangle))}du \\
&= -c_t^{'}(E^*)\frac{\ln(1+\alpha)}{1+\alpha}
\end{aligned}$$

where

$$\begin{aligned}
\alpha &= t\frac{i}{2\pi}\langle v|(I+tC)^{-1}|v\rangle \\
&= -\Omega_t
\end{aligned}$$

The well-known harmonic number generating function is $\sum \mathcal{H}_i t^i = \frac{\ln(1-t)}{t-1}$. Using this, the theorem follows immediately. □

We now proceed to outline an algorithm to compute the Bott-Chern forms for the Todd class of the Euler sequence (whose defining invariant polynomial is $Td(A) = \det(\frac{A}{1-e^{-A}}) = \det(\sum b_q A^q)$). Let $T(A) = \frac{A}{1-e^{-A}}$. Then, it maybe easily proved (in the spirit of lemma 1.2) that,

$$\begin{aligned}
\frac{d}{d\lambda}\ln(Td(\lambda\Theta)) &= \frac{k - \mathrm{tr}(T(-\lambda\Theta))}{\lambda} \\
\ln(Td(\lambda\Theta)) &= \sum_{q=1}\frac{b_q(-1)^q}{q}\lambda^q\mathrm{tr}(\Theta^q) \\
\ln(c(\Theta)) &= -\sum_{q=1}\frac{(-1)^q}{q}\lambda^q\mathrm{tr}(\Theta^q)
\end{aligned}$$

where k is the rank of the vector bundle under consideration. Hence, knowing the Bott-Chern forms for $\ln(c(\Theta))$ would enable us to compute the same for $\ln(Td)$, from which those of Td maybe derived. Indeed, if $0 \to \bar{S} \to \bar{E} \to$

$\bar{Q} \to 0$ is an exact sequence, then, for an invariant polynomial ϕ,

$$\ln(\phi(E)) - \ln(\phi(S \oplus Q)) = \ln\left(1 + \frac{i}{2\pi}\frac{\partial\bar{\partial}\tilde{\phi}}{\phi(S \oplus Q)}\right)$$

$$= \frac{i}{2\pi}\partial\bar{\partial}\left(\sum_{k=0}(-1)^k(\frac{i}{2\pi}\frac{\phi(E) - \phi(S \oplus Q)}{\phi(S \oplus Q)})^k \frac{\tilde{\phi}}{\phi(S \oplus Q)}\right)$$

$$\widetilde{\ln(\phi)} = \tilde{\phi}\frac{\ln\left(\frac{\phi(E)}{\phi(S \oplus Q)}\right)}{\phi(E) - \phi(S \oplus Q)} \tag{2.11}$$

If one knows the Chern classes of S, E and Q, one may compute the Todd classes (in principle) using 2.11. Then, using 2.11, one obtains the Bott-Chern forms for the Todd class. Thus, for the Euler exact sequence, it suffices to know the Bott-Chern forms for the Chern classes and the classes themselves for $\mathcal{O}_E(-1)$, $\pi^*(E^*)$ and $T\mathbb{P} \otimes \mathcal{O}_E(-1)$. The Bott-Chern classes are given by Mourougane's formula [Mou04]. The Chern classes are

$$\begin{aligned}
c(\pi^*(E^*)) &= \pi^*(c(E^*)) \\
c(\mathcal{O}_E(-1)) &= 1 - t\alpha \\
c(T\mathbb{P} \otimes \mathcal{O}_E(-1)) &= \det(I + t(c_{jk} + \frac{i}{2\pi}dz_j \wedge d\bar{z}_k)) \\
&= \Phi(0,t) \\
&= c^{'}(E^*)\frac{1}{1 - \Omega_t} \\
&= c^{'}(E^*)\frac{1}{1 + \frac{i}{2\pi}t(\partial\bar{\partial}\Theta_t - \frac{\Theta_t - 1}{t} + \alpha\Theta_t + \frac{i}{2\pi}\frac{\bar{\partial}\Theta_t\partial\Theta_t}{\Theta_t})}
\end{aligned}$$

where $\alpha(x, [a^*]) = \Omega_{FS}(x, [a^*]) - \frac{\pi^*(\langle\Theta(E^*)(x)a^*, a^*\rangle)}{\|a^*\|^2}$, Ω_{FS} is the Fubini-Study metric on the fibre of $\mathbb{P}(E_x)$, $\Theta_t = \sum_{d=0}(-t)^d \frac{\pi^*(\langle\Theta^d(E^*)(x)a^*, a^*\rangle)}{\|a^*\|^2}$, and the last equality is from [Mou04].

Chapter 3

The analytic aspects

On a compact Kähler manifold, if a closed, real $(1,1)$ form α satisfying $[\alpha] = c_1(V)$ is given, where V is a holomorphic vector bundle, then one may find a metric h so that its first Chern form is α. Indeed, choose any smooth metric h_0 and conformally deform it to $h_0 e^{-f}$; we wish to solve $\mathrm{tr}\left(\frac{\imath(\Theta_0 + i\partial\bar{\partial}f)}{2\pi}\right) = \alpha$. This can be done easily using the d-dbar lemma of Hodge theory.

