Compactness Theorems for Riemannian Manifolds with Boundary and Applications

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Abstract of the Dissertation

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In this thesis we study issues related to the convergence theory of Riemannian manifolds with boundary. First, we establish a compactness theorem for the class of compact, uniformly mean-convex Riemannian manifolds with boundary that satisfy bounds on diameter, area of the boundary, and curvature quantities.

Next, we establish a compactness theorem for manifolds with boundary that have controlled volume growth and integral bounds on curvature quantities. In dimension three, we replace the volume growth assumption with a simpler volume condition at the boundary, provided that an integral norm of the ambient curvature is small.

We use the convergence theory to prove 'geometric stability theorems' for Riemannian 3-manifolds whose ambient curvatures are small. The first theorem applies to 3-manifolds that have Ricci curvature close to 0 (in the pointwise sense) and whose boundaries are Gromov-Hausdorff close to a fixed metric on S^2 with positive curvature. Such manifolds are close (in a Hölder topology)

to the region enclosed by a Weyl embedding of the fixed boundary metric into Euclidean space. This can be thought of as a generalization of a rigidity theorem of Cohn-Vossen–Pogorelov. We then establish stability theorems corresponding to the rigidity theorems of Hopf and Almgren.

The stability theorems have corresponding statements when the Ricci curvature is small in an appropriate integral norm. In particular, we establish a theorem that applies to compact 3-manifolds that have boundary close to the round metric on the sphere and Ricci curvature close to 0 in the L^2 sense. Such manifolds are close (in an appropriate Sobolev space topology) to the unit ball in Euclidean space.

Contents

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Chapter 1

Introduction

In what is now commonly referred to as 'Cheeger-Gromov compactness,' (cf. [GPKS06], [Che70]) one studies the class $\mathcal{M}_{CG}$ of smooth, compact, Riemannian n-manifolds (M, g) satisfying the bounds $|\sec(M)| \leq K$, $\text{diam}(M) \leq D$, and $\text{vol}(M) \geq v_0$. Here $\sec(M)$, $\text{diam}(M)$ and $\text{vol}(M)$ respectively refer to the sectional curvature, diameter and volume. The condition $|\sec(M)| \leq K$ is understood to mean that for each $p \in M$ and each 2-plane $\pi \subset T_pM$, there holds $|\sec_p(\pi)| \leq K$. Equivalently, we may write that $|\sec(g)| \leq K$. Cheeger-Gromov compactness then asserts that to any sequence $(M_i, g_i) \in \mathcal{M}_{CG}$, there is a subsequence (still denoted (M_i, g_i)) and a $C^{1,\alpha}$ Riemannian manifold (M_∞, g_∞) with a $C^{2,\alpha}$ atlas of charts satisfying the following properties. There exist $C^{2,\alpha}$ diffeomorphisms $f_i : M_\infty \to M_i$ such that $f_i^* g_i \to g_\infty$ in $C^{1,\alpha}$ with respect to the fixed $C^{2,\alpha}$ atlas on M_∞. To state this concisely, we say that $\mathcal{M}_{CG}$ is precompact in the $C^{1,\alpha}$ topology. In fact, one can conclude the stronger statement that $\mathcal{M}_{CG}$ is precompact in the weak $L^{2,p}$ topology, any $p < \infty$. Here $C^{k,\alpha}$ refers to the usual Hölder space of functions whose kth derivative is Hölder continuous, and $L^{k,p}$ refers to the Sobolev space of functions with k weak derivatives in L^p.

Let us consider an equivalent formulation of the Cheeger-Gromov compactness theorem. First, we note that the hypotheses $|\sec(g)| \leq K$, $\text{vol}(g) \geq v_0$ and $\text{diam}(g) \leq D$ are diffeomorphism invariant. It is therefore natural to alternatively define $\mathcal{M}_{CG}$ to be the collection of isometry classes $(M, [g])$ satisfying the above bounds. Next, we note that for any sequence $(M_i, [g_i]) \in \mathcal{M}_{CG}$, all but finitely many terms are diffeomorphic to a limit and thus to each other, so that $\mathcal{M}_{CG}$ contains only finitely many diffeomorphism types. We may then restate the Cheeger-Gromov compactness theorem as follows. Associated to the class $\mathcal{M}_{CG}$, there is a finite list of smooth manifolds $(N_1, \ldots, N_l)$ with the following properties. For any $(M, [g]) \in \mathcal{M}_{CG}$, there is a diffeomorphism $f : N_k \to M$, $1 \leq k \leq l$, and an atlas of charts covering N_k under which f^*g

is uniformly controlled in $C^{1,\alpha}$ in terms of the constants that determine the class, i.e. n, D, K, v_0.

Cheeger-Gromov theory is therefore a 'structure theorem' for the class $\mathcal{M}_{CG}$. This provides a natural starting point from which to study Riemannian manifolds with weaker geometric hypothesis, in which case one expects the formation of singularities. Indeed, the structure theory of Riemannian manifolds under various geometric bounds remains a vast and active area of research cf. [CC97], [CC00a], [CC00b], [Che02], [And97], [And01], [NT11], [Col96b], [Fuk91], [SY05], [Per95], [Fuk88], [Fuk87], [Fuk89], [Col96a], [CG86], [CG90], and the surveys [Che10], [Sor12], [And93], [HH97], [Pet97].

It is equally fundamental to study the structure of Riemannian manifolds with boundary under geometric bounds. Comparatively little has been done in this direction, but see for instance [Kod90], [AKK+04], [Sch01], [And12], [Won10], [dMA11], [Whi53]. Let us discuss two of these studies in more detail.

Kodani ([Kod90]) has proven what is perhaps the most direct analogue of Cheeger-Gromov theory for manifolds with boundary. To describe the result, first consider the class of Riemannian n-manifolds with boundary satisfying the conditions

$$\mathrm{vol}(M) \geq V,\ \mathrm{diam}(M) \leq D,$$
$$|\sec(M)| \leq K,\ \lambda_- \leq II \leq \lambda_+.$$

Here II is the second fundamental form of ∂M in M. If λ_- is allowed to be arbitrary, then this class is not even precompact in the C^0 topology. However, Kodani shows that there exists $\lambda^* < 0$, depending upon n, λ_+, K, D, V, and $\mathrm{diam}(\partial M)$, so that if $\lambda_- > \lambda^*$, then any sequence in the class subconverges in the Lipschitz topology to a limiting C^0 Riemannian metric. The precise control required of λ_- is a definite restriction on the applicability of the theorem. Moreover, the regularity of the limiting metric is not optimal. In particular, this convergence result does not imply finiteness of diffeomorphism types. However, since the second fundamental form is a first-order operator, one hopes to gain control of the first derivatives of g and therefore obtain, for instance, an $L^{1,p}$ convergence theory. In fact, the techniques developed in this thesis do show that Kodani's class of manifolds is precompact in the weak $L^{1,p}$ topology and therefore admits only finitely many diffeomorphism types (cf. Chapter 3.1.2).

Extending techniques introduced by Anderson ([And90]), it is shown in [AKK+04] that the class of Riemannian n-manifolds with boundary satisfying

$$\begin{aligned} |\operatorname{ric}(M)| \leq K, &\quad |\operatorname{ric}(\partial M)| \leq K \\ \operatorname{inj}(M) \geq i_0, &\quad \operatorname{inj}(\partial M) \geq i_0, \quad i_b(M) \geq i_0 \\ \operatorname{diam}(M) \leq D, &\quad |H|_{Lip} \leq H_0 \end{aligned}$$

is precompact in the weak C^2_* topology. C^2_* is the Zygmund space intermediate between $C^{1,\alpha}$ and C^2, i_b is the boundary injectivity radius defined in Chapter 2 and $|\cdot|_{Lip}$ is the Lipschitz norm. Weak C^2_* precompactness means that any sequence subconverges in the $C^{1,\alpha}$ topology (any $0 < \alpha < 1$) to a C^2_* limit. The C^2_* limit is necessary for the applications in [AKK+04], therefore Lipschitz control of H is natural in this context. We note that if one strengthens the control of H in our Theorem 1.0.1 from pointwise to Lipschitz, then the techniques in [AKK+04] allow us to obtain weak C^2_* convergence as well. Moreover, if one weakens in the above theorem the control of H from Lipschitz to pointwise, then the techniques developed in Theorem 1.0.1 establish weak $L^{1,p}$ convergence (see also Remark 1.0.1).

Because of the relative dearth of results in the case of manifolds with boundary, it is still worthwhile to better understand the geometric convergence theory of manifolds with boundary, before undertaking the more general study of the types of singularities that can occur under geometric bounds. The present work is concerned with this task.

First, we show that the "almost convexity" condition of Kodani can be replaced with uniform mean-convexity. Let us assume throughout that all manifolds being considered are connected.

Definition 1.0.1. *Write $\mathcal{M}$ for the class of compact Riemannian n-manifolds with connected boundary satisfying*

$$|\sec(M)| \le K,\ |\sec(\partial M)| \le K$$
$$0 < 1/H_0 < H < H_0$$
$$diam(M) \le D,\ area(\partial M) \ge A_0.$$

Theorem 1.0.1. *$\mathcal{M}$ is precompact in the C^α and weak $L^{1,p}$ topologies, for any $0 < \alpha < 1$ and any $p < \infty$. Consequently $\mathcal{M}$ has only finitely many diffeomorphism types.*

Remark 1.0.1. Roughly speaking, the mean curvature controls the extrinsic or 'nontangential' part of the metric, while the (pointwise conformal class of the) intrinsic boundary metric controls the 'tangential part' (cf. [And08]). This leads one to expect, and it turns out to be true, that stronger $L^{2,p}$ control of the tangential part of the metric may be obtained under the hypothesis in Theorem 1.0.1. Alternatively, one could control H in a stronger norm, say $B^{\epsilon,p}(\partial M)$, $0 \le \epsilon \le 1 - 1/p$ (resp. $C^\alpha(\partial M)$, $0 < \alpha < 1$) and expect to obtain $H^{\epsilon+1/p,p}$ (resp. $C^{1+\alpha}$) control over the metric. Here $B^{s,p}$ is a Besov space and $H^{s,p}$ is a Bessel potential space (cf. Section 2.3). We will not pursue these details here. However, let us note that with only minor modification, the techniques used in the proof of Theorem 1.0.1 already give $C^{1,\alpha}$ and weak $L^{2,p}$

compactness results under the additional assumption that $H \in B^{1-1/p,p}(\partial M)$.

The next result is a compactness theorem for manifolds with boundary under L^p bounds on the sectional curvature, for $p > n/2$. Here we also require uniform control over the volume growth at each point p in the manifold.

Definition 1.0.2. *Write $r_\nu^C(p) := r_\nu(p)$ for the volume radius at p, i.e. the largest r so that for any $s < r$ there holds*

$$\frac{vol(B_p(s))}{s^n} \geq C, \tag{1.1}$$

where $B_p(s)$ is the ball of radius s centered at p. If $\partial M \neq \emptyset$, write that $r_\nu \geq C$ if $r_\nu(p) \geq C\, dist(p, \partial M)$. If $\partial M = \emptyset$, write that $r_\nu \geq C$ if $r_\nu(p) \geq C$ for each $p \in M$. The quantity r_ν is called the (interior) volume radius of M.

In this thesis (and in particular in Definition 1.0.3), we will often separately consider inequalities of the form $r_\nu(\partial M) \geq C$, denoting an inequality on the volume radius of the intrinsic boundary manifold, and $r_\nu(M) \geq C$, denoting an inequality on the interior volume radius of the ambient manifold $M \backslash \partial M$.

Definition 1.0.3. *Write $\mathcal{M}_1$ for the class of smooth Riemannian n-manifolds satisfying the following conditions. On the intrinsic Riemannian manifold ∂M require that*

$$diam(\partial M) \leq D, \ ||\sec(\partial M)||_{L^p(\partial M)} \leq K, \ r_\nu(\partial M) \geq r_0, \tag{1.2}$$

where $p > n/2$. On the ambient manifold $M \backslash \partial M$, require that

$$diam(M) \leq D, \ ||\sec(M)||_{L^p(M)} \leq K, \ \ r_\nu(M) \geq r_0. \tag{1.3}$$

Finally, require the following 'extrinsic' boundary conditions. Assume that $r_\nu(p) \geq r_0$ for each $p \in \partial M$, and that the mean curvature H satisfies

$$||H||_{B^{1-1/p,p}(\partial M)} \leq H_0. \tag{1.4}$$

Theorem 1.0.2. *The class $\mathcal{M}_1$ is precompact in the weak $L^{2,p}$ topology.*

Finally, we observe that in dimension $n = 3$ we can get a result that is stronger than Theorem 1.0.2, provided that the L^p norm of the curvature is small. In particular, we will make use of the following theorem to study Riemannian 3-manifolds whose sectional curvature is small in the L^2 sense.

Definition 1.0.4. *For any $p \in \partial M$, write that p satisfies a (v, r) volume condition if there exists an n-dimensional cone $\mathcal{C} \subset T_pM$ with central axis of* *length r such that $\exp_p$ is a diffeomorphism on $\mathcal{C}$ and there holds*

$$\frac{vol(\exp_p(\mathcal{C}))}{r^n} \geq v. \tag{1.5}$$

Note that the boundary injectivity radius can be arbitrarily small even if (M, g) satisfies a uniform (v_0, r_0) volume condition for each $p \in \partial M$.

Definition 1.0.5. *Write $\mathcal{M}_2$ for the class of smooth Riemannian 3-manifolds satisfying the following conditions. On the boundary ∂M require that*

$$diam(\partial M) \leq D, \ ||\sec(\partial M)||_{L^p(\partial M)} \leq K, r_\nu \geq r_0, \tag{1.6}$$

where $p > 3/2$. On the ambient manifold $M \backslash \partial M$, require that

$$diam(M) \leq D, \ ||\sec(M)||_{L^p(M)} \leq \epsilon, \tag{1.7}$$

where ϵ depends on p and D. Finally, require the following 'extrinsic' boundary conditions. There exists $v_0 > 0$ and $r_0 > 0$ so that each $p \in \partial M$ satisfies a (v_0, r_0) volume condition, and the mean curvature H satisfies

$$||H||_{B^{1-1/p,p}(\partial M)} \leq H_0. \tag{1.8}$$

Theorem 1.0.3. *The class $\mathcal{M}_2$ is precompact in the weak $L^{2,p}$ and C^α topologies, $0 < \alpha < 2 - n/p$. In particular, $\mathcal{M}_2$ admits only finitely many diffeomorphism types.*

For the rest of the introduction let us assume that $n = 3$. The first application of our convergence theory is an extension of the rigidity theory of convex isometric immersions to a 'stability theory' for Riemannian 3-manifolds with boundary. To describe the result, let us first recall Cohn-Vossen's rigidity theorem ([CV27], [HH06]). Write (Σ, h_Σ) for a smooth, closed, oriented surface with Gauss curvature $K_\Sigma > 0$. Cohn-Vossen's theorem states that all analytic immersions

$$i : \Sigma \to \mathbb{R}^3$$

of a closed surface Σ with Gauss curvature $K > 0$ have the same image, modulo a rigid motion of $\mathbb{R}^3$. Note that the solution to the Weyl problem (cf. [HH06]) guarantees the existence of such an immersion. We further remark that the condition $K > 0$ implies that $i(\Sigma)$ is convex, and that therefore i is an isometric embedding. Pogorelov ([Pog73]) removed the restriction on the regularity, thus proving the result in the context of convex geometry.

