Hwang–Mok rigidity of cominuscule homogeneous varieties in positive characteristic

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Jan Aleksander Gutt

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Jan Aleksander Gutt

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Jason Starr – Dissertation Advisor
Associate Professor, Department of Mathematics

Samuel Grushevsky – Chairperson of Defence
Associate Professor, Department of Mathematics

Radu Laza
Assistant Professor, Department of Mathematics

Aise Johan de Jong
Professor of Mathematics, Columbia University

This dissertation is accepted by the Graduate School.

Charles Taber
Interim Dean of the Graduate School

Abstract of the Dissertation

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Jun-Muk Hwang and Ngaiming Mok have proved the rigidity of irreducible Hermitian symmetric spaces of compact type under Kaehler degeneration. I adapt their argument to the algebraic setting in positive characteristic, where cominuscule homogeneous varieties serve as an analogue of Hermitian symmetric spaces. The main result gives an explicit (computable in terms of Schubert calculus) lower bound on the characteristic of the base field, guaranteeing that a smooth projective family with cominuscule homogeneous generic fibre is isotrivial. The bound depends only on the type of the generic fibre, and on the degree of an invertible sheaf whose extension to the special fibre is very ample. An important part of the proof is a characteristic-free analogue of Hwang and Mok's extension theorem for maps of Fano varieties of Picard number 1, a result I believe to be interesting in its own right.

To green beans, rosemary, garlic and cherry tomatoes.

Contents

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1

Introduction

1.1 Overview of the work of Hwang and Mok

1.1.1 The study of VMRT

The results of [10] and [11] forming a basis for the generalisation attempted in this dissertation, it is the philosophy of the wider research programme due to Hwang and Mok that informs its overall architecture. We shall thus begin with a review of some of the main concepts, stating the prototypical theorems and reconstructing from [10,11] the sketch of a proof that could serve as a point of departure for our own argument. The lecture notes [9] provide an accessible introduction to this circle of ideas.

By the celebrated theorem of Mori, nonsingular Fano varieties are uniruled, and in fact chain-connected by rational curves. In the particularly simple case of a nonsingular Fano variety X of Picard number 1, one can in fact reduce to a family of irreducible rational curves of minimal degree. Since such curves cannot degenerate to reducible ones, the family is *unsplit*, that is, it becomes proper after taking a quotient by the group of automorphisms of $\mathbb{P}^1$ (or its subgroup fixing the origin $0 \in \mathbb{P}^1$). Associating to a rational curve immersed at a general point $x \in X$ its tangent direction in the projectivised tangent space $\mathbb{P}T_{X,x}$, one obtains a rational map from the space of minimal degree rational curves through x into $\mathbb{P}T_{X,x}$, called the *tangent map* (a theorem of Kebekus [12] shows that in the setting we are to consider, the tangent map is in fact an everywhere-defined, finite morphism). Its closed image, called the variety of minimal rational tangents (VMRT) at x, is the principal object of study in Hwang and Mok's approach.

Since X is chain-connected by minimal rational curves, the VMRT at general points of X connect global information about the geometry of X with the local data of a closed subvariety in a projectivised tangent space, and its infinitesimal variation. More accurately, this is true in characteristic zero, where one can integrate first order infinitesimal data.[1] For example, one can expect certain classification results for complex Fano mani-

[1] In positive characteristic, we will need to replace the VMRT with an object encoding infinite order

folds of Picard number 1, based on the behaviour of VMRT. For the simplest such n-fold, $\mathbb{P}^n_{\mathbb{C}}$, the VMRT are just entire projectivised tangent spaces—that is, there is a rational curve of minimal degree through every point, and in every direction—and indeed $\mathbb{P}^n_{\mathbb{C}}$ is completely characterised by this property, implying Fano index $n+1$. A similar result exists for quadrics, whose VMRT are quadrics themselves, implying Fano index n (cf. [13]). A more general class of complex Fano manifolds of Picard number 1 is provided by Hermitian symmetric spaces of compact type. One application of the study of the VMRT is then the following rigidity theorem [10].

Theorem (Hwang–Mok)**.** *Let $X \to \Delta$ be a proper family of smooth complex manifolds over the unit disc, such that the fibres X_t, $t \neq 0$ are biholomorphic to a fixed irreducible Hermitian symmetric space G/P of compact type. Assume X_0 is Kaehler. Then X_0 is biholomorphic to G/P.*

Its original proof used Ochiai's application of Cartan's equivalence method [15] to a flat L-structure defined by the VMRT on the central fibre X_0 (where L is isogeneous to the Levi factor of P). An underlying prolongation procedure becomes cumbersome in characteristic $p > 0$, requiring an immediate introduction of conditions on p. Fortunately, this approach has since been completely replaced by a more recent 'Cartan-Fubini type' extension theorem [11].

Theorem (Hwang–Mok)**.** *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be complex Fano manifolds of Picard number 1, together with a choice of an irreducible component of rational curves of minimal degree. Let $\mathcal{C} \subset \mathbb{P}T_X$ and $\mathcal{D} \subset \mathbb{P}T_Y$ be the corresponding families of VMRT, and assume that the fibre $\mathcal{C}_x \subset \mathbb{P}T_{X,x}$ at a general point $x \in X$ is positive-dimensional, with generically finite Gauss map (as an embedded projective variety). Let $U \subset X$ and $V \subset Y$ be connected analytic open subsets together with a biholomorphic map $\varphi : U \to V$ such that $\varphi_* : \mathbb{P}T_U \to \mathbb{P}T_V$ maps $\mathcal{C}|_U$ isomorphically onto $\mathcal{D}|_V$. Then φ extends to a biholomorphism $\phi : X \to Y$.*

We shall first sketch an argument reducing the rigidity theorem to the extension theorem, and then describe the main steps in the proof of the latter. Along the way, we will indicate some of the main difficulties that arise upon passage to positive characteristic. The first issue is of course that the notion of a Hermitian symmetric space is a complex-analytic one. The proper algebraic replacement is a *cominuscule homogeneous variety*, defined in Chapter 2. Complex cominuscule homogeneous varieties are precisely irreducible Hermitian symmetric spaces of compact type, and we will from now on use the former notion.

1.1.2 Rigidity theorem

In the setting $X \to \Delta$ of the rigidity theorem, after eliminating the case where G/P is a projective space, Hwang and Mok study specialisations of rational curves of minimal degree on the cominuscule homogeneous general fibre G/P to the central fibre X_0. These are in fact of degree 1 with respect to the ample generator $\mathcal{O}_{G/P}(1)$ of the Picard group of

information.

G/P. Observing that X_0 is also Fano of Picard number one, we have an ample invertible sheaf $\mathcal{O}_X(1)$ on X extending $\mathcal{O}_{G/P}(1)$, and the specialisations of degree one curves on G/P to X_0 are again irreducible rational curves of minimal degree. Now, the space of rational curves of minimal degree through a general point of X_0 is smooth (i.e. all such curves are free; this can fail completely in positive characteristic). In particular, such curves deform to the general fibre: thus the family of degree one rational curves through a general point of X_0 coincides with the family of specialisations of degree one curves from G/P.

For a general section $s : \Delta \to X$ we have a family $M \to \Delta$ such that M_t is the space of degree one rational curves $\mathbb{P}^1_{\mathbb{C}} \to X$ mapping $0 \in \mathbb{P}^1_{\mathbb{C}}$ to $\sigma(t)$, modulo the action of the group of automorphisms of $\mathbb{P}^1_{\mathbb{C}}$ fixing 0. The map $M \to \Delta$ is projective and has smooth fibres. A fundamental result about cominuscule homogeneous varieties states that M_t, $t \neq 0$ is either a Segre variety, or a cominuscule homogeneous variety itself. Dealing separately with the Segre case, one sees that an inductive application of the rigidity theorem allows us to conclude that $M \to \Delta$ is isotrivial (note that M_t have strictly lower dimension than X_t). This shows that the space of degree one curves through a general point of X_0 is *abstractly* biholomorphic to that on the model G/P.

In order to conclude the same about the VMRT (as a subvariety of a projectivised tangent space), Hwang and Mok show that the latter is linearly nondegenerate. The argument uses integrability of the meromorphic distribution defined by the linear span of the VMRT, checking that linear degeneracy leads to the existence of an algebraic foliation of X_0, whose properties would force the Picard number to be greater than 1 (in characteristic $p > 0$ this becomes a statement about a purely inseparable quotient, and we will need some conditions on p to derive a contradiction).

Having shown that the VMRT at a general point of X_0, with its embedding into the projectivised tangent space, is isomorphic to the VMRT at a point of G/P via a linear identification of tangent spaces, there remains one more step needed to satisfy the hypotheses of the extension theorem for X_0 and G/P with their rational curves of degree one (on X_0 we choose the unique dominating component; the VMRT at a general point does satisfy finitness of the Gauss map). It has to be checked that the family of VMRT over a small analytic open subset of X_0 can be identified with the family of VMRT over a biholomorphic analytic open subset of G/P. A natural differential-geometric approach is to associate with it an L-structure.[2] By prolongation theory the latter admits a well-defined notion of curvature, whose vanishing on X_t, $t \neq 0$ extends by continuity to X_0, implying local equivalence of L-structures, and thus of families of VMRT on X_0 and G/P [15]. It has already been pointed out that such differential-geometric machinery is not convenient in positive characteristic. However, as we will explain below, the object we shall use in our version of the extension theorem will be a family of arcs of infinite order, rather than just the VMRT. In that setting, flatness will follow from a general result on the 'moduli' of families of formal arcs on a formal disc (Chapter 2).

[2] A reduction of the frame bundle to a sub-bundle whose structure group is the image of the Levi factor L of P in $\mathsf{GL}(\mathfrak{g}/\mathfrak{p})$.

1.1.3 Extension theorem

Let us now briefly outline the proof of the extension theorem. Recall that we are in the setting of a pair of Fano manifolds X, Y with irreducible components of minimal degree rational curves $\mathcal{M}$, $\mathcal{N}$ and corresponding families of VMRT $\mathcal{C}$, $\mathcal{D}$. The theorem states that an analytic-local biholomorphism $\varphi : U \to V$, compatible with the VMRT, extends to a global biholomorphism $\phi : X \to Y$. The argument consists of several steps.

1. One shows that φ in fact sends holomorphic germs of $\mathcal{M}$-curves to holomorphic germs of $\mathcal{N}$-curves. This relies on differential-geometric methods, and in fact will never hold in positive characteristic: for example, a 'constant' family of subvarieties of the projectivised tangent bundle of a formal disc is not affected by Artin-Schreier type automorphisms, although these will typically not preserve its lift to a family of formal arcs.

   It is thus here that we set our point of departure. Our version of the extension theorem will be a statement about an isomorphism of formal neighbourhoods of general points compatible with families of formal arcs, rather than just the VMRT.

2. A procedure of 'analytic continuation along minimal rational curves' is applied. A rational curve is called *minimal*[3] if its normal bundle, pulled back to the normalisation, splits into line bundles of degrees 0 and 1. General members of $\mathcal{M}$ and $\mathcal{N}$ have this property. The assumption that the VMRT at a general point be positive-dimensional implies that there is at least one summand of degree 1, so that a general $\mathcal{M}$-curve admits a deformation fixing precisely one point. Consider now a general curve C passing through a general point $x_0 \in U$. Its germ at x_0 is sent by φ to a germ of a curve D at $\varphi(x_0) \in V$.

   Choose a point $x \in C$. A deformation of C fixing x induces a deformation of the germ of C at x, that is, a family of germs of $\mathcal{M}$-curves in U. By Step 1, these are sent by φ_* to germs of $\mathcal{N}$-curves in V, thus giving rise to a family of $\mathcal{N}$-curves. By the assumption on generality, these $\mathcal{N}$-curves intersect at a single point $y \in D$. This yields a map $C \to D$ extending $\varphi|_C : C \cap U \to D \cap V$. Furthermore, $C \to D$ can be extended to an open neighbourhood of C swept out by its small deformations.

   There is a Zariski-dense open subset of X that can be covered by chains of general $\mathcal{M}$-curves (of some fixed length) with the first segment passing through x_0. Applying the continuation procedure inductively, we obtain a map into Y from the space parametrising such chains together with the choice of a point on the last segment. Furthermore, the map is constant on small deformations of a chain fixing the marked point. It then follows that the map defined on chains descends to a rational map $\tilde{\phi} : \tilde{X} \dashrightarrow Y$ from a generically étale cover $\tilde{X} \to X$. There is a biholomorphic lift $\tilde{U} \subset \tilde{X}$ of $U \subset X$ such that $\tilde{\phi}$ is defined on $\tilde{U}$ and $\tilde{\phi}|_{\tilde{U}} = \varphi$ via the identification $\tilde{U} \simeq U$. Finally, $\tilde{\phi}$ maps $\mathcal{M}$-curves to $\mathcal{N}$-curves.

[3] 'Standard' in [11].

With analytic neighbourhoods replaced by formal ones, and considering families rather than closed points, this part of the argument works in arbitrary characteristic.

3. One now checks that $\tilde{\phi}$ can in fact be descended to a *birational* map $\phi_0 : X \dashrightarrow Y$. Replacing $\tilde{X}$ with the graph of $\tilde{\phi}$, a careful examination of the construction of $\tilde{\phi}$ shows that $\tilde{X} \to X$ is trivialised over a general $\mathcal{M}$-curve. Then, a genericity argument, together with simply-connectedness of X, rules out ramification in $\tilde{X} \to X$, so that the latter is birational. A similar argument is used to check that $\tilde{X} \dashrightarrow Y$ is unramified in codimension one, so that $\phi_0 : X \dashrightarrow Y$ is birational.

   Simply-connectedness of complex Fano manifolds, being a consequence of their (separable) rational connectedness, will have to be added as a hypothesis in positive characteristic.

4. The open subvariety of X on which ϕ_0 is defined contains a free $\mathcal{M}$-curve, and thus so does its image. It follows that ϕ_0 induces an isomorphism of complements of closed subvarieties of codimension at least two. A standard argument with pluri-anticanonical embeddings shows that in this case ϕ_0 extends to an isomorphism $\phi : X \to Y$.

1.2 Main results

The basic elements of the reasoning laid out in the former section have to be carefully recast in an algebro-geometric language suitable for positive characteristic. Some carry through without much change, some need additional preparation to avoid pathological situations, while others will merely be in a relation of analogy to the actual arguments. In particular, intuitive analytic-local constructions are replaced with somewhat more technical methods of formal geometry.

Chapter 2 sets up the necessary theoretical foundations: spaces of formal arcs, families of rational curves, and cominuscule homogeneous varieties. We work with unparametrised pointed arcs of infinite order, and define corresponding parameter spaces intrinsically, rather than as an inverse limit. Since no convenient reference seems to be available for this setting, we work out some basic properties. Proposition 2.1.11 may be of independent interest here. The section on rational curves is mostly concerned with introducing the notation for different objects associated with a family of curves, and gathering some standard facts. The main reference is [14]. We also cite the theorem of Kebekus [12] in a suitable form. Proposition 2.2.8 is a positive-characteristic analogue of [10, Prop. 13]. The substantial difference is that we need to consider a purely inseparable quotient instead of a foliation by subvarieties. The final section introduces the class of cominuscule homogeneous varieties, their VMRT and some auxiliary intersection numbers.

Chapter 3 states and proves a characteristic-free analogue of the extension theorem. Rather than starting with a single formal isomorphism, we choose to work with an entire bundle parametrising isomorphisms between formal neighbourhoods of points on two varieties X, Y. The bundle admits a natural stratification,[4] restricting to a subscheme cut out by the condition of compatibility with arcs associated with given families $\mathcal{M}$, $\mathcal{N}$ of rational curves satisfying suitable conditions. The central result is Proposition 3.1.5, showing that the latter subscheme admits natural horizontal (generic) trivialisations along $\mathcal{M}$-curves. This plays a role analogous to the analytic continuation discussed in the previous section. Following Step 2 and Step 3 of the original argument, we arrive at Proposition 3.1.7, producing a horizontal (generic) trivialisation over X. Finally, Step 4 leads to Theorem 3.2.1 and its Corollary.

Chapter 4 states and proves our version of the rigidity theorem over an algebraically closed field of positive characteristic. The main problem with applying the strategy outlined in the previous section is potential inseparability of various evaluation maps, with the most serious consequence being failure of smoothness of the space of minimal degree rational curves through a general point of a nonsingular variety, in this case the special fibre of a degeneration. Such behaviour can be ruled out by imposing a lower bound on the characteristic. We are not aware of a universal bound that would not require the knowledge an explicit very ample invertible sheaf on the variety. Hence the main result of this chapter, Theorem 4.4.1, refers to the notion of d-rigidity: we call a cominuscule homogeneous variety G/P d-rigid, if it does not admit nontrivial smooth projective degenerations with $\mathcal{O}_{G/P}(d)$ extending to a very ample invertible sheaf on the special fibre. An inductive application of the Theorem establishes d-rigidity of G/P under the assumption that the characteristic be greater than an explicit integer, depending only on d and G/P, and computable in terms of Schubert calculus on the latter.

1.3 Further directions

It would be interesting to investigate the possibility of applying our method, using formal arcs instead of just VMRT, to other of the multitude of results obtained by Hwang and Mok. Examples include: rigidity of generically étale morphisms to cominuscule homogeneous varieties, Lazarsfeld's problem for morphisms from cominuscule homogeneous varieties, rigidity of non-cominuscule homogeneous varieties (all approachable via the extension theorem, see [9] for a review).

Another outstanding issue is that of improving the bounds on the characteristic in our version of the rigidity theorem. The way we had obtained them being far from subtle, there should be an approach exploiting the particular setting of the degeneration problem to a greater degree. We also do not know how far our bounds are from being effective: a counterexample to rigidity in low characteristic should be enlightening (we know none).

[4] In the sense of an identification of infinitesimally close fibres.

1.4 Conventions and notation

We work over an algebraically closed field k. We will mostly be interested in the case $\operatorname{char} k > 0$, although we do not assume this until Chapter 4. A *presheaf* will mean a presheaf of sets on the category of k-schemes. A *sheaf* will mean a sheaf for the *fpqc* topology. We do not employ any notational convention to distinguish between presheaves, sheaves, formal schemes and schemes.

Given a morphism $X \to S$ of presheaves, $\underline{\mathrm{Aut}}_S X$ denotes the presheaf whose value at T is the set of pairs $(T \to S, \varphi)$ where $\varphi \in \mathrm{Aut}_T X_T$. If $f : Z \to X$ is a morphism of presheaves over S, $\underline{\mathrm{Aut}}_S(X, f)$ denotes the sub-presheaf of $\underline{\mathrm{Aut}}_S X$ whose value at T is the subset of $(T \to S, \varphi)$ such that $\varphi \circ f_T = f_T$. Given a second morphism $Y \to S$ of presheaves, $\underline{\mathrm{Hom}}_S(X, Y)$ denotes the presheaf whose value at T is the set of pairs $(T \to S, \psi)$ where $\psi \in \mathrm{Hom}_T(X_T, Y_T)$. If $g : Z \to Y$ is a morphism of presheaves over S, we let $\underline{\mathrm{Hom}}_S(X, Y; f, g)$ be the sub-presheaf of $\underline{\mathrm{Hom}}_S(X, Y)$ whose value at T is the subset of $(T \to S, \psi)$ such that $\psi \circ f_T = g_T$. There is are sub-presheaves $\underline{\mathrm{Isom}}_S(X, Y) \subset \underline{\mathrm{Hom}}_S(X, Y)$ and $\underline{\mathrm{Isom}}_S(X, Y; f, g) \subset \underline{\mathrm{Hom}}_S(X, Y; f, g)$ whose values at T are restricted to those ψ which are isomorphisms.

Given a morphism $X \to S$ of presheaves, $(X/S)^i$ denotes the i-fold product $X \times_S \cdots \times_S X$, together with the natural morphism to S. For X a presheaf equipped with a pair of structure morphisms $X \rightrightarrows S$, referred to as the left and right structure map, $(S\backslash X/S)^i$ denotes the i-fold product $X \times_S \cdots \times_S X$, together with the pair of structure morphisms into S given by the left structure morphism from the leftmost factor and the right structure morphism from the rightmost factor. Given morphisms $Y \to X \to S$ of presheaves, $\prod(Y/X/S)$ denotes the presheaf whose value at T is the set of pairs $(T \to S, \sigma)$ where $\sigma : X_T \to Y_T$ is a section of the pullback of $Y \to X$. When applied to sheaves, all these constructions yield sheaves.

Given a morphism of sheaves $X \to S$ and a sheaf of groups G over S acting from the right on X, we have the quotient sheaf X/G over S. Its value at T is the set of equivalence classes of diagrams

$$\begin{array}{ccc} T' & \longrightarrow & X \\ \downarrow & & \downarrow \\ T & \longrightarrow & S \end{array}$$

where $T' \to T$ is a covering, and $T' \to X$ is such that the induced morphism $T' \times_T T' \to X \times_S X$ factors through the action morphism $X \times_S G \to X \times_S X$. Two diagrams, the other say with $T \leftarrow T'' \to X$, are identified if $T' \times_T T'' \to X \times_S X$ factors through the action morphism. If instead G acts on X from the left, we have an analogous construction of $G\backslash X$.

A morphism $Y \to X$ of presheaves is *schematic* if for every morphism $T \to X$ from a scheme, the pullback $T \times_X Y$ is a scheme. Furthermore, a schematic morphism $Y \to X$ is *affine* if for every morphism $T \to X$ from a scheme, $T \times_X Y \to T$ is affine.

We will mostly work with locally Noetherian formal schemes. Occasionally, we allow *adic* morphisms $Y \to X$ from a (not necessarily locally Noetherian) formal scheme to a locally Noetherian formal scheme, i.e. such that the pullback of the underlying scheme of X gives an underlying scheme of Y. Such a morphism is in particular schematic, and the ideal of definition in $\mathcal{O}_Y$ is locally finitely generated.

If $X \to S$ is a morphism of locally Noetherian formal schemes, we denote by $(X/S)^\sharp \rightrightarrows X$ the completion of $(X/S)^2$ along the diagonal, together with the induced pair of structure morphisms into X. It is a formal subscheme of $(X/S)^2$, containing X and sharing the same underlying reduced scheme. We sometimes consider $(X/S)^\sharp$ as a bundle over X, using the *left* structure map. In particular, we let $(X/S)^\sharp_x = x^*(X/S)^\sharp$ for a point or geometric point x of X. The relative tangent sheaf $T_{X/S}$ is as usual defined to be the $\mathcal{O}_X$-module dual to $\mathcal{I}/\mathcal{I}^2$, where $\mathcal{I}$ is the ideal of the diagonal in $(X/S)^2$. We let $T_{X/S,x} = x^*T_{X/S}$. Given a sheaf $\mathcal{E}$ over X, an S-stratification on $\mathcal{E}$ is an isomorphism $X^\sharp \times_X \mathcal{E} \to \mathcal{E} \times_X X^\sharp$ satisfying the usual cocycle condition. For example, $X^\sharp \times_X \mathcal{E}$ is canonically stratified. A morphism of stratified sheaves is *horizontal* if it is compatible (in an obvious manner) with the stratifications.

The base $\operatorname{Spec} k$ will be usually omitted from notation, so that $\prod(Y/X) = \prod(Y/X/k)$, $X^i = (X/k)^i$, $X^\sharp = (X/k)^\sharp$, $T_X = T_{X/k}$, $T_{X,x} = T_{X/k,x}$, etc. unless explicitly defined to mean otherwise. We fix an origin 0 in $\mathbb{P}^1$. Given a reduced group scheme G over k, a morphism $X \to S$ together with a G-action is a *G-principal bundle* if $X \times G \to X \times_S X$ is an isomorphsm, and there is an étale cover $S' \to S$ together with a section $S' \to S' \times_S X$.

We will only consider inverse/direct systems indexed by integers. A morphism $X \to S$ of formal schemes is of *pro-finite type* if it is the limit of an inverse system of morphisms $X_i \to S$ of finite type. A scheme is called *pro-algebraic* if it is of pro-finite type over k. An affine group scheme is called *pro-unipotent* if it is the limit of a countable sequence of successive extensions by $\mathbb{G}_a$ (starting with the trivial group). $\mathbb{A}^\infty$ denotes the spectrum of a polynomial algebra in countably infinitely many variables. In particular, a pro-unipotent group is isomorphic to $\mathbb{A}^\infty$ as a scheme.

2

Technical tools

2.1 Formal arcs

2.1.1 Formal discs

The arguments in Chapters 3 and 4 rely on the study of families of formal arcs on a nonsingular variety. A formal arc is a morphism from a one-dimensional formal disc. A formal arc through a closed point x of a nonsingular variety X factors through the completion $\hat{X}$ of X at x, a formal disc of dimension $\dim X$. We thus begin with the description of spaces of morphisms between formal discs. The reader is referred to Section 1.4 for notation and conventions.

Definition. A formal scheme $\hat{X}$ is called a *formal disc* if it is isomorphic to $\operatorname{Spf} k[[z_1, \dots, z_n]]$ for some $n \geq 0$. Its unique k-point is called the *origin* of $\hat{X}$. The integer n is referred to as the *dimension* of $\hat{X}$.

Lemma 2.1.1. *Let $\hat{X}$ and $\hat{Y}$ be formal discs of dimensions $n > 0$ and $m > 0$ with origins x and y. Then there is an isomorphism*

$$\underline{\mathrm{Hom}}(\hat{X}, \hat{Y}; x, y) \simeq \mathbb{A}^\infty$$

such that the tangent map to $\underline{\mathrm{End}}(T_{\hat{X},x}, T_{\hat{Y},y}) \simeq \mathbb{A}^{nm}$ corresponds to a linear projection.

Proof. Fix identifications $\hat{X} = \operatorname{Spf} k[[z_1, \dots, z_n]]$ and $\hat{Y} = \operatorname{Spf} k[[w_1, \dots, w_m]]$. Let B^+ be the set indexing non-constant coefficients of a power series in $k[[z_1, \dots, z_n]]$, and let

$$P = k[a_{i\beta} \mid 1 \leq i \leq m, \beta \in B^+]$$

be a polynomial algebra over k with free generators indexed by $\{1, \dots, m\} \times B^+$. There is a continuous homomorphism

$$\Phi : k[[w_1, \dots, w_m]] \to P[[z_1, \dots, z_n]]$$

such that the β-th coefficient of $\Phi(w_i)$ is $a_{i\beta}$. We claim that

$$\underline{\mathrm{Hom}}(\hat{X}, \hat{Y}; x, y) \simeq \operatorname{Spec} P$$

with the universal morphism $\underline{\mathrm{Hom}}(\hat{X}, \hat{Y}; x, y) \times \hat{X} \to \hat{Y}$ corresponding to

$$\Phi_* : \operatorname{Spf} P[[z_1, \ldots, z_n]] \to \operatorname{Spf} k[[w_1, \ldots, w_m]].$$

Indeed, by the universal property of $\underline{\mathrm{Hom}}(\hat{X}, \hat{Y}; x, y)$, Φ_* induces a morphism

$$\phi : \operatorname{Spec} P \to \underline{\mathrm{Hom}}(\hat{X}, \hat{Y}; x, y)$$

of sheaves. To construct its inverse, we can restrict to affine schemes. Given a k-algebra Q, and an element $f \in \underline{\mathrm{Hom}}(\hat{X}, \hat{Y}; x, y)(\operatorname{Spec} Q)$ defined by

$$f^* : Q[[w_1, \ldots, w_m]] \to Q[[z_1, \ldots, z_n]]$$

we let $\psi(f) : \operatorname{Spec} Q \to \operatorname{Spec} P$ be the morphism such that

$$\psi(f)^* : k[a_{i\beta} \mid 1 \leq i \leq m, \beta \in B^+] \to Q$$

sends $a_{i\beta}$ to the β-th coefficient of $f^* w_i$. This defines a morphism

$$\psi : \underline{\mathrm{Hom}}(\hat{X}, \hat{Y}; x, y) \to \operatorname{Spec} P$$

such that ϕ and ψ are mutual inverses.

Finally, denoting by $j \in B^+$ the coefficient of z_j, we have that the tangent map to

$$\underline{\mathrm{End}}(T_{\hat{X},x}, T_{\hat{Y},y}) \simeq \operatorname{Spec} k[a_{ij} \mid 1 \leq i \leq m, 1 \leq j \leq n]$$

is induced by the natural inclusion $k[a_{ij}] \to k[a_{i\beta}]$. □

Lemma 2.1.2. *Let $\hat{X}$ be a formal disc of dimension $n > 0$, with origin x. Then $\underline{\mathrm{Aut}}(\hat{X}, x)$ is a pro-algebraic affine group scheme. Furthermore, the action of origin-preserving automorphisms on the tangent space at the origin induces an exact sequence*

$$0 \to \mathcal{R}_u \underline{\mathrm{Aut}}(\hat{X}, x) \to \underline{\mathrm{Aut}}(\hat{X}, x) \to \mathsf{GL}(T_{\hat{X},x}) \to 1$$

with a pro-unipotent kernel.