In view of the above, given a (k,k) form η representing the kth Chern character class $[tr\left((\frac{\imath\Theta}{2\pi})^k\right)]$ of a vector bundle, it is very natural to ask whether there is a metric whose induced Chern connection realises $tr\left((\frac{\imath\Theta}{2\pi})^k\right) = \eta$. As phrased, this question seems almost intractable. It is not even obvious as to whether there is *any* connection satisfying this requirement, leave aside a Chern connection. Work along these lines was done by Datta in [Dat04] using the h-principle. Therefore, it is more reasonable to ask whether equality can be realised for the top Chern character form. To restrict ourselves further, we shall ask whether any given metric h_0 may be conformally deformed to $h_0 e^{-\phi}$ so as to satisfy the PDE

$$\mathrm{tr}((i\widetilde{\Theta_0} + i\partial\bar{\partial}\phi)^n) = \eta$$

If $\widetilde{\Theta_0} = \Theta_0 + \omega I$, where, ω is a Kähler form, then the above equation becomes a special case of the equation treated in [Pin12]

$$\alpha_0(\omega + dd^c\phi)^n + \alpha_1 \wedge (\omega + dd^c\phi)^{n-1} + \ldots + \alpha_{n-1} \wedge (\omega + dd^c\phi) = \eta \qquad (3.1)$$

where $d\alpha_i = 0$, and $\displaystyle\int_X \eta = \int_X (\alpha_0\omega^n + \alpha_1 \wedge \omega^{n-1} + \ldots)$. As mentioned earlier, a special case of this equation arose in a very different context as a conjecture of X.X. Chen [Che00]. I state the conjecture here:

Conjecture 3.1. *The following Monge-Ampère-type equation*

$$(\Omega + dd^c\phi)^n = \sum_{i=0}^{n-1} C_i(\Omega + dd^c\phi)^i \wedge \chi^{n-i}$$

where C_i *are integers,* Ω *and* χ *are smooth, closed and positive forms, and* $\displaystyle\int_X \Omega^n = \int_X \sum_{i=0}^{n-1} C_i\Omega^i \wedge \chi^{n-i}$*, has a smooth solution* ϕ*.*

The $n = 2$ case was studied in [Che00]. Some other cases were studied in [Lai11] wherein the equation was reduced to inverse hessian-type equations [FL12].

3.0.4 Standard results used

Our principal tool to study fully nonlinear PDE like equation 3, is the method of continuity (It is like a flow technique. In fact this analogy was exploited more seriously, to great advantage, in [Rub08]). To solve $Lu = f$ where L is a nonlinear operator, one considers the family of equations $Lu_t = \gamma(t)$ where, $\gamma(1) = f$ and $\gamma(0) = g$ such that, at $t = 0$, one has a solution $Lu_0 = g_0$. Then, one proves that the set of $t \in [0, 1]$ for which the equation has a solution is both, open and closed (and clearly non-empty). In order to prove openness, one considers L to be a map between appropriate Banach spaces. Then, the implicit function theorem of Banach spaces proves openness. However, while dealing with equations like Monge-Ampère equations, one has to verify that certain conditions like ellipticity are preserved along the "continuity path". This is crucial because, in order to solve the linearised equation, and to prove that indeed one has a solution in an appropriate Banach space, one needs ellipticity in these cases. In fact, in a few of the cases we shall consider, ellipticity is not preserved and hence, the best we can do is a "short-time" existence result. In order to prove closedness, one needs to prove uniform (i.e. independent of t) *a priori* estimates for u. In our case, we shall need these estimates in $C^{2,\alpha}$ in order to use the Arzela-Ascoli theorem to conclude closedness. These estimates are usually proved by improving on lower order estimates. Once, one produces a $C^{2,\alpha}$ solution, one "bootstraps" the regularity (at teach $t \in [0, 1]$) using the Schauder estimates. The Schauder estimates on a compact manifold (without boundary) are (they can be derived easily using similar interior and boundary ones in domains in $\mathbb{R}^n$ [GT01]).

Theorem 3.2. Schauder *a priori* estimates on a Riemannian manifold*: If* $Lu = f$*, where* L *is a second-order, uniformly elliptic operator with smooth* *coefficients, and u is a $C^{2,\alpha}$ and f is a $C^{0,\alpha}$ function,*

$$\|u\|_{C^{2,\alpha}} \leq C(\|u\|_{C^0} + \|f\|_{C^{0,\alpha}})$$

In order to derive *a priori* estimates, we shall use standard techniques as in [Yau78], [Tia00] for the manifold case, and [CKNS85] for the Euclidean case. The main blackbox is the Evans-Krylov-Safanov theory for proving $C^{2,\alpha}$ estimates from C^2 ones. This requires (apart from uniform ellipticity) concavity of the equation. There is a similar version for the complex case. The real version is :

Theorem 3.3. *Let u be a smooth function on the unit ball satisfying,*

$$F(D^2u, x, Du) = g$$

on the unit ball in $\mathbb{R}^n$ centred at the origin $B(0,1)$ with $u = 0$ on the boundary of the ball. Here, F is a smooth function defined on a convex open set of symmetric $n \times n$ matrices $\times\mathbb{R} \times \mathbb{R}^n$ which satisfies,
a) Uniform ellipticity on solutions *: There exist positive constants λ and Λ so that $0 < \lambda|\xi|^2 \leq F_{ij}(D^2u, x, Du)\xi_i\xi_j \leq \Lambda|\xi|^2$ for all vectors ξ and all u satisfying the equation.*
b) Concavity on a convex open set *: F is a concave function on a convex open set of symmetric matrices (containing D^2u for all solutions u).*
Then, $\|u\|_{C^{2,\alpha}(B(\bar{0},1))} \leq C$ where C and α depend on the first and second derivatives of F, $\|u\|_{C^2(\bar{B})}$, $\|g\|_{C^2(\bar{B})}$, n, λ and Λ.