Cohn-Vossen's rigidity result can be restated, via the developing map, as a theorem about flat 3-manifolds with boundary. Thus if (M_1, g_1) and (M_2, g_2) are compact, simply connected, flat 3-manifolds with isometric boundaries that have positive Gauss curvature, then M_1 is diffeomorphic to M_2 and the metrics g_1, g_2 are in the same isometry class. Such manifolds are therefore 'geometrically rigid.'

Now fix an (Σ, h_Σ), choose an embedding i and write N for the convex solid region bounded by $i(\Sigma)$. Then $N \subset \mathbb{R}^3$ is a smooth, flat manifold with boundary $\partial N = \Sigma$, and Cohn-Vossen's theorem ensures that the isometry class of N does not depend on the choice of immersion. The following theorem is a natural generalization of Cohn-Vossen–Pogorelov's rigidity theorem.

Theorem 1.0.4. *Suppose (M, g) is a compact, oriented, simply connected Riemannian 3-manifold with connected boundary. Write h for the induced metric on ∂M and write K for the Gauss curvature of h. Suppose that $H > 0$ and $K > 0$. To every $\epsilon > 0$ there exists a number $\delta = \delta(\epsilon, \sup H, \sup K, \inf K, \alpha)$ so that if*

$$(h, h_\Sigma)_{GH} < \delta, \quad |\, ric(g)| < \delta$$

then there exists a diffeomorphism $f : N \to M$, and

$$||f^* g - g_{Euc}||_{C^\alpha} \leq \epsilon,$$

where g_{Euc} is the standard Euclidean metric on N and $(\cdot, \cdot)_{GH}$ is the Gromov-Hausdorff distance.

Using the same techniques we can obtain a somewhat different result in the special case that N is a ball. Write B for the unit ball in $\mathbb{R}^3$.

Corollary 1.0.1. *Suppose (M, g) is a compact oriented Riemannian 3-manifold with connected boundary and that $H > 0$. To every $\epsilon > 0$ there exists $\delta = \delta(\epsilon, \sup H, \inf H, \alpha)$ so that if*

$$|K - 1| < \delta, \quad |\, ric(g)| < \delta$$

then there exists a diffeomorphism $f : B \to M$, and

$$||f^* g - g_{Euc}||_{C^\alpha} \leq \epsilon.$$

Let us provide another application of Theorem 1.0.1, motivated by Hopf's rigidity theorem ([Hop51]). Hopf's theorem states that the image of a C^3

isometric immersion

$$i : S^2 \to \mathbb{R}^3$$

of a metric on S^2 with constant mean curvature is a (Euclidean) sphere. Essentially the same proof shows that the image of a C^3 isometric immersion

$$i : S^2 \to \mathbb{H}^3$$

of a metric on S^2 into hyperbolic space is the boundary of a metric ball, provided $H > 2$. Here H is the trace of the second fundamental form with respect to the outward normal, so that in our notation every distance sphere in $\mathbb{H}^3$ has mean curvature $H > 2$. Almgren ([Alm66]) has shown (making use of Hopf's proof) that any analytic minimal immersion of S^2 into S^3 is congruent to the equator. To each of these rigidity theorems we associate a geometric stability theorem.

Theorem 1.0.5. *Let (M, g) be a compact, oriented Riemannian 3-manifold with connected boundary and suppose that $\chi(\partial M) = 2$. To every $\epsilon > 0$ there exists $\delta = \delta(\sup |II|, diam(M), \alpha)$ such that*

i) if

$$|\, ric(M)| \leq \delta, \quad |H - 2| \leq \delta$$

then there exists a diffeomorphism $f : B \to M$ and

$$||f^*g - g_{Euc}||_{C^\alpha} \leq \epsilon.$$

ii) if

$$|\, ric(g) + 2g| \leq \delta, \quad |H - 2\sqrt{2}| \leq \delta$$

then there exists a diffeomorphism $f : B_H \to M$ and

$$||f^*g - g_{-1}||_{C^\alpha} \leq \epsilon,$$

where B_H is a metric ball in hyperbolic space with Gauss curvature of the boundary equal to 1, and g_{-1} is the standard metric of curvature -1.

iii) if

$$|\, ric(g) - 2g| \leq \delta, \quad |H| \leq \delta$$

then there exists a diffeomorphism $f : S^3_+ \to M$ and

$$||f^* g - g_{+1}||_{C^\alpha} \leq \epsilon,$$

where S^3_+ is the upper hemisphere in $S^3 \subset \mathbb{R}^4$ and g_{+1} is the standard metric of curvature $+1$.

The preceding theorems are applications of Theorem 1.0.1. However, it is clear that we could derive similar stability theorems, say for 3-manifolds with sectional curvature close to 0 in the L^2 sense, by making use of Theorem 1.0.3. Instead of doing this in complete generality, let us focus our attention on a similar question that was raised by Sergiu Klainerman in connection with his study of the boudned L^2 conjecture of General Relativity (cf. [KRS]). In an unpublished correspondence with the author's advisor, Klainerman posed the following question.

Question 1.0.1. *Suppose (M, g) is a Riemannian 3-manifold with boundary $(\partial M, h)$, with metric h close to the round metric g_{+1} on S^2. Suppose further that*

$$||\, ric(g)||_{L^2(M)} \leq \epsilon, \tag{1.9}$$

where ϵ is a sufficiently small constant. Is then (M, g) diffeomorphic to the 3-ball B, and g close to the flat metric on B in the $L^{2,2}$ metric in local harmonic coordinates?

Let us first note that there are trivial counterexamples to Question 1.0.1. For instance, let $\tilde{M}$ be a compact, flat, Riemannian 3-manifold, sufficiently scaled so that the unit ball $B \subset \tilde{M}$. Then put $M = \tilde{M} \backslash B$. M is flat and has $\partial M = S^2$, but M is obviously not diffeomorphic to B. This issue cannot be fixed by assuming that M is simply connected, as the following example shows. Consider a sphere $S^3 \subset \mathbb{R}^4$ of sufficiently large radius, i.e. put $\tilde{M} = (S^3, r^2 g_{+1})$, $r >> 1$, and set $M_r = \tilde{M} \backslash B$. If r is sufficiently large, then $||\,\mathrm{ric}(r^2 g_{+1})||_{L^2}$ will be sufficiently small, and the induced metric on ∂M_r will be arbitrarily close to (S^2, g_{+1}). However, this example can be ruled out by assuming that the diameter of M is uniformly bounded, i.e. assuming that ϵ may depend upon $\mathrm{diam}(M)$. Moreover, even though the manifolds M_r do not converge as $r \to \infty$, it is clear that $M_r \to \mathbb{R}^3 \backslash B$ in the smooth pointed topology, as long as the basepoints are chosen to be in the boundary ∂M_r. In fact, if ϵ is allowed to have a few more reasonable dependencies, the proof of the following theorems show that, assuming $\pi_1(M) = 0$, either $M \sim B$ or $M \sim \mathbb{R}^3 \backslash B$.

Theorem 1.0.6. *Suppose that $(M,g) \in \mathcal{M}_2$. To any $\epsilon > 0$ there exists $\delta > 0$ so that if*

$$||h - g_{+1}||_{L^{2,2}(\partial M)} \leq \delta,\ \ ||\sec(g)||_{L^2(M)} \leq \delta,\ \ ||H - 2||_{B^{1/2,2}(\partial M)} \leq \delta, \qquad (1.10)$$

then there exists a diffeomorphism $f : B \to M$ satisfying

$$||f^* g - g_{Euc}||_{L^{2,2}(B)} \leq \epsilon. \qquad (1.11)$$

In case (M,g) is geodesically convex, we can do even better.

Theorem 1.0.7. *Suppose that (M,g) is a compact, geodesically convex manifold with boundary, and fix any $s > 1/2$. To any $\epsilon > 0$, there exists $\delta > 0$, depending only upon ϵ, vol(M), diam(M), and $||H||_{B^{s,2}(\partial M)}$, with the following property. If*

$$||h - g_{+1}||_{L^{2,2}(\partial M)} \leq \delta \text{ and } ||\sec(g)||_{L^2(M)} \leq \delta, \qquad (1.12)$$

then there exists a diffeomorphism $f : B \to M$ satisfying

$$||f^* g - g_{Euc}||_{L^{2,2}(B)} \leq \epsilon. \qquad (1.13)$$

If we take $s = 1/2$ in the above hypothesis, then (M,g) is close to (B, g_{Euc}) in the weak $L^{2,2}$ topology.

Chapter 2

Background Material

2.1 Results from Comparison Geometry

Here we state some preliminary results needed for the proof of Theorem 1.0.1. Let us therefore begin with a few observations about the elements of $\mathcal{M}$. Write $t(x) = \operatorname{dist}(x, \partial M)$ and note that t is smooth off of the cut locus of ∂M. Write

$$\nu = -\operatorname{grad} t$$

so that ν is the outward normal of ∂M. Write $g(S(X), Y)$ for the second fundamental form, thus $S(X) = \nabla_X \nu$. Here ∇ is the usual covariant derivative. The mean curvature H of ∂M is the trace $H := \operatorname{tr} S$. If e_i is a basis for $T_p \partial M$ such that $S(e_i) = \lambda_i e_i$, then the Gauss equation reads

$$\lambda_i \lambda_j = \sec_{\partial M}(e_i, e_j) - \sec_M(e_i, e_j).$$

This implies that each pair $\lambda_i \lambda_j$ is uniformly bounded. Together with the fact that $H = \sum \lambda_i$ is uniformly controlled we see that each λ_i is uniformly controlled as well (in fact $|\lambda_i| \leq \max\{H_0 + 2(n-2)K, 1\}$).

A consequence of the uniform bound on S is that the elements of $\mathcal{M}$ have a uniform upper bound, depending only upon the constants that determine the class $\mathcal{M}$, on the intrinsic diameter of the boundary (see [Won08, Thm 1.1]). Thus we will assume without loss of generality that if $(M, g) \in \mathcal{M}$, then

$$\operatorname{diam}(\partial M) \leq D. \tag{2.1}$$

Write $\operatorname{foc}(\partial M)$ for the focal locus distance of ∂M. Let us recall a basic result from comparison geometry.

Lemma 2.1.1. *Suppose λ and K are any real numbers. Let t_0 be the smallest positive solution to*

$$\cot \sqrt{K} t = \frac{\lambda}{\sqrt{K}} \qquad \text{if } K > 0$$
$$t = \frac{1}{\lambda} \qquad \text{if } K = 0$$
$$\coth \sqrt{-K} t = \frac{\lambda}{\sqrt{-K}} \qquad \text{if } K < 0$$

(a) If $S \leq \lambda$ and $\sec(M) \leq K$, then foc$(\partial M) \geq t_0$.
(b) If $S \geq \lambda$ and $\sec(M) \geq K$, then foc$(\partial M) \leq t_0$.

Write $\text{cut}(\partial M)$ for the cut locus distance of ∂M. Write $\nu(\partial M)$ for the normal bundle of ∂M and define the normal exponential map

$$\exp_\nu : \nu(\partial M) \to M.$$

Define the boundary injectivity radius i_b to be the largest t such that $\exp_\nu$ is a diffeomorphism on $\partial M \times [0, t)$. It is a standard fact that

$$i_b \geq \min\{\text{foc}(\partial M), \text{cut}(\partial M)\}.$$

Let us show, using the proof of [And12, Lemma 2.4], that any $(M, g) \in \mathcal{M}$ has i_b uniformly bounded below.

Lemma 2.1.2. *Suppose $(M, g) \in \mathcal{M}$. Then*

$$i_b(M) \geq \min\{t_0, \frac{2}{(n-1)H_0 K}\} \tag{2.2}$$

for t_0 as defined in Lemma 2.1.1. If in addition ric$(g) \geq 0$, then $i_b(M) \geq t_0$.

Proof. Since $\text{foc}(\partial M) \geq t_0$, it is enough to show that if $\text{cut}(\partial M) < t_0$, then

$$\text{cut}(\partial M) \geq \frac{2}{(n-1)H_0 K}. \tag{2.3}$$

Therefore suppose that $\text{cut}(\partial M) < t_0$. Choose an arclength parametrized geodesic γ realizing the cut locus distance. Thus

$$\gamma : [0, l] \to M$$

is a minimizing geodesic from $p \in \partial M$ to $q \in \partial M$ that is orthogonal to ∂M at the endpoints and such that $\text{Image}(\gamma) \cap \partial M = \{p, q\}$. Consider the index form

$$I(V,W) = \int_0^l g(\nabla_{\gamma'}V, \nabla_{\gamma'}W) - g(R(\gamma', V)W, \gamma')\,dt - g(S(V), W)|_0^l, \quad (2.4)$$

where V and W are vector fields along γ and orthogonal to γ'. Since there are no focal points along γ it follows that

$$I(V,V) \geq 0$$

for any such V. Therefore choose an orthonormal basis e_i for T_pM and define V_i to be the parallel translation of e_i along γ. Then

$$\begin{aligned} 0 \leq \sum_i I(V_i, V_i) &= -\int_0^l \operatorname{ric}(\gamma', \gamma')\,dt - (H(p) + H(q)) \\ &\leq (n-1)lK - 2/H_0, \end{aligned}$$

establishing the first assertion. If in addition $\operatorname{ric}(g) \geq 0$, then

$$0 \leq -2/H_0$$

which contradicts the fact that $\operatorname{cut}(\partial M) < t_0$. □

2.2 Convergence results

As mentioned in the introduction, a sequence (M_i, g_i) of Riemannian manifolds (with or without boundary) converges to (M, g) in the $L^{k,p}$ topology, $k \geq 1$, if $g \in L^{k,p}$ and there exist diffeomorphisms $F_i : M \to M_i$ so that $F_i^* g_i \to g$ in the $L^{k,p}$ topology on M. $L^{k,p}(M)$ is the usual Sobolev space of functions (or tensors) with k weak derivatives in L^p. To be somewhat more precise about the definition of convergence, we require that (M, g) has an $L^{k+1,p}$ atlas of charts in which $F_i^* g_i \to g$ in $L^{k,p}_{\mathrm{loc}}$ in each chart. Similar definitions hold for convergence in other function spaces. For an introduction to the convergence theory of Riemannian manifolds see for instance [GPKS06], [Pet06], [Che10],[HH97], [Sor12].

