

Real rational curves of low degree

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Chapter 1

Introduction

1.1 Real rational curves in the plane

In 1876, Harnack published a paper on the bound on the number of components of a plane projective real algebraic curve [6]. In the year 1900, Hilbert suggested the classification of real, non-singular plane projective real algebraic curves in his famous list of problems [8]. This stimulated research in the topology of real algebraic varieties, particularly the classification of real algebraic curves. Non-singular real algebraic curves in $\mathbb{R}\mathbb{P}^2$ are topologically the union of circles, and Hilbert was interested in which arrangements were possible for a particular degree. The difficulty greatly increases as the degree increases. Klein, Rokhlin, Viro and many others worked on the problem and a complete classification has been obtained up to degree 7. Viro invented a patchworking method which proved to be a very useful combinatorial technique to construct examples of real algebraic curves.

Real algebraic curves can be naturally extended to their complexification and considering this proved very fruitful in understanding them. For instance, Rokhlin realized the importance of curves that divided their complexification in two components, which are called Type I curves. He discovered the Rokhlin's Complex Orientation Formula for Type I curves, which provides a restriction on the type of arrangements possible. Rokhlin also defined the stronger notion of a rigid isotopy which is an isotopy in which the objects are of the same real algebraic class throughout the isotopy. There are examples of curves in $\mathbb{R}\mathbb{P}^2$ that are isotopic but not rigidly isotopic in degree 6.

Real rational curves form an important subclass of real singular curves. Their rigid isotopy classification has been completed up to degree 5. Generally, curves of a particular degree have been classified by considering restrictions on the type of arrangements possible and then constructing examples that

satisfy those restrictions. Indeed, this is how rational curves of degree 4 and 5 were classified. However, in chapter 2 of this thesis we will see an explanation for the rigid isotopy classes of rational curves of degree 4. We will generalize the notion of a real chord diagram by replacing the circle by a sphere. We will define the meaning of real chords and then show that *there is a bijective correspondence between the rigid isotopy classes of generic rational curves of degree 4 and extended chord diagrams with exactly three chords.*

1.2 Real rational knots

A natural generalization of Hilbert’s original problem is the study of co-dimension 1 varieties in \mathbb{RP}^n for a general n , particularly real algebraic hypersurfaces in \mathbb{RP}^3 . They have been classified up to degree 4. Another generalization of Hilbert’s original problem is to classify real algebraic curves of co-dimension 2 in \mathbb{RP}^3 or another smooth projective variety of dimension 3.

1.2.1 Previous work on real rational knots

Viro invented a rigid isotopy invariant of a real rational knot K in \mathbb{RP}^3 , called the encomplexed writhe number [12], denoted by $w(K)$.

Since the number of double points of a real rational plane curve of degree d is bounded by $\frac{(d-1)(d-2)}{2}$, the writhe number of a real rational knot K of degree d satisfies the following restrictions.

$$|w(K)| \leq \frac{(d-1)(d-2)}{2} \tag{1.1}$$

$$w(K) = \frac{(d-1)(d-2)}{2} \pmod{2} \tag{1.2}$$

Bjorklund studied real rational knots in \mathbb{RP}^3 [1]. He proved the following “gluing method” to construct real rational knots in \mathbb{RP}^3 .

Theorem 1. *If two real rational knots of degrees m and n respectively, intersect in only one point p and their tangents at p do not coincide, then there exists a real rational knot of degree $m + n$ that is isotopic to the union of the two original knots, with a small perturbation at p .*

By using the gluing method, he was able to obtain a knot of a given degree with the maximum and minimum possible writhe numbers $\pm \frac{(d-1)(d-2)}{2}$. Indeed, it is possible to construct knots of degree d with each of the writhe numbers permitted by equations 1.1 and 1.2.

He classified, up to rigid isotopy, all real rational knots up to degree 5. The classification can be described as follows.

1. Degree 1: writhe 0 (topologically a line)
2. Degree 2: writhe 0 (topologically a circle)
3. Degree 3: writhe ± 1 (topologically a line and its mirror image)
4. Degree 4:
 - (a) writhe ± 1 (topologically a circle and its mirror image)
 - (b) writhe ± 3 (topologically a two crossing knot and its mirror image)
5. Degree 5:
 - (a) writhe 0 (topologically a line)
 - (b) writhe ± 2 (topologically a line and its mirror image)
 - (c) writhe ± 4 (topologically a trefoil and its mirror image)
 - (d) writhe ± 6 (topologically a projective 5_3 knot and its mirror image)

The classification shows that the writhe number can distinguish rigid isotopy classes of knots of degrees less than 5. This is not true for higher degrees; Bjorklund constructed an example of two non-rigidly isotopic knots of degree 6 which have the same writhe number.

1.2.2 Real rational knots in the 3-sphere

One can consider real rational curves in other 3-manifolds that are also real algebraic varieties, like non-singular quadrics of dimension 3. Among them, the quadric with signature 1, plays an important role. Topologically it is a 3-sphere, which is the traditional ambient space for classical knots. Real algebraic knots in S^3 also arise naturally as the links of singularities of *complex* plane algebraic curves. Chapter 3 describes a classification of real rational knots in the 3-sphere up to degree 6. Apart from the classification, we will also consider the link between rational knots in the 3-sphere and rational knots in \mathbb{RP}^3 .

Chapter 2

Chord diagrams and generic planar rational quartics

2.1 Introduction

All kinds of topological classification problems on real algebraic plane projective curves of degree 4 have been solved, see D.A. Gudkov's survey paper [5]. However, the complexity of the results and just the number of inequivalent curves (117 types of irreducible curves), is a motivation to look for parts of the classification that can help in understanding the entire picture.

Algebraic curves that have only the simplest singularities, i.e. non-degenerate double points, are called *nodal curves*. Rational real nodal curves occupy a very special place in the space of all irreducible real algebraic plane projective curves of degree 4: nodal curves of higher genus can be obtained from them by small perturbations, and curves with more complicated singularities can be obtained from nodal curves by degenerations. Both perturbations and degenerations for the curves under consideration are easy to see, although the complete list of the results is huge.

Topological classifications of nodal rational real algebraic plane projective curves of degree 4 were known, but the result was formulated as just a list of pictures.

The results of this paper provide a key to understanding the list. We prove that the rigid isotopy type of a nodal rational real plane projective curve is determined by the chord diagram which describes the double points and that every chord diagram with at most 3 chords is realized by such a curve. This gives a new elementary proof for the rigid isotopy classification.

2.1.1 Plane projective rational curves

A plane projective curve C is called a rational curve if it can be presented as the image of a regular map of the projective line. Such a map θ is defined by formulae

$$[s : t] \mapsto [p_0(s, t) : p_1(s, t) : p_2(s, t)] \quad (2.1)$$

where p_0, p_1 , and p_2 are homogeneous polynomials of the same degree d with no common root. In this paper we study such curves of degree $d = 4$ defined over the field \mathbb{R} of real numbers.

The set of real points of C is denoted by $\mathbb{R}C$ and the set of complex points of C by $\mathbb{C}C$. $\mathbb{C}C$ is the image of the map $\mathbb{C}\theta : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}C \subset \mathbb{C}\mathbb{P}^2$ defined by (2.1).

2.1.2 Real nodes

If the preimage $\mathbb{C}\theta^{-1}(x) \subset \mathbb{C}\mathbb{P}^1$ of a point $x \in \mathbb{C}C$ consists of two points and the corresponding branches of $\mathbb{C}C$ are transversal, then that point is called a *node*.

A real node $x \in \mathbb{R}C$ is said to be *non-solitary* if $\mathbb{C}\theta^{-1}(x) \subset \mathbb{C}\mathbb{P}^1$ consists of two real points. Then x is an intersection point of two branches of $\mathbb{R}C$ transversal to each other in $\mathbb{R}\mathbb{P}^2$.

A real node $x \in \mathbb{R}C$ is said to be *solitary* if $\mathbb{C}\theta^{-1}(x) \subset \mathbb{C}\mathbb{P}^1$ consists of two imaginary points. Then x is an intersection point of two complex conjugate branches of $\mathbb{C}C$ transversal to each other in $\mathbb{C}\mathbb{P}^2$, but as a point of $\mathbb{R}C$ this point is isolated.

Non-real nodes are of two types: the preimage may consist of points that either belong to the same component of $\mathbb{C}C \setminus \mathbb{R}C$ or to different components.

A generic rational complex plane projective curve of degree d has only nodes as singularities and they are $\frac{(d-1)(d-2)}{2}$ in number. Then its parametrizations $\mathbb{R}\theta$ and $\mathbb{C}\theta$ are generic immersions of the circle $\mathbb{R}\mathbb{P}^1$ to $\mathbb{R}\mathbb{P}^2$ and of the 2-sphere $\mathbb{C}\mathbb{P}^1$ to $\mathbb{C}\mathbb{P}^2$, respectively. In particular, a generic rational complex plane projective curve of degree 4 has 3 nodes.

2.1.3 Chord diagrams

For a generic rational real plane projective curve C , the position of real non-solitary nodes on $\mathbb{R}C$ can be encoded into a graph formed by a circle and a few chords. The circle is a copy of $\mathbb{R}\mathbb{P}^1$ parametrizing C and each chord represents a non-solitary real node of C connecting the corresponding pair of points on the circle. The whole graph is called the *real chord diagram* of C .

In the presence of imaginary nodes, one can enlarge the real chord diagram by incorporating information about the imaginary nodes. The circle is extended to a 2-sphere in which the original circle is the equator, a real solitary node corresponds to a chord with end-points symmetric with respect to the equatorial plane, an imaginary node corresponds to a chord connecting non-symmetric points that do not belong the equator. However, as the curve is real, the set of all end-points of all the chords is invariant under the symmetry in the equatorial plane. A chord with non-symmetric end-points can be of one of two types: it connects either points of the same hemi-sphere or of different hemi-spheres. Such a chord diagram will be called an *extended chord diagrams* of the curve C .

A *rigid isotopy* of a generic real plane projective curve is a continuous one-parameter deformation C_t with $t \in [0, 1]$ consisting of generic real plane projective curves of the same degree. Rigid isotopy is an equivalence relation. A real chord diagram is preserved under a rigid isotopy.

There are two major problems related to using chord diagrams as rigid isotopy invariants. Firstly, it is not known which chord diagrams are realizable by curves of degree d . Secondly, it is not true that two generic real plane projective curves that have the same chord diagram are rigid isotopic.

Below we solve both problems for curves of degree 4.

2.1.4 The main result

Theorem 2. *Any extended chord diagram with at most 3 chords is realizable by a nodal rational real plane projective curve of degree 4. Two nodal rational real plane projective curves of degree 4 with only real nodes are rigidly isotopic if and only if they have the same extended chord diagram.*

Proof. Either all nodes are real or exactly two nodes are imaginary. This is because imaginary nodes must occur in pairs since the curve is real. We first consider the case where all the nodes are real:

Case 1: All nodes are real

Realize a given chord diagram with at most three chords, on the projective line \mathbb{RP}^1 . If the number of chords in it is less than three, then complete it by chords connecting complex conjugate points on the sphere \mathbb{CP}^1 . Let the pairs of end points of the chords be $[\alpha_i : \beta_i]$ and $[\gamma_i : \delta_i]$ for $i = 0, 1, 2$. Then the

polynomials

$$\begin{aligned}
p_0(s, t) &= (\beta_1 s - \alpha_1 t)(\delta_1 s - \gamma_1 t)(\beta_2 s - \alpha_2 t)(\delta_2 s - \gamma_2 t) \\
p_1(s, t) &= (\beta_0 s - \alpha_0 t)(\delta_0 s - \gamma_0 t)(\beta_2 s - \alpha_2 t)(\delta_2 s - \gamma_2 t) \\
p_2(s, t) &= (\beta_0 s - \alpha_0 t)(\delta_0 s - \gamma_0 t)(\beta_1 s - \alpha_1 t)(\delta_1 s - \gamma_1 t)
\end{aligned} \tag{2.2}$$

define a parametrization of a generic real plane projective curve of degree 4 with the nodes at the points $z_0 = [1 : 0 : 0]$, $z_1 = [0 : 1 : 0]$, and $z_2 = [0 : 0 : 1]$ and the prescribed chord diagram. So any chord diagram is realizable as the chord diagram corresponding to rational curve of degree 4.

We now show that chord diagrams are a strong rigid isotopy invariant for curves of degree 4. Let C be a generic real plane projective curve with three real nodes. The nodes cannot be co-linear, because then a line containing them would have an intersection number with C of at least 6, which would contradict Bezout's theorem. Therefore, there is a projective transformation that can transform the curve so that the nodes of C are the points $z_0 = [1 : 0 : 0]$, $z_1 = [0 : 1 : 0]$, and $z_2 = [0 : 0 : 1]$.

Let $[\alpha_i : \beta_i]$ and $[\gamma_i : \delta_i]$ be in the preimage of z_i . Let q_i be the quadratic polynomials defined by

$$\begin{aligned}
q_0(s, t) &= (\beta_0 s - \alpha_0 t)(\delta_0 s - \gamma_0 t) \\
q_1(s, t) &= (\beta_1 s - \alpha_1 t)(\delta_1 s - \gamma_1 t) \\
q_2(s, t) &= (\beta_2 s - \alpha_2 t)(\delta_2 s - \gamma_2 t)
\end{aligned} \tag{2.3}$$

If the points $[\alpha_i : \beta_i]$ and $[\gamma_i : \delta_i]$ are imaginary then they must occur in conjugate pairs, so each q_i is a real polynomial.

The main observation is that when we consider θ explicitly as $[p_0 : p_1 : p_2]$, where p_i are homogeneous polynomials of degree 4, then the fact that $[1 : 0 : 0]$ is a node means that q_0 divides p_1 and p_2 . Similarly, q_1 divides p_0 and p_2 , and q_2 divides p_0 and p_1 . This proves that $p_0 = c_0 q_1 q_2$, $p_1 = c_1 q_0 q_2$, and $p_2 = c_2 q_0 q_1$ where each q_i is quadratic polynomials and each c_i is a constant, which can all be made 1 by a projective transformation that corresponds to scaling the coordinate axes.

Therefore, by a projective transformation, the curve has been transformed to one whose parametrization has coordinates p_0 , p_1 , and p_2 such that $p_0 = q_1 q_2$, $p_1 = q_0 q_2$, and $p_2 = q_0 q_1$, where each q_i is determined by the preimages of the nodes as described above. This is defined explicitly by equations 2.2, where the pair $\{[\alpha_i : \beta_i], [\gamma_i : \delta_i]\}$ is the preimage of each node z_i . This new curve is projectively equivalent to the original curve. What we have proved is that *there is a unique generic degree 4 rational parametrization (up to projective*

transformations) with the given nodes and their preimages. Therefore, to show that two curves are isotopic, we only need to keep a track of their nodes and the pre-images of their nodes.