Proof. Recall first that an endomorphism of a power series algebra is invertible if its linear part is (cf. [2]). It follows that there is a Cartesian diagram

$$\begin{array}{ccc} \underline{\mathrm{Aut}}(\hat{X}, x) & \longrightarrow & \underline{\mathrm{Hom}}(\hat{X}, \hat{X}, x, x) \\ \downarrow & & \downarrow \\ \mathsf{GL}(T_{\hat{X},x}) & \longrightarrow & \underline{\mathrm{End}}(T_{\hat{X},x}) \end{array}$$

so that, by Lemma 2.1.1,

$$\underline{\mathrm{Aut}}(\hat{X}, x) \simeq \mathsf{GL}_n \times_{\mathbb{A}^{n^2}} \mathbb{A}^\infty$$

an affine pro-algebraic group scheme, with an epimorphism onto $\mathsf{GL}(T_{\hat{X},x})$.

Let X_r be the r-th infinitesimal neighbourhood of x in $\hat{X}$, so that $\hat{X} = \varinjlim X_r$. Letting K_r be the kernel in

$$0 \to K_r \to \underline{\mathrm{Aut}}(\hat{X}, x) \to \underline{\mathrm{Aut}}(X_r, x) \to 1$$

we have short exact sequences

$$0 \to K_r/K_{r+1} \to K_1/K_{r+1} \to K_1/K_r \to 0$$

Using the notation of the proof of Lemma 2.1.1, applied to $\underline{\mathrm{Hom}}(\hat{X}, \hat{X}, x, x)$, we have $K_r/K_{r+1} \simeq \prod_{i,\beta} \mathbb{G}_a$ where the product is over $1 \leq i \leq n$ and $\beta \in B^+$ such that $\deg \beta = r$. It follows that K_1/K_r is unipotent, so that

$$\mathcal{R}_u \underline{\mathrm{Aut}}(\hat{X}, x) = \varprojlim K_1/K_r$$

is pro-unipotent. □

Morphisms from formal discs to schemes admit well-behaved parameter spaces as well. The following result is sufficiently general for our purposes.

Lemma 2.1.3. *Let $\hat{X}$ be a formal disc of dimension $n > 0$, with origin x. Let $Y \to \hat{X}$ be an adic morphism from a (not necessarily locally Noetherian) formal scheme, so that the fibre $Y_x = x \times_X Y$ is a scheme. Then $\prod(Y/\hat{X})$ is a scheme, and the morphism $e_x : \prod(Y/\hat{X}) \to Y_x$ given by evaluation at x is affine. If $Y \to \hat{X}$ is of pro-finite type, then so is e_x.*

Proof. If $Y_x = \bigcup U_i$ is a cover by open subschemes, then $\prod(Y/\hat{X}) = \bigcup e_x^{-1}(U_i)$ is a cover by open subfunctors, and it is enough to check that each $e_x^{-1}(U_i)$ is a scheme. We can thus assume Y is affine. Identifying $\hat{X}$ with $\mathrm{Spf}\, k[[z_1, \ldots, z_n]]$, we have $Y = \mathrm{Spf}\, R$ where R is a topological algebra over $k[[z_1, \ldots, z_n]]$, with topology induced by $(z_1, \ldots, z_n)R$. Consider a presentation $R = S/I$ where S is a polynomial algebra over $k[[z_1, \ldots, z_n]]$, possibly of infinite type, with topology induced by $(z_1, \ldots, z_n)S$. Let A be the set indexing free generators y_α of S, and let B be the set indexing coefficients of a power series in $k[[z_1, \ldots, z_n]]$. Let

$$P = k[a_{\alpha\beta} \mid \alpha \in A, \beta \in B]$$

be a polynomial algebra over k with free generators $a_{\alpha\beta}$ indexed by $A \times B$. There is a homomorphism

$$\Phi : S \to P[[z_1, \ldots, z_n]]$$

such that for $\alpha \in A$, $\beta \in B$, the β-th coefficient of $\Phi(y_\alpha)$ is $a_{\alpha\beta}$. Let

$$\Psi : I \times B \to P$$

be the map sending (s, β) to the β-th coefficient of the power series $\Phi(s)$. Finally let $J \subset P$ be the ideal generated by the image $\Psi(I \times B)$. Then Φ factors through

$$\bar{\Phi} : R = S/I \to (P/J)[[z_1, \ldots, z_n]].$$

We claim that

$$\prod(Y/\hat{X}) \simeq \operatorname{Spec} P/J$$

with the universal morphism $\prod(Y/\hat{X}) \times \hat{X} \to Y$ corresponding to

$$\bar{\Phi}_* : \operatorname{Spf}(P/J)[[z_1, \ldots, z_n]] \to \operatorname{Spf} R.$$

Indeed, by the universal property of $\prod(Y/\hat{X})$, $\bar{\Phi}_*$ induces a morphism

$$\phi : \operatorname{Spec}(P/J) \to \prod(Y/\hat{X})$$

of sheaves. To construct its inverse, we can restrict to affine schemes. Given a k-algebra Q, and an element $f \in \prod(Y/\hat{X})(\operatorname{Spec} Q)$ defined by

$$f^* : Q \otimes (S/I) \to Q[[z_1, \ldots, z_n]]$$

we let $\psi(f) : \operatorname{Spec} Q \to \operatorname{Spec} P/J$ be the unique morphism such that

$$\psi(f)^* : k[a_{\alpha\beta} \mid \alpha \in A, \beta \in B]/J \to Q$$

sends $\bar{a}_{\alpha\beta}$ to the β-th coefficient of $f^* y_\alpha$. Note that for each $(s, \beta) \in I \times B$, $f^* s = 0$ so that is β-th coefficient is zero, so that $\psi(f)^*$ takes the generators of J to zero. Hence $\psi(f)^*$ is well-defined. This defines a morphism

$$\psi : \prod(Y/\hat{X}) \to \operatorname{Spec}(P/J)$$

and that ϕ and ψ are mutual inverses.

Finally, if R is countably generated over $k[[z_1, \ldots, z_n]]$, then A and $A \times B$ are countable, so that P/J is countably generated over k. Hence $Y \to \hat{X}$ being pro-finite type implies the same for $\prod(Y/\hat{X}) \to Y_x$. □

Naturally, one would like to have a relative notion of a formal disc. For our purposes, the following is the most convenient (note that in the definition below, dimensions of the fibres are locally constant, i.e. $n_i = n_j$ unless S_i, S_j are disjoint).

Definition. A morphism $\hat{X} \to S$ of locally Noetherian formal schemes, together with a section $S \to \hat{X}$, is called a *bundle of formal discs* if there is a Zariski open cover $S = \bigcup S_i$ such that $S_i \times_S \hat{X} \simeq S_i \times \operatorname{Spf} k[[z_1, \ldots, z_{n_i}]]$ over S_i for some $n_i \geq 0$, and the section $S_i \to S_i \times_S \hat{X}$ corresponds to the pullback of the origin $\operatorname{Spec} k \to \operatorname{Spf} k[[z_1, \ldots, z_{n_i}]]$.

Lemma 2.1.4. *Let $X \to S$ be a smooth morphism of locally Noetherian schemes. Then $(X/S)^\sharp$ together with the diagonal section is a bundle of formal discs over X. If, moreover, S is reduced and $\sigma \in X(S)$ is a section, then $\sigma^*(X/S)^\sharp$ is naturally isomorphic to the completion of X along $\sigma(S)$.*

Proof. Since the question is local, we can assume there is an étale morphism $f : X \to \mathbb{A}^n_S$ and $f \circ \sigma : S \to \mathbb{A}^n_S$ is the zero-section. Since f is étale, it induces an isomorphism $(X/S)^\sharp \simeq f^*(\mathbb{A}^n_S/S)^\sharp$ and an isomorphism of the completion of X along $\sigma(S)$ onto the completion of $\mathbb{A}^n_S$ along $(f \circ \sigma)(S)$. We can thus replace X with $\mathbb{A}^n_S$ and σ with the zero-section. Then $(X/S)^\sharp$ is identified with $S \times (\mathbb{A}^n)^\sharp$, and the completion of X along $\sigma(S)$ is the product of S with the completion of $\mathbb{A}^n$ at 0. We are thus finally reduced to the case $S = \operatorname{Spec} k$, $X = \mathbb{A}^n$, $\sigma = 0$, where the result follows by straightforward inspection. □

The condition that $(X/S)^\sharp$ be a bundle of formal discs is a variant of what in [1] and [7] is caled *lissité differentielle* (it coincides with the latter at least for finite type morphisms of locally Noetherian schemes). We thus make the following definition.

Definition. A morphism $f : X \to S$ of locally Noetherian formal schemes is *differentially smooth (with differential dimension n)* if $(X/S)^\sharp$, together with the diagonal section, is a bundle of formal discs (of dimension n) over X.

By Lemma 2.1.4, a smooth morphism of locally Noetherian schemes is differentially smooth. Conversely, one can show that a flat, finite type differentially smooth morphism of locally Noetherian schemes is smooth [1], although we will not need this fact. At the same time, a bundle $\hat{X} \to S$ of formal discs is differentially smooth since $(\hat{X}/S)^\sharp = (\hat{X}/S)^2$. Sections of a differentially smooth morphism to a formal disc are parametrized by a particularly simple object.

Lemma 2.1.5. *Let $\hat{X}$ be a formal disc with origin x, Y a locally Noetherian formal scheme, and $Y \to \hat{X}$ a differentially smooth morphism with positive differential dimension. Then $\prod(Y/\hat{X}) \to Y_x$ is a Zariski-locally trivial $\mathbb{A}^\infty$-bundle.*

Proof. Note that for any scheme $T \to \hat{X}$, and a morphism $f : T \times \hat{X} \to T \times_{\hat{X}} Y$ over T, the induced morphism $\langle (f \circ x) \times \hat{X}, f \rangle : T \times \hat{X} \to T \times_{\hat{X}} (Y/X)^2$ factors through $T \times_{\hat{X}} (Y/X)^\sharp$. We thus have an isomorphism

$$\begin{aligned} \prod(Y/\hat{X}) &\simeq \underline{\mathrm{Hom}}_Y(Y \times \hat{X}, (Y/\hat{X})^\sharp; Y \times x, \Delta_Y) \\ &\simeq \underline{\mathrm{Hom}}_{Y_x}(Y_x \times \hat{X}, Y_x \times_Y (Y/\hat{X})^\sharp; Y_x \times x, Y_x \times_Y \Delta_Y) \end{aligned}$$

where $\Delta_Y : Y \to (Y/\hat{X})^\sharp$ is the diagonal morphism. By hypothesis, $Y_x \times_Y (Y/\hat{X})^\sharp$ is a bundle of positive-dimensional formal discs over Y_x. Since the problem is local, we can assume that the bundle is trivial, so that

$$\prod(Y/\hat{X}) \simeq Y_x \times \underline{\mathrm{Hom}}(\hat{X}, \hat{Y}; x, y) = Y_x \times \underline{\mathrm{Hom}}(\hat{X}, \hat{Y}; x, y)$$

where $\hat{Y}$ is a formal disc of positive dimension, with origin y. Then, by Lemma 2.1.1, we have that $\prod(Y/\hat{X}) \simeq Y_x \times \mathbb{A}^\infty$ as desired. □

2.1.2 Space of unparametrised pointed arcs

Let $\hat{\mathbb{P}}^1$ denote the completion of $\mathbb{P}^1$ at 0. It is a one-dimensional formal disc, so that in particular we have the pro-algebraic group scheme $\underline{\mathrm{Aut}}(\hat{\mathbb{P}}^1, 0)$. Given a differentially smooth morphism $X \to S$ with positive differential dimension of locally Noetherian formal schemes, we define

$$\underline{\mathrm{Imm}}_S(S \times \hat{\mathbb{P}}^1, X) \subset \underline{\mathrm{Hom}}_S(S \times \hat{\mathbb{P}}^1, X)$$

to be the subsheaf whose value at T/S consists of morphisms $\gamma : T \times \hat{\mathbb{P}}^1 \to X_T$ inducing a nowhere-vanishing map $\gamma' : \mathcal{O}_T \otimes T_{\mathbb{P}^1,0} \to \gamma|^*_{T\times 0} T_{X_T/T}$. That is, $\underline{\mathrm{Imm}}_S(S \times \hat{\mathbb{P}}^1, X)$ is the space of families of parametrised formal arcs in X/S, unramified at the origin 0. The map sending γ to the image of γ' induces a morphism $\underline{\mathrm{Imm}}_S(S \times \mathbb{P}^1, X) \to \mathbb{P}T_{X/S}$ into the projectivised tangent bundle. The automorphism group $\underline{\mathrm{Aut}}(\hat{\mathbb{P}}^1, 0)$ has a natural right action on $\underline{\mathrm{Imm}}_S(S \times \hat{\mathbb{P}}^1, X)$, reparametrising the arc while preserving the tangent direction at 0, and thus compatible with the morphism to $\mathbb{P}T_{X/S}$. We define the space of *unparametrised pointed arcs* in X/S to be the quotient

$$\mathrm{Arc}_{X/S} = \underline{\mathrm{Imm}}_S(S \times \hat{\mathbb{P}}^1, X)/\underline{\mathrm{Aut}}(\hat{\mathbb{P}}^1, 0),$$

in the category of sheaves over $\mathbb{P}T_{X/S}$.

Lemma 2.1.6. *Let $X \to S$ be a differentially smooth morphism with positive differential dimension of locally Noetherian formal schemes. Then $\mathrm{Arc}_{X/S} \to \mathbb{P}T_{X/S}$ is a Zariski-locally trivial $\mathbb{A}^\infty$-bundle.*

Proof. Since the problem is local, we can assume by differential smoothness that there is a positive-dimensional formal disc $\hat{X}$ with origin x, such that $(X/S)^\sharp \simeq X \times \hat{X}$ inducing an isomorphism $\mathbb{P}T_{X/S} \simeq X \times \mathbb{P}T_{\hat{X}}$. Arguing as in the proof Lemma 2.1.5 we then have an isomorphism

$$\underline{\mathrm{Imm}}_S(S \times \hat{\mathbb{P}}^1, X) \simeq X \times \underline{\mathrm{Imm}}(\hat{\mathbb{P}}^1, \hat{X})$$

equivariant under the action of $\underline{\mathrm{Aut}}(\hat{\mathbb{P}}^1, 0)$, and compatible with the morphisms to $\mathbb{P}T_{X/S}$ and $\mathbb{P}T_{\hat{X}}$. It will thus be enough to prove that

$$\mathrm{Arc}_{\hat{X}} \to \mathbb{P}T_{\hat{X}}$$

is a Zariski-locally trivial $\mathbb{A}^\infty$-bundle.

Identifying $\hat{X}$ with the completion of $\mathbb{G}_a \times \cdots \times \mathbb{G}_a$ at identity, we turn it into a formal group scheme. The action of $\hat{X}$ on itself trivialises $\mathrm{Arc}_{\hat{X}}$ and $\mathbb{P}T_{\hat{X}}$ so that

$$\mathrm{Arc}_{\hat{X}} \simeq \hat{X} \times \mathrm{Arc}_{\hat{X},x}, \quad \mathbb{P}T_{\hat{X}} \simeq \hat{X} \times \mathbb{P}T_{\hat{X},x}$$

compatibly with the morphism $\mathrm{Arc}_{\hat{X}} \to \mathbb{P}T_{\hat{X}}$. It is thus enough to check that $\mathrm{Arc}_{\hat{X},x} \to \mathbb{P}T_{\hat{X},x}$ is a Zariski-locally trivial $\mathbb{A}^\infty$-bundle.

Now, identifying $\hat{X} = \mathrm{Spf}\, k[[z_1, \ldots, z_n]]$ so that $\mathbb{P}T_{\hat{X},x} = \mathrm{Proj}\, k[z_1, \ldots, z_n]$, we have that $\mathbb{P}T_{\hat{X},x}$ is covered by open affines $D(z_i)$, and $\mathrm{Arc}_{\hat{X},x}$ is covered by open subfunctors

$U_i = D(z_i) \times_{\mathbb{P}T_{\hat{X},x}} \mathrm{Arc}_{\hat{X},x}$. It will be enough to show that $U_i \simeq D(z_i) \times \mathbb{A}^\infty$, and by permuting the variables we only need to consider $i = n$. Let $\hat{D} = \mathrm{Spf}\, k[[z_1, \dots, z_{n-1}]]$ and identify $\hat{\mathbb{P}}^1 = \mathrm{Spf}\, k[[z_n]]$, inducing isomorphism $\hat{D} \times \hat{\mathbb{P}}^1 \simeq \hat{X}$. There is a natural isomorphism

$$\underline{\mathrm{Hom}}(\hat{\mathbb{P}}^1, \hat{D}) \times \underline{\mathrm{Hom}}(\hat{\mathbb{P}}^1, \hat{\mathbb{P}}^1) \simeq \underline{\mathrm{Hom}}(\hat{\mathbb{P}}^1, \hat{D} \times \hat{\mathbb{P}}^1)$$

which, pulled back to the k-point $\mathrm{id}_{\hat{\mathbb{P}}^1}$ of $\underline{\mathrm{Hom}}(\hat{\mathbb{P}}^1, \hat{\mathbb{P}}^1)$, and restricted over the origin $\bar{x} \in \hat{D}(k)$, gives a monomorphism

$$\tilde{\phi} : \underline{\mathrm{Hom}}(\hat{\mathbb{P}}^1, \hat{D}; 0, \bar{x}) \to D(z_n) \times_{\mathbb{P}T_{\hat{X},x}} \underline{\mathrm{Imm}}(\hat{\mathbb{P}}^1, \hat{X}).$$

Composing with projection to the quotient by $\underline{\mathrm{Aut}}(\hat{\mathbb{P}}^1, 0)$, we obtain

$$\phi : \underline{\mathrm{Hom}}(\hat{\mathbb{P}}^1, \hat{D}; 0, \bar{x}) \to U_n$$

We first check that ϕ is a monomorphism. Consider two T-points of $\underline{\mathrm{Hom}}(\hat{\mathbb{P}}^1, \hat{D}; 0, \bar{x})$, corresponding to morphisms $b, c : T \times \hat{\mathbb{P}}^1 \to T \times \hat{D}$. If $\phi(b) = \phi(c)$, then there is a covering $T' \to T$ and an automorphism $g : T' \times \hat{\mathbb{P}}^1 \to T' \times \hat{\mathbb{P}}^1$ such that $\tilde{\phi}(b)_{T'} \circ g = \tilde{\phi}(c)_{T'}$. But then

$$g = \mathrm{pr}_{T' \times \hat{\mathbb{P}}^1} \circ \tilde{\phi}(b)_{T'} \circ g = \mathrm{pr}_{T' \times \hat{\mathbb{P}}^1} \circ \tilde{\phi}(c)_{T'} = \mathrm{id}_{T' \times \hat{\mathbb{P}}^1}$$

so that $\tilde{\phi}(b)_{T'} = \tilde{\phi}(c)_{T'}$, thus $b_{T'} = c_{T'}$ and $b = c$ by descent.

To check that ϕ is an epimorphism, consider a T-point f of U_n, represented by a covering $T' \to T$ and a morphism $f' : T' \times \hat{\mathbb{P}}^1 \to T' \times \hat{D} \times \hat{\mathbb{P}}^1$ satisfying the condition that

$$\mathrm{pr}_1^* f' = \mathrm{pr}_2^* f' \circ g : T' \times_T T' \times \hat{\mathbb{P}}^1 \to T' \times_T T' \times \hat{D} \times \hat{\mathbb{P}}^1$$

for some automorphism $g : T' \times_T T' \times \hat{\mathbb{P}}^1 \to T' \times_T T' \times \hat{\mathbb{P}}^1$, where $\mathrm{pr}_1, \mathrm{pr}_2 : T' \times_T T' \to T'$ are the two projections. Furthermore, since $\mathrm{pr}_{T' \times \hat{\mathbb{P}}^1} \circ f' : T' \times \hat{\mathbb{P}}^1 \to T' \times \hat{\mathbb{P}}^1$ is unramified at the zero section, it is an automorphism. We then let

$$c' = \mathrm{pr}_{T' \times \hat{D}} \circ f' \circ (\mathrm{pr}_{T' \times \hat{\mathbb{P}}^1} \circ f')^{-1} : T' \times \hat{\mathbb{P}}^1 \to T' \times \hat{D},$$

defining a T'-point of $\underline{\mathrm{Hom}}(\hat{\mathbb{P}}^1, \hat{D}; 0, \bar{x})$. It is immediate that $\tilde{\phi}(c') = f' \circ (\mathrm{pr}_{T' \times \hat{\mathbb{P}}^1} \circ f')^{-1}$, so that $\phi(c') = T' \times_T f$. Pulling c' back by $\mathrm{pr}_1, \mathrm{pr}_2 : T' \times_T T' \rightrightarrows T'$, we have

$$\begin{aligned}
\mathrm{pr}_1^* c' &= \mathrm{pr}_{T' \times_T T' \times \hat{D}} \circ \mathrm{pr}_1^* f' \circ (\mathrm{pr}_{T' \times_T T' \times \hat{\mathbb{P}}^1} \circ \mathrm{pr}_1^* f')^{-1} \\
&= \mathrm{pr}_{T' \times_T T' \times \hat{D}} \circ \mathrm{pr}_2^* f' \circ g \circ g^{-1} \circ (\mathrm{pr}_{T' \times_T T' \times \hat{\mathbb{P}}^1} \circ \mathrm{pr}_2^* f')^{-1} \\
&= \mathrm{pr}_2^* c'
\end{aligned}$$

so that c' descends to a T-point corresponding to $c : T \times \hat{\mathbb{P}}^1 \to T \times \hat{D}$.

We thus have an isomorphism $U_n \simeq \underline{\mathrm{Hom}}(\hat{\mathbb{P}}^1, \hat{D}; 0, \bar{x})$. By Lemma 2.1.1, $U_n \simeq \mathbb{A}^\infty$ and the morphism to $D(z_i) \simeq \mathbb{A}^n$ is a linear projection, so that $U_n \simeq D(z_i) \times \mathbb{A}^\infty$ as a $D(z_i)$-scheme. □

Corollary 2.1.7. *In the setting of Lemma 2.1.6,* $\underline{\mathrm{Imm}}_S(S \times \hat{\mathbb{P}}^1, X) \to \mathrm{Arc}_{X/S}$ *is a Zariski-locally trivial* $\underline{\mathrm{Aut}}(\hat{\mathbb{P}}^1, 0)$*-torsor.*

Proof. A morphism from $\hat{\mathbb{P}}^1$ to X is a formal closed immersion if it is unramified at 0, so that $\underline{\mathrm{Imm}}_S(S \times \hat{\mathbb{P}}^1, X)$ is a pseudo-torsor for $\underline{\mathrm{Aut}}(\hat{\mathbb{P}}^1, 0)$. Zariski-local sections are provided by the morphism $\tilde{\phi}$ of the former proof. □

Remaining in the previous context, we introduce the pointwise space of families of arcs determined by their tangent directions. Considering the fibration $\mathrm{Arc}_{X/S} \to \mathbb{P}T_{X/S}$, let

$$\mathrm{Hilb}\,\mathbb{P}T_{X/S} \leftarrow \mathcal{U} \to \mathbb{P}T_{X/S}$$

be the universal family over the relative Hilbert scheme[1] of the projectivised tangent bundle. We then define $\mathrm{ArcHilb}_{X/S}$ to be the sheaf of sections

$$\mathrm{ArcHilb}_{X/S} = \prod \left(\mathcal{U} \times_{\mathbb{P}T_{X/S}} \mathrm{Arc}_{X/S} \;\; / \; \mathcal{U} \; / \;\; \mathrm{Hilb}\,\mathbb{P}T_{X/S} \right)$$

so that its value at $f : T \to \mathrm{Hilb}\,\mathbb{P}T_{X/S}$ is the set of sections of $T \times_X \mathrm{Arc}_{X/S}$ over the subscheme of $T \times_X \mathbb{P}T_{X/S}$ defining f. In particular, if X is differentially smooth over k, then k-points of $\mathrm{ArcHilb}_X$ correspond to sections of Arc_X over closed subschemes in fibres of $\mathbb{P}T_X$.

Lemma 2.1.8. *Let* $X \to S$ *be a differentially smooth morphism with positive differential dimension of locally Noetherian formal schemes. Then* $\mathrm{ArcHilb}_{X/S} \to \mathrm{Hilb}\,\mathbb{P}T_{X/S}$ *is a pro-finite type affine adic morphism from a formal scheme.*

Proof. By Lemma 2.1.6, $\mathcal{U} \times_{\mathbb{P}T_{X/S}} \mathrm{Arc}_{X/S} \to \mathcal{U}$ is a Zariski-locally trivial $\mathbb{A}^\infty$-bundle. On the other hand, $\mathcal{U} \to \mathrm{Hilb}\,\mathbb{P}T_{X/S}$ is a projective adic morphism of locally Noetherian formal schemes. Identify

$$\mathcal{U} \times_{\mathbb{P}T_{X/S}} \mathrm{Arc}_{X/S} = \mathrm{Spec}_{\mathcal{U}}\, \mathcal{A}$$

where $\mathcal{A}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathcal{U}}$-algebras locally isomorphic to the symmetric algebra of a direct sum of countably many copies of $\mathcal{O}_{\mathcal{U}}$. Then

$$\mathrm{ArcHilb}_{X/S}(T \to \mathrm{Hilb}\,\mathbb{P}T_{X/S}) = \mathrm{Hom}_{\mathcal{U}_T-\mathrm{alg}}(\mathcal{A}_T, \mathcal{O}_{\mathcal{U}_T})$$

Suppose one can show that the functor $\underline{\mathrm{Hom}}_{\mathcal{U}}(\mathcal{A}, \mathcal{O}_{\mathcal{U}})$ sending $T \to \mathrm{Hilb}\,\mathbb{P}T_{X/S}$ to the set of $\mathcal{U}_T$-module morphisms $\mathcal{A}_T \to \mathcal{O}_{\mathcal{U}_T}$ is representable by a formal scheme, affine, adic and of pro-finite type over $\mathrm{Hilb}\,\mathbb{P}T_{X/S}$. Then $\mathrm{ArcHilb}_{X/S}$ is naturally a closed subfunctor of $\underline{\mathrm{Hom}}_{\mathcal{U}}(\mathcal{A}, \mathcal{O}_{\mathcal{U}})$, hence satisfies the same properties, as desired. To simplify notation, the claim that remains to be proven is reformulated as the following Lemma (replacing $\mathrm{Hilb}\,\mathbb{P}T_{X/S}$ with $S, \mathcal{U}$ with X and $\mathcal{A}$ with $\mathcal{E}$). □

Lemma 2.1.9. *Suppose* $X \to S$ *is a flat projective adic morphism of locally Noetherian formal schemes, and* $\mathcal{E}$ *a locally countably generated quasi-coherent sheaf on* X*. Then* $\prod(\mathcal{E}^\vee / X / S)$ *is representable by a formal scheme, affine, adic and of pro-finite type over* S*.*

[1] Note that, by differential smoothness, $\mathbb{P}T_{X/S}$ is locally trivial, so that $\mathrm{Hilb}\,\mathbb{P}T_{X/S}$ is locally just a pullback of $\mathrm{Hilb}\,\mathbb{P}^{n-1}$, where n is the differential dimension of X/S.