It is useful to discuss convergence theory in terms that only refer to the local geometry of M. For this purpose we use terminology first introduced by Anderson in [And90]. Given $p \in M \backslash \partial M$ and a number $Q > 1$, define the $L^{k,p}$ harmonic radius at p, denoted $r_h^Q(p) := r_h(p)$, to be the largest number $r < \operatorname{dist}(p, \partial M)$ satisfying the following conditions. There exists a harmonic coordinate system $\{\phi_i\}$ (i.e. $\Delta_g \phi_i = 0$) centered at p and containing the geodesic r-ball $B_r(p)$ on which there holds, for each multi-index σ with $|\sigma| = l \leq k$,

$$Q^{-1}\delta_{ij} \leq g_{ij} \leq Q\delta_{ij}, \tag{2.1}$$
$$r^{l-n/p}||\partial^\sigma g_{ij}||_{L^p} \leq Q \tag{2.2}$$

where δ_{ij} is the standard Euclidean metric. The conditions (2.1) − (2.2) are invariant under simultaneous rescalings of the metric and the coordinates, so that r_h scales like a distance function, i.e. $r_h(\lambda^2 g) = \lambda r_h(g)$. Throughout the discussion we will assume that Q is fixed, say $Q = 3/2$, so in particular there is no loss of clarity when supressing the dependence of $r_h(p)$ on Q. If $q \in \partial M$, define $r_h^Q(q) := r_h(q)$ to be the largest r such that $r < r_h^{\partial M}(q)$ and such that there exists a harmonic coordinate system centered at q, containing the geodesic r-balls $B_q(r)$ and $B_q^{\partial M}(r)$, and satisfying equations (2.1)-(2.2). Here $r_h^{\partial M}(q)$ refers to the harmonic radius of q in the manifold ∂M, and $B_q^{\partial M}(r)$ is defined to be the ball of radius r in ∂M. A harmonic coordinate system at $q \in \partial M$ is defined to be a system of coordinates $\{\phi_i\}$ so that $\phi_1, \ldots, \phi_{n-1}$ form a harmonic coordinate system at q in ∂M, $\phi_n = 0$ on ∂M and each ϕ_i is harmonic on M. We then define the harmonic radius of M, $r_h(M)$, to be the largest r so that for each p in M, either $r_h(p) > r$ or there exists $q \in \partial M$ with $r_h(q) > 2r$ and $p \in B_r(q)$.

It is clear how to extend the definition of $L^{k,p}$ harmonic radius to other function spaces. We could also consider for instance the $C^{k,\alpha}$ harmonic radius, $k \geq 0$, $0 < \alpha < 1$. By the Sobolev embedding theorem, if $p > n$ then the $L^{k,p}$ harmonic radius controls the $C^{k-1,\alpha}$ harmonic radius, $\alpha \leq n/p$. The harmonic radius of a manifold with boundary was previously defined and studied in [AKK+04], where it is shown that (for instance) if (M, g) is a Riemannian manifold with $g \in L^{1,p}$, then M admits an $L^{2,p}$ atlas of harmonic or boundary harmonic coordinate charts in which $g \in L^{1,p}$ (see also [DK81]).

In order to focus our attention on the local geometry of M near the boundary, we will also define the boundary harmonic radius $r_{bh} := \inf_{q\in\partial M}\{r_h(q)\}$. The boundary harmonic radius retains all of the important properties of the harmonic radius. For instance, it is continuous with respect to the $L^{k,p}$ topology, $k \geq 1$.

Lemma 2.2.1. *Suppose that $(M_i, g_i) \to (M, g_\infty)$ in the $L^{k,p}$ topology, $k \geq 1$. Then*

$$\lim_i r_{bh}(g_i) = r_{bh}(g_\infty).$$

The corresponding result for the $L^{1,p}$ harmonic radius of a complete manifold without boundary is done in [AC92], and the continuity of the C^2_* boundary harmonic radius (under C^2_* convergence) is done in [AKK$^+$04]. The proof for the $L^{k,p}$ boundary harmonic radius is nearly identical to these cases, thus we will not describe it here.

The same method used in the proof of Lemma 2.2.1 also implies that for a fixed manifold (M, g) the function $r_{bh} : \partial M \to \mathbb{R}_+$ is continuous.

The link between the harmonic radius and convergence is established by the following theorem (See for instance [Kas89] and [Pet06, Theorem 72]. The proofs there are about $C^{k,\alpha}$ convergence of manifolds without boundary. However, the Banach-Alaoglu theorem allows the result to be extended to weak $L^{k,p}$ convergence, and we may extend the result to manifolds with boundary by simply including boundary harmonic coordinate charts in the analysis.)

Theorem 2.2.1. *Suppose (M_i, g_i) is a sequence of Riemannian manifolds with boundary such that (for some $k \geq 1$) the $L^{k,p}$ harmonic radius $r_h(g_i) \geq r_0$ and diam$(M_i) \leq D$. Then there exists a smooth manifold with boundary (M, g), $g \in L^{k,p}(M)$, so that (M_i, g_i) subconverges in the weak $L^{k,p}$ topology to (M, g). In particular, the harmonic coordinate charts on M_i subconverge weakly in $L^{k+1,p}$ to harmonic coordinates on M.*

In the statement of Theorem 2.2.1 the diameter bound is used to obtain a uniform upper bound on the number of coordinate charts needed to cover M_i. We could also remove the diameter bound and consider pointed convergence. Given a sequence of points $p_i \in M_i$ we say that (M_i, g_i, p_i) converges to (M, g, p) in the $L^{k,p}$ topology if there exist real numbers $r_k < s_k$, $r_k \to \infty$, and compact sets $U_i \subset M_i$, $V_i \subset M$ so that

$$B_{r_i}(p_i) \subset U_i \subset B_{s_i}(p_i), \quad B_{r_i}(p) \subset V_i \subset B_{s_i}(p)$$

and diffeomorphisms

$$F_i : V_i \to U_i, \quad F_i : V_i \cap \partial M \to U_i \cap \partial M_i$$

so that $F_i^* g_i \to g$ in $L^{k,p}_{\text{loc}}$ and $F_i^{-1}(p_i) \to p$. A similar definition of pointed convergence could also be used to formulate a local version of Theorem 2.2.1.

We will make use of the following result.

Theorem 2.2.2. *Let (M, g) be a compact Riemannian n-manifold with boundary with $|\sec(M)| \leq K$. Suppose that for each $x \in M \backslash \partial M$ and each $r <$ dist$(x, \partial M)$ there holds*

$$vol(B_r(x)) \geq v_0 r^n.$$

Then for any $p < \infty$ there exists a constant $r_0 = r_0(n, K, v_0)$ so that the $L^{2,p}$

harmonic radius $r_h(q)$ satisfies

$$r_h(x) \geq r_0 \, dist(x, \partial M).$$

In the case where $\partial M = \emptyset$, volume comparison implies that it is sufficient to assume that $\mathrm{vol}(M) \geq v_0$ and $\mathrm{diam}(M) \leq D$. In this case Theorem 2.2.2 and Theorem 2.2.1 together imply the usual statement of Cheeger-Gromov compactness.

Proof of Theorem 2.2.2. We will outline the proof of this well-known result. Arguing by contradiction, suppose there were a sequence $(M_i, \tilde{g}_i, x_i)$ satisfying the hypothesis of Theorem 2.2.2 but with

$$\frac{\tilde{r}_i(x_i)}{\mathrm{dist}(x_i, \partial M_i)} \to 0, \tag{2.3}$$

where $\tilde{r}_i(x_i) := \tilde{r}_i := r_h(x_i)$. Consider then the rescaled sequence $(M_i, \frac{1}{\tilde{r}_i^2}\tilde{g}_i) = (M_i, g_i)$. The scaling behavior of the harmonic radius implies that (calculated with respect to g_i) $r_h(x_i) = 1$. Since equation (2.3) is scale-invariant it follows that

$$\mathrm{dist}_{g_i}(x_i, \partial M_i) \to \infty.$$

Thus Theorem 2.2.1 implies that (M_i, g_i, x_i) converges in the pointed weak $L^{2,p}$ topology to a complete Riemannian manifold (M_∞, g_∞, x).

As $|\sec(g_i)| \leq \tilde{r}_i^2 K$ we see that $|\sec(g_i)| \to 0$. In particular, $|\mathrm{ric}(g_i)| \to 0$ in C^0. This allows one to improve the convergence from weak $L^{2,p}$ to (strong) $L^{2,p}$ (see [And90]). Then Lemma 2.2.1 implies that $r_h(g_\infty) = 1$. However, the limit (M_∞, g_∞) is a complete, flat, C^∞ Riemannian manifold and is therefore isometric to a quotient of $\mathbb{R}^n$. The volume growth condition implies that (for all r)

$$\mathrm{vol}(B(x, r)) \geq v_0 r^n.$$

This implies that $(M_\infty, g_\infty) = (\mathbb{R}^n, g_{Euc})$, in contradiction to the fact that $r_h(g_\infty) = 1$. □

2.3 Function Spaces

Let us give precise definitions for the various function spaces to be used, and recall some of the basic theorems relating these spaces (cf. [Ste70], [Tri95]).

Write $S := S(\mathbb{R}^n)$ for the standard Schwartz space of rapidly decreasing, complex-valued functions on $\mathbb{R}^n$, and S' for the space of tempered distributions, i.e. the space of continuous linear functionals on S. Define as usual the

Fourier transform $\mathcal{F}$ on S:

$$(\mathcal{F}f)(\xi) := \int_{\mathbb{R}^n} f(x)e^{ix\cdot\xi}dx. \tag{2.1}$$

The Fourier transform naturally extends to a continuous, injective map $S' \to S'$ defined by

$$(\mathcal{F}f)(u) := f(\mathcal{F}u), \tag{2.2}$$

where $f \in S'$, $u \in S$.

Write $C_0^\infty(\mathbb{R}^n)$ for the collection of smooth, compactly supported, complex-valued functions on $\mathbb{R}^n$. Denote by $L^{k,p}(\mathbb{R}^n)$, $k \in \mathbb{Z}_{\geq 0}$, $p \in \mathbb{R}$, $1 < p < \infty$, by the completion of $C_0^\infty(\mathbb{R}^n)$ with respect to the norm

$$||f||^p_{L^{k,p}(\mathbb{R}^n)} := \sum_{|\alpha|\leq k} \int_{\mathbb{R}^n} |D^\alpha f|^p dx. \tag{2.3}$$

Write p' for the conjugate exponent of p, thus $1/p + 1/p' = 1$. Define $(L^{k,p}(\mathbb{R}^n))^* := L^{-k,p'}(\mathbb{R}^n)$ to be the dual of $L^{k,p}(\mathbb{R}^n)$ with respect to the L^2 inner product, i.e. continuous linear functionals over $L^{k,p}(\mathbb{R}^n)$ satisfying

$$||f||_{L^{-k,p'}(\mathbb{R}^n)} := \sup\{\frac{\int_{\mathbb{R}^n} |f\overline{g}|}{||g||_{L^{k,p}(\mathbb{R}^n)}} : g \in L^{k,p}(\mathbb{R}^n)\,, g \neq 0\}. \tag{2.4}$$

Here we have used the identification $f \leftrightarrow (h \mapsto \int_{\mathbb{R}^n} f\overline{h}dx)$. Alternatively and equivalently, we may define $L^{-k,p'}(\mathbb{R}^n)$ to be the completion of C_0^∞ under the norm (2.4). The spaces $L^{k,p}(\mathbb{R}^n)$ are known as *Sobolev spaces*.

Let us define the *Bessel potential spaces* $H^{s,p}(\mathbb{R}^n)$, $s \in \mathbb{R}$, $1 < p < \infty$. Write $H^{s,p}(\mathbb{R}^n)$ for the set of those $f \in S'$ so that

$$||f||_{H^{s,p}(\mathbb{R}^n)} := ||\mathcal{F}^{-1}(1 + |\xi|^2)^{s/2}\mathcal{F}f||_{L^p(\mathbb{R}^n)} < \infty. \tag{2.5}$$

The norm (2.5) makes $H^{s,p}(\mathbb{R}^n)$ into a Banach space. Equivalently, we may define the space $H^{s,p}(\mathbb{R}^n)$ as follows. Define the operator

$$(1 - \Delta)^{-s/2} := \mathcal{F}^{-1}(1 + |\xi|^2)^{-s/2}\mathcal{F}. \tag{2.6}$$

Then $H^{s,p}(\mathbb{R}^n)$ is the image of $L^p(\mathbb{R}^n)$ under the mapping $(1-\Delta)^{-s/2}$, together with the norm (2.5). We point out that if $s = k \in \mathbb{Z}$, then it is well-known (and not difficult to prove) that

$$H^{k,p}(\mathbb{R}^n) = L^{k,p}(\mathbb{R}^n). \tag{2.7}$$

Thus, the Bessel potential spaces are an equivalent description of the Sobolev spaces, as well as an extension of the Sobolev spaces to arbitrary real-valued differential dimension. Furthermore, we remark that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^{s,p}(\mathbb{R}^n)$, and that $H^{-s,p'}(\mathbb{R}^n)$ is dual to $H^{s,p}(\mathbb{R}^n)$ with respect to the L^2 inner product.

Let Ω denote either the closed upper half plane $\overline{\mathbb{R}^n_+}$ or a bounded domain in $\mathbb{R}^n$ with smooth boundary. For $s \geq 0$, $1 < p < \infty$, we define $H^{s,p}(\Omega)$ by restricting elements of $H^{s,p}(\mathbb{R}^n)$ to Ω. Thus, define

$$||f||_{H^{s,p}(\Omega)} := \inf\{||\tilde{f}||_{H^{s,p}(\mathbb{R}^n)} : \tilde{f} = f \text{ on } \Omega\}. \tag{2.8}$$

We then define $H^{-s,p}(\Omega)$ by dualizing with respect to the L^2 inner product, thus

$$||f||_{H^{-s,p}(\Omega)} := \sup_{g \in H^{s,p'}(\Omega)} \frac{|\int_\Omega f\overline{g}dx|}{||g||_{H^{s,p'}(\Omega)}}. \tag{2.9}$$

Setting $s = k \in \mathbb{Z}$, this provides a description of the Sobolev spaces $L^{k,p}(\Omega)$.