Consider two curves C_1 and C_2 with equivalent chord diagrams. They have the same number of solitary nodes. Denote their explicit parametrizations by $[p_0^1 : p_1^1 : p_2^1]$ and $[p_0^2 : p_1^2 : p_2^2]$ respectively. Use a projective transformation to transform each of the curves in such a way that their nodes are sent to $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$, and the solitary nodes of both curves are sent to the same points. By what we have discussed above, their resulting coordinates are of the form $[q_1^i q_2^i : q_0^i q_2^i : q_0^i q_1^i]$ for $i = 1, 2$, where

$$\begin{aligned} q_0^i(s, t) &= (\beta_0^i s - \alpha_0^i t)(\delta_0^i s - \gamma_0^i t) \\ q_1^i(s, t) &= (\beta_1^i s - \alpha_1^i t)(\delta_1^i s - \gamma_1^i t) \\ q_2^i(s, t) &= (\beta_2^i s - \alpha_2^i t)(\delta_2^i s - \gamma_2^i t) \end{aligned} \tag{2.4}$$

A path $[\alpha_i^t : \beta_i^t]$ between $[\alpha_i^1 : \beta_i^1]$ and $[\alpha_i^2 : \beta_i^2]$ will result in a rigid isotopy as long as the $[\alpha_i^t : \beta_i^t]$ does not coincide with any other roots of any q_i^j . No two of the polynomials q_i^j , for each $j = 1, 2$, can share a root, otherwise all the coordinates will vanish on that point and the curve would not be well defined. No other singularity can be formed in the process because a generic planar rational curve of degree 4 can have only three nodes.

Now we will show that since the chord diagrams are the same, we can find a path between the roots of corresponding q_i^j . Suppose $[\alpha_i^1 : \beta_i^1]$ is imaginary, then so will $[\alpha_i^2 : \beta_i^2]$. Their conjugates would then be $[\delta_i^1 : \gamma_i^1]$ and $[\delta_i^2 : \gamma_i^2]$ respectively. It is always possible to find a path in $\mathbb{C}\mathbb{P}^1 \setminus \mathbb{R}\mathbb{P}^1$ between $[\alpha_i^1 : \beta_i^1]$ and $[\alpha_i^2 : \beta_i^2]$ that does not contain any other root of any q_i^j . The conjugate of this path will be a path between their conjugates $[\delta_i^1 : \gamma_i^1]$ and $[\delta_i^2 : \gamma_i^2]$.

Consider the case where $[\alpha_i^1 : \beta_i^1]$ is real. Since the curves share the same chord diagram, it is also possible to find a path $[\alpha_i^t : \beta_i^t]$ between $[\alpha_i^1 : \beta_i^1]$ and $[\alpha_i^2 : \beta_i^2]$. For the same reason, we can find a path connecting $[\delta_i^2 : \gamma_i^2]$ and $[\delta_i^1 : \gamma_i^1]$, and therefore we obtain a path between the polynomials q_i^1 and q_i^2 , for each i . This defines a path between p_i^1 and p_i^2 , which shows that the curves C_1 and C_2 are in the same rigid isotopic class because we have been able to transform them to curves that have the same nodes and pre-images of the nodes.

Case 2: Two nodes are imaginary

Consider a chord diagram with end points $[\alpha_0, \beta_0]$ and $[\gamma_0 : \delta_0]$ that are imaginary but not conjugate. Since the chord diagram is real, there must be another

chord with end points $[\bar{\alpha}_0, \bar{\beta}_0]$ and $[\bar{\gamma}_0 : \bar{\delta}_0]$. Since the chord diagram has only three chords, the end points of the third chord must be real. Denote them by $[\alpha_1, \beta_1]$ and $[\gamma_1 : \delta_1]$. Then the polynomials

$$\begin{aligned} p_0(s, t) &= (\beta_1 s - \alpha_1 t)(\delta_1 s - \gamma_1 t)(\bar{\beta}_0 s - \bar{\alpha}_0 t)(\bar{\delta}_0 s - \bar{\gamma}_0 t) \\ p_1(s, t) &= (\beta_0 s - \alpha_0 t)(\delta_0 s - \gamma_0 t)(\bar{\beta}_0 s - \bar{\alpha}_0 t)(\bar{\delta}_0 s - \bar{\gamma}_0 t) \\ p_2(s, t) &= (\beta_0 s - \alpha_0 t)(\delta_0 s - \gamma_0 t)(\beta_1 s - \alpha_1 t)(\delta_1 s - \gamma_1 t) \end{aligned} \quad (2.5)$$

define a complex parametrization of a generic complex plane projective curve of degree 4 with the nodes at the points $z_0 = [1 : 0 : 0]$, $z_1 = [0 : 1 : 0]$, and $z_2 = [0 : 0 : 1]$ and the prescribed chord diagram. The parametrization is complex, of a special type: p_1 is real, and $p_2 = \bar{p}_0$. This means that the transformation $[x_0 : x_1 : x_2] \rightarrow [x_0 + x_2 : x_1 : i(x_2 - x_1)]$ transforms it to a real parametrization, but with the double points z_i transformed to $[1 : 0 : -i]$, $[0 : 1 : 0]$, and $[1 : 0 : i]$. Therefore we have produced a real curve with two imaginary double points.

We now show that two curves with imaginary double points that have the same chord diagram are isotopic. First transform each of the curves by a *real* transformation that sends their double points to the points $[1 : 0 : 0]$, $[0 : 1 : i]$, and $[0 : 1 : -i]$. The new curves are isotopic to their original ones.

The invertible projective linear transformation $[x_0 : x_1 : x_2] \rightarrow [x_0 : x_1 + ix_2 : x_1 - ix_2]$ sends the real parametrizations of each curve to complex parametrizations $[p_0^i : p_1^i : p_2^i]$, $i = 1, 2$ such that $p_2^i = \bar{p}_1^i$, and the double points to $[1 : 0 : 0]$, $[0 : 1 : 0]$, and $[0 : 0 : 1]$. Let their pre-images be the pairs $\{[\alpha_0^i : \beta_0^i], [\gamma_0^i : \delta_0^i]\}$, $\{[\alpha_0^i : \bar{\beta}_0^i], [\gamma_0^i : \bar{\delta}_0^i]\}$, and $\{[\alpha_1^i : \beta_1^i], [\gamma_1^i : \delta_1^i]\}$, where $i = 1, 2$. Define the polynomials q_0^i, q_1^i and q_2^i as before, but this time observe that q_2^i 's are real and $q_1^i = \bar{q}_0^i$.

Now we apply the same reasoning as in the case of real double points: the fact that $[1 : 0 : 0]$ is a node means that q_0^i divides p_1^i and p_2^i . Similarly, q_1^i divides p_0^i and p_2^i , and q_2^i divides p_0^i and p_1^i . Again, $p_0^i = c_0^i q_1^i q_2^i$, $p_1^i = c_1^i q_0^i q_2^i$, and $p_2^i = c_2^i q_0^i q_1^i$ and the fact that $p_2^i = \bar{p}_1^i$ and p_0^i is real imposes the condition that $c_2^i = \bar{c}_1^i$ and c_0^i are real. The paths c_0^i and c_1^i make the coefficients 1 while still maintaining the fact that the last two coordinates are conjugates.

As before, since the parametrizations have the same chord diagram and the nodes are the same, there are paths $[\alpha_0^t : \beta_0^t]$ between $[\alpha_0^1 : \beta_0^1]$ and $[\alpha_0^2 : \beta_0^2]$, and $[\gamma_0^t : \delta_0^t]$ between $[\gamma_0^1 : \delta_0^1]$ and $[\gamma_0^2 : \delta_0^2]$. Consequently, there are conjugate paths $[\alpha_0^t : \bar{\beta}_0^t]$ between $[\alpha_0^1 : \bar{\beta}_0^1]$ and $[\alpha_0^2 : \bar{\beta}_0^2]$, and $[\gamma_0^t : \bar{\delta}_0^t]$ between $[\gamma_0^1 : \bar{\delta}_0^1]$ and $[\gamma_0^2 : \bar{\delta}_0^2]$. There is only one real point double point of each curve, so there is path $[\alpha_1^t : \beta_1^t]$ between $[\alpha_1^1 : \beta_1^1]$ and $[\alpha_1^2 : \beta_1^2]$, and $[\gamma_1^t : \delta_1^t]$ between $[\gamma_1^1 : \delta_1^1]$ and $[\gamma_1^2 : \delta_1^2]$. Observe that for each t , p_0^t is real and $p_2^t = \bar{p}_1^t$. So for each t the

complex curve C_t (which has nodes at $[1 : 0 : 0]$, $0 : 1 : 0$, and $[0 : 0 : 1]$) is parametrized by a parametrization defined by the polynomials p_i^t , correspond to a real curve with nodes at $[1 : 0 : -i]$, $[0 : 1 : 0]$, and $[1 : 0 : i]$. The correspondence is given by the map $[x_0 : x_1 : x_2] \rightarrow [x_0 : x_1 + ix_2 : x_1 - ix_2]$. \square

Since there are 9 chord diagrams with at most three chords, we have:

Corollary 3. *There are 9 rigid isotopy classes of real rational curves of degree 4 in \mathbb{RP}^2 with only real nodes as singularities.*

Representatives of each of these rigid isotopy classes and their respective chord diagrams are listed in figure 2.1. The dots represent the positions of the solitary nodes that which will now be deduced. Out of them, there are two chord diagrams that can correspond to curves with imaginary nodes. They are shown in figure 2.1. For each such chord diagram, their nodes may arise out of self intersections of two hemispheres of $\mathbb{CC} \setminus \mathbb{RC}$, or intersections of different hemispheres. So there are four more such curves, and therefore:

Corollary 4. *There are 13 rigid isotopy classes of generic planar rational curves of degree 4.*

2.2 Positions of the solitary nodes

As we have seen, the positions of the nodes and pre-images uniquely determine the rational curve. Therefore, for a curve C with a parametrization θ , we should be able to uniquely determine which components of $\mathbb{RP}^3 \setminus \theta(\mathbb{RP}^1)$ each of the solitary nodes must lie in. We will now determine the component and show that if the curve has more than one solitary node, then all the solitary nodes lie in the same component of $\mathbb{RP}^3 \setminus \theta(\mathbb{RP}^1)$

Consider the curves in figure 2.1 that have solitary nodes. They are in the second row of figure 2.1. Call them C_1, C_2, C_3 , and C_4 and denote their respective parametrizations by $\theta_1, \theta_2, \theta_3$, and θ_4 . Then, $\mathbb{RP}^2 \setminus \theta_2(\mathbb{RP}^1), \mathbb{RP}^2 \setminus \theta_3(\mathbb{RP}^1)$, and $\mathbb{RP}^2 \setminus \theta_4(\mathbb{RP}^1)$ each have only one component that is not a disk.

If a solitary node lay in the interior of a disk, any line passing through it would have to intersect the boundary of the disk twice. So a line joining this solitary node and another node would intersect the curve in at least 5 points counting multiplicity: four from the double intersections with the two nodes defining the line, and at least one more from the intersection with the boundary of the disk at a point that is not a node. So none of the solitary

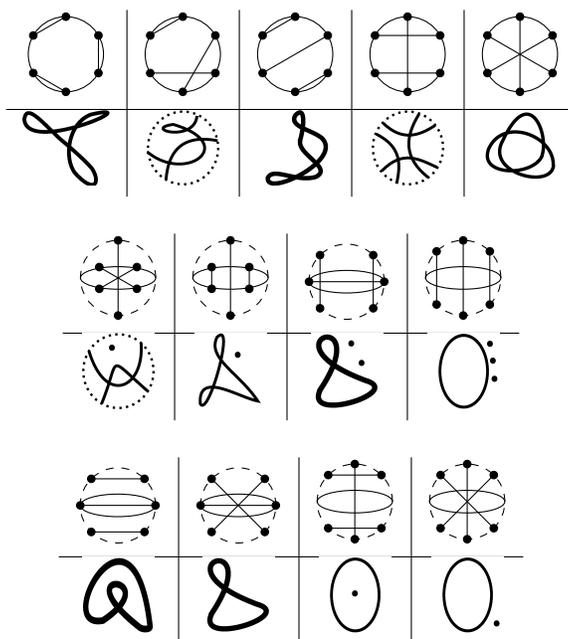


Figure 2.1: Real chord diagrams of planar curves of degree 4 with real nodes. The first row consists of curves with only real non-solitary nodes and so only the equator of the chord diagram is shown. The last row consists of curves with non-real double points.

nodes can lie inside the disk components. Figure 2.2 shows one example of this applied to C_2 .

Since the complements of $\theta_2(\mathbb{RP}^1)$, $\theta_3(\mathbb{RP}^1)$, and $\theta_4(\mathbb{RP}^1)$, each have only one non-disk component, their solitary nodes have to lie in it as is shown for the last three curves of the second row in figure 2.1.

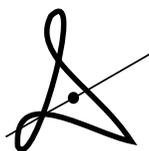


Figure 2.2: An example of an impossible position for a solitary node because the line would intersect the curve in more than 5 points counting multiplicity.

We are only left with the curve C_1 from figure 2.1 and the curves in the last row, which have non-real nodes, from figure 2.1. For the curve C_1 , $\theta_1(\mathbb{RP}^1)$ divides \mathbb{RP}^2 into three components out of which only one is a disk. The same

reasoning as above will show that its node cannot lie in that disk. To deduce which of the other components it lies on, and the position of the solitary nodes in the last row in figure 2.1, we will need to use Rokhlin's Complex Orientation Formula for non-singular curves of Type I which is briefly reviewed in the next section. For more details on Rokhlin's Complex Orientation Formula see [13].

2.2.1 Review of Rokhlin's Complex Orientation Formula

A real algebraic curve A is said to be of *Type I* if $[\mathbb{R}A] = 0 \in H_1(\mathbb{C}A)$, otherwise it is said to be of *Type II*. If A is of Type I, then $\mathbb{R}A$ is the boundary of each of the two halves of $\mathbb{C}A \setminus \mathbb{R}A$ which are conjugate to each other. If the halves intersect, they do so precisely at the solitary nodes.

Each half induces a natural pair of opposite orientations on their common boundary $\mathbb{R}A$. This pair of orientations is called the *complex orientation* of the curve. A rational curve is an example of a Type I curve and its complex orientations coincide with the natural orientations induced on it by its parametrization.

We now consider non-singular real algebraic curves defined by the zero set of a real polynomial. The real zero set $\mathbb{R}A$ of a non-singular real algebraic curve A is a compact one-dimensional manifold and therefore each component is topologically a circle.

If the complement of a component of $\mathbb{R}A$ divides $\mathbb{R}P^2$ it is called an *oval*. Each oval is the boundary of a disk in $\mathbb{R}P^2$; the disk is called the interior of the oval.

