Kähler potentials on the Moduli space of stable parabolic bundles over the Riemann sphere

A Dissertation Presented

by

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to

The Graduate School

in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2013

Stony Brook University

The Graduate School

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Acknowledgements

Writing this work wouldn't have been possible without the support of all the people that surrounded me during it, and from which, in one way or another, I have learned almost everything I know.

To my Professors Leon Takhtajan, Samuel Grushevsky, Alexander Kirillov Jr., Irwin Kra, Dennis Sullivan, Mikhail Lyubich and Jason Star for their help and encouragment.

To my friends Trevor Clark, Benjamim Balsam, Frederico Girao, Somnath Basu, Pedro Solórzano, Andrew Bulawa, Ki Song, Vamsi Pingali, Panagiotis Gianniotis, Ali Aleyasin, Raquel Perales, for all the great moments that we shared over these years.

I would like to thank specially my parents and brothers (which from the distance have always been encouraging me to pursue my dreams), Anna (for teaching me the meaning of happiness), and Shane D'Mello, for being with me during the darkest moments of my recent life.

Stony Brook, NY, August 14th, 2013

A note on terminology

1. If M is a manifold (real or complex), $\widetilde{M}$ will denote its universal cover and C^∞ the sheaf of germs of smooth functions (real or complex valued). When M is complex, $\mathcal{O}$ will denote the sheaf of germs of holomorphic functions on M and Ω^p the sheaf of holomorphic p-forms on M. If $E \to M$ is a complex vector bundle, $\mathcal{A}^p(E)$ will denote the sheaf of E-valued C^∞ p-forms, $\mathcal{A}^{p,q}(E)$ the sheaf of E-valued C^∞ $p+q$ forms of type (p,q) over M and $\mathcal{A}_c^{p,q}(E)$ the sheaf of compactly supported E-valued C^∞ (p,q)-forms. We will simplify the notation of their spaces of global sections as $H^0(M, \mathcal{A}^p(E)) = A^p(E)$, etc. When E is holomorphic, $\Omega^p(E)$ will denote the sheaf of holomorphic E-valued p-forms over M. In particular, $\Omega^0(E) = \mathcal{O}(E)$. If $\{\mathscr{U}_i\}_{i\in\mathscr{I}}$ is a collection of open sets in M, we will denote any finite intersection $\mathscr{U}_{i_1} \cap \mathscr{U}_{i_2} \cap \cdots \cap \mathscr{U}_{i_n}$ by $\mathscr{U}_{i_1 i_2 \cdots i_n}$.

2. The superscript * will be used as a synonym of "dual" (of a vector space V, a representation ρ, a vector bundle E) with two exceptions: to denote the *adjoint* (conjugate transpose) M^* of a complex matrix M, and more generally, for the formal adjoint L^* of an operator L on a Hilbert space. In particular, $\rho^*(\gamma)$ and $\rho(\gamma)^*$ denote different objects. The transpose of a matrix M is denoted as ${}^tM$. Our convention for a Hermitian inner product is to be antilinear in the second entry. On $\mathbb{C}^r$, it is given as $\langle v_1, v_2\rangle = {}^tv_1\overline{v_2} = \mathrm{tr}(v_1 v_2^*)$. On $\mathfrak{gl}(r,\mathbb{C})$, it is given as $\langle M_1, M_2\rangle = \mathrm{tr}(M_1 M_2^*)$. The subscript $_*$ will be used to denote a bundle endowed with a parabolic structure. Parentheses will be used whenever confusion could occur; for instance, $(E_\rho)_*$ denotes the bundle E_ρ together with a parabolic structure, while $E_{(\rho^*)}$ denotes a vector bundle indexed by the representation ρ^*.

3. The variable τ will be used as the complex coordinate on the upper half-plane $\mathbb{H}$ and the variable z will be used as a complex coordinate on $\mathbb{C}$. $\mathrm{d}^2\tau$ (resp. $\mathrm{d}^2 z$) will denote the measure gotten by integration with respect to the 2-form $\left|\frac{\mathrm{d}\tau\wedge\mathrm{d}\bar\tau}{2}\right| := \frac{\sqrt{-1}}{2}\mathrm{d}\tau\wedge\mathrm{d}\bar\tau$ (resp. $\frac{\sqrt{-1}}{2}\mathrm{d}z\wedge\mathrm{d}\bar z$). In particular, the hyperbolic measure on $\mathbb{H}$ is given as $(\mathrm{Im}\tau)^{-2}\mathrm{d}^2\tau$. We will follow the standard notation for the hyperbolic metric on $\mathbb{H}$ as $ds^2 = (\mathrm{Im}\tau)^{-2}|d\tau|^2$. Given a Fuchsian group Γ, the letter F will denote a fundamental domain on $\mathbb{H}$ for it.

4. We will denote the homogeneous space $\mathrm{GL}(r,\mathbb{C})/\mathrm{U}(r)$ of Hermitian and positive-definite matrices by $\mathcal{H}_r$. $\mathrm{B}(r)$ and $\mathrm{N}(r)$ (respectively $\mathfrak{b}(r)$ and $\mathfrak{n}(r)$) will denote the Borel group of complex lower triangular matrices

and its subgroup of unipotent matrices (respectively, the Lie algebras of complex lower triangular and lower diagonal matrices). $\mathcal{F}_r$ will denote the complete flag variety in $\mathbb{C}^r$, given as the homogeneous spaces $\mathrm{U}(r)/\mathrm{T}^r \cong \mathrm{GL}(r, \mathbb{C})/\mathrm{B}(r)$. Notice that this means that we are considering *descending* flags primarily.

Introduction

This dissertation is about parabolic bundles over the Riemann sphere, their moduli, and an unexpected interrelation that a certain physical formalism has with them. The subject has existed for approximately 35 years, but perhaps it hasn't received all the attention it deserves, specially since the core idea can be traced back to the seminal work of André Weil, *Généralisation des fonctions abéliennes* [53].

From its inception in the work of Mehta and Seshadri [37], the study of the subject has depended almost in its entirety on the machinery of algebraic geometry. It is easy to understand this state of affairs considering the dramatic development that algebraic geometry went through in the second half of the twentieth century. Without any doubt, one can say that this point of view is the broadest. However, it should be emphasized that this point of view is not the only one.

It is quite surprising that virtually only one work, that of Takhtajan and Zograf [51], has pursued the classical, complex analytical viewpoint instead. It is illustrative to recall the study of the moduli problem of Riemann surfaces during the twentieth century. The algebraic geometry techniques have prevailed, but the analytical viewpoint, in the form of the Teichmüller theory, has also represented a major development towards its understanding. In conclusion, we could say that, keeping things in proportion, the moduli theory of parabolic bundles has its own "Teichmüller theory" in a very natural way. We follow this analytic approach in the present work. To keep a better idea of this analogy, we could say that stable parabolic bundles are to Riemann surfaces of finite type what stable vector bundles (as in the work of Narasimhan and Seshadri, [40]) are to compact Riemann surfaces.

The analogy is in fact fundamental for us, since our result on the Kähler potential for the Moduli space of stable parabolic bundles is WZNW-equivalent to a result on the Liouville theory on the Teichmüller space $T_{0,n}$ of Riemann surfaces of type (0,n) proved by Takhtajan and Zograf in [55].

Chapter 1 is devoted to the understanding of the notion of "uniformization of a stable parabolic bundle," in analogy to the uniformization of a stable (that is, hyperbolic) Riemann surface. The starting objects of consideration are (1) a Fuchsian group Γ uniformizing $\mathbb{P}^1 \setminus \{z_1, \cdots, z_n\}$ and (2) an irreducible unitary representation $\rho : \Gamma \to \mathrm{U}(r)$ with a fixed system of weights $\mathcal{W}$. The main result there (Theorem 1.5.3) is on the construction of a canonical function $\Upsilon : \mathbb{H} \to \mathrm{GL}(r, \mathbb{C})$ satisfying

$$\Upsilon(\gamma\tau) = \Upsilon(\tau)\rho(\gamma)^{-1} \qquad \forall\gamma \in \Gamma,$$

and having a specific automorphic behaviour near the cusps $\tau_1, \cdots, \tau_n$ of Γ. Once we associate ρ with its corresponding stable parabolic bundle $(E_\rho)_* \to \mathbb{P}^1$, Υ induces a parabolic bundle map

$$\begin{tikzcd}
(\mathbb{H}^+ \times \mathbb{C}^r)_* \arrow[r, "\mathscr{J}"] \arrow[d, "\mathrm{pr}_1"'] & (E_\rho)_* \arrow[d, "\pi"] \\
\mathbb{H}^+ \arrow[r, "J"] & \mathbb{P}^1,
\end{tikzcd}$$

where $\mathbb{H}^+ = \mathbb{H} \cup \{\tau_1, \cdots, \tau_n\}$. Going back to the analogy, the notion of "marked points on the sphere" is replaced by "flags over marked points on the sphere". The trivial connection on $\mathbb{H} \times \mathbb{C}^r$ determined by the DeRham differential can be projected into a flat connection with logarithmic singularities on E_ρ, which is moreover compatible with the degenerate metric arising as the projection of the standard Hermitian metric on $\mathbb{H} \times \mathbb{C}^r$. The specialization to bundles over the sphere determines a canonical system of trivializations (after G. Birkhoff and A. Grothendieck), which in turn allows us to think of a logarithmic connection in terms of the classical notion of a Fuchsian system, and of a degenerate metric in terms of a function on $\mathbb{P}^1$ with values on $\mathcal{H}_r$ and prescribed asymptotics at the marked points.

In Chapter 2 we develop the notion of vector-valued automorphic forms with a representation ρ. Although this notion is a simple generalization of the classical case (and was even considered by H. Petersson), its appearance in the literature remains scarce even today, turning it into an unfortunate rarity. It is shown how the concept, in the special case of weights 0 and 2, fits naturally into the theory of parabolic bundles over Riemann surfaces, since there is a correspondence between the Čech cohomology groups of a parabolic bundle and the spaces of cusp forms (recall that the latter appears when a Fuchsian group has parabolic generators). The Petersson inner product is introduced and it is shown how it corresponds to a particular case of the Hodge inner product for vector bundles, a fact that allows us to relate these notions with the classical harmonic theory on $\mathbb{H}$.

Chapter 3 is substantial for this work. It is divided into two sections. In the first one we provide an explicit construction of the character variety $\mathscr{K}(\mathcal{W})$, whose points parametrize equivalence classes of irreducible unitary representations of Γ with a fixed system of weights, and show how it can be endowed with a natural Kähler structure. The tangent space at a given point gets identified with the first parabolic cohomology group of the representation $\operatorname{Ad}\rho$. In the second section, we introduce a complex analytical construction of the Moduli space $\mathscr{M}(\mathcal{W})$ of stable parabolic bundles over $\mathbb{P}^1$ with the system weights $\mathcal{W}$, by means of a differential equation analogous to the Beltrami differential equation, given as

$$f^{-1} \cdot f_{\bar{\tau}} = \nu$$

where ν is the conjugate transpose of a cusp form of weight 2 for the representation $\operatorname{Ad}\rho_{\mathbb{C}}$ (and by construction, an element of the tangent space at the point $[(E_\rho)_*]$). The two constructions are beautifully connected in terms of the Eichler-Shimura isomorphism, whose proof is also provided. Even though the relation between a point in the moduli space and those in a neighbourhood is complicted in terms of the suitable analogs of a quasiconformal map, to the first order of approximation the relations become particularly simple. We study the first variation of these deformations (they will become fundamental in the final chapter). The chapter is finished with the introduction of a canonical (1,0)-form on certain analytic open set of $\mathscr{M}(\mathcal{W})$, and of the parabolic Narasimhan-Atiyah-Bott (1,1)-form on $\mathscr{M}(\mathcal{W})$. The latter is moreover Kähler, a fact that will be ultimately proved in the last chapter.

As of Chapter 4, an action functional is introduced in the space of degenerate metrics of any given stable parabolic bundle with prescribed asymtotic behavior in the marked points on $\mathbb{P}^1$. The naive choice turns out to be divergent, a problem that is resolved by means of a process of *regularizaton.* Once an additional *topological* term is added to the functional, it is verified that its equations of motion turn out to be

$$(h^{-1}h_z)_{\bar{z}} = 0$$

which, geometrically, means that its associated connection (compatible with the complex structure) is logarithmic. By evaluating the functionals at their extrema, we obtain a function $S : \mathscr{M}(\mathcal{W}) \to \mathbb{R}$. Then, the variational formulas found in Chapter 3 are used to prove the main theorem of this work, which states that *the function -S/2 is a Kähler potential for the Narasimhan-Atiyah-Bott* (1,1)-*form.*

To conclude, there are 3 appendices included for the convenience of the reader. Appendix A gives a brief sumary of some basic results involving logarithms as multivalued functions as well as on the exponential map for the unitary group. Appendix B develops the basic notions of logarithmic connections on a vector bundle over a Riemann surface from the Čech perspective (mostly to fix conventions once and for all) and the closely related theory of Fuchsian systems on the sphere. Finally, Appendix C contains the basic facts about parabolic bundles and stability that will be used throughout this work.

Chapter 1

Parabolic bundles on $\mathbb{P}^1$

Holomorphic vector bundles over the sphere are rigid and do not admit a nontrivial deformation theory [44] and moreover, as a consequence of the Birkhoff-Grothendieck theorem, possess a nontrivial Lie group of automorphisms.[1] The ultimate relevance of parabolic structures in this special case is the introduction of extra parameters giving rise to a Moduli theory. Perhaps it is best to keep in mind the analogy with the Riemann surface situation: the Riemann sphere has a rigid complex structure and a Lie group of automorphisms. The latter disappears after the removal of 3 points, and if one removes more than 3 points, a Moduli theory arises.

1.1 The Riemann sphere as an extension of a Fuchsian model

The Riemann sphere is an exceptional Riemann surface in terms of moduli. It is a curious phenomenon that we can, nevertheless, associate Fuchsian groups uniformizing it as soon as branching is allowed.

Given $\mathscr{D} = \{z_1, \cdots, z_n\} \subset \mathbb{P}^1$, $n \geq 3$, the Riemann surface $X = \mathbb{P}^1 \setminus \mathscr{D}$ has fundamental group with the following presentation:

$$\pi_1(X) \cong \langle \gamma_1, \cdots, \gamma_n | \gamma_1 \cdots \gamma_n \rangle.$$

According to the uniformization theorem, there exist an analytic covering map $J : \mathbb{H} \to X$ and a Fuchsian group $\Gamma = \mathrm{Deck}(\mathbb{H}/X) \cong \pi_1(X)$ satisfying

$$J(\gamma \cdot \tau) = J(\tau) \qquad \forall\, \gamma \in \Gamma, \tau \in \mathbb{H}$$

[1] We will always assume, unless otherwise stated, that the rank of the bundles is grater or equal than 2.

such that the induced map $\Gamma \setminus \mathbb{H} \to X$ is a biholomorphism:

$$\begin{tikzcd} & \mathbb{H} \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "J"] & \\ \Gamma \setminus \mathbb{H} & \cong & X \end{tikzcd}$$

Depending on the situation, we will consider either description of the surface. We can (and should) assume that $z_{n-2} = 0,\ z_{n-1} = 1,\ z_n = \infty$.

The group Γ has the peculiarity of being generated by parabolic elements. Their single fixed points $\tau_1, \cdots, \tau_n \in \mathbb{R} \cup \{\infty\}$ will be called the *cusps* of the covering J, and can be normalized so that $\tau_{n-2} = 0,\ \tau_{n-1} = 1,\ \tau_n = \infty$. We make

$$\mathbb{H}^+ := \mathbb{H} \cup \{\tau_1, \cdots, \tau_n\}.$$

As in [50], we can define a suitable topology and analytic structure on $\mathbb{H}^+$ so that the quotient space $\Gamma \setminus \mathbb{H}^+$ becomes first a Hausdorff and compact space, and secondly, a Riemann surface biholomorphic to $\mathbb{P}^1$. To see this it is useful to consider an explicit form for the generators of Γ as elements of $\mathrm{SL}(2, \mathbb{R})$:

$$\gamma_i = \begin{pmatrix} 1 + \lambda_i \tau_i & -\lambda_i \tau_i^2 \\ \lambda_i & 1 - \lambda_i \tau_i \end{pmatrix}, \quad i \neq n, \qquad \gamma_n = \begin{pmatrix} 1 & \lambda_n \\ 0 & 1 \end{pmatrix}, \tag{1.1}$$

where $\lambda_1, \cdots, \lambda_n$ are real and non-zero. Since each of them is conjugate to the translation $\tau \to \tau + 1$, we can find $\sigma_1, \cdots, \sigma_n \in \mathrm{SL}(2, \mathbb{R})$ such that

$$\sigma_i^{-1} \cdot \gamma_i \cdot \sigma_i = \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix}; \tag{1.2}$$

explicitly

$$\sigma_i = \begin{pmatrix} \sqrt{|\lambda_i|}\tau_i & -\frac{1}{\sqrt{|\lambda_i|}} \\ \sqrt{|\lambda_i|} & 0 \end{pmatrix}, \quad i \neq n, \qquad \sigma_n = \begin{pmatrix} \sqrt{|\lambda_n|} & 0 \\ 0 & \frac{1}{\sqrt{|\lambda_n|}} \end{pmatrix}. \tag{1.3}$$

Now, we consider the usual topology for $\mathbb{H}$, and for each τ_i, a basis of neighbourhoods would be given by the union of $\{\tau_i\}$ and the translates $\sigma_i \cdot \mathbb{H}_\delta$ of the sets $\mathbb{H}_\delta := \{\tau \in \mathbb{H} : \mathrm{Im}(\tau) > \delta > 0\}$ (these are open discs whose boundary is tangent to $\partial\mathbb{H}$ at τ_i for $i \neq n$). Notice that this topology is stronger than the subspace topology of $\mathbb{H}^+ \subset \mathbb{C}$, and in particular, the convergence of a sequence $\{a_m\}_{m=1}^\infty$ to ∞ (which is equivalent to the convergence of a sequence to any other cusp) corresponds to $\mathrm{Im}(a_m) \to \infty$ in the real sense.

For each $\tau \in \mathbb{H}^+$, define

$$\Gamma_\tau := \{\gamma \in \Gamma : \gamma(\tau) = \tau\}.$$

It then follows that $\Gamma_\tau = e$ if $\tau \in \mathbb{H}$ and for $\tau_1, \cdots, \tau_n$, $\Gamma_{\tau_i} = \langle \gamma_i \rangle$, so the stabilizer of each cusp is an infinite cyclic group. The existence of local Fourier series expansions at the cusps of a Fuchsian group is a simple and deep result that will be crucial in our future considerations. Let $\mathscr{V}_i = \{\sigma_i \cdot \mathbb{H}_\delta\} \cup \{\tau_i\}$. Setting

$$q_i := q \circ \sigma_i^{-1}, \qquad q = \exp\left(2\pi\sqrt{-1}\tau\right) \tag{1.4}$$

since q_i extends to τ_i, it follows that $\Gamma_{\tau_i} \setminus \mathscr{V}_i \cong \Delta_\epsilon = \{z \in \mathbb{C} : |z| < \epsilon\}$, where $\epsilon = \exp(-2\pi\delta)$.

Proposition 1.1.1. *There is a $\delta > 0$ for which the maps*

$$\iota_i : \Gamma_{\tau_i} \setminus \mathscr{V}_i \to \Gamma \setminus \mathbb{H}^+.$$

are well-defined and homeomorphisms onto their images.

For a detailed proof, the reader is refered to [50], pp. 10-12. Thus the functions q_i play the role of local coordinates at the cusps on the quotient $\Gamma \setminus \mathbb{H}^+$ and determine the complex structure there. In particular, we can express J as a Fourier series on $\sigma_i \cdot \mathbb{H}_\delta$

$$J(\sigma_i\tau) = \begin{cases} z_i + \displaystyle\sum_{k=1}^{\infty} a_i(k) q^k & \text{if} \quad i \neq n; \\ \\ \displaystyle\sum_{k=-1}^{\infty} a_n(k) q^k & \text{if} \quad i = n. \end{cases} \tag{1.5}$$

It is evident that J extends to $\mathbb{H}^+$ as a branched covering over $\mathbb{P}^1$ having $z_1, \cdots, z_n$ as branch points:

$$\begin{tikzcd} & \mathbb{H}^+ \arrow[dl, "\mathrm{pr}"'] \arrow[dr, "J"] & \\ \Gamma \setminus \mathbb{H}^+ & \cong & \mathbb{P}^1 \end{tikzcd}$$

In fact, it is true that $a_i(1) \neq 0$ for $i < n$ [55]. This is intuitively clear since $q_i(\tau)$ and $J(\tau) - z_i$ should show the same local behaviour.

The following asymptotic formulas for the multivalued behavior of the inverse of J will be particularly useful in the future.

$$\sigma_i(J^{-1}(z)) = \begin{cases} \dfrac{1}{2\pi\sqrt{-1}}\left(\log\left(\dfrac{z-z_i}{a_i(1)}\right) + c_i(z-z_i) + \cdots\right) & \text{if} \quad i \neq n; \\[2ex] \dfrac{1}{2\pi\sqrt{-1}}\left(\log\left(\dfrac{a_n(-1)}{z}\right) + \dfrac{c_n}{z} + \cdots\right) & \text{if} \quad i = n. \end{cases} \tag{1.6}$$

where the coefficients

$$c_i = \begin{cases} -\dfrac{a_i(2)}{a_i(1)^2} & \text{if} \quad i \neq n; \\[2ex] a_n(0) & \text{if} \quad i = n. \end{cases} \tag{1.7}$$

are known as ***accessory parameters*** and appear in the classical theory of uniformization.

1.2 A holomorphic correspondence

Given a unitary representation[2] $\rho : \Gamma \to \mathrm{U}(r)$, we can construct a vector bundle of rank r over $\Gamma \setminus \mathbb{H}$ by defining the following equivalence relation on the trivial bundle $\mathrm{pr}_1 : \mathbb{H} \times \mathbb{C}^r \to \mathbb{H}$:

$$(\tau, v) \sim (\gamma \cdot \tau, \rho(\gamma)v), \qquad \forall \gamma \in \Gamma.$$

The fact that this quotient space is indeed a vector bundle follows from the fact that the action of Γ on $\mathbb{H}$ is free and properly discontinuous. We denote it (more precisely, the total space of it) by $\rho \setminus \mathbb{H} \times \mathbb{C}^r$, and the equivalence class containing an element (τ, v) by $[\tau, v]$. Similarly, its frame bundle is the principal bundle gotten by quotienting the product $\mathbb{H} \times \mathrm{GL}(r, \mathbb{C})$ under the equivalence relation $(\tau, g) \sim (\gamma \cdot \tau, \rho(\gamma)g)$. In particular, the transition functions of such bundles are always the same. The covering map J also allows us to consider them indistinctively as bundles over X.

We can think of $\mathrm{pr} : \mathbb{H} \to \Gamma \setminus \mathbb{H}$ as a principal bundle over $\Gamma \setminus \mathbb{H}$ with structure group Γ if we turn the left action into a right action by $\tau \cdot \gamma := \gamma^{-1} \cdot \tau$. From this point of view, the previous construction lets us interpret a representation of Γ as an "extension of the structure group" of the principal

[2]This construction works for an arbitrary representation, but we are mainly interested in the unitary case, and we should assume so from now on unless stated otherwise.

bundle of a Galois covering.

Since the Riemann surface $\Gamma \setminus \mathbb{H}$ is noncompact, we know by a theorem of Stein that $\rho\setminus\mathbb{H}\times \mathrm{GL}(r,\mathbb{C})$ (and hence $\rho\setminus\mathbb{H}\times\mathbb{C}^r$) are trivial. This fact is equivalent to the existence of a function $\Psi : \mathbb{H} \to \mathrm{GL}(r,\mathbb{C})$ with the representation ρ encoded in its automorphic behaviour. Since the triviality of a principal $\mathrm{GL}(r,\mathbb{C})$-bundle is equivalent to the existence of a holomorphic section on it, let us consider a holomorphic section $s : \Gamma \setminus \mathbb{H} \to \rho \setminus \mathbb{H} \times \mathrm{GL}(r,\mathbb{C})$, and let $F : \mathbb{H} \to \rho \setminus \mathbb{H} \times \mathrm{GL}(r,\mathbb{C})$ be given by

$$F(\tau) = [\tau, I]$$

The following result can be found in [14],[22]. We include it for the convenience of the reader.

Proposition 1.2.1. *There exist a holomorphic function $\Psi : \mathbb{H} \to \mathrm{GL}(r,\mathbb{C})$ satisfying*

$$F(\tau) = s([\tau]) \cdot \Psi(\tau) \tag{1.8}$$

and

$$\Psi(\gamma \cdot \tau) = \Psi(\tau)\rho^{-1}(\gamma) \qquad \forall \gamma \in \Gamma. \tag{1.9}$$

Proof. By considering local trivializations $\phi_i : \pi^{-1}(\mathscr{U}_i) \to \mathscr{U}_i \times \mathrm{GL}(r,\mathbb{C})$ for some open cover $\{\mathscr{U}_i\}$ of $\Gamma \setminus \mathbb{H}$, since

$$[\gamma \cdot \tau, I] = [\tau, \rho^{-1}(\gamma)] = [\tau, I] \cdot \rho^{-1}(\gamma) \tag{1.10}$$

(the last equality meaning the right action of $\mathrm{GL}(r,\mathbb{C})$ on $\rho \setminus \mathbb{H} \times \mathrm{GL}(r,\mathbb{C})$), we have that $F_i := \mathrm{pr}_2 \circ \phi_i \circ F|_{J^{-1}(\mathscr{U}_i)} : \mathrm{pr}^{-1}(\mathscr{U}_i) \to \mathrm{GL}(r,\mathbb{C})$ are holomorphic functions such that the restriction of the products to $\mathrm{pr}^{-1}(\mathscr{U}_{ij})$

$$F_i \cdot F_j^{-1} : \mathrm{pr}^{-1}(\mathscr{U}_{ij}) \to \mathrm{GL}(r,\mathbb{C})$$

are invariant under the action of Γ, and then they define a 2-cocycle on $\{\mathscr{U}_{ij}\}$ which we know is solvable by the triviality of $\rho\setminus\mathbb{H}\times\mathrm{GL}(r,\mathbb{C})$, so if we consider the global holomorphic section s as a collection $\{\mathscr{U}_i, s_i\}$ where $s_i : \mathscr{U}_i \to \mathrm{GL}(r,\mathbb{C})$, we have

$$s_i \cdot s_j^{-1} = F_i \cdot F_j^{-1} \quad \text{on} \quad \mathscr{U}_{ij}.$$

We can now define on $\mathrm{pr}^{-1}(\mathscr{U}_i) \subset \mathbb{H}$ the function

$$\Psi_i := s_i^{-1} \cdot F_i \tag{1.11}$$

and it follows that $\Psi_i = \Psi_j$ on $\mathrm{pr}^{-1}(\mathscr{U}_{ij})$, so the Ψ_i's patch together to define a function $\Psi : \mathbb{H} \to \mathrm{GL}(r, \mathbb{C})$. By construction $s|_{\mathscr{U}_i} \leftrightarrow s_i$ in the same way that $F|_{\mathscr{U}_i} \leftrightarrow F_i$, and we then conclude from (1.11) that

$$F(\tau) = s([\tau]) \cdot \Psi(\tau). \tag{1.12}$$

Now, it readily follows from (1.10) and (1.12) that Ψ satisfies

$$\Psi(\gamma \cdot \tau) = \Psi(\tau)\rho^{-1}(\gamma) \qquad \forall \gamma \in \Gamma.$$

□

The function Ψ establishes the isomorphism $\rho \setminus \mathbb{H} \times \mathbb{C}^r \cong (\Gamma \setminus \mathbb{H}) \times \mathbb{C}^r$ in the following way: we can define a map

$$\mathbb{H} \times \mathbb{C}^r \to (\Gamma \setminus \mathbb{H}) \times \mathbb{C}^r$$

$$(\tau, v) \mapsto ([\tau], \Psi(\tau)v)$$

and then (1.9) implies that it only depends on the equivalence class. The induced bundle map

$$[\tau, v] \mapsto ([\tau], \Psi(\tau)v)$$

will be the bundle isomorphism. Holomorphic sections of a principal bundle correspond tautologically to isomorphisms to the trivial bundle so equation (1.8) is, in a sense, a restatement of this fact given the nature of our bundle as a quotient by the representation ρ.

Definition 1.2.2. *Given a unitary representation $\rho : \Gamma \to \mathrm{U}(r)$, a holomorphic function $f : \mathbb{H} \to \mathbb{C}^r$ is said to be ρ-**automorphic** if it satisfies*

$$f(\gamma\tau) = \rho(\gamma)f(\tau). \tag{1.13}$$

We thus have tautological correspondences

$$\left\{ \begin{array}{c} \text{Holomorphic} \\ \text{sections of} \\ \rho \setminus \mathbb{H} \times \mathbb{C}^r \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \rho\text{-} \\ \text{automorphic} \\ \text{functions} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Holomorphic} \\ \text{functions} \\ X \to \mathbb{C}^r \end{array} \right\} \tag{1.14}$$

the second one depending on the choice of the function Ψ. Namely, any holomorphic function $f : \mathbb{H} \to \mathbb{C}^r$ satisfying

$$f(\gamma\tau) = \rho(\gamma)f(\tau)$$

will descend to the holomorphic section $[\tau] \mapsto [\tau, \Psi(\tau)f(\tau)]$ of $\rho \setminus \mathbb{H} \times \mathbb{C}^r$ and conversely, the pullback $\mathrm{pr}^*(s)$ of a holomorphic section of $\rho \setminus \mathbb{H} \times \mathbb{C}^r$ would be a holomorphic section of the trivial bundle $\mathbb{H} \times \mathbb{C}^r$ and thus a ρ-automorphic function. Now, fix a function Ψ. Since $\Psi(\tau)f(\tau)$ is invariant under the action of Γ then $\Psi(\tau)f(\tau) = g(J(\tau))$ for some holomorphic function on X and conversely, given a holomorphic function $g : X \to \mathbb{C}^r$ the function $f(\tau) = \Psi^{-1}(\tau)g(J(\tau))$ is ρ-automorphic.