Write $\Sigma = \partial\Omega$, and define the Besov space $B^{s,p}(\Sigma)$ $(s > 0,\ p \in (1,\infty))$ to be the collection of restrictions of elements of $H^{s+1/p,\,p}(\Omega)$ to Σ, thus

$$||f||_{B^{s,p}(\Sigma)} := \inf\{||\tilde{f}||_{H^{s+1/p,\,p}(\mathbb{R}^n)} : \tilde{f} = f \text{ on } \Omega\}. \tag{2.10}$$

Define $B^{-s,p}(\Sigma)$ by duality, thus

$$||f||_{B^{-s,p}(\Sigma)} := \sup_{g \in B^{s,p'}(\Sigma)} \frac{|\int_\Sigma f\overline{g}dx|}{||g||_{B^{s,p'}(\Sigma)}}. \tag{2.11}$$

In this thesis we will not formally define or make use of the space $B^{0,p}(\Sigma)$, but we note in passing that $B^{0,p}(\Sigma)$ may be defined, for instance, by using the method of complex interpolation. It then turns out that the spaces $B^{s,p}(\Sigma)$ form complex interpolation scales.

Let us point out the following important characterization of the norm

$B^{s,p}(\mathbb{R}^n)$, valid for those $s > 0$, $s \notin \mathbb{N}$ (cf. [Tri95]).

$$||f||_{B^{s,p}(\mathbb{R}^n)} \sim ||f||_{L^p(\mathbb{R}^n)} + \sum_{|\sigma| \le [s]} \Big(\int_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|D^\sigma f(x) - D^\sigma f(y)|^p}{|x-y|^{n+\{s\}p}} \, dxdy \Big)^{1/p}. \tag{2.12}$$

Here $[\cdot]$ is the floor function and $\{s\}$ denotes the fractional part of s. Thus, the $B^{s,p}$ norm is equivalent to the fractional Sobolev-Slobodeckij norm $W^{s,p}(\mathbb{R}^n)$, provided $s > 0$, $s \notin \mathbb{N}$.

Write M for a compact, differentiable manifold with boundary ∂M. The function spaces above can also be defined on the manifold M, provided that M admits a "sufficiently smooth" atlas of charts. In this thesis, we will make use of the space $B^{s,p}(\partial M)$, $0 < s < 1$, $1 < p < \infty$, but we will not make use of any other globally defined function spaces. In order to avoid technical details that are not relevant to the present work, we will therefore provide a definition for the space $B^{s,p}(\partial M)$, $0 < s < 1$, $1 < p < \infty$, but we will not otherwise discuss the properties of globally defined function theory on M.

Our definition of $B^{s,p}(\partial M)$ is motivated by the relation (2.12). Thus, define $B^{s,p}(\partial M)$, $0 < s < 1$, $1 < p < \infty$, to be the completion of differentiable functions on ∂M under the norm

$$||f||_{B^{s,p}(\partial M)} := ||f||_{L^p(\partial M)} + \Big(\int_{\partial M \times \partial M} \frac{|f(x) - f(y)|^p}{\mathrm{dist}(x,y)^{n+sp}} \, \mathrm{dVol} \Big)^{1/p}. \tag{2.13}$$

Suppose that $f \in B^{s,p}(\partial M)$, that (ϕ, Ω) is a boundary coordinate chart on M, i.e. $\phi : \phi^{-1}(\Omega) \to \Omega \subset \overline{\mathbb{R}^n_+}$, and that $\overline{\Omega}$ is compact. Then it is clear that $f \circ \phi^{-1} \in B^{s,p}(\partial\Omega)$, where $\partial\Omega$ denotes $\Omega \cap \mathbb{R}^{n-1} \subset \overline{\mathbb{R}^n_+}$. Suppose that (ψ, Ω) is another boundary coordinate chart on M. Then the C^1 bound on the transition maps implies that there exists a constant C, depending only upon the differentiable structure of M, so that

$$1/C||f \circ \psi^{-1}||_{B^{s,p}(\partial\Omega)} < ||f \circ \phi^{-1}||_{B^{s,p}(\partial\Omega)} < C||f \circ \psi^{-1}||_{B^{s,p}(\partial\Omega)}. \tag{2.14}$$

2.3.1 Continuous and Compact Embedding Theorems

Let us recall some standard embedding and compactness theorems relating the function spaces described above. First, suppose $1 < p < \infty$, $t \ge 0$, and $s > t + n/p$. Then we have the continuous embedding (cf. [Tri95] and [Roi96])

$$H^{s,p}(\Omega) \subset C^t(\overline{\Omega}). \tag{2.15}$$

Moreover, if t is not an integer, then the embedding (2.15) is continuous for $s = t + n/p$. If

$$1 < p \leq q < \infty,\ -\infty < t \leq s < \infty,\ s - \frac{n}{p} \geq t - \frac{n}{q}, \tag{2.16}$$

then we have the continuous embedding

$$H^{s,p}(\Omega) \subset H^{t,q}(\Omega). \tag{2.17}$$

If

$$1 < p \leq q < \infty,\ -\infty < t \leq s < \infty,\ s - \frac{n-1}{p} \geq t - \frac{n-1}{q}, \tag{2.18}$$

then we have the continuous embedding

$$B^{s,p}(\Sigma) \subset B^{t,q}(\Sigma). \tag{2.19}$$

Finally, let us mention the following generalizations of the Rellich-Kondrachov compactness theorem, which play an important role in the work to follow (cf. [Tay11a], [Tay11b]). If $-\infty < t < s < \infty$, $1 < p < \infty$, then the embeddings

$$H^{s,p}(\Omega) \subset H^{t,p}(\Omega),\ B^{s,p}(\Sigma) \subset B^{t,p}(\Sigma) \tag{2.20}$$

are compact.

Chapter 3

Convergence Theorems for Manifolds with Boundary

3.1 Convergence Under Pointwise Curvature Bounds

3.1.1 Uniformly Mean-Convex Manifolds (Theorem 1.0.1)

The proof of Theorem 1.0.1 proceeds in two basic steps. The first is to control from below the Euclidean volume growth of any point p in the interior of M. The second is to show that the $L^{1,p}$ boundary harmonic radius r_{bh} is bounded from below. These two facts together with Theorem 2.2.2 imply that the harmonic radius $r_h(g)$ is uniformly bounded below, so that Theorem 2.2.1 establishes the result.

Let us begin by controlling the volume of small cylinders in M with base $B \subset \partial M$. Therefore define, for $0 \leq t_1 \leq t_2 < i_b/2$,

$$C(B, t_1, t_2) := \{\exp_\nu(q, t) : t_1 \leq t \leq t_2, \, q \in B\}.$$

Lemma 3.1.1. *Suppose $(M, g) \in \mathcal{M}$ and choose $q \in \partial M$. Suppose there exists $s > 0$ so that $B_q^{\partial M}(s) \subset B$. Then there exists a constant a_0, depending only upon $\mathcal{M}$, so that*

$$vol(C(B, t_1, t_2)) \geq a_0 s^{n-1}(t_2 - t_1).$$

Proof. First note that volume comparison (in ∂M) implies that

$$\text{area}(B) \geq c s^{n-1} \tag{3.1}$$

for some c that only depends on n, $\text{diam}(\partial M)$, and $\text{ric}(\partial M)$. Let B_r be the level set

$$\{\exp_\nu(q,r) : q \in B_q^{\partial M}(s)\}.$$

Since $\mathrm{vol}(C(B,t_1,t_2)) \geq \int_{t_1}^{t_2} \mathrm{area}(B_r)\,dr$, it suffices to show that $\mathrm{area}(B_r)$ is uniformly controlled in terms of $\mathcal{M}$ and $\mathrm{area}(B)$. We note that if another constant c is chosen so that the inequality

$$\frac{1}{K(i_b - t)^2} > \frac{c^2}{(1-c)},$$

holds for all $0 < t < i_b/2$, then

$$H(t) \leq \frac{n-1}{c(i_b - t)}. \tag{3.2}$$

This estimate is proved for instance in Lemma 3.2.2 of [AKK+04] (with a specific choice of c).

Write $A(r) := \mathrm{area}(B_r)$ and write $A_0 := \mathrm{area}(B)$. Consider the first variation of area

$$A'(r) = -\int_{B_r} H d\mu_r$$

where $d\mu_r$ is the volume form on B_r. Together with (3.2) this gives a differential inequality for the area

$$A'(r) \geq -\int_{B_r} \frac{(n-1)}{c(i_b - r)} d\mu_r = -\frac{(n-1)}{c(i_b - r)} A(r) \tag{3.3}$$

This implies that

$$A(r) \geq A_0 (i_b)^{-\frac{n-1}{c}} (i_b - r)^{\frac{n-1}{c}}. \tag{3.4}$$

Since i_b only depends on $\mathcal{M}$ (Lemma 2.1.2), $A(r)$ only depends on $\mathcal{M}$ and $\mathrm{area}(B)$. □

Proposition 3.1.1. *There exists a constant $v_0 = v_0(\mathcal{M})$ so that for each $x \in (M,g) \in \mathcal{M}$ and each $r < dist(x, \partial M)$,*

$$vol(B_r(x)) \geq v_0 r^n.$$

Proof. Put $r_x = \mathrm{dist}(x, \partial M)$. Choose an arclength parametrized minimizing geodesic γ from x to $q \in \partial M$. Consider the unique point y in the image of γ satisfying $r_y = \min(r_x, i_b/2)$. Write Σ_t for the level set of constant t from ∂M and suppose that $t \leq i_b/2$. Hessian comparison implies that there exists a constant

$C = C(\mathcal{M})$ so that if $\mathrm{dist}_{\partial M}(p,q) \le \epsilon$, then $\mathrm{dist}_{\Sigma_t}(\exp_\nu(p,t), \exp_\nu(q,t)) \le C\epsilon$. Put

$$B := B^{\partial M}_{\frac{r_y}{4C}}(q), \tag{3.5}$$

i.e. the $(r_y/4C)$-ball about q in ∂M. From the triangle inequality we see that

$$C(B, 3/4r_y, r_y) \subset B_{r_y}(y) \subset B_{r_x}(x), \tag{3.6}$$

where $C(B, 3/4r_y, r_y)$ is the cylinder defined in Lemma 3.1.1. Then by Lemma 3.1.1 there exists $a_0 = a_0(\mathcal{M})$ so that

$$\mathrm{vol}(B_{r_x}(x)) \ge \mathrm{vol}(C(B, 3/4r_y, r_y)) \ge a_0 r_y^n. \tag{3.7}$$

Thus either

$$\mathrm{vol}(B_{r_x}(x)) \ge a_0 r_x^n$$

or

$$\mathrm{vol}(B_{r_x}(x)) \ge \frac{a_0 i_b^n}{2^n}.$$

In either case (and we remark again that i_b is uniformly bounded below, cf. Lemma 2.1.2), volume comparison applied to the ball $B_{r_x}(x)$ establishes the desired result. □

It remains to show that the boundary harmonic radius is bounded from below.

Proposition 3.1.2. *For large enough p (depending only upon n) there exists $r_0 > 0$, depending only upon the constants that determine the class $\mathcal{M}$, so that for any $(M, g) \in \mathcal{M}$, the $L^{1,p}$ boundary harmonic radius $r_{bh} \ge r_0$.*

Proof. We proceed by contradiction. If the conclusion were false, then there exists a sequence $(M_i, \tilde{g}_i) \in \mathcal{M}$ so that $\tilde{r}_i = r_{bh}(g_i) \to 0$. Choose points $p_i \in \partial M_i$ satisfying $r_{bh}(p_i) = \tilde{r}_i$ and consider the normalized sequence

$$(M_i, g_i, p_i) := (M_i, \frac{1}{\tilde{r}_i^2}\tilde{g}_i, p_i).$$

Note that $r_i = r_{bh}(g_i) = 1$ and so Theorems 2.2.1 and 2.2.2 imply that the sequence (M_i, g_i, p_i) converges weakly in $L^{1,p}$ to a limit (M_∞, g_∞, p). Write $\tilde{h}_i$ and h_i for the metrics on ∂M_i induced respectively from $\tilde{g}_i$ and g_i. Due to the scaling properties of the various quantities we see that

$$\begin{aligned} |\sec(h_i)| &\le \tilde{r}_i^2 K \to 0 \\ H_i &\le \tilde{r}_i H_0 \to 0 \\ |\sec(g_i)| &\le \tilde{r}_i^2 K \to 0 \\ i_b(g_i) &= \frac{1}{\tilde{r}_i} i_b(\tilde{g}_i) \to \infty. \end{aligned}$$

The metrics $\tilde{h}_i$ have bounded curvature, diameter and volume so that passing to a subsequence if necessary we can assume (as in the proof of Lemma 2.2.2) that $(\partial M_i, h_i, p_i)$ converges to $(\mathbb{R}^{n-1}, g_{Euc}, 0)$ in $C^{1,\alpha}$ and $L^{2,p}$. The limit (M_∞, g_∞) satisfies (in the $L^{1,p}$ sense)

$$\operatorname{ric}(h_\infty) = 0 \tag{3.8}$$

$$H_\infty = 0 \tag{3.9}$$

$$\operatorname{ric}(g_\infty) = 0. \tag{3.10}$$

As noted in [AKK$^+$04], when expressed in boundary harmonic coordinates on (M_∞, g_∞) one obtains the system of equations (writing $g := g_\infty$ for the moment)

$$\Delta g^{in} = B^{in}(g, \partial g) - 2\operatorname{ric}(g)^{in} = B^{in}(g, \partial g) \tag{3.11}$$

$$\partial_\nu g^{nn} = -2(n-1)H g^{nn} = 0 \tag{3.12}$$

$$\partial_\nu g^{\alpha n} = -(n-1)H g^{\alpha n} + \frac{1}{2\sqrt{g^{nn}}} g^{\alpha k}\partial_k g^{nn} = \frac{1}{2\sqrt{g^{nn}}} g^{\alpha k}\partial_k g^{nn} \tag{3.13}$$

$$\Delta g_{\alpha\beta} = B_{\alpha\beta}(g, \partial g) - 2\operatorname{ric}(g)_{\alpha\beta} = B_{\alpha\beta}(g, \partial g) \tag{3.14}$$

$$g_{\alpha\beta}|_{\partial M} = h_{\alpha\beta} \tag{3.15}$$

for $1 \le i, j \le n$, $1 \le \alpha, \beta \le n-1$, and ∂_ν denoting the normal derivative. Here $B(g, \partial g)$ is a polynomial in g and is quadratic in ∂g. We remark again that these equations must be (initially) interpreted in the $L^{1,p}$ sense even though we did not write them in this way.