A pair of ovals is called *injective* if one lies in the interior of the other. A pair of injective ovals bounds an annulus. If the orientation of the ovals is compatible with the orientation induced on the boundary of the annulus, it is called a *positive pair*, otherwise it is called a *negative pair*. The number of positive pairs is denoted by Π^+ and the number of negative pairs is denoted by Π^- . Rokhlin proved that non-singular real planar curves of Type I of even degree d and with l components satisfy the following formula known as *Rokhlin's Complex Orientation Formula* [10]

$$2(\Pi^+ - \Pi^-) = l - \frac{d^2}{4} \tag{2.6}$$

For a singular real curve A with only double points, this formula has been extended in the following way as described in [11]: The complex orientation defines a unique way to smoothen the singularities of $\mathbb{R}A$ in a way that is compatible with a chosen complex orientation. An example of such a pertur-

bation is show in figure 2.3. Denote this smoothened curves as $\widetilde{\mathbb{R}A}$. $\widetilde{\mathbb{C}A}$ may still have non-real singularities. Let σ denote those imaginary nodes that arise owing to the intersection of different halves of $\mathbb{R}A \setminus \mathbb{C}A$. As before, denote the number of components of $\widetilde{\mathbb{C}A}$ by l , the number of its positive pairs by Π^+ and the number of negative pairs as Π^- . Positive and negative pairs are defined as before. In that case the Rokhlin's Complex Orientation Formula extends to:

$$2(\Pi^+ - \Pi^-) = l - \frac{d^2}{4} + \sigma \quad (2.7)$$

2.2.2 The rest of the curves

Orient $\mathbb{R}C_1$. This orientation corresponds to one of its complex orientations, which is equivalent to choosing one half of the complexification. It is possible to smoothen the real part of the curve in such a way that the node changes to an oval and the curve continues to remain of Type I (see section 2.2 of [11]). There is only one way to resolve the crossings so that they conform to the complex orientation that will be induced. This, as shown in figure 2.3, will give rise to a non-singular degree 4 planar curve with two ovals. One of the ovals is the result of the perturbation of the solitary node.

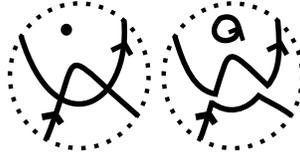


Figure 2.3: C_1 and its perturbation with a complex orientation

If the pair of ovals is not injective, then the number of positive pairs of ovals Π^+ and the number of negative pairs of ovals Π^- will be 0. But the right hand side of formula 2.6 should be -2 , because $d = 4$, $\sigma = 0$, and $l = 2$. There is only one possible component of $\mathbb{R}\mathbb{P}^2 \setminus \theta_1(\mathbb{R}\mathbb{P}^1)$ in which the solitary node can lie in, so that on perturbing the resulting ovals will be nested. The only possible position for the solitary node is shown in figures 2.3 and 2.1.

Rokhlin's Complex Orientation Formula also determines which of the curves in the last row of figure 2.1 have nodes which are the result of a self intersection of a hemisphere and which are the result of two hemispheres intersecting.

Perturbing the third curve in the last row results in a negative pair of ovals and therefore $2(\Pi^+ - \Pi^-) = 2$. Equation 2.7, then, forces $\sigma = 0$, which means that each imaginary node is the result of a the self intersection of a hemisphere.

On the other hand, perturbing the last curve in the last row results in no nested ovals and therefore $2(\Pi^+ - \Pi^-) = 0$. Equation 2.7, then, forces $\sigma = 2$, which means that each imaginary node is owing to the intersection of distinct hemispheres.

This completes the proof of the rigid isotopy classification of all generic degree 4 projective planar rational curves, including the positions of the isolated nodes.

Chapter 3

Real rational knots in the sphere

3.1 Introduction

3.1.1 Background

In the topology of real algebraic varieties the most traditional objects of study were either non-singular varieties (like surfaces) or subvarieties of co-dimension one (curves in the 2-space or surfaces in the 3-space). There were few papers devoted to the topology of curves in the 3-space [1] [12], and there the ambient 3-space was the real projective space \mathbb{RP}^3 . From a purely topological point of view, the curves are knots in \mathbb{RP}^3 . There were also papers devoted to polynomial knots in \mathbb{R}^3 [3] [9].

In topology, the more traditional objects are classical knots which are curves in the 3-sphere S^3 . The sphere S^3 appears as the set of real points of algebraic 3-folds. The simplest of them is the quadric in \mathbb{RP}^4 defined by the equation $x_1^2 + x_2^2 + x_3^2 + x_4^2 = x_0^2$. Following notations used in topology, we denote this quadric by S^3 .

In this paper we will study the simplest real algebraic knots in S^3 : non-singular real rational curves of low degrees. A curve $C \subset S^3$ is rational if there exists a regular map $k : \mathbb{RP}^1 \rightarrow \mathbb{RP}^4$ which defines an isomorphism $k : \mathbb{RP}^1 \rightarrow C$. Such a map k is defined by formulas $([s : t] \rightarrow [k_0(s, t) : k_1(s, t) : k_2(s, t) : k_3(s, t) : k_4(s, t)])$ where k_0, \dots, k_4 are real homogeneous polynomials of degree d with no common root.

A map k is not unique, but C defines k up to real projective transformations of the source \mathbb{RP}^1 . We will call k a (rational) **parametrization** of C .

The same polynomials define a regular map $\mathbb{C}k : \mathbb{CP}^1 \rightarrow \mathbb{CP}^4$, a parametrization of the set $\mathbb{C}C$ of complex points of C .

A rational curve C is called a **real rational knot** if C has no singular points (neither real nor complex: i.e. its parametrization $\mathbb{C}k : \mathbb{C}\mathbb{P}^1 \rightarrow \mathbb{C}\mathbb{P}^4$ has no double points and at least one of the partial derivatives is non-zero).

A rational curve $C \in S^3$ with singularities is called a singular real rational knot. Real rational knots may be considered from the perspectives of knot theory where the major equivalence relation comes from isotopy. Recall that knots $k_1, k_2 \in S^3$ are called isotopic if they can be included in a one parameter continuous family $k_t \in S^3$ of classical knots.

In our real algebraic setup, where all knots are real algebraic curves, the notion of isotopy is modified accordingly to a one-parameter family of non-singular real algebraic curves, that is a path in the space of such curves. Such a path is called a **rigid isotopy**. Observe that a rigid isotopy of real algebraic knots can be lifted to a continuous family of parametrizations. In particular, the degree of the knots (which is the degree polynomials forming the parametrization) is preserved under a rigid isotopy. The notion of rigid isotopy can be defined in a similar way in a more general situation and real algebraic varieties of several classical types were classified up to rigid isotopy (see for example, [4]).

In [12], Viro defined the **encomplexed writhe number** for real rational knots in $\mathbb{R}\mathbb{P}^3$. It is a rigid isotopy invariant. The definition can be used to define the writhe number of a knot in the sphere as follows: given a knot C in the 3-sphere, choose a point of projection p on the sphere which is not on the knot and project the knot to $\pi_p(C)$ in $\mathbb{R}\mathbb{P}^3$. The encomplexed writhe number of C is defined as the writhe number of $\pi_p(C)$. Note that this is independent of the point of projection because the image of knots under two different such projections are rigidly isotopic in $\mathbb{R}\mathbb{P}^3$ and so they will have the same writhe number.

3.1.2 Main results

An odd degree rational curve in $\mathbb{R}\mathbb{P}^4$ must intersect a generic linear hypersurface in an odd number of real points to preserve the mod 2 intersection number. This is impossible for odd degree knots contained in the sphere because there are real planes disjoint from the real sphere. Therefore: *all real rational knots in the sphere are of even degree*. We will prove the following rigid isotopy classification for knots of degrees 2, 4, and 6 in S^3 :

Theorem 5. *The rigid isotopy classes of knots of degrees less than 6 can be described as follows:*

1. *Degree 2: there is one rigid isotopy class; it has writhe number 0. All knots are topologically isotopic to the unknot.*

2. Degree 4: there are two rigid isotopy classes; they have writhe numbers ± 2 . Mirror images of each knot lie in separate classes. Each of them are topologically isotopic to the unknot.
3. Degree 6: there is one rigid isotopy class with each of the writhe numbers $0, \pm 2, \text{ or } \pm 4$. Their smooth isotopy classes may be described as follows:
 - (a) writhe 0: unknot
 - (b) writhe 2: unknot
 - (c) writhe -2 : unknot
 - (d) writhe 4: trefoil
 - (e) writhe -4 : trefoil

In fact for degrees 2 and 4 we will prove a stronger result that knots in the same rigid isotopy class are projectively equivalent: any two of them can be transformed from one to the other by a projective linear transformation in \mathbb{RP}^4 that preserves S^3 .

Besides merely classifying them, we will also relate them to knots in the projective space. Indeed, we will use the classification of knots in the projective space to derive our classification, although we will need to extend the classification in the projective space to pairs of knots and planes that intersect in a special way.

To construct examples, we will note that for each degree d , there are $(m, d/2)$ -torus knots for any m which is coprime to $d/2$. We will also observe that Björklund's method [1] of constructing real rational knots can be lifted to the sphere to provide an exact counterpart to his method of construction. This will at first seem to contradict the classification of curves in S^3 which does not allow the figure eight knot to be realized by a degree 6 curve. However, section 3.4.1 will demonstrate why we need to be careful when constructing curves in S^3 because we are not permitted to use general conics (as Björklund had used in \mathbb{RP}^3 setting), only circles. Indeed, the trefoil in the sphere cannot be constructed by perturbing the configuration that Björklund used, but uses a different configuration.

Björklund [1] had classified real rational knots in \mathbb{RP}^3 up to degree 5. We can partially add to that classification by classifying singular knots of degree 6 in \mathbb{RP}^3 with four double points, by showing that such knots are characterized, up to rigid isotopy and mirror reflection, by their chord diagrams just as it was done in Chapter 3. There is also some hope of this method providing some information about knots of degree 8 in S^3 .

3.1.3 Outline of the method

There are several difficulties when trying to classify real rational knots in S^3 up to rigid isotopy. In [1], linear transformations were used to simplify knots in \mathbb{RP}^3 ; for knots in S^3 , we can only use linear transformations that preserve the sphere. This does not help except in the simplest cases like degrees 4. In fact the method of classification of degree 4 knots in S^3 is very similar to the approach used in [1] to classify rational curves of degree 3 in \mathbb{RP}^3 , which suggests a connection between knots in S^3 and knots in \mathbb{RP}^3 of lower degree.

The connection is owing to the stereographic projection. If one considers knots with one singularity, the degree d knots in S^3 with the north pole as its only double point, relate to degree $d - 2$ knots in \mathbb{RP}^3 which intersect the plane at infinity in $d - 2$ points, $d - 4$ of which lie on a conic of signature 0. More generally we may consider pairs of knots and planes as described in the next two paragraphs.

Following the notation in [1], let \mathcal{K}_d denote the space of real rational knots of degree d in \mathbb{RP}^3 . $\mathbb{P}\text{Gr}(3, \mathbb{R}^4)$ is the grassmanian of 2-planes in \mathbb{RP}^3 . Consider the space of pairs $(C, X) \in \mathcal{K}_d \times \mathbb{P}\text{Gr}(3, \mathbb{R}^4)$, where C is a real rational knot of degree d in \mathbb{RP}^3 and X is a real plane in \mathbb{RP}^3 . Denote by $\mathcal{P}_{d,m}$ the subspace of those pairs such that $\mathbb{R}C$ and $\mathbb{R}X$ intersect in exactly m real points, where $m = 0$ or $m = 2$.

Define $\mathcal{P}'_{d,m} \subset \mathcal{P}_{d,m}$ to be the pairs (C, X) , such that $d - 4$ points of intersection of $\mathbb{C}C$ and $\mathbb{C}X$ lie on a conic in X of signature 0.

In theorem 9 of section 3.2.3, we will prove that the space of rational knots of degree d in S^3 with exactly one double point, is a double covering of $\mathcal{P}'_{d-2,m}$ where $m = 0$ if the double point is solitary, and $m = 2$ if it is not. The two curves in each fibre of the double covering are rigidly isotopic to the mirror images of each other.

It will be difficult to extend an isotopy of knots in \mathbb{RP}^3 to an isotopy of pairs in $\mathcal{P}'_{d,m}$ because of the condition imposed on the intersection of the knot and the plane. However, in section 10, we will show that when $d \leq 6$, then $\mathcal{P}'_{d,m}$ is a strong deformation retract of $\mathcal{P}_{d,m}$. This simplifies the restriction on the intersection, however maintaining the right number of real intersection points is still difficult. For degree 6, we will be able to overcome this difficulty because they relate to degree 4 knots in \mathbb{RP}^3 , which always lie on a quadric. Knowing the bi-degree of the degree 4 knot on the quadric will make it easier to classify the pairs of knots and planes which intersect in a fixed number of real points.

While this will classify degree 6 knots with one double point, we obtain the non-singular knots by perturbing these. By using the Borel-Moore homology, we will that the number of rigid isotopy components cannot exceed 5. In order

to obtain this bound, we will be faced with another difficulty: that the space of rational knots of degree 6 in S^3 is a connected manifold. This problem is trivial for knots in a projective space, since the space of knots of a fixed degree in \mathbb{RP}^3 is itself a projective space. We will solve it for rational knots in S^3 in section 3.3.4. Finally, we will construct 5 rational knots of degree 6 in the sphere with different writhe numbers, to obtain a representative for each of the 5 rigid isotopy classes.

3.2 Degrees 2 and 4

3.2.1 Degree 2 knots

Degree 2 knots in the sphere also lie on a two dimensional plane in \mathbb{RP}^4 because any three points on the curve define a pencil of planes; it has to lie on the intersection of the planes from the pencil because it intersects each plane in more than 2 points. The intersection is a two dimensional plane on which our knot lies and so it is a planar conic which is non-empty. It is well known that non-empty non-singular planar conics are projectively equivalent and so the knot is rigidly isotopic to the unknot.

3.2.2 Degree 4 knots

Lemma 6. *A degree 4 knot in S^3 lies on a linear hypersurface of \mathbb{RP}^4 if and only if it has a double point.*

Proof. Degree 4 knots in the sphere with double points, lie on a linear hypersurface. This is because any double point on the knot along with three other points define a linear hypersurface in \mathbb{RP}^4 which intersects the knot in five points (counting multiplicity, because of the double point). So the singular knot is forced to lie on the linear hypersurface.

Conversely, if a curve lies on a linear hypersurface in \mathbb{RP}^4 , then it must be singular. This can be seen by projecting it to \mathbb{RP}^3 via the stereographic projection, from a point on the linear hypersurface but not on the curve. It will project to a degree 4 curve on a plane. The complexification of this rational curve must have 3 double points because any planar rational curve of degree d must have $\frac{(d-1)(d-2)}{2}$ double points. Since the curve is real, any imaginary double points must occur in pairs. Therefore, at least one double point is real and the original curve is not a knot. \square

Observe that changing the parametrization of a knot C parametrized by $k : \mathbb{RP}^1 \rightarrow S^3 \subset \mathbb{RP}^4$ means composing k by a transformation of \mathbb{RP}^1 , namely

a transformation in $\mathbb{P}GL(2, \mathbb{R})$ which has two connected components that correspond to transformations that either retain or reverse the orientations respectively. Therefore changing the parametrization provides a knot which is either the same or reverses its orientation.

Projective transformations that preserve the 3-sphere form the projectivization of the transformations that preserve the bilinear form $x_1^2 + x_2^2 + x_3^2 + x_4^2 - x_0^2$, namely the indefinite orthogonal group $O(4, 1)$. $O(4, 1)$ has four components but after projectivizing it has two components. One component contains the identity whereas the other component contains the reflection. Denote the projectivization of $O(4, 1)$ by $\mathbb{P}O(4, 1)$. Given any curve C in S^3 , its image under a transformation $A \in \mathbb{P}O(4, 1)$ is either isotopic to itself or to its mirror image.