The complex, interior version (that we need) is :

Theorem 3.4. *Let u be a C^4 function on the unit ball in $\mathbb{C}^n$ satisfying*

$$F(u_{i\bar{j}}, z, \bar{z}) = 0$$

for a $C^{2,\beta}$ function $F(x, p, \bar{p})$ satisfying,
a) Uniform ellipticity on solutions *: There exist positive constants λ and Λ so that $0 < \lambda|\xi|^2 \leq F_{i\bar{j}}(dd^cu, z, \bar{z})\xi_i\xi_j \leq \Lambda|\xi|^2$ for all vectors ξ and all u satisfying the equation.*
b) Concavity on a convex open set *: F is a concave function on a convex open set of hermitian matrices (containing $u_{i\bar{j}}$ for all solutions u).*
Then, $\|u\|_{C^{2,\alpha}(B(0,\frac{1}{2}))} \leq C$ where, C and α depend on λ, Λ, n, and $\|u_{i\bar{j}}\|_{C^0(\bar{B})}$ and uniform bounds on the first and second derivatives of F evaluated at u.

The proofs are standard [GT01], [Siu87], [Blo12], [Kaz85]. Usually, one proves these estimates when the PDE is concave on all symmetric matrices.

In Monge-Ampère equations, one needs a weaker requirement of being concave on a convex open set of symmetric matrices [CKNS85] (the proofs go through easily with this requirement).

To conclude, we add a few words about uniqueness. The usual technique for demonstrating uniqueness (due to Calabi) of $Lu = f$ is to assume two solutions u_1 and u_2, and to write $0 = Lu_1 - Lu_2 = \int_0^1 \frac{dL}{dt}(tu_2 + (1-t)u_1)dt$. If the integrand is an elliptic operator, by the maximum principle, $u_1 = u_2$.

3.0.5 A general result

We state a somewhat general theorem about uniqueness, openness and C^0 estimates. The proof is quite standard (adapted largely from [Tia00] which is in turn based on [Yau78]). Although the theorem is folklore, we haven't found the precise statement (in this level of generality) in the literature on the subject. In what follows, positivity of (p,p) forms is strong positivity. Let $\mathcal{B}$ be the product of Banach submanifolds of forms wherein, an element of $\mathcal{B}$ is of the form $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, \phi)$ where α_i are $C^{1,\beta}$ (i,i), closed forms and ϕ is a $C^{3,\beta}$ function satisfying $\displaystyle\int_M \phi = 0$, $n\alpha_0(\omega + dd^c\phi)^{n-1} + (n-1)\alpha_1 \wedge (\omega + dd^c\phi)^{n-2} + \ldots + \alpha_{n-1} > 0$ and $\displaystyle\int_X (\sum_i \alpha_i \wedge \omega^{n-i}) \neq 0$. Also, let $\tilde{\mathcal{B}}$ be the Banach submanifold of $C^{1,\beta}$ top forms γ with $\displaystyle\int_X \gamma = 1$ and $\gamma > 0$.

Theorem 3.5. *If $\alpha_0\omega^n + \alpha_1\omega^{n-1} + \ldots > 0$, $\eta > 0$ and $d\alpha_i = 0$, then, any smooth solution ϕ of 3 satisfying $\int_X \phi\omega^n = 0$ and $\kappa \geq K\omega^{n-1}$ where $K > 0$ and $\displaystyle\sum_k (\alpha_k(\omega + dd^c\phi)^{n-k} - \alpha_k\omega^{n-k}) = \kappa \wedge dd^c\phi$, is bounded a priori: $\|\phi\|_{C^0} \leq C_\eta$. Also, if $\alpha_i > 0\ \forall i$ and if there exists a smooth solution ϕ such that $\omega + dd^c\phi > 0$, it is unique (upto a constant) among all such solutions; In addition, the mixed derivatives of ϕ are bounded a priori $\|\phi\|_{C^{1,1}} \leq C_\eta$.*

The map $T : \mathcal{B} \to \tilde{\mathcal{B}}$ defined by $T(\alpha_0, \alpha_1, \ldots, \phi) = \frac{\sum_i \alpha_i \wedge (\omega + dd^c\phi)^{n-i}}{\int_X (\sum_i \alpha_i \wedge \omega^{n-i})}$ is open and so is the restriction of T to a subspace defined by fixing the α_i. Also, a level set of this map is locally a graph with ϕ being a function of the α_i.

When $n = 2$, and $\alpha_0 = 1$, $\eta - \alpha_2 + \frac{\alpha_1^2}{4} > 0$, there exists a unique, smooth solution to 3 satisfying $\omega + dd^c\phi + \frac{\alpha_1}{2} > 0$.

Remark 3.0.1. In particular, if $\alpha_i = \omega^i$ for some i and all the other α_j are small enough, then, by the solution of the k-Hessian equations [HMW10], [DK12] we have a smooth solution of equation 3.

Remark 3.0.2. One may formulate a version of the same problem locally as a Dirichlet problem on a pseudoconvex domain in $\mathbb{C}^n$. In this context, we note that viscosity solutions to the Dirichlet problem exist by [HL11] and [CKNS85].