Since the group Γ has a canonical presentation, a unitary representation of Γ is determined, up to conjugation, by a collection of n diagonal matrices $W_i = \mathrm{diag}(\alpha_{i1}, \cdots, \alpha_{ir})$, with $0 \leq \alpha_{i1} \leq \cdots \leq \alpha_{ir} < 1$, and unitary matrices U_i such that

$$U_i \exp(2\pi\sqrt{-1}W_i)U_i^{-1} = M_i = \rho(\gamma_i). \tag{1.15}$$

Notice that the matrices U_i are determined only up to left multiplication by an element of the centralizer of $e^{2\pi\sqrt{-1}W_i}$, that is, $[U_i]$ is an element of the flag variety $\mathrm{U}(r)/Z(e^{2\pi\sqrt{-1}W_i})$.

Definition 1.2.3. *We will call the n-tuple $\mathcal{W} = (W_1, \cdots, W_n)$ the **system of weights** of the representation ρ. A system of weights is **generic** if all the inequalities defining it are strict.*

It is the case that the automorphic behaviour of the function Ψ near the cusp τ_i is completely determined by $\rho(\gamma_i)$.

Proposition 1.2.4. *For a sufficiently large δ, the restriction of Ψ to $\sigma_i \cdot \mathbb{H}_{\delta_i}$ has the form*

$$\Psi(\sigma_i\tau) = \Phi_i(q) \cdot q^{-W_i}U_i^{-1} \tag{1.16}$$

where the Φ_i are holomorphic in $\Delta_\epsilon \setminus \{0\}$.

Proof. The proof is simple and can also be found in [14]. The function $E_i(\sigma_i\tau) := q^{-W_i}U_i^{-1}$ is defined on $\sigma_i \cdot \mathbb{H}_\delta$ and satisfies

$$E_i(\gamma_i\sigma_i\tau) = E_i(\sigma_i(\sigma_i^{-1}\gamma_i\sigma_i)\tau) = E_i(\sigma_i(\tau + 1)) = E_i(\sigma_i\tau)\rho(\gamma_i)^{-1},$$

so it has the same automorphic behaviour than Ψ with respect to γ_i. Therefore

$$\Psi|_{\sigma_i\cdot\mathbb{H}_{\delta_i}} \cdot E_i^{-1}$$

is a function invariant under the action of γ_i. We define Φ_i to be this function. □

Remark 1. Starting from a representation $\rho : \Gamma \to \mathrm{U}(r)$, the construction of the function Ψ relied on 2 objects: 1) the locally constant function $F(\tau) = [\tau, I]$

and 2) a global section of the principal bundle $\rho \setminus \mathbb{H} \times \mathrm{GL}(r, \mathbb{C})$. However, the choice of the later is completely arbitrary and since $\Gamma \setminus \mathbb{H} \cong X$ is not compact the space of such holomorphic sections is not even finite dimensional. We can expand the invariant local terms as a Fourier series

$$\Phi_i(\sigma_i \tau) = \sum_{k=-\infty}^{\infty} C_i(k) q^k \tag{1.17}$$

but nothing concrete can be concluded about the type of singularities that could occur. The goal of the next sections is to determine how prescribing these singularities leads to the notion of extensions of our quotients.

1.3 Extensions of the vector bundle $\rho \setminus \mathbb{H} \times \mathbb{C}^r$

From an algebro-geometric perspective, the construction of the Riemann surface $\Gamma \setminus \mathbb{H}^+$ can be understood as an extension of the scheme structure of $\Gamma \setminus \mathbb{H}$ to the cusps: the sheaf of holomorphic functions is defined there in terms of the functions q_i as a local basis. The idea turns out to extrapolate naturally to the vector bundle level and provide extensions of $\rho \setminus \mathbb{H} \times \mathbb{C}^r$.

One of such extensions, that we should call the *canonical extension*, was introduced by Deligne [20] and later consider by Mehta and Seshadri [37] in the general context of a punctured Riemann surface of genus g, where they show the correspondence between these and the so-called *parabolic semistable* bundles (see section C.1 in the appendix for details). To do this we need to consider the correspondence between holomorphic vector bundles over X and locally-free sheafs of $\mathcal{O}_X$-modules. The sheaf defining $\rho \setminus \mathbb{H} \times \mathbb{C}^r$ is given by the correspondence (1.14), so it is only left to extend it in a suitable way to the neighbourhoods of the cusps.

We explain this in more detail for the purpose of completion. We should keep in mind, however, that we are just dealing with complex manifolds, and that these ideas could be easily restated from a purely analytic perspective, as we will do immediatly after.

The trivial bundle over $\mathbb{H}$ corresponds to the free sheaf $\mathcal{O}(\mathbb{H} \times \mathbb{C}^r)$ from which we can induce the direct image sheaf $\mathrm{pr}_*(\mathcal{O}(\mathbb{H} \times \mathbb{C}^r))$ on $\Gamma \setminus \mathbb{H}$ under the map pr (for an arbitrary sheaf $\mathcal{S}$ over $\mathbb{H}$, this is defined as $\mathrm{pr}_*(\mathcal{S})(\mathscr{U}) := \mathcal{S}(\mathrm{pr}^{-1}(\mathscr{U}))$ for any open set $\mathscr{U} \subset \Gamma \setminus \mathbb{H}$). This can be thought of intuitively as the "sheaf of multivalued-sections on $\Gamma \setminus \mathbb{H}$", and it possesses a subsheaf $\mathrm{pr}_*^{\Gamma}(\mathcal{O}(\mathbb{H} \times \mathbb{C}^r))$ whose sections correspond to ρ-automorphic functions on $\mathbb{H}$ and is clearly locally-free. It is the latter that corresponds to the quotient bundle $\rho \setminus \mathbb{H} \times \mathbb{C}^r$ over $\Gamma \setminus \mathbb{H}$.

The extension would be a prescription of the desired behaviour of the sections of this sheaf at the points $\tau_1, \cdots, \tau_n$. Proposition 1.1.1 characterizes the local behaviour of the projection $\mathrm{pr} : \mathbb{H}^+ \to \Gamma \setminus \mathbb{H}^+$, and moreover, implies that near the cusps τ_i a Γ-invariant function would be expressable in the form

$$f_i(\sigma_i \tau) = U_i q^{W_i} g_i(q)$$

for some local function g_i, holomorphic on Δ_ϵ^* and with a potential singularity of arbitrary type at $q = 0$. This motivates us to define the extension of the sheaf $\mathrm{pr}_*^\Gamma(\mathcal{O}(\mathbb{H} \times \mathbb{C}^r))$ at the point τ_i, $1 \leq i \leq n$, in terms of the basis

$$\{q_i^{\alpha_{i1}}, \cdots, q_i^{\alpha_{ir}}\},$$

where $\mathrm{diag}(\alpha_{i1}, \cdots, \alpha_{ik}) = W_i$. In other words, the local holomorphic sections would be the f_i's for which g_i is holomorphic at τ_i. This condition is in turn equivalent to

$$\lim_{\tau \to i\infty} \| f(\sigma_i \tau) \| < \infty \tag{1.18}$$

The previous sheaf could be generalized if we let the local bases to be instead

$$\left\{q_i^{\alpha_{i1}+n_{i1}}, \cdots, q_i^{\alpha_{ir}+n_{ir}}\right\},$$

for an arbitrary collection of integers $\{n_{ij}\}$, thus allowing the holomorphic sections to correspond to meromorphic functions with a certain bound on its zeros or poles at τ_i. In a series of papers [14]-[15] on the Riemann-Hilbert problem, Andrei Bolibrukh introduced a generalization of this last idea (for a general, not necessarely unitary representation) and a family of extensions of the bundle $\rho \setminus \mathbb{H} \times \mathbb{C}^r$ to the whole sphere on which a "Fuchsian linear system" could be naturally defined. Such system, which in geometric terms is a special type of singular connection, would be introduced and studied in section 1.7.

We now proceed to restate the previous ideas in a more analytical setup. In section 1.4 we will find the transition functions of these extensions for two canonical coverings of $\Gamma \setminus \mathbb{H}^+$ giving yet another equivalent way of defining them.

Let's start by noticing that the quotient $\Gamma \setminus \mathbb{H}^+$ can be equivalently described as the identification

$$(\Gamma \setminus \mathbb{H}) \sqcup_{q_i} \Delta_\epsilon \tag{1.19}$$

since the functions q_i are biholomorphisms from $\Gamma_{\tau_i} \setminus \sigma_i \cdot \mathbb{H}_\delta$ onto the punctured disks $\Delta_\epsilon \setminus \{0\}$. We can carry over the same idea to the bundle level and analogously provide a collection of extensions of the bundle $\rho \setminus \mathbb{H} \times \mathbb{C}^r$.

Let's first consider a collection $\mathscr{N} = \{N_1, \cdots, N_n\}$ of diagonal $r \times r$ matrices of integers. For a sufficiently small neighbourhood $\sigma_i^{-1} \cdot \mathbb{H}_\delta$ of the cusps τ_i, $i = 1, \cdots, n$, consider the functions

$$h_i : \mathrm{pr}^{-1}(\Gamma_{\tau_i} \setminus (\sigma_i \cdot \mathbb{H}_\delta)) \to \Delta_\epsilon \times \mathbb{C}^r;$$

$$[\sigma_i \tau, v] \mapsto (q, q^{-(W_i+N_i)} U_i^{-1} v)$$

which are biholomorphic maps [3] onto their images $(\Delta_\epsilon \setminus \{0\}) \times \mathbb{C}^r$. We can use these functions to identify the fibers of $\rho \setminus \mathbb{H} \times \mathbb{C}^r$ around the punctured neighbourhoods $\Gamma_{\tau_i} \setminus \sigma_i \cdot \mathbb{H}_\delta$ with the corresponding fibers of the trivial bundles $\Delta_\epsilon \times \mathbb{C}^r$.

For notational simplicity, we will denote the bundle $\rho \setminus \mathbb{H} \times \mathbb{C}^r$ by E_0 assuming that the representation ρ under consideration is implicit from the context.

Definition 1.3.1. *For each choice of a collection of diagonal matrices of integers $\mathcal{N} = \{N_1, \cdots, N_n\}$, we define an extension of $\rho \setminus \mathbb{H} \times \mathbb{C}^r = E_0$ to $\Gamma \setminus \mathbb{H}^+$ with total space*

$$E_{\rho,\mathcal{N}} := (\rho \setminus \mathbb{H} \times \mathbb{C}^r) \sqcup_{h_i} (\Delta_\epsilon \times \mathbb{C}^r)$$

we call the extension corresponding to $N_1 = \cdots = N_n = 0$ ***the canonical extension****, and we denote it by E_ρ.*

Remark 2. In [16], A. A. Bolibrukh proves that these extensions are indeed all possible analytic extensions of the bundle E_0 and they possess a canonical logarithmic connection (see Appendix B). The signs of the exponents in the functions h_i are the opposite from Bolibrukh [14]. The reason for his convention is the wish to apply vector bundle techniques in the study of Fuchsian systems on $\mathbb{P}^1$, and historically these are defined by an equation $(\mathrm{d} - \Omega)w = 0$ so that the local behaviour of a solution in a neighbourhood of a singular point is a multivalued function of the form

$$\phi_i(z) e^{E_i \log(z - z_i)}$$

(see proposition 1.1.1, page 13). This makes necessary to consider clockwise oriented loops around the singularities as generators of the fundamental group in order to obtain a monodromy representation. On the other hand, a connection on a vector bundle is defined so that its parallel sections are given locally

[3]with inverse $h_i^{-1}(q, v) = \left[\frac{1}{2\pi\sqrt{-1}} \mathscr{L}(q, 0), U_i e^{(W_i+N_i)\sigma_i^{-1}\mathscr{L}(q,0)} v\right]$; see appendix A.1 for details.

by functions s_i satisfying equations of the form $(\mathrm{d} + A_i)s_i = 0$, so in the end Ω and A_i would be matrix-valued 1-forms with the same local behaviour, that is, the correspondence between Fuchsian linear systems on $\mathbb{P}^1$ and logarithmic connections modulo gauge transformations on a vector bundle over $\mathbb{P}^1$ is consistent after the choice of the canonical Birkhoff-Grothendieck trivializations (cf. [30]).

1.4 Transition functions for the extensions $E_{\rho,\mathcal{N}}$

By its very definition, the bundle E_0 can be prescribed by a collection of constant transition functions. Given that the bundles $E_{\rho,\mathcal{N}}$ are an extension to $\Gamma \setminus \mathbb{H}^+$ of E_0, it should be possible to give a collection of transition functions for the former containing the constant transition functions of the latter as a subcollection.

Let us start by considering a finite covering $\{\mathscr{U}_i\}_{i=1}^m$ of $\Gamma\setminus\mathbb{H}^+$ "fine enough" in the following sense:

1. The subcollection $\{\mathscr{U}_i\}_{i>n}$ is a covering of $\Gamma \setminus \mathbb{H}$,
2. Any nonempty intersection is contractible,
3. For $1 \leq i, j \leq n$, $\tau_i \in \mathscr{U}_i$ and $\mathscr{U}_{ij} = \emptyset$,
4. $\exists \mathscr{V}_1, \cdots, \mathscr{V}_m \subset \mathbb{H}^+$ connected such that, if $i > n$, then $\mathscr{V}_i \cong \mathrm{pr}(\mathscr{V}_i) = \mathscr{U}_i$, and if $1 \leq i \leq n$, then $\Gamma_{\tau_i} \setminus \mathscr{V}_i \cong \mathrm{pr}(\mathscr{V}_i) = \mathscr{U}_i$ and moreover, $\sigma_i \cdot \mathscr{V}_n = \mathscr{V}_i$.

For the neighbourhoods of the cusps $\tau_1, \cdots \tau_n$ we shall fix a reference point $\tau_{i_0} \in \mathscr{V}_i$, $\tau_{i_0} \neq \tau_i$.

Proposition 1.4.1. *Given a collection $\mathcal{N} = \{N_1, \cdots, N_n\}$, the transition functions of the extension $E_{\rho,\mathcal{N}} \to \Gamma \setminus \mathbb{H}^+$ with respect to the aforementioned open cover are given, up to equivalence, by*

$$g_{ij} = \begin{cases} \rho(\gamma_{ij}) & \textit{if} \quad i, j > n; \\ \\ q_i^{-(W_i+N_i)} U_i^{-1} & \textit{if} \quad 1 \leq i \leq n, j > n. \end{cases} \tag{1.20}$$

In particular, a collection of transition functions of the bundle $E_0 = \rho \setminus \mathbb{H} \times \mathbb{C}^r$ are given by restriction of the former to the cover $\{\mathscr{U}_i\}_{i>n}$.

Remark 3. These transition functions (up to the sign of the exponents) are described by A. Bolibrukh in [16].

Proof. It is easy to see that that the previous conditions imply that for $i \neq j$, if non-empty, $\mathrm{pr}^{-1}(\mathscr{U}_{ij}) \cap \mathscr{V}_k$ is connected. We define $\mathscr{W}_{ij,k} := \mathrm{pr}^{-1}(\mathscr{U}_{ij}) \cap \mathscr{V}_k$. Now, denote by γ_{ij} the element in Γ such that $\gamma_{ij} \cdot \mathscr{W}_{ij,j} = \mathscr{W}_{ji,i}$. It follows that if $\mathscr{U}_{ijk} \neq \emptyset$, then the set $\gamma_{jk} \cdot \mathscr{W}_{ik,k} \cap \mathscr{W}_{ij,j}$ is nonempty and biholomorphic to $\mathscr{U}_{ijk}$ under pr.

Thus $\{\mathscr{U}_{ij}, \gamma_{ij}\}_{i,j>n}$ is a collection of transition functions for $\mathrm{pr} : \mathbb{H} \to \Gamma \backslash \mathbb{H}$ thought as a principal Γ-bundle. It follows that $\{\mathscr{U}_{ij}, \rho(\gamma_{ij})\}_{i,j>n}$ would be the corresponding collection of transition functions for $\rho \setminus \mathbb{H} \times \mathbb{C}^r$.

To complete the collection we need to consider a function Ψ as in proposition 1.2.1. Then Ψ^{-1} is a ρ-automorphic function, and we can prescribe a meromorphic frame on $E_{\rho,\mathcal{N}} \to \Gamma \setminus \mathbb{H}^+$ by choosing local representatives $\{\mathscr{U}_i, s_i\}_{i=1}^m$ with respect to the open cover $\{\mathscr{U}_i\}_{i=1}^m$ in the following way:

1. For $i > n$, we let $s_i(q) := \Psi^{-1}(\tau)|_{\mathscr{V}_i}$.
2. If $1 \leq i \leq n$, then after taking into consideration the local basis of sections, we make $s_i(q_i) := q_i^{N_i} \Phi_i^{-1}(q_i)$.

On any nonempty intersection $\mathscr{U}_{ij}$, the matrix-valued functions s_i, s_j would be holomorphic and invertible. By construction this would possibly happen only for the cases 1) $i, j > n$ (known already), 2) $1 \leq i \leq n$ and $j > n$. For the second case we find

$$\begin{aligned} g_{ij} &= q_i^{N_i} \Phi_i^{-1}|_{\mathscr{U}_{ij}} \cdot \Psi|_{\mathscr{W}_{ij,j}} &= q_i^{-(W_i+N_i)} U_i^{-1}|_{\mathscr{W}_{ij,j}} \\ & &= q_i^{-(W_i+N_i)} U_i^{-1}|_{\mathscr{U}_{ij}}. \end{aligned}$$

The last equation is justified since $\mathscr{W}_{ij,j}$ is simply-connected and there is a unique $n_{ij} \in \mathbb{Z}$ such that

$$\sigma_i^{-1}\tau = \mathscr{L}(q_i, n_{ij})$$

is a biholomorphism between $\mathscr{U}_{ij}$ and $\mathscr{W}_{ij,j}$ (see Appendix A.1).

To finish, we verify that the cocycle condition is satisfied. Triple intersections could possibly occur in 2 cases: 1) $i, j, k > n$, and 2) $i \leq n,\ j, k > n$. Since

$$\gamma_{ik} = \gamma_{ij} \cdot \gamma_{jk}$$

then the cocycle condition is satisfied in the first case, and for the second case, if $\mathscr{U}_{ijk} \neq \emptyset$, we have $\gamma_{jk} = \gamma_i^{n_{ij}-n_{ik}}$. But by the definition of $\mathscr{L}$ we know that

$$\mathscr{L}(q, n_{ij}) = \gamma_n^{n_{ij}-n_{ik}} \cdot \mathscr{L}(q, n_{ik}) = \mathscr{L}(q, n_{ik}) + n_{ij} - n_{ik},$$

consecuently,

$$g_{ij}|_{\mathscr{U}_{ijk}} = g_{ik}|_{\mathscr{U}_{ijk}} \cdot \rho^{-1}(\gamma_{jk})$$

and we conclude that the cocycle condition is satisfied in the second case as well.

□

We have mentioned before that the bundle $\rho \setminus \mathbb{H} \times \mathbb{C}^r$ is isomorphic to the trivial bundle $(\Gamma \setminus \mathbb{H}) \times \mathbb{C}^r$. Thus the transitions functions previously found should be equivalent to the identity when $i, j > n$, that is, there should exist holomorphic functions $g_i : U_i \to \mathrm{GL}(r, \mathbb{C})$ such that

$$\rho(\gamma_{ij}) = g_i g_j^{-1} \qquad \text{when } i, j > n$$

In particular, the functions determined by the equation $g_i([\tau]) := \Psi^{-1}(\tau)|_{\mathscr{V}_i}$ satisfy these properties. If we complete this set by the constant functions $g_1 = \mathrm{Id}, \cdots, g_n = \mathrm{Id}$, we would find an equivalent set of transition functions given by $g_{ij} = g_i \tilde{g}_{ij} g_j^{-1}$. Since $\tilde{g}_{ij} = \mathrm{Id}$ when $i, j > n$, we shall modify the cover by replacing all the sets $\mathscr{U}_i$ with $i > n$ by their union $\mathscr{U}_0 = \cup_{i>n} \mathscr{U}_i = \Gamma \setminus \mathbb{H}$. In this case the only nonempty intersections are of the form $\mathscr{U}_{i0}$. If we take proposition 1.2.4 into account, we conclude the following

Corollary 1.4.2. *Let $\{\mathscr{U}_i\}_{0 \le i \le n}$ be the open cover of $\Gamma \setminus \mathbb{H}^+$ given by $\mathscr{U}_0 = \Gamma \setminus \mathbb{H}$, $\mathscr{U}_i = \Gamma_{\tau_i} \setminus \mathscr{V}_i$ for $i = 1, \cdots, n$. For any choice of a ρ-automorphic function $\Psi : \mathbb{H} \to \mathrm{GL}(r, \mathbb{C})$ with local expressions $\Psi|_{\mathscr{V}_i} = \Phi_i q_i^{-(W_i + N_i)} U_i^{-1}$, the functions*

$$g_{0i} = \Phi_i \qquad i = 1, \cdots, n \tag{1.21}$$

give transition functions for the bundle $E_{\rho, \mathcal{N}}$.

From now on we will concentrate in the canonical extension E_ρ.

Definition 1.4.3. *Given a unitary representation $\rho : \Gamma \to \mathrm{U}(r)$, the **contragradient** (or dual) representation $\rho^* : \Gamma \to \mathrm{U}(r)$ is defined as $\rho^* = {}^t\rho^{-1}$ and the **adjoint** representation is the composion of ρ with the adjoint representation of $\mathrm{U}(r)$ in $\mathfrak{u}(r)$, $\mathrm{Ad}\,\rho := \mathrm{Ad} \circ \rho$. Its complexification $(\mathrm{Ad} \circ \rho) \otimes_{\mathbb{R}} \mathbb{C} = \rho^* \otimes \rho$ will be denoted by $\mathrm{Ad}\,\rho_{\mathbb{C}}$.*

In a similar fashion we can associate the corresponding canonical extensions to each of these. One should expect a relationship between these bundles; this one is however, not as one would expect at first.

Proposition 1.4.4. *Assume that the system of weights of ρ is generic. There is an isomorphism*

$$(E_\rho)^* \cong \mathcal{O}(n) \otimes E_{\rho^*}$$

Proof. Since ρ is unitary, ρ^* is also unitary and its generators satisfy

$$\rho^*(\gamma_i) = {}^t\rho(\gamma_i)^{-1} = \overline{\rho(\gamma_i)},$$

Since the weights of ρ are generic, the weights $\mathcal{W}' = (W_1', \cdots, W_n')$ of ρ^* satisfy $W_i' = I - W_i$ after a permutation . This implies that with respect to the cover consider in the beginning of this section, the vector bundle E_{ρ^*} would be determined by the transition functions

$$g'_{ij} = \begin{cases} \overline{\rho(\gamma_{ij})} & \text{if} \quad i, j > n; \\ \\ q_i^{-(I-W_i)}\overline{U}_i^{-1} & \text{if} \quad 1 \leq i \leq n, j > n. \end{cases} \tag{1.22}$$

On the other hand, if a vector bundle E is determined by transition functions $\{g_{ij}\}$, the dual bundle E^* is determined by the transition functions $\{{}^t(g_{ij})^{-1}\}$. Since the base of the bundles is $\mathbb{P}^1$, the divisor $\mathscr{D}$ is linearly equivalent to the divisor $n\infty$, which implies that (1.22) are equivalent to

$$g'_{ij} = \begin{cases} \overline{\rho(\gamma_{ij})} & \text{if} \quad i, j > n; \\ \\ q_i^{W_i}\overline{U}_i^{-1} & \text{if} \quad 1 \leq i \leq n-1, j > n, \\ \\ q_n^{(W_n - nI)}\overline{U}_n^{-1} & \text{if} \quad i = n, j > n. \end{cases} \tag{1.23}$$

Comparing (1.20) and (1.23), we obtain the claim. □

The next theorem was proved by A. Grothendieck [25] for arbitrary reductive groups, although its proof was know to G. Birkhoff already in 1913 [10] (before vector bundles were even defined!).

Theorem 1.4.5. (Birkhoff-Grothendieck) *Any holomorphic vector bundle $E \to \mathbb{P}^1$ of rank r splits as a direct sum of line bundles*

$$E \cong \mathcal{O}(a_1) \oplus \cdots \oplus \mathcal{O}(a_r)$$

where $a_1 \geq a_2 \geq \cdots \geq a_r$ are unique up to a possible permutation.

This means that with respect to an open cover of $\mathbb{P}^1$ given by the affine charts $\{\mathbb{C}_0, \mathbb{C}_\infty\}$, with coordinates $z \in \mathbb{C}_0$ and $\zeta \in \mathbb{C}_\infty$ related by $\zeta = 1/z$, the transition functions of this bundle are equivalent to the ones of the form

$$g_{0\infty}(z) = z^N$$

with $N = \operatorname{diag}(a_1, \cdots, a_r)$. We will call this diagonal matrix of integers the *Birkhoff-Grothendieck type* of the bundle.

The next proposition relates algebraic information (the space of invariants of ρ) and analytic information (the space of holomorphic sections of E_ρ). The isomorphism was first described by Mehta and Seshadri in [37] for parabolic bundles over Riemann surfaces of arbitrary genus.

Proposition 1.4.6. *There is an isomorhism*

$$(\mathbb{C}^r)^\rho \cong H^0(\mathbb{P}^1, \mathcal{O}(E_\rho)) \tag{1.24}$$

between the subspace of invariant vectors in $\mathbb{C}^r$ under ρ and the space of holomorphic sections of E_ρ.

Proof. Given an eigenvector v of ρ with eigenvalue 1, we can define a global section of E_0 as the constant section defined by v, which clearly extends to the cusps. Therefore we have an injective map $(\mathbb{C}^r)^\rho \to H^0(\mathbb{P}^1, \mathcal{O}(E_\rho))$. Assume now that s is a holomorphic section of E_ρ. Its pull-back to $\mathbb{H} \times \mathbb{C}^r$ is a ρ-automorphic function f all of whose Fourier expansions 1.17 are holomorphic as power series in q. This implies that the function $\|f\|$ descends to a subharmonic function on $\mathbb{P}^1 \setminus \mathscr{D}$ bounded on $\mathscr{D}$, therefore a constant. This in turn implies that f, and hence s is a constant, and the map is also surjective. □

Since unitary representations are totally reducible and $\dim(\mathbb{C}^r)^\rho \leq r$, we conclude

Corollary 1.4.7. $E_\rho \cong \bigoplus_{i=1}^{r} \mathcal{O}(a_j)$ *where* $a_j \leq 0$, $j = 1, \cdots, r$. *If ρ is irreducible, then $a_j < 0$ $\forall j$. If moreover, the system of weights $\mathcal{W}$ is generic, then $-n < a_j < 0$.*

Remark 4. Later on we will prove that $\sum_{j=1}^{r} a_j = \deg(E_\rho) = -\sum_{i=1}^{n}\sum_{j=1}^{r} \alpha_{ij}$ (Proposition B.1.7, cf. [37]) Among all the bundles of fixed degree d and rank r over $\mathbb{P}^1$, there is a particular splitting that will be of interest to us, and is completely determined by these. By writing $d = \deg(E) = ar + b = b(a+1) + (r-b)a$ for $0 \leq b < r$, we see that there is a unique splitting satisfying the condition $|a_j - a_k| \leq 1$ for all $1 \leq j, k \leq r$, namely,

$$\mathcal{O}(a+1)^b \oplus \mathcal{O}(a)^{r-b}$$

We will call such bundles *evenly split*.

1.5 A canonical $\mathbf{GL}(r, \mathbb{C})$-valued ρ-automorphic function

Now we give an application of the previous results into the construction of a canonical ρ-automorphic function with the virtue of "uniformizing" the vector bundle E_ρ. In order to do that we need one more technical lemma.

Definition 1.5.1. *A **monopole** is a matrix-valued rational function $P(z)$ on $\mathbb{P}^1$, holomorphic and holomorphically invertible in $\mathbb{P}^1 \setminus \{\infty\}$.*

In other words, $P(z) \in \mathrm{GL}(r, \mathbb{C}[z])$. Notice that this implies that the determinant of $P(z)$, being holomorphic and different from zero on $\mathbb{P}^1 \setminus \{\infty\}$, is a constant, and therefore $P^{-1}(z)$ is also a monopole.

Lemma 1.5.2. (permutation lemma). *Let $\zeta^{-N}\Phi(\zeta)$ be a meromorphic function on $\mathbb{C}_\infty$ with Φ holomorphic and invertible everywhere and N a diagonal matrix of integers. There exist a monopole $P(z)$ and another function $\Phi'(\zeta)$ holomorphic and invertible on $\mathbb{C}_\infty$ such that on $\mathbb{C}_0 \cap \mathbb{C}_\infty$,*

$$P \cdot \zeta^{-N} \cdot \Phi = \Phi' \cdot \zeta^{-N'} \tag{1.25}$$

where N' is gotten from N by a possible permutation of its diagonal elements.

Proof. We are only interested here in the case where $N = \mathrm{diag}(a+1, \cdots, a+1, a, \cdots, a)$, that is, when N is the Birkhoff-Grothendieck type of an evenly split bundle. For a proof of the general case see [30], pp. 282-284, [13]. If we express Φ in block form

$$\Phi(\zeta) = \begin{pmatrix} Q_1(\zeta) & Q_2(\zeta) \\ Q_3(\zeta) & Q_4(\zeta) \end{pmatrix}$$

then it follows that

$$\zeta^{-N} \cdot \Phi(\zeta) \cdot \zeta^{N} = \begin{pmatrix} Q_1(\zeta) & \zeta^{-1} Q_2(\zeta) \\ \zeta Q_3(\zeta) & Q_4(\zeta) \end{pmatrix}.$$

By hypothesis $Q_4(0)$ is an invertible matrix. If we define

$$P(z) = \begin{pmatrix} \mathrm{Id} & -Q_2(0) Q_4(0)^{-1} z \\ 0 & \mathrm{Id} \end{pmatrix},$$

it readily follows that $\Phi' = P \cdot \zeta^{-N} \cdot \Phi \cdot \zeta^{N}$ is indeed a holomorphic function in ζ. $\square$

Theorem 1.5.3. *Let ρ be a unitary representation, and let N is the Birkhoff-Grothendieck type of E_ρ. There exists a unique function $\Upsilon : \mathbb{H} \to \mathrm{GL}(r, \mathbb{C})$ satisfying $\Upsilon(\gamma\tau) = \Upsilon(\tau)\rho(\gamma)^{-1}$ for all $\gamma \in \Gamma$ such that*

$$\Upsilon(\sigma_i\tau) = \begin{cases} \left(\sum_{k=0}^{\infty} C_i(k)q^k\right) q^{-W_i}U_i^{-1} & \textit{for} \quad i < n; \\ \\ \left(I + \sum_{k=1}^{\infty} C_n(k)q^k\right) q^{-(W_n+N')}U_n^{-1} & \textit{for} \quad i = n. \end{cases} \tag{1.26}$$

with $C_i(0) \in \mathrm{GL}(r, \mathbb{C})$, $\quad i = 1, \cdots, n-1$ and N' is a diagonal matrix differing from N by a possibly nontrivial permutation.