Recall that a finite set of points in $\mathbb{R}P^n$ is said to be in *general position* if there is no subset with $n + 1$ points that all lie on a linear hypersurface. The following lemma will show that any projective transformation will do:

Lemma 7. *A transformation in $\mathbb{P}GL_5(\mathbb{R})$ that takes at least 14 points in general position in S^3 to points within S^3 is in $\mathbb{P}O(4, 1)$.*

Proof. The images of the points are also in general position because the inverse of the transformation would pull back any linear relations satisfied by the images. The images lie in the image of the sphere (another quadric), but are given to be in the original sphere. The sphere and its image must coincide because 14 points in general position uniquely define a quadric in $\mathbb{R}P^4$. So the transformation preserves S^3 . \square

Theorem 8. *Any two degree 4 real rational knots in the 3-sphere are projectively equivalent. Therefore, there are two rigid isotopy classes which are mirror images of each other.*

Proof. Let $k_1, k_2 : \mathbb{R}P^1 \rightarrow S^3 \subset \mathbb{R}P^4$ each denote the parametrization of two knots C_1 and C_2 of degree 4. Distinct points on C_1 or C_2 are now in general position in $\mathbb{R}P^4$ because a linear hypersurface of $\mathbb{R}P^4$ containing more than five of these points would contain the curve for intersecting it in more than four points. This would contradict lemma 6.

If the i th coordinate of a parametrization k of a real rational knot is the polynomial $\sum_{j=0}^d \alpha_j^i s^j t^{d-j}$, then let A_k denote the matrix of coefficients (α_i^j) of k . Note that knot defined by k not lying on a linear hypersurface of $\mathbb{R}P^4$ is equivalent to $A_k \in \mathbb{P}GL_5(\mathbb{R})$. Therefore, $A_{k_1} A_{k_2}^{-1} \in \mathbb{P}GL_5(\mathbb{R})$. Observe that $A_{k_1} A_{k_2}^{-1} k_2 = k_1$. Therefore, they are projectively equivalent.

If $A_{k_1} A_{k_2}^{-1}$ lies in the component of $\mathbb{P}O(4, 1)$ that contains the identity, then the knots defined by k_1 and k_2 are rigidly isotopic, otherwise k_1 is isotopic to the mirror image of k_2 . \square

3.2.3 The connection between real rational knots in S^3 and real rational knots in \mathbb{RP}^3

The idea of using the fact that non-planar curves are non-singular, is borrowed from Björklund's paper [1]. The behaviour of degree 4 knots in S^3 is very similar to degree 3 knots in \mathbb{RP}^3 . This connection will be made explicit by using the stereographic projection. Since we are concerned with the stronger notion of rigid isotopy, we will need to keep a track of the intersection of the curve with the blow up of the point of projection.

Consider the sphere $S^3 \subset \mathbb{RP}^4$ and a point p on S^3 . A linear hypersurface in \mathbb{RP}^4 which is disjoint from S^3 can be treated as a copy of \mathbb{RP}^3 . The projection map $\pi_p : S^3 \setminus p \rightarrow \mathbb{RP}^3$ is defined by mapping any point α to the intersection of the plane with the unique line joining p and α . The line is unique because any line can transversally intersect a quadric in only two points, one of which is p .

The projection can be extended to the complexification $\mathbb{C}S^3 \subset \mathbb{C}\mathbb{P}^4$. If a point α on S^3 is such that the line joining it with p is not tangential to S^3 , then the line does not intersect any other points of S^3 . Therefore, π_p is a bijection on the set $S^3 \setminus \mathbb{C}T_p$, where T_p is the tangent plane to S^3 at p . It is a rational map on S^3 that blows up p to a plane X_p in \mathbb{RP}^3 .

$\mathbb{C}_p \cap \mathbb{C}S^3$ is a quadric with a singularity at p . It is therefore a complex 2-dimensional cone with an apex at p . This cone intersects the plane of projection \mathbb{RP}^2 in an empty conic lying on X_p . Recall that an empty conic is a conic defined by a polynomial with real coefficients but which does not have any real zeros.

The complexification $\mathbb{C}C$ of any degree d rational curve C , parametrized by k , intersects $\mathbb{C}T_p$ in d points (counting multiplicity). These d points of intersection must lie on the cone defined by $\mathbb{C}_p \cap \mathbb{C}S^3$. If p does not lie on the curve C , the closure of $\pi_p(\mathbb{C}C) \setminus \mathbb{C}T_p$ intersects $\mathbb{C}X_p$ in d points that all lie on the empty conic which is the projection of the cone $\mathbb{C}_p \cap \mathbb{C}S^3$. We define the image $\pi_p(C)$ of the curve C under the rational map π_p as the rational curve defined by the closure of $\pi_p(\mathbb{C}C \setminus \mathbb{C}T_p)$. Since the image intersects the plane X_p in d points, the degree of this rational curve is d . π_p is a bijection between real rational curves in S^3 of degree d and real rational curves in \mathbb{RP}^3 of degree d which intersect the blow-up of p in points that lie on an empty conic.

By choosing the point of projection to be on the curve C , the degree of $\pi_p(C)$ can be made to drop. If the curve C passes through p , then T_p intersects C at p with multiplicity 2, then $d-2$ of them intersect T_p in imaginary points. The closure of $\pi_p(\mathbb{C}C \setminus \mathbb{C}T_p)$ intersects $\mathbb{C}X_p$ in $d-1$ points and is therefore a curve of degree $d-1$. $d-2$ of these points of intersection lie on the empty conic which is the projection of the cone $\mathbb{C}_p \cap \mathbb{C}S^3$. π_p is a bijection between real

rational curves in S^3 of degree d and real rational curves in $\mathbb{R}\mathbb{P}^3$ of degree $d-1$ which intersects the blow-up of p in d points, $d-1$ of which lie on an empty conic.

The branch of the curve around p gets mapped to a branch that intersects the plane at infinity once. This point of intersection is determined by the direction of the tangent line to the curve at p .

It will prove more useful to consider the case where the point p is a double point.

Theorem 9. *The space of degree d knots in S^3 with one double point is a double covering of $\mathcal{P}'_{d-2,k}$, where $k = 0$ if the double point is solitary, and $k = 2$ if its not. Given a pair $(C, X) \in \mathcal{P}_{d-2,k}$, the two lifts of it are related: they are each isotopic to the mirror image of the other.*

Proof. If p is a double point of the curve C , then T_p intersects two branches of C at p , each with multiplicity 2. In that case, $d-4$ of them intersect T_p in imaginary points. The closure of $\pi_p(\mathbb{C}C) \setminus \mathbb{C}T_p$ intersects $\mathbb{C}X_p$ in $d-2$ points, and is therefore of degree $d-2$. $d-4$ of them lie on the empty conic which is the projection of the cone $\mathbb{C}_p \cap \mathbb{C}S^3$. This will help us to relate rational knots of degree d with one double point, to rational curves of degree $d-2$ in $\mathbb{R}\mathbb{P}^3$ that intersect the blow up of the double point in $d-2$ points, $d-4$ of which lie on an empty conic. \square

Simplifying the hypothesis for degree 6

The main difficulty is in maintaining the right intersection for between a knot and a plane belonging to a pair in $\mathbb{P}_{4,k}$. Recall that we had defined the space $\mathcal{P}'_{4,k}$ to be the space of those pairs (C, k) , where C intersects k in k real points, but the imaginary points of intersection need not lie on the empty conic.

Theorem 10. *$\mathcal{P}'_{4,k}$ is a deformation retract of $\mathcal{P}_{4,k}$, where $k = 0$ or $k = 2$. In other words: there is a retraction map $r : \mathcal{P}_{4,k} \rightarrow \mathcal{P}'_{4,k}$ which is homotopic to the identity.*

Proof. For each line we will need a fixed real point that does not lie on it. But it is possible to define a continuous map θ from the space of all real lines at the plane at infinity to the space of all real points at the plane at infinity such that each line is taken to a point that does not lie on it.

Define a homotopy of maps $[0, 1] \times \mathcal{P}'_{4,k} \rightarrow \mathcal{P}_{4,k}$ as follows: given a curve C in $\mathcal{P}_{4,k}$, it intersects the plane at infinity in a conjugate pair z and \bar{z} which will define a real line l . $\mathbb{C}l$ intersects the standard empty conic in another conjugate pair w and \bar{w} where we let w denote the point that is in the same

component of $\mathbb{C}l \setminus \mathbb{R}l$ as z . By the previous paragraph, $\theta(l)$ is a point in the plane at infinity. Define T_t to be the unique real linear transformation that fixes the real points $[1 : 0 : 0 : 0]$ and $\theta(l)$, and maps $z \rightarrow (1 - t)z + tw$ and $\bar{z} \rightarrow (1 - t)\bar{z} + t\bar{w}$. Observe that T_0 is the identity map, and T_1 takes $z \rightarrow w$ and $\bar{z} \rightarrow \bar{w}$. Therefore the map $r : \mathcal{P}_{4,k} \rightarrow \mathcal{P}'_{4,k}$ defined by $r(C) := T_1(C)$ is a retract which is homotopic to the identity. \square

Therefore, to prove that two degree 6 curves in S^3 , each with one double point, are isotopic, we only need to find a path between their stereographic projections in the space $\mathcal{P}_{4,k}$.

Remark 11. We can use the stereographic projection to classify the degree 4 knots in S^3 by connecting them with degree 3 knots in $\mathbb{R}\mathbb{P}^3$.

A degree 4 knot in S^3 can be projected by a point on the knot to a knot in $\mathbb{R}\mathbb{P}^3$ which intersects the plane at infinity in one real and a pair of conjugate imaginary points. The imaginary points lie on the empty quadric. Any degree 3 curve in $\mathbb{R}\mathbb{P}^3$ intersects the plane at infinity in one real and a pair of conjugate imaginary points. The conjugate imaginary points can always be made to lie on the standard empty conic and therefore any degree 3 curve in $\mathbb{R}\mathbb{P}^3$ may be lifted to S^3 by the stereographic projection. Therefore, the isotopy classes of degree 4 knots in S^3 correspond to the isotopy classes of degree 3 knots in $\mathbb{R}\mathbb{P}^3$. Since there are two isotopy classes of degree 3 knots in $\mathbb{R}\mathbb{P}^3$, there are two isotopy classes of degree 4 knots in S^3 [1].

3.3 Degree 6

To classify degree 6 knots in S^3 with one double point, we can relate them to pairs of degree 4 knots and planes in $\mathbb{R}\mathbb{P}^3$. Björklund [1] had classified degree 4 curves by first classifying the singular ones and then perturbing them. But this cannot be used to extend this to the space of pairs of knots and planes, so we will consider the fact that degree 4 curves lie on a quadric. But first we will show how theorem 9 can be used to obtain the degree 4 curves with one double point.

Change bases so that the double point is $[1 : 0 : 0 : 0]$ as the images of either $[0 : 1]$ and $[1 : 0]$, or $[1 : i]$ and $[1 : -i]$, depending on whether it is real or imaginary. Project from this to the plane defined by $x_0 = 0$. This projection will produce only finitely many double points because the knot is on a quadric and so a double point can only occur when its inverse under the projection is a line on the quadric that passes through $[1 : 0 : 0 : 0]$. At most two lines can pass through $[1 : 0 : 0 : 0]$, namely the ones from the ruling of the quadric.

As before the projection is a curve degree $2 = 4 - 2$, because the common factors of either st or $s^2 + t^2$ will pull out. Change the basis so that this degree 2 curve is the standard $[s : t] \rightarrow [s^2 : st : t^2]$. Since the knot lies on the cone over this curve with the apex as the double point, it must be of the form $[s^3t : s^2t^2 : st^3 : p_4[s : t]]$ or $[s^2(s^2 + t^2) : st(s^2 + t^2) : t^2(s^2 + t^2) : p_4[s : t]]$ where $p_4[s : t]$ is a polynomial of degree 4.

In the case where they are real double points, this polynomial may be reduced to either $s^4 + t^4$, $s^4 - t^4$, $-s^4 + t^4$, or $-s^4 - t^4$ because all the other coefficients may be brought to zero without forming a double point as the first three coordinates are distinct anyway. We only need to avoid s or t dividing it or else the degree will drop; the presence of both s^4 and t^4 prevent this.

In the case where they are solitary double points, we only need to avoid multiples of $s^2 + t^2$ which form a co-dimension two hypersurface in the space of polynomials of degree 4. This can therefore be reduced to a standard form of $[s^2(s^2 + t^2) : st(s^2 + t^2) : t^2(s^2 + t^2) : s^4]$. Therefore the walls are:

$$\begin{aligned} & [s^3t : s^2t^2 : st^3 : s^4 + t^4] \\ & [s^3t : s^2t^2 : st^3 : s^4 - t^4] \\ & [s^3t : s^2t^2 : st^3 : -s^4 + t^4] \\ & [s^3t : s^2t^2 : st^3 : -s^4 - t^4] \\ & [s^2(s^2 + t^2) : st(s^2 + t^2) : t^2(s^2 + t^2) : s^4] \end{aligned}$$

These were precisely the walls considered by Björklund [1], and he also observed that the first and the third are rigidly isotopic. By perturbing them he showed that *knots of degree 4 were either the unknot and its image (with writhes +1 or -1) or the two crossing knot and its mirror image (with writhes +3 or -3)*.

3.3.1 Classification of $\mathcal{P}_{4,k}$

Degree 4 curves in \mathbb{RP}^3 must lie on a quadric surface because we can always find a quadric surface passing through any 9 points on the curve. The curve can intersect the quadric in more than 8 points only if it lies on it. To classify $\mathcal{P}_{4,k}$, it will prove essential to understand how the knots of degree 4 lie on this quadric surface.

In the complex case, there is only one non-singular quadric. In the real case, there are two non-singular quadrics, which are the sphere and hyperboloid.

By a change of coordinates, the hyperboloid can be defined by as the zero set of the polynomial $z_0z_3 - z_1z_2 = 0$. It can also be realized as the image of

the embedding, $\theta : \mathbb{RP}^1 \times \mathbb{RP}^1 \rightarrow \mathbb{RQ} \subset \mathbb{RP}^3$, which is defined explicitly by the map

$$([x_0 : x_1], [x_2 : x_3]) \rightarrow [x_0x_2 : x_0x_3 : x_1x_2 : x_1x_3]$$

This map can be extended to the complexification, which is known as the Segre embedding.

For a fixed $[\alpha : \beta]$, the image of $([\alpha : \beta], [x_2 : x_3])$ is a line $L_{[\alpha:\beta]}$. Similarly, for a fixed $[x_0 : x_1]$, the image of $([x_0 : x_1], [\alpha : \beta])$ is a line $L'_{[\alpha:\beta]}$. The lines $L_{[\alpha:\beta]}$ form a family of rulings F , while the lines $L'_{[\alpha:\beta]}$ form a family of rulings F' . If two lines are from the same family they are disjoint, and if they are different families they intersect in one point. The homology classes $[\mathbb{R}L]$ and $[\mathbb{R}L']$ with representatives L and L' from different families, generate $H_1(\mathbb{R}Q) = \mathbb{Z} \oplus \mathbb{Z}$. Similarly the homology classes $[\mathbb{C}L]$ and $[\mathbb{C}L']$ with representatives L and L' from different families, generate $H_2(\mathbb{C}Q) = \mathbb{Z} \oplus \mathbb{Z}$.