Proof: The proof of this theorem is similar to the one for the usual Monge-Ampère equation [Tia00].
The C^0 estimate : As usual, without loss of generality, we may change the normalisation to $\sup\phi = -1$ i.e. we may add $-1-\sup\phi$ to ϕ. Indeed, if the new ϕ has a C^0 estimate, then, $\int_X \phi = 0$ yields the desired C^0 estimate. This means, we just have to find a lower bound on ϕ. Certainly ϕ has an L^1 bound [Tia00]. Let $\phi = -\phi_-$ (so that $\phi_- \geq 1$). Subtracting $\Theta = \sum_k \alpha_k \wedge \omega^{n-k}$ and then, multiplying the equation by ϕ_-^p and integrating, we have (here $\eta = e^f\Theta$),

$$
\begin{aligned}
-\int \phi_-^p dd^c\phi_- \wedge \kappa &= \int \phi_-^p (e^f - 1)\Theta \\
\int \phi_-^p (e^f - 1)\Theta &\leq c\|\phi_-\|_{L^p}^p \\
-\int \phi_-^p dd^c\phi_- \wedge \kappa &= \int d(\phi_-^p) \wedge d^c\phi_- \wedge \kappa \\
&= c\int d(\phi_-^{\frac{p+1}{2}}) \wedge d^c(\phi_-^{\frac{p+1}{2}}) \wedge \kappa \\
&\geq C\|\nabla(\phi_-^{\frac{p+1}{2}})\|_{L^2}^2 \\
&\geq C_1\left(\int \phi_-^{\frac{(p+1)n}{n-1}}\right)^{\frac{n-1}{n}} - C_2\int \phi_-^{p+1}
\end{aligned}
$$

where the last inequality follows from the Sobolev embedding theorem. Upon rearranging, we have

$$
\|\phi_-\|_{L^{(p+1)(n)/(n-1)}} \leq (C(p+1))^{\frac{1}{p+1}}\|\phi_-\|_{L^{p+1}}
$$

The Moser-iteration procedure gives $\sup|\phi| \leq C\|\phi\|_{L^2}$. If we prove that the right hand side is controlled by the L^1 norm of ϕ, we will be done. Indeed,

$$
\begin{aligned}
C\|\phi\|_{L^1} &\geq \int \phi(1-e^f)\Theta \\
&\geq C\|\nabla\phi\|_{L^2}^2 \\
&\geq C\|\phi - \langle\phi\rangle\|_{L^2}^2 \\
\|\phi\|_{L^2} &\leq C(\|\phi\|_{L^1} + 1)
\end{aligned}
$$

where, we have used the Poincaré inequality. Hence, proved.

Uniqueness : If ϕ_1 and ϕ_2 are two solutions, upon subtraction we have,

$$\sum_i \alpha_i \wedge ((\omega + dd^c\phi_1)^{n-i} - (\omega + dd^c\phi_2)^{n-i}) = 0$$

$$\Rightarrow \int_0^1 \sum_k k\alpha_k \wedge (\omega + dd^c\phi_1 + tdd^c(\phi_2 - \phi_1))^{n-k-1} dt \wedge dd^c(\phi_2 - \phi_1) = 0$$

Thus, by the maximum principle, $\phi_2 - \phi_1$ is a constant.

The mixed derivatives estimate: When $\alpha_i > 0$,

$$\eta \geq \alpha_{n-1} \wedge (\omega + dd^c\phi) > C(\mathrm{tr}(\omega + dd^c\phi))$$

where $C > 0$. Since $0 < \omega + dd^c\phi$, the eigenvalues of $dd^c\phi$ are bounded above. Thus, the mixed second derivatives of ϕ are bounded. Note that, by the Schauder estimate [MJ08], the first derivatives are bounded as well.

Openness : The map T is smooth. Its Gâteaux derivative is $DT(0,0,\ldots,0,\chi) = (n\alpha_0(\omega + dd^c\phi)^{n-1} + (n-1)\alpha_1 \wedge (\omega + dd^c\phi)^{n-2} + \ldots + \alpha_{n-1}) \wedge dd^c\chi$. It is clearly a bounded surjection (by the Schauder theory) onto its image if $n\alpha_0(\omega + dd^c\phi)^{n-1} + (n-1)\alpha_1 \wedge (\omega + dd^c\phi)^{n-2} + \ldots + \alpha_{n-1} > 0$. If DT is restricted to vectors of the form $(0,0,\ldots,0,\chi)$, then, it is a Banach space isomorphism. Hence, by the implicit function theorem of Banach manifolds, openness is guaranteed. In fact, it also guarantees that, on a level set, ϕ can be solved for (locally), in terms of α_i.

The n=2 case : The equation we have is equivalent to

$$(\omega + dd^c\phi + \frac{\alpha_1}{2})^2 = \eta - \alpha_2 + \frac{\alpha_1^2}{4}$$

This is just the usual Monge-Ampère equation and hence we are done.

3.0.6 Proof of theorem 1.4

Uniqueness is proven as before. We shall only prove existence. Let $Lu = \det(D^2u) + \Delta u$. To this end, we use the method of continuity. Consider the equation

$$\begin{aligned} Lu_t &= tf + (1-t)L\phi \\ u_t|_{\partial B} &= 0 \\ \phi &= \frac{3}{2}\sum x_i^2 - \frac{3}{2} \end{aligned} \tag{3.2}$$

When $t = 0$, it has a smooth solution, namely, ϕ.