Proof. Consider any function Ψ as in Proposition 1.2.1. Its local Fourier expansions (1.17) Φ_i at the cusps satisfy $\Phi_i = g_{0i} \circ J$ for some holomorphic functions $g_{0i} : \mathscr{U}_i \setminus \{z_i\} \to \mathrm{GL}(r, \mathbb{C})$ which give transition functions for the bundle E_ρ in the convering $\{\mathscr{U}_0, \cdots, \mathscr{U}_n\}$ by Corollary 1.4.2. We can assume without loss of generality that $\cup_{i=0}^{n-1}\mathscr{U}_i = \mathbb{C}_0$ and $\mathscr{U}_n = \mathbb{C}_\infty$.

By the Birkhoff-Grothendieck theorem, the cocycle $\{\mathscr{U}_{0i}, g_{0i}\}$ splits in the canonical form

$$g_{0i} = \begin{cases} g_0 \cdot g_i^{-1} & \text{on } \mathscr{U}_{0i},\ i = 1, \cdots, n-1; \\ \\ g_0 \cdot z^N \cdot g_n^{-1} & \text{on } \mathscr{U}_{0n}. \end{cases}$$

for some holomorphic functions $g_i : \mathscr{U}_i \to \mathrm{GL}(r, \mathbb{C})$. The function $\Psi' = \left(g_0^{-1} \circ J\right) \cdot \Psi$ has new Fourier expansions Φ_i' satisfying the conditions (1.26) for $i = 1, \cdots, n-1$. In order to obtain the asymptotic condition at ∞ as well, we construct a monopole $P(z)$ such that $P \cdot \zeta^{-N} \cdot g_n^{-1} = g_n' \cdot \zeta^{-N'}$ with g_n' holomorphic in $\mathbb{C}_\infty$. We then define

$$\Upsilon(\tau) = (P \circ J) \cdot \Psi'.$$

To show uniqueness, assume there are two functions Υ_1, Υ_2 satisfying the above conditions. Then the function $\Upsilon_1 \cdot \Upsilon_2^{-1}$ would be invariant under the action of Γ, thus descending to a holomorphic function $Q : X \to \mathrm{GL}(r, \mathbb{C})$ so that $Q \circ J = \Upsilon_1 \cdot \Upsilon_2^{-1}$ and that extends holomorphically to $z_1, \cdots, z_{n-1}$ since it is bounded there. By hypothesis $\det(Q) \neq 0$ on $\mathbb{P}^1 \setminus \{\infty\}$ (since potentially N' could be a nontrivial permutation of N) which implies that $\det(Q)$ cannot have either a zero or a pole at ∞ . Therefore $\det(Q)$ is a nonzero constant on the whole $\mathbb{P}^1$. Each one of the component functions Q_{ij}, $1 \leq i, j \leq r$ is a rational function on $\mathbb{P}^1$. If one of them has a pole at ∞ then it follows from the determinant condition that another one should have a zero at ∞. But then the latter would have a positive number of zeros and no poles on $\mathbb{P}^1$, a contradiction. Therefore none of the functions Q_{ij} could have a pole on $\mathbb{P}^1$, which forces them to be constant, which also gives $N_1' = N_2'$. Since by construction $\Upsilon_1 \cdot \Upsilon_2^{-1}(\infty) = I$, it follows that $\Upsilon_1 = \Upsilon_2$. $\square$

Remark 5. In the case E_ρ is evenly split, the proof of Lemma 1.5.2 gives that indeed $N' = N$. Moreover, if we consider the matrix $C_n(1)$ in block form

$$C_n(1) = \begin{pmatrix} C_1 & C_2 \\ C_3 & C_4 \end{pmatrix}$$

where C_1 is $b \times b$, the evenly split property also implies that the Fourier series expansion at ∞ of Υ can be rewritten as

$$\Upsilon(\sigma_n \tau) = q^{-N} \left(C + o(q) \right) q^{-W_n} \tag{1.27}$$

where

$$C = \begin{pmatrix} I & 0 \\ C_3 & I \end{pmatrix}.$$

Theorem 1.5.4. *Assume that E_ρ is an evenly split bundle. Then the function Υ defines a holomorphic bundle map*

$$\begin{tikzcd} \mathbb{H}^+ \times \mathbb{C}^r \arrow[r, "\mathscr{J}"] \arrow[d, "\mathrm{pr}_1"'] & E_\rho \arrow[d, "\mathrm{pr}"] \\ \mathbb{H}^+ \arrow[r, "J"] & \mathbb{P}^1 \end{tikzcd} \tag{1.28}$$

Proof. We have already seen in section 1.2 that any $\mathrm{GL}(r, \mathbb{C})$-valued ρ-automorphic map induces a bundle map

$$\begin{tikzcd} \mathbb{H} \times \mathbb{C}^r \arrow[r, "\mathscr{J}"] \arrow[d, "\mathrm{pr}_1"] & X \times \mathbb{C}^r \arrow[d, "\mathrm{pr}_1"] \\ \mathbb{H} \arrow[r, "J"] & X. \end{tikzcd}$$

The asymptotic expansions 1.26 imply that this extends to the cusps $\tau_1, \cdots, \tau_{n-1}$

giving an intermediate extension

$$\begin{tikzcd} \mathbb{H}^+ \setminus \{\infty\} \times \mathbb{C}^r \arrow[r, "\mathscr{J}"] \arrow[d] & \mathbb{C}_0 \times \mathbb{C}^r \arrow[d, "\mathrm{pr}_1"] \\ \mathbb{H}^+ \setminus \{\infty\} \arrow[r, "J"] & \mathbb{C}_0 \end{tikzcd}$$

to the local trivialization of E_ρ on $\mathbb{C}_0$. The two local trivializations of the Birkhoff-Grothendieck splitting are related as $(z, z^N v) \sim (1/z, v)$. The Fourier series expansion in Remark 5 now implies that the map also extends holomorphically to the local trivialization on $\mathbb{C}_\infty$. $\square$

1.6 Endomorphisms and the parabolic structure

The contents of this section are described in [37] as a motivation for the introduction of parabolic structures. We include them here for the convenience of the reader. Going back to Proposition 1.4.6, we see that the space of invariants of the representation $\operatorname{Ad}\rho_{\mathbb{C}}$ is always at least 1 dimensional, containing the multiples of the identity. On the other hand, a bundle morphism between two extensions E_{ρ_1}, E_{ρ_2} for two inequivalent representations should be able to distinguish them. Recall that two unitary representations ρ_1, ρ_2 are said to be *unitary equivalent* if they possess a unitary intertwiner, i.e. if there exist a unitary matrix U such that $\rho_1 \cdot U = U \cdot \rho_2 \cdot$.

Keeping (1.18) in mind, we give the following provisional definition: a *parabolic endomorphism* of E_ρ is a $(\operatorname{Ad}\rho_{\mathbb{C}})$-automorphic function f on $\mathbb{H}$ bounded at the cusps, that is, a holomorphic section of $E_{\operatorname{Ad}\rho_{\mathbb{C}}}$. In terms of its local Fourier series expansions

$$f(\sigma_i \tau) = U_i q^{W_i} \left(\sum_{k=0}^{\infty} B_i(k) q^k \right) q^{-W_i} U_i^{-1}, \tag{1.29}$$

the boundedness condition translates into

$$\left(B_i(0)\right)_{jk} = 0 \quad \text{whenever} \quad \alpha_{ij} - \alpha_{ik} < 0. \tag{1.30}$$

In particular, when the system of weights is generic, the matrices $B_i(0)$ turn out to be lower triangular.

Proposition 1.6.1. *The bundles E_{ρ_1}, E_{ρ_2} are equivalent (in the sense of* *parabolic automorphisms) if and only if ρ_1, ρ_2 are equivalent.*

Proof. This is a corollary of Proposition 1.4.6 for the representation $\rho_1^* \otimes \rho_2$, since unitary intertwiners correspond to its invertible invariants. □

Remark 6. A more general result can be proved in fact: given two unitary representations (not necessarily of the same rank),

$$\mathrm{Hom}(\rho_1, \rho_2) \cong \mathrm{Par\ Hom}(E_{\rho_1}, E_{\rho_2}) \tag{1.31}$$

where the first space corresponds to the intertwiners, and the second one is interpreted as the space of $(\rho_1^* \otimes \rho_2)$-automorphic functions bounded at the cusps. Its proof is based on the same argument for the proof of Proposition 1.4.6.

There is yet another way to describe the parabolic endomorphism property, giving rise to the general notion of a parabolic structure (see Appendix C). Observe that an endomorphism f will satisfy the condition (1.30) if and only if it preserves the descending flags defined over each one of the fibers $\{\tau_i\} \times \mathbb{C}^r$ by $[U_i]$. Thus the descending flags $[U_i]$ on the fibers over the cusps become a fundamental structure to consider.

Now, we can clearly descend these flags to the bundle E_ρ to the fibers over the points $z_1, \cdots, z_n$. We can give a more specific description of them by using our uniformization map Υ and the trivializations over $\mathbb{C}_0$, $\mathbb{C}_\infty$. On $\mathbb{C}_0$, the descending flags at the fibers over $z_1, \cdots, z_{n-1}$ are given as $[C_i(0)]$. The normalization of the function Υ at ∞ implies that over $\mathbb{C}_\infty$, the descending flag at z_n is the standard one, determined by $[I]$. The parabolic structure is completeted by adding the weights W_i to each flag $[C_i(0)]$.

1.7 The canonical connection

The trivial bundle $\mathbb{H} \times \mathbb{C}^r$ has the canonical Hermitian metric $|\mathrm{d}\tau|^2$ and moreover, the deRham differential operator d_τ defines a tautological Hermitian holomorphic (flat) connection compatible with the complex structure. One could think it is an excess to think about such simple structures in this intricate way. However, since the operations $(\tau, v) \mapsto (\gamma \cdot \tau, \rho(\gamma)v)$ for any $\gamma \in \Gamma$ are isometries of the fibers when ρ is unitary and this operations commute with d_τ for arbitrary sections of $\mathbb{H} \times \mathbb{C}^r$, we conclude that

1. The Hermitian metric $I|\mathrm{d}\tau|^2$ descends to a Hermitian metric on $\rho\backslash\mathbb{H}\times\mathbb{C}^r$.

2. The deRham differential d_τ descends to a Hermitian holomorphic connection on $\rho \setminus \mathbb{H} \times \mathbb{C}^r$.

Two natural questions are: how do these structures extend to the bundle E_ρ? What is their explicit form in the canonical trivializations? To answer them, we consider once again the Birkhoff-Grothendieck local trivializations and the inclusion $X \hookrightarrow \mathbb{C}_0$. The bijective correspondence between the sheaf of Γ-invariant holomorphic sections of $\mathbb{H} \times \mathbb{C}^r$ and the sheaf of holomorphic sections of $X \times \mathbb{C}^r$ also follows at the level of holomorphic forms, and in general we have an inclusion

$$\iota_p : \Omega^p(X \times \mathbb{C}^r) \hookrightarrow \Omega^p(\mathbb{H} \times \mathbb{C}^r)$$

Since $X \times \mathbb{C}^r$ is trivial and $X \subset \mathbb{C}_0$, any holomorphic connection on it would be of the form $\mathrm{d}_z + A(z)$, where $A(z)$ is a holomorphic matrix-valued 1-form on X. Then the canonical connection on $X \times \mathbb{C}^r$ is determined by the commutativity of the following diagram:

$$\begin{tikzcd} \mathcal{O}(X \times \mathbb{C}^r) \arrow[r, "\mathrm{d}_z + A"] \arrow[d, "\iota_0"'] & \Omega^1(X \times \mathbb{C}^r) \arrow[d, "\iota_1"] \\ \mathcal{O}(\mathbb{H} \times \mathbb{C}^r) \arrow[r, "\mathrm{d}_\tau"] & \Omega^1(\mathbb{H} \times \mathbb{C}^r). \end{tikzcd} \tag{1.32}$$

Proposition 1.7.1. *The matrix valued 1-form $A(z)$ is holomorphic on X and satisfies*

$$J^*(A) = -\mathrm{d}_\tau \Upsilon \cdot \Upsilon^{-1} \tag{1.33}$$

Proof. If $z \mapsto (z, s(z))$ is a section of $X \times \mathbb{C}^r$ and f the corresponding Γ-invariant function, we know that $f(\tau) = \Upsilon^{-1}(\tau) s(J(\tau))$. Therefore

$$\Upsilon^{-1} \cdot ((\mathrm{d}_z + A)s) \circ J = \mathrm{d}_\tau(\Upsilon^{-1} \cdot (s \circ J))$$

or

$$\mathrm{d}_\tau(s \circ J) + (A \circ J)(s \circ J) = \Upsilon \cdot \mathrm{d}_\tau(\Upsilon^{-1} \cdot (s \circ J))$$

but

$$\Upsilon \cdot \mathrm{d}_\tau(\Upsilon^{-1} \cdot (s \circ J)) = \mathrm{d}_\tau(s \circ J) - (\mathrm{d}_\tau \Upsilon \cdot \Upsilon^{-1})(s \circ J)$$

since $z = J(\tau)$. □

Lemma 1.7.2. *Let $f(\tau)\mathrm{d}\tau$ be a holomorphic differential on $\mathbb{H}$, invariant under the action of Γ (so that $f(\tau)$ is an automorphic form of weight 2, see Chapter*

2) and having meromorphic q-series expansions at each cusp τ_i of the form

$$f(\sigma_i\tau)\sigma_i'(\tau) = \frac{b_i}{q^{m_i}}(1+o(q)), \qquad i=1,\cdots,n.$$

Then $f(\tau)\mathrm{d}\tau$ is the pull-back under J of a meromorphic differential $R(z)\mathrm{d}z$ on $\mathbb{P}^1$ with singularities at $z_1,\cdots,z_n$ so that in a neighbourhood of z_i, $i=1,\cdots,n-1$

$$R(z) = \frac{1}{2\pi\sqrt{-1}}\frac{b_i a_i(1)^{m_i}}{(z-z_i)^{m_i+1}}(1+o(z-z_i))$$

and near infinity

$$R(z) = -\frac{1}{2\pi\sqrt{-1}}\frac{b_n z^{m_n-1}}{a_n(-1)^{m_n}}(1+o(1/z))$$

Proof. By definition, $J^*(R(z)\mathrm{d}z) = R(J(\tau))\mathrm{d}J$ and the series expansions (1.5) imply that

$$\mathrm{d}J(\sigma_i\tau) = \begin{cases} 2\pi\sqrt{-1}\left(\displaystyle\sum_{k=1}^{\infty} k a_i(k) q^k\right)\mathrm{d}\tau & \text{if} \quad i=1,\cdots,n-1; \\ \\ 2\pi\sqrt{-1}\left(\displaystyle\sum_{k=-1}^{\infty} k a_n(k) q^k\right)\mathrm{d}\tau & \text{if} \quad i=n. \end{cases}$$

The result is now a straightforward computation. □

Proposition 1.7.3. *The matrix-valued 1-form $A(z)$ is rational on $\mathbb{P}^1$ with simple poles at $z_1,\cdots,z_n$.*

Proof. The Fourier expansions (1.26) of Υ imply that near the cusps τ_i, $i=1,\cdots,n-1$

$$-\mathrm{d}_\tau\Upsilon(\sigma_i\tau)\cdot\Upsilon^{-1}(\sigma_i\tau) = 2\pi\sqrt{-1}\left(C_i(0)W_iC_i(0)^{-1}+o(q)\right)\mathrm{d}\tau$$

and near infinity,

$$-\mathrm{d}_\tau\Upsilon\cdot\Upsilon^{-1}(\sigma_n\tau) = 2\pi\sqrt{-1}\left(W_n+N+o(q)\right)\mathrm{d}\tau$$

It follows from lemma 1.7.2 that on a neighbourhood of z_i, $i=1,\cdots,n-1$,

$$A(z) = \left(\frac{C_i(0)W_iC_i(0)^{-1}}{z-z_i}+o(1)\right)\mathrm{d}z \tag{1.34}$$

and at infinity

$$A(z) = -\left(\frac{W_n + N}{z} + o\left(\frac{1}{z^2}\right)\right) \mathrm{d}z \tag{1.35}$$

Therefore the holomorphic parts of $A(z)$ have to vanish by Liouville's theorem, and

$$A(z) = \left(\sum_{i=1}^{n-1} \frac{A_i}{z - z_i}\right) \mathrm{d}z \tag{1.36}$$

where $A_i = C_i(0)W_iC_i(0)^{-1}$. □

Corollary 1.7.4. *The cannonical connection extends to a logarithmic connection on the bundle E_ρ. With respect to the Birkhoff-Grothendieck local trivializations, it has residues $C_i(0)W_iC_i(0)^{-1}$ along the fibers at $z_1, \cdots, z_n$.*

Proof. We have seen that on $\mathbb{C}_0$, $\theta_0 = A(z)$. On $\mathbb{C}_0 \cap \mathbb{C}_\infty$,

$$\theta_\infty(\zeta) = \zeta^N \theta_0 \zeta^{-N} - \mathrm{d}\zeta^N \cdot \zeta^{-N} = \left(\frac{W_n}{\zeta} + o(1)\right) \mathrm{d}\zeta$$

and the claim follows. □

Corollary 1.7.5. *The degree of the bundle E_ρ is equal to minus the sum of the weights $\{\alpha_{ij}\}$.*

Proof. From proposition B.1.7 we have that

$$\deg(E_\rho) = -\sum_{i=1}^{n} \mathrm{tr}(W_i) = -\sum_{i=1}^{n}\sum_{j=1}^{r} \alpha_{ij}. \tag{1.37}$$

This also follows from the residue theorem applied to the matrix-valued 1-form $A(z)$ on $\mathbb{P}^1$, since its residue at ∞ is $A_n = W_i + N$. Then

$$\sum_{i=1}^{n} A_i = 0 \tag{1.38}$$

implies that $\mathrm{tr}(N) = -\sum_{i=1}^{n}\sum_{j=1}^{r} \alpha_{ij}$. □

Corollary 1.7.6. *The bundle E_ρ is trivial if and only if the representation ρ is trivial.*

Proof. Since $\forall\ i, j$, $\alpha_{ij} \geq 0$, a necessary condition for E to be trivial is that $\alpha_{ij} = 0\ \forall i, j$, but this is also sufficient since in this case the representation is the trivial one. □

Remark 7. It turns out that we can interpret all of our construction in purely classical terms. If we consider the Fuchsian linear system on the sphere

$$\left(\frac{\mathrm{d}}{\mathrm{d}z}+\sum_{i=1}^{n-1}\frac{A_i}{z-z_i}\right)\mathscr{Y}=0, \tag{1.39}$$

then the multivalued function $\mathscr{Y}(z)$ satisfying $\mathscr{Y}\circ J=\Upsilon$ is a fundamental matrix for it. In conclusion, we have related the unitary case of the *classical Riemann-Hilbert problem*, that requires to find, given a unitary representation of $\pi_1(\mathbb{P}^1\setminus\mathscr{D})$, a Fuchsian linear system on $\mathbb{P}^1$ having it as its monodromy group, with the theory of parabolic bundles on $\mathbb{P}^1$ and their *uniformization*. In this sense we have produced a canonical injective correspondence

$$\{\text{Unitary representations } \rho \text{ of } \pi_1(X)\}\hookrightarrow\{\text{Fuchsian systems on } \mathbb{P}^1\}.$$

which is easily seen to only depend on the equivalence class of each representation. We will refer to is as the *Riemann-Hilbert map*.

The idea of using vector bundle techniques in the Riemann-Hilbert problem goes back to H. Röhrl [43]. The main idea is to first solve the problem *locally*, a trivial process, since the matrix valued functions

$$\frac{U_iW_iU_i^{-1}}{z-z_i}$$

are readily seen to have monodromy groups generated by $\rho(\gamma_i)$. A simple computation shows that if we make

$$\begin{aligned}\theta_0 &= A(z), && \text{on} \quad \mathscr{U}_0=X,\\ \theta_i &= \frac{U_iW_iU_i^{-1}}{z-z_i}\mathrm{d}z, && \text{on} \quad \mathscr{U}_i, \quad i=1,\cdots,n,\end{aligned}$$

the compatibily condition $\theta_i=g_{i0}\theta_0g_{i0}^{-1}-(\mathrm{d}g_{i0})g_{i0}^{-1}$ holds on each $\mathscr{U}_{i0}$ for the transition functions (1.21) and the matrix valued forms patch up to define a logarithmic connection on the bundle E_ρ.

Proposition 1.7.7. *The function*

$$h_0=(\mathscr{Y}\mathscr{Y}^*)^{-1}$$

determines a Hermitian metric on $X\times\mathbb{C}^r$ compatible with the canonical connection. It extends to a degenerate metric on E_ρ with the following asymptotic *behaviour at the punctures* $z_1, \cdots, z_{n-1}$ *on* $\mathbb{C}_0$

$$h_0(z) = (C_i(0)^{-1})^* |z - z_i|^{2W_i} C_i(0)^{-1} + o(|z - z_i|), \tag{1.40}$$

and at z_n *on* $\mathbb{C}_\infty$

$$h_\infty(\zeta) = |\zeta|^{2W_n} + o(|\zeta|). \tag{1.41}$$

Proof. The values of the function h are Hermitian and positive-definite matrices on X. A straightforward computation shows that there

$$h^{-1}\partial h = (\mathscr{Y}\mathscr{Y}^*)\partial(\mathscr{Y}\mathscr{Y}^*)^{-1} = -(\mathrm{d}\mathscr{Y})\mathscr{Y}^{-1} = A(z).$$

The asymptotics (1.40) now follow from the asymptotic behaviour of $\mathscr{Y}$ on $z_1, \cdots, z_{n-1}$. The asymptotics (1.41) follow after using formula (B.3) for $g_{0\infty}(\zeta) = \zeta^{-N}$. $\square$

Chapter 2

Automorphic forms and unitary representations

The notions introduced in the following sections are both a generalization to the vector-valued case and a specialization to the Fuchsian, torsion free-case of the theory of automorphic forms as appears in Kra [33]. The case $r = 1$ was already considered in H. Petersson's seminal work [45]. Our main goal is to relate such theory with a suitable version of the harmonic theory on parabolic bundles. As before we consider a unitary representation $\rho : \Gamma \to \mathrm{U}(r)$, not necessarily irreducible.

2.1 Hilbert spaces of automorphic forms

Definition 2.1.1. *A measurable ρ-automorphic form of weight $(2p, 2q)$ is an equivalence class (modulo functions vanishing a.e.) of measurable functions $f : \mathbb{H} \to \mathbb{C}^r$ satisfying*

$$f(\gamma\tau)\gamma'(\tau)^p\overline{\gamma'(\tau)^q} = \rho(\gamma)f(\tau) \qquad \forall\gamma \in \Gamma. \tag{2.1}$$

We will abbreviate the weight $(2p, 0)$ simply as $2p$. We would be mostly interested in the case when $p, q \in \{0, 1\}$ (see section 2.2).

Definition 2.1.2. *The* ***Petersson inner product*** *[46] of two measurable ρ-automorphic forms of weight $(2p, 2q)$ is defined as*

$$\langle f_1, f_2\rangle_P = \iint\limits_F \mathrm{tr}(f_1(\tau)f_2(\tau)^*)(\mathrm{Im}\tau)^{2p+2q-2}\, d^2\tau \tag{2.2}$$

As it is customary, $L^2_{p,q}(\mathbb{H},\rho)$ will denote the Hilbert space of measurable ρ-automorphic forms of weight $(2p, 2q)$ and finite norm.

We are ultimately interested in holomorphic (or antiholomorphic) automorphic forms of weight $2p$, the former being the classical case. When such is the case, the transformation rule (2.1) implies that near the cusps we have the analytic Fourier series expansions

$$f(\sigma_i\tau)\sigma_i'(\tau)^p = U_i q^{W_i}\left(\sum_{k=-\infty}^{\infty} b_i(k)q^k\right). \tag{2.3}$$

It is intuitively clear that it is desirable to have a better understanding of the singular growth at the cusps. This motivates the following definitions.

Definition 2.1.3. *A ρ-automorphic form of weight $2p$ is called* ***regular*** *if it is holomorphic and for every $i = 1, \cdots, n$, $\lim_{\tau\to\tau_i} f(\tau)$ is finite. It is called a* ***cusp form*** *if furthermore*

$$\lim_{\tau\to\tau_i} f(\tau) = 0.$$

We verify that in terms of its local expansions, f is regular if $b_i(k) = 0$ for $k < 0$ and a cusp form if moreover $(b_i(0))_j = 0$ whenever $\alpha_{ij} = 0$.

We will denote the vector space of regular ρ-automorphic forms of weight $2p$ by $\mathfrak{M}_{2p}(\Gamma,\rho)$, and the space of cusps forms by $\mathfrak{S}_{2p}(\Gamma,\rho)$. We have seen that $\mathfrak{M}_0(\Gamma,\rho)$ is naturally isomorphic to $(\mathbb{C}^r)^\rho$ and this is trivial it the ρ is irreducible. It is not at all clear that if $p = 1$, these spaces are nontrivial and even in that case, if their intersection with $L^2_2(\mathbb{H},\rho)$ would be nontrivial.

Proposition 2.1.4. *The integral (2.2) is convergent whenever $f_1 \in \mathfrak{M}_2(\Gamma,\rho)$ and $f_2 \in \mathfrak{S}_2(\Gamma,\rho)$.*

Proof. The fundamental region F of Γ can be chosen to be bounded by $2n$ geodesic segments, and the integral (2.2) can be decomposed as a sum of $n+1$ integrals by partitioning the fundamental region F into the sets $F_i = F\cap\sigma_i\cdot\mathbb{H}_\delta$ and $F\setminus\cup_i F_i$ for some $\delta \gg 1$. The integral is obviously convergent in $F\setminus\cup_i F_i$, being a compact set, so it remains to verify its convergence on each F_i. The Fourier series expansions (2.3) of f_1, f_2 are absolutely and uniformly convergent on compact sets, hence we obtain

$$
\begin{aligned}
&\iint\limits_{F_i} \operatorname{tr}(f_1 f_2^*)\mathrm{d}^2\tau \\
= &\lim_{a\to\infty}\int_\delta^a \int_0^1 \operatorname{tr}\left(|q|^{2W_i}\sum_{k,l\geq 0} b_i^1(k)(b_i^2(l))^*\right) q^k \overline{q}^l \mathrm{d}x\mathrm{d}y \\
= &\sum_{k,l\geq 0}\sum_{j=1}^r \left(b_i^1(k)\right)_j \overline{\left(b_i^2(l)\right)_j}\left(\int_\delta^\infty e^{-2\pi(k+l+2\alpha_{ij})y}\mathrm{d}y\right)\left(\int_0^1 e^{2\pi\sqrt{-1}(k-l)x}\mathrm{d}x\right) \\
= &\sum_{k\geq 0}\sum_{j=1}^r C_{ijk}\left(b_i^1(k)\right)_j \overline{\left(b_i^2(k)\right)_j},
\end{aligned}
$$

where $C_{ijk} = e^{-2\pi(k+\alpha_{ij})\delta}/4\pi(k+\alpha_{ij})$. Notice that this would be ill-defined if $k=0$ and $\alpha_{ij}=0$, a possibility that is ruled out by hypothesis. The remaining series is absolutely convergent by comparison with the series $\sum_{k\geq 0} C_{ijk}$ for each $j = 1, \cdots, r$. □

Since the uniform limit of a sequence of analytic functions is analytic, we readily conclude:

Corollary 2.1.5. *$\mathfrak{S}_2(\Gamma,\rho)$ is a closed subspace of $L_2^2(\mathbb{H},\rho)$, consisting of the holomorphic automorphic forms of weight 2 of bounded norm.*

It should not be surprising to encounter a relation between the spaces of cusp forms and the analytic structure of the bundle E_ρ. The latter is encoded in the mutually isomorphic Čech and Dolbeaut cohomologies theories associated to it. In the simplest case when the representation is trivial we can assume without loss of generality that $r=1$. Then there is a simple correspondence between automorphic forms and meromorphic differentials on the quotient surface $S \cong \Gamma \setminus \mathbb{H}^+$, and it turns out that the cusp form condition corresponds to the holomorphicity of the coresponding differentials, which represent canonically the space $H^1(S)$ by Kodaira-Serre duality. The following theorem is a generalization of this correspondence for an arbitrary unitary representation ρ.

Theorem 2.1.6. *There is an isomorphism $H^1(E_\rho) \cong \mathfrak{S}_2(\Gamma,\rho^*)$. In particular, the Hilbert space of cusp forms of weight 2 is finite dimensional.*

Proof. We begin by noting that if $f(\tau)$ is an automorphic form of weight 2 for the representation ρ^*, then $\phi = f(\tau)d\tau$ is a vector-valued holomorphic differential on $\mathbb{H}$ satisfying $\gamma^*(\phi) = \rho^*(\gamma)\phi \ \forall\gamma \in \Gamma$. The representation ρ^* has weights $W_i' = \mathrm{diag}(\alpha_{i1}', \cdots, \alpha_{ir}')$ where

$$\alpha_{ij}' = \begin{cases} \alpha_{ij} & \text{if} \quad \alpha_{ij} = 0, \\ \\ 1 - \alpha_{ij} & \text{if} \quad \alpha_{ij} > 0. \end{cases}$$

By Kodaira-Serre duality we know that $H^1(E_\rho) \cong H^0((E_\rho)^* \otimes K)$. Notice that as opposed to the local system case, $(E_\rho)^*$ is in general not isomorphic to E_{ρ^*} as the transition functions of the former are $g_{ij}^* = {}^t g_{ij}^{-1}$ where g_{ij} are given in (1.20) (but, nevertheless, $(E_\rho)^*|_X \cong E_{\rho^*}|_X$, cf. Proposition 2.9).

If f is a cusp form for ρ^*, since $\mathrm{d}q = 2\pi\sqrt{-1}q\mathrm{d}\tau$, the form $\phi = f\mathrm{d}\tau$ would have Fourier series expansions near each cusp τ_i

$$\phi = \overline{U_i} q^{W_i'} \left(\sum_{k=0}^{\infty} b_i(k) q^k \right) \frac{\mathrm{d}q}{q} = \overline{U_i} q^{-W_i} \left(\sum_{k=0}^{\infty} b_i'(k) q^k \right) \mathrm{d}q$$

where the last equality follows from the cusp form condition $(b_i(0))_j = 0$ whenever $\alpha_{ij}' = \alpha_{ij} = 0$. Recalling once again the transition functions 1.20, it follows that the cusp form condition is precisely the holomorphicity condition of sections of $(E_\rho)^* \otimes K$ in the neighbourhoods $\{\mathscr{V}_i\}$ of the cusps.