By a change of coordinates, a real sphere in the projective space can be defined to satisfy the equation $x_1^2 + x_2^2 = x_0^2 - x_3^2$. The rulings on it are imaginary and can be realized as follows. Each side of $x_1^2 + x_2^2 = x_0^2 - x_3^2$ can be factored to give $(x_1 + ix_2)(x_1 - ix_2) = (x_0 - x_3)(x_0 + x_3)$. So the bijection $[x_0 : x_1 : x_2 : x_3] \rightarrow [x_1 + ix_2 : x_0 - x_3 : x_0 + x_3 : x_1 - ix_2]$ takes the sphere to a complex quadric which satisfies the equation $z_0z_3 = z_1z_2$ and is the image of the Segre embedding. This is a bijection on the complexification of the sphere; when the bijection is restricted to the real part, the coordinates of the image also satisfy the conditions that z_1 and z_2 are real, and that $z_3 = \bar{z}_0$. The defined in the previous paragraph will define a ruling. Observe that in this case the two families of rulings are conjugate to each other.

There are also two singular quadrics: a pair of planes and a cone; the latter has only one singular point whereas the former has infinitely many.

We now revise the concept of a bi-degree of a rational curve, but more specifically for real rational curves.

Lemma 12. *Given a curve $f : \mathbb{RP}^1 \rightarrow Q \subset \mathbb{RP}^3$, there is a map $\tilde{f} : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1 \times \mathbb{RP}^1$ such that $\theta \circ \tilde{f} = f$ as is in the diagram:*

$$\begin{array}{ccc} \mathbb{RP}^1 & & \\ \tilde{f} \downarrow & \searrow f & \\ \mathbb{RP}^1 \times \mathbb{RP}^1 & \xrightarrow{\theta} & Q \end{array}$$

Proof. Let $[s : t] \rightarrow [p_0(s, t) : p_1(s, t) : p_2(s, t) : p_3(s, t)]$ denote the parametrization of a degree d curve that lies on the quadric defined by $X_0X_3 - X_1X_2 = 0$.

Therefore $p_0p_3 = p_1p_2$ and so p_0p_3 and p_1p_2 have the same factors. Let q_0 be the factor of highest degree common to p_0 and p_1 (it must be real since imaginary roots occur in pairs), and denote its degree by m . Therefore $q_2 = p_0/q_0$ and $q_3 = p_1/q_0$ are degree $d - m$ real polynomials.

By definition, q_2 divides p_0 , and therefore it divides p_1p_2 . If it shares a common factor r with p_1 then q_0r is a factor common to p_0 and p_1 , contradicting the fact that q_0 was the highest common factor. Therefore, q_2 is co-prime to q_1 and so it must divide p_2 and $q_1 = p_2/q_2$ is a degree $d - (m - d) = m$ real polynomial.

This way $p_0 = q_0q_2$, $p_1 = q_0q_3$, and $p_2 = q_1q_2$. We only have to check that $p_3 = q_1q_3$, but $q_1q_3 = p_1p_2/(q_0q_2) = p_1p_2/p_0$. Since $p_0p_3 = p_1p_2$, we get $p_3 = q_1q_3$.

q_0 and q_1 have the same degree m ; q_2 and q_3 have the same degree $d - m$. Therefore the map defined by coordinates $[q_0 : q_1]$ and $[q_2 : q_3]$ are well defined. \square

Definition 13. Given a rational map $f : \mathbb{RP}^1 \rightarrow Q \subset \mathbb{RP}^3$, consider its corresponding map $\tilde{f} : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1 \times \mathbb{RP}^1$ as defined in the previous lemma. Fix an orientation of each \mathbb{RP}^1 in both the domain and co-domain and let π_1 and π_2 denote the projections onto the first and second coordinates respectively. $\pi_i \circ \tilde{f} : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$, for $i = 1, 2$, are maps with degrees $m = \deg(\pi_1 \circ \tilde{f})$ and $n = \deg(\pi_2 \circ \tilde{f})$. Then the **real bi-degree** of the curve is defined to be (m, n) .

The maps in lemma 12 can be extended to the complexification. We can therefore define the complex bi-degree of the curve as follows:

Definition 14. Given a rational map $f : \mathbb{CP}^1 \rightarrow Q \subset \mathbb{CP}^3$, consider its corresponding map $\tilde{f} : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1 \times \mathbb{CP}^1$ as defined in the previous lemma. If $m = \deg(\pi_1 \circ \tilde{f})$ and $n = \deg(\pi_2 \circ \tilde{f})$, Then the **complex bi-degree** of the curve is defined to be (m, n) .

Remark 15. The complex bi-degree of the curve as defined above is the same as the degrees of the homogeneous polynomials defining the map \tilde{f} . This means that if the complex bi-degree of the curve f is (m, n) , then $m + n =$ the degree of the rational map f . Note that m or n may be negative when (m, n) denotes the real bi-degree but not when it denotes the complex bi-degree.

Lemma 16. A curve C in a quadric surface Q cannot have bi-degree $(2, 2)$.

Proof. This easily follows from the adjunction formula [7] $g = mn - m - n + 1$ for a genus g curve of bi-degree (m, n) embedded in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Curves of bi-degree $(2, 2)$ would have to have genus 1 and therefore cannot be rational.

However, in this special case, it may also be seen by the following elementary means.

The stereographic projection from a point on Q which is not on the curve, will project C to a planar rational curve of degree 4.

Note that the tangent plane to the point of projection intersects the quadric in a pair of lines. By the definition of the bi-degree, the curve intersects each of these lines in a pair of points. Each pair projects to the same point at the plane at infinity, giving rise to two nodes (one for each line) at the line at infinity in \mathbb{RP}^2 . So it projects to a degree 4 rational curve with only two nodes, which is impossible because a rational curve with only nodes as its singularity has to have exactly $\frac{(d-1)(d-2)}{2}$ nodes. \square

Lemma 17. *A degree 4 real rational curve C cannot lie in a real sphere.*

Proof. The complexification of the sphere is ruled by two families of complex lines. The two families are conjugate to each other. If the complexification $\mathbb{C}C$ of a real curve C intersects a line from one of these families in m points, then it must also intersect the conjugate of that line in m points. This means that it would have to have bi-degree (m, m) . This is impossible for degree 4 real rational curves which cannot have bi-degree $(2, 2)$ because of lemma 16. \square

We now note two lemmas that will help us when classifying degree 4 curves by reducing the number of real bi-degree cases to be checked to merely two cases.

Lemma 18. *Consider a curve of real bi-degree (m, n) in a curve in the hyperboloid Q defined by $x_0x_3 = x_1x_2$. Reflection along the plane $x_1 = x_2$ preserves the hyperboloid, but interchanges the family of rulings and therefore changes the curve to one of bi-degree (n, m) .*

Proof. When the Segre embedding $\theta : \mathbb{RP}^1 \times \mathbb{RP}^1 \rightarrow Q \subset \mathbb{RP}^3$ is composed with the map $r : \mathbb{RP}^1 \times \mathbb{RP}^1 \rightarrow \mathbb{RP}^1 \times \mathbb{RP}^1$ which interchanges coordinates, it reverses the families. Explicitly: $([x_0 : x_1], [x_2 : x_3]) \rightarrow ([x_2 : x_3], [x_0 : x_1]) \rightarrow [x_0x_2 : x_1x_2 : x_0x_3 : x_1x_3]$. But this is the same as the embedding $([x_0 : x_1], [x_2 : x_3]) \rightarrow [x_0x_2 : x_0x_3 : x_1x_2 : x_1x_3]$ composed by a reflection in the plane given by $x_1 = x_2$. \square

Lemma 19. *Consider a curve C of real bi-degree (m, n) in the hyperboloid Q defined by $x_0x_3 = x_1x_2$. Reversing the orientation of one coordinate, results in a curve \bar{C} of real bi-degree $(-m, n)$, which is projectively equivalent to C by a transformation which is in the component of $\mathbb{P}GL_4(\mathbb{R})$ containing the identity. Therefore \bar{C} is rigidly isotopic to C .*

Proof. Reversing the orientation of one coordinate is equivalent to composing the Segre embedding $\theta : \mathbb{RP}^1 \times \mathbb{RP}^1 \rightarrow Q \subset \mathbb{RP}^3$ with the map $r : \mathbb{RP}^1 \times \mathbb{RP}^1 \rightarrow \mathbb{RP}^1 \times \mathbb{RP}^1$, given explicitly by $([x_0 : x_1], [x_2 : x_3]) \rightarrow ([x_0 : -x_1], [x_2 : x_3]) \rightarrow [x_0x_2 : -x_1x_2 : x_0x_3 : -x_1x_3]$ which is the same as the embedding $([x_0 : x_1], [x_2 : x_3]) \rightarrow [x_0x_2 : x_0x_3 : x_1x_2 : x_1x_3]$ composed by the projective transformation $[x_0 : x_1 : x_2 : x_3] \rightarrow [x_0 : -x_1 : x_2 : -x_3]$. This projective transformation is in the same component of $\mathbb{P}GL_4(\mathbb{R})$ that contains the identity. \square

Lemma 20. *All degree 4 rational curves in the hyperboloid are rigidly isotopic to either one of these curves, or their reflections:*

$$\begin{aligned} & [s(s^3 + t^3) : s(s^3 - t^3) : t(s^3 + t^3) : t(s^3 - t^3)] \\ & [s^2(s^2 - 4t^2) : st(s^2 - t^2) : st(s^2 - 4t^2) : t^2(s^2 - t^2)] \end{aligned}$$

Proof. We know that the curve must have bi-degree $(1, 3)$ or $(3, 1)$. If it has bi-degree $(3, 1)$, consider its reflection with bi-degree $(1, 3)$. After a change of coordinates, its parametrization $k : \mathbb{RP}^1 \times \mathbb{RP}^1$ may be explicitly defined as:

$$[s : t] \rightarrow ([s : t], [p_0 : p_1])$$

where each p_i has degree 3. In that case k is always an embedding, no matter what p_i 's are as long as they are of degree 3, since the linear part has no double points. Therefore, to ensure that a homotopy is an isotopy, we only need to avoid a common factor in the coordinate polynomials defining the second coordinate of k . As long as this is ensured, no singularities can result owing to the linear part. Therefore moving a root of p_0 will define a rigid isotopy as long it does not meet with a root of p_1 . Similarly, moving a root of p_1 will define a rigid isotopy as long it does not meet with a root of p_0 .

There are three cases to be examined:

Case 1 (p_0 and p_1 have imaginary roots): Then the polynomials are each characterized by a pair of conjugate imaginary roots and therefore by an element of $\mathbb{CP}^1 \setminus \mathbb{RP}^1$. It is easy to shift the complex roots of p_0 to $[1 : \omega]$ and $[1 : -\omega^2]$ and the roots of p_1 to $[1 : -\omega]$ and $[1 : \omega^2]$. Thereafter, there is no obstruction to shifting the real roots of p_0 and p_1 to $[1 : -1]$ and $[1 : 1]$, thereby showing that all curves of this bi-degree form one isotopy class.

$$[s : t] \rightarrow ([s : t], [s^3 + t^3 : s^3 - t^3])$$

Case 2 (either p_0 or p_1 has imaginary roots): Assume that p_0 has imaginary

roots, and p_1 has all real roots, then the only real root of p_0 divides \mathbb{RP}^1 into only two components. At least two of the roots of p_1 are forced to lie in the same component and therefore may be combined to form a double root and then converted to an imaginary pair, reducing it to the above case.

Case 3 (p_0 and p_1 have only real roots): The roots of p_0 divides \mathbb{RP}^1 into three components. If at least two of the roots of p_1 lie in the same component, then this pair can be combined to form an imaginary pair, thereby reducing this to the above case.

So finally the only case left is where both polynomials have real roots, and none of the roots of p_1 lie in the same component of $\mathbb{RP}^1 \setminus \{\text{roots of } p_0\}$. We may move the roots so that the roots of p_0 are $[1, -2], [0 : 1], [1, 2]$ while the roots of p_1 are $[1, -1], [1 : 0], [1, 1]$, giving rise the following map:

$$[s : t] \rightarrow ([s : t], [s(s^2 - 4t^2) : t(s^2 - t^2)])$$

Composing each of the above four curves on $\mathbb{RP}^1 \times \mathbb{RP}^1$ with the Segre map $([x_0 : x_1], [x_2 : x_3]) \rightarrow [x_0x_2 : x_0x_3 : x_1x_2 : x_1x_3]$ gives a parametrization for each of the four isotopy classes of degree 4 up to mirror reflection.

They are as follows:

$$\begin{aligned} & [s(s^3 + t^3) : s(s^3 - t^3) : t(s^3 + t^3) : t(s^3 - t^3)] \\ & [s^2(s^2 - 4t^2) : st(s^2 - t^2) : st(s^2 - 4t^2) : t^2(s^2 - t^2)] \end{aligned}$$

□

When the stereographic projection is applied to a knot of degree 6 in S^3 with one double point, it is sent to a real rational knot in \mathbb{RP}^3 , which intersects the plane at infinity in two real points if the double point was real and only imaginary points if the double point was solitary.

A non-singular knot in \mathbb{RP}^3 may lie on a cone (if it lies on a plane it would have to be singular). However, the following lemma reduces its study to only those knots which lie on a hyperboloid.

Lemma 21. *Consider a pair $(C, X) \in \mathcal{P}_{4,k}$ of a real rational knot C and a plane X so that the $\mathbb{C}C \cap \mathbb{C}X$ has four distinct points. C can be perturbed by an arbitrary small perturbation to lie on a hyperboloid and still intersect the plane $\mathbb{C}X$ in four distinct points and the same number of real points.*

Proof. If C lies on a cone, use a projective transformation to transform the cone to one defined by $x_0x_2 - x_1^2 = 0$. So a parametrization $[p_0 : p_1 : p_2 : p_3]$ of a curve on this cone is $p_0p_2 = p_1^2$. This means that the roots of p_1 determine the roots of p_0 and p_2 while p_3 can be arbitrary; therefore the dimension of

the curves on this cone are 10. Since the space of cones is of dimension 8, the total dimension of the space of curves lying on a cone is 18 and therefore of codimension one in the space of all curves of degree 4 in \mathbb{RP}^3 .

So by small perturbation, the curve can be made to lie on a hyperboloid. The perturbation may be chosen small enough to not change the real intersection and ensure that the points in the intersection of the complexification are distinct. \square

We now use our study of knots of degree 4 in \mathbb{RP}^3 to study knots of degree 6 in the sphere with one double point.