Openness: Let $\Omega \subset C_0^{2,\alpha}(\bar{B})$ be the set of u such that $D^2u > 3$ (where the subscript 0 indicates vanishing on the boundary). This is an open subset. Define $T : \Omega \to C_0^{0,\alpha}$ to be $T(u_t) = \det(D^2u_t) + \Delta u_t$. If u_s is a solution of 3.2, then, it is easy to see that DT_{u_s} is a linear isomorphism. Hence, by the inverse function theorem of Banach manifolds, we see that the set of t for which there is a solution is open.

Closedness: Suppose there is a sequence $t_i \to t$ such that there are smooth solutions u_{t_i} satisfying $D^2u > 3$. Then, we wish to prove that a subsequence of the u_{t_i} converges to a smooth solution u_t in the $C^{2,\beta}$ topology. This requires *apriori* estimates (the convergence following from the Arzela-Ascoli theorem). We shall prove the same for the equation 1.4. We just have to prove the $C^{2,\alpha}$ estimate in order to ensure smoothness (by the Schauder theory).

C^0 *estimate*: Note that $\Delta u \leq f$. Hence, for $A >> 1$, $0 > f - \Delta(A\sum x_i^2) = \Delta(u - A\sum x_i^2)$. The minimum principle implies that $u \geq A\sum x_i^2 - A$. Since, $\Delta u > 9$, $u \leq 0$ by the maximum principle. thus, $\|u\|_{C^0} \leq C$.

C^1 *estimate*: Differentiating both sides using the operator D, $\mathrm{tr}((\mathrm{Hess}u)^{-1}D^2w) + \Delta w = Df$ where $w = Du$. Just as before, by adding or subtracting a large multiple of $\sum x_i^2$ to w and using the maximum principle, we see that $\|Du\|_{C^0}$ is controlled by its supremum on the boundary. The tangential boundary derivatives are 0. Since, $A\sum x_i^2 - A \leq u \leq 0$, $|\frac{\partial u}{\partial n}| \leq 2A$. Hence, $\|u\|_{C^1} \leq C$.

C^2 *estimate*: Since $\Delta u \leq Lu \leq f$ and $\Delta u > 0$, $\|u_{ij}\|_{C^0} \leq C$. Hence, $\|u\|_{C^2} \leq C$.

$C^{2,\alpha}$ *estimate*: So far, we haven't used anything about the sequence except that $D^2u_{t_i} > 0$. This will change presently. For any function $F : \mathbb{R} \to \mathbb{R}$, $F(\det(D^2u) + \Delta u) = F(f)$. If we choose the function appropriately, then the resulting equation will be a concave, uniformly elliptic Monge Ampère PDE to which we may apply the Evans-Krylov theory to extract a $C^{2,\alpha}$ estimate.

We claim that, the function $F(x) = \int_{36}^{x} e^{-\frac{t^2}{2}}dt$ is such that, $g(\lambda_1, \lambda_2, \lambda_3) = F(\sum \lambda_i + \lambda_1\lambda_2\lambda_3)$ has a uniformly positive gradient and is concave if $\lambda_i > 3$. By using the C^2 estimate and theorem 3.3, we have the desired estimate.

We shall prove the aforementioned fact: Let $x = \sum \lambda_i + \lambda_1\lambda_2\lambda_3$. We see that $\frac{\partial g}{\partial \lambda_i}|_{D^2u} = e^{-\frac{x^2}{2}}(1 + \frac{\lambda_1\lambda_2\lambda_3}{\lambda_i}) > e^{-\frac{f^2}{2}}$ and is less than $1 + 3f$ where we

have evaluated the derivative at the eigenvalues of the Hessian of a solution of equation 1.4. Hence, it is uniformly elliptic.
If $(v_1, v_2, v_3) \in \mathbb{R}^3$, then $-v_iv_j\frac{\partial^2 g}{\partial x_i \partial x_j} = e^{-\frac{x^2}{2}}(x(v_1(1+\lambda_2\lambda_3) + v_2(1+\lambda_3\lambda_1) + v_3(1+\lambda_1\lambda_2))^2) - 2e^{-\frac{x^2}{2}}(v_1v_2\lambda_3 + v_2v_3\lambda_1 + v_3v_1\lambda_2)$, which is in turn equal to $e^{-\frac{x^2}{2}}(v_1^2\alpha + \beta v_1 + \gamma) \geq 0 \Leftrightarrow \beta^2 - 4\alpha\gamma \leq 0$ and

$$\begin{gathered}\frac{\beta^2 - 4\alpha\gamma}{4} = \\ (v_2(x(1+\lambda_2\lambda_3)(1+\lambda_1\lambda_3) - \lambda_3) + v_3(x(1+\lambda_1\lambda_2)(1+\lambda_2\lambda_3) - \lambda_2))^2 \\ -x(1+\lambda_2\lambda_3)^2(v_2^2x(1+\lambda_1\lambda_3)^2 + v_3^2x(1+\lambda_1\lambda_2)^2 \\ +2v_2v_3(x(1+\lambda_1\lambda_3)(1+\lambda_1\lambda_2) - \lambda_1)) = \tilde{\alpha}v_2^2 + \tilde{\beta}v_2 + \tilde{\gamma} \leq 0\end{gathered}$$

with the last inequality holding if and only if $\tilde{\alpha} \leq 0$ and $\tilde{\beta}^2 - 4\tilde{\alpha}\tilde{\gamma} \leq 0$. Let us assume (without loss of generality) that $v_3 \neq 0$ and that $\lambda_1 < \lambda_2 < \lambda_3$.