Proof. Since the problem is local on the base, we can assume S is Noetherian. Then X is quasi-compact, and we can filter $\mathcal{E}$ by *coherent* subsheaves $\mathcal{E}_1 \subset \mathcal{E}_2 \subset \dots$ so that $\bigcup \mathcal{E}_i = \mathcal{E}$. By [6, Thm. 7.7.6], there is a coherent sheaf[2] $\mathcal{Q}_i$ on S such that

$$\prod(\mathcal{E}_i^\vee / X / S) = \operatorname{Spec}_S \operatorname{Sym}_S \mathcal{Q}_i$$

where the relative Spec construction over the formal scheme S is understood as locally taking formal spectra with the adic topology induced from $\mathcal{O}_S$. Now, $\mathcal{E}^\vee = \varprojlim \mathcal{E}_i^\vee$, so that

$$\prod(\mathcal{E}/X/S) = \varprojlim \operatorname{Spec}_S \operatorname{Sym}_S \mathcal{Q}_i = \operatorname{Spec}_S \varinjlim \operatorname{Sym}_S \mathcal{Q}_i$$

This is a formal scheme, affine, adic and of pro-finite type over S. $\square$

2.1.3 Families of arcs on a formal disc

We would now like to describe families of pointed arcs on a formal disc, up to manageable equivalence. Suppose $\hat{X}$ is a formal disc of positive dimension, with origin x. The action of $\underline{\operatorname{Aut}}(\hat{X}, x)$ on $\hat{X}$ lifts to $\operatorname{ArcHilb}_{\hat{X}}$, so that we can consider $\underline{\operatorname{Aut}}(\hat{X}, x)$ acting on the space of sections of $\operatorname{ArcHilb}_{\hat{X}}$ over $\hat{X}$. The morphism $\operatorname{ArcHilb}_{\hat{X}} \to \operatorname{Hilb} \mathbb{P}T_{\hat{X}}$ induces a morphism on the spaces of sections and, upon evaluation at x, a morphism

$$\prod \left(\operatorname{ArcHilb}_{\hat{X}} / \hat{X}\right) \to \operatorname{Hilb} \mathbb{P}T_{\hat{X},x}.$$

Lemma 2.1.10. *Let $\hat{X}$ be a formal disc of positive dimension, with origin x. Then $\prod \left(\operatorname{ArcHilb}_{\hat{X}} / \hat{X}\right) \to \operatorname{Hilb} \mathbb{P}T_{\hat{X},x}$ is a pro-finite type affine morphism of schemes.*

Proof. By Lemma 2.1.8, $\operatorname{ArcHilb}_{\hat{X}} \to \operatorname{Hilb} \mathbb{P}T_{\hat{X}}$ is a pro-finite type affine adic morphism from a formal scheme, and thus so is its composite with projection to $\hat{X}$. We then have by Lemma 2.1.3 that $\prod(\operatorname{ArcHilb}_{\hat{X}} / \hat{X}) \to \operatorname{ArcHilb}_{\hat{X},x}$ is a pro-finite type affine morphism of schemes. Again by Lemma 2.1.8, the projection $\operatorname{ArcHilb}_{\hat{X},x} \to \operatorname{Hilb} \mathbb{P}T_{\hat{X},x}$ is a pro-finite type affine morphism of schemes. Hence the composite $\prod(\operatorname{ArcHilb}_{\hat{X}} / \hat{X}) \to \operatorname{Hilb} \mathbb{P}T_{\hat{X},x}$ is a pro-finite type affine morphism of schemes. $\square$

The projection to $\operatorname{Hilb} \mathbb{P}T_{X,x}$ is preserved by the action of the pro-unipotent subgroup $\mathcal{R}_u\underline{\operatorname{Aut}}(\hat{X}, x)$ of automorphisms inducing identity on $T_{\hat{X},x}$.

Proposition 2.1.11. *Let $\hat{X}$ be a formal disc of positive dimension, with origin x. Then the orbits of k-points under the $\mathcal{R}_u\underline{\operatorname{Aut}}(\hat{X}, x)$-action on $\prod(\operatorname{ArcHilb}_{\hat{X}} / \hat{X})$ are closed.*

[2] While the result in [6] is stated over a base scheme, it extends easily to the formal case. Letting $\mathcal{I}$ be an ideal of definition of S, we have suitable coherent sheaves $\mathcal{Q}_i^{(r)}$ on $S_r = \operatorname{Spec}_S \mathcal{O}_S / \mathcal{I}^{r+1}$. By the universal property of the functor of sections, there are isomorphisms $\mathcal{Q}_i^{(r+1)}|_{S_r} \to \mathcal{Q}_i^{(r)}$, so that $\mathcal{Q}_i = \varprojlim \mathcal{Q}_i^{(r)}$ is a coherent sheaf on S. Since its pullback to any scheme factors through one of $\mathcal{Q}_i^{(r)}$, it has the desired property.

Proof. As we can restrict to fibres over k-points of $\operatorname{Hilb}\mathbb{P}T_{\hat{X},x}$, the Proposition is an immediate corollary of Lemma 2.1.10 and the following pro-algebraic version of Borel's fixed point theorem. $\square$

Lemma 2.1.12. *Suppose a pro-unipotent group scheme acts on an affine pro-algebraic scheme. Then the orbits of k-points are closed.*

Proof. Let U be a pro-unipotent group scheme acting on $\operatorname{Spec} A$. Since U is a limit of extensions by $\mathbb{G}_a$, we have $U \simeq \mathbb{A}^\infty$ as a scheme, so that $k[U] \simeq k[u_1, u_2, \dots]$ is a polynomial algebra in countably infinitely many variables. Let

$$\rho : A \to A[u_1, u_2, \dots]$$

be the action morphism.

We claim that for any finite collection $a_1, \dots, a_n \in A$ there is a finitely generated subalgebra $A_0 \subset A$ containing $a_1, \dots, a_n$ such that ρ restricts to an action morphism $A_0 \to A_0[u_1, u_2, \dots]$. Indeed, let A_0 be the subalgebra generated by the coefficients of the polynomials $\rho(a_1), \dots, \rho(a_n) \in A[u_1, u_2, \dots]$ (these include in particular $a_1, \dots, a_n$). We have a commutatitive diagram

$$\begin{tikzcd} k[a_1, \dots, a_n] \arrow[r, "\rho"] \arrow[d, "\rho"'] & A_0[v_1, v_2, \dots] \arrow[d, "\rho\otimes 1"] \\ A_0[u_1, u_2, \dots] \arrow[r, "1\otimes\mu"] & A[u_1, u_2, \dots][v_1, v_2, \dots] \end{tikzcd}$$

where $\mu : k[u_1, u_2, \dots] \to k[u_1, u_2, \dots][v_1, v_2, \dots]$ is the multiplication morphism in U. The bottom morphism factors through $A_0[u_1, u_2, \dots][v_1, v_2, \dots]$, so that for each $\rho(a_i)$, a polynomial in the $v_1, v_2, \dots$, we have that ρ sends its coefficients into $A_0[u_1, u_2, \dots]$. Since these coefficients generate A_0, the claim follows.

Now, we can filter A by a sequence of finitely generated algebras $A_1 \subset A_2 \subset \dots$, so that $A = \bigcup A_i$, and the action of U descends to each A_i. Recalling that U is a limit of extensions by $\mathbb{G}_a$, we can assume that for each $n > 0$,

$$U \simeq \operatorname{Spec} k[u_1, u_2, \dots] \to \operatorname{Spec} k[u_1, \dots, u_n]$$

is naturally an epimorphism onto a unipotent algebraic group. For each i, choose the largest $n > 0$ such that u_n appears in the polynomials $\rho(A_i) \subset A_i[u_1, u_2, \dots]$, and set $U_i = \operatorname{Spec} k[u_1, \dots, u_n]$. Then U_i is naturally a unipotent algebraic quotient of U, and the action of U on $\operatorname{Spec} A_i$ factors through U_i. Since $\operatorname{Spec} A = \varprojlim \operatorname{Spec} A_i$, the action of U on $\operatorname{Spec} A$ factors through $\varprojlim U_i$. We assume without loss of generality that $U = \varprojlim U_i$.

Fix an orbit map $\varphi : U \to \operatorname{Spec} A$, given by pullback of the action morphism $U \times \operatorname{Spec} A \to \operatorname{Spec} A$ to a k-point. For each i consider the commutative diagram

$$\begin{tikzcd} U \arrow[r, "\varphi"] \arrow[d, "\theta_i"'] & \operatorname{Spec} A \arrow[d, "\pi_i"'] \\ U_i \arrow[r, "\varphi_i"] & \operatorname{Spec} A_i \end{tikzcd}$$

where π_i and θ_i are the natural projections, and φ_i the factorisation of $\pi_i \circ \varphi$. By Borel's Fixed Point Theorem, $D_i = \varphi_i(U_i)$ is closed in $\operatorname{Spec} A_i$. Let $D = \bigcap \pi_i^{-1} D_i$, a closed subset of $\operatorname{Spec} A$. We then have that φ factors through D, and it remains to check surjectivity. Given $x \in D$, consider the pullback

$$\begin{tikzcd} V_i \arrow[r] \arrow[d] & U \arrow[d, "\varphi_i \circ \theta_i"'] \\ \operatorname{Spec} \kappa(x) \arrow[r, "\pi_i \circ x"] & \operatorname{Spec} A_i \end{tikzcd}$$

defining a nonempty closed subscheme $V_i \subset U \otimes \kappa(x)$. Noting that $V_1 \supset V_2 \supset \dots$, we let $V = \bigcap V_i$, a closed subset of $U \otimes \kappa(x)$. Since $U \otimes \kappa(x)$ is affine, and thus quasi-compact, V is nonempty. Hence its image in U is nonempty, so that $x \in \varphi(U)$. $\square$

We remark that in the setting of Proposition 2.1.11 the U_i and $\operatorname{Spec} A_i$ of the former proof can be explicitly constructed by filtering $\hat{X}$ and $\hat{\mathbb{P}}^1$ by infinitesmial neighbourhoods of their origins (cf. the proof of Lemma 2.1.2): this induces a natural presentation of the action of $\mathcal{R}_u \underline{\mathrm{Aut}}(\hat{X}, x)$ on $\prod(\mathrm{ArcHilb}_{\hat{X}} / \hat{X})$ as a limit of an inverse system of actions of unipotent algebraic groups on affine algebraic schemes.

2.2 Rational curves

2.2.1 Space of unparametrised pointed curves

We now turn to the study rational curves on a family of projective varieties. Let $X \to S$ be a smooth morphism of Noetherian schemes. The space of morphisms $\underline{\mathrm{Hom}}_S(\mathbb{P}^1_S, X)$ is a locally Noetherian S-scheme. Relevant classes of its geometric points are defined as follows.

Definition. Let $\operatorname{Spec} L \to \underline{\mathrm{Hom}}_S(\mathbb{P}^1_S, X)$ be a geometric point, corresponding to a morphism $f : \mathbb{P}^1_L \to X$. Then f is called *free* (resp. *minimal*[3]) if $f^* T_{X/S}$ is a direct sum of invertible sheaves of non-negative degree (resp. a direct sum of $\mathcal{O}(2)$ and invertible sheaves of degree 0 or 1).

There is an open subscheme

$$\underline{\mathrm{Hom}}_{S,\mathrm{bir}}(\mathbb{P}^1_S, X) \subset \underline{\mathrm{Hom}}_S(\mathbb{P}^1_S, X)$$

whose geometric points correspond to morphisms $\mathbb{P}^1_L \to X$ birational to their images. We let $\underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1_S, X)$ be its normalisation. The right action of $\underline{\mathrm{Aut}}(\mathbb{P}^1)$ on $\underline{\mathrm{Hom}}_S(\mathbb{P}^1_S, X)$ restricts to $\underline{\mathrm{Hom}}_{S,\mathrm{bir}}(\mathbb{P}^1_S, X)$ and lifts to $\underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1_S, X)$. The quotient morphisms

$$\underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1_S, X) \to \underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1_S, X) / \underline{\mathrm{Aut}}(\mathbb{P}^1, 0)$$

[3] Called 'standard' in [10].

and

$$\underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1_S, X) \to \underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1_S, X)/\underline{\mathrm{Aut}}(\mathbb{P}^1)$$

are principal bundles [14].

Definition. Given a smooth morphism $X \to S$ of Noetherian schemes, a *family of rational curves* on X/S is a closed $\underline{\mathrm{Aut}}(\mathbb{P}^1)$-invariant subscheme of $\underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}$.

Note that the definition does not assume irreducibility. However, most families we consider will be in fact irreducible components of $\underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1, X)$. Given a family $\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1_S, X)$, we introduce the following notation for the quotients:

$$\mathcal{M}_0 = \mathcal{M}/\underline{\mathrm{Aut}}(\mathbb{P}^1)$$

$$\mathcal{M}_1 = \mathcal{M}/\underline{\mathrm{Aut}}(\mathbb{P}^1, 0)$$

where $\mathcal{M} \to \mathcal{M}_0$, resp. $\mathcal{M} \to \mathcal{M}_1$, is an $\underline{\mathrm{Aut}}(\mathbb{P}^1)$-principal bundle, resp. an $\underline{\mathrm{Aut}}(\mathbb{P}^1, 0)$-principal bundle. It follows that $\mathcal{M}_1 \to \mathcal{M}_0$ is a $\mathbb{P}^1$-bundle. We also consider the associated $\mathbb{P}^1$-bundle

$$\mathcal{M}_2 = \mathcal{M} \times^{\underline{\mathrm{Aut}}(\mathbb{P}^1,0)} \mathbb{P}^1.$$

Let $\mathrm{ev} : \mathcal{M} \times \mathbb{P}^1 \to X$ be the evaluation morphism. The composite $\mathrm{ev} \circ 0_{\mathcal{M}}$ descends to a structure map $\mathcal{M}_1 \to X$. Its composite with the projection $\mathcal{M}_2 \to \mathcal{M}_1$ yields a morphism $\mathcal{M}_2 \to X$, which we consider as the *left* structure map. On the other hand, ev descends to a morphism $\mathcal{M}_2 \to X$, which we consider as the *right* structure map. We thus have a double fibration $\mathcal{M}_2 \rightrightarrows X$, and think of $\mathcal{M}_2$ as the space of unparametrised 2-pointed $\mathcal{M}$-curves. Completion of $\mathcal{M}_2$ along the 0-section $\mathcal{M}_1 \to \mathcal{M}_2$ gives a bundle of formal discs $\hat{\mathcal{M}}_2 \to \mathcal{M}_1$. We also define the products

$$\mathcal{M}_2^i = (X \backslash \mathcal{M}_2 / X)^i$$

together with morphisms $\mathcal{M}_2^i \rightrightarrows X$ as introduced in 1.4, and think of them as spaces of 2-pointed i-chains of $\mathcal{M}$-curves. Since $\mathcal{M} \to \mathcal{M}_0$ is a principal $\underline{\mathrm{Aut}}(\mathbb{P}^1)$-bundle, we have natural isomorphisms

$$\mathcal{M}_1 \simeq \mathcal{M} \times^{\underline{\mathrm{Aut}}(\mathbb{P}^1)} \mathbb{P}^1, \quad \mathcal{M}_2 \simeq \mathcal{M} \times^{\underline{\mathrm{Aut}}(\mathbb{P}^1)} (\mathbb{P}^1 \times \mathbb{P}^1).$$

The transposition on $\mathbb{P}^1 \times \mathbb{P}^1$ induces an involution $\mathcal{M}_2 \to \mathcal{M}_2$ over $\mathcal{M}_0$, swapping the two marked points (i.e. the two structure maps to X).

There are open subschemes $\mathcal{M}^{\mathrm{free}}$, $\mathcal{M}^{\mathrm{min}}$, $\mathcal{M}^{\mathrm{arc}}$ of $\mathcal{M}$ such that the curves corresponding to their geometric points are free, resp. minimal, resp. unramified at 0. Since $\mathcal{M}^{\mathrm{free}}$ and $\mathcal{M}^{\mathrm{min}}$ are $\underline{\mathrm{Aut}}(\mathbb{P}^1)$-equivariant, they descend to open subschemes $\mathcal{M}_1^{\mathrm{free}}$, $\mathcal{M}_0^{\mathrm{free}}$ and $\mathcal{M}_1^{\mathrm{min}}$, $\mathcal{M}_0^{\mathrm{min}}$. Since $\mathcal{M}^{\mathrm{arc}}$ is $\underline{\mathrm{Aut}}(\mathbb{P}^1, 0)$-invariant, it descends to an open subscheme $\mathcal{M}_1^{\mathrm{arc}}$. The inclusion $\hat{\mathbb{P}}^1 \to \mathbb{P}^1$ induces morphisms

$$\mathcal{M}_1^{\mathrm{arc}} \to \mathrm{Arc}_{X/S} \to \mathbb{P}T_{X/S}$$

Finally, we let

$$\mathcal{M}_2^{\text{free}} = \mathcal{M}_1^{\text{free}} \times_{\mathcal{M}_1} \mathcal{M}_2, \qquad \mathcal{M}_2^{i,\text{free}} = (X\backslash\mathcal{M}_2^{\text{free}}/X)^i.$$

We remark that if $\mathcal{M}$ is an irreducible component of $\underline{\text{Hom}}^n_{\text{bir}}(\mathbb{P}^1, X)$, then $\mathcal{M}_1^{\text{free}} \to X$ and both $\mathcal{M}_2^{\text{free}} \rightrightarrows X$ are smooth [14].

2.2.2 Rational curves on a variety

Suppose now X is an irreducible variety, and $\mathcal{M} \subset \underline{\text{Hom}}^n_{\text{bir}}(\mathbb{P}^1, X)$ a family of rational curves. We will say that $\mathcal{M}$ is *dominating* if $\mathcal{M}_1 \to X$ is dominant. We will say that $\mathcal{M}$ is *unsplit* if $\mathcal{M}_0$ is proper. We will say that X is *chain-connected* by $\mathcal{M}$-curves if the induced morphism $\mathcal{M}_2^i \to X \times X$ is dominant for some $i \geq 0$. In this case

$$\coprod \mathcal{M}_2^i \rightrightarrows X$$

is a transitive category-scheme, with composition given by concatenation of chains. Given an object over X together with an isomorphism of its two pullbacks along $\mathcal{M}_2 \rightrightarrows X$, we obtain an action of $\coprod \mathcal{M}_2^i$, i.e. a 'parallel transport' along chains. We will use this technique in Chapter 3 to extend ∞-jets of morphisms between varieties chain-connected by suitable families of rational curves. A simpler application is the following Proposition.

Lemma 2.2.1. *Let X be a nonsingular variety, $\mathcal{L}$ an invertible sheaf on X, and $\mathcal{M} \subset \underline{\text{Hom}}^n_{\text{bir}}(\mathbb{P}^1, X)$ a family of rational curves such that the pullback of $\mathcal{L}$ by the generic $\mathcal{M}$-curve $\mathbb{P}^1 \otimes k(\mathcal{M}) \to X$ is trivial. Then the two pullbacks of $\mathcal{L}$ along $\mathcal{M}_2 \rightrightarrows X$ are isomorphic.*

Proof. Considering the $\underline{\text{Aut}}(\mathbb{P}^1, 0)$-equivariant diagram

$$\mathcal{M} \underset{0_{\mathcal{M}}}{\overset{p_1}{\leftrightarrows}} \mathcal{M} \times \mathbb{P}^1 \xrightarrow{\text{ev}} X$$

we have that $\text{ev}^*\mathcal{L} \simeq p_1^*\mathcal{K}$ for some $\underline{\text{Aut}}(\mathbb{P}^1, 0)$-equivariant invertible sheaf $\mathcal{K}$ on $\mathcal{M}$. There is then an $\underline{\text{Aut}}(\mathbb{P}^1, 0)$-equivariant isomorphism

$$\text{ev}^*\mathcal{L} \simeq p_1^*\mathcal{K} = (0_{\mathcal{M}} \circ p_1)^* p_1^*\mathcal{K} \simeq (0_{\mathcal{M}} \circ p_1)^* \text{ev}^*\mathcal{L} = (\text{ev} \circ 0_{\mathcal{M}} \circ p_1)^*\mathcal{L}$$

descending to $e_1^*\mathcal{L} \simeq e_0^*\mathcal{L}$ where $e_0, e_1 : \mathcal{M}_2 = \mathcal{M} \times^{\underline{\text{Aut}}(\mathbb{P}^1,0)} \mathbb{P}^1 \to X$ are the left and right morphisms to X. $\square$

Proposition 2.2.2. *Suppose X is a nonsingular projective variety, and $\mathcal{M} \subset \underline{\text{Hom}}^n_{\text{bir}}(\mathbb{P}^1, X)$ a connected unsplit family of rational curves, such that X is chain-connected by $\mathcal{M}$-curves. Then X has Picard number 1.*

Proof. We are going to show that if a divisor D on X intersects some (hence every) $\mathcal{M}$-curve trivially, then it is numerically trivial. Indeed, we have by Lemma 2.2.1 that there is an isomorphism of the two pullbacks of $\mathcal{O}(D)$ along $\mathcal{M}_2 \rightrightarrows X$. It then follows that there is an isomorphism of the pullbacks of $\mathcal{O}(D)$ along $\mathcal{M}_2^i \rightrightarrows X$ for each $i \geq 0$. Let i be such that $\mathcal{M}_2^i \to X \times X$ is dominant, hence surjective. Fix a closed point $x \in X$. Then the right evaluation morphism $e : x \times_X \mathcal{M}_2^i \to X$ is surjective, and $e^*\mathcal{O}(D)$ is trivial. Since the components of $\underline{\mathrm{Hom}}(\mathbb{P}^1, X)$ are quasi-projective, it follows by properness that $\mathcal{M}_1$ is projective. Now, for any irreducible curve $C \subset X$, there is an irreducible curve $\tilde{C} \subset x \times_X \mathcal{M}_2^i$ surjecting onto C. Since $\deg_{\tilde{C}} e^*\mathcal{O}(D) = 0$, we have $(\deg e|_{\tilde{C}})(C.D) = 0$, so that $C.D = 0$ as desired. $\square$

We say that an irreducible family of rational curves $\mathcal{M}$ on X is of degree d with respect to an invertible sheaf $\mathcal{L}$ if the pullback of $\mathcal{L}$ by the generic $\mathcal{M}$-curve $\mathbb{P}^1 \otimes k(\mathcal{M}) \to X$ has degree d. We say that $\mathcal{M}$ is of minimal degree with respect to $\mathcal{L}$ if there is no component of $\underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ of lower degree with respect to $\mathcal{L}$. Unsplit families arise from curves of minimal degree with respect to an ample invertible sheaf (essentially unique, *a posteriori*, in case of a chain-connected nonsingular variety).

Lemma 2.2.3. *Suppose X is a nonsingular projective variety, $\mathcal{L}$ an ample invertible sheaf, and $\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ an irreducible family of rational curves of minimal degree with respect to $\mathcal{L}$. Then $\mathcal{M}$ is unsplit.*

Proof. We use the valuative criterion of properness. Let T be a spectrum of a discrete valuation ring, with closed point t_0 and generic point t_1, and let $f_1 : t_1 \to \mathcal{M}_0$ be a morphism. The image of $t_1 \times_{\mathcal{M}_0} \mathcal{M}_1 \to t_1 \times X$ defines a morphism $t_1 \to \operatorname{Hilb} X$, extending by properness of the Hilbert scheme to $\tilde{f} : T \to \operatorname{Hilb} X$. Let $C \subset T \times X$ be the pullback of the universal family, so that $p_1 : C \to T$ is a flat family whose fibres are rational cycles of dimension one. Write $[t_0 \times_T C] = \sum_{i=1}^r a_i[C_i]$ where C_i are integral rational curves. Since $\mathcal{M}$ has minimal degree with respect to $\mathcal{L}$, we have

$$\deg_{C_j} p_2^*\mathcal{L} \geq \sum_{i=1}^r a_i \deg_{C_i} p_2^*\mathcal{L}$$

for each j. Hence $r = 1$, $a_1 = 1$, and $t_0 \times_T C$ is an integral rational curve. It follows that the normalization $\nu : \tilde{C} \to C$ is a $\mathbb{P}^1$-bundle over T. After unramified base change $T' \to T$, we have $T' \times_T \tilde{C} \simeq T' \times \mathbb{P}^1$. Then

$$\mathrm{id}_{T'} \times (p_2 \circ \nu) : T' \times_T \tilde{C} \simeq T \times \mathbb{P}^1 \to T' \times X$$

induces a morphism $T' \to \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$, necessarily factoring through $\mathcal{M}$. Its composite with $\mathcal{M} \to \mathcal{M}_0$ gives

$$f' : T' \to \mathcal{M}_0.$$

Letting $q_1, q_2 : T' \times_T T' \to T'$ be the two projections, consider

$$\delta f' = \langle q_1^* f', q_2^* f' \rangle : T' \times_T T' \to \mathcal{M}_0 \times \mathcal{M}_0.$$

Observing that $f'|_{t_1 \times_T T'}$ is the pullback of f_1, we have that $\delta f'|_{t_1 \times_T (T' \times_T T')}$ factors through the diagonal $\mathcal{M}_0 \to \mathcal{M}_0 \times \mathcal{M}_0$. But then, by separatedness of $\mathcal{M}_0$, so does entire $\delta f'$. Hence f' descends along the étale surjection $T' \to T$ to $f : T \to \mathcal{M}_0$ such that $f|_{t_1} = f_1$. $\square$

Lemma 2.2.4. *Let X be a nonsingular projective variety and $\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ an irreducible component such that the generic $\mathcal{M}$-curve $f : \mathbb{P}^1 \otimes k(\mathcal{M}) \to X$ is free. Let $D \subset X$ be a reduced closed subscheme of codimension 1, and $W \subset X$ a closed subscheme of codimension 2. Then f^*D is reduced, and f^*W is empty.*

Proof. By [14, Cor. 3.5.4], the evaluation morphism $\mathrm{ev} : \mathcal{M}^{\mathrm{free}} \times \mathbb{P}^1 \to X$ is smooth. $\square$

We end this subsection showing that one can often restrict to chains of free curves.[4]

Lemma 2.2.5. *Suppose X is a nonsingular projective variety of Picard number 1, and $\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ an irreducible component such that the generic $\mathcal{M}$-curve is free. Then $\mathcal{M}_2^{i,\mathrm{free}} \to X \times X$ is dominant for some $i \geq 0$.*

Proof. Let $\mathcal{M}_2^{i,\mathrm{free}} = \bigcup M_j$ be the irreducible components. For each j, let $W_j \subset X \times X$ be the closed image subscheme of M_j under $\mathcal{M}_2 \to X \times X$. Choose j_0 such that W_{j_0} has maximal dimension among all M_j. Set $M = M_{j_0}$, $W = W_{j_0}$, with the pair of projections $W \rightrightarrows X$.

Let x be the generic point of X and set $W_x = x \times_X W$. Let η be the generic point of W_x. By construction, $\eta \times_X \mathcal{M}_2^{\mathrm{free}} \to X$ factors through W_x (for otherwise $\dim W_j > \dim W$ for some j, a contradiction). Since by freeness $X \leftarrow \mathcal{M}_2^{\mathrm{free}}$ is smooth, we have that $W_x \times_X \mathcal{M}_2^{\mathrm{free}}$ is the closure of $\eta \times_X \mathcal{M}_2^{\mathrm{free}}$ so that $W_x \times_X \mathcal{M}_2^{\mathrm{free}} \to X$ factors through W_x as well.

As a closed subscheme of $X \otimes \kappa(x)$, W_x defines an x-point of $\mathrm{Hilb}\, X$ which, by properness of the Hilbert scheme, extends to

$$q : U \to \mathrm{Hilb}\, X$$

over an open subscheme $U \subset X$ whose complement has codimension at least 2 in X. It will be enough to show that q is constant. Note that it is constant at least on the fibres of $X \leftarrow W$, in particular $q|_{W_x}$ factors through $q|_x$. Letting $f : \mathbb{P}^1 \otimes k(\mathcal{M}) \to X$ be the generic $\mathcal{M}$-curve, we have by freeness that $f(0) = x$, so that by the previous paragraph f factors through W_x. On the other hand, since $X \setminus U$ has codimension at least 2, f factors through U by Lemma 2.2.4. It then follows that for any ample invertible sheaf $\mathcal{L}$ on $\mathrm{Hilb}\, X$, the pullback $f^*q^*\mathcal{L}$ is trivial. Since X has Picard number 1, it follows that $\mathcal{L}$ is numerically trivial, so that q must be a constant morphism. But $W_x \to X$ is dominant, so that q must factor through the k-point of $\mathrm{Hilb}\, X$ corresponding to entire X. $\square$

Lemma 2.2.6. *Suppose X is a nonsingular projective variety, and $\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ an irreducible component such that the generic $\mathcal{M}$-curve is free, and the generic fibre of $\mathcal{M}_1^{\mathrm{free}} \to X$ is geometrically connected. Then:*

[4] I have learned this argument from Jason Starr.

1. $X \leftarrow \mathcal{M}_2^{i,\text{free}}$ *has geometrically integral generic fibre,*

2. $\mathcal{M}_2^{i+1,\text{free}} \to \mathcal{M}_2^{i,\text{free}}$ *is a smooth surjection,*

3. *if the generic $\mathcal{M}$-curve is minimal (resp. unramified at 0), then the generic point of $\mathcal{M}_2^{i,\text{free}}$ is a chain of minimal (resp. unramified at 0) curves*

for all $i \geq 0$.

Proof. Note that by freeness $X \leftarrow \mathcal{M}_1^{\text{free}}$ is smooth, so that, being geometrically connected, its generic fibre is geometrically integral. The same holds for $X \leftarrow \mathcal{M}_2^{\text{free}}$, since $\mathcal{M}_2^{\text{free}} \to \mathcal{M}_1^{\text{free}}$ is a $\mathbb{P}^1$-bundle. Assume by induction that $X \leftarrow \mathcal{M}_2^{i,\text{free}}$ has geometrically integral generic fibre. By freeness, $\mathcal{M}_2^{i,\text{free}} \to X$ is dominant. On the other hand, $X \leftarrow \mathcal{M}_2^{\text{free}}$ is smooth with geometrically connected generic fibre. Hence

$$\mathcal{M}_2^{i+1,\text{free}} = \mathcal{M}_2^{i,\text{free}} \times_X \mathcal{M}_2^{\text{free}}$$

has geometrically integral generic fibre, and the projection $\mathcal{M}_2^{i+1,\text{free}} \to \mathcal{M}_2^{i,\text{free}}$ is dominant and smooth. To check surjectivity, it is enough to note that it has an obvious section, duplicating the rightmost link in the i-chain.