3.3.2 Degree 6 knots in S^3 with a real double point

The real part of a real curve in \mathbb{RP}^3 is called affine if there is a real plane which is disjoint from it. We first prove the following lemma that will be used later:

Lemma 22. *A real curve of real bi-degree $(1, 3)$ on a hyperboloid in \mathbb{RP}^3 cannot be affine.*

Proof. Any plane X intersects Q in either an ellipse or a distinct pair of lines. In either case, the bi-degree of $\mathbb{R}Q \cap \mathbb{R}X$ is $(\pm 1, \pm 1)$ and it represents the homology class $\pm[L_1] + \pm[L_2]$, where $[L_1]$ and $[L_2]$ are the generators of $H_1(Q)$ and are each represented by a line in the ruling. The homology class represented by the curve of bi-degree $(1, 3)$ is $[L_1] + 3[L_2]$, so the intersection number $([L_1] + 3[L_2]) \circ (\pm[L_1] + \pm[L_2]) = \pm[L_1] \circ [L_2] \pm 3[L_1] \circ [L_2]$, so the intersection number is either ± 2 or ± 4 . \square

Lemma 23. *Consider a rational curve C of real bi-degree $(1, 3)$ in a hyperboloid Q . All real lines l such that $\mathbb{C}l$ intersects $\mathbb{C}C$ in two (conjugate) imaginary points lie in the same component of $\mathbb{RP}^3 \setminus \mathbb{R}Q$.*

Proof. Consider two lines l_1 (resp. l_2), such that $\mathbb{C}l_1$ (resp. $\mathbb{C}l_2$) intersect $\mathbb{C}C$ (parameterized by k) in the points $k(z_1)$ and $k(\bar{z}_1)$ (resp. $k(z_2)$ and $k(\bar{z}_2)$). We may assume that z_1 and z_2 lie in the same component of $\mathbb{CP}^1 \setminus \mathbb{RP}^1$, because if not, we may replace z_2 by its conjugate. So there is a path z_t of imaginary points in one half of $\mathbb{CP}^1 \setminus \mathbb{RP}^1$, connecting z_1 and z_2 . The conjugate \bar{z}_t of the path connects \bar{z}_1 and \bar{z}_2 . If a line l_t , connecting z_t and \bar{z}_t , intersects $\mathbb{C}Q$ in more than two points, it would have to lie on $\mathbb{C}Q$ and would be one of the rulings. This is impossible since by definition, l_t intersects the curve in at least two imaginary points but the curve has real bi-degree $(1, 3)$. Therefore the line $\mathbb{R}l_t$ never intersects $\mathbb{R}Q$ and therefore $\mathbb{R}l_1$ and $\mathbb{R}l_2$ must lie in the same component. \square

Lemma 24. *Consider a line l intersecting a non-singular real rational curve C of bi-degree $(1, 3)$ in a hyperboloid Q in only a pair of conjugate imaginary points. It defines a pencil of planes containing l . Any plane from that pencil intersects C in two more distinct real points.*

Proof. The pencil may be parametrized by \mathbb{RP}^1 and each plane may be denoted by X_t , for some $t \in \mathbb{RP}^1$.

Since the knot is not affine (lemma 22), $\mathbb{R}X_t$ intersects $\mathbb{R}C$. X_t already intersects the C in two imaginary points, so there are exactly two real points of intersection. We will now show that these points have to be distinct.

If a plane, say X_0 , intersects C in a real point with multiplicity two then for a small ϵ , the intersections $\mathbb{R}X_t \cap \mathbb{R}C$ for all $t \in (-\epsilon, \epsilon)$ cannot be real because then the intersection of the planes with the curve would be tracing out local branches of the curve that are intersecting. We know that this curve does not have crossings. But then, since the knot is affine, $\mathbb{R}X_t \cap \mathbb{R}C$ cannot be purely imaginary for any t . \square

Lemma 25. *Consider a homotopy of rational maps $q_t : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ of degree 3. If there is a point w_0 such that $q_0(w_0) = z_0$ such that $\#(q_0^{-1}(z_0)) = 1$, then for each t there is a point w_t such that $q_t(w_t) = z_t$ such that $\#(q_t^{-1}(z_t)) = 1$.*

Proof. Consider the largest interval I_0 containing w_0 such that $q_0 \upharpoonright_{I_0}$ is injective. This interval must remain non-empty throughout the isotopy because for the smallest t_0 for which $I_{t_0} = \emptyset$, the end-points of the interval would merge to give point z_{t_0} such that $\#(q_{t_0}^{-1}(q_{t_0}(z_{t_0}))) = 4$ which is impossible for a rational map of degree 3. So at each stage we may continuously choose a point z_t from I_t . \square

Lemma 26. *Consider two pairs (C_0, X_0) and (C_1, X_1) in $\mathcal{P}_{4,2}$. If C_0 and C_1 are rigidly isotopic, then there is there is a path in $\mathcal{P}_{4,2}$ joining (C_0, X_0) and (C_1, X_1) .*

Proof. We will solve these individually for the two types of real bi-degrees. Denote the path of rational knots connecting C_0 and C_1 by C_t , where $0 \leq t \leq 1$. Let k_t denote a parametrization of C_t , for each t , and $\mathbb{R}X_0 \cap \mathbb{R}C_0 = \{k_0(z), k_0(\bar{z}), k_0(\lambda_1), k_0(\lambda_2)\}$, for some z which is imaginary and some λ_i which are real. Keep z fixed throughout the isotopy and define the line l_t to be the line defined by the two imaginary points $k_t(z)$ and $k_t(\bar{z})$. The points $k_t(z)$ and $k_t(\bar{z})$ must be distinct, otherwise it would give rise to a solitary double point. Now consider the two bi-degree cases separately:

Real bi-degree $(1, 3)$: For each t , l_t does not intersect Q in any more points, otherwise it would be forced to lie on Q and would have to be one of the rulings

that intersect the curve in one real and two imaginary points. That is impossible because this curve is of *real* bi-degree $(1, 3)$. Therefore, by lemma 24, for each t , the unique plane X_t defined by the line l_t and $k_t(\lambda_1)$ must intersect the curve in two distinct real points.

Using lemma 23, one can find a path of lines l_t between the line l_0 and one which is defined by the conjugate pair in the intersection $\mathbb{C}X_1 \cap \mathbb{C}C_1$. Again by lemma 24, the plane X_t defined by each line l_t and the fixed real point $k_t(\lambda_1)$ will intersect it in distinct points. Finally move λ_1 to a point which is in the preimage of the real point of intersection $\mathbb{R}X_1 \cap \mathbb{R}C_1$. Once again lemma 24 ensures that the curve and the plane will intersect in *distinct* real points.

Real bi-degree (1, 1): Each plane X_t containing l_t will intersect the curve in the two real points $k_t(\lambda_t)$ and $k_t(\lambda'_t)$. Since the curve is of degree 4, the point λ_t can always be continuously chosen so the plane X_t defined by it and the line l_t intersects the curve in *distinct* points. This can be seen as follows: Parametrize the pencil of planes containing a line l by $\mathbb{R}\mathbb{P}^1$. For each $s \in \mathbb{R}\mathbb{P}^1$, denote the plane by X_s . If there is a plane X_{s_0} that intersects the knot in a one double point, then for every s small interval around s_0 , it is impossible that X_s intersects k in only real points because that will trace out two intersecting branches of the curve. This means that the double point must arise because X_{s_0} is tangent to the knot. Supposing X_{s_1} is another plane tangential to the knot, then let I denote the interval component of $\mathbb{R}\mathbb{P}^1 \setminus \{s_0, s_1\}$ for which X_s intersects the knot in real points for all $s \in I$. In that case, for each $s \in I$, X_s is intersecting the curve in distinct real points that coincide at the for X_{s_0} and X_{s_1} . The two real intersection points trace out a closed loop. Since the real part of a rational curve is connected, this loop must be the entire real curve. Since the rational curve is infinite, this interval must also be infinite, and therefore throughout the isotopy of lines, one may continuously choose a plane that intersects the knot in *distinct* real points.

Consider the smallest t_0 for which the line l_{t_0} lies on Q . X_{t_0} will then be tangent to Q and would intersect Q in a pair of lines, one of which intersects the curve in one real point, and another which intersects the curve in a real point and a conjugate pair.

The pullback of the line l_{t_0} by the Segre embedding may be explicitly given by the set $([x_0 : x_1], [\alpha_{t_0} : \beta_{t_0}])$ where $[\alpha_{t_0} : \beta_{t_0}]$ is fixed. Its points of intersections with the curve are given by those points $[p_0(s_i, t_i) : p_1(s_i, t_i)]$ ($i = 1, \dots, 3$) such that $[p_2(s_i, t_i) : p_3(s_i, t_i)] = [\alpha_{t_0} : \beta_{t_0}]$. Since the line intersects in two imaginary points and one real point, two of the $[s_i : t_i]$ are conjugate imaginary. By lemma 25, one can extend $[\alpha_{t_0}, \beta_{t_0}]$ throughout the isotopy to $[\alpha_t, \beta_t]$ so that the line l_t defined by $([x_0 : x_1], [\alpha_t : \beta_t])$ intersects the curve in an imaginary pair and one real point. The line from the other

family of rulings that passes through $[\alpha_t, \beta_t]$ intersects the curve in only one point, so together the two lines, one from each ruling, define a plane X_t which intersects the curve in two real and two imaginary points.

By lemma 20, we know that k_0 is isotopic to the curve

$$[s(s^3 + t^3) : s(s^3 - t^3) : t(s^3 + t^3) : t(s^3 - t^3)]$$

(or its mirror image) and by what we have just proved, the plane X_0 can be isotoped to a plane X_0 which intersects the curve $[s(s^3 + t^3) : s(s^3 - t^3) : t(s^3 + t^3) : t(s^3 - t^3)]$ in two distinct real points throughout the isotopy.

By the same reasoning, k_1 can also be isotoped to the curve $[s(s^3 + t^3) : s(s^3 - t^3) : t(s^3 + t^3) : t(s^3 - t^3)]$ and the plane X_1 to X_1 so that it intersects the curve in two distinct real points throughout the isotopy. Observe that X_0 and X_1 are tangential to the ambient hyperboloid, but there the curve $[s(s^3 + t^3) : s(s^3 - t^3) : t(s^3 + t^3) : t(s^3 - t^3)]$ intersects every line of its ruling in just one real point. So every plane tangent to the hyperboloid intersects the curve in two real points and therefore X_0 and X_1 may be moved to each other via planes that intersect the curve in two real points. \square

A degree four knot in \mathbb{RP}^3 which intersects the plane at infinity in two real points is either the unknot or the two crossing knot. Therefore, by the previous theorem,

Theorem 27. *Singular knots in S^3 with one double point form a wall between two unknots or a wall between the unknot and the trefoil.*

3.3.3 Degree 6 knots in S^3 with a solitary double point

Given a knot, if there is a plane disjoint from it, then it intersects the complexification of the knot in two pairs of conjugate imaginary points. Each pair defines a real line. Pick one of these real lines. It does not intersect the real part of the curve and indeed it will be shown that such a line can always be chosen along an isotopy. We will need the following lemma.

Lemma 28. *Consider a homotopy of rational maps $f_t : \mathbb{CP}^1 \rightarrow \mathbb{CP}^1$, each of degree 3. For a fixed open set $U \subset \mathbb{CP}^1$, consider an $z_0 \in f_t^{-1}(U)$. It is possible to continuously extend z_0 throughout the isotopy, so that for each t , $x_t \in f_t^{-1}(U)$*

Proof. Fix $w = f(z_0)$. For each t , there are three points in $f_t^{-1}(w)$, out of which one can choose the point z_t which will continuously extend z_0 . \square

Lemma 29. *Consider a pair (C, l) of a degree 4 knot and a line, such that Cl intersects $\mathbb{C}C$ in a conjugate pair of imaginary points but is disjoint from the real point of the knot. Suppose there is an isotopy C_t of the knot, where $C_0 = C$, then this isotopy can be extended to a pair (C_t, l_t) , where for each t , Cl_t intersects $\mathbb{C}C_t$ in a pair of conjugate imaginary points but no real points.*

Proof. A degree 4 knot lies on a hyperboloid. The line that is defined, already intersects the knot and therefore the hyperboloid in two imaginary points. If it is not already transversal to the to the hyperboloid, perturb it so that it is. It cannot intersect the hyperboloid (and therefore not even the knot lying on it) in any more points and therefore is disjoint from the real part of C .

Let k_t be the parametrization of C . We need to find a path z_t so that and the line l_t be defined by the points $k(z_t)$ and $k(\bar{z}_t)$ does not intersect the curve in any more points. This means that the line l_t must never be the real ruling. In other words, $[p_2(z) : p_3(z)]$ must never be real in the parametrization $([s : t], [p_2 : p_3])$, where the p_i 's are of degree 3, because the ruling is defined by $([x_0 : x_1], [\alpha : \beta])$ for some real α and β and it intersects the curve when for those values of z such that $[p_2(z) : p_3(z)] = [\alpha : \beta]$. But the previous lemma shows that this is always possible. \square

If the line defined by a pair of conjugate imaginary points, $k(z)$ and $k(\bar{z})$, is such that no real point of the curve lies on it, then each real point and this line defines a plane belonging to the pencil of curves that is parametrized by $\mathbb{R}P^1$. This defines a map $\theta : \mathbb{R}P^1 \rightarrow \mathbb{R}P^1$.

Lemma 30. *If θ is surjective, it is a two sheeted covering of degree 2.*

Proof. The image of a point by θ corresponds to a plane from the pencil that passes through the point. Since the degree is 4 it intersects one more point which, will turn out to be distinct. If the points coincide, then since the map is surjective, the preimage of a small enough neighbourhood of the pencil will be two intervals that intersect each other at one point. This is impossible since the knot does not have real double points.

Since each plane from the pencil intersects the curve in two real points, the preimage of a small enough interval around the point will be a disjoint union of two intervals and will therefore be evenly covered. \square

Corollary 31. *Consider two isotopic knots C_1 and C_2 . If there is a plane X_1 disjoint from $\mathbb{R}C_1$, then an isotopy of knots C_t between C_1 and C_2 , can be extended to a pair of (C_t, X_t) , where X_t is disjoint from $\mathbb{R}k_t$.*

Proof. If there does not exist a plane disjoint from a knot k , then the map θ corresponding to it must be surjective and hence a two sheeted covering of

degree 2. The map corresponding to any knot isotopic to C will also have degree 2 and will therefore be surjective.

The planes which do intersect the curve in real points form an interval in the pencil. The complement is also an interval, which we have just shown is non-empty throughout the isotopy and therefore it is possible to always continuously choose a plane from it which will not intersect the knot in any real points. \square

Although the plane that is obtained at the end of the isotopy is disjoint from k_2 , it need not be the one that we want, but we also have:

Lemma 32. *If there are two planes X_1 and X_2 that intersect a knot in only imaginary points, then they are connected by a path of planes X_t that are all disjoint from the real part of the knot.*

Proof. The two planes intersect in a line that define a pencil of planes. Again, we may define a map $\theta : \mathbb{RP}^1 \rightarrow \mathbb{RP}^1$ by taking each point on the real part of the knot to the plane defined by it and the line. The image is a connected interval that is a proper subset of \mathbb{RP}^1 because the two given planes do not intersect the knot in real points. The complement of a connected interval in \mathbb{RP}^1 is an interval and therefore one can choose path to connect the planes in the pencil. \square

Observe that this wall is the same as one of the earlier walls because we can isotope the knot so that the solitary double point becomes a cusp and thereafter a non-solitary double point. This can be seen by considering its corresponding pair of the real degree 4 rational curve and the plane that intersects the curve in only imaginary points. Move the plane so that it is tangential to the curve and thereafter intersects it in a real pair. At the stage that it is tangential to the curve is when the curve lifts to one in the sphere that has a cusp.