$$\begin{aligned}\tilde{\alpha} &= (x(1+\lambda_2\lambda_3)(1+\lambda_1\lambda_3) - \lambda_3)^2 - x^2(1+\lambda_2\lambda_3)^2(1+\lambda_1\lambda_3)^2 \\ &= \lambda_3^2 - 2\lambda_3x(1+\lambda_2\lambda_3)(1+\lambda_1\lambda_3) \\ &\leq -2x\lambda_3^2(\lambda_1+\lambda_2+\lambda_1\lambda_2\lambda_3) \leq -2x\lambda_3^2\frac{2x}{3}\end{aligned}$$

$$\begin{aligned}\frac{\tilde{\gamma}}{v_3^2} &= (x(1+\lambda_2\lambda_1)(1+\lambda_2\lambda_3) - \lambda_2)^2 - x^2(1+\lambda_2\lambda_3)^2(1+\lambda_1\lambda_2)^2 \\ &\leq -\frac{4x^2\lambda_2^2}{3}\end{aligned}$$

$$\begin{aligned}\frac{\tilde{\beta}}{2v_3} &= (x(1+\lambda_2\lambda_3)(1+\lambda_1\lambda_2) - \lambda_2)(x(1+\lambda_2\lambda_3)(1+\lambda_1\lambda_3) - \lambda_3) \\ &- x(1+\lambda_2\lambda_3)^2(x(1+\lambda_1\lambda_3)(1+\lambda_1\lambda_2) - \lambda_1)\end{aligned}$$

$$\begin{aligned}\left(\frac{\tilde{\beta}}{2v_3}\right)^2 &= (x(1+\lambda_2\lambda_3)(\lambda_2+\lambda_3-\lambda_1+\lambda_1\lambda_2\lambda_3) - \lambda_2\lambda_3)^2 \\ &\leq x^4(1+\lambda_2\lambda_3)^2 \\ \frac{\tilde{\beta}^2 - 4\tilde{\alpha}\tilde{\gamma}}{4v_3^2} &\leq 0\end{aligned}$$

Hence proved.

Remark : Writing equation 3 for $n = 3$ and $\alpha_0 = 1$ we have,

$$(\omega + dd^c\phi)^3 + \alpha_1(\omega + dd^c\phi)^2 + \alpha_2(\omega + dd^c\phi) = \eta$$
$$\Rightarrow (\omega + \frac{\alpha_1}{3} + dd^c\phi)^3 + (\alpha_2 - \frac{\alpha_1^2}{3})(\omega + \frac{\alpha_1}{3} + dd^c\phi) = \eta - \frac{2\alpha_1^3}{27} + \frac{\alpha_1\alpha_2}{3}$$

A local, real version of a special case of the above is equation 1.4.

3.0.7 Proof of theorem 1.5

Once again, we apply the method of continuity. We shall impose several conditions on C (as we go along). It should be large enough so that, whenever $\beta > C\omega_0$, $\beta^3 > 3\alpha^2\beta$ (Indeed, if $K > 0$ and $B > 0$ are given, $\det(A) > K\mathrm{tr}(BA)$ for sufficiently large $A > 0$). Obviously, at $t = 0$, $u = 0$ solves the equation. Openness and uniqueness, follow from theorem 3.5. As before, if $t_i \to t$ is a sequence such that there exist smooth solutions u_i satisfying $\omega + dd^c u_i > C\omega_0$, then, we shall prove that a subsequence converges to a smooth solution u in the $C^{2,\beta}$ topology. As usual, we need *apriori* estimates for this.

The C^0 and the mixed derivative estimates follow directly from theorem 3.5. We have to prove the $C^{2,\alpha}$ estimate (thus proving existence and smoothness as before). It suffices to prove a local (interior) estimate. We shall accomplish this via the complex version of the (interior) Evans-Krylov theory done in [Blo12] and [Siu87].

The local (in a ball) version of the equation is

$$\begin{aligned} \det(\phi_{i\bar{j}}) + \mathrm{tr}(B^{-1}[\phi_{i\bar{j}}]) &= f \\ \phi_{i\bar{j}} &> C > 1 \\ f &> C^3 + 9\|B^{-1}\|^2 C \end{aligned} \tag{3.3}$$

where $B^{-1}_{i\bar{j}} = \det(\alpha)[\alpha]^{-1}_{i\bar{j}}$. We claim that the function $g(A) = F(\det(A) + \mathrm{tr}(B^{-1}A))$ from hermitian matrices satisfying $A > CId$ to $\mathbb{R}$ (where $F(x) = \int_c^x e^{-\frac{t^2}{2}} dt$) is concave and uniformly elliptic. Let the eigenvalues of A be λ_1, λ_2 and λ_3. The uniform ellipticity is trivial (as in the proof of theorem 1.4). The concavity is also somewhat similar to theorem 1.4, but requires some modification. Indeed (here V is an arbitrary hermitian matrix and

$x = \det(A) + \mathrm{tr}(B^{-1}A)),$

$$\begin{aligned} g^{''}(V,V) &= g''(x)(\det(A)\mathrm{tr}(A^{-1}V) + \mathrm{tr}(B^{-1}V))^2 \\ &+ g'(x)(-\det(A)\mathrm{tr}((A^{-1}V)^2) + \det(A)(\mathrm{tr}(A^{-1}V))^2) \end{aligned}$$