□

The following generalization of the classical Riemann-Roch theorem to vector bundles of arbitrary rank is due to André Weil [53].

Theorem 2.1.7. (Riemann-Roch for vector bundles) *Let $E \to S$ be a holomorphic vector bundle of rank r and degree d over a compact Riemann surface of genus g. Then*

$$h^0(E) - h^1(E) = d + r(1 - g),$$

where $h^i(E) = \mathrm{dim} H^i(S, E)$.

Corollary 2.1.8. *The following identity holds (wtih respect to the weights of ρ) :*

$$\mathrm{dim}\mathfrak{S}_2(\Gamma, \rho^*) = \mathrm{dim}(\mathbb{C}^r)^\rho + \sum_{i=1}^{n} \sum_{j=1}^{r} \alpha_{ij} - r. \tag{2.4}$$

Proposition 2.1.9. *If two regular automorphic forms f and f' satisfy $b_i(0) = b_i(0)'$ for $i = 1, \cdots, n$, then $f = f'$.*

Proof. The difference $f - f'$ is a regular automorphic form without constant terms on its local expansions. The product

$$\Upsilon(f - f')\mathrm{d}\tau$$

is invariant under the action of Γ and thus corresponds to a vector valued 1-form $R(z)\mathrm{d}z$ on $\mathbb{P}^1$, holomorphic on $\mathbb{P}^1 \setminus \{\infty\}$ by Lemma 1.7.2. Moreover, it has the following asymptotics near infinity

$$R(z) = \left(I + o\left(\frac{1}{z}\right)\right) z^{N'-2I}\left(b_n + o\left(\frac{1}{z}\right)\right)$$

Since the coefficients of N' are nonpositive, it follows that $R(z)$ is bounded on $\mathbb{P}^1$, hence a constant. Since $\lim_{z\to\infty} R(z) = 0$, we obtain the claim. $\square$

2.2 Harmonic theory and automorphic forms

By their very definition, the Riemann surface $X = \Gamma \setminus \mathbb{H}$ and the bundle $E_0 = \rho\backslash\mathbb{H}\times\mathbb{C}^r$ over it possess natural hermitian metrics that allow to introduce an inner product and a $*$-operator on the spaces $A_c^{p,q}(E_0)$, $p, q = 0, 1$. The last one is defined by means of the generalized wedge product

$$\wedge : A^{p,q}(E_0) \otimes A^{p',q'}(E_0^*) \to A^{p+p',q+q'}(X).$$

The ***Hodge inner product*** is defined for $s_1, s_2 \in A_c^{p,q}(E_0)$ as

$$\langle s_1, s_2\rangle_H = \int_X s_1 \wedge *s_2, \tag{2.5}$$

and allows us to define Hilbert spaces $\mathscr{H}^{p,q}(E_0)$ as the completion of $A_c^{p,q}(E_0)$. The ***Hodge Laplacian*** is defined in terms of the operator $\bar\partial$ and its formal adjoint $\bar\partial^* = - * \bar\partial *$ as

$$\Delta = \bar\partial\bar\partial^* + \bar\partial^*\bar\partial.$$

A form is $s \in \mathscr{H}^{p,q}(E_0)$ called ***harmonic*** if $\Delta s = 0$. We denote the space of harmonic (p, q)-forms with values on E_0 by $\mathcal{H}^{p,q}(E_0)$.

Since all these stuctures on E_0 are induced from those on $\mathbb{H}\times\mathbb{C}^r$, we might as well describe them in terms of the corresponding ones in the latter. This has the advantage of simplifying calculations since $\mathbb{H}$ consists of a single global chart. The pull-back $\mathrm{pr}^*(s)$ of a section $s \in A^{p,q}(E_0)$ would be a $\mathbb{C}^r$-valued (p, q)-form on $\mathbb{H}$ that by virtue of the global coordinate τ corresponds to an automorphic function function of weight $(2p, 2q)$. Since $*\mathrm{pr}^*(s) = \mathrm{pr}^*(*s)$ we have

$$\int_X s_1 \wedge *s_2 = \int_F \mathrm{pr}^*(s_1) \wedge *\mathrm{pr}^*(s_2) = \iint_F \mathrm{tr}(f_1(\tau) f_2(\tau)^*)(\mathrm{Im}\tau)^{2p+2q-2}\mathrm{d}^2\tau, \tag{2.6}$$

and we conclude that the Hodge inner product coincides with the Petterson inner product when $p, q \in \{0, 1\}$, and hence $\mathscr{H}^{p,q}(E_0) \cong L^2_{p,q}(\mathbb{H}, \rho)$.

At the level of functions of finite norm satisfying the automorphicity condition (2.1), the operator $\bar{\partial}$ is given as $\frac{\partial}{\partial \bar{\tau}}$ and since

$$\langle \bar{\partial} f_1, f_2 \rangle = \langle f_1, \bar{\partial}^* f_2 \rangle$$

we conclude from (2.6) and integration by parts that $\bar{\partial}^* = -(\mathrm{Im}\tau)^2 \frac{\partial}{\partial \tau}$. Notice that only one of the two summands of Δ will make sense depending on the values of q as X is 1-dimensional: when $q = 0$, $\Delta = \bar{\partial}^* \bar{\partial}$ and when $q = 1$, $\Delta = \bar{\partial} \bar{\partial}^*$. By standard properties of the adjoint of an operator this implies that

$$\mathcal{H}^{p,0}(E_0) = \mathrm{Ker}(\bar{\partial}), \qquad \mathcal{H}^{p,1}(E_0) = \mathrm{Ker}(\bar{\partial}^*), \tag{2.7}$$

and their elements correspond to holomorphic (resp. antiholomorphic) functions satisfying $f(\gamma\tau)\gamma'(\tau)^p = \rho(\gamma) f(\tau)$ (resp. $f(\gamma\tau)\gamma'(\tau)^p \overline{\gamma'(\tau)} = \rho(\gamma) f(\tau)$).

Finally, there are projection maps

$$P_{p,q} : \mathscr{H}^{p,q}(E_0) \to \mathcal{H}^{p,q}(E_0),$$

given explicitly as

$$P_{p,0} = I - \bar{\partial}^* \circ \Delta^{-1} \circ \bar{\partial} \qquad P_{p,1} = I - \bar{\partial} \circ \Delta^{-1} \circ \bar{\partial}^* \tag{2.8}$$

Remark 8. It follows from the transition functions (1.20) that $E_0^* = (\rho \setminus \mathbb{H} \times \mathbb{C}^r)^* \cong \rho^* \setminus \mathbb{H} \times \mathbb{C}^r$. Moreover, *if ρ is unitary, then $\rho^* = \overline{\rho}$.* Usual Hilbert space techniques (the Riesz representation theorem) gives us a complex isomorphism analogous to the *Kodaira-Serre duality* in the case of unitary local systems:

$$\mathcal{H}^{p,q}(E_0) \cong \mathcal{H}^{1-p,1-q}(E_0^*)^* \qquad \text{for } p, q \in \{0, 1\} \tag{2.9}$$

There are two cases to consider: when $p = q = 0$, the map

$$f(\tau) \mapsto \langle \cdot, \overline{f(\tau)(\mathrm{Im}\tau)^2} \rangle_H,$$

Gives the isomorphism $\mathcal{H}^{0,0}(E_0) \cong \mathcal{H}^{1,1}(E_0^*)^*$ as a consecuency of the relation

$$\mathrm{Im}(\gamma(\tau)) = |\gamma'(\tau)| \mathrm{Im}(\tau), \qquad \forall \gamma \in \mathrm{PSL}(2, \mathbb{R}),\ \tau \in \mathbb{H}.$$

When $p=1, q=0$ the map

$$f(\tau) \mapsto \langle \cdot, \overline{f(\tau)} \rangle_H$$

gives the isomorphism $\mathcal{H}^{1,0}(E_0) \cong \mathcal{H}^{0,1}(E_0^*)^*$.

Theorem 2.2.1. *We have canonical isomorphisms*

$$\mathcal{H}^{0,0}(E_0) \cong (\mathbb{C}^r)^{\rho}, \qquad \mathcal{H}^{1,1}(E_0) \cong (\mathbb{C}^r)^{(\rho^*)}, \tag{2.10}$$

$$\mathcal{H}^{1,0}(E_0) \cong \mathfrak{S}_2(\Gamma, \rho), \qquad \mathcal{H}^{0,1}(E_0) \cong \overline{\mathfrak{S}_2(\Gamma, \rho^*)}. \tag{2.11}$$

Proof. As a consequence of (2.7), the proof of the first isomorphism in (2.10) was given in the proof of Proposition 1.4.6 and the second one follows from $\mathcal{H}^{1,1}(E_0) \cong \mathcal{H}^{0,0}(E_0^*)$. The first isomorphism in (2.11) is essentially a consequence of the matching inner products 2.2 and 2.5, while the second follows after duality is considered. □

Remark 9. It should be noted that our proof admits a straightforward generalization to the case of a unitary local system on a Kähler manifold. The classical result of Kodaira-Serre was proven for compact Kähler manifold, and it wasn't until very recently that a proof for a general complex manifold was given [19].

Remark 10. The Hodge theorem asserts that on a hermitian vector bundle over a compact complex manifold Dolbaut cohomology classes admit a unique harmonic representative. In our case the metrics on $\mathbb{P}^1$ and E_ρ degenerate at the cusps $\tau_1, \cdots, \tau_n$, but miracluosly, by putting together Theorems 2.1.6 and 2.2.1, we can conclude that a suitable analog of it still holds. The $(0,1)$-case is the most relevant for our purposes:

$$H^{0,1}_{\bar\partial}(E_\rho) \cong H^1(E_\rho) \cong \mathfrak{S}_2(\Gamma, \rho^*) \cong \mathcal{H}^{1,0}(E_0^*) \cong \mathcal{H}^{0,1}(E_0). \tag{2.12}$$

We have identified three different incarnations of the weight 2 cusp form condition: the finiteness of the Petterson inner product for holomorphic automorphic forms (Proposition 2.1.4), the holomorphicity of the sections of a bundle (Theorem 2.1.6) and the harmonicity of an automorphic form (Theorem 2.2.1).

2.3 A canonical regular automorphic form of weight 2

Let us consider now a unitary representation ρ with a generic system of weights and the induced representation $\operatorname{Ad}\rho_{\mathbb{C}}$. As for the case of parabolic endomorphisms, an arbitrary regular automorphic form of weight 2 for the representation $\operatorname{Ad}\rho_{\mathbb{C}}$ would possess local Fourier series expansions of the same nature than those of (1.29), namely

$$f(\sigma_i\tau)\sigma_i'(\tau) = U_i q^{W_i}\left(\sum_{k=0}^{\infty} B_i(k)q^k\right) q^{-W_i}U_i^{-1}. \tag{2.13}$$

where the matrices $B_i(0)$ are lower triangular.

In the next proposition we construct a canonical element of $\mathfrak{M}_2(\Gamma, \operatorname{Ad}\rho_{\mathbb{C}}) \setminus \mathfrak{S}_2(\Gamma, \operatorname{Ad}\rho_{\mathbb{C}})$ when the .

Proposition 2.3.1. *If E_ρ is an evenly split bundle, then the function $\mathscr{A} = -\Upsilon^{-1}\cdot\Upsilon'$ is a regular automorphic form of weight 2 for the representation $\operatorname{Ad}\rho_{\mathbb{C}}$ with*

$$B_i(0) = \begin{cases} 2\pi\sqrt{-1}W_i & i = 1, \cdots, n-1, \\ 2\pi\sqrt{-1}(W_n + M) & i = n, \end{cases}$$

where

$$M = \begin{pmatrix} (a+1)I & 0 \\ -C_3 & aI \end{pmatrix}.$$

Proof. The result is a direct consequence of the local expansions of the function Υ at the cusps. We verify for $i = 1, \cdots, n-1$,

$$\begin{aligned}
&-(\Upsilon^{-1}\Upsilon')(\sigma_i\tau)\sigma_i'(\tau) \\
&= -2\pi\sqrt{-1}U_i q^{W_i}\left(\sum_{k=0}^{\infty} C_i(k)q^k\right)^{-1}\cdot\left(\sum_{k=0}^{\infty} C_i(k)(kI - W_i)q^k\right) q^{-W_i}U_i^{-1} \\
&= U_i q^{W_i}\left(2\pi\sqrt{-1}W_i + \sum_{k=1}^{\infty} B_i(k)q^k\right) q^{-W_i}U_i^{-1}.
\end{aligned}$$

For $i = n$, we get from the asymptotics (1.27)

$$
\begin{aligned}
&-(\Upsilon^{-1}\Upsilon')(\sigma_n(\tau))\sigma_n'(\tau) \\
&= 2\pi\sqrt{-1}U_n q^{W_n}\left(C + \sum_{k=1}^{\infty} C_n'(k)q^k\right)^{-1} \cdot \\
&\left(CW_n + NC + \sum_{k=1}^{\infty}(NC_n'(k) + C_n'(k)W_n - kC_n'(k))q^k\right) q^{-W_n}U_n^{-1} \\
&= U_n q^{W_n}\left(2\pi\sqrt{-1}(W_n + C^{-1}NC) + \left(\sum_{k=1}^{\infty} B_n(k)q^k\right)\right) q^{-W_n}U_n^{-1}
\end{aligned}
$$

and it is readily verified that $M = C^{-1}NC$. □

Remark 11. The function $\operatorname{tr}(\mathscr{A})$ is a regular (scalar) automorphic form of weight 2 corresponding to a meromorphic differential on $\mathbb{P}^1$ with simple poles in $\mathscr{D}$ and holomorphic everywhere else. The residue theorem yields once again the result

$$\sum_{i=1}^{n} \operatorname{tr}(W_i) = -\sum_{i=1}^{n} a_i.$$

As a corollary of Proposition (2.1.9), it follows that this is the only regular automorphic form with these features. There is yet another way to see this, since the function $\mathscr{A}$ is some sort of "lifting" of the canonical connection, in the sense that the following identity holds:

$$\Upsilon \cdot \mathscr{A} \cdot \Upsilon^{-1} = \left(\sum_{i=1}^{n-1} \frac{A_i}{z - z_i}\right) \circ J. \tag{2.14}$$

We could extend this idea and establish a projection from the space of regular $\operatorname{Ad}\rho$-automorphic forms of weight 2 nd the space of matrix valued-rational functions on $\mathbb{P}^1$, but this idea doesn't seem to give any new or useful information, esentially because we don't know anything about the matrices $C_i(0)$, besides their invertibility.

Chapter 3

Deformation theory and the Moduli space

The main goal of this chapter is twofold: on one hand we provide a global construction of the Moduli space of stable parabolic bundles of parabolic degree 0 over $\mathbb{P}^1$ with a fixed set of weights by incarnating it as a character variety (the Mehta-Seshadri theorem), and on the other hand, we consider local aspects specializing the infinitesimal deformation theory to the language of complex analysis on the upper half plane by drawing an analogy with the construction of the Teichmüller space as in the work of Ahlfors and Bers.

3.1 The character variety

3.1.1 Group cohomology and the character variety

We proceed to construct the space of equivalence classes of irreducible unitary representations of Γ for a fixed set of weights $\mathcal{W} = \{W_1, \cdots, W_n\}$ in the spirit of [39],[40]. Despite the differences under consideration (a purely parabolic Fuchsian group, appearance of weights), the fact that the generators of Γ satisfy only one relation ensures that their techniques apply almost word-by-word once the proper analogies are made. In particular we will see how we can endow this space with the structure of a Kähler manifold.

We should give a word of caution before starting. Unlike the higher genus case, it is not necessarely true that unitary representations exist for an arbitrary system of weights. P. Belkale [7] and I. Biswas [11] independently found a collection of inequalities on the system of weights that give necessary and sufficient conditions for the existence of representations. We will call those systems *admissible*.

Given an admisible set of weights $\mathcal{W} = \{W_1, \cdots, W_n\}$ we define the map

$$\varphi : \underbrace{\mathrm{U}(r) \times \cdots \times \mathrm{U}(\mathrm{r})}_{n \text{ times}} \to \mathrm{SU}(r)$$

$$(U_1, \cdots, U_n) \mapsto \prod_{i=1}^{n} U_i e^{2\pi\sqrt{-1}W_i} U_i^{-1}$$

This map is invariant under the action of the group $(\mathrm{T}^r)^n$ given as

$$(U_1, \cdots, U_n) \mapsto (U_1 D_1, \cdots, U_n D_n), \quad D_i \in \mathrm{T}^r$$

and thus descends to a map

$$\psi : \underbrace{\mathcal{F}_r \times \cdots \times \mathcal{F}_r}_{n \text{ times}} \to \mathrm{SU}(r).$$

Before continuing we make a short digression. Given a group G and an arbitrary (real or complex) finite dimensional representation $\rho : G \to \mathrm{GL}(V)$, it is possible to associate a cochain complex giving rise to a cohomology theory (see for instance [50], Chapter 8). Let $C^i(G, V)$ the vector space of maps from i copies of G into V (in particular $C^0(G, V) = V$) and define $d^i : C^i(G, V) \to C^{i+1}(G, V)$ by $d^0 u(g) = (\rho(g) - I)u$ and

$$\begin{aligned} d^i u(g_1, \cdots, g_{i+1}) &= \rho(g_1) u(g_2, \cdots, g_{i+1}) \\ &\quad + \sum_{j=1}^{i} (-1)^j u(g_1, \cdots, g_{j-1}, g_i g_{j+1}, g_{j+2}, \cdots, g_{i+1}) \\ &\quad + (-1)^{i+1} u(g_1, \cdots, g_i). \end{aligned}$$

for $i > 0$. We are mostly interested in the cases $i = 0, 1$. Since $B^0(G, V) = 0$, we have

$$H^0(G, V) = \{v \in V | \rho(g)v = v \quad \forall g \in G\} = V^G, \tag{3.1}$$

and

$$Z^1(G, V) = \{u : G \to V \mid u(g_1 g_2) = u(g_1) + \rho(g_1) u(g_2) \quad \forall g_1, g_2 \in G\}, \tag{3.2}$$

$$B^1(G, V) = \{u : G \to V \mid u(g) = (\rho(g) - I)v\}. \tag{3.3}$$

The space $H^0(G, V)$ has a very natural interpretation, namely, that of *the space of invariants of* ρ. However, The relations (3.2), (3.3) might appear artificial at first glance, but there is an instance where its nature is truly revealed.

Recall that any Lie group possesses a natural adjoint representation on its Lie algebra, which in the case of subgroups of $\mathrm{GL}(V)$ reduces to conjugation in the corresponding Lie subalgebra of $\mathfrak{gl}(V) \cong \mathrm{End}(V) \cong V^* \otimes V$. The composition of ρ with the adjoint representation gives a new representation $\mathrm{Ad}\,\rho : G \to \mathrm{GL}(\mathfrak{gl}(V))$.

Proposition 3.1.1. *Let U be a subgroup of $\mathrm{GL}(V)$ and $\rho_t : G \to U$ be a real 1-parameter family of representations of G with $\rho_0 = \rho$. Then*

$$z := \left(\frac{d\rho_t}{dt}\rho_t^{-1}\right)\Big|_{t=0} : G \to \mathfrak{u}$$

where $\mathfrak{u} = \mathrm{Lie}(U) \subset \mathfrak{gl}(V)$, is a 1-cocycle for $\mathrm{Ad}\,\rho$. If in particular $\rho_t = M_t^{-1} \cdot \rho \cdot M_t$ for a smooth path M_t in U with $M_0 = I$, then u is a coboundary for $\mathrm{Ad}\,\rho$.

Proof. Let $\dot\rho = \frac{d\rho_t}{dt}\big|_{t=0}$. It readily follows from the chain rule that

$$\begin{aligned} z(g_1g_2) = \dot\rho(g_1g_2)\rho(g_1g_2)^{-1} &= \dot\rho(g_1)\rho(g_1)^{-1} + \rho(g_1)\left(\dot\rho(g_2)\rho(g_2)^{-1}\right)\rho(g_1)^{-1} \\ &= z(g_1) + \mathrm{Ad}\,\rho(g_1)(z(g_2)). \end{aligned}$$

Now, if $\rho_t = M_t^{-1} \cdot \rho \cdot M_t$, then $\dot M = \frac{dM_t}{dt}\big|_{t=0} \in \mathfrak{u}$ and $z(g) = \mathrm{Ad}\,\rho(g)\left(\dot M\right) - \dot M$. □

Corollary 3.1.2. *Let Γ be a Fuchsian group, $\rho : \Gamma \to \mathrm{U}(r)$ and $\gamma_1, \cdots, \gamma_n$ a set of parabolic generators for it. The space of deformations of ρ preserving the conjugacy classes $[\rho(\gamma_i)]$ is in correspondence with the space of $\mathrm{Ad}\,\rho$ 1-cocycles satisfying*

$$z(\gamma_i) = \mathrm{Ad}\,\rho(\gamma_i)(X_i) - X_i.$$

for some $X_i \in \mathfrak{u}(r)$.

Proof. A 1-parameter deformation ρ_t would preserve the conjugacy classes of the parabolic generators if and only if $\forall i$,

$$\rho_t(\gamma_i) = V_i(t)^{-1}\rho(\gamma_i)V_i(t).$$

with $V_i(t) \in \mathrm{U}(r)$ and $V_i(0) = I$. Let $X_i = \dfrac{dV_i}{dt}\Big|_{t=0}$. Then

$$z(\gamma_i) = \left(\dot\rho\rho^{-1}\right)(\gamma_i) = \mathrm{Ad}\,\rho(\gamma_i)(X_i) - X_i.$$

□

This motivates the following definition: a 1-cocycle $z \in H^1(\Gamma, \rho)$ is called *parabolic* if $z(\gamma_i) = (\rho(\gamma_i) - I)v$ for every parabolic generator $\gamma_i \in \Gamma$. We denote the subspace of all parabolic cocycles by $Z^1_P(\Gamma, \rho)$ and let the *first Parabolic cohomology, or Eichler cohomology group* be

$$H^1_P(\Gamma, \rho) = Z^1_P(\Gamma, \rho)/B^1(\Gamma, \rho). \tag{3.4}$$

It is immediate to see that the parabolic cohomology can be characterized as the kernel of the map

$$\varphi : H^1(\Gamma, \rho) \to \bigoplus_{i=1}^{n} H^1(\Gamma_i, \rho) \qquad [z] \mapsto ([z(\gamma_1)], \cdots, [z(\gamma_n)]) .$$

where $\Gamma_i = \langle \gamma_i \rangle$. The subject has been intensively studied in [49], [54], [50], [33] (cf. [42]) and there exist several variations of it (like considering either real or complex coefficients), although it was Martin Eichler [21] who first realized its importance in connection with the theory of Eichler integrals that we will discuss in the next section.

At the moment we are assuming complex coefficients since unitary representations are often though of a complex modules. There is a major exception, though, which incidentally is the most important example to us: the associated adjoint representation $\operatorname{Ad}\rho = \operatorname{Ad} \circ \rho$ which is real by its very definition.

Proposition 3.1.3. *Assume that none of the weights of ρ is equal to 0. Then there is an isomorphism*

$$H^1_P(\Gamma, \rho) \cong H^1(\Gamma, \rho).$$

Proof. It is enough to prove the isomorphism at the level of 1-cocyles. In order to prove the nontrivial contention, consider any $z \in Z^1(\Gamma, \rho)$, and observe that for each parabolic generator γ_i, the matrix

$$\rho(\gamma_i) - I = U_i(e^{2\pi\sqrt{-1}W_i} - I)U_i^{-1}$$

is invertible as no weight is equal to 0. If we make $v_i = (\rho(\gamma_i) - I)^{-1} z(\gamma_i)$, then $z(\gamma_i) = (\rho(\gamma_i) - I)v_i$, and the claim follows. □

It is not difficult to see that conversely, given any 1-cocycle of $\operatorname{Ad}\rho$, there exist a path of representations of G giving rise to it (for instance, we can construct any 1-coboundary in terms of one-parameter subgroups of U) . It is clear that two representations $\rho_1, \rho_2 : G \to U$ are equivalent if and only if they are conjugated in U, so the 1-coboundaries of $\operatorname{Ad}\rho$ correspond to the trivial deformations of ρ. The conclusion from Proposition 3.1.1 and these observations is that the space $H^1(G, \operatorname{Ad}\rho)$ is the right candidate to model a "tanget space" at a given representation ρ up to equivalence if only we could make sense of the "space of regresentations of G in V modulo conjugation" as a smooth manifold. When $G = \Gamma$ and $U = \mathrm{U}(r)$, the space of local deformations would specialize to $H^1_P(\Gamma, \operatorname{Ad}\rho)$ since we are interested on preserving the weights at the parabolic generators, and the set of rank r-unitary representations of Γ with fixed weights $\mathcal{W}$ corresponds to $\psi^{-1}(I)$ under the obvious map

$$[U_i] \mapsto U_i e^{2\pi\sqrt{-1}W_i} U_i^{-1} = \rho(\gamma_i).$$

but it is not a manifold in general. However, its restriction to the set irreducible representations is a manifold as a consequence of the following two propositions and the implicit function theorem.

Lemma 3.1.4. *A representation $\rho : \Gamma \to \mathrm{U}(r)$ is irreducible if and only if the map*

$$\kappa : \underbrace{\mathfrak{u}(r) \times \cdots \times \mathfrak{u}(r)}_{n\ times} \to \mathfrak{su}(r),$$

$$(X_1, \cdots, X_n) \mapsto \sum_{i=1}^{n} \operatorname{Ad}\rho(\gamma_1 \cdots \gamma_{i-1}) \left([X_i, \rho(\gamma_i)]\rho(\gamma_i)^{-1}\right)$$

is surjective.

Proof. By Schur's lemma, a representation $\rho : \Gamma \to \mathrm{U}(r)$ is irreducible if and only if the only matrices in $\mathfrak{u}(r)$ that commute with all $\rho(\gamma_i)$ are the multiples of the identity, which is equivalent to $H^0(\Gamma, \operatorname{Ad}\rho) = 0$. As a consequence, ρ is irreducible if and only if no nonzero matrix $X \in \mathfrak{su}(r)$ would commute with all $\rho(\gamma_i)$. Since

$$\begin{aligned} & \operatorname{tr}\left(\sum_{i=1}^{n} [\rho(\gamma_i), \operatorname{Ad}\rho(\gamma_1 \cdots \gamma_i)^{-1}(X)]\rho(\gamma_i)^{-1} X_i\right) \\ = \quad & \operatorname{tr}\left(X \sum_{i=1}^{n} \operatorname{Ad}\rho(\gamma_1 \cdots \gamma_{i-1}) \left([X_i, \rho(\gamma_i)]\rho(\gamma_i)^{-1}\right)\right) \end{aligned}$$

for $X_1, \cdots, X_n \in \mathfrak{u}(r)$ arbitrary and the Killing form $\langle X, Y\rangle_K := -\operatorname{tr}(XY)$ defines an inner product in $\mathfrak{su}(r)$, we conclude that there is $X \neq 0$ in the orthogonal complement of the image of κ in $\mathfrak{su}(r)$ if and only

$$[\rho(\gamma_i), \operatorname{Ad}\rho(\gamma_1 \cdots \gamma_i)^{-1}(X)] = 0 \Leftrightarrow [\operatorname{Ad}\rho(\gamma_1 \cdots \gamma_{i-1})(\rho(\gamma_i)), X] = 0$$

for all i. It is easy to verify that the centralizer in $\mathfrak{su}(r)$ of $\{\rho(\gamma_i)\}_{i=1}^n$ coincides with the centralizer of $\{\operatorname{Ad}\rho(\gamma_1\cdots\gamma_{i-1})(\rho(\gamma_i))\}_{i=1}^n$ due to the only relation satisfied by the generators of Γ. Therefore κ is not surjective if and only if ρ is not irreducible. $\square$

Proposition 3.1.5. *ψ has maximal rank at a point $([U_1],\cdots,[U_n])\in\psi^{-1}(I)$ if and only if the induced representation ρ is irreducible.*

Proof. A tangent vector at $([U_1],\cdots,[U_n])\in\psi^{-1}(I)$ is given as an n-tuple of matrices $(X_1,\cdots,X_n)\in\mathfrak{u}(r)^n$ with zeros in their diagonals. Once we identify $T_g\mathrm{SU}(r)\cong\mathfrak{su}(r)$ under right translation by g^{-1}, the differential of ψ at the ith flag component is given explicitly as

$$\operatorname{Ad}\rho(\gamma_1\cdots\gamma_{i-1})\left([\mathrm{d}U_iU_i^{-1},\rho(\gamma_i)]\rho(\gamma_i)^{-1}\right)$$

as a simple computation shows. Thus

$$\mathrm{d}\psi(X_1,\cdots,X_n)=\sum_{i=1}^n\operatorname{Ad}\rho(\gamma_1\cdots\gamma_{i-1})\left([X_i,\rho(\gamma_i)]\rho(\gamma_i)^{-1}\right)$$

and the result follows from Lemma 3.1.4. $\square$

Remark 12. Proposition 3.1.1 implies that $\operatorname{Ker}(\mathrm{d}\psi|_{([U_1],\cdots,[U_n])})\cong Z_P^1(\Gamma,\operatorname{Ad}\rho)$ regardless of the irreducibility of ρ. The isomorphism is given explicitly by defining

$$z(\gamma_i)=[X_i,\rho(\gamma_i)]\rho(\gamma_i)^{-1},\qquad i=1,\cdots,n,$$

and forcing the cocycle condition to arbitrary products of the generators. The only relation $z(\gamma_1\cdots\gamma_n)=0$ would then follow from the Kernel condition since

$$z(\gamma_1\cdots\gamma_n)=\sum_{i=1}^n\operatorname{Ad}\rho(\gamma_1\cdots\gamma_{i-1})\left([X_i,\rho(\gamma_i)]\rho(\gamma_i)^{-1}\right).$$

Definition 3.1.6. *The **character variety** $\mathscr{K}(\mathcal{W})=\operatorname{Hom}_{\mathcal{W}}(\Gamma,\mathrm{U}(r))/\mathrm{U}(r)$ is the set of equivalence classes of irreducible unitary representations of Γ with fixed weights $\mathcal{W}$.*

Corollary 3.1.7. *$\mathscr{K}(\mathcal{W})$ possesses the structure of a smooth manifold of real dimension $n(r^2-r)-2(r^2-1)$. The tanget space at a given equivalence class $[\rho]$ is isomorphic to $H_P^1(\Gamma,\operatorname{Ad}\rho)$.*

Proof. We already know that $\mathscr{P}=\psi^{-1}(I)$, the set of irreducible unitary representations of Γ with fixed weights $\mathcal{W}$ is a smooth manifold. Now, $\mathrm{PSU}(r)$ acts freely and properly by conjugation on $\mathscr{P}$ since it is a compact group.