For the last statement, it is enough to note that projection $\mathcal{M}_2^{i,\text{free}} \to \mathcal{M}_2$ to the rightmost factor is dominant for all $i \geq 0$: indeed the generic point of $\mathcal{M}_2$ lifts to a point in $\mathcal{M}_2^{i,\text{free}}$ corresponding to a chain consisting of i copies of the single generic $\mathcal{M}$-curve. □

2.2.3 The tangent map and VMRT

Recall that given a nonsingular projective variety X and a family $\mathcal{M}$ of rational curves on X, we have morphisms

$$\mathcal{M}_1^{\text{arc}} \to \text{Arc}_X \to \mathbb{P}T_X.$$

The composite will be called the *tangent map*. The results of Kebekus [12, Thm. 3.3 and 3.4] describe[5] the tangent map at the generic point of X.

Proposition 2.2.7 (Kebekus). *Let X be a nonsingular projective variety with generic point x, $\mathcal{L}$ an ample invertible sheaf on X, and $\mathcal{M}$ a dominating irreducible family of rational curves on X of degree 1 with respect to $\mathcal{L}$ (hence unsplit by Lemma 2.2.3). Then:*

1. $x^*\mathcal{M}_1^{\text{arc}} = x^*\mathcal{M}_1$

2. $x^*\mathcal{M}_1 \to \mathbb{P}T_{X,x}$ *is finite.*

[5] There are concerns about the proof in [12] when the degree of the family with respect to the ample invertible sheaf is divisible by p. This is clearly not the case here.

Under the hypotheses of Proposition 2.2.7, we call the closed image scheme of $x^*\mathcal{M}_1 \to \mathbb{P}T_{X,x}$ the *variety of $\mathcal{M}$-rational tangents at x*. Denote by m_1 the generic point of $\mathcal{M}_1$. If the generic $\mathcal{M}$-curve is minimal, then $x^*\mathcal{M}_1 \to \mathbb{P}T_{X,x}$ unramified at m_1, and induces a well-defined morphism (cf. [10])

$$\mathbb{P}T_{\mathcal{M}_1/X,m_1} \to \mathbb{P}\Lambda^2 T_{X,x}.$$

The remainder of this subsection will be occupied by a proof of the following Proposition, an analogue of [10, Prop. 13].

Proposition 2.2.8. *Assume* $\operatorname{char} k = p > 0$. *Let X be a nonsingular projective Fano variety of Picard number* 1 *and with* $\operatorname{index}(X) < p$. *Let $\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ an irreducible component of degree* 1. *Denote by x the generic point of X, and by m_1 the generic point of $\mathcal{M}_1$. Assume that:*

1. *The generic $\mathcal{M}$-curve is minimal.*
2. *The generic fibre of $\mathcal{M}_1 \to X$ is geometrically irreducible.*
3. *There is $i \geq 0$ such that $\mathcal{M}_2^{i,\mathrm{free}}$ contains a subscheme separably dominating $X \times X$.*
4. *Letting $\mathcal{D}_x \subset T_{X,x}$ be the linear span of the variety of $\mathcal{M}$-rational tangents at x, the image of the natural morphism*

$$\mathbb{P}T_{\mathcal{M}_1/X,m_1} \to \kappa(m_1) \otimes_{\kappa(x)} \mathbb{P}\Lambda^2 T_{X,x}$$

spans $\kappa(m_1) \otimes_{\kappa(x)} \Lambda^2 \mathcal{D}_x$.

Then the variety of $\mathcal{M}$-rational tangents at x is linearly nondegenerate in $\mathbb{P}T_{X,x}$.

Proof. Note that Proposition 2.2.7 applies, so that hypothesis 4 makes sense. We extend $\mathcal{D}_x \subset T_{X,x}$ to a saturated subsheaf $\mathcal{D} \subset T_X$, a sub-bundle away from codimension 2. Note that $\operatorname{rk} \mathcal{D} > 0$. The argument relies on the existence of a height one purely inseparable quotient of X associated with $\mathcal{D}$. The proof of integrability is essentially due to Hwang and Mok. In characteristic p we need p-closedness as well (see [18] for a related argument).

Lemma 2.2.9. *$\mathcal{D}$ is integrable and p-closed.*

Proof. By [4, Lemma 4.2], we need to check that the maps

$$\theta : \Lambda^2 \mathcal{D} \to T_X/\mathcal{D}, \quad \theta(\xi_1 \wedge \xi_2) = [\xi_1, \xi_2] + \mathcal{D}$$

$$\phi : F^*\mathcal{D} \to T_X/\mathcal{D}, \quad \phi(1 \otimes \xi) = \xi^p + \mathcal{D}$$

are zero, where $F : X \to X$ is the absolute Frobenius morphism, θ is $\mathcal{O}_X$-linear, and ϕ is $\mathcal{O}_X$-linear if $\theta = 0$. Let $\bar{m}_1$ be the geometric generic point of $\mathcal{M}_1$, and $\bar{x}$ the geometric generic point of X. The rational curve

$$f : \mathbb{P}^1_{\bar{m}_1} \simeq \bar{m}_1 \times_{\mathcal{M}_1} \mathcal{M}_2 \to X$$

is minimal and unramified at 0, factors through the locus over which $\mathcal{D} \subset T_X$ is a subbundle, and sends 0 to $\bar{x}$. Fix a splitting

$$f^*T_X = \mathcal{O}(2) \oplus \mathcal{O}(1)^{\mathrm{index}(X)-2} \oplus \mathcal{O}^{\dim(X)+1-\mathrm{index}(X)}$$

and a nonzero vector $u \in \kappa(\bar{m}_1) \otimes_{\kappa(x)} T_{X,x}$ in the image of $df|_0$. Note that $\mathcal{M}_2^{\mathrm{free}} \times_X \bar{x}$ is nonsingular, so that given a nonzero vector

$$v \in \kappa(\bar{m}_1) \otimes_{\kappa(x)} T_{X,x} \simeq f|_0^* T_X$$

contained in the $\mathcal{O}(1)^{\mathrm{index}(X)-2}$ summand, one can find a morphism

$$\gamma_v : \operatorname{Spf} \kappa(\bar{m}_1 \times \bar{x})[[t]] \to \mathcal{M}_2^{\mathrm{free}} \times_X \bar{x}$$

over $\bar{x}$ such that

$$(p \circ \gamma_v)(0) = \bar{m}_1 \times \bar{x}, \quad d(e_1\gamma_v)|_0(\frac{\partial}{\partial t}) = v \otimes 1$$

where $p : \mathcal{M}_2^{\mathrm{free}} \times_X \bar{x} \to \mathcal{M}_1^{\mathrm{free}} \times \bar{x}$ is the natural projection, and $e_1 : \mathcal{M}_2^{\mathrm{free}} \times_X \bar{x} \to X \times \bar{x}$ is the *left* evaluation map. That is, γ_v is a deformation of a 2-pointed rational curve, fixing the second marked point, with $f \otimes 1$ as the central curve, and $v \otimes 1$ as the tangent vector to the corresponding deformation of the first marked point. Considering $\mathrm{pr}_1 \circ p \circ \gamma_v : \operatorname{Spf} \kappa(\bar{m}_1 \times \bar{x})[[t]] \to \mathcal{M}_1$, we have a formal family of parametrised rational curves

$$\delta_v : \operatorname{Spf} \kappa(\bar{m}_1 \times \bar{x})[[t]] \times_{\mathcal{M}_1} \mathcal{M}_2 \to X$$

such that $d\delta_v$ factors through $\delta_v^*\mathcal{D}$, and $d\delta_v|_{(0,0)}$ is an isomorphism onto the base-change of the span of u and v in $\kappa(\bar{m}_1) \otimes_{\kappa(x)} T_{X,x}$. In particular, restricting to $\hat{\mathcal{M}}_2 \subset \mathcal{M}_2$ gives an unramified morphism from a formal disc of dimension two:

$$\operatorname{Spf} \kappa(\bar{m}_1 \times \bar{x})[[t]] \times_{\mathcal{M}_1} \hat{\mathcal{M}}_2 \to X,$$

tangent to $\mathcal{D}$ everywhere, and to the span of u and v at the origin. It follows that

$$(\kappa(\bar{m}_1) \otimes_{\kappa(x)} \theta_x)(u \wedge v) = 0.$$

But then, since v was arbitrary, θ_x vanishes on any element in the linear span of the image of $\mathbb{P}T_{\mathcal{M}_1/X,m_1} \to \mathbb{P}\Lambda^2 T_{X,x}$. Hence, by hypothesis 4 of the Proposition, $\theta_x = 0$, and finally $\theta = 0$ by construction of $\mathcal{D}$.

We proceed to show vanishing of ϕ, which we now know to be $\mathcal{O}_X$-linear. Consider an unramified morphism $h : \bar{m}_1 \times \hat{W} \to X$ from the base-change of a formal disc $\hat{W}$ of dimension $\dim(X) - 1$, such that the image of dh at the origin is transverse to f. By smoothness of $\mathcal{M}_1 \to X$ at the generic point, there is a lift $\tilde{h} : \bar{m}_1 \times \hat{W} \to \mathcal{M}_1$ of h, and a formal family of parametrised rational curves

$$g : \bar{m}_1 \times \hat{W} \times \mathbb{P}^1 \simeq \bar{m}_1 \times \hat{W} \times_{\mathcal{M}_1} \mathcal{M}_2 \to X$$

whith f as the central fibre. Restricting to $\hat{\mathbb{P}}^1 \subset \mathbb{P}^1$, we obtain an unramified morphism from the base-change of a formal disc of dimension $\dim(X)$

$$\hat{g} : \bar{m}_1 \times \hat{W} \times \hat{\mathbb{P}}^1 \to X.$$

Identifying $\hat{\mathbb{P}}^1 = \operatorname{Spf} k[[t]]$, we have

$$(\hat{g}^*\phi)(d\hat{g}(1 \otimes 1 \otimes \frac{\partial}{\partial t})) = 0.$$

But the smallest subspace of $T_{X,x}$ whose pullback by $\hat{g}$ contains $d\hat{g}(1 \otimes 1 \otimes \frac{\partial}{\partial t})$ is precisely $\mathcal{D}_x$. Hence $\phi_x = 0$, and finally $\phi = 0$ by construction of $\mathcal{D}$. □

It follows that $\mathcal{D}$ defines a height one purely inseparable morphism $\pi : X \to Y$ to a normal variety, flat away from codimension two, and factoring the geometric Frobenius $F_X : X \to X'$. There is an exact sequence (cf. [4])

$$0 \to \mathcal{D} \to T_X \to \pi^* T_Y \xrightarrow{\pi^*\delta} \pi^*\sigma^*\mathcal{D} \to 0 \tag{2.1}$$

where $\sigma : Y \to X$ is a composite of the natural morphism $Y \to X'$ factoring F_X with the projection $X' \to X$, and $\delta : T_Y \to \sigma^*\mathcal{D}$ is an $\mathcal{O}_Y$-module morphism. In particular, $\sigma \circ \pi$ is the absolute Frobenius $X \to X$. This leads to an equality

$$(1-p)c_1(\mathcal{D}) - \operatorname{index}(X) + c_1(\pi^* T_Y) = 0. \tag{2.2}$$

Let $f : \mathbb{P}^1 \otimes k(\mathcal{M}) \to X$ be the generic $\mathcal{M}$-curve. By definition of $\mathcal{D}$, $\pi \circ f$ is everywhere ramified, so that we have a commutative diagram

$$\begin{array}{ccc} \mathbb{P}^1 \otimes k(\mathcal{M}) & \xrightarrow{f} & X \\ {\scriptstyle F_{\mathbb{P}^1}\otimes 1}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathbb{P}^1 \otimes k(\mathcal{M}) & \xrightarrow{\bar{f}} & Y \end{array}$$

where $F_{\mathbb{P}^1} : \mathbb{P}^1 \to \mathbb{P}^1$ is the geometric Frobenius.

Lemma 2.2.10. $c_1(\pi^* T_Y) \geq p$.

Proof. It will be enough to check that $H^0(\mathbb{P}^1, f^*\pi^*\omega_Y) = 0$. Using $\omega_X = \pi^!\omega_Y$ (cf. [8]) and the fact that $\mathbb{P}^1$ is Frobenius-split, we have

$$\begin{aligned} 0 = H^0(\mathbb{P}^1, f^*\omega_X) &= H^0(\mathbb{P}^1, f^*\pi^!\omega_Y) = H^0(\mathbb{P}^1, f^* \mathcal{H}om_Y(\pi_*\mathcal{O}_X, \omega_Y)) \\ &= \operatorname{Hom}_{\mathbb{P}^1}(F_{\mathbb{P}^1 *}\mathcal{O}_{\mathbb{P}^1}, \bar{f}^*\omega_Y) \supset H^0(\mathbb{P}^1, \bar{f}^*\omega_Y) = H^0(\mathbb{P}^1, f^*\pi^*\omega_Y). \end{aligned}$$ □

By (2.2), Lemma 2.2.10 and the hypothesis on the index of X, it follows that

$$c_1(\mathcal{D}) = \frac{c_1(\pi^* T_Y) - \operatorname{index}(X)}{p-1} > 0. \tag{2.3}$$

The sequence (2.1) restricts to

$$0 \to f^*\mathcal{D} \to f^*T_X \to F^*_{\mathbb{P}^1}\bar{f}^*T_Y \xrightarrow{F^*_{\mathbb{P}^1}\bar{f}^*\delta} F^*_{\mathbb{P}^1}f^*\mathcal{D} \to 0$$

where exactnes on the left follows from the fact that the generic $\mathcal{M}$-curve, being free, factors through the locus over which $\mathcal{D}$ is a sub-bundle of T_X. In particular, we have a short exact sequence

$$0 \to f^*\mathcal{D} \to f^*T_X \to F^*_{\mathbb{P}^1} \ker \bar{f}^*\delta \to 0 \tag{2.4}$$

Since f is by hypothesis minimal, there is a splitting

$$f^*T_X = \overbrace{\mathcal{O}(2) \oplus \mathcal{O}(1)^{\mathrm{index}(X)-2}}^{\mathcal{T}^+} \oplus \overbrace{\mathcal{O}^{\dim(X)+1-\mathrm{index}(X)}}^{\mathcal{T}^0} \tag{2.5}$$

By definition of $\mathcal{D}$, we have $\mathcal{T}_+ \subset f^*\mathcal{D}$. Identifying $f^*T_X/\mathcal{T}^+$ with $\mathcal{T}^0$, let

$$\mathcal{D}^0 = f^*\mathcal{D}/\mathcal{T}^+ \subset f^*T_X/\mathcal{T}^+ = \mathcal{T}^0,$$

a sub-bundle of the trivial bundle $\mathcal{T}^0$ over $\mathbb{P}^1 \otimes k(\mathcal{M})$. There is a splitting $\mathcal{D}^0 = \bigoplus \mathcal{O}(d_i)$ where all $d_i \leq 0$ since $\mathcal{D}^0$ is a sub-bundle of the trivial bundle $\mathcal{T}^0$. On the other hand, (2.4) yields a short exact sequence

$$0 \to \mathcal{D}^0 \to \mathcal{T}^0 \to F^*_{\mathbb{P}^1} \ker \bar{f}^*\delta \to 0$$

so that

$$c_1(\mathcal{D}^0) = -pc_1(\ker \bar{f}^*\delta)$$

and the inequality (2.3) gives

$$0 < c_1(f^*\mathcal{D}) = \mathrm{index}(X) - pc_1(\ker \bar{f}^*\delta).$$

Since $\mathrm{index}(X) < p$, it follows that $c_1(\ker \bar{f}^*\delta) \leq 0$ and thus $c_1(\mathcal{D}^0) = \sum d_i \geq 0$. But $d_i \leq 0$, so that necessarily $d_i = 0$ for all i, i.e. $\mathcal{D}^0$ is a trivial bundle. We can then assume that

$$f^*\mathcal{D} = \overbrace{\mathcal{O}(2) \oplus \mathcal{O}(1)^{\mathrm{index}(X)-2}}^{\mathcal{T}^+} \oplus \overbrace{\mathcal{O}^{\mathrm{rk}(\mathcal{D})+1-\mathrm{index}(X)}}^{\mathcal{D}^0} \tag{2.6}$$

compatibly with the decomposition (2.5). This leads to the following property, which will allow us to use induction on generic chains.

Lemma 2.2.11. *Let m_2 be the generic point of $\mathcal{M}_2$, and $e_1, e_2 : m_2 \to X$ the two projections. Then*

$$de_1^{-1}(e_1^*\mathcal{D}) = de_2^{-1}(e_2^*\mathcal{D})$$

as subspaces of $T_{\mathcal{M}_2,m_2}$.

Proof. Immediate by (2.6). □

Let m_2^i be the generic point of $\mathcal{M}_2^{i,\text{free}}$, and $e_1^i, e_2^i : m_2^i \to X$ the two projections. By hypotheses 1–2 of the Proposition, Lemma 2.2.6 applies, so that

$$m_2^{i+1} \in m_2^i \times_X m_2.$$

Then Lemma 2.2.11 and induction on i give

$$(de_1^i)^{-1}(e_1^{i*}\mathcal{D}) = (de_2^i)^{-1}(e_2^{i*}\mathcal{D})$$

on m_2^i for all $i \geq 0$. The equality extends to $\mathcal{M}_2^{i,\text{free}}$, so that given a subscheme $W \subset \mathcal{M}_2^{i,\text{free}}$ dominating $X \times X$, we have that $W \to X \times X$ is separable only if $\mathcal{D} = T_X$. This concludes the proof of Proposition 2.2.8. □

2.3 Cominuscule homogeneous varieties

2.3.1 Classification and properties

We will devote the last section of this chapter to a review of properties of the class of homogeneous varieties to be considered in Chapter 4. It is, with exception of one sub-class, closed under the operation of taking the variety of minimal degree rational tangents through a point. This forms the basis of an inductive argument in the proof of the Rigidity Theorem. Recall that we work over the algebraically closed field k. Most of the material here is standard.

Lemma 2.3.1 (cf. [17], Lemma 2.2)**.** *Let $X = G/P$, where G is a connected, simply connected simple algebraic group, and P a maximal reduced parabolic subgroup. Then the following are equivalent:*

1. *The unipotent radical $\mathcal{R}_u P$ is abelian.*

2. *For a suitable choice of a maximal torus and Borel subgroup, $P = P_\alpha$ is a standard maximal reduced parabolic associated with a simple root α occuring with coefficient 1 in the simple root decomposition of the highest positive root.*

Definition. A homogeneous variety X satisfying either of the equivalent hypotheses in Lemma 2.3.1 is called *cominuscule*.

In the following, we will always assume that a cominuscule variety is presented as $X = G/P$ as in Lemma 2.3.1, and that furthermore a maximal torus T and a Borel subgroup B have been chosen so that $P = P_\alpha$ is a standard maximal parabolic as in hypothesis 2 of the Lemma. We let Φ denote the root system of G, Φ^+ the set of positive roots (so that $\mathcal{R}_u B$ is generated by root spaces $U_{-\beta}$ with $\beta \in \Phi^+$), and Δ the set of simple roots (so that in particular $\alpha \in \Delta$). Let ω be the fundamental weight corresponding to α, and denote by $\Phi_\alpha^+ \subset \Phi^+$ the subset on which ω is zero. Let $L_\alpha = P_\alpha / \mathcal{R}_u P_\alpha$ be the Levi factor of P_α, together with an inclusion $L_\alpha \subset P_\alpha$ splitting the projection, and let L_α^{ss}

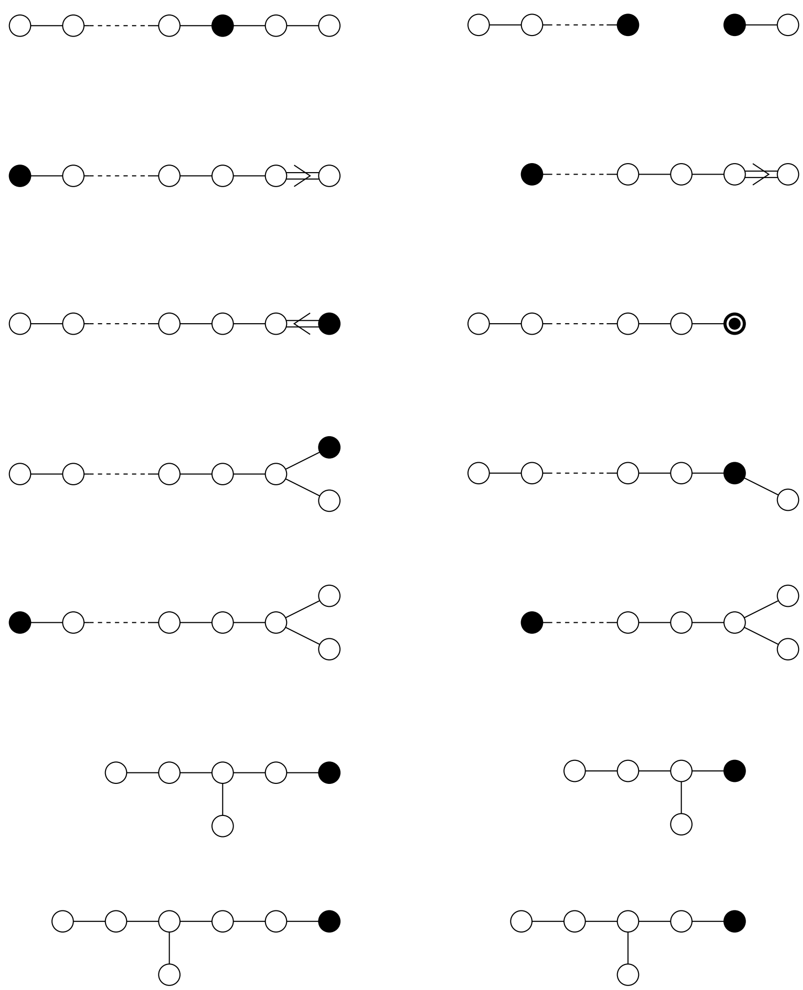

Figure 2.1: Marked Dynkin diagrams corresponding to cominuscule homogeneous varieties (left) and their varieties of minimal rational tangents (right). From top to bottom: Grassmannian, odd-dimensional quadric, symplectic Grassmannian, orthogonal Grassmiannian, even-dimensional quadric, Cayley plane, Freudenthal variety.

be its semisimple part. L_α contains T as a maximal torus, and the set of positive roots of L_α^{ss} is identified with $\Phi_\alpha^+ \setminus \Phi_\alpha^+$. In particular, $\Delta \setminus \{\alpha\}$ is the set of simple roots of L_α^{ss}. We have $L_\alpha = P_\alpha \cap P_\alpha^+$ where P_α^+ is the opposite parabolic. We associate with X the Dynkin diagram of Φ with a marked node corresponding to α. A complete classification of cominuscule varieties in terms of their diagrams is then given in the left column of Figure 2.1 (the right column lists Dynkin diagrams of L_α^{ss}, where the marking will be explained in the next subsection).

Lemma 2.3.2. *Let $X = G/P_\alpha$ be cominuscule. Then X is a simply-connected rational Fano variety with* $\mathrm{Pic}(X) \simeq \mathbb{Z}$ *generated by a very ample invertible sheaf $\mathcal{O}_X(1)$.*

Proof. The projection $G \to X$ induces an open immersion $\mathcal{R}_u P_\alpha^+ \to X$, where $\mathcal{R}_u P_\alpha^+ \simeq \mathbb{A}^{\dim X}$ as a variety. It follows that X is rational, hence simply connected. Now, invertible sheaves on X are equivalent to descent data on G, i.e. homomorphisms $P_\alpha \to \mathbb{G}_m$. Every such homomorphism factors through L_α and is trivial on L_α^{ss}, hence factors through the rank one torus $L_\alpha / L_\alpha^{ss} \simeq \mathbb{G}_m$.

It follows that $\mathrm{Pic}(X) \simeq \mathbb{Z}$, generated by the invertible sheaf $\mathcal{O}_X(1)$ associated with the representation $P_\alpha \to \mathbb{G}_m$ whose restriction to T is ω. More precisely, given an invertible sheaf $\mathcal{L}_\lambda$ on X, with λ the associated weight of $T \subset P_\alpha$, we have

$$\mathcal{L}_\lambda \simeq \mathcal{O}_X(\langle \lambda, \alpha^\vee \rangle).$$

This in particular shows that

$$c_1(T_X) = \sum_{\beta \in \Phi^+ \setminus \Phi_\alpha^+} \langle \beta, \alpha^\vee \rangle = \langle 2\rho, \alpha \rangle - \sum_{\beta \in \Phi_\alpha^+} \langle \beta, \alpha^\vee \rangle$$

where ρ is the half sum of all positive roots. Since ρ is dominant, the first term is non-negative. On the other hand, the second term is negative, being given by the evaluation on $\alpha^\vee$ on a positive combination of simple roots in $\Delta \setminus \{\alpha\}$. Hence $c_1(T_X) > 0$ and X is Fano. Very-ampleness of $\mathcal{O}_X(1)$ follows from [16, Thm. 1]. $\square$

We list the basic information in Table 2.1. Here $\mathsf{LG}(n, 2n)$, resp. $\mathsf{OG}(n, 2n)$, is the Grassmannian of maximal isotropic subspaces in k^{2n} equipped with a standard symplectic form, resp. inner product; $Q \subset \mathbb{P}^n$ denotes a quadric hypersurface; the Cayley plane parametrises rays through idempotent elements in the Albert algebra, while the Freudenthal variety parametrises rays through strictly regular elements in the Freudenthal Triple System associated with the Albert algebra (cf. [5]). We will refer to $X \subset \mathbb{P}H^0(X, \mathcal{O}_X(1))^\vee$ as a *minimally embedded* cominuscule variety.

2.3.2 Varieties of line tangents

We continue the notation of the previous subsection, with $X = G/P_\alpha$ a cominuscule variety. Let $I(\alpha) \subset \Delta$ be the set of simple roots corresponding to nodes adjacent to α in the Dynkin diagram of Φ, including α. There is a corresponding standard reduced

G	α	$X = G/P_\alpha$	embedding	$\dim(X)$	$\operatorname{index}(X)$
A_n	$\alpha_i \sim \alpha_{n-i}$	$\mathsf{Gr}(i, n+1)$	Plücker	$i(n+1-i)$	$n+1$
B_n	α_1	$Q \subset \mathbb{P}^{2n}$	standard	$2n-1$	$2n-1$
C_n	α_n	$\mathsf{LG}(n, 2n)$	Plücker	$n(n+1)/2$	$n+1$
D_n	$\alpha_n \sim \alpha_{n-1}$	$\mathsf{OG}(n, 2n)$	Plücker	$n(n-1)/2$	$2n-2$
D_n	α_1	$Q \subset \mathbb{P}^{2n-1}$	standard	$2n-2$	$2n-2$
E_6	$\alpha_6 \sim \alpha_1$	Cayley plane		16	12
E_7	α_7	Freudenthal variety		27	18

Table 2.1: Cominuscule varieties: root system of G, simple root defining P_α, root system of the semisimple part of the Lévi factor, type of $X = G/P_\alpha$, embedding by $|\mathcal{O}_X(1)|$, dimension and Fano index. We use Bourbaki's ordering of simple roots.

parabolic $P_{I(\alpha)}$ in G, and its intersection $P_{I(\alpha)} \cap L_\alpha^{\mathrm{ss}}$ is a standard reduced parabolic in L_α^{ss} corresponding to $I(\alpha) \setminus \{\alpha\} \subset \Delta \setminus \{\alpha\}$. The latter is the set of marked nodes in the right column of Figure 2.1. It follows that $P_{I(\alpha)}$ is the normaliser of U_α in P_α, and $P_{I(\alpha)} \cap L_\alpha^{\mathrm{ss}}$ is the stabiliser in L_α^{ss} of the line $(\mathfrak{g}_\alpha + \mathfrak{p})/\mathfrak{p}$ in $\mathfrak{g}/\mathfrak{p}$.

We now consider rational curves on X of degree 1 with respect to $\mathcal{O}_X(1)$. These are simply lines in $\mathbb{P}H^0(X, \mathcal{O}_X(1))^\vee$ contained in X, and we will refer to them this way. We will use the notation of Section 2.2.