We wish to prove that $g^{''}(V,V) < 0$ for every hermitian V. Let's diagonalise the positive-definite form B^{-1}, i.e. $PB^{-1}P^\dagger = I$ for some matrix P. Define $\tilde{A} = (P^\dagger)^{-1}AP^{-1}$ and $\tilde{V} = (P^\dagger)^{-1}VP^{-1}$. Now, using a unitary matrix U, we may diagonalise $\tilde{A}$ i.e. $\tilde{\tilde{A}} = U\tilde{A}U^\dagger = \mathrm{diag}(a_1, a_2, a_3)$ where $a_1 \leq a_2 \leq a_3$ and $\tilde{\tilde{V}} = U\tilde{V}U^\dagger$. This implies that $\det(\tilde{A})\det(B) = \det(A)$ and $\mathrm{tr}(\tilde{A}) = \mathrm{tr}(B^{-1}A)$. Let $\tilde{\tilde{V}}_{ii} = v_i$. Hence,

$$g^{''}(V,V) = -xe^{-\frac{x^2}{2}}(\det(B)a_1a_2a_3(\sum \frac{v_i}{a_i}) + \sum v_i)^2$$
$$+e^{-\frac{x^2}{2}}((\sum \frac{v_i}{a_i})^2 - (\sum \frac{v_i^2}{a_i^2} + 2(\frac{|v_{12}|^2}{a_1a_2} + \frac{|v_{23}|^2}{a_2a_3} + \frac{|v_{13}|^2}{a_1a_3})))$$
$$\leq -xe^{-\frac{x^2}{2}}(\sum v_i(\det(B)\frac{a_1a_2a_3}{a_i} + 1))^2$$
$$+2\det(B)e^{-\frac{x^2}{2}}(v_1v_2a_3 + v_2v_3a_1 + v_3v_1a_2) = e^{-\frac{x^2}{2}}(Pv_1^2 + Qv_1 + R)$$

where,

$$P = -x(\det(B)a_2a_3 + 1)^2 \leq 0$$
$$Q = 2(\det(B)(v_2a_3 + v_3a_2) - x(\det(B)a_2a_3 + 1)(v_2(\det(B)a_1a_3 + 1)$$
$$+v_3(\det(B)a_1a_2 + 1))))$$
$$R = 2\det(B)v_2v_3a_1 - x(v_2(\det(B)a_1a_3 + 1) + v_3(\det(B)a_1a_2 + 1))^2$$

as before, we want $Q^2 - 4PR < 0$. Assume (without loss of generality) that $v_3 = 1$.

$$\frac{Q^2 - 4PR}{4} = Jv_2^2 + Kv_2 + L$$

where,

$$
\begin{aligned}
J &= \det(B)a_3(\det(B)a_3 - 2x(\det(B)a_2a_3 + 1)(\det(B)a_1a_3 + 1)) \\
&< \det(B)^2 a_3(1 - 2a_1a_2\det(B)) < 0 \\
K &= 2(a_2a_3(\det(B))^2 - x(\det(B)a_2a_3 + 1) \\
&\times((\det(B))^2 a_1a_2a_3 + \det(B)(a_2 + a_3 - a_1))) \\
L &= \det(B)a_2(\det(B)a_2 - 2x(\det(B)a_2a_3 + 1)(\det(B)a_1a_2 + 1))
\end{aligned}
$$

where the first inequality follows from the assumption that $\det(A) = \det(B)a_1a_2a_3$ which is larger than $3\text{tr}(B^{-1}A) > 3\sum a_i$. Hence,

$$
\begin{aligned}
\frac{K^2}{4} &= (a_2a_3(\det(B))^2 - x\det(B)(\det(B)a_2a_3 + 1) \\
&\times(\det(B)a_1a_2a_3 + (a_2 + a_3 - a_1)))^2 \leq x^4(\det(B)a_2a_3 + 1)^2\det(B)^2 \\
J &\leq -2x\det(B)^2 a_3^2(\det(B)a_1a_2a_3 + a_1 + a_2) \\
&\leq -2x\det(B)^2 a_3^2 \frac{2x}{3} \\
K &\leq -2x\det(B)^2 a_2^2 \frac{2x}{3}
\end{aligned}
$$

Thus, $K^2 - 4JL < 0$ implying that g is concave. The $C^{2,\alpha}$ estimate follows from theorem 3.4.

3.0.8 Proof of theorem 1.3

We use the method of continuity again. As before, openness follows easily using the Implicit function theorem on Banach spaces. Here, we prove only the *a priori* estimates. Smoothness follows by bootstrapping, as indicated earlier. Lastly, we shall also prove the uniqueness of convex solutions.
C^0 *estimate*: Since u is convex, its maximum is attained on the boundary and hence, $u \leq 0$. Let $\phi = \frac{\mu}{2}r^2 - \frac{\mu}{2}$ where $\mu > 0$, and $\mu^3 - 3\mu > \max f$; Then subtracting $\det(D^2u) - \Delta u$ from $\det(D^2\phi) - \Delta\phi$, we have (assume that the eigenvalues of D^2u are λ_i),

$$
\begin{aligned}
L(\phi - u) &= \det(D^2\phi) - \det(D^2u) - \Delta(\phi - u) \\
&= (\mu - \lambda_1)(\frac{\mu^2 + \frac{\mu}{2}(\lambda_2 + \lambda_3) + \lambda_2\lambda_3}{3} - 1) + (\mu - \lambda_2)\ldots \\
&= \mu^3 - \mu - f > 0
\end{aligned}
$$