The quotient $\mathrm{PSU}(r) \setminus \mathscr{P} = \mathscr{K}(\mathcal{W})$ is then a smooth manifold with $\mathscr{P}$ as principal $\mathrm{PSU}(r)$-bundle over it. Since the tangent space along the fibers consisting of the Kernel of the differential of the projection at a given point $([U_1], \cdots, [U_n])$ is isomorphic to $B^1(\Gamma, \mathrm{Ad}\,\rho)$, it follows from Corollary 3.1.2 that the tangent space over $[([U_1], \cdots, [U_n])]$ in the quotient would be isomorphic to $H^1_P(\Gamma, \mathrm{Ad}\,\rho)$. As for the dimension, the implicit function theorem gives

$$\dim(\mathscr{P}) = \dim(\mathcal{F}_r)^n - \dim(\mathrm{SU}(r)) = n(r^2 - r) - (r^2 - 1),$$

and since $\mathscr{P}$ is a principal $\mathrm{PSU}(r)$-bundle,

$$\dim(\mathscr{K}(\mathcal{W})) = \dim(\mathscr{P}) - \dim(\mathrm{PSU}(r)) = n(r^2 - r) - 2(r^2 - 1).$$

□

3.1.2 The Eichler-Shimura isomorphism

We now connect the constructions of section 3.1.1 with the developed theory of automorphic forms of weight 2.

Fix any reference point $\tau_0 \in \mathbb{H}$. For any $f \in \mathfrak{S}_2(\Gamma, \rho)$, let $\phi = f(\tau)\mathrm{d}\tau$ and define

$$F_v(\tau) = \int_{\tau_0}^{\tau} \phi + v$$

for any $v \in \mathbb{C}^r$. This gives a generalization of an *Eichler integral* . Then clearly $F_v(\tau)$ is well-defined and holomorphic on $\mathbb{H}$. Moreover,

$$F_v(\gamma\tau) = \int_{\gamma\tau_0}^{\gamma\tau} \phi + \int_{\tau_0}^{\gamma\tau_0} \phi + v = \rho(\gamma)F_v(\tau) + z_v(\gamma) \tag{3.5}$$

where

$$z_v(\gamma) = \int_{\tau_0}^{\gamma\tau_0} \phi + (I - \rho(\gamma))v.$$

Note that

$$z_v(\gamma_1\gamma_2) = z_v(\gamma_1) + \rho(\gamma_1)z_v(\gamma_2)$$

so that $z_v(\gamma) \in Z^1(\Gamma, \rho)$.

Definition 3.1.8. *Let p be a nonnegative integer. An* ***Eichler integral of weight*** $-2p$ ***with the representation*** ρ *is a meromorphic function $\mathscr{E} : \mathbb{H} \to \mathbb{C}^r$ satisfying for any $\gamma \in \Gamma$*

$$\mathscr{E}(\gamma\tau)/\gamma'(\tau)^p - \rho(\gamma)\mathscr{E}(\tau) = p_\gamma(\tau) \tag{3.6}$$

where the entries of $p_\gamma(\tau)$ are polynomials of degree at most $2p$.

Thus, the F_v are Eichler integrals of weight 0. Moreover, consider any other $v' \in \mathbb{C}^r$. Then

$$z_v(\gamma) - z_{v'}(\gamma) = (I - \rho(\gamma))(v - v') \in B^1(\Gamma, \rho),$$

and similarly if we change the base point. Finally, we notice from the q-series expansions (2.3) at each cusp of $f(\tau)$ that

$$F_v(\sigma_i \tau) = U_i q^{W_i} \left(\sum_{k=0}^{\infty} b_i'(k) q^k \right) \Bigg|_{\tau_0}^{\tau} \qquad (b_i'(k))_j = \frac{(b_i(k))_j}{\alpha_{ij} + k} \tag{3.7}$$

the cusp form condition guarantees that if $\alpha_{ij} + k = 0$ then $(b_i(k))_j = 0$, so the last expression is meaningful and obviously convergent. As an immediate consequence we obtain that for each $i = 1, \cdots, n$, $\lim_{\tau \to \tau_i} F_v(\tau)$ exists, where the limit is taken over a fundamental region containing τ_0. We define

$$F_v(\tau_i) = \lim_{\tau \to \tau_i} F_v(\tau).$$

These values are specially important since $\gamma_i \tau_i = \tau_i$, hence

$$z_v(\gamma_i) = (I - \rho(\gamma_i)) F_v(\tau_i), \tag{3.8}$$

i.e. z_v is a parabolic cocycle. Our conclusion is that there is canonical $\mathbb{C}$-linear map $L : \mathfrak{S}_2(\Gamma, \rho) \to H^1_P(\Gamma, \rho)$, with both domain and range being finite dimensional. In fact, it is easy to prove that this is injective: a cusp form $f(\tau)$ would belong to its kernel if its associated cocycle $z_v(\gamma)$ is a coboundary, that is, for any given v there is a vector $v_0 \in \mathbb{C}^r$ such that

$$v_0 = F_v(\tau_i), \qquad i = 1, \cdots, n.$$

this means that such coboundary canbe chosen to be 0. Now, this implies that F_v is an $\operatorname{Ad}\rho$-automorphic function bounded at the cusps, hence an endomorphism of E_ρ, hence a multiple of the identity. But this could only happen if $f = 0$.

It would be natural to try to understand this map in more detail. This has been done extensively, but there is a particular case that will be important to us. This requires to consider *real* representations and their complexifications simultaneously. The reader would notice that this is the case for $\operatorname{Ad}\rho$ and $\operatorname{Ad}\rho_{\mathbb{C}}$, which leads to Shimura's generalization of Eichler's theorem [50].

Theorem 3.1.9 (Eichler-Shimura)**.** *Let ρ be a real representation of Γ on a compact subgroup of* $\mathrm{GL}(r,\mathbb{R})$*. The map*

$$L_{ES} : \mathfrak{S}_2(\Gamma, \rho_{\mathbb{C}}) \to H^1_P(\Gamma, \rho), \qquad f(\tau) \mapsto [\mathrm{Re}(z_v(\gamma))]$$

is an isomorphism of real vector spaces.

Proof. We provide a proof when the real representation is $\mathrm{Ad}\,\rho$ for some irreducible unitary representation ρ. We calculated in section 3.1.1 (indirectly in Corollary 3.1.7) the real dimension of $H^1_P(\Gamma, \mathrm{Ad}\,\rho)$ to be $n(r^2-r)-2(r^2-1)$ and the complex dimension of $\mathfrak{S}_2(\Gamma, \mathrm{Ad}\,\rho_{\mathbb{C}})$ to be equal to $n(r^2-r)/2-(r^2-1)$ (indirectly in Corollary 2.1.8). Therefore it is enough to prove that the map is injective. Notice that since the isomorphism

$$\mathfrak{gl}(r,\mathbb{C}) \cong \mathfrak{u}(r) \otimes_{\mathbb{R}} \mathbb{C},$$

is relalized by expressing an arbitrary complex square matrix M as $(M - M^*)/2 + \sqrt{-1}((M+M^*)/2\sqrt{-1})$, the real part of a cocycle $z_v(\gamma)$ associated to the a cusp form $f(\tau)$ would correspond to the class of the unitary cocycle $(z_v - z_v^*)/2$.

Consider now a cusp form $f(\tau)$ lying in the kernel of L_{ES}. This means that the original $\mathrm{Ad}\,\rho_{\mathbb{C}}$ cocycle z_v satisfies $z_v = z_v^*$, that is, each one of the matrices $F_v(\tau_i)$ is self-adjoint. This readily implies that

$$L_{ES}\left(\sqrt{-1}f(\tau)\right) = \sqrt{-1}z_v,$$

but on the other hand, the complexification of the map L_{ES} is just the map $f(\tau) \mapsto z_v(\gamma)$. This implies that $[z_v] \equiv 0$, but we have seen that this in turn implies that $f(\tau) = 0$. This concludes the proof. □

Notice that the same argument showing injectivity would hold for an arbitrary unitary representation and the proof would be complete if we only had a way to verify that the dimensions of both spaces coincide in general. Since we won't need this result, we won't dwell on it here.

The reader should keep in mind that the theorem holds for arbitrary Fuchsian groups of the first kind with parabolic generators (*H groups* in Lehner's convention, [34]), and in more generality for cusp forms of higher weight once the appropriate representations are considered (Shimura [50] discusses this case for real coefficients, while Kra [33] generalizes the scalar case to arbitrary Kleinian groups).

3.1.3 The canonical Kähler structure

The Eichler-Shimura isomorphism allows us to introduce a complex structure on the real vector space $H^1_P(\Gamma, \operatorname{Ad}\rho)$. We could go beyond and, as in [39], prove that this defines an integrable almost complex structure on $\mathscr{K}(\mathcal{W})$, and furthermore introduce a Hermitian inner product determining a Kähler metric as in [38], but we take a different and more powerful approach instead that allows us to do these two constructions at once (eventually we will discuss the generalization of the Narasimhan Kähler metric in the case of parabolic bundles, see section 3.2.5). It is possible to introduce an intrinsic Kähler structure on $\mathscr{K}(\mathcal{W})$ using the machinery of Kähler reduction that we now discuss. We start by noticing that the *compact coadjoint orbit* nature of the complete flag variety $\mathcal{F}_r$ defines a canonical projective structure on it [17]. In particular $\mathcal{F}_r$ is a Kähler manifold. $\mathrm{PSU}(r)$ acts properly on the left and moreover, this action preserves the Kähler structure of $\mathcal{F}_r$. The same observations hold for the product space $\mathcal{F}_r \times \cdots \times \mathcal{F}_r$, of course.

The group $\mathrm{PSU}(r)$ acts freely and properly on $\mathrm{SU}(r)$ by means of the adjoint action. A further inspection of the map

$$\psi : \mathcal{F}_r \times \cdots \times \mathcal{F}_r \to \mathrm{SU}(r)$$

shows that it is equivariant with respect to these actions, and hence it behaves *analogously* to a moment map. This observation makes the consideration of a prospective Kähler structure on $\mathscr{K}(\mathcal{W})$ very natural. we should mention that the so-called Lie group valued moment maps (like the function ψ) have been studied by Alekseev, Malkin and Meinrenken in [2]. For our purposes, it is enough to revise the Kähler reduction apparatus as is presented in [18].

Definition 3.1.10. *Let M be a Kähler manifold and G a compact Lie group acting on M and preserving the Kähler structure. Given an equivariant smooth map $\phi : M \to \mathfrak{g}^*$ (where $\mathfrak{g} = \mathrm{Lie}(G)$), we call $y \in \mathfrak{g}^*$ a* ***good value*** *if the following conditions are satisfied:*

1. *$\phi^{-1}(y)$ is a smooth manifold.*

2. *for every $x \in \phi^{-1}(y)$, $T_x\phi^{-1}(y) = \mathrm{Ker}(\mathrm{d}\phi_x)$.*

3. *The projection $\pi_y : \phi^{-1}(y) \to G_y \backslash \phi^{-1}(y)$, where G_y is the isotropy group of y, is a smooth submersion.*

Theorem 3.1.11 (Kähler reduction)**.** *Let M be a Kähler manifold with Kähler structure determined by the pair (g, ω), and let G be a compact Lie group* *acting on M and preserving the Kähler structure. For every good value of $y \in \mathfrak{g}^*$, there is a unique Kähler structure (g_y, ω_y) on $G_y \setminus \phi^{-1}(y)$ defined by the conditions:*

1. *$\pi_y^*(\omega_y)$ is the pull-back of ω to $\phi^{-1}(y)$.*

2. *π_y is a Riemannian submersion.*

Remark 13. One has to require the action of G to be *Poisson* as well, but this is automatically satisfied when the group G is compact.

The important observation here is that the details of the proof, which can be found in [18], do not use any special properties of $\mathfrak{g}^*$, but instead rely on (1) the equivariance of ϕ, and (2) the analytical properties of clean values. In particular, the theorem will still be true if a moment map is replaced by a Lie group-valued moment map (that is, an equivariant map with values on a Lie group). Notice that we can always transform a moment map into a Lie group valued moment map by dualizing and exponentiating.

In the case of interest, $\exp^{-1}(I)$ is a discrete set on $\mathfrak{su}(r) \cong \mathfrak{su}(r)^*$. It is not hard to prove that $\psi^{-1}(I)$ is connected, and from these considerations it would follow that $\psi^{-1}(I) = \phi^{-1}(0)$, where $\psi = \exp \circ \phi$. We thus see that checking that I is a clean value of ψ is equivalent to checking 0 is a clean value of ϕ. Now, we have proved in section 3.1.1 that I is a clean value of the map ψ, and as a corollary we conclude that the manifold $\mathscr{K}(\mathcal{W}) = \mathrm{PSU}(r) \setminus \mathscr{P}$ acquires a canonical structure of a Kähler manifold.

3.2 The complex analytic theory of moduli of stable parabolic bundles

The goal of this section is to give a construction of the moduli space $\mathscr{M}(\mathcal{W})$ of stable parabolic bundles over $\mathbb{P}^1$ with a given system of weights $\mathcal{W}$ over the set $\mathscr{D} = \{z_1, \cdots, z_n\}$ as a Kähler manifold in a way analogous to the construction of the Teichmüller spaces. We know from section C.4 that the infinitesimal deformations of a given parabolic bundle E_* are parametrized by the space $H^1(\text{Par End}(E_*))$ and therefore the tangent space to the corresponding point on any potential moduli space containing it would be modeled by it. In the case this bundle is parabolic stable, the Mehta-Seshadri theorem (theorem C.3.2) asserts that $E_* \cong (E_\rho)_*$ for some irreducible unitary representation ρ with weights $\mathcal{W}$, and we have seen that in the latter, the space $H^1(\text{Par End}((E_\rho)_*))$ is isomorphic to $H^1(E_{\operatorname{Ad}\rho_{\mathbb{C}}})$. In particular, there would be an isomorphism of Kähler manifolds $\mathscr{M}(\mathcal{W}) \cong \mathscr{K}(\mathcal{W})$.

From now on, we will assume that ρ is an irreducible unitary representation with generic weights. The representation $\operatorname{Ad}\rho$ (and its complexification) is also unitary and splits as a direct sum of two irreducibles, one of them being trivial and corresponding to the multiples of the identity while the other consists of the traceless $r \times r$ matrices. Now, for every $i = 1, \cdots, n$, the eigenvectors of $\operatorname{Ad}\rho(\gamma_i)$ are the matrices $U_i E_{jk} U_i^{-1}$ with eigenvalues $e^{2\pi\sqrt{-1}(\alpha_{ij}-\alpha_{ik})}$ (recall that $(E_{jk})_{lm} = \delta_{jl}\delta_{km}$). The weights of $\operatorname{Ad}\rho(\gamma_i)$ are seen to be given as

$$\beta_{ijk} = \begin{cases} \alpha_{ij} - \alpha_{ik} & \text{if} \quad j \geq k, \\ \\ 1 + \alpha_{ij} - \alpha_{ik} & \text{if} \quad j < k. \end{cases}$$

From this we readily see that regardless of the system of weights characterizing the unitary representation ρ, the degree of the bundle $E_{\operatorname{Ad}\rho_{\mathbb{C}}}$ only depends on the rank r and the number parabolic generators n, since

$$\deg(E_{\operatorname{Ad}\rho_{\mathbb{C}}}) = -\sum_{i=1}^{n}\sum_{j,k=1}^{r} \beta_{ijk} = -n(r^2 - r)/2. \tag{3.9}$$

Recall that $H^0(E_{\operatorname{Ad}\rho_{\mathbb{C}}}) = \mathfrak{gl}(r, \mathbb{C})^{\operatorname{Ad}\rho_{\mathbb{C}}}$, and ρ is irreducible if and only if $\dim \mathfrak{gl}(r, \mathbb{C})^{\operatorname{Ad}\rho_{\mathbb{C}}} = 1$. Therefore, using the Riemann-Roch theorem for vector bundles (Theorem 2.1.7), we conclude

Corollary 3.2.1. *If $E_* \to \mathbb{P}^1$ is parabolic stable, then*

$$\dim H^0(\text{Par End}(E_*)) = 1, \tag{3.10}$$

and

$$\dim H^1(\text{Par End}(E_*)) = n\left(\frac{r^2-r}{2}\right) - (r^2-1). \tag{3.11}$$

3.2.1 The differential equation

We will now draw an analogy between the infinitesimal deformation theory and the Teichmüller theory, as it is considered in [51]. We can materialize the elements of the space $H^1(E_{\operatorname{Ad}\rho_{\mathbb{C}}})$ in terms of the Dolbeault isomorphism as vector bundle-valued differential forms of type (0,1), but for every class in $H^{0,1}_{\bar\partial}(E_{\operatorname{Ad}\rho_{\mathbb{C}}})$ there are infinitely many representatives to choose from. The isomorphism $H^{0,1}_{\bar\partial}(E_{\operatorname{Ad}\rho_{\mathbb{C}}}) \cong \overline{\mathfrak{S}_2(\Gamma, \operatorname{Ad}\rho_{\mathbb{C}})}$ allows us to choose a canonical harmonic representative on every equivalence class.[1] Thus an element of it corresponds to an antiholomorphic function $\nu : \mathbb{H} \to \mathfrak{gl}(r,\mathbb{C})$ satisfying

$$\nu(\gamma\tau)\overline{\gamma'(\tau)} = \rho(\gamma)\nu(\tau)\rho(\gamma)^{-1} \tag{3.12}$$

and the corresponding behaviour at the cusps.

In the standard (nonparabolic) deformation theory of a holomorphic vector bundle E whose holomorphic structure is determined by a Cauchy-Riemann operator $\bar\partial$, the 1-cocycles $\theta^{0,1}$ representing elements in $H^{0,1}_{\bar\partial}(\operatorname{End}(E))$ have the interpretation of deformations of the complex structure of the form $\bar\partial + \theta^{0,1}$ and the trivial deformations correspond to pull-backs of $\bar\partial$ under an automorphism of E, taking the form $f^{-1}\circ\bar\partial\circ f = \bar\partial + f^{-1}\bar\partial f$.

If we now consider the bundle $\mathbb{H}\times\mathbb{C}^r$ as a uniformization of a vector bundle E, the Cauchy-Riemann operator on the latter would be the projection of standard Cauchy-Riemann operator on $\mathbb{H}\times\mathbb{C}^r$. Since this operator doesn't admit nontrivial deformations, it becomes apparent that we can relate the local deformations of the holomorphic structure on E with the solutions of the equation

$$f^{-1}\cdot f_{\bar\tau} = \nu \tag{3.13}$$

for a suitable automorphic form ν of weight (0,1). We start with a preparatory lemma.

Lemma 3.2.2. *Let ν be an antiholomorphic $\operatorname{Ad}\rho_{\mathbb{C}}$-automorphic form. Any antiholomorphic solution f of (3.13) satisfies*

$$f(\gamma\tau) = \chi(\gamma)f(\tau)\rho(\gamma)^{-1}, \qquad \forall\gamma\in\Gamma,$$

for some representation $\chi : \Gamma \to \mathrm{GL}(r,\mathbb{C})$.

[1] Notice that $\operatorname{Ad}\rho_{\mathbb{C}}$ is *self-dual*, i.e. $(\operatorname{Ad}\rho)^* = \operatorname{Ad}\rho$.

Proof. Since for an arbitrary solution f we have that $f^{-1}f_{\bar{\tau}} = \nu$, in particular the representation ρ will appear in $f_{\bar{}}$ as a right automorphic factor. Let f be antiholomorphic and consider the function $g(\tau,\gamma) = f(\gamma\tau)\rho(\gamma)f(\tau)^{-1}$. It is also antiholomorphic, and moreover,

$$\begin{aligned} g_{\bar{\tau}}(\tau,\gamma) &= f_{\bar{\tau}}(\gamma\tau)\rho(\gamma)f(\tau)^{-1}\overline{\gamma'(\tau)} - f(\gamma\tau)\rho(\gamma)f(\tau)^{-1}f_{\bar{\tau}}(\tau)f(\tau)^{-1} \\ &= f(\gamma\tau)\rho(\gamma)\nu(\tau)f(\tau)^{-1} - f(\gamma\tau)\rho(\gamma)\nu(\tau)f(\tau)^{-1} \equiv 0, \end{aligned}$$

which implies that $g(\cdot,\gamma)$ is holomorphic for each fixed $\gamma \in \Gamma$, hence harmonic and thus a constant $\chi(\gamma)$. To prove that the collection of constants $\{\chi(\gamma)|\gamma \in \Gamma\}$ defines a representation of Γ, notice that

$$\begin{aligned} \chi(\gamma_1\gamma_2) &= f(\gamma_1\gamma_2\tau)\rho(\gamma_1\gamma_2)f(\tau)^{-1} \\ &= (f(\gamma_1\gamma_2\tau)\rho(\gamma_1)f(\gamma_2\tau)^{-1}) \cdot (f(\gamma_2\tau)\rho(\gamma_2)f(\tau)^{-1}) \\ &= \chi(\gamma_1)\chi(\gamma_2). \end{aligned}$$

□

Lemma 3.2.3. *Let $\epsilon \in \mathbb{D}$. The differential equation*

$$f^{-1} \cdot f_{\bar{\tau}} = \epsilon\nu$$

admits a unique solution $f(\tau,\epsilon)$ normalized so that $f(\tau_0,\epsilon) = I$ which is antiholomorphic in τ and holomorphic in ϵ.

The proof of a slightly more general result can be found in [36] and won't be presented here.

Theorem 3.2.4 (Takhtajan-Zograf, [51])**.** *For each $\nu \in \overline{\mathfrak{S}_2(\Gamma, \operatorname{Ad}\rho_{\mathbb{C}})}$ and $\epsilon \in \mathbb{C}$ sufficiently close to 0, there is a unique solution $f^{\epsilon\nu} : \mathbb{H} \to \mathrm{GL}(r,\mathbb{C})$ of the differential equation*

$$f^{-1} \cdot f_{\bar{\tau}} = \epsilon\nu \tag{3.14}$$

satisfying

1. *$f^{\epsilon\nu}(\gamma\tau) = \rho^{\epsilon\nu}(\gamma)f(\tau)\rho(\gamma)^{-1} \quad \forall\gamma \in \Gamma$, where $\rho^{\epsilon\nu} : \Gamma \to \mathrm{U}(r)$ is irreducible;*

2. *$\det f^{\epsilon\nu}(\tau_0) = 1$ for some fixed $\tau_0 \in \mathbb{H}$;*

3. *$f^{\epsilon\nu}$ is regular at the cusps, that is*

$$\lim_{\tau\to\tau_i} (f^{\epsilon\nu}(\tau))_{jk} < \infty, \qquad i = 1,\cdots,n, \quad 1 \le j,k \le r.$$

Proof. The differential equation (3.14) has a unique antiholomorphic solution $f_1^{\epsilon\nu}$ satisfying $f_1^{\epsilon\nu}(\tau_0) = I$. The regularity at the cusps of $f_1^{\epsilon\nu}$ follows as a consequence of the cusp form condition on ν^*. Moreover, it is immediate to see that an arbitrary solution of it is of the form $f_2^{\epsilon\nu} f_1^{\epsilon\nu}$ with $f_2^{\epsilon\nu}$ holomorphic. We know from Lemma 3.2.2 that $f_1(\gamma\tau) = \chi^{\epsilon\nu}(\gamma) f(\tau)\rho(\gamma)^{-1}$ and moreover, the dependence on ϵ is holomorphic as a consequence of Lemma 3.2.3. In this way we construct a 1-dimensional holomorphic family of parabolic bundles $\mathcal{F}$ with the property that $\mathcal{F}|_{\{\epsilon\}\times\mathbb{P}^1} = (E_{\chi^{\epsilon\nu}})_*$ and moreover, $\mathcal{F}|_{\{0\}\times\mathbb{P}^1} = (E_\rho)_*$. The openness of parabolic stability implies that that if ϵ is sufficiently small, the bundle $(E_{\chi^{\epsilon\nu}})_*$ would be parabolic stable, therefore $(E_{\chi^{\epsilon\nu}})_* \cong (E_{\rho^{\epsilon\nu}})_*$ for some irreducible unitary representation $\rho^{\epsilon\nu}$. This last condition is equivalent to the existence of a holomorphic map $f_2^{\epsilon\nu} : \mathbb{H} \to \mathrm{GL}(r, \mathbb{C})$ satisfying $f_2^{\epsilon\nu}(\gamma\tau) = \rho^{\epsilon\nu}(\gamma) f_2^{\epsilon\nu}(\tau)\chi^{\epsilon\nu}(\gamma)^{-1}$ which is furthermore regular at each of the cusps. we finish the proof by letting

$$f^{\epsilon\nu} = f_2^{\epsilon\nu} \cdot f_1^{\epsilon\nu}.$$

□

The function $f^{\epsilon\nu}$ plays the role of a deformation mapping (analogous to a quasiconformal mapping) and induces a parabolic bundle map $F^{\epsilon\nu}$ by requiring the commutativity of the diagram

$$\begin{tikzcd}
\mathbb{H}^+ \times \mathbb{C}^r \arrow[r, "f^{\epsilon\nu}"] \arrow[d, "\mathscr{J}"'] & \mathbb{H}^+ \times \mathbb{C}^r \arrow[d, "\mathscr{J}_{\epsilon\nu}"] \\
E_\rho \arrow[r, "F^{\epsilon\nu}"] & E_{\rho^{\epsilon\nu}}.
\end{tikzcd}$$

Or equivalently,

$$\begin{tikzcd}
(\tau, v) \arrow[r, mapsto] \arrow[d, mapsto] & (\tau, f^\nu(\tau)v) \arrow[d, mapsto] \\
(J(\tau), \Upsilon(\tau)v) \arrow[r, mapsto] & (J(\tau), w),
\end{tikzcd}$$

with $w = (\Upsilon_{\epsilon\nu} f^{\epsilon\nu})(\tau)v = ((F^{\epsilon\nu} \circ J)\Upsilon)(\tau)v$, or simply,

$$\Upsilon_{\epsilon\nu} \cdot f^{\epsilon\nu} = (F^{\epsilon\nu} \circ J) \cdot \Upsilon \qquad \text{on} \quad \mathbb{H}. \tag{3.15}$$

In general, $f^{\epsilon\nu}$ is a smooth function of $\epsilon, \bar{\epsilon}$. We will discuss this in more detail in the next section.

The pull-back $(F^{\epsilon\nu})^*$ of any object or structure on $E_{\rho^{\epsilon\nu}}$ to an object or structure on E_ρ can be understood at the level of $\mathbb{H}^+ \times \mathbb{C}^r$ in terms of the function $f^{\epsilon\nu}$. For example, the corresponding $\rho^{\epsilon\nu}$-automorphic function g

associated to a section of $E_{\rho^{\epsilon\nu}}$ transforms into the ρ-automorphic function $(f^{\epsilon\nu})^{-1} \cdot g$, and similarly for parabolic endomorphisms. Of special importance are the holomorphic structures on E_ρ, $E_{\rho^{\epsilon\nu}}$, which are determined by the corresponding projections of the Cauchy-Riemann operator $\bar{\partial}$ on $\mathbb{H}^+ \times \mathbb{C}^r$ (with the obvious extension at the cusps). by pulling back, the holomorphic structure of the latter on E_ρ becomes the projection of

$$\bar{\partial}_{\epsilon\nu} = \mathrm{Ad}\left((f^{\epsilon\nu})^{-1}\right)\left(\bar{\partial}\right) = \bar{\partial} + \epsilon\nu \mathrm{d}\bar{\tau}. \tag{3.16}$$

A similar argument holds for the canonical connections and their corresponding holomorphic parts. The following lemma will be useful to understand this better.

Lemma 3.2.5. *Let $\theta^{1,0} \in A^{1,0}(\mathbb{H} \times \mathbb{C}^r)$ and $\theta^{0,1} \in A^{0,1}(\mathbb{H} \times \mathbb{C}^r)$. The operator $\partial + \theta^{1,0}$ on $\mathbb{H} \times \mathbb{C}^r$ is a holomorphic connection with respect to the holomorphic structure $\bar{\partial} + \theta^{0,1}$ if and only if*

$$\partial\theta^{0,1} + \bar{\partial}\theta^{1,0} + [\theta^{1,0}, \theta^{0,1}] = 0, \tag{3.17}$$

where $[\theta^{1,0}, \theta^{0,1}] = \theta^{1,0} \wedge \theta^{0,1} + \theta^{0,1} \wedge \theta^{1,0}$ is the graded commutator on matrix-valued 1-forms.

Proof. Let $s : \mathbb{H} \to \mathbb{C}^r$ be any smooth function such that $(\bar{\partial} + \theta^{0,1})s = 0$. We have

$$\begin{aligned}(\bar{\partial} + \theta^{0,1})(\partial + \theta^{1,0})s &= -(\partial + \theta^{1,0})\bar{\partial}s + (\bar{\partial}\theta^{1,0} + \theta^{0,1} \wedge \theta^{1,0})s + \theta^{0,1} \wedge \partial s \\ &= (\partial\theta^{0,1} + \theta^{1,0} \wedge \theta^{0,1})s + (\bar{\partial}\theta^{1,0} + \theta^{0,1} \wedge \theta^{1,0})s,\end{aligned}$$

since $\partial + \theta^{1,0}$ is a holomorphic connection if and only if the previous expression vanishes for any admissible s, the claim follows. □

We defined the canonical connection on E_ρ as an extension of the projection of the De Rham differential. We can also pull-back the canonical connection corresponding to the bundle $E_{\rho^{\epsilon\nu}}$ onto E_ρ and this would be given as the projection of the operator

$$\mathrm{Ad}\left((f^{\epsilon\nu})^{-1}\right)(\mathrm{d}) = \mathrm{d} + \theta$$

where

$$\theta = (f^{\epsilon\nu})^{-1}\mathrm{d}f^{\epsilon\nu}.$$

The functoriality of the pull-back implies that this is not only compatible with the holomorphic structure $\bar{\partial} + \epsilon\nu\mathrm{d}\bar{\tau}$ but also its $(1,0)$-part is a holomorphic connection for it: since $\partial(\epsilon\nu \mathrm{d}\bar{\tau}) = 0$, by lemma 3.2.5 we only have to verify that

$$((f^{\epsilon\nu})^{-1} f^{\epsilon\nu}_{\tau})_{\bar{\tau}} - [(f^{\epsilon\nu})^{-1} f^{\epsilon\nu}_{\tau}, (f^{\epsilon\nu})^{-1} f^{\epsilon\nu}_{\bar{\tau}}] = 0,$$

but this equation is clearly equivalent to the equation $((f^{\epsilon\nu})^{-1} f^{\epsilon\nu}_{\bar{\tau}})_{\tau} = (\epsilon\nu)_{\tau} = 0$.