Let us show that (M_∞, g_∞) is a smooth Riemannian manifold with smooth metric tensor (in fact it turns out that $(M_\infty, g_\infty) = (\mathbb{R}^n_+, g_{Euc})$). As remarked above, $h_{\infty\alpha\beta} \in C^\infty$ so that [AKK$^+$04, Proposition 5.2.2] applied to equations (3.14)-(3.15) shows that $g_{\alpha\beta} \in C^{1,\epsilon}$ for some $\epsilon > 0$. Similarly we can obtain $C^{1,\epsilon}$ regularity for the other metric components by applying [AKK$^+$04, Proposition 5.4.1] to the equations (3.11)-(3.13). In [And08] it is shown that, in harmonic coordinates, the system

$$\Phi(g) = \mathrm{ric}(g) \tag{3.16}$$
$$B_1(g) = H \tag{3.17}$$
$$B_2(g) = [h] \tag{3.18}$$

is an elliptic boundary value problem. Here $[h]$ is the pointwise conformal class of h. Since $\mathrm{ric}(g) = 0$, $H = 0$ and $[h] = [g_{Euc}]$ (and since $g_{ij} \in C^1$), elliptic regularity (see for instance [Mor08, Theorem 6.8.3]) shows that $g_\infty \in C^\infty$ in a neighborhood of ∂M_∞. The corresponding interior estimates are well known (see for instance [And90]).

Let us show that the convergence is in the strong $L^{1,p}$ topology. We will establish this by separately obtaining estimates for the Dirichlet problem (3.14)-(3.15) and the Neumann problem (3.11)-(3.13). The estimates themselves will be established as a consequence of the Fredholm property for distributional linear elliptic boundary value problems (cf. [Roi96]) and potential theory for the laplacian ([MT05], [MT00]).

Write Ω for the interior of the upper half of the unit ball in $\mathbb{R}^n$ and write $\Sigma = \overline{\Omega} \cap \mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1} = \{(x_1, \ldots, x_{n-1}, 0)\}$. Consider the elliptic equations

$$L := \sum_{|\sigma| \leq 2} a_\sigma \partial^\sigma \text{ on } \Omega, \tag{3.19}$$
$$B := \sum_{|\sigma| \leq m} b_\sigma \partial^\sigma \text{ on } \Sigma, \tag{3.20}$$

where σ is a multi-index, $m = 0$ or $m = 1$ is fixed, $b_\sigma \in C^{0,\beta}(\Sigma)$, and $a_\sigma \in C^{0,\alpha}(\Omega)$. As a consequence of the Fredholm property established in [Roi96, Theorem 4.1.5], we may immediately deduce the following local boundary estimates. Write $\Omega' \subset \Omega$ for a domain compactly contained in $\Omega \cup \Sigma$.

Proposition 3.1.3. *Suppose that $\alpha > |s-2|$, $\beta > |s-m-1/p|$, and $u \in H^{s,p}(\Omega)$ $(p > 1)$. Then there exists a constant C, independent of u, so that*

$$||u||_{H^{s,p}(\Omega')} \leq C\big(||Lu||_{H^{s-2,p}(\Omega)} + ||Bu||_{B^{s-m-1/p,p}(\Sigma)} + ||u||_{L^p(\Omega)}\big).$$

Making use of the potential theory developed in [MT05], [MT00], we obtain another boundary estimate which will also be of use.

Proposition 3.1.4. *Suppose that g is a $C^{0,\alpha}$ Riemannian metric on $\Omega \cup \Sigma$ and that $\Omega \cup \Sigma$ is contained in a boundary harmonic coordinate chart. Suppose* *that $u \in L^{1,p}(\Omega)$ is a solution to*

$$\Delta_g u = f,\ \partial_\nu u = \phi,$$

where $f \in L^p(\Omega)$ and $\phi \in L^q(\Sigma)$, $\frac{nq}{n-1} \geq p > n$. Choose $\epsilon \geq 0$ satisfying $\epsilon < \alpha$ and $\epsilon < 1 - n/p$. Then there exists a constant C, independent of u, so that

$$||u||_{L^{1,nq/(n-1)}(\Omega')} \leq C\big(||f||_{H^{-\epsilon,p}(\Omega)} + ||\phi||_{L^q(\Sigma)} + ||u||_{H^{-\epsilon,p}(\Omega)}\big).$$

Moreover, one has the following estimate on the non-tangential maximal function of the gradient (cf. [MT99, Appendix A])

$$||(\text{grad } u)^*||_{L^q(\Sigma)} \leq C||\phi||_{L^q(\Sigma)}. \tag{3.21}$$

Proof. It suffices to assume Ω is a domain with smooth boundary $\partial\Omega = \Sigma$. Then the local version (i.e. Proposition 3.1.4) follows from a standard argument using a cutoff function (cf. the proof of [AKK$^+$04, Theorem 5.4.1]). Write V for a nonzero bump function $V \geq 0$ supported in Ω and write $\mathcal{O}$ for an open neighborhood containing Ω. Extend g to a C^α metric on $\mathcal{O}$ and extend f by setting $f \equiv 0$ on $\mathcal{O} \backslash \Omega$. Define the operator

$$L := \Delta_g - V.$$

Setting $\tilde{f} := f - uV$ we get $Lu = \tilde{f}$. Without loss of generality assume that $\mathcal{O}$ has smooth boundary $\partial\mathcal{O}$. It follows that there exists a unique solution $v \in L^{2,p}(\mathcal{O})$ satisfying $Lv = \tilde{f}$ and $v|_{\partial\mathcal{O}} = 0$. More generally, the maximum principle implies that the dirichlet problem

$$u \mapsto (Lu, u|_{\partial\mathcal{O}}) \tag{3.22}$$

is an injective map $L^{2,p}(\mathcal{O}) \to L^p(\mathcal{O}) \times B^{2-1/p,p}(\partial\mathcal{O})$. In fact, since L is Fredholm and formally self-adjoint, it follows that this assignment is an isomorphism. Then [Roi96, Theorem 4.1.5] implies that (3.22) is an isomorphism when viewed as a map $H^{2-\epsilon,p}(\mathcal{O}) \to H^{-\epsilon,p}(\mathcal{O}) \times B^{2-\epsilon-1/p,p}(\partial\mathcal{O})$. This implies the estimate

$$||v||_{H^{2-\epsilon,p}(\Omega')} \leq C||Lv||_{H^{-\epsilon,p}(\Omega)}. \tag{3.23}$$

Set $w = u - v$. Since $H^{2-\epsilon,p}(\Omega) \subset C^{2-\epsilon-n/p}(\Omega \cup \Sigma)$ it follows that $\partial_\nu v$ is Hölder continuous. In particular we see that $\partial_\nu w \in L^q(\Sigma)$ and $Lw = 0$. Then the proof of [AKK+04, Proposition 5.5.1] shows that

$$||w||_{H^{1,nq/(n-1)}(\Omega)} \leq C||\phi||_{L^q(\Sigma)}, \tag{3.24}$$

$$||(\text{grad } w)^*||_{L^q(\Sigma)} \leq C||\phi||_{L^q(\Sigma)}. \tag{3.25}$$

The hypothesis on ϵ implies that $H^{2-\epsilon,p}(\Omega) \subset L^{1,nq/(n-1)}(\Omega)$. Together with equations (3.23) and (3.24), this establishes the result. □

Let us fix a term (M_i, g_i, p_i) and set $\Delta_{g_i} := \Delta_i$. Suppress the subscript on g_i for the moment, setting $g := g_i$. From the definition of pointed convergence we may assume that the metric g is defined on a region (containing p) of fixed but arbitrarily chosen size in M_∞. In what follows we assume that $\Omega \cup \Sigma$ is simultaneously contained in the image of boundary harmonic coordinate charts for g and g_∞, and we work in boundary harmonic coordinates for g. Write g_{ij} and $g_{\infty ij}$ for the coordinate expressions of g and g_∞.

Consider first the 'tangential' components $g_{\alpha\beta}$, $1 \leq \alpha, \beta \leq n-1$. If s satisfies $1 + n/p < s < 2$, then Proposition 3.1.3 implies that

$$||g_{\alpha\beta}||_{H^{s,p}(\Omega)} \leq C\big(||\Delta_g g_{\alpha\beta}||_{H^{s-2,p}(\Omega)} + ||h_{\alpha\beta}||_{H^{s-1/p,p}} + ||g_{\alpha\beta}||_{H^{s-2,p}(\Omega)}.\big) \tag{3.26}$$

Noting that

$$\Delta_g g_{\alpha\beta} = B_{\alpha\beta}(g, \partial g) - 2\,\mathrm{ric}(g)_{\alpha\beta} \tag{3.27}$$

we see that in order to control $g_{\alpha\beta}$ in $H^{s,p}(\Omega)$ it is sufficient to control $B_{\alpha\beta}(g,\partial g)$ in $H^{s-2,p}(\Omega)$. Since $B_{\alpha\beta}(g,\partial g)$ is quadratic in the derivatives of g, the Hölder inequality implies that $B_{\alpha\beta}(g,\partial g)$ is controlled in $L^{p/2}$. Choosing s small enough so that $L^{p/2}(\Omega) \subset H^{s-2,p}(\Omega)$, it follows that $g_{\alpha\beta}$ is uniformly controlled in $H^{s,p}(\Omega)$.

Let us consider then the 'non-tangential' components g^{in}. From Proposition 3.1.4 we see that g^{nn} is uniformly bounded in $L^{1,q}(\Omega)$, $q < \infty$. Moreover, since the metric is smooth we have $(\text{grad } g^{nn})^* = \text{grad } g^{nn}$, so that in the given harmonic coordinate system the boundary estimate of Proposition 3.1.4 implies uniform control of ∂g^{nn} in $L^q(\Sigma)$, $q < \infty$. Similarly for the components $g^{\alpha n}$ we have the estimate

$$\begin{aligned} ||g^{\alpha n}||_{L^{1,qn/(n-1)}(\Omega)} \leq & C\big(||\Delta_g g^{\alpha n}||_{H^{-\epsilon,p}(\Omega)} + ||-(n-1)H + \frac{1}{2\sqrt{g^{nn}}} g^{\alpha k}\partial_k g^{nn}||_{L^q(\Sigma)} \\ & + ||g^{\alpha n}||_{H^{-\epsilon,p}(\Omega)}\big), \end{aligned} \tag{3.28}$$

any $0 < \epsilon < 1 - n/p$. As noted above, that the term $||\Delta_g g^{\alpha n}||_{H^{-\epsilon,p}(\Omega)}$ is bounded follows from the fact that $L^{p/2}(\Omega) \subset H^{-\epsilon,p}(\Omega)$ if ϵ is chosen close enough to $1 - n/p$. Control of the other terms comes from the fact that $|H|_{L^\infty} \sim 0$ and the $L^q(\Sigma)$ bound on ∂g^{nn}. Thus we see that $g^{\alpha n}$ is uniformly controlled in $L^{1,q}(\Omega)$ and $\partial g^{\alpha n}$ is uniformly controlled in $L^q(\Sigma)$, $q < \infty$. In particular, we now have uniform control of g_{ij} in $C^\beta(\Omega)$, any $\beta < 1$. This allows us to apply Proposition 3.1.3 with $1 < s < 1 + 1/p$. We then obtain the estimate

$$\begin{aligned}||g^{nn}||_{H^{s,p}(\Omega)} \leq & C\big(||B^{nn}(g, \partial g) - 2\,\mathrm{ric}^{nn}(g^{nn})||_{H^{s-2,p}(\Omega)} \\ & + ||-2(n-1)H||_{B^{s-1-1/p,p}(\Sigma)} + ||g^{nn}||_{H^{s-2,p}(\Omega)}\big).\end{aligned} \tag{3.29}$$

As before we note that s can be chosen large enough so that $B^{nn}(g, \partial g) \subset H^{s-2,p}(\Omega)$; thus the first term of the right hand side of (3.29) is controlled. Since $s - 1 - 1/p < 0$ it follows that $L^\infty(\Sigma) \subset B^{s-1-1/p,p}(\Sigma)$, and thus g^{nn} is uniformly controlled in the norm $H^{s,p}(\Omega)$. The components $g^{\alpha n}$ are estimated similarly, making use of the fact that $\partial g^{nn} \in L^q(\Sigma)$, $q < \infty$. From this we readily conclude that, passing to a subsequence if necessary, the pointed sequence (M_i, g_i, p_i) converges weakly in the $H^{s,p}$ topology to (M_∞, g_∞, p). Due to the compact inclusion $H^{s,p}(\Omega) \subset H^{1,p}(\Omega) = L^{1,p}(\Omega)$, it follows that (M_i, g_i, p_i) converges strongly in the $L^{1,p}$ topology.

From Lemma 2.2.1 we see that $r_{bh}(g_\infty) = 1$. We will derive a contradiction from this by showing that $(M_\infty, g_\infty) = (\mathbb{R}^n_+, g_{Euc})$. We have already seen that the boundary $(\partial M_\infty, h_\infty) = (\mathbb{R}^{n-1}, g_{Euc})$. The fact that $i_b(M_i) \to \infty$ and the definition of pointed convergence shows that M is diffeomorphic to $\mathbb{R}^n_+$. Since M is flat, the Gauss constraint equation reads

$$||S||^2 = H^2 = 0.$$

On whatever interval boundary normal coordinates exist we have the equation

$$\nabla_{\partial t} S - S^2 = 0. \tag{3.30}$$

The initial condition $S(0) = 0$ implies that $S \equiv 0$ and thus the metric is Euclidean on this interval. On the other hand it is easy to see that boundary normal coordinates exist for all $t > 0$, since M is flat and diffeomorphic to $\mathbb{R}^n_+$. □

3.1.2 Almost Convex Manifolds (A Refinement of a Theorem of Kodani)

As in the introduction we consider Kodani's class of almost convex manifolds $\mathcal{M}_{Kod}$, defined to be the class of Riemannian n-manifolds with boundary satisfying the conditions

$$\text{vol}(M) \geq V,\, \text{diam}(M) \leq D,$$
$$|\sec(M)| \leq K,\, \lambda^* \leq II \leq \lambda_+,$$

where $\lambda^* < 0$ depends upon n, λ_+, K, D, V, and $\text{diam}(\partial M)$. Kodani's theorem ([Kod90]) states that any sequence in $\mathcal{M}_{Kod}$ subconverges in the Lipschitz topology to a limiting C^0 Riemannian metric. Here we note that, as a corollary to Theorem 1.0.1, we can improve the convergence of Kodani's theorem to C^α and weak $L^{1,p}$. In particular, our improvement shows that the class $\mathcal{M}_{Kod}$ admits only finitely many diffeomorphism types. This fact does not directly follow from Kodani's result.