Proposition 2.3.3. *Let $X = G/P_\alpha$ be a cominuscule variety with origin $x \in X(k)$. Then:*

1. *Lines on X form an irreducible component $\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$.*
2. *The natural action of G on $\mathcal{M}_1$ is transitive, and we have $\mathcal{M}_1 \simeq G/P_{I(\alpha)}$.*
3. *All lines are minimal, and the tangent map $\mathcal{M}_1 \to \mathbb{P}T_X$ is a closed immersion.*
4. *$x \times_X \mathcal{M}_1 \simeq L_\alpha^{\mathrm{ss}}/(P_{I(\alpha)} \cap L_\alpha^{\mathrm{ss}})$, and the embedding $x \times_X \mathcal{M}_1 \to \mathbb{P}T_{X,x}$ is defined by the invertible sheaf associated with $-\alpha|_{T \cap L_\alpha^{\mathrm{ss}}}$.*

Proof. This is essentially a corollary of the Main Theorem in [3]. More explicitly, let $\mathsf{SL}_2 \hookrightarrow G$ be the subgroup corresponding to α, so that the maximal torus of SL_2 maps to T, and the positive root subgroup of SL_2 maps to the root subgroup $U_\alpha \subset G$ associated with α. Then $\mathsf{SL}_2 \cap P_\alpha$ is a Borel subgroup of SL_2, and the inclusion $\mathsf{SL}_2 \hookrightarrow G$ descends to a rational curve $c_\alpha : \mathbb{P}^1 \to X$ of degree 1 with respect to $\mathcal{O}_X(1)$. The Main Theorem in [3] states that every line in X is a G-translate of $c_\alpha(\mathbb{P}^1)$. In fact, by construction, every parametrised line $c : \mathbb{P}^1 \to X$ is of the form $g \circ c_\alpha$ for some $g \in G(k)$.

It follows that lines on X are free (in fact minimal), and form a single irreducible component $\mathcal{M}$, which is reduced (in fact nonsingular), and thus G-homogeneous. Likewise for $\mathcal{M}_1$, which is additionally proper, hence a quotient of G by a parabolic subgroup. Since $c_\alpha(\mathbb{P}^1) = \overline{U_\alpha P_\alpha}/P_\alpha$, we have that the stabiliser of $[c_\alpha] \in \mathcal{M}_1$ in P_α is $P_{I(\alpha)}$.

G	α	L_α^{ss}	$x \times_X \mathcal{M}_1$	embedding	r
A_n	α_i	$\mathsf{A}_{i-1} \times \mathsf{A}_{n-i}$	$\mathbb{P}^{i-1} \times \mathbb{P}^{n-i}$	$\lvert\mathcal{O}(1,1)\rvert$	$\min(i, n+1-i)$
B_n	α_1	B_{n-1}	$Q \subset \mathbb{P}^{2n-2}$	$\lvert\mathcal{O}(1)\rvert$	2
C_n	α_n	A_{n-1}	$\mathbb{P}^{n-1}$	$\lvert\mathcal{O}(2)\rvert$	n
D_n	α_n	A_{n-1}	$\mathsf{Gr}(2, n)$	$\lvert\mathcal{O}(1)\rvert$	$\lfloor \frac{n}{2} \rfloor$
D_n	α_1	D_{n-1}	$Q \subset \mathbb{P}^{2n-3}$	$\lvert\mathcal{O}(1)\rvert$	2
E_6	α_6	D_5	$\mathsf{OG}(5, 10)$	$\lvert\mathcal{O}(1)\rvert$	2
E_7	α_7	E_6	Cayley plane	$\lvert\mathcal{O}(1)\rvert$	3

Table 2.2: Varieties of line tangents and length of connecting chains.

Obviously now $\mathcal{M}_1 \to \mathbb{P}T_X$ is a closed immersion, so that $x \times_X \mathcal{M}_1$ embeds into $\mathbb{P}T_{X,x} \simeq \mathbb{P}(\mathfrak{g}/\mathfrak{p})$ as $L_\alpha^{\mathrm{ss}}/L_\alpha^{\mathrm{ss}} \cap P_{I(\alpha)}$. The tangent direction to $c_\alpha(\mathbb{P}^1)$ at x is the image of the root subspace $\mathfrak{g}_\alpha$ in $T_{X,x} \simeq \mathfrak{g}/\mathfrak{p}$. It follows that $L_\alpha^{\mathrm{ss}} \cap P_{I(\alpha)}$ acts on the fibre of $\mathcal{O}_{\mathbb{P}T_{X,x}}(-1)$ at $[c_\alpha] \in \mathbb{P}T_{X,x}$ via $\alpha|_{T \cap L_\alpha^{\mathrm{ss}}}$. Hence $\mathcal{O}(1)$ restricts on $x \times_X \mathcal{M}_1$ to the invertible sheaf associated with $-\alpha|_{T \cap L_\alpha^{\mathrm{ss}}}$. □

Note that $x \times_X \mathcal{M}_1 \to \mathbb{P}T_{X,x}$ is a closed immersion onto the variety of line tangents at x. Associating with $L_\alpha^{\mathrm{ss}}/(P_{I(\alpha)} \cap L_\alpha^{\mathrm{ss}})$ the Dynkin diagram of the root system L_α^{ss}, and marking nodes corresponding to $I(\alpha) \setminus \{\alpha\}$, gives the right column in Figure 2.1. The weight $-\alpha|_{T \cap L_\alpha^{\mathrm{ss}}}$ defining the embedding is a combination of fundamental weights of L_α^{ss}, and the coefficients determine the multiplicity of marking on corresponding nodes (this is simply 1 for all marked nodes, except for $\mathsf{LG}(n, 2n)$ where $-\alpha|_{T \cap L_\alpha^{\mathrm{ss}}} = 2\omega_{n-1}$). We list this information in Table 2.2.

Corollary 2.3.4. *The variety of line tangents at the origin of a cominuscule homogeneous variety is one of the following:*

1. *a Segre variety,*
2. *a Veronese variety of degree* 2*, or*
3. *a minimally embedded cominuscule variety.*

2.3.3 Chains of lines

As in the proof of Proposition 2.3.3, we consider the subgroup $\mathsf{SL}_2 \subset G$ with root subgroups $U_{\pm\alpha}$, and the line $c_\alpha : \mathbb{P}^1 \to X$. Note that in particular the open immersion $\mathcal{R}_u P_\alpha^+ \simeq \mathbb{A}^{\dim X} \to X$ restricts to $U_\alpha \simeq \mathbb{A}^1 \to c_\alpha(\mathbb{P}^1)$. The pullback of the P_α-principal bundle $G \to X$ by c_α gives a subvariety

$$c_\alpha^* G = \mathsf{SL}_2 \cdot P_\alpha = \overline{U_\alpha P_\alpha}$$

in G, a P_α-principal bundle over $\mathbb{P}^1$. We will inductively construct morphisms q^i fitting into a commutative diagram

$$\begin{tikzcd}
(\overline{U_\alpha P_\alpha})^i \arrow[r, "q^i"] \arrow[d] & {[c_\alpha] \times_{\mathcal{M}_1} \mathcal{M}_2^i} \arrow[d] \\
G \arrow[r] & X
\end{tikzcd}$$

where the left vertical arrow is an i-fold product in G, the right vertical arrow is the right structure morphism, and $[c_\alpha] \in \mathcal{M}_1(k)$ is the image of $c_\alpha \in \mathcal{M}(k)$. By abuse of notation, define $[c_\alpha] \times_{\mathcal{M}_1} \mathcal{M}_2^0$ to be $\operatorname{Spec} k$, with both structure maps to X given by x. Then $q^0 = \mathrm{id}_{\operatorname{Spec} k}$. Suppose by induction that q^i has been defined and makes the above diagram commute. We then let q^{i+1} be the composite

$$\overline{U_\alpha P_\alpha} \times (\overline{U_{-\alpha} P_\alpha})^i \xrightarrow{\mathrm{id} \times q^i} \overline{U_\alpha P_\alpha} \times [c_\alpha] \times_{\mathcal{M}^1} \mathcal{M}_2^i \xrightarrow{\langle \varphi \circ \mathrm{pr}_1, \psi \rangle} [c_\alpha] \times_{\mathcal{M}^1} \mathcal{M}_2 \times_X \mathcal{M}_2^i$$

where $\psi : G \times \mathcal{M}_2^i \to \mathcal{M}_2^i$ is the action morphism, while $\phi : \overline{U_\alpha P_\alpha} \to \mathcal{M}_2$ sends g to $[gc_\alpha, (gc_\alpha)^{-1}(x)] \in \mathcal{M}_2$. It is clear by construction that q^{i+1} makes the corresponding diagram commute.

Lemma 2.3.5. *q^i is a composite of P_α-principal bundles.*

Proof. There is a Cartesian diagram

$$\begin{tikzcd}
\overline{U_\alpha P_\alpha} \times [c_\alpha] \times_{\mathcal{M}^1} \mathcal{M}_2^i \arrow[r, "\mathrm{pr}_1"] \arrow[d, "\langle \varphi \circ \mathrm{pr}_1, \psi \rangle"'] & U_{-\alpha} P_\alpha \arrow[d] \\
{[c_\alpha] \times_{\mathcal{M}_1} \mathcal{M}_2 \times_X \mathcal{M}_2^i} \arrow[r, "c_\alpha^{-1} \circ e_2 \circ \mathrm{pr}_1"] & \mathbb{P}^1.
\end{tikzcd}$$

where in the bottom horizontal arrow pr_1 projects to $[c_\alpha] \times_{\mathcal{M}_1} \mathcal{M}_2$, and $e_2 : \mathcal{M}_2 \to X$ is the right structure morphism. Since the right vertical arrow is a P_α-principal bundle, so is $\langle \varphi \circ \mathrm{pr}_1, \psi \rangle$. Assuming by induction that q^i is a composite of P_α-principal bundles, so is $q^{i+1} = (\mathrm{id} \times q^i) \circ \langle \varphi \circ \mathrm{pr}_1, \psi \rangle$. $\square$

Proposition 2.3.6. *Given a cominuscule variety $X = G/P$ with $\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ the component of lines, let $r+1$ be the number of P-P double cosets in G. Then $\mathcal{M}_2^i \to X \times X$ is separably surjective for $i \geq r$.*

Proof. As before, we can assume $P = P_\alpha$ for a suitable choice of a maximal torus and a Borel subgroup. Consider first the action morphism

$$\theta^i : G \times [c_\alpha] \times_{\mathcal{M}_1} \mathcal{M}_2^i \to \mathcal{M}_2^i.$$

There is a Cartesian diagram

$$\begin{tikzcd}
G \times [c_\alpha] \times_{\mathcal{M}_1} \mathcal{M}_2^i \arrow[r] \arrow[d, "\theta^i"'] & G \arrow[d, "{[c_\alpha]}"] \\
\mathcal{M}_2^i \arrow[r] & \mathcal{M}_1
\end{tikzcd}$$

where the bottom horizontal arrow is a composite of projection to the first segment $\mathcal{M}_2^i \to \mathcal{M}_2$ and the natural map $\mathcal{M}_2 \to \mathcal{M}_1$, while the right vertical arrow is a $P_{I(\alpha)}$-principal bundle. Hence θ^i is a $P_{I(\alpha)}$-principal bundle.

We now have a commutative diagram

$$\begin{array}{ccc} G \times (\overline{U_\alpha P_\alpha})^i & \xrightarrow{\theta^i \circ (\mathrm{id} \times q^i)} & \mathcal{M}_2^i \\ {\scriptstyle \mathrm{id} \times \mu^i}\downarrow & & \downarrow \\ G \times G & \longrightarrow & X \times X \end{array}$$

where the left vertical arrow is given by multiplication $\mu^i : (\overline{U_\alpha P_\alpha})^i \to G$, while the right vertical arrow is the evaluation morphism. It will be enough to check that μ^i is separably surjective. By maximality of P_α it follows that μ^i is surjective for i sufficiently large. On the other hand, its image is a union of P_α-P_α double cosets, and its dimension stabilises once the image of μ^{i+1} coincides with that of μ^i. Since $U_\alpha P_\alpha$ is strictly larger than the first double coset P_α, it follows that μ^i is surjective for $i \geq r$. For separability of μ^i it is enough to notice that the normaliser of U_α in P_α is the reduced parabolic $P_{I(\alpha)}$ stabilising $[c_\alpha]$. □

The numbers r appearing in the Proposition are listed in Table 2.2, following [17]

2.3.4 Further properties

Let $H = c_1(\mathcal{O}_X(1))$ be the hyperplane class for the minimal embedding. We will use the notation

$$\delta_X = [X] \cdot H^{\dim X},$$

for the degree of the minimal embedding. Consider the total evaluation morphism

$$\nu^i : \mathcal{M}_2^i \to X^{i+1}$$

for $i \geq r$ as in Proposition 2.3.6. Letting $p_j^i : X^{i+1} \to X$, $0 \leq j \leq i$ be the natural projections, we define the following intersection number:

$$\delta_X(i) = \nu_*^i[\mathcal{M}_2^i] \cdot p_0^{i*}[x] \cdot \left(\sum_{j=1}^{i-1} p_j^{i*} H \right)^{d_i} \cdot p_i^{i*}[x]$$

where d_i is the dimension of the generic fibre of $\mathcal{M}_2^i \to X \times X$. We will express $\delta_X(i)$ in terms of Schubert calculus on X.

Denote by $V \subset \mathbb{P}T_{X,x}$ the variety of line tangents at x, and by $C_x \subset X$ the subvariety swept by lines through x, the closure in X of the image of $P_\alpha U_\alpha P_\alpha$ under the projection $\pi : G \to X$. Letting $\tilde{C}_x \to C_x$ be the blowing-up at x, we have a natural commutative diagram

$$\begin{array}{ccccc} x \times_X \mathcal{M}_1 & \longleftarrow & x \times_X \mathcal{M}_2 & \longrightarrow & X \\ \simeq\downarrow & & \simeq\downarrow & & \uparrow \\ V & \longleftarrow & \tilde{C}_x & \longrightarrow & C_x \end{array}$$

where the left horizontal arrows are Zariski-locally trivial $\mathbb{P}^1$-bundles.

The morphism $G \times C_x \to X \times X$ sending (g, c) to (gx, gc) factors through a closed immersion $G \times^{P_\alpha} C_x \to X \times X$, and we let $C \subset X \times X$ be its image. It follows that $C = G \cdot (x \times C_x)$, with the diagonal action of G on $X \times X$. Define $C^i = (X \backslash C / X)^i$, naturally a closed subvariety of X^{i+1}. There is a commutative diagram

$$\begin{array}{ccc} C^{i+1} & \longrightarrow & X^{i+1} \\ \downarrow & & \downarrow \\ C^i & \longrightarrow & X^i \end{array}$$

where the horizontal arrows are the natural closed immersions, the right vertical arrow is the projection onto the first i factors, and the left vertical arrow is a Zariski-locally trivial C_x-fibration. It follows that C^i is an iterated C_x-fibration.

Lemma 2.3.7. *The total evaluation map $\nu^i : \mathcal{M}_2^i \to X^{i+1}$ factors through a birational morphism onto C^i.*

Proof. That ν^i factors through a proper surjective morphism onto C^i follows by induction on i. Since $\mathcal{M}$-curves are projective lines, the restriction of ν^i over the complement of the diagonals in X^{i+1} is a proper monomorphism, i.e. a closed immersion. □

It follows that

$$\delta_X(i) = [C^i] \cdot p_0^{i*}[x] \cdot \left(\sum_{j=1}^{i-1} p_j^{i*} H \right)^{d_i} \cdot p_i^{i*}[x].$$

Assuming $i \geq r$ as in Proposition 2.3.6, we have

$$d_i = \dim C^i - 2 \dim X = i \dim C_x - \dim X = i \dim V + i - \dim X.$$

Letting $\{[X_\sigma]\}_{\sigma \in S}$ be an additive basis of the Chow ring $A^*(X)$ consisting of closed Schubert varieties, and identifying $A^*(X^{i+1})$ with $A^*(X)^{\otimes(i+1)}$, we can write

$$[C] = \sum a^{\sigma\tau} [X_\sigma] \otimes [X_\tau]$$

where $(a^{\sigma\tau})$ is a symmetric integer matrix. Then

$$[C^i] = \sum a^{\rho_0 \cdots \rho_i} [X_{\rho_0}] \otimes \cdots \otimes [X_{\rho_i}]$$

with

$$a^{\rho_0 \cdots \rho_i} = \sum a^{\sigma_0 \tau_0} \cdots a^{\sigma_i \tau_i} \delta^{\rho_0}_{\sigma_0} \mu^{\rho_1}_{\tau_0 \sigma_1} \cdots \mu^{\rho_{i-1}}_{\tau_{i-1} \sigma_i} \delta^{\rho_i}_{\tau_i}$$

where $(\mu^\rho_{\sigma\tau})$ are the Littlewood-Richardson coefficients: $[X_\sigma] \cdot [X_\tau] = \sum \mu^\rho_{\sigma\tau} [X_\rho]$.

Letting $\xi \in S$ be the unique element with $[X_\xi] = [X]$, we have

$$p_0^{i*}[x] \cdot [C^i] \cdot p_i^{i*}[x] = \sum a^{\xi \rho_1 \cdots \rho_{i-1} \xi} [x] \otimes [X_{\rho_1}] \otimes \cdots \otimes [X_{\rho_{i-1}}] \otimes [x]$$

so that

$$\begin{aligned}
\delta_X(i) &= \sum_{\substack{\rho_1,\ldots,\rho_{i-1}\in S \\ b_1+\cdots+b_{i-1}=d_i}} N(d_i; b_1, \ldots, b_{i-1}) a^{\xi\rho_1\cdots\rho_{i-1}\xi} \prod_{j=1}^{i-1} [X_{\rho_j}] \cdot H^{b_j} \\
&= \sum_{\rho_1,\ldots,\rho_{i-1}} N(i \dim V + i - \dim X; \dim X_{\rho_1}, \ldots, \dim X_{\rho_{i-1}}) a^{\xi\rho_1\cdots\rho_{i-1}\xi} \prod_{j=1}^{i-1} \deg X_{\rho_j}
\end{aligned}$$

where $N(d;\underline{b})$ are the multinomial coefficients: $(\sum x_j)^d = \sum_{\underline{b}} N(d;\underline{b}) \prod x_j^{b_j}$. It follows that $\delta_X(i)$ can be computed in terms of the matrix $(a_{\sigma\tau})$ and the degrees and dimensions of Schubert varieties X_σ.

In order to apply Proposition 2.2.8 in Chapter 4, we will also need the following property. The proof by Hwang and Mok [10, Proposition 14] goes without change.

Lemma 2.3.8. *Assume* $\operatorname{char} k \neq 2$. *Then* $(X, \mathcal{M})$ *satisfy hypothesis (4) of Proposition 2.2.8.*

3

The extension theorem

3.1 Morphisms of varieties with curves

3.1.1 Jets of étale morphisms

Given a nonsingular variety X, let $\Delta_X : X \to X^\sharp$ denote the diagonal section. Since $X^\sharp$ is a bundle of formal discs, the sheaf $\underline{\mathrm{Aut}}_X(X^\sharp; \Delta_X)$ is an affine group scheme over X, a Zariski-locally trivial twisted form of the group of origin-preserving automorphisms of a formal disc of dimension $\dim(X)$. Given another nonsingular variety Y with $\dim(Y) = \dim(X)$, let

$$X \xleftarrow{\mathrm{pr}_X} X \times Y \xrightarrow{\mathrm{pr}_Y} Y$$

be the two projections, and consider the sheaves

$$\begin{aligned} \mathcal{F}_{X,Y} &= \underline{\mathrm{Hom}}_{X\times Y}(\mathrm{pr}_X^* X^\sharp, \mathrm{pr}_Y^* Y^\sharp; \mathrm{pr}_X^*\Delta_X, \mathrm{pr}_Y^*\Delta_Y) \\ \mathcal{E}_{X,Y} &= \underline{\mathrm{Isom}}_{X\times Y}(\mathrm{pr}_X^* X^\sharp, \mathrm{pr}_Y^* Y^\sharp; \mathrm{pr}_X^*\Delta_X, \mathrm{pr}_Y^*\Delta_Y) \end{aligned}$$

over $X \times Y$. Since $\mathrm{pr}_X^* X^\sharp$ and $\mathrm{pr}_Y^* Y^\sharp$ are bundles of formal discs of equal dimension, it follows that $\mathcal{F}_{X,Y}$ is a Zariski-locally trivial $\mathbb{A}^\infty$-bundle over $X \times Y$, while $\mathcal{E}_{X,Y}$ is a Zariski-locally trivial right torsor for $\underline{\mathrm{Aut}}_X(X^\sharp; \Delta_X) \times Y$, and a Zariski-locally trivial left torsor for $X \times \underline{\mathrm{Aut}}_Y(Y^\sharp; \Delta_Y)$.

Lemma 3.1.1. *$\mathcal{F}_{X,Y}/X$ is equipped with a natural stratification, restricting to $\mathcal{E}_{X,Y}/X$.*

Proof. Let $\epsilon : \mathcal{F}_{X,Y} \times_X X^\sharp \to Y^\sharp$ be the universal map. Consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{F}_{X,Y} \times_X X^\sharp \times_X X^\sharp & \xrightarrow{\mathrm{id}\times p_{13}} & \mathcal{F}_{X,Y} \times_X X^\sharp & \xrightarrow{\epsilon} & Y^\sharp \\ {\scriptstyle \mathrm{id}\times\Delta_X}\uparrow & & & & \uparrow{\scriptstyle \Delta_Y} \\ \mathcal{F}_{X,Y} \times_X X^\sharp & \xrightarrow{\epsilon} & Y^\sharp & \xrightarrow{p_2} & Y \end{array}$$

where $p_2 : Y^\sharp \to Y$ is the right projection, and $p_{13} : X^\sharp \times_X X^\sharp \to X^\sharp$ is the leftmost-rightmost projection. By the universal property of $\mathcal{F}_{X,Y}$, the top composite $\epsilon \circ (\mathrm{id} \times p_{13})$ defines a morphism fitting into a diagram

$$\begin{array}{ccc} \mathcal{F}_{X,Y} \times_X X^\sharp & \longrightarrow & \mathcal{F}_{X,Y} \\ \downarrow & & \downarrow \\ X^\sharp & \longrightarrow & X \end{array}$$

where the bottom horizontal arrow is the *right* structure map. This gives the desired morphism $\mathcal{F}_{X,Y} \times_X X^\sharp \to X^\sharp \times_X \mathcal{F}_{X,Y}$ over $X^\sharp$. Since $\mathcal{E}_{X,Y}$ is an open subscheme of $\mathcal{F}_{X,Y}$, it inherits a stratification. $\square$

The sheaf $\mathcal{E}_{X,Y}$ parametrises ∞-jets of formally-étale maps $X \to Y$. Such maps induce isomorphisms of spaces of arcs: the universal map

$$\Phi_{X,Y} : \mathcal{E}_{X,Y} \times_X X^\sharp \to \mathcal{E}_{X,Y} \times_Y Y^\sharp$$

lifts to produce a commutative diagram

$$\begin{array}{ccc} \mathcal{E}_{X,Y} \times_X X^\sharp \times_X \mathrm{Arc}_X & \xrightarrow{\tilde{\Phi}_{X,Y}} & \mathcal{E}_{X,Y} \times_Y Y^\sharp \times_Y \mathrm{Arc}_Y \\ \downarrow & & \downarrow \\ \mathcal{E}_{X,Y} \times_X X^\sharp & \xrightarrow{\Phi_{X,Y}} & \mathcal{E}_{X,Y} \times_Y Y^\sharp \end{array}$$

where both horizontal arrows are isomorphisms.

We now want to restrict to maps preserving families of arcs induced by families of rational curves. Let us for convenience introduce the following notion.

Definition. A *good family* on a nonsingular projective variety X is a irreducible component $\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X)$ such that the generic $\mathcal{M}$-curve is minimal and unramified at 0, and the generic fibre of $\mathcal{M}_1^{\mathrm{free}} \to X$ is geometrically irreducible and positive-dimensional.

Given a nonsingular variety with good family $(X, \mathcal{M})$, we will denote by $\hat{\mathcal{M}}_1 \subset \mathrm{Arc}_X$ the closure of the image of the generic point of $\mathcal{M}_1$ under the natural monomorphism $\mathcal{M}_1^{\mathrm{arc}} \to \mathrm{Arc}_X$ (in particular, $\mathcal{M}_1$ and $\hat{\mathcal{M}}_1$ are birational). Given another such pair $(Y, \mathcal{N})$, with $\dim(Y) = \dim(X)$, consider the subsheaf $\mathcal{C}_{X,Y} \subset \mathcal{E}_{X,Y}$ defined by

$$\mathcal{C}_{X,Y}(T) = \{\varphi \in \mathcal{E}_{X,Y}(T) \mid \varphi^* \tilde{\Phi}_{X,Y} : T \times_X X^\sharp \times_X \hat{\mathcal{M}}_1 \xrightarrow{\simeq} T \times_Y Y^\sharp \times_Y \hat{\mathcal{N}}_1\}.$$

It is a closed subscheme of $\mathcal{E}_{X,Y}$, parametrising ∞-jets of formally étale maps $X \to Y$ inducing isomorphisms of $\hat{\mathcal{M}}_1$ onto $\hat{\mathcal{N}}_1$. We will also use the 'generic locus'

$$\mathcal{C}^\circ_{X,Y} = x \times_X \mathcal{C}_{X,Y} \times_Y y$$

where x, y are the generic points of X, Y.

Lemma 3.1.2. *The stratification on $\mathcal{E}_{X,Y}/X$ restricts to $\mathcal{C}_{X,Y}/X$ and $\mathcal{C}^{\circ}_{X,Y}/X$.*

Proof. It is enough to notice that $\tilde{\Phi}_{X,Y}$ is horizontal with respect to the pullbacks of the canonical stratifications on $X^{\sharp} \times_X \mathrm{Arc}_X$ and $Y^{\sharp} \times_Y \mathrm{Arc}_Y$. $\square$

Lemma 3.1.3. *$\mathcal{C}^{\circ}_{X,Y}$ is dense in $\mathcal{C}_{X,Y}$.*

Proof. Let $c_0 \in \mathcal{C}_{X,Y}$ be a point with $\mathrm{pr}_X(c_0) = x_0$. Let $I \subset \mathcal{O}_{X,x_0}$ be the kernel of $\mathcal{O}_{X,x_0} \to \mathrm{pr}_{X*}\mathcal{O}_{\mathcal{C}_{X,Y},c_0}$. By Lemma 3.1.2, $I \subset \cap_r \mathfrak{m}^r_{x_0} = 0$, so that $x \times_X \operatorname{Spec} \mathcal{O}_{\mathcal{C}_{X,Y},c_0}$ is dense in $\operatorname{Spec} \mathcal{O}_{\mathcal{C}_{X,Y},c_0}$.

Now let $c \in x \times_X \operatorname{Spec} \mathcal{O}_{\mathcal{C}_{X,Y},c_0}$ be a point with $\mathrm{pr}_Y(c) = y_0$. Let $J \subset \mathcal{O}_{Y,y_0}$ be the kernel of $\mathcal{O}_{Y,y_0} \to \mathrm{pr}_{Y*}\mathcal{O}_{\mathcal{C}_{X,Y},c}$. Since the definition of $\mathcal{C}_{X,Y}$ is symmetric in X, Y, Lemma 3.1.2 gives a stratification on $\mathcal{C}_{X,Y}/Y$. Hence $J \subset \cap_r \mathfrak{m}^r_{y_0} = 0$, so that $\operatorname{Spec} \mathcal{O}_{\mathcal{C}_{X,Y},c} \times_Y y$ is dense in $\operatorname{Spec} \mathcal{O}_{\mathcal{C}_{X,Y},c}$. $\square$

Lemma 3.1.4. *There is a natural commutative diagram*

$$\begin{tikzcd}
\mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X m_1 \arrow[r] \arrow[d] & \mathcal{C}^{\circ}_{X,Y} \times_Y Y^{\sharp} \times_Y n_1 \arrow[d] \\
\mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X \hat{\mathcal{M}}_1 \arrow[r, "\tilde{\Phi}_{X,Y}"] & \mathcal{C}^{\circ}_{X,Y} \times_Y Y^{\sharp} \times_Y \hat{\mathcal{N}}_1
\end{tikzcd}$$

where m_1, n_1 are the generic points of $\mathcal{M}_1$, $\mathcal{N}_1$.