Let us now denote by A, resp. $A^{\epsilon\nu}$ the corresponding logarithmic connections on E_ρ, resp. $E_{\rho^{\epsilon\nu}}$.

$$A \circ J + \Upsilon \left((f^{\epsilon\nu})^{-1} f^{\epsilon\nu}_{\tau}\right) \Upsilon^{-1} = \left((F^{\epsilon\nu})^{-1} F^{\epsilon\nu}_{\tau}\right) \circ J + \left((F^{\epsilon\nu})^{-1} A^{\epsilon\nu} F^{\epsilon\nu}\right) \circ J \quad (3.18)$$

3.2.2 Variational formulas

We have encountered several families of functions depending on a moduli local parameter ϵ. Even though their behaviour on such parameter could be rather complicated, a surprising amount of information is encoded in their first variations. It is convenient to introduce the following notation: if the dependence of a family f^ϵ is smooth, we make

$$\dot{f}_+ = \frac{\partial f^\epsilon}{\partial \epsilon}\Big|_{\epsilon=0}, \qquad \dot{f}_- = \frac{\partial f^\epsilon}{\partial \bar{\epsilon}}\Big|_{\epsilon=0},$$

while if the dependence is complex analytic, we simply make

$$\dot{f} = \frac{\partial f^\epsilon}{\partial \epsilon}\Big|_{\epsilon=0}.$$

The following lemma is of paramounth importance. All the variational formulas we will encounter are a consequence of it. The case for stable vector bundles over Riemann surfaces of genus greater than 1 was proved in [56] and later reconsidered in [51] for stable parabolic bundles.

Lemma 3.2.6 (Vanishing of the first variation of the hermitian metric under deformations)**.** *The function $f^{\epsilon\nu}$ satisfies*

$$\frac{\partial}{\partial \epsilon}\left((f^{\epsilon\nu})^* f^{\epsilon\nu}\right)\Big|_{\epsilon=0} = \frac{\partial}{\partial \bar{\epsilon}}\left((f^{\epsilon\nu})^* f^{\epsilon\nu}\right)\Big|_{\epsilon=0} = 0.$$

Proof. The function $\Phi = \frac{\partial}{\partial \epsilon}\left((f^{\epsilon\nu})^* f^{\epsilon\nu}\right)|_{\epsilon=0}$ is smooth and satisfies $\Phi(\gamma\tau) = \mathrm{Ad}(\rho(\gamma))(\Phi(\tau))$. Since $f^0_1 = f^0_2 = I$, we also have that

$$\Phi = (\dot{f}_1)_+ + (\dot{f}_2)_+ + (\dot{f}_1)^*_- + (\dot{f}_2)^*_-,$$

with each of the summands being either holomorphic or antiholomorphic.

Hence Φ is harmonic and bounded at the cusps, thus a constant. Finally, we observe that $\frac{\partial}{\partial \bar{\epsilon}}\left((f^{\epsilon\nu})^* f^{\epsilon\nu}\right)|_{\epsilon=0} = \Phi^* = 0$. $\square$

Corollary 3.2.7. *The functions $\dot{f}_+$, $\dot{f}_-$ satisfy*

$$\left(\dot{f}_+\right)_{\bar{\tau}} = \nu, \qquad \left(\dot{f}_-\right)_{\bar{\tau}} = 0, \tag{3.19}$$

$$\left(\dot{f}_+\right)_{\tau} = 0, \qquad \left(\dot{f}_-\right)_{\tau} = -\nu^*.$$

Proof. Clearly $\dot{f}_+ + \dot{f}_-^* = \frac{\partial}{\partial \epsilon}\left((f^{\epsilon\nu})^* f^{\epsilon\nu}\right)|_{\epsilon=0} = 0$. The result now follows from the identities $\frac{\partial}{\partial \epsilon}\left((f^{\epsilon\nu})^{-1} f^{\epsilon\nu}_{\bar{\tau}}\right)|_{\epsilon=0} = \left(\dot{f}_+\right)_{\bar{\tau}}$, $\frac{\partial}{\partial \bar{\epsilon}}\left((f^{\epsilon\nu})^{-1} f^{\epsilon\nu}_{\bar{\tau}}\right)|_{\epsilon=0} = \left(\dot{f}_-\right)_{\bar{\tau}}$ $\square$

For the second time, Eichler integrals of weight 0 appear naturally in the infinitesimal deformation theory of stable parabolic bundles. Let us let

$$\mathscr{E}_+ = \Upsilon^{-1} \cdot \dot{\Upsilon}_+, \qquad \mathscr{E}_- = \Upsilon^{-1} \cdot \dot{\Upsilon}_-, \tag{3.20}$$

Lemma 3.2.8. *$\mathscr{E}_+$ and $\mathscr{E}_-$ are Eichler integrals of weight 0 with the representation $\operatorname{Ad}\rho$.*

Proof. we know that for sufficiently small values of ϵ and $\forall \gamma \in \Gamma$, $\Upsilon_{\epsilon\nu}(\gamma\tau) = \Upsilon_{\epsilon\nu}(\tau)\rho^{\epsilon\nu}(\gamma)^{-1}$. Thus

$$\dot{\Upsilon}_{\pm}(\gamma\tau) = \dot{\Upsilon}_{\pm}(\tau)\rho(\gamma)^{-1} - \Upsilon(\tau)\rho(\gamma)^{-1}\dot{\rho}(\gamma)_{\pm}\rho(\gamma)^{-1},$$

and consequently,

$$\mathscr{E}_{\pm}(\gamma\tau) = \operatorname{Ad}\,\rho(\gamma)\left(\mathscr{E}_{\pm}(\tau)\right) - \dot{\rho}(\gamma)_{\pm}\rho(\gamma)^{-1}$$

$\square$

Remark 14. Notice that the unitarity of the deformations of ρ implies that

$$\left(\dot{\rho}_+(\gamma) \cdot \rho(\gamma)^{-1}\right)^* = -\dot{\rho}_-(\gamma) \cdot \rho(\gamma)^{-1} \qquad \forall \gamma \in \Gamma.$$

and none of these is, in principle, an element of $Z^1_P(\Gamma, \operatorname{Ad}\rho)$ but instead of $Z^1_P(\Gamma, \operatorname{Ad}\rho_{\mathbb{C}})$.

We can use these facts to study the first variation of the function $F^{\epsilon\nu}$. A direct computation from 3.15 gives

$$\Upsilon\left(\mathscr{E}_+ + \dot{f}_+\right)\Upsilon^{-1} = \dot{F}_+ \circ J, \tag{3.21}$$

$$\Upsilon\left(\mathscr{E}_- + \dot{f}_-\right)\Upsilon^{-1} = \dot{F}_- \circ J. \tag{3.22}$$

The left hand side of 3.22 is holomorphic in τ as a consequence of corollary 3.2.7, and moreover, the automorphic behaviour of $F^{\epsilon\nu}$ implies that the function $\mathscr{E}_- + \dot{f}_-$ is $\mathrm{Ad}\,\rho$-automorphic and bounded at the cusps, i.e. an endomorphism of E_ρ, thus a multiple of the identity. The normalization of $F^{\epsilon\nu}$ now implies that

$$\dot{F}_- = 0. \tag{3.23}$$

In general we can write

$$F^{\epsilon\nu} = I + \epsilon\dot{F} + o(|\epsilon|^2).$$

It is easy to see that if $\mathscr{E}$ is an Eichler integral of weight $-2p$ with a representation ρ, then $\mathscr{E}^{(2p+1)}$ is an automorphic form of weight $2p+2$ with the representation ρ. In particular, $\mathscr{E}'_\pm$ are automorphic forms of weight 2 with the representation $\mathrm{Ad}\,\rho$. We readily get

$$\mathscr{E}'_- = \nu^*. \tag{3.24}$$

The one-parameter family of uniformization maps $\Upsilon_{\epsilon\nu}$ generates a family of hermitian metrics on the bundles $E_{\rho^{\epsilon\nu}}$ given as $h^{\epsilon\nu} = (\mathscr{Y}_{\epsilon\nu}\mathscr{Y}^*_{\epsilon\nu})^{-1}$ where $\mathscr{Y}_{\epsilon\nu} \circ J = \Upsilon_{\epsilon\nu}$.

Proposition 3.2.9. *The function $\dot{F}$ satisfies*

$$\dot{F} = -h^{-1}\dot{h}_+, \tag{3.25}$$

and moreover,

$$(\dot{F} \circ J)_{\bar{\tau}} = \Upsilon \cdot \nu \cdot \Upsilon^{-1}. \tag{3.26}$$

Proof. Indeed, since $(\Upsilon^{-1})^*(f^{\epsilon\nu})^* f^{\epsilon\nu}\Upsilon^{-1} = ((F^{\epsilon\nu})^* h^{\epsilon\nu} F^{\epsilon\nu}) \circ J$, it follows from lemma 3.2.6 and $F^0 = I$ that

$$0 = h\dot{F} + \dot{h}_+,$$

which gives equation (3.25). Equation (3.26) is an immediate consequence of (3.21) and Corollary 3.2.7. □

To conclude this section, we observe that if we let $\mathscr{A}_{\epsilon\nu} = -\Upsilon^{-1}_{\epsilon\nu} \cdot (\Upsilon_{\epsilon\nu})_\tau$, its first variation with respect to $\bar{\epsilon}$ is related to $\mathscr{E}_-$ as

$$\dot{\mathscr{A}}_- = [\mathscr{A}, \mathscr{E}_-] - \mathscr{E}'_-. \tag{3.27}$$

3.2.3 Canonical coordinates on $\mathscr{M}(\mathcal{W})$

Let $\mathscr{M}(\mathcal{W})$ denote, for a moment, the set of stable parabolic bundles over $\mathbb{P}^1$ with fixed weights $\mathcal{W}$. Our goal here is to introduce complex coordinates that will turn $\mathscr{M}(\mathcal{W})$ into a complex manifold biholomorphic to the character variety $\mathscr{K}(\mathcal{W})$.

We shall first construct a smooth family $\mathcal{E}$ of parabolic bundles parametrized by $\mathscr{P}$ analogous to the one appearing in [39], and calculate its infinitesimal deformation map [41]. To begin, consider the space $\mathscr{P} \times \mathbb{H} \times \mathbb{C}^r$ as a vector bundle over $\mathscr{P} \times \mathbb{H}$ with the following left action of Γ:

$$\gamma \cdot (\rho, \tau, v) = (\rho, \gamma \cdot \tau, \rho(\gamma)v), \qquad \gamma \in \Gamma.$$

The space of orbits is a vector bundle over $\mathscr{P} \times X$, which after the canonical extensions yields a vector bundle $\mathcal{E}$ over $\mathscr{P} \times \mathbb{P}^1$, with the property that for each $\rho \in \mathscr{P}$, $\mathcal{E}|_{\{\rho\}\times\mathbb{P}^1} \cong E_\rho$.

Proposition 3.2.10. *The infinitesimal deformation map*

$$\mathscr{I}_\rho : T_\rho\mathscr{P} \to H^1(\text{Par End}((E_\rho)_*))$$

of the family $\mathcal{E}$ is the composition of the maps

$$T_\rho\mathscr{P} \to Z^1_P(\Gamma, \operatorname{Ad}\rho) \to H^1_P(\Gamma, \operatorname{Ad}\rho) \to \mathfrak{S}_2(\Gamma, \operatorname{Ad}\rho_{\mathbb{C}}) \to H^1(\text{Par End}((E_\rho)_*)).$$

In particular, it is surjective.

Proof. The infinitesimal deformation map $\mathscr{I}_\rho$ of the family $\mathcal{E}$ is defined by assigning to a tangent vector $v \in T_\rho\mathscr{P}$ the Čech 1-cocycle on a cover $\{\mathscr{U}_i\}_{i=1}^m$ of $\mathbb{P}^1$, considered as a local section of $E_{\operatorname{Ad}\rho_{\mathbb{C}}}|_{\mathscr{U}_{ij}}$ in terms of the trivialization at $\mathscr{U}_i$, given as $\dot{g}_{ij}(\rho) \cdot g_{ij}(\rho)^{-1}$, where

$$\dot{g}_{ij}(\rho) = (\mathrm{d}g_{ij}(\rho'))\,|_{\rho'=\rho}(v)$$

and $\{g_{ij}(\rho)\}$are the transition functions of $\mathcal{E}$ over the cover $\{\mathscr{U}_i \times \mathscr{M}(\mathcal{W})\}_{i=1}^m$.

Now, recall the transition functions (1.20) over the cover $\{\mathscr{U}_i\}_{i=1}^m$ of $\mathbb{P}^1$ specialized to the representation $\operatorname{Ad}\rho_{\mathbb{C}}$. Using duality and following the proof of Theorem 2.1.6 we see that these can be calculated in terms of the cusp form arising from $L_{ES}^{-1}([z])$, where $z = \delta(v) \in Z^1_P(\Gamma, \operatorname{Ad}\rho)$, and δ is the map that identifies $T_\rho\mathscr{P}$ and $Z^1_P(\Gamma, \operatorname{Ad}\rho)$ following Proposition 3.1.1. □

This theorem implies that we have overdetermined the possible infinitesimal deformations of a given stable parabolic bundle, a fact that we know from the Mehta-Seshadri theorem. We proceed to introduce a system of coordinates in a neighbourhood of a point of $\mathscr{M}(\mathcal{W})$ analogous to the Bers coordinates on Teichmüller spaces.

Let us call $\mathscr{F}$ the map

$$\mathscr{F}:\mathscr{M}(\mathcal{W})\to\mathscr{K}(\mathcal{W}),$$

given by sending an equivalence class of stable parabolic bundles $[E_*]$ to the equivalence class of unitary representations $[\rho]$ so that $[E_*]=[(E_\rho)_*]$.

Theorem 3.2.11. *With respect to the complex coordinates* $(\epsilon_1,\cdots,\epsilon_s)$*, given by choosing an arbitrary basis* $\{\nu_1,\cdots,\nu_s\}\subset\overline{\mathfrak{S}_2(\Gamma,\operatorname{Ad}\rho_{\mathbb{C}})}$*, the differential of* $\mathscr{F}$ *at the point* $[(E_\rho)_*]$ *is given by*

$$\mathrm{d}\mathscr{F}_{[(E_\rho)_*]}(\nu_i)=-2L_{ES}(\nu_i^*).\tag{3.28}$$

Proof. Let $\epsilon=x+\sqrt{-1}y$. By restriction of scalars, a complex basis $\{\nu_1,\cdots,\nu_s\}$ corresponds to the real basis $\left\{\nu_1,\sqrt{-1}\nu_1,\cdots,\nu_s,\sqrt{-1}\nu_s\right\}$. If we now interpret any given ν, $\sqrt{-1}\nu$ as tangent vectors at $[(E_\rho)_*]$, it follows that their image under the differential $\mathrm{d}\mathscr{F}$ would be equal to the classes of the unitary parabolic cocycles $\rho_x|_{\epsilon=0}\cdot\rho^{-1}$, $\rho_x|_{\epsilon=0}\cdot\rho^{-1}$.

Now, we have seen in section 3.2.2 that whenever $\nu\in\overline{\mathfrak{S}_2(\Gamma,\operatorname{Ad}\rho_{\mathbb{C}})}$, the Eichler integral associated to ν^* is equal to $\mathscr{E}_-$ (equation (3.24)), and its parabolic 1-cocycle is $z_{\nu^*}=-\dot{\rho}_-\cdot\rho^{-1}$. It follow from remark 14 that

$$L_{ES}(\nu^*)=\frac{z_{\nu^*}-z_{\nu^*}^*}{2}=-\frac{1}{2}(\rho_x|_{\epsilon=0})\cdot\rho^{-1}$$

and

$$L_{ES}\left(-\sqrt{-1}\nu^*\right)=\frac{z_{\nu^*}+z_{\nu^*}^*}{2\sqrt{-1}}=-\frac{1}{2}(\rho_y|_{\epsilon=0})\cdot\rho^{-1}.$$

This concludes the proof. □

Remark 15. Theorem 3.2.11 shows indirectly that the almost complex structure on $\mathscr{K}(\mathcal{W})$ induced by the Eichler-Shimura isomorphism (in the same spirit of [39]) would be integrable provided that the complex coordinates on $\mathscr{M}(\mathcal{W})$ define a holomorphic structure. This is guaranteed by the next result, which can be found in [51] and will be of paramount importance.

Theorem 3.2.12. *Let* $\mathscr{I}:(\mathbb{D}_\delta)^s\subset\overline{\mathfrak{S}_2(\Gamma,\operatorname{Ad}\rho_{\mathbb{C}})}\to\mathscr{M}(\mathcal{W})$ *be the local embedding given by* $\epsilon\nu\mapsto[(E_{\rho^{\epsilon\nu}})_*]$*. Then* $d\mathscr{I}$ *is given as*

$$d\mathscr{I}(\mu)=\mathrm{P}_{\operatorname{Ad}\rho_{\mathbb{C}}^{\epsilon\nu}}\left(f^{\epsilon\nu}\mu(f^{\epsilon\nu})^{-1}\right).\tag{3.29}$$

where $\mathrm{P}_{\operatorname{Ad}\rho_{\mathbb{C}}^{\epsilon\nu}}$ *is the projection operator defined in section 2.2.*

3.2.4 The evenly split locus

So far, we don't know much about the splitting type of the vector bundles E_ρ. H. Röhrl proved in [44] that on any holomorphic family $\mathcal{F} \to \mathscr{M} \times \mathbb{P}^1$ of vector bundles over $\mathbb{P}^1$ parametrized by a complex manifold $\mathscr{M}$, the Birkhoff-Grothendieck theorem still holds in the complement of a codimension 1 analytic set in $\mathscr{M}$ (which could be empty). However, as we have discussed, there are many potential splittings. Our next goal is to determine the generic splitting that the stable parabolic bundles have.

Lemma 3.2.13. *Let $\mathcal{F}$ be a family of vector bundles over $\mathbb{P}^1$ parametrized by a complex manifold $\mathscr{M}$. If $E = \mathcal{F}|_{\{x_0\}\times\mathbb{P}^1}$ is evenly split for some $x_0 \in \mathscr{M}$, then there exist a neighbourhood $x_0 \in \mathscr{U}$ so that $\mathcal{F}|_{\mathscr{U}\times\mathbb{P}^1}$ is evenly split.*

Proof. Recall that the evenly split property is equivalent to the vanishing of $H^1(\mathrm{End}(E))$. the result then follows from the upper semicontinuity of cohomology. $\square$

Theorem 3.2.14. *The generic member of $\mathscr{M}(\mathcal{W})$ is evenly split.*

Proof. It is a well-known fact that every vector bundle E over a Riemann surface which is generated by its global sections admits an exact sequence

$$0 \to \mathcal{O}^{r-1} \to E \to \det(E) \to 0.$$

(a proof of this fact can be found in [3]).

If $E_\rho = \oplus_{j=1}^r \mathcal{O}(a_j)$, $a_1 \geq \cdots \geq a_r$, then $E_\rho \otimes \mathcal{O}(-a_r)$ is generated by its global sections. Therefore, after twisting again we get a short exact sequence

$$0 \to \mathcal{O}(a_r)^{r-1} \to E_\rho \to L \to 0,$$

where $L = \det(E_\rho) \otimes \mathcal{O}(-(r-1)a_r)$. Thus E_ρ belongs to the holomorphic and irreducible family

$$\mathrm{Ext}(L, \mathcal{O}(a_r)^{r-1}).$$

Let $d = \deg(E_\rho) = ar + b$. This is also the case for the bundle

$$\mathcal{O}(a+1)^b \oplus \mathcal{O}(a)^{r-b}$$

and as a consequence of Lemma 3.2.13, there exists a stable parabolic bundle which is also evenly split. The theorem now follows by applying Lemma 3.2.13 once again to the family of stable parabolic bundles parametrized by $\mathscr{M}(\mathcal{W})$. $\square$

We will call *evenly split locus* the analytic open set in $\mathscr{M}(\mathcal{W})$ consisting of stable parabolic bundles which are evenly split, and denote it by $\mathscr{U}(\mathcal{W})$.

3.2.5 The Kähler metric on $\mathscr{M}(\mathcal{W})$

We have learned that, when incarnated as a character variety, the Moduli space $\mathscr{M}(\mathcal{W})$ acquires a natural Kähler structure. There is another way to induce this structure in terms of the identification $T_{[(E_\rho)_*]}\mathscr{M}(\mathcal{W}) \leftrightarrow \overline{\mathfrak{S}_2(\Gamma, \operatorname{Ad}\rho_{\mathbb{C}})}$. If we denote the holomorphic tanget vector corresponding to μ by $\frac{\partial}{\partial\epsilon(\mu)}$, we define a (1,1)-form Ω which at $[(E_\rho)_*] \in \mathscr{M}(\mathcal{W})$ is given as

$$\Omega\left(\frac{\partial}{\partial\epsilon(\mu)}, \frac{\partial}{\partial\overline{\epsilon(\nu)}}\right) = \frac{\sqrt{-1}}{2}\langle\mu,\nu\rangle_P, \tag{3.30}$$

If follows from Theorem 3.2.12 that the previous prescription trully defines a smooth (1,1)-form on $\mathscr{M}(\mathcal{W})$. One of the goals of the next chapter would be to prove that this form is indeed Kähler. We will call it the *parabolic Narasimhan-Atiyah-Bott metric*.

Keeping in mind the analogy with the Teichmüller theory, the reader would not be surprised with such a choice. This metric is nothing but a siutable analog of the Weil-Petersson metric in the context of parabolic bundles. An analogous expression for the case of stable parabolic bundles over Riemann surfaces of genus $g > 1$ appeared first in [38], and later in [5].

3.2.6 A canonical section of $T^*\mathscr{U}(\mathcal{W})$

We have learned in section 2.3 that whenever the vector bundle E_ρ is evenly split, the $\operatorname{Ad}\rho_{\mathbb{C}}$-automorphic form $\mathscr{A}$ is regular. Since the inner product of regular automorphic forms and cusp forms is always well-defined (Proposition 2.1.4), this implies that we can construct a canonical 1-form $\mathscr{Q}$ over the evenly-split locus $\mathscr{U}(\mathcal{W}) \subset \mathscr{M}(\mathcal{W})$ defined over each tangent space $T_{[(E_\rho)_*]}\mathscr{U}(\mathcal{W})$ as the linear functional

$$\mathscr{Q}(\nu) = \langle\mathscr{A}, \nu^*\rangle_P = \iint\limits_F \operatorname{tr}(\mathscr{A}\nu)\mathrm{d}^2\tau. \tag{3.31}$$

As with the Narasimhan-Atiyah-Both (1,1)-form, the smoothness of $\mathscr{Q}$ follows Theorem 3.2.12. The goal of the next chapter will be to understand the relation between these structures.

Chapter 4

The WZNW functional on $\mathscr{M}(\mathcal{W})$

In this chapter, the reader should keep in mind the work of Takhtajan and Zograf [55], since all of our results are WZNW-parallel to the ones on the Liouville theory once the proper analogies are given. For every Riemann surface of type (0,n), they construct an action functional on the space of metrics with prescribed singularities at the set of points $\mathscr{D}$, whose Euler-Lagrange equation turns out to be the Liouville equation, and whose solution corresponds to the hyperbolic metric on $\mathbb{P}^1 \setminus \mathscr{D}$ with cusp singularities on $\mathscr{D}$. Their main result states that each one of these functionals evaluated at their extrema gives rise to a function on Teichmüller space $T_{0,n}$ which is a generating function for the accessory parameters of the uniformization problem on $\mathbb{P}^1 \setminus \mathscr{D}$ and a Kähler potential for the Weil-Petersson metric on $T_{0,n}$. The main difference that the reader should keep in mind is that in their work, the set of points $\mathscr{D}$ is *not fixed* (in fact, such points constitute the Moduli parameters).

4.1 The space of singular Hermitian metrics on E_ρ

Consider the space $C^\infty_{[\rho]}(\mathbb{P}^1 \setminus \mathscr{D}, \mathcal{H}_r)$ of smooth maps $h : \mathbb{P}^1 \setminus \mathscr{D} \to \mathcal{H}_r$ having asymptotic behaviour

$$h(z) \simeq \begin{cases} \left(C_i(0)^*\right)^{-1} |z - z_i|^{2W_i} C_i(0)^{-1} & \text{as} \quad z \to z_i, \quad i = 1 \cdots n-1, \\ |z|^{-2(W_n+N)} & \text{as} \quad z \to \infty. \end{cases} \tag{4.1}$$

Our aim is to construct a functional $S : C^\infty_{[\rho]}(\mathbb{P}^1 \setminus \mathscr{D}, \mathcal{H}_r) \to \mathbb{R}$. Let

$$X^\lambda = \mathbb{C} \setminus \left(\bigcup_{i=1}^{n-1} \{|z - z_i| < \lambda\} \cup \{|z| > 1/\lambda\} \right).$$

Our first observation is that if $0 < \lambda' < \lambda$ are taken to be sufficiently small, the asymptotics (4.1) imply

$$h^{-1}h_z \simeq \begin{cases} \dfrac{C_i(0)W_iC_i(0)^{-1}}{z - z_i} & \text{as} \quad z \to z_i, \quad i = 1, \cdots, n-1, \\ -\dfrac{W_n + N}{z} & \text{as} \quad z \to \infty, \end{cases} \tag{4.2}$$

$$h^{-1}h_{\bar{z}} \simeq \begin{cases} \dfrac{C_i(0)W_iC_i(0)^{-1}}{\bar{z} - \bar{z}_i} & \text{as} \quad z \to z_i, \quad i = 1, \cdots, n-1 \\ -\dfrac{W_n + N}{\bar{z}} & \text{as} \quad z \to \infty. \end{cases} \tag{4.3}$$

hence

$$\operatorname{tr}(h^{-1}h_z h^{-1}h_{\bar{z}}) \simeq \begin{cases} \dfrac{\operatorname{tr}(W_i^2)}{|z - z_i|^2} & \text{as} \quad z \to z_i, \quad i = 1, \cdots, n-1 \\ \dfrac{\operatorname{tr}((W_n + N)^2)}{|z|^2} & \text{as} \quad z \to \infty. \end{cases}$$

Thus

$$\int\limits_{X^{\lambda'} \setminus X^\lambda} \operatorname{tr}(h^{-1}h_z h^{-1}h_{\bar{z}}) \mathrm{d}^2 z \simeq 2\pi(\ln(\lambda) - \ln(\lambda')) \sum_{j=1}^{r} \left(\sum_{i=1}^{n-1} \alpha_{ij}^2 + (\alpha_{nj} + a_j)^2 \right).$$

In particular, the integral of $\operatorname{tr}(h^{-1}h_z h^{-1}h_{\bar{z}})$ on X^λ would be divergent when $\lambda \to 0$. In order to fix this, we define the *regularized kinetic term* of our action functional as the limit

$$S_0[h] = \lim_{\lambda \to 0} S_0^\lambda[h],$$

where

$$S_0^\lambda[h] = \int\limits_{X^\lambda} \operatorname{tr}(h^{-1}h_z h^{-1}h_{\bar{z}}) \mathrm{d}^2 z + 2\pi \ln(\lambda) \sum_{j=1}^{r} \left(\sum_{i=1}^{n-1} \alpha_{ij}^2 + (\alpha_{nj} + a_j)^2 \right).$$

We would like to determine the first variation of the functional S_0. For this let us consider the linear space of Lie algebra-valued smooth maps $u : \mathbb{P}^1 \to n(r) \rtimes \mathbb{R}^r$ with support contained in X^λ for some $\lambda > 0$, and consider the increments be^{tu} (or equivalently $e^{tu^*}he^{tu}$) that consequently preserve the asymptotics (4.1). The first variation is the linear functional on this space defined as

$$\delta S_0[b;u] = \lim_{\lambda\to 0}\left(\frac{d}{dt}\Big|_{t=0} S_0^\lambda[be^{tu}]\right). \tag{4.4}$$

Proposition 4.1.1. *The Euler-Lagrange equations of S_0 are*

$$(h^{-1}h_z)_{\bar z} = \frac{1}{2}[h^{-1}h_z, h^{-1}h_{\bar z}], \tag{4.5}$$

known as equations of principal chiral fields.