Theorem 3.1.1. *The class $\mathcal{M}_{Kod}$ is precompact in the C^α and weak $L^{1,p}$ topologies, any $p < \infty$. In particular, $\mathcal{M}_{Kod}$ admits only finitely many diffeomorphism types.*

Proof. We first note that the proof of Theorem 1.0.1 applies to the class $\mathcal{M}^*$ of manifolds with boundary satisfying

$$\text{area}(\partial M) > a_0,\ |\sec(\partial M)| < K \tag{3.31}$$
$$\text{diam}(M) \leq D,\ |\sec(M)| \leq K \tag{3.32}$$
$$|H| < H_0,\ i_b > i_0. \tag{3.33}$$

In fact, essentially the first step in the proof is to make this reduction. We further note that Kodani shows that each $(M, g) \in \mathcal{M}_{Kod}$ satisfies $i_b > i_0 > 0$, and moreover each $p \in M$ satisfies $\text{inj}(p) \geq i_0 \,\text{dist}(p, \partial M)$, where i_0 only depends upon the constants that determine the class $\mathcal{M}_{Kod}$. Finally, the Gauss equation shows that $|\sec(g)| \leq K$. Thus, in order to show that $\mathcal{M}_{Kod} \subset \mathcal{M}^*$, it suffices to show that $\text{area}(\partial M) \geq a_0$ for each $(M, g) \in \mathcal{M}_{Kod}$.

As noted above, ∂M admits a collar neighborhood $\mathcal{C}$ of radius i_0, so that there exists a ball $B_p(r) \subset \mathcal{C}$ of fixed but definite radius r, say $r > i_0/4$. Then the Riemannian manifold $(B_p(r), g)$ has bounded diameter, sectional curvature, interior injectivity radius, and satisfies $\text{inj}(p) \geq i_0/4$. In particular, $\text{vol}(B)$ is uniformly boudned below. On the other hand, a theorem of Heintze-Karcher ([HK78]) shows that $\text{vol}(\mathcal{C}) \leq C\,\text{area}(\partial M)$, where C only depends upon i_b, the uniform curvature bound for M, and the uniform bound on the second fundamental form of M. These two facts together imply that $\text{area}(\partial M)$ is uniformly bounded below. □

3.2 Convergence Under L^p curvature Bounds

3.2.1 Convergence of 3-Manifolds with Boundary (Theorem 1.0.3)

Proof of Theorem 1.0.3. Write $r_h(p)$ for the $L^{2,p}$ harmonic radius at $p \in M$, and define $r_{bh} := r_{bh}(g) := \inf_{p \in \partial M} r_h(p)$. In order to establish Theorem 1.0.3 it suffices to show that there exists a constant $C_1 > 0$, $C_1 = C_1(H_0, C, v_0, r_0, D, A_0)$ so that if $(M, g) \in \mathcal{M}_2$, then for each $p \in M \backslash \partial M$ there holds

$$r_h(p) \geq C_1 \operatorname{dist}(p, \partial M), \tag{3.1}$$

$$r_{bh} \geq C_1. \tag{3.2}$$

Let us first establish equation (3.2). Arguing by contradiction, if equation (3.2) were false then there exists a pointed sequence $(M_i, \tilde{g}_i, p_i) \in \mathcal{M}_2$ with $p_i \in \partial M_i$, $r_h(p_i) = r_{bh}$ and $r_h(p_i) \to 0$. Consider then the rescaled sequence (M_i, g_i, p_i), $g_i = \frac{1}{r_{bh}^2} \tilde{g}_i$. The scaling properties of the hypothesis then imply that

$$|\sec(\partial M_i)| \to 0, \ ||\sec(M_i)||_{L^2(M_i)} \to 0.$$

In case $p \leq 2$, the scaling behavior of H does not imply that $||H||_{L^p(\partial M_i)} \to 0$. However, a standard embedding theorem shows that $||H||_{L^q(\partial M)}$ is uniformly contolled for some $q > 2 \geq p$, so that $||H||_{L^q(\partial M_i)} \to 0$. Next choose a fixed but arbitrary scale R and consider a region $\mathcal{R} \subset B_{p_i}(R)$. We have that $L^q(\mathcal{R}) \subset L^p(\mathcal{R})$ independent of i, so that

$$||H||_{L^p(\mathcal{R})} \to 0 \tag{3.3}$$

and thus

$$||H||_{B^{1-1/p,\,p}(\mathcal{R})} \to 0. \tag{3.4}$$

In order to show that (M_i, g_i, p_i) admits an $L^{2,p}$ limit, it suffices to show that, for each fixed $R > 0$ and any $p \in B_R(p_i)$, the harmonic radius $r_h(p)$ is uniformly bounded below, independent of i. Since the boundary harmonic radius $r_{bh}(q) \geq 1$ for each $q \in \partial M$, it suffices to control $r_h(p)$ for those p satisfying $p \in B_R(p_i)$ and $\operatorname{dist}(p, \partial M_i) > 1$. Thus, choose such a point p and consider a unit-speed minimizing geodesic segment γ from ∂M_i to p, i.e. assume that $\gamma(0) \in \partial M_i$, $\gamma(l) = p$, $\operatorname{dist}(\gamma(t), \partial M_i) = t$ and that

$l = \mathrm{dist}(p, \partial M)$. Put $q = \gamma(1/2)$. Since $r_{bh}(M_i) = 1$ it follows that $B_{1/2}(q)$ satisfies $\mathrm{vol}(B_{1/2}(q)) \geq C$, where C is a constant that can be chosen independent of i. Therefore the triangle inequality shows that $\mathrm{vol}(B_l(p)) \geq C$. Now Lemma 3.13 of [And97] shows that the volume radius $r_\nu(p)$ is bounded below by a constant that only depends on R, $||\,\mathrm{ric}(g_i)||_{L^2}$, and C. From here a standard blowup argument as in Theorem 2.2.2 shows that $r_h(p)$ is bounded below (cf. also [Pet97]).

Since the $L^{2,p}$ harmonic radius of (M_i, g_i, p_i) is uniformly controlled in a definite scale, it follows that there exists an $L^{2,p}$ Riemannian manifold $(M_\infty, g_\infty, p_\infty)$ with

$$(M_i, g_i, p_i) \to (M_\infty, g_\infty, p_\infty) \tag{3.5}$$

in the weak $L^{2,p}$ topology. The remarks above imply that $(M_\infty, g_\infty, p_\infty)$ satisfies $||H||_{L^p(\partial M_\infty)} = 0$ and thus $H = 0$. Moreover, since definition 1.0.4 is scale-invariant, we see that every point $p \in \partial M_\infty$ satisfies a (v_0, r) volume condition, any $r < \infty$. Parallel to the proof of Theorem 1.0.4 we see that $(M_\infty, g_\infty, p_\infty)$ is a flat C^∞ Riemannian manifold with boundary.

Let us show that (M_i, g_i, p_i) converges in the (strong) $L^{2,p}$ topology. In case $g_i \in C^1(M)$, we could make use of well-known $L^{2,p}$ elliptic estimates for boundary value problems (cf. [ADN59]). However, g_i need not be C^1 if $p < 3$. Nevertheless, making use of the Maz′ya- Shaposhnikova theory of Sobolev Multipliers (cf. [MS85]), let us show that it is sufficient to assume that $g \in L^{2,p}(M)$, $p > 3/2$.

As before, write Ω for the interior of the upper half of the unit ball in $\mathbb{R}^n$ and write $\Sigma = \overline{\Omega} \cap \mathbb{R}^{n-1}$, where $\mathbb{R}^{n-1} = \{(x_1, \ldots, x_{n-1}, 0)\}$. We are interested here in the case $n = 3$, but the estimate to follow is of course valid for any n. Consider the elliptic equations

$$L := \sum_{|\sigma| \leq 2} a_\sigma \partial^\sigma \text{ on } \Omega, \tag{3.6}$$

$$B := \sum_{|\sigma| = 1} b_\sigma \partial^\sigma \text{ on } \Sigma, \tag{3.7}$$

where σ is a multi-index, $b_\sigma \in L^{2,p}(\Omega)$, and $a_\sigma \in L^{2,p}(\Omega)$, $p > n/2$. Write $\Omega' \subset \Omega$ for a domain compactly contained in $\Omega \cup \Sigma$.

Proposition 3.2.1. *Suppose that $u \in H^{2,p}(\Omega)$ $(p > n/2)$. Then there exists a constant C, independent of u, so that*

$$||u||_{H^{2,p}(\Omega')} \leq C\big(||Lu||_{L^p(\Omega)} + ||Bu||_{B^{1-1/p,p}(\Sigma)} + ||u||_{L^p(\Omega)}\big).$$

Proof. Inspecting the proof of the Fredholm property [Roi96, Theorem 4.1.5] (see also [Roi96, Remark 2.3.1] and [Roi96, Corollary 2.5.1]), it is sufficient to show that multiplication by b_σ is a continuous mapping $B^{1-1/p,p}(\Sigma) \to B^{1-1/p,p}(\Sigma)$, satisfying in addition (for some $\epsilon > 0$) the inequality

$$||Bu||_{B^{1-1/p,p}(\Sigma)} \leq c_1 ||b_\sigma||_{C^0(\Sigma)} ||u||_{L^{2,p}(\Omega)} + c_2 ||u||_{H^{2-\epsilon,p}(\Omega)}, \tag{3.8}$$

where c_1 does not depend upon b_σ (and, of course, c_1 and c_2 are independent of u). We remark that equation (3.8) is a crucial inequality involved in the well-known method of 'freezing the coefficients'. Denote by f an element of $B^{1-1/p,p}(\Sigma)$ and consider its extension (still denoted f) to an element of $B^{1,p}(\Omega)$. Then $b_\sigma f$ is an extension of the product (on Σ) to Ω and it follows from [MS85, Sections 2.2.9 and 2.3] that $b_\sigma f \in L^{1,p}(\Omega)$. Since

$$||b_\sigma f||_{B^{1-1/p,p}(\Sigma)} = \inf_g ||g||_{L^{1,p}(\Omega)}, \tag{3.9}$$

where the infimum is taken over all $L^{1,p}(\Omega)$ functions g extending $b_\sigma f$ from Σ to Ω, it follows that $b_\sigma f \in B^{1-1/p,p}(\Sigma)$. In order to establish inequality (3.8), consider first the expression $||Db_\sigma f||_{L^p(\Omega)}$, where Db_σ is a derivative of b_σ in a fixed but arbitrary direction. Making use of Hölder's inequality and standard embedding theorems, we see that

$$||Db_\sigma f||_{L^p(\Omega)} \leq ||Db_\sigma||_{L^{2p}(\Omega)} ||f||_{L^{2p}(\Omega)} \leq C ||Db_\sigma||_{L^{1,p}(\Omega)} ||f||_{H^{1-\epsilon,p}(\Omega)}, \tag{3.10}$$

where $\epsilon > 0$ is chosen to satisfy the condition $H^{1-\epsilon,p}(\Omega) \subset L^{2p}(\Omega)$. Note that the condition $H^{1-\epsilon,p}(\Omega) \subset L^{2p}(\Omega)$ is satisfied if $\epsilon \leq 1 + n/2p - n/p$, thus ϵ can be chosen to be positive due to the fact that $p > n/2$. Using an argument as above together with the Leibniz rule, we conclude that

$$||b_\sigma f||_{L^{1,p}(\Omega)} \leq ||b_\sigma||_{C^0(\Omega)} ||f||_{L^{1,p}(\Omega)} + C ||f||_{H^{1-\epsilon,p}(\Omega)}, \tag{3.11}$$

where C depends on $||b_\sigma||_{L^{2,p}(\Omega)}$. This implies that

$$||b_\sigma f||_{B^{1-1/p,p}(\Sigma)} \leq ||b_\sigma||_{C^0(\Sigma)} ||f||_{B^{1-1/p,p}(\Sigma)} + C ||f||_{B^{1-1/p-\epsilon,p}(\Sigma)}. \tag{3.12}$$

Equation (3.8) then follows from equation (3.12) and the fact that $u \mapsto Du|_\Sigma$ is a continuous mapping $L^{2,p}(\Omega) \to B^{1-1/p,p}(\Sigma)$. □

Let us fix a term (M_i, g_i, p_i) and set $\Delta_{g_i} := \Delta_i$. Suppress the subscript on g_i for the moment, setting $g := g_i$. From the definition of pointed convergence we may assume that the metric g is defined on a region (containing p) of fixed but arbitrarily chosen size in M_∞. In what follows we assume that $\Omega \cup \Sigma$ is simultaneously contained in the image of boundary harmonic coordinate charts for g and g_∞, and we work in boundary harmonic coordinates for g. Write g_{ij} and $g_{\infty ij}$ for the coordinate expressions of g and g_∞.

A simple calculation using the fact that (in the boundary harmonic coordinates $\{x^i\}$) $\partial_\nu = \frac{\operatorname{grad} x^n}{||\operatorname{grad} x^n||}$ shows that

$$\partial_\nu = \frac{g^{ni}}{\sqrt{g^{nn}}}\frac{\partial}{\partial x^i}. \tag{3.13}$$

Therefore the Maz′ya-Shaposhnikova theory on Sobolev Multipliers cited above implies that the boundary operator $B(u) = \partial_\nu u$ has coefficients that are restrictions of functions in $L^{2,p}(\Omega)$. Thus Proposition 3.2.1 and equations (3.11)-(3.13) imply the estimate

$$\begin{aligned} ||g^{nn} - g^{nn}_\infty||_{L^{2,p}(\Omega)} \leq C\big(||\Delta_i(g^{nn} - g^{nn}_\infty)||_{L^p(\Omega)} + ||-2(n-1)Hg^{nn}||_{H^{1-1/p,p}(\Sigma)} \\ + ||g^{nn}||_{L^p(\Omega)}\big). \end{aligned}$$

Here and above $n = 3$, but we continue to use the variable in order to emphasize the fact that this portion of the proof is valid an all dimensions. A well-known argument making use of the formula $\Delta_i g^{nn} = B^{nn}(g, \partial g) - 2\operatorname{ric}(g)^{nn}$, the fact that $\operatorname{ric}(g) \to 0$, and the Hölder inequality (cf. [Pet97]) shows that

$$||\Delta_i(g^{nn} - g^{nn}_\infty)||_{L^p(\Omega)} \to 0. \tag{3.14}$$

The fact that $||H||_{B^{1-1/p,p}(\Sigma)} \to 0$, together with the fact that multiplication by g^{nn} is a continuous operator $B^{1-1/p,p} \to B^{1-1/p,p}$ (this follows from the Maz′ya-Shaposhnikova theory of Sobolev Multipliers mentioned above, together with the definition of the norm $B^{1-1/p,p}(\Sigma)$) implies that

$$||-2(n-1)Hg^{nn}||_{H^{1-1/p,p}(\Sigma)} \to 0. \tag{3.15}$$

Thus the component g^{nn} converges strongly in $L^{2,p}(\Omega)$. The same argument shows that, in order to establish strong convergence $g^{\alpha n} \to g^{\alpha n}_\infty$, it is sufficient to show that

$$||\frac{1}{2\sqrt{g^{nn}}} g^{\alpha k}\partial_k g^{nn} - \frac{1}{2\sqrt{g^{nn}_\infty}} g^{\alpha k}_\infty \partial_k g^{nn}_\infty||_{B^{1-1/p,p}(\Sigma)} \to 0. \tag{3.16}$$

Consider first the expression

$$||\frac{1}{2\sqrt{g^{nn}}} g^{\alpha k}(\partial_k g^{nn} - \partial_k g^{nn}_\infty)||_{B^{1-1/p,\,p}(\Sigma)}. \tag{3.17}$$

Since $2p > n$ and $g \in L^{2,p}(\Omega)$ it follows from [MS85, Section 2.3] that $\frac{1}{2\sqrt{g^{nn}}}g^{\alpha k} \in L^{2,p}(\Omega)$. Also from [MS85] it therefore follows that multiplication by $\frac{1}{2\sqrt{g^{nn}}}g^{\alpha k}$ is a bounded operator on $B^{1-1/p,p}(\Sigma)$. Due to the strong convergence established above, we see that equation (3.17) tends to 0 as $i \to \infty$.