We have proved that:

Theorem 33. *In the space of rational curves of degree 6 in S^3 , there are 4 rigid isotopy classes of curves that have only one double point as a singularity.*

3.3.4 The space of real rational curves in S^3

The space of rational curves in \mathbb{RP}^3 is a projective space and therefore a closed manifold. When we consider those curves in \mathbb{RP}^4 whose image lies in S^3 , the situation is no longer straightforward. However, we will prove that this space is also a closed manifold. This will enable us to use the fact that the top homology is $\mathbb{Z}/2$ and the Poincare duality to find a bound on the number

of components. We will then show that in the case of degree 6, the bound obtained is sharp.

The space of real rational curves in S^3 is a manifold

We will need two lemmas from linear algebra to prove the following theorem which states that the space of degree 6 rational curves in S^3 is a manifold.

Lemma 34. *If h_0, h_1, \dots, h_d are linearly independent polynomials of degree $\leq d$, then the determinant of the following matrix will be non-zero:*

$$\begin{pmatrix} h_0(t_1) & h_1(t_1) & \dots & h_d(t_1) \\ h_0(t_2) & h_1(t_2) & \dots & h_d(t_2) \\ \vdots & \dots & \dots & \vdots \\ h_0(t_{d+1}) & h_1(t_{d+1}) & \dots & h_d(t_{d+1}) \end{pmatrix}$$

as long as all the t_i 's are distinct.

Proof. No linear combination of $\{h_0, h_1, \dots, h_d\}$ can result in the zero polynomial since they are linearly independent. However, if the columns of this matrix are linearly dependant, it is equivalent to a linear combination of h_i 's having $d + 1$ roots, which is impossible for a polynomial of degree less than $d + 1$. \square

Lemma 35. *Let $p_0(t)$ and $p_1(t)$ be two polynomials of degree d that do not share a root. Then the set*

$$\{p_0(t), tp_0(t), \dots, t^{d-1}p_0(t), p_1(t), tp_1(t), \dots, t^{d-1}p_1(t), t^d p_1(t), \dots, t^{d+k} p_1(t)\}$$

is linearly independent in the space of polynomials of degree $2d + k$.

Proof. Let $p_0(t) = a_0 + a_1 t + \dots + a_d$ and $p_1(t) = b_0 + b_1 t + \dots + b_d$ where $a_d \neq 0 \neq b_d$. The set of polynomials

$$\{p_0(t), tp_0(t), \dots, t^{d-1}p_0(t), p_1(t), tp_1(t), \dots, t^{d+k-1}p_1(t), t^{d+k} p_1(t)\}$$

is linearly independent if and only if the following matrix is non-singular:

$$\begin{pmatrix} a_0 & a_1 & \cdots & a_{d-1} & a_d & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_0 & a_1 & \cdots & a_{d-1} & a_d & 0 & 0 & 0 & 0 & 0 \\ & & & & & \vdots & & & & & \\ 0 & 0 & 0 & 0 & 0 & a_0 & a_1 & \cdots & a_{d-1} & a_d & 0 \\ b_0 & b_1 & \cdots & b_{d-1} & b_d & 0 & 0 & 0 & 0 & 0 & \vdots & 0 \\ 0 & b_0 & b_1 & \cdots & b_{d-1} & b_d & 0 & 0 & 0 & 0 & \vdots & 0 \\ & & & & & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & b_0 & b_1 & \cdots & b_{d-1} & b_d & 0 \\ & & & & & \vdots & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & b_0 & b_1 & \cdots & b_{d-1} & b_d \end{pmatrix}$$

Observe that the submatrix which is obtained by removing the last row and column is the Sylvester matrix whose determinant is the resultant $R(p_0, p_1)$ which is non-zero because the p_0 and p_1 do not share a root. The determinant of the original matrix is $\pm b_d^{k_1} R(p_0, p_1)$ which is non-zero because $b_d \neq 0$. \square

Theorem 36. *The space of real parametrizations of rational curves of degree 6 in \mathbb{RP}^4 that lie in the sphere, is a manifold of dimension 21 (co-dimension 13).*

Proof. Denote any rational parametrization explicitly by $[p_0 : p_1 : p_2 : p_3 : p_4]$, where each p_i is a homogeneous polynomial of the same degree. Let $q(t) = p_1(t)^2 + p_2(t)^2 + p_3(t)^2 + p_4(t)^2 - p_0(t)^2$. The space of all rational parametrizations in \mathbb{RP}^4 is the projective space \mathbb{RP}^{34} . We will restrict attention to each chart, defined by one coefficient of one of the polynomials, and dividing all the coordinate polynomials by that coefficient. We will prove that for each chart, we can find a submersion to \mathbb{RP}^{13} whose pullback of 0 will be parametrizations of curves that lie in S^3 .

Fix points t_1, t_2, \dots, t_{13} in \mathbb{RP}^1 . Define the map $\theta : \mathbb{R}^{34} \rightarrow \mathbb{R}^{13}$ that takes the parametrization $[p_0 : p_1 : p_2 : p_3 : p_4]$ (restricted to a chart) to $(q(t_1), q(t_2), \dots, q(t_{13}))$. The inverse image of $(0, 0, \dots, 0) \in \mathbb{R}^{13}$ is precisely the set that we want because any degree 6 rational curve the intersects the sphere in 13 distinct points must lie on the sphere.

Represent a parametrization in \mathbb{RP}^{34} by coordinates $[a_0 : a_2 : \dots : a_{34}]$ where a_0, \dots, a_6 are the coefficients of p_0 , a_7, \dots, a_{13} are the coefficients of p_1 , and so on. Each $q(t_i)$ is a quadratic polynomial in a_i for a fixed t_i .

It is enough to show that for any arbitrary point in this space, the differential $d\theta$ is surjective. $d\theta$ is a 13×34 matrix where the rows are of the form

$(\frac{\partial q(t_i)}{\partial a_j})_{(i,j)}$. By using the chain rule on each entry it is easy to see that if a_i is the l th coefficient of p_k , the matrix is $(\pm 2p_k(t_j)t_j^l)_{(i,j)}$. The sign is $-$ when $k = 6$.

We now have to find a minor of the above matrix which has a non-zero determinant. Choose two p_{i_1} and p_{i_2} which do not share a root and whose coefficients are not the with the coefficient defining the chart. Consider the following 13×13 minor:

$$\begin{pmatrix} p_{i_1}(t_1) & \dots & t_1^5 p_{i_1}(t_1) & p_{i_2}(t_1) & \dots & t_1^6 p_{i_2}(t_1) \\ \vdots & \dots & \dots & \dots & \dots & \vdots \\ p_{i_1}(t_{13}) & \dots & t_{13}^5 p_{i_1}(t_{13}) & p_{i_2}(t_{13}) & \dots & t_{13}^6 p_{i_2}(t_{13}) \end{pmatrix}$$

Lemma 35 shows that the polynomials $\{p_{i_1}(t), \dots, t^5 p_{i_1}(t), p_{i_2}(t), \dots, t^6 p_{i_2}(t)\}$ are linearly independent, and lemma 34 shows that the above matrix is non-singular.

If the point $[a_0 : a_2 : \dots : a_{34}]$ corresponds to a parametrization $[p_0 : p_1 : p_2 : p_3 : p_4]$ where two coordinate polynomials share a root, it means that the curve intersects the co-dimension two linear subspace defined by two coordinates of \mathbb{RP}^4 equal zero. By the transversality theorem, it is possible to find a diffeomorphism (for instance, a rotation) that preserves S^3 and takes the curve to one which does not intersect any of the co-dimension 2 hypersurfaces defined by $x_i = 0$ and $x_j = 0$. Simply compose this diffeomorphism with the map θ to define a new map which will be regular when restricted to a small enough neighbourhood of $[a_0 : a_2 : \dots : a_{34}]$ in a chart containing it. \square

Remark 37. The above theorem easily generalizes to the curves of higher degrees.

The space of real rational curves in S^3 is connected

We do this in two steps because it is easier to prove the theorem for singular curves:

Lemma 38. *The space of real rational curves of degree 6 in S^3 with at least one double point, is connected.*

Proof. Given two curves in S^3 with at least one double point, use a linear transformation so that they have a common double point. Project each of these curves from their common double point. Denote the projections by C_1 and C_2 and their respective parametrizations by k_1 and k_2 . They each intersect the plane at infinity in at least one conjugate pair say $k_i(z_i)$ and $k_i(\bar{z}_i)$.

We know that each rational curve of degree 6 with one double point in S^3 corresponds to a pair (C, P) where C is a degree 4 real rational curve in \mathbb{RP}^3 and P is a plane that intersects k in 4 points such that at least two of them form a conjugate imaginary pair.

Consider a path k_t between k_1 and k_2 and a path z_t between z_1 and z_2 . Extend the path to the path of pairs (k_t, P_t) where each P_t is continuously chosen so that it contains $k_t(z_t)$ and $k_t(\bar{z}_t)$. Now make the plane P_t coincide with the original plane at infinity. At each stage we get a pair that will be pulled back to a degree 6 curve in the sphere. \square

Lemma 39. *Given a curve k_1 in the space of real rational curves in S^3 , there is a curve with one double point in its path connected component.*

Proof. Project the curve from a point on it to a degree 5 curve in S^3 . The projection will be a curve which intersects the plane at infinity in one real point and two conjugate pairs of imaginary points that lie on the empty conic. Denote the parametrization of the curve by $k : \mathbb{RP}^1 \rightarrow \mathbb{RP}^3$, and fix the imaginary pairs $k(z_i) = v_i$ and $k(\bar{z}_i) = \bar{v}_i$ for some fixed z_i where $i = 1, 2$. Fix two distinct points λ_1 and λ_2 in \mathbb{RP}^1 , then we can always find a curve so that $k(\lambda_1) = w_1$ and $k(\lambda_2) = w_2$ for any pair w_1 and w_2 , while still maintaining $k(z_i) = v_i$ and $k(\bar{z}_i) = \bar{v}_i$ so that it lies on the imaginary conic at infinity and the curve can be lifted back to S^3 . Now given a path w_t between w_1 to w_2 , for each w_t there is a k_t such that $k_t(\lambda_1) = w_t = k_t(\lambda_2)$, $k_t(z_i) = v_i$ and $k_t(\bar{z}_i) = \bar{v}_i$ and to create a double point. \square

By the previous two lemmas we obtain the theorem:

Theorem 40. *The space of real rational curves of degree 6 in the sphere is connected.* \square

We now prove that:

Lemma 41. *The space of rational curves with exactly one double point is a manifold of dimension 20.*

Proof. Denote the space of rational curves with at least one double point as D and the space of rational curves with more than one double point as D' , then

$$D \setminus D' \cong S^3 \times D_p$$

where D_p denotes the space of curves with exactly one double point at the north pole p . This equivalence map can be seen as follows: S^3 may be treated as the unit quaternions with the north pole p as the identity quaternion. The quaternions act transitively on S^3 and extend linearly to \mathbb{R}^4 . The map is

then defined by sending a curve k with one double point at q to the pair (q, k_0) where k_0 is the action of A_q on k , where A_q is the quaternion that takes q to p .

So it is enough to show that D_p is a manifold. Via the stereographic projection, this space is isomorphic to the space of knots of degree 4 in \mathbb{RP}^3 that intersects the plane at infinity in four points, two of which lie on the empty conic. The curves that intersect the plane at infinity in the points $[0 : 0 : 1 : i]$ and $[0 : 0 : 1 : -i]$ is a manifold of dimension 17, which will be denoted by K_i . Fix a real point λ_1 on the plane at infinity and one point λ_2 outside it. Given any conjugate pair z and \bar{z} on the empty conic, there is unique transformation T_z which fixes λ_1 and λ_2 and takes $z \rightarrow i$ and $\bar{z} \rightarrow -i$. For an open neighbourhood U around z in the empty conic, the map and a neighbourhood V around a chosen knot, T_z gives an isomorphism

$$V \cong U \times K_i$$

that is defined by taking a knot k which intersects the empty conic in z and \bar{z} to $(z, T_z(k))$. \square

Computing a bound on the number of components

Let X denote the space of parametrizations of real rational knots of degree 6 in S^3 . Its dimension is 21. Denote by D , the co-dimension 1 subspace of parametrizations which have at least one singularity.

Since X is a manifold, by Poincare duality:

$$H^0(X \setminus D; \mathbb{Z}/2) = H_{21}^{BM}(X \setminus D; \mathbb{Z}/2)$$

By the long exact sequence of Borel-Moore homology [2],

$$\dim H_{21}^{BM}(X \setminus D; \mathbb{Z}/2) \leq \dim H_{21}^{BM}(X; \mathbb{Z}/2) + \dim H_{20}^{BM}(D; \mathbb{Z}/2) \quad (3.1)$$

$\dim H_{21}^{BM}(X; \mathbb{Z}/2) = 1$ since X is a connected manifold of dimension 21 any manifold is $\mathbb{Z}/2$ -orientable. Consider the space of curves which have only one double point or a cusp as a singularity. Denote the complement of this space in the space of curves with singularities to be D' ; it is of co-dimension 2. Then

$$\dim H_{20}^{BM}(D; \mathbb{Z}/2) \leq \dim H_{20}^{BM}(D'; \mathbb{Z}/2) + \dim H_{20}^{BM}(D \setminus D'; \mathbb{Z}/2) \quad (3.2)$$

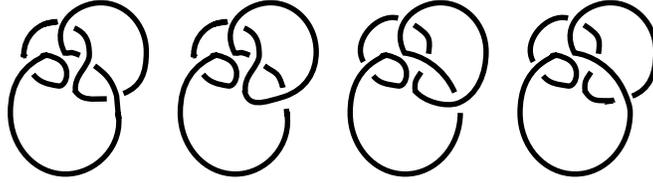


Figure 3.1: Knots of degree 6 with writhe 4, 2, 0, and 2. The first is topologically isotopic to the trefoil and the other two are topologically isotopic to the unknot. The successive perturbations are formed by a vertical rotation of the innermost circle.

Again by duality, since $D \setminus D'$ was shown to be a manifold by lemma 41,

$$H_{20}^{BM}(D \setminus D'; \mathbb{Z}/2) = H^0(D \setminus D'; \mathbb{Z}/2)$$

which we know has rank 4, and D' has co-dimension 2 so $\dim H_{20}^{BM}(D'; \mathbb{Z}/2) = 0$. Therefore, by the inequality 3.1,

$$H^0(X \setminus D; \mathbb{Z}/2) \leq 1 + 4 = 5$$

We will now try and obtain representatives for these five components.