Proof. This is a routine computation. The following formulas will be useful:

$$h^{-1}h_z = b^{-1}(b_zb^{-1} + (b_{\bar z}b^{-1})^*)b, \qquad h^{-1}h_{\bar z} = b^{-1}(b_{\bar z}b^{-1} + (b_zb^{-1})^*)b,$$

$$(be^{tu})_z(be^{tu})^{-1} = b_zb^{-1} + tbu_zb^{-1}, \qquad (be^{tu})_{\bar z}(be^{tu})^{-1} = b_zb^{-1} + tbu_{\bar z}b^{-1}$$

then

$$\begin{aligned}
\delta S_0^\lambda[b;u] \quad = \quad & \int\limits_{X^\lambda} \mathrm{tr}\,\big((bu_zb^{-1} + (bu_{\bar z}b^{-1})^*)(b_{\bar z}b^{-1} + (b_zb^{-1})^*) \\
& \qquad + (b_zb^{-1} + (b_{\bar z}b^{-1})^*)(bu_{\bar z}b^{-1} + (bu_zb^{-1})^*)\big)\,\mathrm{d}^2z \\
= \quad & \int\limits_{X^\lambda} \mathrm{tr}\,\big(u_zh^{-1}h_{\bar z} + u_{\bar z}h^{-1}h_z + (u_zh^{-1}h_{\bar z} + u_{\bar z}h^{-1}h_z)^*\big) \\
= \quad & -\int\limits_{X^\lambda} \mathrm{tr}\,\big(u\left((h^{-1}h_{\bar z})_z + (h^{-1}h_z)_{\bar z}\right) + u^*\left((h^{-1}h_{\bar z})_z + (h^{-1}h_z)_{\bar z}\right)^*\big) \\
& -\frac{\sqrt{-1}}{2}\int\limits_{\partial X^\lambda} \mathrm{tr}\,\big(uh^{-1}h_{\bar z} + (uh^{-1}h_z)^*\big)\,\mathrm{d}z \\
& -\frac{\sqrt{-1}}{2}\int\limits_{\partial X^\lambda} \mathrm{tr}\,\big(uh^{-1}h_z + (uh^{-1}h_{\bar z})^*\big)\,\mathrm{d}\bar z.
\end{aligned}$$

The vanishing of u near the punctures implies that the line integrals at ∂X^λ also vanish for sufficiently small λ. Letting $h^{-1}\delta h = u + h^{-1}u^*h$, we can rearrange the above equations after the limit is taken as

$$\begin{aligned}\delta S_0[b;u] &= -\int\limits_{\mathbb{C}} \operatorname{tr}\left(h^{-1}\delta h\left((h^{-1}h_{\bar z})_z + (h^{-1}h_z)_{\bar z}\right)\right) \mathrm{d}^2 z \\ &= -\int\limits_{\mathbb{C}} \operatorname{tr}\left(h^{-1}\delta h\left(2(h^{-1}h_z)_{\bar z} + [h^{-1}h_{\bar z}, h^{-1}h_z]\right)\right) \mathrm{d}^2 z.\end{aligned}$$

We thus see that the functional S_0 has the solutions of (4.5) satisfying the asymptotics (4.1) as its critical points. $\square$

4.2 Cholesky decomposition and the WZNW term

Recall that a matrix h is Hermitian and positive definite, i.e. $h \in \mathcal{H}_r$ if and only if $h = M^*M$ for some invertible matrix M. Since the Gram-Schmidt process on the columns of M allows us to express $M = Ub$ where U is unitary and b upper triangular with positive diagonal terms (a particular case of the Iwasawa decomposition) we conclude that h factors in the form $h = b^*b$ where b is upper triangular with positive diagonal terms (hence invertible). This factorization is known as the *Cholesky decomposition*, even though it was apparently known to Jacobi (and later rediscovered by Toeplitz, see [52]) . It is easy to see that this decomposition is in fact unique. Since b can be factored as pd with p unipotent and d positive diagonal, we see that $\mathcal{H}_n$ acquires the structure of a group in terms of the Cholesky decomposition isomorphic to the semidirect product $N(r) \rtimes (\mathbb{R}^+)^r$, with the action of $(\mathbb{R}^+)^r$ on $N(r)$ being conjugation. Since this factorization gives global canonical coordinates on $\mathcal{H}_r$, we could consider the functions b as the primary parameters in S_0. One could equivalently factor h in the form c^*ac with c unipotent and a diagonal and positive, i.e.

$$b = \sqrt{a}c.$$

The factorization $h = c^*ac$ has the advantage of admiting a particular explicit (yet cumbersome) expression in terms of the coefficients of h that we now describe. Let M be a $r \times r$ matrix and s an integer between 1 and r. Consider arbitrary collections of integers

$$1 \le k_1 < k_2 < \cdots < k_s < r, \qquad 1 \le l_1 < l_2 < \cdots < l_s < r$$

and let $M^{k_1\cdots k_s}_{l_1\cdots l_s}$ be the $s\times s$ matrix constructed from M as

$$(M^{k_1\cdots k_s}_{l_1\cdots l_s})_{ij} = (M)_{k_i l_j}.$$

Define, for $j \leq k$,

$$P_{jk} = \sum_{l_1<\cdots<l_j} \overline{\det\left(M^{l_1\cdots l_j}_{1\cdots j-1 j}\right)} \det\left(M^{l_1\cdots l_j}_{1\cdots j-1 k}\right) \tag{4.6}$$

Notice that if M is invertible, then P_{jj} is real and positive for $j = 1, \cdots, r$ by the standard properties of the minors of a matrix.

Lemma 4.2.1. *If $h = M^*M$ for $M \in \mathrm{GL}(r,\mathbb{C})$, the factorization $h = c^*ac$ is given by the following explicit formulas*

$$(a)_{jj} = \frac{P_{jj}}{P_{j-1j-1}} \tag{4.7}$$

$$(c)_{jk} = \frac{P_{jk}}{P_{jj}} \tag{4.8}$$

It is not hard to prove this lemma using induction and the recursive formulas defining the Cholesky decomposition (following the Gram-Schimdt process) but since this is rather tedious we will omit the proof. A proof of an equivalent result can be found in [23]. As an application, we can use formulas (4.7),(4.8) and the expansions (4.1) to provide asymptotic expansions near the points $z_1, \cdots, z_n$ for the matrix-valued functions b factoring the singular metrics h.

Consider the 3-form $\Theta = \mathrm{tr}(\theta\wedge\theta\wedge\theta)$ on $\mathrm{GL}(r,\mathbb{C})$ defined in terms of the Maurer-Cartan form $\theta = g^{-1}\cdot \mathrm{d}g$. Since θ satisfies the Maurer-Cartan equation $\mathrm{d}\theta + \theta\wedge\theta = 0$, it follows that Θ is closed. Indeed,

$$\mathrm{d}\Theta = -\mathrm{d}(\mathrm{tr}(\mathrm{d}\theta\wedge\theta)) = -\mathrm{tr}(\mathrm{d}\theta\wedge\mathrm{d}\theta) = -\mathrm{tr}(\theta\wedge\theta\wedge\theta\wedge\theta) = 0.$$

The unitary group $\mathrm{U}(r)$ is a deformation retract of $\mathrm{GL}(r,\mathbb{C})$ which implies that the nonzero cohomology groups of the homogeneous space $\mathcal{H}_r$ vanish. In particular, the projection of Θ to $\mathcal{H}_r$ is exact. By a slight abuse of notation, we will also denote this projection by Θ. This fact can be shown explicitly with the aid of the Cholesky decomposition.

Lemma 4.2.2. *Let $\theta_1 = \mathrm{d}b\cdot b^{-1}$. The projection of Θ to $\mathcal{H}_r$ is given as*

$$\Theta = 3\mathrm{d}(\mathrm{tr}(\theta_1\wedge\theta_1^*))$$

Proof.

$$\theta = h^{-1}\mathrm{d}h = b^{-1}(\mathrm{d}b \cdot b^{-1} + (\mathrm{d}b \cdot b^{-1})^*)b$$

$$\mathrm{d}\theta_1 = \theta_1^2, \qquad \mathrm{d}\theta_1^* = -(\theta_1^*)^2$$

$$\begin{aligned} \Theta = \mathrm{tr}\left((\theta_1 + \theta_1^*)^3\right) &= 3\mathrm{tr}(\theta_1^2 \wedge \theta_1^* + \theta_1 \wedge (\theta_1^*)^2) \\ &= 3\mathrm{tr}(\mathrm{d}\theta_1 \wedge \theta_1^* - \theta_1 \wedge \mathrm{d}\theta_1^*) \\ &= \mathrm{d}(3\mathrm{tr}(\theta_1 \wedge \theta_1^*)). \end{aligned}$$

□

The form Θ is a generator of $H^3(\mathrm{GL}(r,\mathbb{C}),\mathbb{Z})$. The classical Wess-Zumino-Novikov-Witten term (abbreviated WZNW) is defined as a multivalued functional by considering cycles $h : \mathbb{P}^1 \to \mathrm{GL}(r,\mathbb{C})$ as boundaries of arbitrary 3-chains over which Θ can be integrated. Since the difference of any 2 3-chains is a 3-cycle, the value of such integrals would be determined up to an integral multiple of a given constant. In our situation (a coset model in physics terminology) every 3-cycle on $\mathcal{H}_r$ is a boundary and we can define the analog of the WZNW term on the space $C^\infty_{[\rho]}(\mathbb{P}^1 \setminus \mathscr{D}, \mathcal{H}_r)$ explicitly as

$$\begin{aligned} W[b] &= \frac{1}{2\sqrt{-1}} \int_{\mathbb{P}^1} \mathrm{tr}(\theta_1 \wedge \theta_1^*) && (4.9) \\ &= \int_{\mathbb{P}^1} \mathrm{tr}\left((b_{\bar z} b^{-1})(b_{\bar z} b^{-1})^* - (b_z b^{-1})(b_z b^{-1})^*\right) \mathrm{d}^2 z. && (4.10) \end{aligned}$$

Proposition 4.2.3. *The Euler-Lagrange equations of $W[b]$ are*

$$[h^{-1}h_z, h^{-1}h_{\bar z}] = 0. \tag{4.11}$$

Proof. To determine the first variation of W we use once again the identities in the begining of the proof of proposition 4.5.

$$\begin{aligned}
\delta W^{\lambda}[b;u] &= \int_{\mathbb{P}^1} \mathrm{tr}\,\Big(bu_{\bar z}b^{-1}(b_{\bar z}b^{-1})^* + b_{\bar z}b^{-1}(bu_{\bar z}b^{-1})^* \\
&\quad -bu_zb^{-1}(b_zb^{-1})^* - b_zb^{-1}(bu_zb^{-1})^*\Big)\,\mathrm{d}^2z \\
&= \int_{\mathbb{P}^1} \mathrm{tr}\,\Big(u\left((h^{-1}h_{\bar z} - b^{-1}b_{\bar z})_z - (h^{-1}h_z - b^{-1}b_z)_{\bar z}\right) \\
&\quad +u^*\left((h^{-1}h_{\bar z} - b^{-1}b_{\bar z})_z - (h^{-1}h_z - b^{-1}b_z)_{\bar z}\right)^*\Big)\,\mathrm{d}^2z \\
&= -\int_{X^\lambda} \mathrm{tr}\,\Big(u\left([h^{-1}h_z, h^{-1}h_{\bar z}] - [b^{-1}b_z, b^{-1}b_{\bar z}]\right) \\
&\quad + u^*\left([h^{-1}h_z, h^{-1}h_{\bar z}] - [b^{-1}b_z, b^{-1}b_{\bar z}]\right)^*\Big)\,\mathrm{d}^2z
\end{aligned}$$

Notice that since $b^{-1}b_z$, $b^{-1}b_{\bar z}$ are upper triangular, their commutator is upper diagonal. Since upper diagonal matrices form an ideal in the algebra of upper triangular matrices, we conclude

$$\delta W[b;u] = -\int_{\mathbb{C}} \mathrm{tr}\,\left(h^{-1}\delta h[h^{-1}h_z, h^{-1}h_{\bar z}]\right)\mathrm{d}^2z$$

$\square$

Definition 4.2.4. *The* ***regularized WZNW functional*** *on the space $C^\infty_{[\rho]}(\mathbb{P}^1\setminus\mathscr{D}, \mathcal{H}_r)$ is defined as $S := S_0 + W$.*

Corollary 4.2.5. *The critical points of the WZNW functional on the space $C^\infty_{[\rho]}(\mathbb{P}^1 \setminus \mathscr{D}, \mathcal{H}_r)$ correspond to the solutions of the equation*

$$\left(h^{-1}h_z\right)_{\bar z} = 0 \tag{4.12}$$

satisfying the asymptotics (4.1).

We should call this equation the *matrix Laplace equation*. We can say more about its solutions.

Proposition 4.2.6. *The matrix Laplace equation has at most one solution satisfying the asymptotics (4.1).*

Proof. Assume that h, h' are hermitian, positive definite functions on $\mathbb{P}^1 \setminus \mathscr{D}$ solving (4.12) with asymptotics (4.1). Since $h^{-1}h_z$, $h'^{-1}h'_z$ are holomorphic and have asymptotics (4.2), it follows that $h^{-1}h_z = h'^{-1}h'_z = \sum_{i=1}^{n-1} \frac{A_i}{z-z_i}$. This and the hermitian property imply that the function $h'h^{-1}$ is holomorphic and antiholomorphic on $\mathbb{P}^1 \setminus \mathscr{D}$, hence a constant. Since $\lim_{z\to z_i}(h'h^{-1})(z) = I \ \forall i$, we conclude that $h = h'$. $\square$

4.3 The main theorems

Once uniqueness of the solution of equation (4.12) has been proved, we can define a function on $\mathscr{M}(r, \mathcal{W})$ by evaluating the functional S at each corresponding critical point, which we will denote as S by a slight abuse of notation. Doubts might be raised on whether this crude function could have any interesting properties at all. The next theorems reveal the deep relation of it and the geometry of $\mathscr{M}(r, \mathcal{W})$.

Theorem 4.3.1. *The function $S : \mathscr{M}(\mathcal{W}) \to \mathbb{R}$ satisfies*

$$\partial S|_{\mathscr{U}(\mathcal{W})} = -2\mathscr{Q} \tag{4.13}$$

that is, for every $\nu \in \mathfrak{S}_2(\Gamma, \operatorname{Ad}\rho)^*$ and ϵ its local coordinate,

$$\left.\frac{\partial S}{\partial \epsilon}\right|_{\epsilon=0} = -2 \iint_F (\mathscr{A}\nu)\, d^2\tau.$$

Proof. To prove the result, we start by showing that $\left.\frac{\partial S^\lambda}{\partial \epsilon}\right|_{\epsilon=0} \to \mathscr{Q}(\nu)$ as

$\lambda \to 0$ in the sense of pointwise convergence.

$$
\begin{aligned}
\frac{\partial S_0^\lambda}{\partial \epsilon} &= \int\limits_{X^\lambda} \operatorname{tr}\left((h^{-1}h_z)_\epsilon h^{-1}h_{\bar z} + h^{-1}h_z(h^{-1}h_{\bar z})_\epsilon\right) \mathrm{d}^2 z \\
&= \int\limits_{X^\lambda} \operatorname{tr}\left(\left((h^{-1}h_\epsilon)_z + [h^{-1}h_z, h^{-1}h_\epsilon]\right) h^{-1}h_{\bar z}\right. \\
&\quad \left. + h^{-1}h_z\left((h^{-1}h_\epsilon)_{\bar z} + [h^{-1}h_{\bar z}, h^{-1}h_\epsilon]\right)\right) \mathrm{d}^2 z \\
&= \int\limits_{X^\lambda} \operatorname{tr}\left((h^{-1}h_\epsilon)_{\bar z} h^{-1}h_z + (h^{-1}h_\epsilon)_z h^{-1}h_{\bar z}\right) \mathrm{d}^2 z \\
&= \int\limits_{X^\lambda} \operatorname{tr}\left((h^{-1}h_\epsilon)_{\bar z} h^{-1}h_z\right) \mathrm{d}^2 z - \int\limits_{X^\lambda} \operatorname{tr}\left(h^{-1}h_\epsilon (h^{-1}h_{\bar z})_z\right) \mathrm{d}^2 z \\
&\quad + \frac{\sqrt{-1}}{2} \int_{\partial X^\lambda} \operatorname{tr}\left(h^{-1}h_\epsilon h^{-1}h_{\bar z}\right) \mathrm{d}\bar z \\
&= 2\int\limits_{X^\lambda} \operatorname{tr}\left((h^{-1}h_\epsilon)_{\bar z} h^{-1}h_z\right) \mathrm{d}^2 z - \int\limits_{X^\lambda} \operatorname{tr}\left((h^{-1}h_\epsilon)[h^{-1}h_{\bar z}, h^{-1}h_z]\right) \mathrm{d}^2 z \\
&\quad + \frac{\sqrt{-1}}{2} \int\limits_{\partial X^\lambda} \operatorname{tr}\left(h^{-1}h_\epsilon h^{-1}\mathrm{d}h\right)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial W}{\partial \epsilon} &= \int\limits_{\mathbb{P}^1} \mathrm{tr}\left((b_{\bar z}b^{-1})_\epsilon (b_{\bar z}b^{-1})^* + (b_{\bar z}b^{-1})(b_{\bar z}b^{-1})^*_\epsilon\right. \\
&\quad \left. - (b_z b^{-1})_\epsilon (b_z b^{-1})^* - (b_{\bar z}b^{-1})(b_{\bar z}b^{-1})^*_\epsilon\right) \mathrm{d}^2 z \\
&= \int\limits_{\mathbb{P}^1} \mathrm{tr}\left(\left((b_\epsilon b^{-1})_{\bar z} + [b_\epsilon b^{-1}, b_{\bar z}b^{-1}]\right)(b_z b^{-1})^*\right. \\
&\qquad + b_{\bar z}b^{-1}\left((b_{\bar\epsilon} b^{-1})_{\bar z} + [b_{\bar\epsilon} b^{-1}, b_{\bar z}b^{-1}]\right)^* \\
&\qquad - \left((b_\epsilon b^{-1} + [b_\epsilon b^{-1}, b_z b^{-1}])_z\right)(b_z b^{-1})^* \\
&\qquad \left. - b_z b^{-1}\left((b_{\bar\epsilon} b^{-1})_z + [b_{\bar\epsilon b^{-1}}, b_z b^{-1}]\right)^*\right) \mathrm{d}^2 z \\
&= \int\limits_{\mathbb{P}^1} \mathrm{tr}\left(b_\epsilon b^{-1}\left([b_{\bar z}b^{-1}, (b_{\bar z}b^{-1})^*]\right.\right. \\
&\qquad \left.\left. + [(b_z b^{-1})^*, b_z b^{-1}] + [(b_z b^{-1})^*, (b_{\bar z}b^{-1})^*]\right)\right) \mathrm{d}^2 z \\
&\quad + \int\limits_{\mathbb{P}^1} \mathrm{tr}\left((b_{\bar\epsilon} b^{-1})^*\left([b_{\bar z}b^{-1}, b_z b^{-1}]\right.\right. \\
&\qquad \left.\left. + [(b_{\bar z}b^{-1}), (b_{\bar z}b^{-1})^*] + [(b_z b^{-1})^*, b_z b^{-1}]\right)\right) \mathrm{d}^2 z \\
&= \int\limits_{\mathbb{P}^1} \mathrm{tr}\left(h^{-1}h_\epsilon [h^{-1}h_{\bar z}, h^{-1}h_z]\right) \mathrm{d}^2 z
\end{aligned}
$$

The last equality follows from the identity

$$
\begin{aligned}
b[h^{-1}h_{\bar z}, h^{-1}h_z]b^{-1} &= [b_{\bar z}b^{-1}, b_z b^{-1}] + [b_{\bar z}b^{-1}, (b_{\bar z}b^{-1})^*] \\
&\quad + [(b_z b^{-1})^*, b_z b^{-1}] + [(b_z b^{-1})^*, (b_{\bar z}b^{-1})^*]
\end{aligned}
$$

and the fact that the product of an upper triangular matrix and the commutator of upper triangular matrices is upper diagonal.

Putting the terms together, after consideration of Lemma 3.2.9 we obtain

$$
\begin{aligned}
\left.\frac{\partial S^\lambda}{\partial \epsilon}\right|_{\epsilon=0} &= 2\int\limits_{X^\lambda} \operatorname{tr}\Big((h^{-1}\dot{h}_+)_{\bar{z}} h^{-1} h_z\Big)\, \mathrm{d}^2 z + \frac{\sqrt{-1}}{2}\int\limits_{\partial X^\lambda} \operatorname{tr}\Big(h^{-1}\dot{h}_+ h^{-1}\mathrm{d}h\Big) \\
&= -2\int\limits_{X^\lambda} \operatorname{tr}\Big((\dot{F})_{\bar{z}} h^{-1} h_z\Big)\, \mathrm{d}^2 z - \frac{\sqrt{-1}}{2}\int\limits_{\partial X^\lambda} \operatorname{tr}\Big(\dot{F} h^{-1}\mathrm{d}h\Big) \\
&= I_1 + I_2.
\end{aligned}
$$

We consider each of these integrals independently. The change of variables formulas $(F^{\epsilon\nu}_{\bar{z}} \circ J)\,\overline{J'} = (F^{\epsilon\nu} \circ J)_{\bar{\tau}}$, $((h^{-1}h_z) \circ J)\, J' = (h \circ J)^{-1}(h \circ J)_\tau$ and equations 3.26 and 2.14 give

$$
\begin{aligned}
I_1 &= -2\int\limits_{F^\lambda} \operatorname{tr}\Big((\dot{F} \circ J)_{\bar{\tau}} (h \circ J)^{-1}(h \circ J)_\tau\Big)\, \mathrm{d}^2\tau \\
&= -2\int\limits_{F^\lambda} \operatorname{tr}\big((\Upsilon \nu \Upsilon^{-1})(\Upsilon \mathscr{A} \Upsilon^{-1})\big)\, \mathrm{d}^2\tau \\
&= -2\int\limits_{F^\lambda} \operatorname{tr}(\mathscr{A}\nu)\, \mathrm{d}^2\tau
\end{aligned}
$$

where $F^\lambda = J^{-1}(X^\lambda) \cap F$. Notice that $I_4 = \mathscr{Q}(\nu) - I_1 \simeq o(1)$ by the cusp form condition.

The identity (3.25) guarantees that in a neighbourhood of the points $z_1, \cdots, z_n$, the function $\dot{F}$ behaves like

$$\dot{F}(z) \simeq f(|z - z_i|)$$

and similarly for h. Since

$$\mathrm{d}h \simeq g(|z - z_i|)\left(\frac{\mathrm{d}z}{z - z_i} + \frac{\mathrm{d}\bar{z}}{\overline{\overline{z - z_i}}}\right) \qquad \text{as} \quad z \to z_i,$$

we conclude that

$$I_2 \simeq o(1) \qquad \text{as} \quad \lambda \to 0.$$

Putting everything together, we have concluded that

$$\left.\frac{\partial S^\lambda}{\partial \epsilon}\right|_{\epsilon=0} = -2\mathscr{Q}(\nu) + o(1)$$

Now, the function $\Upsilon_{\epsilon\nu}$ is differentiable in ϵ (in particular this also holds for its Fourier coefficients $C_i(0)_{\epsilon\nu}$). Since every single factor in the integrals I_2, I_3 and I_4 depends directly on the derivative at of $\Upsilon_{\epsilon\nu}$ at 0, we conclude that such remainders can be estimated uniformly in a neighbourhood of $[E_\rho]$ in $\mathscr{M}(\mathcal{W})$. Of course, the same will hold if we let $\nu = \sum_{i=1}^{s} \epsilon_i \nu_i$ for an arbitrary basis $\{\nu_1, \cdots, \nu_s\}$ of $\overline{\mathfrak{S}_2(\Gamma, \operatorname{Ad}\rho)}$. $\square$

Theorem 4.3.2. *The linear functional $\mathscr{Q}$ satisfies*

$$\bar{\partial}\mathscr{Q} = 2\sqrt{-1}\Omega|_{\mathscr{U}(\mathcal{W})}. \tag{4.14}$$

Proof. Let us consider two arbitrary (but different) forms μ, ν as tanget vectors at $[(E_{\rho^{\epsilon\nu}})_*]$, and let ϵ be an arbitrary complex number belonging to a sufficiently small open disk.

Recall that by their very definition, the local coordinates $\{\epsilon_1, \cdots, \epsilon_s\}$ corresponding to the choice of a basis $\{\nu_1, \cdots, \nu_s\}$ of $\overline{\mathfrak{S}_2(\Gamma, \operatorname{Ad}\rho)}$ determine a local trivialization of the tangent and cotangent bundles, and in this case, the linear map identifying the tangent space at $[(E_{\rho^{\epsilon\nu}})_*]$ with the tangent space at $[(E_\rho)_*]$ is given by (3.29). That is why it is enough evaluate $\mathscr{Q}$ at the vector field $Y_{\mu,\nu}$ over the complex curve $\epsilon \mapsto [(E_{\rho^{\epsilon\nu}})_*]$ give as

$$Y_{\mu,\nu}|_{[(E_{\rho^{\epsilon\nu}})_*]} = P\left(f^{\epsilon\nu}\mu(f^{\epsilon\nu})^{-1}\right),$$

since this way we obtain the coordinate function corresponding to $\mathrm{d}\epsilon$.

As before, let

$$\mathscr{Q}^\lambda(Y_{\mu,\nu}) = \iint\limits_{F^\lambda} \operatorname{tr}\left(\mathscr{A}_{\epsilon\nu} P\left(f^{\epsilon\nu}\mu(f^{\epsilon\nu})^{-1}\right)\right) \mathrm{d}^2\tau$$

A direct computation shows that

$$\begin{aligned}
\left.\frac{\partial \mathscr{Q}^\lambda(Y_{\mu,\nu})}{\partial\bar{\epsilon}}\right|_{\epsilon=0} &= \iint\limits_{F^\lambda} \operatorname{tr}\Big(\dot{\mathscr{A}}_-\mu + \mathscr{A}[\dot{f}_-, \mu]\Big)\, \mathrm{d}^2\tau \\
&= \iint\limits_{F^\lambda} \operatorname{tr}\left(\left([\mathscr{A}, \mathscr{E}_-] - \mathscr{E}_-'\right)\mu - \mathscr{A}[\mathscr{E}_-, \mu]\right) \mathrm{d}^2\tau \\
&= -\iint\limits_{F^\lambda} \operatorname{tr}\left(\mu\nu^*\right) \mathrm{d}^2\tau,
\end{aligned}$$

where the last two equations follow as a consequence of (3.27), (3.22) and

(3.24), respectively. From here we conclude

$$\left.\frac{\partial \mathscr{Q}_{\epsilon\nu}(\mathrm{Y}_{\mu,\nu})}{\partial \bar{\epsilon}}\right|_{\epsilon=0} = -\iint\limits_F \mathrm{tr}(\mu\nu^*)\mathrm{d}^2\tau = 2\sqrt{-1}\Omega\left(\frac{\partial}{\partial\epsilon(\mu)}, \overline{\frac{\partial}{\partial\epsilon(\nu)}}\right).$$

□

Recall that, given a Kähler manifold (M, ω), a *Kähler potential* is a function $u : M \to \mathbb{R}$ that satisfies (either locally or globally)

$$\frac{\sqrt{-1}}{2}\partial\bar{\partial}u = \omega. \tag{4.15}$$

Putting together the Theorems 4.3.1 and 4.3.2, we obtain as a corollary that

$$\partial\bar{\partial}S|_{\mathscr{U}(\mathcal{W})} = 4\sqrt{-1}\Omega|_{\mathscr{U}(\mathcal{W})}, \tag{4.16}$$

that is, *over the evenly split locus in* $\mathscr{M}(\mathcal{W})$, *the function* $-S/2$ *is a Kähler potential for the parabolic Narasimhan-Atiyah-Bott (1,1)-form.*

Appendix A

Logarithms and uniformization

A.1 The Logarithm function

The real logarithm function is defined to be the inverse function of the exponential function, but in the complex case the later is not injective. By restricting the domain we define the complex logarithm as a power series around 1 with radius of convergence 1

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} z^n$$

and the value of it at any other complex number $z \neq 0$ is then gotten by considering analytic continuations. Recalling that the notion of analytic continuation relies on the choice of a path and that another possible definition of $\log(x)$ when x is real and positive is in terms of the definite integral

$$\log(x) = \int_1^x \frac{\mathrm{d}x}{x},$$

we can then try to make the continuations more explicit by considering the integral

$$\int_\gamma \frac{\mathrm{d}z}{z}$$

for any continuous path $\gamma : [0,1] \to \mathbb{C}^*$ with $\gamma(0) = 1$. Since $f(z) = 1/z$ is holomorphic in $\mathbb{C}^*$, this integral only depends on the homotopy class of the path. However, if we allow the paths to wind around 0 we would get a "multivalued function", since $\mathbb{C}^*$ is not simply connected[1]. Riemann was the first to realize that we could make an unambiguous definition of such functions in terms of the universal cover of the domain. In this particular case, the universal cover of $\mathbb{C}^*$ is $\mathbb{C}$ and one possible covering map[2] is given by mapping $\tau \in \mathbb{C}$ to $\exp(2\pi\sqrt{-1}\tau)$. This function is automorphic with respect to the cyclic group Γ_∞ generated by the parabolic transformation $\tau \mapsto \tau + 1$, and has the additional property that its restriction to the upper half plane $\mathbb{H}$ maps it onto the punctured unit disk $\mathbb{D}^*$. Since the action of Γ_∞ on $\mathbb{C}$ (and $\mathbb{H}$) is free, holomorphic and properly discontinuous, it follows that $\Gamma_\infty \setminus \mathbb{C} \cong \mathbb{C}^*$, $\Gamma_\infty \setminus \mathbb{H} \cong \mathbb{D}^*$.

We can see now that if we fix points $\tau_0 \in \mathbb{C}$, $z_0 = \exp(2\pi\sqrt{-1}\tau_0)$ and we consider only the paths starting at z_0, there will be a unique lift to of this path to $\mathbb{C}$ starting at τ_0. In particular, for a loop γ at z_0, the corresponding lift would start at τ_0 and end at the point $\tau_0 + n$ for some $n \in \mathbb{Z}$ where n is the *winding number of* γ,

$$n(\gamma) := \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{\mathrm{d}z}{z}$$

As it was pointed out before, n only depends on the homotopy class of γ, $[\gamma] \in \pi_1(\mathbb{C}^*, z_0)$[3].

We can characterize the function that sends the ending point z of a path γ to the ending point of its lift in the following simple way: First let's notice that the ending point z of a path with 0 winding number starting at z_0 would not lie in the ray through 0 and z_0 and that the lift of this path starting at τ_0 would have ending point $\tau_0 + \frac{1}{2\pi\sqrt{-1}} \log(z/z_0)$ [4]. Since the lift of a loop at z_0 with winding number n would be a path ending at $\tau_0 + n$ and any path γ in $\mathbb{C}^*$ starting at z_0 and ending at some arbitrary point z is homotopic to the product of a loop γ_n at z_0 with winding number $n(\gamma)$ and a path γ_0 joining z_0 and z with 0 winding number, the lift of γ at τ_0 will be a path ending at

[1] Of course we can avoid this by defining $\log(z)$ (as it is customarily done) in a maximal simply connected subdomain of $\mathbb{C}^*$, for instance, the slitted plane $\mathbb{C} \setminus \{az_0 : a \geq 0\}$ for any $z_0 \neq 0$.

[2] This covering is analytic, and any other analytic covering will differ from this by pre-composition with a biholomorphism of $\mathbb{C}$.

[3] From now on we will slightly abuse notation and use the symbol γ to denote indistinctively 1) a loop in a complex domain, 2) the homotopy class of this loop, 3) the corresponding deck transformation in the universal cover. The precise meaning should be clear from the given context.

[4] Or in other words, the logarithm function is well-defined for the choice of the branch $\mathbb{C} \setminus \{az_0 : a \geq 0\}$; the factor $\frac{1}{2\pi\sqrt{-1}}$ comes from the established choice of the uniformization function.

$\tau_0 + \frac{1}{2\pi\sqrt{-1}}\log(z/z_0) + n$.