Next consider the expression

$$||\partial_k g_\infty^{nn}\big(\frac{1}{2\sqrt{g^{nn}}}g^{\alpha k} - \frac{1}{2\sqrt{g_\infty^{nn}}}g_\infty^{\alpha k}\big)||_{B^{1-1/p,p}(\Sigma)}. \tag{3.18}$$

The term $\partial_k g_\infty^{nn}$ is a fixed C^∞ function on Σ and as before we note that

$$\frac{1}{2\sqrt{g^{nn}}}g^{\alpha k} - \frac{1}{2\sqrt{g_\infty^{nn}}}g_\infty^{\alpha k} \in L^{2,p}(\Omega). \tag{3.19}$$

Due to the compact inclusion $B^{2-1/p,p}(\Sigma) \subset\subset B^{1-1/p,p}(\Sigma)$ it therefore follows that

$$||\frac{1}{2\sqrt{g^{nn}}}g^{\alpha k} - \frac{1}{2\sqrt{g_\infty^{nn}}}g_\infty^{\alpha k}||_{B^{1-1/p,p}(\Sigma)} \to 0. \tag{3.20}$$

This shows that $g^{\alpha n} \to g_\infty^{\alpha n}$ in $L^{2,p}(\Omega)$.

Next let us consider the metric components $g_{\alpha\beta}$. Standard estimates for the Dirichlet problem (that only require C^0 regularity of the metric coefficients, cf. [Mor08]) imply that

$$\begin{aligned}||g_{\alpha\beta} - g_{\infty\alpha\beta}||_{L^{2,p}(\Omega)} \leq C\big(||\Delta_i(g_{\alpha\beta} - g_{\infty\alpha\beta})||_{L^p(\Omega)} + ||h_{\alpha\beta} - h_{\infty\alpha\beta}||_{B^{2-1/p,p}(\Sigma)} \\ + ||g_{\alpha\beta}||_{L^p(\Omega)}\big).\end{aligned}$$

In this case the strong $L^{2,p}(\Omega)$ convergence $h_{\alpha\beta} \to h_{\infty\alpha\beta}$ and the arguments above show that $g_{\alpha\beta} \to g_{\infty\alpha\beta}$ in $L^{2,p}(\Omega)$.

A simple linear algebra argument allows us to express the components $g_{\alpha n}$ as an algebraic combination of terms of the form $g_{\alpha\beta}$ and $g^{\alpha n}$ (cf. [AKK+04, Lemma 2.1.4]). Then the strong $L^{2,p}(\Omega)$ convergence of $g_{\alpha\beta}$ and $g^{\alpha n}$, together with the fact that $L^{2,p}(\Omega)$ is closed under multiplication, shows that $g_{\alpha n} \to g_{\infty\alpha n}$ in $L^{2,p}(\Omega)$.

The continuity of the $L^{2,p}$ harmonic radius then implies that $r_h(p_\infty) = 1$. In order to derive a contradiction from this, let us show that (M_∞, g_∞) is isometric to $(\mathbb{R}^n_+, g_{Euc})$.

To begin, define $V_r(p_i) := B_r(p_i) \cap \exp_{p_i}(\mathcal{C})$, where $\mathcal{C}$ is the cone in $T_{p_i}M$ guaranteed by definition 1.0.4. Provided that $r < r_0$, the fact that $\sec_{\tilde{g}_i}(M_i) \sim 0$ and the proof of the volume comparison theorem in Lemma 3.13 of [And97] (cf. also [PW97]) implies that (calculated with respect to $\tilde{g}_i$)

$$\frac{\mathrm{vol}(V_r(p_i))}{r^n} \geq C, \tag{3.21}$$

any $r < r_0$. Since the inequality is scale invariant, this implies that $(M_\infty, g_\infty, p_\infty)$ satisfies inequality (3.21) about p_∞ for any $r < \infty$. In particular, we note that

$$\frac{\mathrm{vol}(B_r(p_\infty))}{r^n} \geq C, \tag{3.22}$$

any $r < \infty$.

Standard Cheeger-Gromov theory applied to the sequence $(\partial M_i, h_i, p_i)$ shows that $(\partial M_\infty, h_\infty)$ is isometric to $(\mathbb{R}^{n-1}, g_{Euc})$. Moreover, as noted above we have that $\sec(g_\infty) = 0$ and $H = 0$. The Gauss constraint equation then implies that the second fundamental form $II = 0$.

Thus (M_∞, g_∞) is a flat, smooth, Riemannian manifold with totally geodesic boundary such that each boundary component is isometric to $\mathbb{R}^{n-1}$. Standard comparison geometry shows that ∂M_∞ has no focal points in M_∞. If ∂M_∞ has no cut points, then the normal exponential map induces an isometry from M_∞ to $\mathbb{R}^n_+$. Therefore suppose the cut locus distance is finite. Choose a point $q \in \partial M_\infty$ and suppose that γ is a geodesic intersecting ∂M_∞ orthogonally at $q = \gamma(0)$. Write $t_f < \infty$ for the first time (after $t = 0$) that γ intersects ∂M_∞. Since ∂M_∞ is totally geodesic, we may consider two disjoint copies of M_∞ and identify their boundaries to form the double $\mathcal{D}(M)$, a complete, flat, C^2 (and by elliptic regularity, C^∞) Riemannian manifold without boundary. The geodesic γ constructed above then extends to a closed geodesic in $\mathcal{D}(M)$, possibly with a non-smooth point at $\gamma(t_f)$. However, since smooth, flat, Riemannian manifolds are characterized as quotients of $\mathbb{R}^n$, we see that the extension of γ is smooth at the point $\gamma(t_f)$. It follows that $\mathcal{D}(M)$ is a nontrivial quotient of $\mathbb{R}^n$. However, it is clear $\mathcal{D}(M)$ also satisfies inequality (3.21), which shows that $\mathcal{D}(M) = \mathbb{R}^n$. This contradiction shows that M_∞ has no cut locus, and we may therefore conlude that $(M_\infty, g_\infty) = (\mathbb{R}^n, g_{Euc})$. This establishes equation (3.2).

Let us demonstrate that equation (3.1) is true. The proof is essentially already contained in our analysis of the blowup argument above. For convenience, we present it again here. Suppose that $(M, g) \in \mathcal{M}_2$. Since $r_{bh} \geq C_1$, it follows in particular that for any point $p \in M$ with $\mathrm{dist}(p, \partial M) \leq C_1/2$ we have (setting $r_p = \mathrm{dist}(p, \partial M)$)

$$\mathrm{vol}(B_p(r_p)) \geq C r_p^n, \tag{3.23}$$

for some constant C depending only upon the constants that determine the class $\mathcal{M}_2$. Since we have already shown that $r_{bh} \geq C_1$, it is sufficient to show that if $r_p \geq C_1/2$, then $r_h(p) \geq C$, for some new constant C depending on C_1 (and, of course, the constants that determine the class $\mathcal{M}_2$). Thus, suppose $p \in M$ satisfies $r_p \geq C_1/2$. Choose a geodesic γ from p to ∂M that minimizes the distance to the boundary, and choose a point $q \in \text{Image}(\gamma)$ so that $r_q = C_1/2$. The triangle inequality implies that $B_q(C_1/2) \subset B_p(r_p)$. Therefore equation (3.23) shows that

$$\text{vol}(B_p(r_p)) \geq C, \tag{3.24}$$

for some constant C. Since $|| \sec(g)||_{L^2(B_p(r_p))} \leq \epsilon$, the local volume comparison result of Lemma 3.13 of [And97] implies that the volume radius $r_\nu(p)$ is uniformly bounded below. As before, this implies that $r_h(p)$ is uniformly bounded below. □

3.2.2 Convergence of n-manifolds with Boundary (Theorem 1.0.2)

In the proof of Theorem 1.0.3, the restrictions to $n = 3$ and $|| \sec(g)||_{L^p(M)} \leq \epsilon$ are only used to control the volume radius $r_\nu(p)$ uniformly from below. Since Theorem 1.0.2 already assumes uniform boundedness of the volume radius, we see that Theorem 1.0.2 follows immediately from the proof of Theorem 1.0.3.

Chapter 4

Geometric Stability Theorems

4.1 Generalization of Cohn-Vossen–Pogorelov's Rigidity Theorem

4.1.1 Theorem 1.0.4

Proof of Theorem 1.0.4. Suppose that (M_i, g_i) is a sequence of compact, oriented, simply connected Riemannian 3-manifolds and write h_i for the induced metric on ∂M_i. To prove Theorem 1.0.4 it is enough to show that if

$$\begin{aligned} \mathrm{ric}(g_i) &\to 0, \\ 0 < H_i &< H_0, \\ 0 < 1/K_0 < K_i &< K_0, \\ h_i &\overset{GH}{\to} h \end{aligned}$$

then (M_i, g_i) subconverges to (N, g_{Euc}) in the C^α topology. Since $|\,\mathrm{ric}(g)|$ controls $|\sec(g)|$ in dimension 3, we may assume that $\sec(g_i) \to 0$. We will first show that for large i, $(M_i, g_i) \in \mathcal{M}$. If $\{e_1, e_2\}$ is an orthonormal basis so that $S(e_k) = \lambda_k e_k$ at some point p, then the Gauss equation gives

$$\lambda_1 \lambda_2 = K_i - \sec_{M_i}(e_1, e_2) > 0. \tag{4.1}$$

Since $H_i = \lambda_1 + \lambda_2 > 0$ we see that λ_1 and λ_2 are both positive. The upper bound on H then implies that each λ_k is uniformly bounded above, while the inequality $K_i > 1/K_0$ shows that λ_k is uniformly bounded below. Thus there is a constant $\widetilde{H}_0 > 0$ so that

$$1/\widetilde{H}_0 < \lambda_k < \widetilde{H}_0.$$

In particular, each M_i is uniformly convex when i is large.

Let us find an upper bound for $\operatorname{diam}(M_i)$. Myers' theorem implies that the diameter of ∂M_i is bounded above. Let k be a negative lower bound for $\sec(g_i)$. If i is large enough then $-k$ can be chosen small enough so that

$$\frac{1}{\widetilde{H}_0\sqrt{-k}} > 1.$$

Thus there exists a positive solution t_0 to the equation

$$\coth(\sqrt{-k}t) = \frac{1}{\widetilde{H}_0\sqrt{-k}}$$

and by Lemma 2.1.1 it follows that $\operatorname{foc}(\partial M_i) \leq t_0$. To any $p \in M_i$ there exists a length minimizing geodesic from p to ∂M_i that meets ∂M_i orthogonally. Since normal geodesics do not minimize distance to the boundary past the first focal point it follows that $\operatorname{dist}(p, \partial M_i) \leq t_0$. Thus $\operatorname{diam}(M_i)$ is bounded above by $\operatorname{diam}(\partial M_i) + 2t_0$.

The Gauss-Bonnet theorem and the inequality $1/K_0 < K_i < K_0$ implies that $\operatorname{area}(\partial M_i)$ is uniformly bounded below when i is large enough.

Therefore eventually $(M_i, g_i) \in \mathcal{M}$ for appropriately defined constants. Theorem 1.0.1 then shows that, passing to a subsequence if necessary, (M_i, g_i) converges to an $L^{1,p}$ limit (M_∞, g_∞). Moreover, by Theorem 2.2.2 the metrics g_i converge in the weak $L^{2,p}_{loc}$ topology on the interior $M_\infty \backslash \partial M_\infty$. Write h_∞ for the boundary metric of g_∞. Applying Cheeger-Gromov compactness to the sequence $(\partial M_i, h_i)$ we can assume that $h_i \to h_\infty$ in the $C^{1,\alpha}$ or weak $L^{2,p}$ topology. Since $h_i \to h$ in the Gromov-Hausdorff topology we see that $(\partial M_\infty, h_\infty)$ and (Σ, h) are isometric as metric spaces. Since ∂M_∞ is orientable and admits a metric of positive curvature it follows that ∂M_∞ is diffeomorphic to S^2.

The condition $\sec(g_i) \to 0$ implies that $\sec(g_\infty) = 0$ in the $L^{2,p}$ sense in (compact regions of) the interior of M_∞. Elliptic regularity then implies that the interior of M_∞ is C^∞ and flat.

Since M_∞ is simply connected we may consider the developing map

$$\xi : M_\infty \backslash \partial M_\infty \to \mathbb{R}^3.$$

The limit M_∞ has an $L^{2,p}$ atlas of coordinate charts and therefore ξ extends to an $L^{2,p}$ (and thus $C^{1,\alpha}$) isometric immersion $M_\infty \to \mathbb{R}^3$. Let us show that the interior of $\xi(M_\infty)$ is a convex region in $\mathbb{R}^3$. Fix $q_1, q_2 \in \xi(M_\infty)$ and choose p_1, p_2 so that $\xi(p_i) = q_i$. Choose a compact set $\Omega \subset M_\infty$ so that $p_i \in \Omega$ and $\operatorname{dist}(\Omega, \partial M_\infty) = \epsilon$ for some small $\epsilon > 0$. From Theorem 2.2.2 it follows that

$g_i \to g_\infty$ in $C^{1,\alpha}$ on Ω. Let γ_i be a geodesic with respect to the metric g_i from p_1 to p_2. We may choose ϵ small enough so that Image$(\gamma_i) \subset \Omega$ for large i. The γ_i subconverge continuously (see [Pet87]) to a g_∞-geodesic γ from p_1 to p_2. The metric g_∞ is a flat metric on a simply connected region, therefore $\xi(\gamma)$ is just the straight line segment from q_1 to q_2.