Proof. Recall that $m_1 \to \hat{\mathcal{M}}_1$ and $n_1 \to \hat{\mathcal{N}}_1$ are inclusions of generic points, while the bottom horizontal arrow is an isomorphism. The field extensions $\kappa(m_1)/\kappa(x)$ and $\kappa(n_1)/\kappa(y)$ are separable by freeness of m_1, n_1. It follows that for every $c \in C^{\circ}_{X,Y}$ the induced isomorphism

$$\kappa(c) \otimes_{\kappa(x)} (x \times_X \hat{\mathcal{M}}_1) \xrightarrow{\tilde{\Phi}_{X,Y}} \kappa(c) \otimes_{\kappa(y)} (y \times_Y \hat{\mathcal{N}}_1)$$

sends $c \times_x m_1$ to $c \times_y n_1$. Since $\mathcal{C}^{\circ}_{X,Y} \times_Y n_1$ is a localisation of $\mathcal{C}^{\circ}_{X,Y} \times_Y \hat{\mathcal{N}}_1$, it follows that

$$\mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X \hat{\mathcal{M}}_1 \xrightarrow{\tilde{\Phi}_{X,Y}} \mathcal{C}^{\circ}_{X,Y} \times_Y Y^{\sharp} \times_Y \mathcal{N}_1$$

sends $\mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X m_1$ into $\mathcal{C}^{\circ}_{X,Y} \times_Y Y^{\sharp} \times_Y n_1$. $\square$

3.1.2 Parallel transport along curves

Note that given a morphism $f : T \to X$, the stratification on $\mathcal{C}_{X,Y}/X$ induces one on the pullback $f^*\mathcal{C}_{X,Y}/T$:

$$\begin{tikzcd}
T^{\sharp} \times_T f^*\mathcal{C}_{X,Y} \arrow[r, equal] & T^{\sharp} \times_X \mathcal{C}_{X,Y} \arrow[r, equal] & T^{\sharp} \times_{X^{\sharp}} (X^{\sharp} \times_X \mathcal{C}_{X,Y}) \arrow[d, "\simeq"'] \\
f^*\mathcal{C}_{X,Y} \times_T T^{\sharp} \arrow[r, equal] & \mathcal{C}_{X,Y} \times_X T^{\sharp} \arrow[r, equal] & (\mathcal{C}_{X,Y} \times_X X^{\sharp}) \times_{X^{\sharp}} T^{\sharp}.
\end{tikzcd}$$

The key point of this section is then the following 'parallel transport' result, an analogue of Hwang and Mok's analytic continuation along rational curves [11].

Proposition 3.1.5. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be a pair of nonsingular varieties with good families, such that* $\dim(X) = \dim(Y)$. *Let m_2 be the generic point of $\mathcal{M}_2$ together with evaluation maps $m_2 \rightrightarrows X$. Then there is a natural isomorphism*

$$\mathcal{C}^{\circ}_{X,Y} \times_X m_2 \to m_2 \times_X \mathcal{C}^{\circ}_{X,Y}$$

horizontal over m_2 with respect to the induced stratifications.

Proof. Let $\tilde{m}_1 \in \mathcal{M}_2$ be the image of m_1 under the zero-section $\mathcal{M}_1 \to \mathcal{M}_2$. Set

$$M = \operatorname{Spec} \mathcal{O}_{\mathcal{M}_2, \tilde{m}_1}, \quad \hat{M} = \operatorname{Spf} \hat{\mathcal{O}}_{\mathcal{M}_2, \tilde{m}_1}$$

so that M is the spectrum of a discrete valuation ring with closed point $\tilde{m}_1$ and generic point m_2, and $\hat{M}$ is its completion. We will first construct a morphism

$$\psi : \mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X M \to \mathcal{N}_2.$$

extending the canonical top horizontal arrow in the diagram

$$\begin{array}{ccc}
\mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X \hat{M} & \xrightarrow{\hat{\psi}} & \hat{\mathcal{N}}_2 \\
\downarrow & & \downarrow \\
\mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X \mathrm{Arc}_X \times_X X^{\sharp} & \longrightarrow & \mathrm{Arc}_Y \times_Y Y^{\sharp}
\end{array}$$

where the bottom hotizontal arrow is induced by $\tilde{\Phi}$ and Φ, while the vertical arrows are induced by the natural maps to the spaces of arcs and by evaluation at the second marked point.

With $(\mathcal{M}_2/X)^2$ denoting the fibre product of $\mathcal{M}_2$ with itself with respect to the *right* structure maps into X, consider the natural diagram

$$\mathcal{M}_2 \underset{\Delta}{\overset{q_1}{\leftrightarrows}} (\mathcal{M}_2/X)^{\sharp} \xrightarrow{q_2} X^{\sharp} \times_X \mathcal{M}_2$$

of morphisms over $X^{\sharp}$. Its pullback by the rightmost structure map $\mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \to X$ gives

$$\mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X M \underset{\tilde{\Delta}}{\overset{\tilde{q}_1}{\leftrightarrows}} \mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X (M/X)^{\sharp} \xrightarrow{\tilde{q}_2} \mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X X^{\sharp} \times_X M.$$

Let $\pi : \mathcal{M}_2 \to \mathcal{N}_1$, $\varpi : \mathcal{N}_2 \to \mathcal{N}_1$ and $p_{13} : X^{\sharp} \times_X X^{\sharp} \to X^{\sharp}$ denote the natural projections. By Lemma 3.1.4, we have the composite

$$\begin{aligned}
\nu : \mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X (M/X)^{\sharp} \quad & \xrightarrow{\tilde{q}_2} \quad \mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X X^{\sharp} \times_X M \\
& \xrightarrow{p_{13*}} \quad \mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X M \\
& \xrightarrow{\pi_*} \quad \mathcal{C}^{\circ}_{X,Y} \times_X X^{\sharp} \times_X m_1 \\
& \xrightarrow{\tilde{\Phi}_{X,Y}} \quad \mathcal{C}^{\circ}_{X,Y} \times_Y Y^{\sharp} \times_Y n_1 \\
& \xrightarrow{\mathrm{pr}_{n_1}} \quad n_1.
\end{aligned}$$

Consider now the pullback diagram

$$\begin{array}{ccccc}
\tilde{\Delta}^*\nu^*\mathcal{N}_2 & \longrightarrow & \nu^*\mathcal{N}_2 & \longrightarrow & \mathcal{N}_2 \\
{\scriptstyle h}\downarrow & & \downarrow & & {\scriptstyle \varpi}\downarrow \\
\mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X M & \xrightarrow{\tilde{\Delta}} & \mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X (M/X)^\sharp & \xrightarrow{\nu} & \mathcal{N}_1
\end{array}$$

where the vertical arrows are $\mathbb{P}^1$-bundles. Viewing $\mathcal{N}_2 \to \mathcal{N}_1$ as the universal $\mathcal{N}$-curve in Y, let $N_{\mathcal{N}_2/Y}$ be the universal normal sheaf on $\mathcal{N}_2$. Identifying $T_{M/X}$ with the pullback by Δ of the relative tangent sheaf of q_1, the pullback $\tilde{\Delta}^* d\nu$ defines by adjunction a map

$$g : r_1^* T_{M/X} \to r_2^* N_{\mathcal{N}_2/Y}$$

of locally free sheaves on $\tilde{\Delta}^*\nu^*\mathcal{N}_2$, where.

$$M \xleftarrow{r_1} \tilde{\Delta}^*\nu^*\mathcal{N}_2 \xrightarrow{r_2} \mathcal{N}_2,$$

are the natural projections.

Lemma 3.1.6. *The zero-locus of g is the graph of a morphism*

$$\psi : \mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X M \to \mathcal{N}_2$$

lifting $\nu \circ \tilde{\Delta}$ and extending $\hat{\psi}$.

Proof. Let $\mathcal{C}^\circ_{X,Y}$ be the zero-locus of g. Since h is proper, so it $h|_{\mathcal{C}^\circ_{X,Y}}$. Since the generic $\mathcal{N}$-curve is minimal, the ideal sheaf $\mathcal{I}_{\mathcal{C}^\circ_{X,Y}}$ of $\mathcal{C}^\circ_{X,Y}$ in $\tilde{\Delta}^*\nu^*\mathcal{N}_2$ splits along the fibres of the $\mathbb{P}^1$-bundle h into invertible sheaves with degrees in $\{-1, 0\}$, so that $R^1 h_* \mathcal{I}_{\mathcal{C}^\circ_{X,Y}} = 0$ and the natural map

$$\mathcal{O}_{\mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X M} \to h_* \mathcal{O}_{\mathcal{C}^\circ_{X,Y}}$$

is surjective. It follows that every geometric fibre of $h|_{\mathcal{C}^\circ_{X,Y}}$ is either empty, a single reduced point, or a whole $\mathbb{P}^1$.

Consider the pullback

$$\begin{array}{ccccc}
\iota^*\tilde{\Delta}^*\nu^*\hat{\mathcal{N}}_2 & \xrightarrow{\tilde{\iota}} & \tilde{\Delta}^*\nu^*\hat{\mathcal{N}}_2 & \longrightarrow & \hat{\mathcal{N}}_2 \\
\downarrow & & \downarrow & & {\scriptstyle \varpi}\downarrow \\
\mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X \hat{M} & \xrightarrow{\iota} & \mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X M & \xrightarrow{\nu\circ\tilde{\Delta}} & \mathcal{N}_1
\end{array}$$

where ι is the natural monomorphism. By construction, the graph of

$$\hat{\psi} : \mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X \hat{M} \to \hat{\mathcal{N}}_2$$

factors through $\mathcal{C}^\circ_{X,Y}$. Since the geometric generic fibre of $\mathcal{M}_1^{\mathrm{free}} \to X$ is positive-dimensional, the geometric fibres of $h|_{\mathcal{C}^\circ_{X,Y}}$ over $\mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X \tilde{m}_1$ are single reduced points, so that the restriction $\mathcal{C}^\circ_{X,Y} \cap \iota^*\tilde{\Delta}^*\nu^*\mathcal{N}_2$ actually coincides with the graph of $\hat{\psi}$. In particular, $\iota^* h|_{\mathcal{C}^\circ_{X,Y}}$ is an isomorphism. Since ι is an epimorphism of formal schemes, and ϖ is a $\mathbb{P}^1$-bundle, it follows that $\tilde{\iota}$ is an epimorphism of formal schemes. Thus $h|_{\mathcal{C}^\circ_{X,Y}}$ is a closed immersion, adic and admitting a section over ι, hence an isomorphism. □

We have thus constructed the map ψ, which will allow us to produce a morphism $\mathcal{C}^\circ_{X,Y} \times_X M \to M \times_X \mathcal{C}^\circ_{X,Y}$ whose restriction over m_2 gives the isomorphism announced in the Proposition. By freeness of the generic $\mathcal{M}$-curve, we can choose an isomorphism

$$\rho : \mathcal{C}^\circ_{X,Y} \times_X M \times_X X^\sharp \to \mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X M$$

over $\mathcal{C}^\circ_{X,Y} \times X$ (leftmost-rightmost structure map). Let ϕ be the composite

$$\phi : \mathcal{C}^\circ_{X,Y} \times_X M \times_X X^\sharp \xrightarrow{\rho} \mathcal{C}^\circ_{X,Y} \times_X X^\sharp \times_X M \xrightarrow{\psi} \mathcal{N}_2 \to Y$$

where the rightmost arrow is the right structure map. Consider now the pair

$$\mathcal{C}^\circ_{X,Y} \times_X M \times_X X^\sharp \underset{\phi}{\overset{s^*\phi}{\rightrightarrows}} Y$$

where $s = \Delta_X \circ p_1 : X^\sharp \to X^\sharp$ is the 'retraction onto origin'. These induce a morphism

$$\theta = \langle \mathrm{id}, s^*\phi, \phi \rangle : \mathcal{C}^\circ_{X,Y} \times_X M \times_X X^\sharp \to \mathcal{C}^\circ_{X,Y} \times_X M \times_Y Y^\sharp$$

together with the map

$$[\theta] : \mathcal{C}^\circ_{X,Y} \times_X M \to M \times_X \mathcal{F}_{X,Y}$$

defined by the universal property of $\mathcal{F}_{X,Y}$.

Now, since ψ is an extension of $\hat{\psi}$, there is a commutative diagram

$$\begin{tikzcd}
\mathcal{C}^\circ_{X,Y} \times_X \hat{M} \times_X X^\sharp \arrow[r, "\theta"] \arrow[d, "\mathrm{id} \times (p_{13} \circ e)"'] & \mathcal{C}^\circ_{X,Y} \times_X \hat{M} \times Y^\sharp \arrow[d, "\mathrm{id} \times \mathrm{pr}_{Y^\sharp}"] \\
\mathcal{C}^\circ_{X,Y} \times_X X^\sharp \arrow[r, "\Phi_{X,Y}"] & \mathcal{C}^\circ_{X,Y} \times_Y Y^\sharp
\end{tikzcd}$$

where $e : \hat{M} \to \hat{X}$ is the restriction of the structure morphism $\mathcal{M}_2 \to X \times X$. Hence the restriction of $[\theta]$ to $\mathcal{C}^\circ_{X,Y} \times_X \hat{M}$ is a pullback of the stratifying isomorphism

$$\mathcal{C}^\circ_{X,Y} \times_X X^\sharp \to X^\sharp \times_X \mathcal{C}^\circ_{X,Y},$$

and in particular it is horizontal and factors through $\hat{M} \times_X \mathcal{C}^\circ_{X,Y}$. Since $\mathcal{C}^\circ_{X,Y} \times_X \hat{M}$ does not factor through any proper subscheme of $\mathcal{C}^\circ_{X,Y} \times_X M$, it follows that $[\theta]$ factors through the open subscheme $M \times_X \mathcal{E}_{X,Y} \subset M \times_X \mathcal{F}_{X,Y}$, through the closed subscheme $M \times_X$

$\mathcal{C}_{X,Y} \subset M \times_X \mathcal{E}_{X,Y}$, and finally through the 'generic locus' $M \times_X \mathcal{C}^\circ_{X,Y} \subset M \times_X \mathcal{C}_{X,Y}$. Letting

$$\tau : \mathcal{C}^\circ_{X,Y} \times_X M \to M \times_X \mathcal{C}^\circ_{X,Y}$$

be the map induced by the point-swapping involution on $M \subset \mathcal{M}_2$, we have a pair of morphisms

$$\mathcal{C}^\circ_{X,Y} \times_X M \underset{\tau[\theta]\tau}{\overset{[\theta]}{\rightleftarrows}} M \times_X \mathcal{C}^\circ_{X,Y}$$

such that $\tau[\theta]$ and $[\theta]\tau$ restrict to identity over $\mathcal{C}^\circ_{X,Y} \times_X \hat{M}$ and $\hat{M} \times_X \mathcal{C}^\circ_{X,Y}$. Hence the above morphisms are mutual inverses, and their restriction over m_2 gives the isomorphism announced in the Proposition, thus concluding its proof. □

3.1.3 Induction and descent

We can now use Proposition 3.1.5 inductively to trivialise $\mathcal{C}^\circ_{X,Y}$ along generic chains of $\mathcal{M}$-curves. Under suitable conditions, the trivialisation descends generically to the base.

Proposition 3.1.7. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be a pair of nonsingular varieties with good families such that* $\dim(X) = \dim(Y)$. *Suppose that X is simply-connected, of Picard number* 1. *Let ξ be the generic point of $X \times X$. Then there is a natural isomorphism*

$$\mathcal{C}^\circ_{X,Y} \times_X \xi \to \xi \times_X \mathcal{C}^\circ_{X,Y}$$

horizontal over ξ.

We will need the following bit of commutative algebra.

Lemma 3.1.8. *Let L/K be a finitely generated field extension, and $K^{s,L}$ the separable algebraic closure of K in L. Then the following is an equaliser diagram:*

$$K^{s,L} \to L \rightrightarrows \widehat{L \otimes_K L}$$

where we complete at the diagonal ideal.

Proof. Let $K' \subset L$ be the equaliser of $L \rightrightarrows \widehat{L \otimes_K L}$. We first observe that the claim is true in the following cases:

1. L/K purely transcendental: then $L \otimes_K L \to \widehat{L \otimes_K L}$ is injective, so that $K' = K$.
2. L/K purely inseparable: then $L \otimes_K L$ is Artinian, hence already complete, and we argue as above.
3. L/K separable algebraic: then $L \otimes_K L$ is a product of finitely many copies of L, so that the diagonal ideal is idempotent and $K' = L$.

In the general case, by (2) we can assume that L/K is separably generated, so that there is an intermediate extension $K \subset L_0 \subset L$ such that L/L_0 is separable algebraic and L_0/K purely transcendental. Suppose $x \in K'$. Since K' is invariant under the Galois group of L/L_0, the conjugates of x are also contained in K'. It follows that the coefficients of the minimal polynomial of x in L_0 are contained in $K' \cap L_0$, and thus in K by (1). Hence x is separable algebraic over K, and thus $K' \subset K^{s,L}$. The converse follows by (3). $\square$

Proof of Proposition 3.1.7. Let m_2^i the generic point of $\mathcal{M}_2^{i,\text{free}}$. By Lemma 2.2.6, $m_2^{i+1} \in m_2^i \times_X m_2$, so that Proposition 3.1.5 and induction on i gives a horizontal isomorphism

$$\theta^i : \mathcal{C}^\circ_{X,Y} \times_X m_2^i \to m_2^i \times_X \mathcal{C}^\circ_{X,Y}$$

over m_2^i. By Lemma 2.2.5 we can choose i such that m_2^i maps to $\xi \in X \times X$. By horizontality of θ^i, the pullbacks

$$\mathcal{C}^\circ_{X,Y} \times_X (m_2^i/\xi)^\sharp \rightrightarrows \mathcal{C}^\circ_{X,Y}$$

of $\mathrm{pr}_{\mathcal{C}^\circ_{X,Y}} \circ \theta^i$ by $(m_2^i/\xi)^\sharp \rightrightarrows m_2^i$ coincide. Hence, by Lemma 3.1.8, θ^i descends to a morphism

$$\bar{\theta} : \mathcal{C}^\circ_{X,Y} \times_X \tilde{\xi} \to \tilde{\xi} \times_X \mathcal{C}^\circ_{X,Y}$$

where $\tilde{\xi}$ is the spectrum of the separable algebraic closure of $\kappa(\xi)$ in $\kappa(m_2^i)$. Being an algebraic subextension of a finitely generated extension, $\kappa(\tilde{\xi})/\kappa(\xi)$ is finite. Horizontality and invertibility of $\bar{\theta}$ follows from that of θ^i by descent.

To show that $\bar{\theta}$ is in fact defined over $\kappa(\xi)$, we first consider a geometric generic point $\bar{\zeta}$ of $X \times \mathcal{M}_0$ and the corresponding rational curve

$$f : \mathbb{P}^1_{\bar{\zeta}} \to \bar{\zeta} \times X \to X \times X.$$

Lemma 3.1.9. *Let $W \subset f^*\tilde{\xi}$ be a connected component. Then $f^*\bar{\theta} : \mathcal{C}^\circ_{X,Y} \times_X W \to W \times_X \mathcal{C}^\circ_{X,Y}$ descends along $W \to f^*\xi$.*

Proof. Let $\bar{\vartheta} = \mathrm{pr}_2 \circ \bar{\theta} : \mathcal{C}^\circ_{X,Y} \times_X \tilde{\xi} \to \mathcal{C}^\circ_{X,Y}$. Recall that in the Proof of Proposition 3.1.5 we have actually constructed an isomorphism $\theta : \mathcal{C}^\circ_{X,Y} \times_X M \to M \times_X \mathcal{C}^\circ_{X,Y}$ horizontal over $M = \operatorname{Spec} \mathcal{O}_{\mathcal{M}_2,\tilde{m}_1}$, where $\tilde{m}_1$ is the image of m_1 under the zero-section $\mathcal{M}_1 \to \mathcal{M}_2$. Consider the diagram with Cartesian squares

$$\begin{array}{ccccc}
\mathcal{C}^\circ_{X,Y} \times_X (X\backslash\tilde{\xi})^\sharp \times_{X\times X} \hat{M} & \longrightarrow & \mathcal{C}^\circ_{X,Y} \times_X (X\backslash\tilde{\xi})^2 \times_{X\times X} M & \xrightarrow{\Theta} & \mathcal{C}^\circ_{X,Y} \times_X M \times_X \mathcal{C}^\circ_{X,Y} \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{C}^\circ_{X,Y} \times_X (X\backslash\tilde{\xi})^\sharp & \longrightarrow & \mathcal{C}^\circ_{X,Y} \times_X (X\backslash\tilde{\xi})^2 & \xrightarrow{\bar{\vartheta}\times\bar{\vartheta}} & \mathcal{C}^\circ_{X,Y} \times \mathcal{C}^\circ_{X,Y}
\end{array}$$

The left square is induced by the natural inclusion $(X\backslash\tilde{\xi})^\sharp \to (X\backslash\tilde{\xi})^2$. By horizontality, the top horizontal composite factors through the graph of θ.

Identify $\mathbb{P}^1_{\bar{\zeta}}$ and $\mathbb{P}^1_{\bar{\zeta}} \times_{\bar{\zeta}} \mathbb{P}^1_{\bar{\zeta}}$ with, respectively, $\bar{\zeta} \times_{\mathcal{M}_0} \mathcal{M}_1$ and $\bar{\zeta} \times_{\mathcal{M}_0} \mathcal{M}_2$. Consider $M_{\bar{\zeta}} = \bar{\zeta} \times_{\mathcal{M}_0} M$ as a subscheme of $\mathbb{P}^1_{\bar{\zeta}} \times_{\bar{\zeta}} \mathbb{P}^1_{\bar{\zeta}}$. The curve f is identified with the natural morphism $\bar{\zeta} \times_{\mathcal{M}_0} \mathcal{M}_1 \to X \times X$ induced by $\bar{\zeta} \to X$ and $\mathcal{M}_1 \to X$. Let $W \subset f^*\tilde{\xi}$ be an irreducible component. Pulling back the top row of the above diagram, we have that the composite

$$\mathcal{C}^\circ_{X,Y} \times_X (W/\bar{\zeta})^\sharp \to \mathcal{C}^\circ_{X,Y} \times_X (W/\bar{\zeta})^2 \to \mathcal{C}^\circ_{X,Y} \times \mathcal{C}^\circ_{X,Y}$$

factors through the pullback of the diagram of θ by $(W/\bar{\zeta})^\sharp \to M_{\bar{\zeta}}$. Hence so does the right arrow itself, and in particular the restriction

$$\mathcal{C}^\circ_{X,Y} \times_X (W/\mathbb{P}^1_{\bar{\zeta}})^2 \to \mathcal{C}^\circ_{X,Y} \times \mathcal{C}^\circ_{X,Y}$$

factors through the diagonal. Hence $f^*\bar{\vartheta} : \mathcal{C}^\circ_{X,Y} \times W \to \mathcal{C}^\circ_{X,Y}$ descends along $W \to f^*\xi$, and so does $f^*\bar{\theta}$. □

Continuing the proof of the Proposition, fix a separable closure $\kappa(\bar{\xi}^s)$ of $\kappa(\xi)$. The Galois group $\mathrm{Gal}(\bar{\xi}^s/\xi)$ acts on the set E of isomorphisms $\mathcal{C}^\circ_{X,Y} \times_X \bar{\xi}^s \to \bar{\xi}^s \times_X \mathcal{C}^\circ_{X,Y}$ horizontal over $\bar{\xi}^s$. By the first part of the proof, there is an element $\bar{\theta} \in E$ whose stabiliser in $\mathrm{Gal}(\bar{\xi}^s/\xi)$ is of finite index. Letting $\eta \simeq \bar{\zeta} \otimes k(t)$ be the generic point of $\mathbb{P}^1_{\bar{\zeta}}$, we have an extension $\kappa(\eta)/\kappa(\xi)$. We can lift it to $\kappa(\bar{\eta}^s)/\kappa(\bar{\xi}^s)$ where $\bar{\eta}^s$ is a separable closure of η. It then follows by Lemma 3.1.9 that the stabiliser of $\bar{\theta}$ in $\mathrm{Gal}(\bar{\xi}^s/\xi)$ contains the image of $\mathrm{Gal}(\bar{\eta}^s/\eta)$.

Let $\Gamma \to X \times X$ be a normal Galois cover corresponding to the stabiliser of $\bar{\theta}$, so that $\bar{\theta}$ is defined over the generic point of Γ, and $f^*\Gamma$ is trivial by the previous paragraph. We want to show that Γ itself is trivial. Since X is simply-connected, it will be enough to show that $\Gamma \to X \times X$ is étale. Assuming the opposite, we have by the classical purity theorem that it is ramified over a divisor $D \subset X \times X$. Since the problem is symmetric under the transposition on $X \times X$, we can assume that D is not a pullback of a divisor from the first factor. Since X has Picard number 1, the pullback f^*D is positive, and there is a lift $\tilde{f} : \mathbb{P}^1_{\bar{\zeta}} \to \Gamma$ of f intersecting the ramification divisor. It follows that f is tangent to D at the intersection points. But by Lemma 2.2.4 a generic $\mathcal{M}$-curve intersects D transversely, a contradiction. Hence Γ is trivial, $\bar{\theta}$ is invariant under $\mathrm{Gal}(\bar{\xi}^s/\xi)$, and thus finally defined over ξ. □

3.2 Extension

Theorem 3.2.1. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be a pair of simply-connected, nonsingular projective Fano varieties of Picard number 1 and equal dimensions, together with good families of rational curves. Let K be an algebraically closed field, and $\bar{c} : \mathrm{Spec}\, K \to \mathcal{C}^\circ_{X,Y}$ a geometric point. Then there is an isomorphism*

$$\phi : X \otimes K \to Y \otimes K$$

extending the canonical isomorphism $\bar{c}^*\Phi : \bar{c}^*X^\sharp \to \bar{c}^*Y^\sharp$.

Proof. By Lemma 2.2.5, $\mathcal{M}_2^{i,\text{free}} \to X \times X$ is dominant for some $i > 0$. Let $X_K = X \otimes K$ and $Y_K = Y \otimes K$ with generic points x_K, y_K. Then, by Proposition 3.1.7, there is a horizontal section

$$\sigma : \eta_{X_K} \to \mathcal{C}^{\circ}_{X,Y} \otimes K.$$

Its composite with projection to Y_K extends to a morphism

$$\phi_0 : X_K \setminus W \to Y_K, \qquad \operatorname{codim}_{X_K} W \geq 2$$

whose restriction to x_K is formally étale and induces an isomorphism $x_K^* X^\sharp \times_X \hat{\mathcal{M}}_1 \simeq \phi_0|_{x_K}^* Y^\sharp \times_Y \hat{\mathcal{N}}_1$. It follows that ϕ_0 is dominant and generically étale. We will now show that it is actally étale on entire $X_K \setminus W$.

Indeed, let $\bar{\zeta}$ be a geometric generic point of $\mathcal{M}_1 \otimes K$. Then the generic $\mathcal{M}$-curve $f : \mathbb{P}^1_\zeta \to X_K$ factors through $X_K \setminus W$ (by Lemma 2.2.4), and $\phi_0 \circ f : \mathbb{P}^1_\zeta \to Y_K$ is a generic $\mathcal{N}$-curve (since its restriction to $\hat{\mathbb{P}}^1_\zeta$ maps, as an unramified morphism from a formal disc, to the generic point of $\hat{\mathcal{N}}_1$). Now, if ϕ_0 is not étale, then, by the classical purity theorem, it is ramified over a divisor $D \subset Y_K$. Since Y has Picard number 1, $\phi_0 \circ f$ intersects D. But since $\phi_0 \circ f$ is free, the intersection is transverse (again by Lemma 2.2.4). It then follows that f does not intersect the ramification divisor in $X_K \setminus W$ – a contradiction.

Hence ϕ_0 is étale. Furthermore, since $\phi_0 \circ f$ is free, it follows that the complement of the image of ϕ_0 has codimension at least 2 in Y_K (Lemma 2.2.4. It then follows by simply-connectedness of Y_K that ϕ_0 is an isomorphism onto its image. Now, since X and Y are Fano, we can find an integer $d > 0$ such that $-dK_X$ and $-dK_Y$ are both very ample. Being an isomorphism of open subsets whose complements have codimension at least 2, ϕ_0 induces an isomorphism of Picard groups and of spaces of global sections for any invertible sheaf. Using the differential $d\phi_0$ to identify $\phi_0^* K_Y$ with K_X, we have a diagram

$$\begin{tikzcd}
X_L \arrow[r, dotted, "\phi_0"] \arrow[d] & Y_L \arrow[d] \\
\mathbb{P}H^0(X_K, \mathcal{O}(-dK_X))^\vee \arrow[r] & \mathbb{P}H^0(Y_K, \mathcal{O}(-dK_Y))^\vee
\end{tikzcd}$$

where the vertical arrows are the projective embeddings indced by $-dK_X$ and $-dK_Y$. Hence ϕ_0 extends to an isomorphism $\phi : X_K \to Y_K$.