The first three knots shown in figure 3.3.4 are not isotopic to each other because they have distinct writhe numbers: 4, 2, and 0. There is a wall separating the knots with writhe 4 and 2 and this is the wall obtained by the pull back in the sphere of the two crossing knot of degree 4 in \mathbb{RP}^3 .

Observe that the trefoil T^+ and its mirror image T^- lie in separate components and therefore account for two of the rigid isotopy classes. They can be obtained by perturbing the pullback of the two-crossing knot.

Now consider the pullback of the unknot. Perturbing it will lead to knots of different writhe numbers and therefore one of them, denoted by U_2^+ , has got to have a non-zero writhe. Its mirror image, denoted by U_2^- , would have a negative writhe number and therefore will lie in a separate component. Together, they account for two more components.

Since we need to account for only one more component, the other knot formed by a pullback of the unknot, and denoted by U_0 , is isotopic to its mirror reflection. It must have writhe number 0 and therefore U_2^+ , and U_2^- must have writhe numbers +2 and -2 respectively.

1. A knot of writhe number 2 which is topologically isotopic to the trefoil (and its mirror image)

2. A knot of writhe number 2 which is topologically isotopic to the unknot (and its mirror image)
3. A knot of writhe number 0 which is topologically isotopic to the unknot. It is rigidly isotopic to its mirror image.

and that the two walls divide the knots of different writhes.

3.4 Constructing examples of knots in S^3

Before demonstrating why Björklund's [1] method of constructing examples applies to curves in the sphere, we note that following simple lemma which proves the existence of torus knots in rational curves of each degree d .

Lemma 42. *There exist $(m, d/2)$ -torus knots in S^3 of degree d for each m which is coprime to $d/2$.*

Proof. Treat the real 3-sphere as a subset of \mathbb{C}^2 defined by $|z_1|^2 + |z_2|^2 = 1$. It is easy to see that the curve defined by $t \rightarrow (e^{mit}, e^{dit/2})$ (where $m < d/2$) is a rational immersion of degree d . It is an embedding if and only if $(m, d/2)$ are co-prime. \square

One method of constructing real rational knots in the sphere is to pull back knots from the projective space and perturb them. However this cannot be done directly because in applying theorem 9 to curves of degree 8 in S^3 with one double point, we see that they correspond to curves of degree 6 in \mathbb{RP}^3 which intersect the plane at infinity in 6 points, 4 of which lie in the empty conic. The 4 intersection points of the curve with the plane at infinity that lie on the empty conic are in conjugate pairs. Although they define a real pencil of conics, not every real pencil of conics may contain the empty conic, for example: the pencil joining $x_0^2 - x^2 + x_3^2$, and $x_0^2 - x^2 + 2x_3^2$. This is because any linear combination will always result in the coefficients of x_0 and x_1 having different signs.

However, Björklund [1] showed that a real rational knot in \mathbb{RP}^3 can be obtained from a union of two real rational knots that intersect each other in one point. He specifically showed that if the parametrizations of the rational curves are $[p_0 : p_1 : p_2 : p_3]$, and $[q_0 : q_1 : q_2 : q_3]$, then the perturbed union is defined by the rational curve given by $(p_1/p_0 + q_1/q_0, p_2/p_0 + q_2/q_0, p_3/p_0 + q_3/q_0)$ for points where p_0 is not zero. The perturbed union is then extended by continuity to the whole projective space. The degree of the resulting curve is the sum of the degrees of the original curves.

It will be useful to define the perturbed union in terms of projective coordinates: $[p_0q_0 : p_1q_0 + q_1p_0 : p_2q_0 + q_2p_0 : p_3q_0 + q_3p_0]$. Observe that the intersections of each knot with the plane at infinity do not change after the perturbation because for any $[s : t]$ such that $q(s, t) = 0$, $[p_0q_0 : p_1q_0 + q_1p_0 : p_2q_0 + q_2p_0 : p_3q_0 + q_3p_0] = [0 : q_1p_0 : q_2p_0 : q_3p_0] = [0 : q_1 : q_2 : q_3]$. The same happens for any $[s : t]$ such that $p(s, t) = 0$. Therefore in order to construct curves of degree 8 which intersect the plane at infinity such that four points of the intersection points lie on the empty conic, we can perturb the union of two curves of degrees m and n (where $m + n = 8$) which individually satisfy this condition and which intersect in one point. Perturbing will not disrupt their intersection with the plane at infinity. In fact, we only need to take care of one point of each conjugate pair because the other will also have to lie on it since the empty conic and the curve are both real.

The above was done to construct singular knots by using knots of lower degrees in the projective space. But it is not too difficult to see that we also have the following counterpart to Björklund's theorem for knots in spheres:

Theorem 43. *If there are two knots in the sphere with degrees m and n which intersect in only one point, they can be perturbed slightly to form a knot of degree $m + n$ in the sphere which is isotopic to the union of the original knots outside a small neighbourhood of the intersection.*

Proof. Project the union of the knots to \mathbb{RP}^3 by a point not on either of the knots. This will result in knots in \mathbb{RP}^3 that intersect the plane at infinity in conjugate imaginary points, all lying on the sphere. If the parametrizations of the projected knots are $[p_0 : p_1 : p_2 : p_3]$, and $[q_0 : q_1 : q_2 : q_3]$, then the perturbed union defined by the rational curve parametrization given by $[p_0q_0 : p_1q_0 + q_1p_0 : p_2q_0 + q_2p_0 : p_3q_0 + q_3p_0]$ preserves the points at infinity. So this new knot also intersects the plane at infinity in only imaginary points at the empty conic and can therefore be lifted via the stereographic projection to S^3 . \square

The following lemma will be useful in constructing many examples:

Lemma 44. *A degree 2 curve C in S^3 projects under the stereographic projection π_p to a degree 2 curve $\pi_2(C)$ in \mathbb{RP}^3 if and only if $\pi_2(C)$ is a circle.*

Proof. $\pi_2(C)$ is a conic that lies on a plane which can be transformed via an orthogonal transformation to the plane defined by $x_0 = 0$. The plane defined by $x_0 = 0$ intersects the empty conic at the plane at infinity in the conjugate pair $[0 : 0 : 1 : i]$ and $[0 : 0 : 1 : -i]$. $\pi_2(C)$ would be the image under the stereographic projection of a curve of the same degree if and only if its two points of intersection with the plane at infinity lie on the empty conic. Since



Figure 3.2: Figure-eight knot in S^3 realized by a degree 8 rational curve.

its ambient plane intersects the conic in $[0 : 0 : 1 : i]$ and $[0 : 0 : 1 : -i]$, $\pi_2(C)$ would have to contain these two points. It is easy to see that conics that lie on the plane $x_0 = 0$ and pass through these two points are circles. Indeed, circles may be characterized by conics which pass through this conjugate pair.

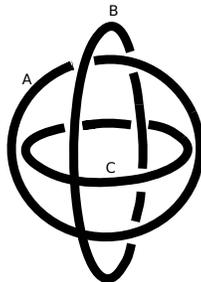
Note that the “only if” part is easy because the stereographic projection is conformal. For the converse, we needed to show that the pullback was of the same degree. \square

Corollary 45. *Any knot in $\mathbb{R}P^3$ that can be constructed inductively by using only circles, can be pulled back to via the stereographic projection, to a knot of the same degree in S^3 .*

Figure 3.4 shows an example of the figure-eight knot of degree 8 constructed in S^3 . The knot is constructed in $\mathbb{R}P^3$ using only circles. Therefore, by the previous corollary, it is possible to lift it back to a degree 8 curve in the sphere.

The fact that we are allowed to use only circles (and not general conics) is a strong restriction. It prevents us from constructing the wall between the trefoil and figure eight knot as is demonstrated in the following section. The impossibility of this construction was already predicted by the classification.

3.4.1 Example of a construction impossible in S^3

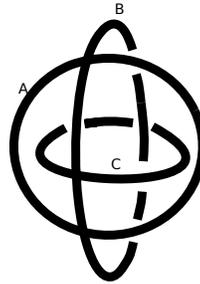


Björklund constructed the figure of eight knot and the trefoil knot by perturbing the above union of ellipses. Given that the method of constructing curves is possible for curves in S^3 , it seems that such a construction must be possible in the sphere too. The following remarks will illustrate where the problem lies.

If such a configuration came from the sphere via the stereographic projection, then the conics would have to be circles because the only conics in the sphere are circles and the stereographic projection takes circles to circles.

In the figure, Circle B meets Circle C in the plane containing it but also meets that plane in another point. It will be shown that if Circle C is large enough to contain this other point in the interior component, it will be too large to remain unlinked with Circle A

The following lemma will make it more helpful to consider the above configuration with a minor change: keeping the point of intersection of circle A and circle B fixed, circle A is shrunk to create another intersection point, like this:

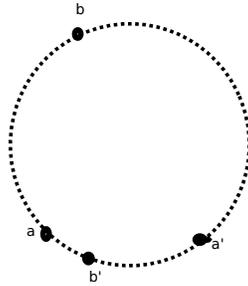


If the first configuration of circles is possible, so will the second configuration, but the second configuration will be easier to prove impossible because now the unions of circles A and B lie on a sphere:

Lemma 46. *The union of two circles in \mathbb{RP}^3 that intersect in two points, lies on a sphere.*

Proof. There is a sphere containing the two intersection points and one additional point from each of the two circles (since they are only four in number). The sphere shares three points with each circle and therefore contains them. □

The plane containing circle C will intersect the sphere containing circles A and B in following way:



Here a and a' are the intersections of circle A with the plane, while b and b' are intersections of the circle B with the plane. The dotted circle is the intersection of the sphere with the plane.

Points a and a' divide the dotted circle into two arcs, one that contains b and the other that contains b' .

Circle C passes through b' and will intersect the dotted circle in one more point. If that point is on the arc aa' containing the point b then circle C and circle A would have a non-zero linking number, which we know is not the case. Therefore both points of intersection of circle C with the dotted circle lie on the arc aa' containing b' . The arc of the dotted circle that is in the interior of circle C must therefore be disjoint from the arc aa' containing b . This proves that b cannot lie in the interior of circle C which is a contradiction.

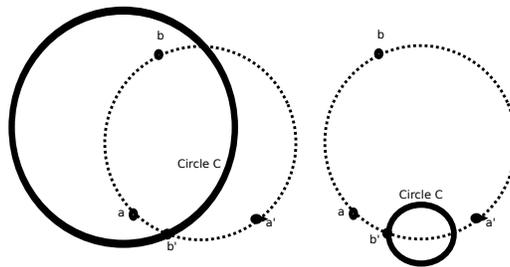


Figure 3.3: Circle C will either link with circle A or not contain b . In the second case the arc contained in the interior of Circle C is disjoint from the arc aa' containing b .

3.5 Degree 6 knots in \mathbb{RP}^3

Consider a real rational curve of degree 6 in \mathbb{RP}^3 with four real double points. It is parametrized by a map $\theta : \mathbb{RP}^1 \rightarrow \mathbb{RP}^3$ that can be extended to its

complexification $\theta : \mathbb{CP}^1 \rightarrow \mathbb{CP}^3$. If the preimage of a double point is a real pair, then the double point is a self intersection; if the preimage of a double point is a conjugate imaginary pair, then the double point is solitary.

Associated to each curve, is a chord diagram defined as follows. Represent \mathbb{RP}^1 by a circle and mark on it the preimages of the non-solitary double points. Connect the pair in the pre-image of each double point by a chord. But first:

Lemma 47. *A degree 6 curve in \mathbb{RP}^3 cannot have more than four double points.*

Proof. If a degree 6 curve has five double points, then choosing these double points and four more points on the curve will define a quadric that would intersect the curve in at least 13 points (counting multiplicity). So the curve would have to lie on the quadric because a transversal intersection can have no more than $2 \times 6 = 12$ points. By a small perturbation, one can ensure that it lies on a non-singular quadric.

The stereographic projection from one of these double points would project the curve to a degree 4 planar rational curve with four nodes, which is impossible. \square

Theorem 48. *Degree 6 curves in \mathbb{RP}^3 with four real double points rational curves with equivalent chord diagrams are rigidly isotopic up to mirror reflection.*

Proof. Transform the curve by a projective transformation that takes the four double points to the points $z_0 = [1 : 0 : 0 : 0]$, $z_1 = [0 : 1 : 0 : 0]$, $z_2 = [0 : 0 : 1 : 0]$, and $z_3 = [0 : 0 : 0 : 1]$ such that if there is only one solitary double point then that is sent to z_0 and if there are two solitary double points then they are sent to z_0 and z_1 and so on.

Let $[\alpha_i : \beta_i]$ and $[\gamma_i : \delta_i]$ be in the preimage of z_i . Let q_i be the quadratic polynomial $(\beta_i s - \alpha_i t)(\delta_i s - \gamma_i t)$. If $[\alpha_i : \beta_i]$ and $[\gamma_i : \delta_i]$ are imaginary then they must be in conjugate pairs and so q_i 's are always real polynomials.

If θ is given explicitly by $[p_0 : p_1 : p_2 : p_3]$ (where p_i are homogeneous polynomials of degree 4), then the fact that $[1 : 0 : 0 : 0]$ is a double point means that q_0 divides p_1 , p_2 and p_3 . Similarly, q_1 divides p_0 , p_2 and p_3 ; q_2 divides p_0 , p_1 and p_3 and q_3 divides p_0 , p_1 and p_2 .

This proves that $p_0 = c_0 q_1 q_2 q_3$, $p_1 = c_1 q_0 q_2 q_3$, $p_2 = c_2 q_0 q_1 q_3$, and $p_3 = c_3 q_0 q_1 q_2$ where the c_i 's are constants which can all be made ± 1 by a projective transformation.

Therefore, by a projective transformation, the curve has been transformed to one whose parametrization has coordinates p_0 , p_1 , p_2 and p_3 such that $p_0 = \pm 1 q_1 q_2 q_3$, $p_1 = \pm 1 q_2 q_3$, $p_2 = \pm 1 q_1 q_3$, and $p_3 = \pm 1 q_1 q_2$. This will be rigidly isotopic to the original curve.

No two of the polynomials q_i can share a root, otherwise all the coordinates will vanish on that point and the curve would not be well defined. Moving a root of a q_i will result in a rigid isotopy as long as it does not coincide with another root of a q_i , in the process. No other double points can be formed in the process because of the previous lemma.

Suppose the conjugate pair of the preimage of one of the solitary double points is x and \bar{x} , this conjugate pair may be moved to any other conjugate pair without passing through any other root of the q_i 's. This is because x is in one half of $\mathbb{CP}^1 \setminus \mathbb{RP}^1$ and can be moved to any other point that lies in that half minus any other imaginary roots of q_i 's. Its conjugate will be moved accordingly.

Since there is no obstruction to moving the conjugate imaginary pairs, we only need to keep a track of the non-solitary double points. They can be moved as long as they do not coincide with another real root of a q_i . In other words, the roots of q_i can be moved to any configuration having an equivalent chord diagram. \square

Remark 49. The above method will also prove that rigid isotopy class of degree 4 rational curves in \mathbb{RP}^2 with three real double points. This has been in the first chapter.

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