We see that, since $\mu^2 > 3$, hence, L is an elliptic operator acting on $\phi - u$ with $L(\phi - u) > 0$. It is in fact uniformly elliptic on the ball because $f > 0$ on $\bar{B}$. So, by the maximum principle, $\phi < u$. This gives us a C^0 estimate on u.
C^1 *estimate*: As before, we shall use ellipticity and the maximum principle. We differentiate the equation and obtain,

$$\det(D^2u)\mathrm{tr}((D^2u)^{-1}D^2u_i) - \Delta u_i = f_i$$
$$H(u_i) = f_i$$

where, $H(u_i) = \det(D^2u)\mathrm{tr}((D^2u)^{-1}D^2u_i) - \Delta u_i$. Subtracting from $H(\phi)$, we have,

$$H(\phi) - H(u_i) = \mu(\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 - 3) - f_i \tag{3.4}$$

Using the equation itself, we see that, $\lambda_1\lambda_2 - 1 = \frac{f+\lambda_2+\lambda_1}{\lambda_3}$. Hence, the right hand side of 3.4 is bounded below by $6\mu - f_i$ which is positive for large enough μ. It is also clear that, $H(\phi - u_i)$ is an elliptic operator, and hence, by the maximum principle, $\phi - u_i$ is bounded above by its value on the boundary. Applying the same argument to $H(u_i) + H(\phi)$, we see that, u_i is controlled by its boundary values. On the boundary, the same argument of [CKNS85] as in theorem 1.4 proves the result.
C^2 *estimate*: For future use, notice that, atleast two of the eigenvalues of D^2u are larger than 1. Taking derivatives of the equation we have (let u_0 be the minimum of u),

$$\det(D^2u)\mathrm{tr}((D^2u)^{-1}D^2u_i) - \Delta u_i = f_i$$
$$\det(D^2u)\mathrm{tr}((D^2u)^{-1}D^2\Delta u) - \Delta\Delta u = \Delta f + \sum_i \det(D^2u)\mathrm{tr}(((D^2u)^{-1}D^2u_i)^2)$$
$$- \det(D^2u)\sum_i (\mathrm{tr}((D^2u)^{-1}D^2u_i))^2$$

Let $A = \det(D^2u)(D^2u)^{-1} - I$. Consider $g = \Delta u + \mu(u - u_0) > 0$ (we shall choose the constant $\mu > 0$ later. It can depend on $\|u\|_{C^1}$ and other constants). Notice that, if g is bounded, then, so is Δu and thus, D^2u is bounded. At the maximum of g (if it occurs in the interior), $(\Delta u)_i = -\mu u_i$ and $\mathrm{tr}(AD^2g) \leq 0$. This implies,

$$\begin{aligned} 0 &\geq \Delta f + \mu\mathrm{tr}(AD^2u) - \frac{|-\mu\nabla u + \nabla f|^2}{\Delta u + f} \\ &\geq C_1(\mu) - \frac{C_2(\mu)}{\Delta u + f} + (2\Delta u + 3f)\mu \end{aligned}$$

Hence, Δu is bounded at that point. Thus, g is bounded at that point. This implies that Δu is bounded everywhere. If the maximum of g occurs on the boundary (call the max g_0), we shall have to analyse it separately. Let $\tilde{g} = g + g_0(1-2r^2)$. Clearly, the maximum of $\tilde{g}$ has to occur in the interior. There, $D\tilde{g} = 0$ and $\mathrm{tr}(AD^2\tilde{g}) \leq 0$. Hence (here, we assume that, $\mathrm{tr}(A) = \sum(\lambda_i\lambda_j - 1)$ and that g_0 are sufficiently large compared to constants; If not, we are done),

$$\begin{aligned}
0 &\geq -C_1(\mu) + \mu \mathrm{tr}(A)g_0 - C_2 g_0 - (C_4 g_0 + C_6(\mu))\mathrm{tr}(A) - C_5\frac{g_0^2}{\Delta u + f} \\
&\geq -\tilde{C}_1(\mu) + \mu \mathrm{tr}(A)g_0 - \tilde{C}_2 g_0 - \tilde{C}_4 g_0 \mathrm{tr}(A) - \tilde{C}_5 g_0 \\
&\geq -E_1(\mu) + (\mu g_0 - E g_0)\mathrm{tr}(A)
\end{aligned}$$

Choosing $\mu > E$, we see that, g_0 is bounded. Notice that, this also implies a lower bound on D^2u. This is because, $M\lambda_i > \lambda_1\lambda_2\lambda_3 > f$.
$C^{2,\alpha}$ *estimate* : Notice that, the set Y of positive, symmetric matrices satisfying $\det(A) - \mathrm{tr}(A) > 0$ is a convex open set (lemma 4.16 of [Kry95]). Also, our equation maybe written as $-1 = -\frac{\Delta u}{\det(D^2u)} - \frac{f}{\det(D^2u)} = F(D^2u, x)$ which is certainly concave on Y by the same lemma in [Kry95]. It is uniformly elliptic on solutions as long as the eigenvalues of the Hessian are bounded below and above (which they are, by the C^2 estimates). Theorem 3.3 yields the desired estimates.
Uniqueness : If u_1 and u_2 are two convex solutions of the equation $F(u) = -1$ (as above), then, upon subtraction, $0 = \int_0^1 \mathrm{tr}((-I + \det(D^2u_t)(D^2u_t)^{-1})D^2(u_2 - u_1))dt = L(u_2 - u_1)$ where, L is elliptic. By the maximum principle, $u_1 = u_2$.

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