It remains to check that ϕ is an extension of $\bar{c}^*\Phi$. Consider the lift of ϕ to a morphism $\tilde{\phi}$, horizontal over $\mathcal{M}_2^i$ and fitting into a commutative diagram

$$\begin{array}{ccc}
\bar{c} \times_X \mathcal{M}_2^i & \xrightarrow{\tilde{\phi}} & \bar{c} \times_X \mathcal{M}_2^i \times_X \mathcal{C}_{X,Y} \\
\downarrow & & \downarrow \\
\bar{c} \times X & \xrightarrow{\phi} & \bar{c} \times Y
\end{array}$$

where the left vertical arrow is induced by the right structure map $\mathcal{M}_2^i \to X$. Recall that in the proof of Proposition 3.1.5 we have constructed an isomorphism

$$\mathcal{C}^\circ_{X,Y} \times_X M \to M \times_X \mathcal{C}^\circ_{X,Y}$$

where $M = \operatorname{Spec} \mathcal{O}_{\mathcal{M}_2, \tilde{m}_1}$ and $\tilde{m}_1$ is the image of m_1 under the zero-section $\mathcal{M}_1 \to \mathcal{M}_2$. Note that its restriction over $\tilde{m}_1$ is the identity on $\mathcal{C}^\circ_{X,Y} \times_X m_1$. By induction, we have an isomorphism

$$\tilde{\theta}^i : \mathcal{C}^\circ_{X,Y} \times_X (X\backslash M/X)^i \to (X\backslash M/X)^i \times_X \mathcal{C}^\circ_{X,Y}$$

extending θ^i of the proof of Proposition 3.1.7. It follows that we have a commutative diagram

$$\begin{tikzcd}
\bar{c} \times_X m_2^i \arrow[r] \arrow[d] & \bar{c} \times_X \mathcal{M}_2^i \arrow[d, "\tilde{\phi}"'] \\
\bar{c} \times_X (X\backslash M/X)^i \arrow[r, "\bar{c}^*\tilde{\theta}^i"] & \bar{c} \times_X \mathcal{M}_2^i \times_X \mathcal{C}_{X,Y}
\end{tikzcd}$$

i.e. $\tilde{\phi}$ and $\bar{c}^*\tilde{\theta}^i$ agree on $\bar{c} \times_X m_2^i$. Then, by irreducibility and reducedness of $\bar{c} \times_X \mathcal{M}_2^{i,\text{free}}$ (cf. Lemma 2.2.6), they agree on $(X\backslash M/X)^i \subset \mathcal{M}_2^i$. In particular, the composite

$$\bar{c} \times_X m_1 \xrightarrow{\bar{c}^*\langle \tilde{m}_1, \ldots, \tilde{m}_1 \rangle} \bar{c} \times_X \mathcal{M}_2^i \xrightarrow{\tilde{\phi}} \bar{c} \times_X \mathcal{M}_2^i \times_X \mathcal{C}_{X,Y} \to \bar{c} \times_X \mathcal{C}_{X,Y}$$

factors through the diagonal embedding $\bar{c} \to \bar{c} \times_X \mathcal{C}_{X,Y}$. Since $\tilde{\phi}$ is a horizontal lift of ϕ, it follows that we have a commutative diagram

$$\begin{tikzcd}
(\bar{c} \times_X X^\sharp) \times_{X \times X} \mathcal{M}_2^i \arrow[r, "\tilde{\phi}"] \arrow[d] & \bar{c} \times_X \mathcal{M}_2^i \times_X \mathcal{C}_{X,Y} \arrow[d] \\
\bar{c} \times_X X^\sharp \arrow[r, "\bar{c}^*\Phi"] & \bar{c} \times Y
\end{tikzcd}$$

Since the left vertical arrow is an epimorphism of formal schemes, it follows that the composite

$$\bar{c} \times_X X^\sharp \to \bar{c} \times X \xrightarrow{\phi} \bar{c} \times Y$$

coincides with $\bar{c}^*\Phi$. $\square$

Corollary 3.2.2. *Let $(X, \mathcal{M})$ and $(Y, \mathcal{N})$ be a pair of simply-connected, nonsingular projective Fano varieties of Picard number 1 and equal dimensions, together with good families of rational curves. Let K be an algebraically closed field, and $\bar{x}_0 : \operatorname{Spec} K \to X$, $\bar{y}_0 : \operatorname{Spec} K \to Y$ a pair of geometric points such that there is an isomorphism $\bar{x}_0^* X^\sharp \simeq \bar{y}_0^* Y^\sharp$ identifying $\bar{x}_0^* X^\sharp \times_X \hat{\mathcal{M}}_1$ with $\bar{y}_0^* Y^\sharp \times_Y \hat{\mathcal{N}}_1$. Then there is an isomorphism $X \simeq Y$ identifying $\mathcal{M}$ with $\mathcal{N}$.*

Proof. By Lemma 3.1.3, $\mathcal{C}^\circ_{X,Y}$ is nonempty, so that there is an algebraically closed extension L/K and a geometric point $\bar{c} : \operatorname{Spec} L \to \mathcal{C}^\circ_{X,Y}$, inducing by Theorem 3.2.1 an isomorphism

$$\phi_L : X \otimes L \to Y \otimes L$$

identifying $\hat{\mathcal{M}}_1 \otimes L$ with $\hat{\mathcal{N}}_1 \otimes L$ and thus $\mathcal{M} \otimes L$ with $\mathcal{N} \otimes L$. Since X and Y are algebraic, there is a subalgebra $A \subset L$, of finite type over k, and such that ϕ_L is the base-change of an isomorphism

$$\phi_A : X \otimes A \to Y \otimes A$$

identifying $\mathcal{M} \otimes A$ with $\mathcal{N} \otimes A$. Restricting ϕ_A over a closed point of $\operatorname{Spec} A$ yields the desired isomorphism $X \simeq Y$. $\square$

4

The rigidity theorem

4.1 The setup

Definition. Let G/P be a cominuscule variety. A *smooth projective degeneration* of G/P is a smooth, projective morphism $X \to S$ such that S is the spectrum of a discrete valuation ring over k, with residue field k and fraction field F, and the geometric generic fibre of X is isomorphic to $G/P \otimes \bar{F}$.

Hwang and Mok [10] show that, over $k = \mathbb{C}$, every smooth projective degeneration of a cominuscule variety G/P is an isotrivial fibration. Assuming from now on that k is of characteristic $p > 0$, we want to find conditions on p guaranteeing an analogous rigidity result. However, we will need to introduce an additional parameter.

Definition. Let G/P be a cominuscule variety, and d a positive integer. We will say that G/P is *d-rigid* if every smooth projective degeneration $X \to S$ of G/P, such that there exists a very ample invertible sheaf on X retricting to $\mathcal{O}_{G/P}(d) \otimes \bar{F}$ on the geometric generic fibre, is necessarily an isotrivial fibration.

Since isotriviality can be checked after faithfully flat base change, we can restrict to smooth projective degenerations with trivial generic fibres. Furthermore, since the group $\underline{\mathrm{Aut}}(G/P)$ is smooth, isotriviality is equivalent to the central fibre being isomorphic to G/P.

It the following we will fix a cominuscule variety G/P and a smooth projective degeneration $X \to S$ with trivial generic fibre. Denote with s_1, resp. s_0, the generic, resp. special, point of S. Let $X_1 = s_1 \times_S X$, $X_0 = s_0 \times_S X$. Recall that the Picard group of G/P is generated by an ample invertible sheaf $\mathcal{O}_{G/P}(1)$. We let $\mathcal{O}_X(1)$ be the unique extension of $\mathcal{O}_{G/P}(1) \otimes F$ to an invertible sheaf on X, and $\mathcal{O}_{X_0}(1)$ its restriction to the central fibre. Note that the restriction map

$$\operatorname{Pic} X \to \operatorname{Pic} X_1 \simeq \mathbb{Z}$$

is an isomorphism, so that $\mathcal{O}_X(1)$ is ample by projectivity of $X \to S$. We also let

$$\mathcal{M} \subset \underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1_S, X)$$

be the closed subscheme, flat over S, such that $s_1 \times_S \mathcal{M} \subset s_1 \times \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, G/P)$ is the component of lines on G/P. The fibre

$$s_0 \times_S \mathcal{M} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X_0)$$

is connected, but it may be in general reducible. Since every component is an irreducible family of rational curves of degree 1 with respect to $\mathcal{O}_{X_0}(1)$, hence unsplit by Lemma 2.2.3, it follows that $s_0 \times_S \mathcal{M}_0$ is proper. So is then $\mathcal{M}_0 \to S$.

Let us state some immediate properties of X_0.

Lemma 4.1.1. *The central fibre X_0 is simply-connected, Fano, of Picard number 1, and chain-connected by $(s_0 \times_S \mathcal{M})$-curves.*

Proof. Every finite étale cover $\tilde{X}_0 \to X_0$ deforms to a finite étale cover $\tilde{X} \to X$, restricting to $\tilde{X}_1 \to X_1$ at the generic fibre. By simply-connectedness of G/P, $\tilde{X}_1 \otimes \bar{F} \to X_1 \otimes \bar{F}$ is trivial. After a finite separable base change $T \to S$ it follows that $T \times_S \tilde{X} \to T \times_S X$ is trivial, and thus so is the restriction $\tilde{X}_0 \to X_0$ over a closed point of T above s_0. Hence X_0 is simply-connected.

The relative anticanonical sheaf $\omega^{-1}_{X/S}$ is isomorphic to $\mathcal{O}_X(\mathrm{index}(G/P))$, hence ample, so that in particular X_0 is Fano. Since $\mathcal{M}_0 \to S$ is proper, $\mathcal{M}^i_2 \to X \times_S X$ is surjective for i as in Proposition 2.3.6. Hence X_0 is chain-connected by $(s_0 \times_S \mathcal{M})$-curves, and furthermore of Picard number 1 by Proposition 2.2.2. □

4.2 Curves at the generic point

4.2.1 Smoothness

In order to proceed with the proof of rigidity, we first need to establish smoothness of the family of $\mathcal{M}$-curves through the generic point of X_0. Equivalently, we check that every $\mathcal{M}$-curve through the generic point of X_0 is free, a result that comes for free in characteristic zero (cf. [14]). Let us begin with a converse, stating that $\mathcal{M}$ contains all free rational curves of degree one:

Lemma 4.2.1. *Let $W \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X_0)$ be an irreducible component of degree 1 with respect to $\mathcal{O}_X(1)$, and such that the generic W-curve is free. Then $W \subset \mathcal{M}$.*

Proof. Let $f : \mathbb{P}^1 \otimes k(W) \to X_0$ be the generic W-curve. By freeness, $H = \underline{\mathrm{Hom}}^n_{S,\mathrm{bir}}(\mathbb{P}^1_S, X) \to S$ is smooth at η_W, so that in particular $\mathrm{Spec}\, \mathcal{O}_{H,\eta_W} \to S$ is faithfully flat. Since $\deg f^*\mathcal{O}_X(1) = 1$, it follows that $s_1 \times_S \mathrm{Spec}\, \mathcal{O}_{H,\eta_W}$ is a family of lines on $X_1 \simeq s_1 \times (G/P)$ and thus $\mathrm{Spec}\, \mathcal{O}_{H,\eta_W} \subset \mathcal{M}$. Hence $\eta_W \in \mathcal{M}$ and $W \subset \mathcal{M}$. □

Letting x_1, resp. x_0, be the generic point of X_1, resp. X_0, observe that $\mathcal{O}_S \subset \mathcal{O}_{X,x_0}$ is an unramified extension of discrete valuation rings, inducing separably generated extensions of residue and fraction fields.

Lemma 4.2.2. $\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1 \to \operatorname{Spec} \mathcal{O}_{X,x_0}$ *is flat.*

Proof. Since $\mathcal{M} \to S$ is flat, and $\mathcal{M} \to \mathcal{M}_1$ is a principal bundle, $\mathcal{M}_1 \to S$ is flat. That is, multiplication by the uniformizer of $\mathcal{O}_S$ is injective in the local rings of $\mathcal{M}_1$. But the uniformizer of $\mathcal{O}_S$ is also a uniformizer in $\mathcal{O}_{X,x_0}$. $\square$

Singular points of $x_0 \times_X \mathcal{M}_1$ can be detected by inseparability of their residue fields viewed as extensions of $\kappa(x_0)$:

Lemma 4.2.3. *Let $y \in x_0 \times_X \mathcal{M}_1$ be a point such that $\kappa(y)/\kappa(x_0)$ is separably generated. Then $x_0 \times_X \mathcal{M}_1 \to x_0$ is smooth at y.*

Proof. Let $f : \mathbb{P}^1 \otimes \overline{\kappa(y)} \to X_0$ be the corresponding rational curve. By separability of $\kappa(y)/\kappa(x_0)$, the map $H^0(\mathbb{P}^1 \otimes \overline{\kappa(y)}, f^*T_{X_0}) \to f|_0^* T_{X_0}$ is surjective, so that f is free. Hence $\mathcal{M}_1 \to X_0$ is smooth at y (cf. [14, Cor. 3.5.4], noting that $\mathcal{M}_1$ is the universal $\mathbb{P}^1$-bundle over $\mathcal{M}_0$). $\square$

Let now $\bar{x}_1$, resp. $\bar{x}_0$, be the geometric generic point of X_1, resp. X_0. We will identify a condition on the characteristic p ensuring that $\bar{x}_0 \times_X \mathcal{M}_1$ is a nonsingular variety. The idea is to examine the degrees of the components of its singular locus in a suitable projective embedding.

Lemma 4.2.4. *Let Z be an irreducible component of the singular locus of $\bar{x}_0 \times_X \mathcal{M}_1$. Then* $\operatorname{length} \mathcal{O}_{Z,\eta_Z}$ *is divisible by p.*

Proof. This is an immediate consequence of Lemma 4.2.3. $\square$

The idea of the following crucial lemma is due to Fedor Bogomolov.[1]

Lemma 4.2.5. *Let $Y \subset \mathbb{P}^N$ be a closed subscheme of degree e and pure dimension n. Assume that the singular locus Y^{sing} is zero-dimensional. Then* $\operatorname{length} Y^{\text{sing}} \leq e(e-1)^n$.

Proof. We claim that there is an n-dimensional subspace $V \subset H^0(Y, \mathcal{O}(e-1))$ such that the base locus of the linear system $|V|$ zero-dimensional and contains Y^{sing}. We will then have

$$\operatorname{length} Y^{\text{sing}} \leq Y \cdot ((d-1)H)^n = e(e-1)^n,$$

where H is the hyperplane class in $\mathbb{P}^N$.

To prove the claim, we first note that for every closed point $y \in Y \setminus Y^{\text{sing}}$ there exists a linear projection $\pi_y : \mathbb{P}^N \dashrightarrow \mathbb{P}^{n+1}$ such that $\pi_y(Y)$ is a degree d hypersurface, and π_y is an immersion on some open neighbourhood of y. Let $f_y \in H^0(\mathbb{P}^{n+1}, \mathcal{O}(e))$ be the homogeneous polynomial cutting out $\pi_y(Y)$, and $f_{y,i} \in H^0(\mathbb{P}^{n+1}, \mathcal{O}(e-1))$ its

[1] Explained to me by Jason Starr.

partial derivatives. Let $U \subset H^0(Y, \mathcal{O}(e-1))$ be the subspace generated by $\pi_y^* f_{y,i}$ for all $0 \le i \le n+1$ and all $y \in Y \setminus Y^{\text{sing}}$. For each $y \in Y \setminus Y^{\text{sing}}$ we have that not all $\pi_Y^* f_{y,i}$ vanish at y, but all vanish on Y^{sing}. It follows that the base locus of $|U|$ contains Y^{sing} and has the same reduced structure. In particular, it is zero-dimensional, so that we can find an n-dimensional subspace $V \subset U$ such that the base-locus of $|V|$ is zero-dimensional. $\square$

Lemma 4.2.6. *Let $m = \dim(\bar{x}_1 \times_X \mathcal{M}_1)$. Suppose there is a degree e projective embedding $\bar{x}_0 \times_X \mathcal{M}_1 \hookrightarrow \bar{x}_0 \times \mathbb{P}^N$. Assume $p > e(e-1)^m$. Then $\bar{x}_0 \times_X \mathcal{M}_1$ is smooth.*

Proof. Suppose $\bar{x}_0 \times_X \mathcal{M}_1 \subset \bar{x}_0 \times \mathbb{P}^N$ is not smooth. Let Z be an irreducible component of maximal dimension of the singular locus of $\bar{x}_0 \times_X \mathcal{M}_1$. By Bertini's Theorem, there is a linear subspace $\Lambda \subset \bar{x}_0 \times_X \mathbb{P}^N$ with $\operatorname{codim} \Lambda = \dim Z$ such that $Y = \Lambda \cap (\bar{x}_0 \times_X \mathcal{M}_1)$ is a degree e subscheme of pure dimension $m - \dim Z$, whose singular locus is zero-dimensional and contains $\Lambda \cap Z$, a nonempty zero-dimensional subscheme of length divisible by p. It then follows by Lemma 4.2.5 that

$$0 < \operatorname{length}(\Lambda \cap Z) \le e(e-1)^{m - \dim Z} \le e(e-1)^m < p,$$

a contradiction. $\square$

We now need an expression for the integer e in Lemma 4.2.6. By Proposition 2.2.7, $x_0 \times_X \mathcal{M}_1^{\text{arc}} = x_0 \times_X \mathcal{M}_1$, so that $\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1^{\text{arc}} = \operatorname{Spec} \mathcal{O}_{X,x_0} \times \mathcal{M}_1^{\text{arc}}$, and we have morphisms

$$\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1 \to \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \operatorname{Arc}_{X/S} \to \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathbb{P}T_{X/S}$$

whose composite, the *tangent map*, is a finite morphism, and an isomorphism over X_1.

Lemma 4.2.7. *Suppose $\mathcal{O}_X(d)$ is very ample on X. Let $\tau : x_0 \times_X \mathcal{M}_1 \to \mathbb{P}T_{X_0,x_0}$ be the tangent map over X_0. Then $\tau^* \mathcal{O}_{\mathbb{P}T_{X_0,x_0}}\left(\frac{d(d+1)}{2}\right)$ is very ample on $x_0 \times_X \mathcal{M}_1$.*

Proof. Consider the projective embedding $\iota : X_0 \to \mathbb{P}^N = \mathbb{P}H^0(X_0, \mathcal{O}_{X_0}(1))$ defined by the complete linear system $|\mathcal{O}_{X_0}(1)|$. Let $\mathcal{N} \subset \underline{\operatorname{Hom}}^n_{\text{bir}}(\mathbb{P}^1, \mathbb{P}^N)$ be the component of degree d with respect to $\mathcal{O}_{\mathbb{P}^N}(1)$, so that there is a natural commutative diagram

$$\begin{array}{ccccc}
x_0 \times_X \mathcal{M}_1 & \xrightarrow{\tau} & x_0 \times_X \mathbb{P}T_{X_0} & \longrightarrow & x_0 \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{N}_1^{\text{arc}} & \xrightarrow{\theta} & \mathbb{P}T_{\mathbb{P}^N} & \longrightarrow & \mathbb{P}^N
\end{array}$$

where the vertical arrows are induced by ι, and θ is the tangent morphism. By linearity of $\iota_* : \mathbb{P}T_{X_0,x_0} \to \mathbb{P}T_{\mathbb{P}^N,\iota(x_0)}$ and $\operatorname{Aut}(\mathbb{P}^N)$-equivariance of $\mathcal{N}_1$, it will be enough to check that, for a point $q \in \mathbb{P}^N(k)$, the pullback $\theta^* \mathcal{O}_{\mathbb{P}T_{\mathbb{P}^N,q}}(d(d+1)/2)$ is very ample on $q \times_{\mathbb{P}^N} \mathcal{N}_1^{\text{arc}}$.

Let $\mathfrak{m}$ be the maximal ideal in $\mathcal{O}_{\mathbb{P}^N,q}$, and define

$$D = \operatorname{Spec} \mathcal{O}_{\mathbb{P}^N,q}/\mathfrak{m}^{d+1}, \quad P = \operatorname{Spec} k[t]/(t^{d+1}), \quad A = \underline{\mathrm{Imm}}(P, D; 0, q)/\operatorname{Aut}(P, 0)$$

where $0 \in P(k)$. There are natural morphisms

$$q \times_{\mathbb{P}^N} \mathrm{Arc}_{\mathbb{P}^N} \to A \to \mathbb{P}T_{\mathbb{P}^N,q}$$

where the tangent map $A \to \mathbb{P}T_{\mathbb{P}^N,q}$ is a Zariski-locally trivial affine space bundle, and can be viewed as a quotient of $q \times_{\mathbb{P}^N} \mathrm{Arc}_{\mathbb{P}^N}$ parametrising d-th jets of immersed arcs through $q \in \mathbb{P}^N$. Since $\mathcal{N}$ parametrises rational curves of degree d on $\mathbb{P}^N$, the composite

$$q \times_{\mathbb{P}^N} \mathcal{N}_1^{\mathrm{arc}} \xrightarrow{\vartheta} q \times_{\mathbb{P}^N} \mathrm{Arc}_{\mathbb{P}^N} \to A$$

is a locally closed immersion, factoring θ, so that it will now be enough to show that $\vartheta^* \mathcal{O}_{\mathbb{P}T_{\mathbb{P}^N,q}}(d(d+1)/2)$ is very ample on A.

Let $T \subset A \times D$ be the universal family over A, so that $\mathcal{O}_T$ is a sheaf of infinitesimal extensions of $\mathcal{O}_A$. The evaluation morphism $T \to D$ induces an epimorphism

$$\mathcal{O}_A \otimes \mathcal{O}_{\mathbb{P}^N,q}/\mathfrak{m}^{d+1} \to \mathcal{O}_T \to 0$$

of filtered $\mathcal{O}_A$-algebras, with filtrations induced by $\mathfrak{m}$ and the ideal sheaf $\mathcal{I} \subset \mathcal{O}_T$ of the zero-section $A \to T$. Since $\mathcal{I}$ is a locally free $\mathcal{O}_A$-module, the map

$$\mathcal{O}_A \otimes \mathfrak{m}/\mathfrak{m}^{d+1} \to \mathcal{I} \to 0$$

induces a morphism

$$f : A \to \mathsf{Gr}(\dim(\mathfrak{m}/\mathfrak{m}^{d+1}) - d, \dim(\mathfrak{m}/\mathfrak{m}^{d+1}))$$

factoring the natural immersion $A \to \mathrm{Hilb}_D$. It follows that $\det \mathcal{I}$ is very ample on A. Since A is an affine space bundle over $\mathbb{P}T_{\mathbb{P}^N,q}$, it follows that $\vartheta^* : \operatorname{Pic} \mathbb{P}T_{\mathbb{P}^N,q} \to \operatorname{Pic} A$ is an isomorphism, so that $\det \mathcal{I} = \vartheta^* \mathcal{O}_{\mathbb{P}T_{\mathbb{P}^N,q}}(e)$ for some $e > 0$, and one can determine e by computing the intersection of $c_1(\mathcal{I})$ with a curve. Let $c : \mathbb{P}^1 \to A$ be a rational curve such that $\vartheta \circ c$ is a line in $\mathbb{P}T_{\mathbb{P}^N,q}$. Then

$$c^* \mathcal{I} \simeq \mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(d)$$

so that

$$e = \deg c^* \det \mathcal{I} = \frac{d(d+1)}{2}.$$

Hence finally $\vartheta^* \mathcal{O}_{\mathbb{P}T_{\mathbb{P}^N,q}}(\frac{d(d+1)}{2})$ is very ample on A. □

We have thus arrived at the following condition for smoothness of $\bar{x}_0 \times_X \mathcal{M}_1$.

Proposition 4.2.8. *Let $m = \dim(\bar{x}_1 \times_X \mathcal{M}_1)$ and let δ be the degree of $\bar{x}_1 \times_X \mathcal{M}_1$ with respect to the embedding in $\bar{x}_1 \times_X \mathbb{P}T_{X/S}$. Suppose $\mathcal{O}_X(d)$ is very ample on X. Assume*

$$p > \left(\frac{d(d+1)}{2}\right)^m \delta \left(\left(\frac{d(d+1)}{2}\right)^m \delta - 1\right)^m .$$

Then $\bar{x}_0 \times_X \mathcal{M}_1$ is smooth.

Proof. By Lemma 4.2.7, the pullback of $\mathcal{O}_{\mathbb{P}T_{X/S}}(d(d+1)/2)$ by the tangent map is very ample on $x_0 \times_X \mathcal{M}_1$, and thus on $\operatorname{Spec} \mathcal{O}_{X_0,x_0} \times_X \mathcal{M}_1$. Then, by Lemma 4.2.2, the degree of the corresponding projective embedding of $\bar{x}_0 \times_X \mathcal{M}_1$ is $e = \left(\frac{d(d+1)}{2}\right)^m \delta$. Applying Lemma 4.2.6, the claim follows. $\square$

4.2.2 Segre case

The preceding subsection gives a condition under which the space of $\mathcal{M}$-curves through the generic point of X_0 is a smooth degeneration of the space of $\mathcal{M}$-curves through the generic point of X_1. As indicated in Corollary 2.3.4, in case of G being of type A_n, the tangent map indentifies the space of $\mathcal{M}$-curves through any point of X_1 with a Segre subvariety of the projectivised tangent space. Smooth degenerations in such situation are described by the following.

Lemma 4.2.9. *Let $f : Y \to S$ be a smooth morphism whose geometric generic fibre is isomorphic to $(\mathbb{P}^a \times \mathbb{P}^b) \otimes \bar{F}$. Suppose f factors through a finite morphism $\varphi : Y \to \mathbb{P}_S^{ab+a+b}$ whose restriction to the geometric generic fibre of f is a Segre embedding. Then f is an isotrivial fibration.*

Proof. Set $Y_1 = s_1 \times_S Y$, $Y_0 = s_0 \times_S Y$. After faithfully flat base change, we can assume that $Y_1 \simeq (\mathbb{P}^a \times \mathbb{P}^b) \otimes F$. Let

$$P \subset \mathrm{Hilb}_{Y/S}$$

be the closed subscheme, flat over S, such that $P_1 = s_1 \times_S P \simeq \mathbb{P}_F^a$ is the component parametrising subspaces of the form $\{*\} \times \mathbb{P}^b \subset Y_1$. Let $\Lambda_P \subset P \times_S Y$ be the universal family, so that $s_1 \times_S \Lambda_P \simeq P_1 \times \mathbb{P}^b$. We will show that $\Lambda_P \to P$ is an isotrivial bundle of projective spaces. By flatness of the universal family, and by smoothness of PGL_{b+1}, it will be enough to check that for each closed point $p \in P_0 = s_0 \times_S P$, the fibre $p \times_P \Lambda_P$ is isomorphic to $\mathbb{P}^b$. Given $p \in P_0$, there is a finite flat base change $T \to S$ to the spectrum of a discrete valuation ring with closed point t_0 and generic point t_1, together with a section $\sigma : T \to T \times_S P$ such that $\sigma(t_0) = p$. The composite

$$\sigma|_{t_1}^* \Lambda_P \to t_1 \times_S Y \xrightarrow{\varphi} t_1 \times \mathbb{P}^{ab+a+b}$$

is a family of linear subspaces over t_1, thus defining a t_1-point of the appropriate Grassmannian. By properness of the Grassmannian, the t_1-point extends to a T-point, and thus defines a family

$$\bar{\Lambda}_P \subset \mathbb{P}_T^{ab+a+b}$$

of linear subspaces, flat over T. In fact, $\bar{\Lambda}_P \simeq \mathbb{P}^b_T$. The inclusion $t_1 \times_T \bar{\Lambda}_P \subset t_1 \times_S Y$ extends to a rational map $i : \bar{\Lambda}_P \dashrightarrow Y$ such that $\varphi \circ i$ is the identity on $\bar{\Lambda}_P$. Since φ is finite, it follows that i is in fact regular, hence a closed immersion. Thus $\sigma^* \Lambda_P$ and $\bar{\Lambda}_P$ are closed subschemes of $T \times_S Y$, flat over T, and with identical restrictions over t_1 – hence $\sigma^* \Lambda_P = \bar{\Lambda}_P$, and in particular $p^* \Lambda_P \simeq \mathbb{P}^b$.

We have thus defined the subscheme $P \subset \mathrm{Hilb}_{Y/S}$ such that the universal family $\Lambda_P \to P$ is an isotrivial $\mathbb{P}^b$-bundle. Symmetrically, we define a closed subscheme $Q \subset \mathrm{Hilb}_{Y/S}$, flat over S, such that $Q_1 = s_1 \times_S Q \simeq \mathbb{P}^b_F$ is the component parametrising subspaces of the form $\mathbb{P}^a \times \{*\} \subset Y_1$, and with the universal family $\Lambda_Q \to Q$ being an isotrivial $\mathbb{P}^a$-bundle. We now claim that the arrows in the projection diagram

$$P \times_S Q \leftarrow \Lambda_P \times_Y \Lambda_Q \to Y$$

are isomorphisms. Since they do become isomorphisms after restriction over s_1, it will be enough to check that their restrictions over s_0 are bijective on closed points. Let us refer to the closed fibres of $s_0 \times_S \Lambda_P \to s_0 \times_S P$ (resp. $s_0 \times_S \Lambda_Q \to s_0 \times_S Q$) as P-planes (resp. Q-planes) in Y_0. We then need to check that all P-planes (resp. Q-planes) are disjoint, and that a P-plane intersects a Q-plane in a single point. Using smoothness of Y, this follows by intersection theory, specialising relevant classes from the generic fibre. Finally, we have $Y \simeq P \times_S Q$, where the double fibration $P \leftarrow P \times_S Q \to Q$ is a pair of bundles of projective spaces. It thus follows that $P \simeq \mathbb{P}^a_S$, $Q \simeq \mathbb{P}^b_S$, so that $Y \simeq \mathbb{P}^a_S \times_S \mathbb{P}^b_S$. □

4.2.3 General case

Recall the description of $\bar{x}_1 \times_X \mathcal{M}_1$ given in Proposition 2.3.3, Corollary 2.3.4 and Table 2.2. It is isomorphic to $\bar{x}_1 \times V$, where V is the variety of line tangents at the origin of G/P: either a Segre variety, or degree 2 Veronese, or a minimally embedded cominuscule variety. Combining the results of the two preceding subsections, we are ready to state a condition on the characteristic p ensuring that the space of $\mathcal{M}$-curves through the generic point does not degenerate.