The convexity of M_∞ implies that ξ is an embedding. Composing the restriction of ξ to ∂M_∞ with a distance preserving bijection $(\partial M_\infty, h_\infty) \to (S^2, h)$, we obtain an isometric embedding (of metric spaces) from (S^2, h) to $(\mathbb{R}^3, g_{Euc})$ whose image bounds a convex connected open set.

A theorem of Pogorelov [Pog73, Thm 3.1.6] implies that $\xi(\partial M_\infty)$ differs from N by a rigid motion of $\mathbb{R}^3$. Composing ξ by this rigid motion, we obtain the required $C^{1,\alpha}$ isometry between M_∞ and N. □

4.1.2 Corollary 1.0.1

Proof of Corollary 1.0.1. Consider a sequence (M_i, g_i) satisfying

$$\begin{aligned} K_i &\to 1 \\ \sec(g_i) &\to 0 \\ 0 < 1/H_0 < H &< H_0. \end{aligned}$$

As demonstrated in the proof of Theorem 1.0.4, $M_i \in \mathcal{M}$ for large i. Thus Theorem 1.0.1 shows that there exists a flat limit (M_∞, g_∞) in which $g_\infty \in L^{1,p}$ on M_∞ and $g_\infty \in C^\infty$ in the interior. Since $K_i \to 1$ and ∂M_i is oriented, it follows that $\partial M_i = S^2$ and that the boundary metrics are tending to the round sphere in the $C^{1,\alpha}$ topology, any $0 < \alpha < 1$. In particular, Theorem 1.0.4 then shows that the universal cover $(\overline{M}_\infty, \overline{g}_\infty)$ is isometric to the (Euclidean) unit ball in $\mathbb{R}^3$. Since the projection map is an isometry, (M_∞, g_∞) satisfies, in the $L^{1,p}$ sense, the curvature conditions

$$\begin{aligned} K &= 1, \\ \operatorname{ric}(g_\infty) &= 0, \\ S &= \operatorname{Id}. \end{aligned}$$

Similar to the analysis in Theorem 1.0.1, elliptic regularity implies that (M_∞, g_∞) is a smooth Riemannian manifold. Then it is easy to deduce from the above curvature conditions that (M_∞, g_∞) is isometric to the unit ball. □

4.2 Generalization of Hopf's Rigidity Theorem

4.2.1 Theorem 1.0.5

Proof. Let us begin by proving part i of Theorem 1.0.5. Suppose (M_i, g_i) is a sequence of compact, oriented Riemannian 3-manifolds with $\chi(\partial M_i) = 2$, $H_i \to 2$, and $\text{ric}(g_i) \to 0$. Write h_i for the induced metrics on ∂M_i. It is sufficient to show that if $|S_i| \leq C$ and $\text{diam}(M_i) \leq D$, then (M_i, g_i) subconverges to (B, g_{Euc}) in the C^α topology, where B is the unit ball. Let us show that $g_i \in \mathcal{M}$ for large enough i. As before we may assume that $\sec(g_i) \to 0$. The Gauss equation then implies that $K_i = \sec_{\partial M_i}(g_i)$ is bounded. The Gauss-Bonnet theorem

$$\int_{\partial M_i} K_i = 4\pi$$

shows that area(∂M_i) is uniformly bounded below. By Theorem 1.0.1 it follows that (M_i, g_i) subconverges in the weak $L^{1,p}$ topology to an $L^{1,p}$ limit (M_∞, g_∞). The limit (M_∞, g_∞) satisfies weakly the equations

$$H = 2, \qquad \text{ric}(g_\infty) = 0.$$

As in the proof of Theorem 1.0.1 we can use the fact that $h_\infty \in L^{2,p}$ and $H \in C^\infty$ to conclude that $g_\infty \in C^{1,\epsilon}$ for some $\epsilon > 0$. It follows that the developing map from the universal cover

$$\xi : \overline{M}_\infty \to \mathbb{R}^3 \tag{4.1}$$

is a $C^{2,\epsilon}$ isometric immersion. Restricting ξ to $\partial\overline{M}_\infty$ we get a $C^{2,\epsilon}$ isometric immersion of a closed 2-manifold of genus 0 and with $H = 2$ into $\mathbb{R}^3$. As mentioned in the introduction, Hopf's rigidity theorem implies that if ξ were C^3, then $\xi(\partial\overline{M}_\infty)$ is a sphere. It is straightforward to show (Lemma 4.2.1) that C^3 may be replaced with $C^{2,\alpha}$. Therefore $\xi(\partial\overline{M}_\infty)$ is a sphere and it follows that $K = 1$ in the $L^{2,p}$ sense. Elliptic regularity applied to the system (see the equations (3.11)-(3.15) and the arguments nearby)

$$K = 1 \quad H = 2 \quad \text{ric}(g) = 0 \tag{4.2}$$

implies that $(\overline{M}_\infty, \overline{g}_\infty)$ (and thus (M_∞, g_∞)) is a smooth Riemannian 3-manifold with boundary. This, together with the equations (4.2), implies that (M_∞, g_∞) is isometric to (B, g_{Euc}).

The other cases follow from similar arguments. Let us briefly outline their proofs. Suppose that (M_i, g_i) is a sequence with $\chi(\partial M_i) = 2$, $H_i \to 2\sqrt{2}$ and $\sec(g_i) \to -1$, and suppose that $\mathrm{diam}(M_i) \leq D$ and $|S_i| \leq C$. Then Theorem 1.0.1 shows that $(M_i, g_i) \to (M_\infty, g_\infty)$ in the weak $L^{1,p}$ topology, where (M_∞, g_∞) is an $L^{1,p}$ Riemannian manifold satisfying

$$H = 2\sqrt{2}, \quad \sec(g_\infty) = -1.$$

Elliptic regularity then shows that $g \in C^{1,\epsilon}$ for some $\epsilon > 0$, and thus the developing map

$$\xi : \overline{M}_\infty \to \mathbb{H}^3$$

induces a $C^{2,\epsilon}$ isometric immersion of $S^2 = \partial M_\infty$ into $\mathbb{H}^3$. The proof of Lemma 4.2.1 shows that the second fundamental form of ξ is constant, i.e. $S = \sqrt{2}\,\mathrm{Id}$. This implies that $\mathrm{Image}(\xi)$ is a distance sphere in $\mathbb{H}^3$. In particular $K = 1$, where K is the Gauss curvature of g_∞. From elliptic regularity we conclude that (M_∞, g_∞) is a C^∞ Riemannian manifold, and the curvature conditions

$$K = 1, \quad H = 2\sqrt{2}, \quad \sec(g_\infty) = -1$$

imply that (M_∞, g_∞) is isometric to a metric ball in $\mathbb{H}^3$ with boundary isometric to the Euclidean sphere.

Now suppose that (M_i, g_i) is a sequence with $\sec(g_i) \to 1$, $H_i \to 0$, $\chi(\partial M_i) = 2$, $\mathrm{diam}(M_i) \leq D$ and $|S_i| \leq C$. In this case the sequence (M_i, g_i) is not contained in $\mathcal{M}$ (since $H_i \to 0$). However, if $\sec(g_i)$ is close enough to 1 and H_i is close enough to 0, then the proof of Lemma 2.1.2 implies that $i_b(g_i)$ is uniformly bounded below. Then the proof of Theorem 1.0.1 shows that the sequence (M_i, g_i) converges weakly in $L^{1,p}$ to an $L^{1,p}$ limit (M_∞, g_∞) of constant curvature 1. The developing map induces a $C^{2,\epsilon}$ minimal isometric immersion of S^2 into S^3. Almgren has shown in [Alm66] that if this immersion were analytic, then its image would be congruent to the equator. His proof is based on the vanishing of the same holomorphic quadratic differential defined in Lemma 4.2.1. Therefore essentially the same proof as in Lemma 4.2.1 shows that 'analytic' may be replaced with '$C^{2,\epsilon}$'. This implies that M_∞ has constant Gauss curvature $K = 1$. Elliptic regularity implies that (M_∞, g_∞) is a C^∞ manifold with

$$K = 1, \quad H = 0, \quad \sec(g_\infty) = 1$$

and thus M_∞ is isometric to the upper hemisphere of (S^3, g_{+1}).

□

Lemma 4.2.1. *Suppose (Σ, g) is a closed surface of genus 0 and $\xi : \Sigma \to \mathbb{R}^3$ is* *a $C^{2,\alpha}$ isometric immersion. Suppose that $H = c > 0$. Then $\xi(\Sigma)$ is a sphere.*

Proof. We will verify one of Hopf's classical proofs, applying only a minor modification. First note that g induces a complex structure on Σ, so that Σ is a Riemann surface. If u and v are isothermal coordinates on Σ, define the complex parameters

$$w = u + iv, \qquad \overline{w} = u - iv.$$

Write L, M, and N for the coefficients of the second fundamental form in the coordinates u, v and write the metric

$$g = E(du^2 + dv^2).$$

Write k_1 and k_2 for the principal curvatures. Define the complex function

$$\phi(w, \overline{w}) = \frac{L - N}{2} - iM.$$

From the formulas

$$K = k_1 k_2 = \frac{LN - M^2}{E^2}$$

and

$$H = (k_1 + k_2) = \frac{L + N}{E}$$

we see that

$$\frac{2|\phi|}{E} = |k_1 - k_2|$$

so that the zeroes of ϕ correspond to the umbilic points of Σ. Since $\xi \in C^{2,\alpha}$, we see that $E \in C^{1,\alpha}$ and $L,\ N,\ M \in C^{\alpha}$. Therefore the Codazzi equations hold weakly

$$L_v - M_u = \frac{E_v H}{2}$$
$$M_v - N_u = -\frac{E_u H}{2}$$

A simple calculation gives

$$\Big(\frac{L - N}{2}\Big)_u + M_v = \frac{EH_u}{2} \tag{4.3}$$
$$\Big(\frac{L - N}{2}\Big)_v - M_u = -\frac{EH_v}{2} \tag{4.4}$$

As H is constant, it follows that these are the Cauchy-Riemann equations for ϕ. Equations (4.3)-(4.4) form an elliptic system and from the regularity theory for weak solutions to elliptic systems (cf. [Mor08, Thm. 6.4.4]) it follows that ϕ is complex-analytic. Therefore ϕdw^2 is a holomorphic quadratic differential. It is a consequence of Liouville's theorem that there are no nontrivial holomorphic quadratic differentials on a Riemann surface of genus 0, so that $\phi \equiv 0$. Therefore every point of M is umbillic, so that $S = k\,\mathrm{Id}$ for some $k : \Sigma \to \mathbb{R}$. The Codazzi equations show that $Dk = 0$ in the distributional sense, so that $k = const = c/2$. Identifying ξ with the position vector of the immersion and n for the unit normal, we get in local coordinates the system of equations

$$(n + c/2\xi)_u = 0$$
$$(n + c/2\xi)_v = 0.$$

These equations only require two derivatives of the immersion, so they are valid classically. This gives the usual proof that $\xi(M)$ is a sphere. □

4.3 Remarks on a Question of Klainerman

Proof of Theorem 1.0.6. Suppose $(M_i, g_i) \in \mathcal{M}_2$ is a sequence with

$$||\sec(g_i)||_{L^2(M)} \to 0, \; ||H_i - 2||_{B^{1/2,2}(\partial M_i)} \to 0, \text{ and } h_i \to g_{+1} \tag{4.1}$$

in the $L^{2,2}$ topology. To establish the result it is sufficient to prove that (M_i, g_i) subconverges in the $L^{2,2}$ topology to (B, g_{Euc}). Theorem 1.0.3 shows that this sequence subconverges in the weak $L^{2,2}$ topology. However, the same argument, together with equation (4.1), shows that the convergence is actually in the $L^{2,2}$ topology. Thus there exists an $L^{2,2}$ limit (M_∞, g_∞) satisfying $(\partial M_\infty, h_\infty) = (S^2, g_{+1})$, $||\sec(g_\infty)|| = 0$, and $H = 2$. Elliptic regularity on the limit implies that (M_∞, g_∞) is a C^∞ Riemannian manifold with boundary. In particular, M_∞ is flat, $H = 2$ in the classical sense, and $(\partial M_\infty, h_\infty) = (S^2, g_{+1})$. The Gauss equation, together with the condition that $H = 2$ implies that either $II = h_\infty$ or $II = -h_\infty$. Since $h_\infty = g_{+1}$, the Riccati equation then implies that either $(M_\infty, g_\infty) = (B, g_{Euc})$ or $(M_\infty, g_\infty) = (\mathbb{R}^3 \backslash B, g_{Euc})$. The fact that the diameter of M_∞ is uniformly bounded rules out the latter case. □

Proof of Theorem 1.0.7. Consider a geodesically convex sequence (M_i, g_i) with uniformly bounded volume, diameter, and with uniform control over the norm $||H_i||_{B^{1/2,2}(\partial M_i)}$. Since M_i is convex, volume comparison implies that the volume radius of each point (even those points on the boundary) is bounded below. Then the proof of Theorem 1.0.3 shows that (M_i, g_i) has an $L^{2,2}$ limit (M_∞, g_∞) satisfying $(\partial M_\infty, g_\infty) = (S^2, g_{+1})$ and $\sec(g_\infty) = 0$. Elliptic regularity on the interior shows that M_∞ is smooth away from the boundary. Thus we may consider the developing map on the universal cover, which induces a $C^{1,\alpha}$ isometric immersion $\partial M_\infty \to \mathbb{R}^3$. Since each (M_i, g_i) is geodesically convex, and geodesics converge to geodesics under C^α convergence, it follows that M_∞ and its universal cover are convex as well. This implies that the developing map is an embedding, and that the image of ∂M_∞ in $\mathbb{R}^3$ bounds a convex region. The rigidity theorem of Pogorelov–Cohn-Vossen then implies that the image of ∂M_∞ is a sphere. From this we deduce that the mean curvature H of M_∞ satisfies $H = 2$. Elliptic regularity shows that M_∞ is smooth, and we conclude from the boundary and curvature conditions that $(M_\infty, g_\infty) = (B, g_{Euc})$. This shows that (M_i, g_i) converges to (B, g_{Euc}) in the weak $L^{2,2}$ topology. To establish strong convergence, we note that the weak convergence $H_i \rightharpoonup 2$ in $B^{1/2,2}$ and the compact embedding $B^{s,2} \subset\subset B^{1/2,2}$ implies that $H_i \to 2$ in the $B^{1/2,2}$ topology. Then the proof of Theorem 1.0.3 establishes strong convergence of (M_i, g_i) as desired. $\square$

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