Proposition 4.2.10. *Let V be the variety of line tangents at the origin of G/P. Suppose $\mathcal{O}_X(d)$ is very ample on X, and assume one of the following holds.*

1. *G is of type A_n, P is associated to α_i, and*

$$p > \left(\frac{d(d+1)}{2}\right)^{n-1} \binom{n-1}{i-1} \left(\left(\frac{d(d+1)}{2}\right)^{n-1} \binom{n-1}{i-1} - 1 \right)^{n-1}$$

2. *G is of type C_n and*

$$p > (d(d+1))^{n-1} \left((d(d+1))^{n-1} - 1 \right)^{n-1}$$

3. *G is of one of remaning types, V is $\frac{d(d+1)}{2}$-rigid, and*

$$p > \left(\frac{d(d+1)}{2}\right)^m (\deg V)(\left(\frac{d(d+1)}{2}\right)^m (\deg V - 1)^m$$

where $m = \dim V$ and $\deg V$ is the degree of V embedded in the projectivised tangent space at the origin of G/P.

Then $\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1 \to \operatorname{Spec} \mathcal{O}_{X,x_0}$ is an isotrivial V-bundle.

Proof. We first use Proposition 4.2.8 to conclude smoothness of $\bar{x}_0 \times_X \mathcal{M}_1$, and thus – by Lemma 4.2.2 – of $\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1 \to \operatorname{Spec} \mathcal{O}_{X,x_0}$. The conditions on p correspond precisely to the hypothesis of Proposition 4.2.8, where in cases (1) and (2) we use well-known expressions for degrees of Segre and Veronese varieties.

Then, to check isotriviality, after a faithfully flat base-change and replacing k with $\kappa(\bar{x}_0)$, we may replace $\operatorname{Spec} \mathcal{O}_{X,x_0}$ with S. We thus obtain a smooth projective morphism $Y \to S$ whose generic fibre is isomorphic to a base-change of V. Now, depending on the type of G, we use:

1. Lemma 4.2.9 with the finite morphism $Y \to \mathbb{P}_S^{\dim(G/P)-1}$ induced by the tangent map;
2. well-known rigidity of $\mathbb{P}^{n-1}$;
3. $\frac{d(d+1)}{2}$-rigidity of V, where $\mathcal{O}_Y(d(d+1)/2)$ is very ample by Lemma 4.2.7. □

4.3 Arcs at the generic point

4.3.1 Minimality

We have so far described, under suitable conditions, the abstract space of $\mathcal{M}$-curves through the generic point, concluding that it does not degenerate in the central fibre. In order to apply the main Theorem of Chapter 3, we need a similar result for the embedding into the space of arcs. This will be achieved in three steps: we first check that, after discarding non-dominant components of $s_0 \times_X \mathcal{M}$, we are left with an irreducible component whose generic member is minimal; next, we obtain a condition on the characteristic p ensuring that the corresponding variety of rational tangents at the generic point of X_0 is linearly nondegenerate; finally, we check that the corresponding family of arcs through a formal neighbourhood of the generic point is isomorphic to that on G/P.

Lemma 4.3.1. *Assume $\bar{x}_0 \times_X \mathcal{M}_1$ is smooth. Then there is a unique irreducible component $\mathcal{M}_*$ of $s_0 \times_X \mathcal{M}$ such that $\mathcal{M}_*$ is a dominating family of rational curves on X_0. Furthermore, $\mathcal{M}_*$ is an irreducible component of $\underline{\operatorname{Hom}}^n_{\operatorname{bir}}(\mathbb{P}^1, X_0)$, and the generic $\mathcal{M}_*$-curve is free.*

Proof. Existence and uniqueness of $\mathcal{M}_*$ follows from smoothness, and thus irreducibility of $x_0 \times_X \mathcal{M}_1$. By Lemma 4.2.1, M contains every irreducible component of $\underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, X_0)$ whose generic member is free, hence is a component itself. Since $x_0 \times_X \mathcal{M}_{*,1} = x_0 \times_X \mathcal{M}_1$, the generic $\mathcal{M}_*$-curve is free. $\square$

In the remainder of this section we will assume the hypotheses of Proposition 4.2.10, so that in particular $\bar{x}_0 \times_X \mathcal{M}_1$ is smooth and Lemma 4.3.1 applies. We use our standard notation $\mathcal{M}_{*,1}$, $\mathcal{M}^i_{*,2}$, etc. Note that by Lemma 2.2.5, $\mathcal{M}^{i,\mathrm{free}}_{*,2} \to X_0 \times X_0$ is dominant for some $i \geq 0$, so that X_0 is chain-connected by $\mathcal{M}_*$-curves.

By smoothness of $\bar{x}_0 \times_{X_0} \mathcal{M}_{*,1}$, the generic $\mathcal{M}_*$-curve is free. In fact, our hypotheses on p imply more.

Lemma 4.3.2. *The generic $\mathcal{M}_*$-curve is minimal.*

Proof. Let $(V, \mathcal{L})$ be the variety of line tangents at the origin o in G/P, together with the very ample invertible sheaf defining the embedding $V \subset \mathbb{P}T_{G/P,o}$ into the projectivised tangent space. More concretely, for G of type A_n, V is a Segre variety with $\mathcal{L} = \mathcal{O}(1,1)$; for G of type C_n, V is a Veronese variety with $\mathcal{L} = \mathcal{O}(2)$; for remaining types, V is cominuscule with $\mathcal{L} = \mathcal{O}(1)$. By Proposition 4.2.10, we have a commutative diagram with Cartesian squares

$$\begin{array}{ccccc} V \otimes \kappa(\bar{x}_0) & \longrightarrow & \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1 & \longleftarrow & V \otimes \kappa(\bar{x}_1) \\ {\scriptstyle \mathfrak{d}}\big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle |\mathcal{L} \otimes \kappa(\bar{x}_1)|} \\ \bar{x}_0 \times_X \mathbb{P}T_{X/S} & \longrightarrow & \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathbb{P}T_{X/S} & \longleftarrow & \bar{x}_1 \times_X \mathbb{P}T_{X/S} \end{array}$$

where $\mathfrak{d} \subset |\mathcal{L} \otimes \kappa(\bar{x}_0)|$ is a linear subsystem. In particular, it follows that the tangent morphism $\bar{x}_0 \times_X \mathcal{M}_{*,1} \to \bar{x}_0^* \mathbb{P}T_{X_0}$ factors as

$$\bar{x}_0 \times_X \mathcal{M}_{*,1} \simeq V \otimes \kappa(\bar{x}_0) \xrightarrow{|\mathcal{L}|} \mathbb{P}T_{G/P,o} \otimes \kappa(\bar{x}_0) \dashrightarrow \bar{x}_0^* \mathbb{P}T_{X/S}$$

where the rightmost arrow is a linear projection onto a subspace. Now, the hypotheses of Proposition 4.2.10 ensure that the degree of the embedding defined by $|\mathcal{L}|$ is less than p. It follows that the tangent morphism, being finite, is generically unramified, so that the generic $\mathcal{M}_*$-curve $f : \mathbb{P}^1 \otimes k(\mathcal{M}_*) \to X_0$ does not admit nontrivial infinitesimal deformations fixing $f(0)$ and the tangent direction in $f|_0^* \mathbb{P}T_{X_0}$. Hence f is minimal. $\square$

4.3.2 Linear nondegeneracy

We are going to derive linear nondegeneracy of the variety of $\mathcal{M}_*$-rational tangents at x_0 from Proposition 2.2.8. That will require X_0 to be *separably* connected by chains of free $\mathcal{M}_*$-curves, a condition we shall satisfy by constructing a projective embedding of a suitable blow-down of the generic fibre of $\mathcal{M}^{i,\mathrm{free}}_{*,2} \to X_0 \times X_0$, and imposing its degree as another bound on the characteristic p. As a consequence, it will follow that the space of $\mathcal{M}$-curves through the generic point of X_0, the corresponding variety of rational tangents, and the tangent map between the two, are isomorphic to those on G/P.

Recall that by Lemma 2.2.5, $\mathcal{M}_{*,2}^{i,\text{free}} \to X_0 \times X_0$ is dominant for some $i \geq 0$. Letting ξ_1, resp. ξ_0, be the generic point of $X_1 \times_{s_1} X_1$, resp. $X_0 \times X_0$, observe that $\mathcal{O}_S \subset \mathcal{O}_{(X/S)^2,\xi_0}$ is an unramified extension of discrete valuation rings, inducing separably generated extensions of residue and fraction fields.

Lemma 4.3.3. *Let r_0 be the smallest $i > 0$ such that $\mathcal{M}_{*,2}^{i,\text{free}} \to X_0 \times X_0$ is dominant. Then:*

1. *$r_0 \leq \dim(G/P)$;*

2. *The total evaluation morphism $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0,\text{free}} \to \xi_0 \times_{X_0^2} X_0^{r_0+1}$ is quasi-finite.*

3. *$\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0,\text{free}}$ is a dense open subscheme of an irreducible component of $\xi_0 \times_{(X/S)^2} \mathcal{M}_2^{r_0}$;*

Proof. 1. Let $n = \dim(G/P) = \dim X_0$. Denote by d_i the dimension of the closed image of $\mathcal{M}_{*,2}^{i,\text{free}} \to X_0 \times X_0$. We then have that the sequence d_i is nondecreasing, $d_0 = n$, and $d_{i+1} = d_i$ if and only if $d_i = 2n$. We then have $d_{r_0} = d_n = 2n$, so that $r_0 \leq n$.

2. Suppose not, so that there is a point $\vec{x} \in \xi_0 \times_{X_0^2} X_0^{r_0+1}$ with a positive-dimensional fibre in $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0,\text{free}}$. By Bend-and-Break, this can only happen if $\vec{x}$ factors through one of the diagonals in $X_0^{r_0+1}$. It follows that a chain corresponding to a point in the fibre contains a segment whose two marked points coincide. Removing the segment, we obtain a chain of length $r_0 - 1$, corresponding to a point in $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0-1,\text{free}}$. This contradicts minimality of r_0.

3. Since $\mathcal{M}_*^{\text{free}}$ is open $\mathcal{M}_*$, we have that $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0,\text{free}}$ is open in $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0}$. Since $\bar{x}_0 \times_{X_0} \mathcal{M}_{*,1}$ is smooth and connected, we have by Lemma 2.2.6 that $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0,\text{free}}$ is irreducible, and thus its closure in $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0}$ is an irreducible component. Finally, $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0}$ is a union of irreducible components of $\xi_0 \times_{(X/S)^2} \mathcal{M}_2^{r_0}$.

□

Lemma 4.3.4. *Let V be the variety of line tangents at the origin of G/P, and assume $p > \delta_{G/P}(r_0)$ (cf. 2.3.4). Then $\mathcal{M}_{*,2}^{r_0,\text{free}} \to X_0 \times X_0$ is separably dominant.*

Proof. Let $\bar{\mathcal{M}}_2^{r_0} \subset \operatorname{Spec} \mathcal{O}_{(X/S)^2,\xi_0} \times_{(X/S)^2} \mathcal{M}_2^{r_0}$ be the flat limit of $\xi_1 \times_{(X/S)^2} \mathcal{M}_2^{r_0}$ over $\operatorname{Spec} \mathcal{O}_{(X/S)^2,\xi_0}$. Note that $\xi_0 \times_{(X/S)^2} \bar{\mathcal{M}}_2^{r_0}$ contains $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0,\text{free}}$. Consider the total evaluation morphism

$$\nu : \bar{\mathcal{M}}_2^{r_0} \to \operatorname{Spec} \mathcal{O}_{X/S^2,\xi_0} \times_{(X/S)^2} (X/S)^{r_0+1}.$$

Letting $p_j : (X/S)^{r_0+1} \to X$, $0 \le j \le r_0$, be the natural projections, we have an invertible sheaf

$$\mathcal{L} = \bigotimes_{j=1}^{r_0-1} \nu^* p_j^* \mathcal{O}_X(1)$$

on $\operatorname{Spec} \mathcal{O}_{(X/S)^2,\xi_0} \times_{(X/S)^2} \mathcal{M}_2^{r_0}$. By the definition in 2.3.4,

$$(\bar{\xi}_1 \times_{(X/S)^2} \mathcal{M}_2^{r_0}) \cdot c_1(\mathcal{L})^{\dim \bar{\xi}_1^* \mathcal{M}_2^{r_0}} = \delta_{G/P}(r_0)$$

holds on the geometric generic fibre, so that

$$(\bar{\xi}_0 \times_{(X/S)^2} \bar{\mathcal{M}}_2^{r_0}) \cdot c_1(\mathcal{L})^{\dim \bar{\xi}_0^* \mathcal{M}_2^{r_0}} = \delta_{G/P}(r_0)$$

holds on the geomteric special fibre.

Let now $W \subset \xi_0 \times_{(X/S)^2} \bar{\mathcal{M}}_2^{r_0}$ be the closure of $\xi_0 \times_{X_0^2} \mathcal{M}_{*,2}^{r_0,\text{free}}$, an irreducible component. Suppose $\kappa(\eta_W)/\kappa(\xi_0)$ is not separably generated. Then local rings of generic points of $\bar{\xi}_0 \times_{\xi_0} W$ have length divisible by p, so that

$$(\bar{\xi}_0 \times_{\xi_0} W) \cdot c_1(\mathcal{L})^{\dim \bar{\xi}_0^* W} \in p\mathbb{Z}.$$

By Lemma 4.3.3, the restriction of ν to W is generically finite, and the above intersection number is positive. We then have

$$\delta_{G/P}(r_0) \ge (\bar{\xi}_0 \times_{\xi_0} W) \cdot c_1(\mathcal{L})^{\dim \bar{\xi}_0^* W} \ge p.$$

But $p > \delta_{G/P}(r_0)$, a contradiction. It follows that $\mathcal{M}_{*,2}^{r_0,\text{free}} \to X_0 \times X_0$, dominant by Lemma 4.3.3, is separable. $\square$

Proposition 4.3.5. *Let $(V, \mathcal{L})$ be the variety of line tangents at the origin of G/P, and the invertible sheaf defining the projective embedding into the projectivised tangent space at the origin. Assume the hypotheses of Proposition 4.2.10, and furthermore $p > \delta_{G/P}(\dim(G/P))$ and $p > \operatorname{index}(G/P)$. Then the tangent morphism*

$$\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1 \to \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathbb{P}T_{X/S},$$

a map between an isotrivial V-bundle and a trivial projective space bundle, is defined by the complete linear system associated with $\mathcal{L}$.

Proof. Recall the factorisation

$$\bar{x}_0 \times_X \mathcal{M}_{*,1} \simeq V \otimes \kappa(\bar{x}_0) \xrightarrow{|\mathcal{L}|} \mathbb{P}T_{G/P,o} \otimes \kappa(\bar{x}_0) \dashrightarrow \bar{x}_0 \times_X \mathbb{P}T_{X/S}$$

of the tangent morphism $\bar{x}_0 \times_X \mathcal{M}_{*,1} \to \bar{x}_0 \times_X \mathbb{P}T_{X_0}$ (cf. the proof of Lemma 4.3.2). It will be enough to show that the linear projection corresponding to the dashed arrow is an isomorphism, i.e. that the image of the tangent morphism is linearly nondegenerate in $\bar{x}_0^* \mathbb{P}T_{X/S}$. We apply Proppsition 2.2.8 to X_0 and $\mathcal{M}_*$. Note that $\operatorname{index}(X_0) = \operatorname{index}(G/P)$. Hypothesis (1) is satisfied by Lemma 4.3.2. Hypothesis (2) is satisfied by smoothness of $\bar{x}_0 \times_{X_0} \mathcal{M}_{*,1}$. Hypothesis (3) is satisfied by Lemma 4.3.4. Hypothesis (4) is satisfied by Lemma 2.3.8, using the above factorisation. $\square$

4.3.3 Identification with model

We now wish to extend Proposition 4.3.5 to a statement about the family of arcs through a formal neighbourhood of the generic point, essentially saying that this family cannot acquire 'curvature' when specialising to X_0. The extension theorem of Chapter 3 will then apply immediately.

It will be convenient to introduce the following notation:

$$Y = G/P, \quad y \in Y \text{ origin}, \quad \hat{Y} = y \times_Y Y^\sharp \text{ completion at } y$$

$$\mathcal{N} \subset \underline{\mathrm{Hom}}^n_{\mathrm{bir}}(\mathbb{P}^1, Y) \text{ lines}, \quad V = y \times_Y \mathcal{N}_1 \hookrightarrow \mathbb{P}T_{Y,y} \text{ line tangents at } y.$$

In addition to the hypotheses of Proposition 4.2.8, we now assume those of Proposition 4.3.5, so that

$$\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1 \to \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathbb{P}T_{X/S}$$

is a closed immersion, étale locally isomorphic to the embedding

$$V \to \mathbb{P}T_{Y,y}.$$

Lemma 4.3.6. *There is a faithfully flat base-change $T \to \operatorname{Spec} \mathcal{O}_{X,x_0}$ such that T is the spectrum of a discrete valuation ring over $\kappa(\bar{x}_0)$, with the latter as its residue field, and there is a commutative diagram*

$$\begin{tikzcd} T \times_X \mathcal{M}_1 \arrow[r] \arrow[d, "\simeq"] & T \times_X \mathbb{P}T_{X/S} \arrow[d, "\simeq"] \\ T \times V \arrow[r] & T \times \mathbb{P}T_{Y,y} \end{tikzcd}$$

where the vertical arrows are isomorphisms, and the horizontal arrows are the natural tangent embeddings.

Proof. Immediate by the preceding paragraph. $\square$

Since $\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1 = \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1^{\mathrm{arc}}$ (Proposition 2.2.7), it follows that the tangent map, a closed immersion, factors through the arc space:

$$\begin{tikzcd} & \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathrm{Arc}_{X/S} \arrow[d] \\ \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1 \arrow[r] \arrow[ur] & \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathbb{P}T_{X/S} \end{tikzcd}$$

where the diagonal arrow is an isomorphism onto $\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \hat{\mathcal{M}}_1$. Since $\operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathcal{M}_1$ is flat over $\operatorname{Spec} \mathcal{O}_{X,x_0}$, the above diagram defines a section

$$\sigma_X : \operatorname{Spec} \mathcal{O}_{X,x_0} \to \operatorname{Spec} \mathcal{O}_{X,x_0} \times_X \mathrm{ArcHilb}_{X/S}.$$

Note that $\mathcal{N}_1$ defines a corresponding section $\sigma_Y : Y \to \mathrm{ArcHilb}_Y$.

Proposition 4.3.7. *There is a faithfully flat base-change $T \to \operatorname{Spec} \mathcal{O}_{X,x_0}$ such that T is the spectrum of a discrete valuation ring over $\kappa(\bar{x}_0)$, with the latter as its residue field, and there is a commutative diagram*

$$\begin{array}{ccc} T \times_X (X/S)^\sharp & \xrightarrow{\mathrm{id}_T \times \sigma_X} & T \times_X (X/S)^\sharp \times_X \mathrm{ArcHilb}_{X/S} \\ \downarrow \simeq & & \downarrow \simeq \\ T \times \hat{Y} & \xrightarrow{\mathrm{id}_T \times \sigma_Y} & T \times \hat{Y} \times_Y \mathrm{ArcHilb}_Y \end{array}$$

where the vertical arrows are isomorphisms.

Proof. Let T be as in Lemma 4.3.6, together with the isomorphism

$$\bar{\phi} : T \times_X \mathbb{P}T_{X/S} \to T \times \mathbb{P}T_{Y,y}$$

identifying pullbacks of $\mathcal{M}_1$ and V. Let

$$\phi : T \times_X (X/S)^\sharp \to T \times \hat{Y}$$

be any isomorphism of bundles of formal discs whose restriction to the first infinitesimal neighbourhood of T in $T \times_X (X/S)^\sharp$, viewed as an isomorphism of pullbacks of tangent bundles, projectivises to $\bar{\phi}$. There is a natural lift of ϕ to a commutative diagram

$$\begin{array}{ccc} T \times_X (X/S)^\sharp \times_X \mathrm{ArcHilb}_{X/S} & \xrightarrow{\tilde{\phi}} & T \times \hat{Y} \times_Y \mathrm{ArcHilb}_Y \\ \downarrow & & \downarrow \\ T \times_X (X/S)^\sharp & \xrightarrow{\phi} & T \times \hat{Y} \end{array}$$

where both horizontal arrows are isomorphisms. The section σ_X then induces a commutative diagram

$$\begin{array}{ccc} T \times_X (X/S)^\sharp \times_X \mathrm{ArcHilb}_{X/S} & \xrightarrow{\tilde{\phi}} & T \times \hat{Y} \times_Y \mathrm{ArcHilb}_Y \\ {\scriptstyle \mathrm{id}_T \times \sigma_X} \uparrow & & \uparrow {\scriptstyle \sigma_X^\phi} \\ T \times_X (X/S)^\sharp & \xrightarrow{\phi} & T \times \hat{Y} \end{array}$$

where both vertical arrows are sections. We now have a pair of sections

$$T \times \hat{Y} \underset{\mathrm{id}_T \times \sigma_Y}{\overset{\sigma_X^\phi}{\rightrightarrows}} T \times \hat{Y} \times_Y \mathrm{ArcHilb}_Y$$

thus defining a pair morphisms

$$T \underset{t_Y}{\overset{t_X}{\rightrightarrows}} \prod (\mathrm{ArcHilb}_{\hat{Y}} / \hat{Y})$$

where t_Y is constant, i.e. factors through a k-point, which we will denote with the same symbol. By construction, the restriction of t_X to the generic point of T factors through the $\mathcal{R}_u\underline{\mathrm{Aut}}(\hat{Y}, y)$-orbit of t_Y. Hence, by Proposition 2.1.11, t_X itself factors through the same orbit. It follows that, possibly after a further faithfully flat base change, there is $g : T \to \mathcal{R}_u\underline{\mathrm{Aut}}(\hat{Y}, y)$ such that $g\phi$ gives the desired isomorphism. □

Corollary 4.3.8. *Under the hypotheses of Propositions 4.2.8 and 4.2.10, X_0 is isomorphic to G/P.*

Proof. By the above Proposition, there is an isomorphism $\bar{x}_0^* X_0^\sharp \to \bar{x}_0 \times \hat{Y}$ identifying $\bar{x}_0^* X_0^\sharp \times_X \hat{\mathcal{M}}_1$ with $\bar{x}_0 \times \hat{Y} \times_Y \hat{\mathcal{N}}_1$. Let $\bar{y}_0 : \operatorname{Spec} \kappa(\bar{x}_0) \to Y$ be the geometric point factoring through y. Note that $\mathcal{M}_*$ and $\mathcal{N}$ are *good families* in the language of Chapter 3. Hence, applying Corollary 3.2.2 to the pair $(X_0, \mathcal{M}_*)$, $(Y, \mathcal{N})$, and geometric points $\bar{x}_0$, $\bar{y}_0$, we have an isomorphism $X \simeq Y$. □

4.4 Conclusion

Corollary 4.3.8 essentially concludes the proof of a rigidity theorem for G/P, establishing its d-rigidity under the hypotheses we have been gradually introducing. Combining these, and abstracting from the particular situation $X \to S$, we can restate the main result of this chapter as follows.

Theorem 4.4.1. *Let G/P be a cominuscule homogeneous variety, and $d > 0$ an integer. Assume one of the following holds:*

1. *G is of type A_n, P is associated to α_i, and*

$$p > \max\{ \left(\frac{d(d+1)}{2}\right)^{n-1} \binom{n-1}{i-1} \left(\left(\frac{d(d+1)}{2}\right)^{n-1} \binom{n-1}{i-1} - 1 \right)^{n-1}, \; n+1,\; \delta_{G/P}(i(n+1-i))\}$$

2. *G is of type C_n,*

$$p > \max\{(d(d+1))^{n-1} \left((d(d+1))^{n-1} - 1\right)^{n-1},\; n+1,\; \delta_{G/P}(n(n+1)/2)\}$$

3. *G is of one of remaining types, its variety V of line tangents at the origin is $\frac{d(d+1)}{2}$-rigid, and*

$$p > \max\{ \left(\frac{d(d+1)}{2}\right)^{m} \delta_V \left(\left(\frac{d(d+1)}{2}\right)^{m} \delta_V - 1 \right)^{m},\; \operatorname{index}(G/P),\; \delta_{G/P}(\dim(G/P))\}$$

   where $m = \dim V$.

Then G/P is d-rigid.

Proof. It is enough to show isotriviality for every smooth projective degeneration $X \to S$ of G/P with trivial generic fibre and very ample $\mathcal{O}_X(d)$. In such situation, the hypotheses of the Theorem imply those of Propositions 4.2.8 and 4.2.10, so that $X_0 \simeq G/P$ by Corollary 4.3.8. $\square$

Note that for G not of type A_n, C_n, the Theorem derives d-rigidity of G/P from $\frac{d(d+1)}{2}$-rigidity of its variety of line tangents, a cominuscule homogeneous variety of lower dimension (i.e. with lower rank of the corresponding simple algebraic group). Given an integer d, this allows one to obtain a lower bound on p guaranteeing d-rigidity by applying the Theorem inductively, eventually terminating at a cominuscule homogeneous variety for a group of type A_n or C_n.[2]

[2] Unfortunately, the bounds obtained in subsequent steps of the induction tend to grow due to replacing d with $\frac{d(d+1)}{2}$, the expression appearing in Lemma 4.2.7. This situation would be greatly improved if one could prove the Lemma with d in place of the former expression. An observation due to David Jensen indicates that this is indeed possible.

Bibliography

[1] *Revêtements étales et groupe fondamental*. Springer-Verlag, Berlin, 1971. Séminaire de Géométrie Algébrique du Bois Marie 1960–1961 (SGA 1), Dirigé par Alexandre Grothendieck. Augmenté de deux exposés de M. Raynaud, Lecture Notes in Mathematics, Vol. 224.

[2] N. Bourbaki. *Algebra. II. Chapters 4–7*. Elements of Mathematics (Berlin). Springer-Verlag, Berlin, 1990. Translated from the French by P. M. Cohn and J. Howie.

[3] A. M. Cohen and B. N. Cooperstein. Lie incidence systems from projective varieties. *Proc. Amer. Math. Soc.*, 126(7):2095–2102, 1998.

[4] T. Ekedahl. Foliations and inseparable morphisms. In *Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985)*, volume 46 of *Proc. Sympos. Pure Math.*, pages 139–149. Amer. Math. Soc., Providence, RI, 1987.

[5] J. C. Ferrar. Strictly regular elements in Freudenthal triple systems. *Trans. Amer. Math. Soc.*, 174:313–331 (1973), 1972.

[6] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. II. *Inst. Hautes Études Sci. Publ. Math.*, (17):91, 1963.

[7] A. Grothendieck. Éléments de géométrie algébrique. IV. Étude locale des schémas et des morphismes de schémas IV. *Inst. Hautes Études Sci. Publ. Math.*, (32):361, 1967.

[8] R. Hartshorne. *Algebraic geometry*. Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[9] J.-M. Hwang. Geometry of minimal rational curves on Fano manifolds. In *School on Vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000)*, volume 6 of *ICTP Lect. Notes*, pages 335–393. Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.

[10] J.-M. Hwang and N. Mok. Rigidity of irreducible Hermitian symmetric spaces of the compact type under Kähler deformation. *Invent. Math.*, 131(2):393–418, 1998.

[11] J.-M. Hwang and N. Mok. Cartan-Fubini type extension of holomorphic maps for Fano manifolds of Picard number 1. *J. Math. Pures Appl. (9)*, 80(6):563–575, 2001.

[12] S. Kebekus. Families of singular rational curves. *J. Algebraic Geom.*, 11(2):245–256, 2002.

[13] S. Kobayashi and T. Ochiai. Characterizations of complex projective spaces and hyperquadrics. *J. Math. Kyoto Univ.*, 13:31–47, 1973.

[14] J. Kollár. *Rational curves on algebraic varieties*, volume 32 of *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]*. Springer-Verlag, Berlin, 1996.

[15] T. Ochiai. Geometry associated with semisimple flat homogeneous spaces. *Trans. Amer. Math. Soc.*, 152:159–193, 1970.

[16] S. Ramanan and A. Ramanathan. Projective normality of flag varieties and Schubert varieties. *Invent. Math.*, 79(2):217–224, 1985.

[17] R. Richardson, G. Röhrle, and R. Steinberg. Parabolic subgroups with abelian unipotent radical. *Invent. Math.*, 110(3):649–671, 1992.

[18] M. Shen. Foliations and rational connectedness in positive characteristic. *J. Algebraic Geom.*, 19(3):531–553, 2010.