Definition A.1.1. *We define a function* $\mathscr{L} : \mathbb{C}^* \times \mathbb{Z} \to \mathbb{C}$ *by*

$$\mathscr{L}(z, n) := \tau_0 + \frac{1}{2\pi\sqrt{-1}} \int_\gamma \frac{dz}{z}$$

where γ *is any path starting at* z_0 *and ending at* z *with* $n(\gamma) = n$. *This function satisfies*

$$\exp(2\pi\sqrt{-1}\mathscr{L}(z, n)) = z. \tag{A.1}$$

Remark 16. The introduction of the function $\mathscr{L}$ is done as a way of keeping track of the "multivaluedness" of the logarithm. By its very definition, it also satisfies

$$\mathscr{L}(z, n+1) = \mathscr{L}(z, n) + 1$$

Moreover, the restriction of it to the subdomain $\mathbb{D}^* \times \mathbb{Z}$ has the upper half plane $\mathbb{H}$ as its image, and the restriction of it to $U \times \{0\}$ where U is any simply connected subdomain of $\mathbb{C}^*$ gives a suitable branch of the usual logarithmic function (after multiplication of the factor $2\pi\sqrt{-1}$).

We are now ready to give a cleaner version of a fundamental definition for our future constructions:

Definition A.1.2. *For any matrix* $r \times r$ *matrix* A *with complex entries, we define the function*

$$z^A : \mathbb{C}^* \times \mathbb{Z} \to \mathbb{C}$$

as

$$(z, n) \mapsto \exp(2\pi\sqrt{-1}A\mathscr{L}(z, n)).$$

We define $(z - z_i)^A$ *in a similar way.*

It should be emphasized that this definition depends on the choice of the points $\tau_0 \in \mathbb{C}$, $z_0 \in \mathbb{C}^*$ with $\exp(2\pi\sqrt{-1}\tau_0) = z_0$.

A.2 The logarithm of a unitary matrix

Given a $r \times r$ unitary matrix M, we would like to find all possible solutions of the equation

$$\exp(A) = M \tag{A.2}$$

we would call any such solution *a logarithm* of M.

By the fact M is unitary we know it is diagonalizable and its eigenvalues are unitary complex numbers. If U is the matrix whose columns correspond to a basis of normalized eigenvectors of M, then U is unitary and $M = UDU^*$, where D is the diagonal matrix of eigenvalues of M. Now, since the operation of conjugation commutes with the exponential, we can reduce our problem to the particular case

$$\exp(A) = D \tag{A.3}$$

where D is unitary and diagonal. Furthermore, since a unitary complex number can be uniquely expressed in the form $\exp(2\pi\sqrt{-1}\lambda)$ where $\lambda \in [0, 1)$, we can assume that

$$D = \exp(2\pi\sqrt{-1}\Lambda), \quad \Lambda = \mathrm{diag}(\lambda_1, \cdots, \lambda_r), \quad 0 \leq \lambda_1 \leq \cdots \leq \lambda_r < 1. \tag{A.4}$$

We will call $2\pi\sqrt{-1}\Lambda$ the *canonical solution* of the system (A.2). First we prove the following

Lemma A.2.1. *A $r \times r$ matrix A is diagonalizable if and only if* $\exp(A)$ *is diagonalizable.*

Proof. One direction is obvious since conjugation commutes with exponentiation. Let's now assume that $\exp(A)$ is diagonalizable. We can assume without loss of generality that A is in its Jordan canonical form, and then $\exp(A)$ would be upper triangular (with each triangular block having the same element along the diagonal, namely the exponential of the corresponding eigenvalue of A), but such a matrix can de diagonalizable just if it is already diagonal, and we conclude that A is diagonal. $\square$

Proposition A.2.2. *A matrix A is is skew-Hermitian if and only if* $\exp(A)$ *is unitary.*

Proof. One direction is clear since $A + A^* = 0$ implies $\mathrm{Id} = \exp(A + A^*) = \exp(A)\exp(A^*) = \exp(A)\exp(A)^*$ (since A and $A^* = -A$ commute).

Now if we assume that $\exp(A)$ is unitary then A has to be diagonalizable by lemma (A.2.1) which in turn implies that A is normal, i.e. $AA^* = A^*A$. Thus

$$\mathrm{Id} = \exp(A)\exp(A)^* = \exp(A + A^*)$$

but we know that $A = UBU^*$, where B is diagonal and U unitary so $A^* = U\Lambda^*U^*$, and then

$$A + A^* = U(B + B^*)U^* = 0$$

since B is the matrix of eigenvalues of A, which are imaginary since their exponentials are unitary complex numbers. $\square$

From the previous proposition we know that it is enough to restrict the matrix A to the skew-Hermitian case. These are precisely the diagonalizable matrices with purely imaginary eigenvalues. Hence if we assume that $A = U(2\pi\sqrt{-1}\Lambda')U^*$ and $\exp(A) = D$, then from (A.4) we conclude that up to a permutation, $\Lambda' = \Lambda + N$, where Λ is te canonical solution and $N = \mathrm{diag}(n_1, \cdots, n_r), \quad n_i \in \mathbb{Z}$.

Now, it is still possible that there is a unitary matrix U that commutes with Λ, and in this case $A = 2\pi\sqrt{-1}U(\Lambda + N)U^*$ would be the most general type of a solution (provided U does not commute with N). Hence we can state:

Corollary A.2.3. *Every solution of the equation* (A.3) *is of the form*

$$A = 2\pi\sqrt{-1}U(\Lambda + N)U^*,$$

where N is a diagonal matrix of integers, and U is a unitary matrix that commutes with Λ.

We notice that if in particular the elements of the canonical solution are all different, $0 \leq \lambda_1 < \lambda_2 < \cdots < \lambda_r < 1$, then any other solution would have to differ from the canonical solution by a diagonal matrix of multiples of $2\pi\sqrt{-1}$.

Appendix B

Singular metrics and connections

B.1 The Čech point of view

The Čech conception of vector bundles in terms of transition functions allows us to have a simple understanding of the different constructions we can associate to it by 1) first classifying them on trivial bundles and then 2) understanding the corresponding compatibility condition for its form on different patches. The formulas developed hereafter, though simple in nature, could be easily confused; Special emphasis is made on their differences with the ones appearing when changes of frames are considered (cf. [24]).

Given a complex vector bundle $E \to M$, *a connection* is a $\mathbb{C}$-linear mapping $\nabla : \mathcal{A}^0(E) \to \mathcal{A}^1(E)$ satisfying Leibniz rule:

$$\nabla(fs) = \mathrm{d}f \otimes s + f\nabla(s), \qquad \forall f \in C^\infty(\mathscr{U}),\ s \in \mathcal{A}(E)(\mathscr{U}),\ \mathscr{U} \subset M, \tag{B.1}$$

and its *curvature* is the resulting map ∇^2 after extending as a graded derivation. It can be readily verified that the latter map is $C^\infty(M)$-linear, or equivalently is defined as the action of a section of $\wedge^2(M) \otimes \mathrm{End}(E)$. Similarly, the *difference* of two connections is $C^\infty(M)$-linear:

$$(\nabla_1 - \nabla_2)(fs) = f(\nabla_1 - \nabla_2)(s),$$

and the space of connections on E form an affine space parametrized by $\mathcal{A}^1(E)$. If $E = M \times \mathbb{C}^r$, exterior differentiation defines a connection, so any connnection will be of the form $\mathrm{d} + \theta$ with $\theta \in A^1(M \times \mathbb{C}^{r\times r})$. Thus, locally, a connection is an operator of the form $\mathrm{d} + \theta$, $\theta \in \mathcal{A}^1(\mathrm{End}(E))(\mathscr{U})$ and equivalently, it can be defined in terms of the Čech cocycle $\{\mathscr{U}, g_{ij}\}$ as a collection $\{\mathscr{U}_i, \theta_i\}$,

$\theta_i \in \mathcal{A}^1(\mathrm{End}(E))(\mathscr{U}_i)$, such that on $\mathscr{U}_{ij}$,

$$\theta_i = g_{ij}\theta_j g_{ij}^{-1} - (\mathrm{d}g_{ij})g_{ij}^{-1}. \tag{B.2}$$

since sections of E correspond to collections of local functions $\{s_i : \mathscr{U}_i \to \mathbb{C}^r\}$ such that on $\mathscr{U}_{ij}$, $s_i = g_{ij}s_j$, and (B.2) ensures that on $\mathscr{U}_{ij}$,

$$\nabla|_{\mathscr{U}_i}(s_i)|_{\mathscr{U}_{ij}} = \nabla|_{\mathscr{U}_i}(g_{ij}s_j)|_{\mathscr{U}_{ij}} = g_{ij}\nabla|_{\mathscr{U}_j}(s_j)|_{\mathscr{U}_{ij}}.$$

The following is a classical result.

Theorem B.1.1. *Let E be a complex vector bundle over a smooth manifold M. Then the following are equivalent:*

1. *E is isomorphic to the quotient $\rho \setminus (\widetilde{M} \times \mathbb{C}^r)$, for a given representation $\rho : \pi_1(M) \to \mathrm{GL}(r, \mathbb{C})$.*

2. *E is defined by a set of constant transition functions.*

3. *E admits a connection with vanishing curvature*

For the proof, the reader can consult [31] (although the equivalence of 1 and 2 follows from a direct generalization of the construction given in proposition 1.4.1). The word *flat* is used to denote such a bundle and such a connection. This connection is precisely the one for which the local trivializations giving the constant transition functions correspond to a frame of parallel sections. The nature of it is rather transparent: a section of a bundle E isomorphic to a quotient by a reprensentation ρ is equivalent to a a function $f : \widetilde{M} \to \mathbb{C}^r$ satisfying $f(\gamma x) = \rho(\gamma)f(x)$, and the exterior derivative $\mathrm{d}f$ would also have the same automorphic property and thus correspond to an element of $\mathcal{A}^1(E)$. It is readily seen that this procedure indeed defines a connection on E. In particular, the bundle $\rho \setminus \mathbb{H} \times \mathbb{C}^r \to X$ constructed before admits a flat connection.

Definition B.1.2. *A* ***holomorphic connection*** *on a holomorphic vector bundle $E \to M$ is a $\mathbb{C}$-linear map of sheaves $\nabla : \mathcal{O}(E) \to \Omega^1(E)$, which satisfies the Leibniz rule*

$$\nabla(fs) = \partial f \otimes s + f\nabla(s), \qquad \forall f \in \mathcal{O}(\mathscr{U}),\ s \in \mathcal{O}(E)(\mathscr{U}),\ \mathscr{U} \subset M,$$

We should emphasize that such connections do not necessarily exist in general (cf. [3]). The following proposition will throw some light on this issue when M is a Riemann surface, but before, one more definition.

Definition B.1.3. *We say that a connection* ∇ *on a holomorphic vector bundle is* ***compatible with the complex structure*** *if* $\nabla^{0,1} = \bar{\partial}$*, where* $\nabla^{0,1}$ *denotes the* $(0,1)$*-part of* ∇*.*

Proposition B.1.4. *Given a holomorphic vector bundle* E *over a Riemann surface* S*, there is bijective correspondence between holomorphic connections on* E *and flat connections on* E *compatible with the complex structure*

Proof. A holomorphic connection ∇ on E satisfies $\nabla^{0,1} = \bar{\partial}$ and is thus compatible with the complex structure. It is given locally for a sufficiently small $\mathscr{U} \subset S$ as $\mathrm{d} + \theta$, where θ is a $\Omega^1(\mathscr{U})$-valued $r \times r$ matrix. Since S is 1-dimensional and θ holomorphic, $\mathrm{d}\theta - \theta \wedge \theta = 0$ so ∇ is flat. Conversely, if ∇ is a flat connection compatible with the complex structure we have that $\nabla = \nabla^{1,0} + \bar{\partial}$ and $\nabla^2 = 0$. Looking at the $(1,1)$-part we have $\nabla^{1,0}\bar{\partial} + \bar{\partial}\nabla^{1,0} = 0$, so if $s \in \mathcal{O}(E)$

$$0 = (\nabla^{1,0}\bar{\partial} + \bar{\partial}\nabla^{1,0})s = \bar{\partial}(\nabla^{1,0}s)$$

this is, $\nabla^{1,0}s \in \Omega^1(E)$. $\square$

We are led to conclude that $\rho \setminus \mathbb{H} \times \mathbb{C}^r$ admits a canonical holomorphic connection. Since we are ultimately interested in the bundle E_ρ, it would be desirable to extend this holomorphic connection to it. This, however, couldn't possibly be a holomorphic extension, since otherwise we would conclude that E_ρ is itself a quotient by a representation of the fundamental group of $\mathbb{P}^1$ which is trivial and hence the trivial bundle. We are undoubtedly required to relax and extend our definitions.

Definition B.1.5. *Let* $D = \sum V_i$ *be a smooth divisor in* M*. A* ***logarithmic connection*** *on a holomorphic vector bundle* $E \to M$ *is a* $\mathbb{C}$*-linear map of sheaves* $\nabla : \mathcal{O}(E) \to \mathcal{O}(E) \otimes \Omega^1(\mathrm{Log}(D))$ *satisfying the Leibniz rule, where* $\Omega^1(\mathrm{Log}(D))$ *is the sheaf of rational 1-forms on* M *with logarithmic poles on* D*.*

In terms of the Čech cocycle $\{\mathscr{U}_{ij}, g_{ij}\}$, a logarithmic connection is given by a collection $\{\mathscr{U}_i, \theta_i\}$ where the θ_i are meromorphic matrix-valued 1-forms on $\mathscr{U}_i$ with simple poles in $D \cap \mathscr{U}_i$ related by the rule (B.2). More generally, a *meromorphic connection* is one whose singularities are poles of higher order. We are only interested in the former.

A meromorphic frame $\{\mathscr{U}_i, \phi_i\}$ of E defines a logarithmic connection in a canonical way:

$$-\partial(\phi_i)\phi_i^{-1} = -\partial g_{ij} g_{ij}^{-1} + g_{ij}(-\partial(\phi_j)\phi_j^{-1})g_{ij}^{-1} \qquad \text{on } U_{ij},$$

thus the collection $\{\mathscr{U}_i, \theta_i\}$, where $\theta_i = -\partial(\phi_i)\phi_i^{-1}$, defines a logarithmic connection on E. Given that on $\mathscr{U}_i$

$$(\partial + \theta_i)\phi_i = 0,$$

we see that $\{\mathscr{U}_i, \phi_i\}$ is a frame of parallel sections of the induced connection.

Proposition B.1.6. *A frame of parallel sections of a Logarithmic connection is uniquely determined up to right multiplication by an invertible matrix.*

Proof. If two frames differ by right multiplication of an element of $\mathrm{GL}(r, \mathbb{C})$, $\phi_{i2} = \phi_{i1}M$, then $-\partial(\phi_{i2})\phi_{i2}^{-1} = -\partial(\phi_{i1})\phi_{i1}^{-1}$ and similarly, if for two meromorphic frames the induced connections coincide, then

$$\begin{aligned}
\partial(\phi_{i1}^{-1}\phi_{i2}) &= \partial(\phi_{i1}^{-1})\phi_{i2} + \phi_{i1}^{-1}\partial(\phi_{i2}) \\
&= -\phi_{i1}^{-1}\partial(\phi_{i1})\phi_{i1}^{-1}\phi_{i2} + \phi_{i1}^{-1}\partial(\phi_{i2}) \\
&= 0
\end{aligned}$$

therefore $\phi_{i2} = \phi_{i1}M$, $M \in \mathrm{GL}(r, \mathbb{C})$. □

In terms of the transition functions, a Hermitian metric on a bundle E is given by a collection of hermitian and positive definite functions $h_i : \mathscr{U}_i \to \mathcal{H}_n$ satisfying

$$h_j = g_{ij}^* h_i g_{ij}. \tag{B.3}$$

If the eigenvalues of the functions h_i are allowed to become zero along a divisor in M, the metric is called *pseudo-Hermitian*. It is easy to see that given a pseudo-Hermitian metric in a holomorphic vector bundle, there exist a unique metric-compatible connection, which we will refer to as a metric connection. It is explicitly given as

$$\theta_i = h_i^{-1}\partial h_i. \tag{B.4}$$

Given a logarithmic connection, consider a meromorphic frame of parallel sections. Then the functions

$$h_i = (\phi_i \phi_i^*)^{-1}$$

are hermitian and satisfy

$$h_j = (\phi_j \phi_j^*)^{-1} = (g_{ij}^{-1}\phi_i\phi_i^* g_{ij}^{*-1})^{-1} = g_{ij}^* h_i g_{ij}.$$

Also, it is immediately verified that

$$h_i^{-1}\partial h_i = -\partial(\phi_i)\phi_i^{-1} = \theta_i.$$

However, it is not clear at all that the eigenvalues of these hermitian functions should be nonnegative (and then define a hermitian metric making the Logarithmic connection a metric connection). Since for any divisor D in a complex manifold M, $M \setminus D$ is connected, the eigenvalue functions will have either positive or negative values in the complement of the divisor defined by the functions $\{h_i\}$.

Given a logarithmic connection on a vector bundle, we can associate another logarithmic connection on the determinant bundle called the ***trace connection***. Since the transition functions of the latter are $\{\mathscr{U}_{ij}, \det(g_{ij})\}$, it follows from the Liouville-Ostrogradski formula that the trace connection is given by the collection of 1-forms $\{\mathscr{U}_i, \mathrm{tr}(\theta_i)\}$. The residues of these 1-forms are well-defined as a number associated to a point in S.

Proposition B.1.7. *Given a logarithmic connection ∇ on a vector bundle E over a compact Riemann surface S, we have the equality*

$$\sum_{x \in S} \mathrm{Res}_x \nabla = -\deg(E) \tag{B.5}$$

Proof. If E is a line bundle, its degree is equal to the sum of the orders of any meromorphic section s. Given a section s, let us consider a cover $\{U_i\}$ of S such that the zeros and poles of s do not lie simultaneously in two different open sets of the cover. The section is equivalent to a collection $\{U_i, s_i\}$. As a consequence of the previous convention we have

$$\deg(E) = \sum_i \mathrm{ord}_{x \in U_i}(s_i) = \sum_i \mathrm{Res}_x(\mathrm{d}(s_i)s_i^{-1}) \tag{B.6}$$

Now, since on U_{ij} $s_i = g_{ij}s_j$, we have that

$$-\mathrm{d}(s_i)s_i^{-1} = g_{ij}(-\mathrm{d}(s_j)s_j^{-1})g_{ij}^{-1} - \mathrm{d}(g_{ij})g_{ij}^{-1}$$

and therefore the collection $\{U_i, -\mathrm{d}(s_i)s_i^{-1}\}$ determines a logarithmic connection ∇_s on E and by definition the right hand side of (B.6) is equal to minus the sum of the residues of ∇_s. Then

$$\sum_{x \in S} \mathrm{Res} \nabla_s = -\deg(E).$$

Since the difference of two connections corresponds to a 1-form on S and the sum of the residues of any 1-form on a compact Riemann surface is equal to zero, the result is true as well for any meromorphic connection ∇.

By definition, $\deg(E) = \deg(\Lambda^r E)$, and the result follows from the Liouville-

Ostrogradski formula for the trace connection on the determinant bundle and the previous result. □

B.2 Linear systems and monodromy

For any meromorphic 1-form ω on a compact Riemann surface

$$\sum_{p\in S} \mathrm{Res}_p \omega = 0$$

Consider a matrix valued 1-form ω on $\mathbb{P}^1$ with only simple poles. Make the singular points $z_1, \ldots, z_n$ and moreover assume none of them is ∞. Define

$$R_i = \mathrm{Res}_{z_i}\omega$$

Consider the new form (on the finite plane first)

$$\omega'(z) = \omega(z) - \sum_{i=1}^{n} \frac{R_i \mathrm{d}z}{z - z_i}$$

such that ω' is holomorphic on $\mathbb{C}$. Now make $\zeta = 1/z$;

$$\omega'(\zeta) = \omega(\zeta) - \sum_{i=1}^{n} \frac{R_i \mathrm{d}\zeta}{\zeta(1 - z_i\zeta)} = \omega(\zeta) + (\sum_{i=1}^{n} R_i)\frac{\mathrm{d}\zeta}{\zeta} + \sum_{i=1}^{n} \frac{R_i z_i \mathrm{d}\zeta}{1 - z_i\zeta}$$

so the form ω' is holomorphic also at ∞. This implies that $\omega' \equiv 0$. Hence

$$\omega(z) = \sum_{i=1}^{n} \frac{R_i \mathrm{d}z}{z - z_i}.$$

The same calculation shows that any form $\sum_{i=1}^{n} \frac{R_i \mathrm{d}z}{z - z_i}$ such that $\sum_{i=1}^{n} R_i = 0$ is holomorphic at ∞. This defines a linear system on $\mathbb{P}^1$ by

$$\mathrm{d}w = \Omega w,$$

which is known as a *Fuchsian system*. It is a classical result that every Fuchsian system is regular and that it determines a representation of $\mathbb{P}^1 \setminus \{z_1, \cdots, z_n\}$ into $\mathrm{GL}(r, \mathbb{C})$.

The monodromy of such a system is not easy to find. However, if the matrices R_i commute, the system has a fundamental solution given by

$$Y(z) = \Pi_{i=1}^{n}(z - z_i)^{R_i}$$

and monodromy

$$M_i = \exp(2\pi\sqrt{-1}R_i).$$

Definition B.2.1. *A regular linear system (not necessarily Fuchsian) is **reducible** if the matrix-valued function defining it has the form*

$$A(z) = \begin{pmatrix} A' & * \\ 0 & A'' \end{pmatrix} \tag{B.7}$$

for some square matrix-valued functions A', A''.

Proposition B.2.2. *Given an r-dimensional representation ρ of $\pi_1(X)$, the associated linear system is irreducible if and only if its monodromy representation is irreducible.*

Proof. If the system is reducible and $A(z)$ has the form (B.7), then the solutions of the system determined by A' are naturally included in the space of solutions of our system. The monodromy ρ' of this subsystem is a subrepresentation of ρ. Conversely, if the monodromy representation is reducible then it contains a nontrivial subrepresentation of strictly smaller dimension $r' < r$. We can then assume without any loss of generality that ρ is in block upper triangular form,

$$\rho(\gamma_i) = M_i = \begin{pmatrix} M_i' & M_i''' \\ 0 & M_i'' \end{pmatrix}$$

so the subrepresentation ρ' is given by $\rho'(\gamma_i) = M_i'$. We claim that the function Ψ corresponding to this system is equivalent to a block-upper triangular one. In principle it has the block form

$$\Psi = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

with $C \neq 0$. We can prove this by constructing a block-upper triangular function Ψ' with the same monodromy ρ and regular single valued parts at $z_1, \cdots, z_{n-1}$, which by uniqueness will differ from Ψ by left multiplication of an invertible matrix M. We first associate the function $\Psi_1 : \mathbb{H} \to \mathrm{GL}(r', \mathbb{C})$ arising from ρ' . This will constitute the first block of Ψ'. Although the matrices $\{M_i''\}$ do not form a subrepresentation of ρ, they form a representation of $\pi_1(X)$ which we will denote by ρ''. We can associate as well a fucntion Ψ_2

having ρ'' as its monodromy. We consider the function

$$\Psi' = \begin{pmatrix} \Psi_1 & B \\ 0 & \Psi_2 \end{pmatrix}$$

A simple calculation shows that it has the same monodromy as Ψ. Since $\det(\Psi) = \det(\Psi')\det(\Psi'')$, the single valued parts are regular at $z_1, \cdots, z_{n-1}$ as well. The uniqueness of Ψ implies that $\Psi = M\Psi'$ for some $M \in \mathrm{GL}(r, \mathbb{C})$. $\square$

Proposition B.2.3. *If the representation ρ is unitary and reducible, the function Ψ is equivalent to a block-diagonal function. The associated matrix-valued function $A(z)$ has the form*

$$\begin{pmatrix} A' & 0 \\ 0 & A'' \end{pmatrix} \tag{B.8}$$

Proof. This case is simpler to deal with, since a reducible unitary representation is a direct sum $\rho \cong \rho_1 \oplus \rho_2$. Each one of the subrepresentations has an associated matrix valued function Ψ_1, Ψ_2. The function

$$\Psi' = \begin{pmatrix} \Psi_1 & 0 \\ 0 & \Psi_2 \end{pmatrix}$$

has ρ as its monodromy representation. By the uniqueness of the function Ψ we conclude that $\Psi = M\Psi'$ for some $M \in \mathrm{GL}(r, \mathbb{C})$. The associated fuchsian linear system on X would then be conjugated to one of the form (B.8) $\square$

Appendix C

Parabolic structures on vector bundles

C.1 The category of parabolic bundles

Definition C.1.1. *Let S be a compact Riemann surface and $E \to S$ a holomorphic vector bundle of rank r. Given a finite set $\mathscr{D} = \{z_1, \cdots, z_n\} \subset S$, a **parabolic structure** of E at $\mathscr{D}$ consists of*

1. *decreasing flags at each $\pi^{-1}(z_i)$*

$$E_{z_i} = F^1_{z_i} \supset F^2_{z_i} \supset \cdots \supset F^{l_i+1}_{z_i} = \{0\}$$

2. *constants $0 \leq \alpha_{i1} < \cdots < \alpha_{il_i} < 1$, and multiplicities $m_{i1}, \cdots, m_{il_i}$ such that $m_{ij} = dim(F^j_{z_i}/F^{j+1}_{z_i})$ (hence $\sum_{j=1}^{l_i} m_{ij} = r$), called the **system of weights** of the flags. We call α_{ij} the **weight** of $F^j_{z_i}$.*

*A **parabolic bundle** E_* is a holomorphic vector bundle E endowed with a parabolic structure. The **parabolic degree** is defined as*

$$\text{Par deg}(E_*) = \deg(E) + \sum_{i=1}^{n} \sum_{j=1}^{l_i} m_{ij}\alpha_{ij}. \tag{C.1}$$

Definition C.1.2. *Given two parabolic vector bundles E_*, E'_* over S with weights $\{\alpha_{ij}\}$, $\{\alpha'_{ij}\}$ a **parabolic homomorphism** is a homomorphism $T \in Hom(E, E')$ with the additional property that*

$$T(F^j_{z_i}) \subset F^{k+1}_{z_i} \qquad \textit{whenever} \quad \alpha_{ij} > \alpha'_{ik}.$$

In particular, the notion of isomorphism makes sense only if the systems of weights coincide.

C.2 Induced parabolic structures

To every parabolic bundle E_* we can associate the subsheaf of the sheaf of endomorphisms of E consisting of the local germs of parabolic endomorphisms. It turns out to be locally free and thus a correspondent vector bundle Par End(E) whose global sections correspond to the parabolic endomorphisms of E_*. We can induce a canonical parabolic structure on Par End(E) and thus turn it into a parabolic bundle. On each fiber $\pi^{-1}(z_i)$,

As it has been remarked in section 1.6, the following identity holds by definition

$$\text{Par End}(E_\rho)_* \cong (E_{\text{Ad}\,\rho})_* \,. \tag{C.2}$$

A subbundle E' of a parabolic bundle E_* acquires a natural parabolic structure by restriction: the flag of E'_* at z_i is simply

$$E'_{z_i} = F^1_{z_i} \cap E'_{z_i} \supset F^2_{z_i} \cap E'_{z_i} \supset \cdots \supset F^{l_i+1}_{z_i} \cap E'_{z_i} = \{0\}.$$

Of course, the lenght of this flag might be smaller than that of the flag of E_* as different intersections could coincide. We take care of this by associating to $F^j_{z_i} \cap E'_{z_i}$ the weight α_{ik} of the largest k such that $F^j_{z_i} \cap E'_{z_i} \subset F^k_{z_i}$. In the same way we can define a canonical parabolic structure on any quotient E/E' by considering the quotient flag

$$(E/E')_{z_i} = F^1_{z_i}/E'_{z_i} \supset F^2_{z_i}/E'_{z_i} \supset \cdots \supset F^{l_i+1}_{z_i}/E'_{z_i} = \{0\}.$$

Once again, several quotients could coincide and give the same quotient flag element. If $F^{j-1}_{z_i}/E'_{z_i} = F^j_{z_i}/E'_{z_i}$ but $F^{j+1}_{z_i}/E'_{z_i}$ is a proper subset we asign the weight α_{ij} to $F^j_{z_i}/E'_{z_i}$

Remark 17. In their paper [37], Mehta and Seshadri define an *exact sequence of parabolic bundles*

$$0 \to E'_* \to E_* \to E''_* \to 0$$

being an exact sequence of bundles in the usual sense, such that E'_* is a parabolic subbundle of E_* and E''_* a parabolic quotient bundle of E_*. It follows that the parabolic bundles E'_*, E_* and $(E/E')_*$ form an exact sequence of parabolic bundles. However, there is an important subtlety here: the notion of splitting (and direct sum) does not follow straighforwardly if a sequence splits in the usual bundle sense. The problem is that since direct sums and intersections do not commute in general as operations on vector spaces, even if $E \cong E' \oplus E''$, we cannot recover the flags of E_* from those of E'_* and E''_*.

Since neither Mehta and Seshadri, nor Biswas give a precise definition of the parabolic structure in a direct sum of parabolic bundles, we propose the following one, which in the case of a reducible representation $\rho = \rho_1 \oplus \rho_2$ induces the same parabolic structure on $(E_\rho)_*$ and $(E_{\rho_1})_* \oplus (E_{\rho_2})_*$.

Assume we have 2 parabolic bundles E'_*, E''_* with parabolic structures over the same set of points $z_1, \cdots, z_n$ such that the underlying bundles E', E'' are subbundles of E and satisfy $E \cong E' \oplus E''$. We will induce a decreasing flag and weights at every parabolic vertex z_i in the following way: the first element will be $F^1_{z_i} = E_{z_i} \cong E'_{z_i} \oplus E''_{z_i}$ with weight $\alpha_{i1} = \mathrm{Min}(\{\alpha'_{ik}\}_{k=1}^{l'_i} \cup \{\alpha''_{ik}\}_{k=1}^{l''_i})$. Now, assuming we have constructed the jth element of the flag, we construct the $(j+1)$th element by first considering the set of weights minus $\{\alpha_{i1}, \cdots, \alpha_{ij}\}$, this is,

$$\{\{\alpha'_{ik}\}_{k=1}^{l'_i} \cup \{\alpha''_{ik}\}_{k=1}^{l''_i}\} \setminus \{\alpha_{i1}, \cdots, \alpha_{ij}\}$$

the subspace $F^{j+1}_{z_i}$ will be

$$F^{j+1}_{z_i} = F'^{k'}_{z_i} \oplus F''^{k''}_{z_i}$$

where k', resp. k'' are the minimum indices in the sets of weights $\{\alpha'_{ik}\}_{k=1}^{l'_i} \setminus \{\alpha_{i1}, \cdots, \alpha_{ij}\}$, resp. $\{\alpha''_{ik}\}_{k=1}^{l''_i} \setminus \{\alpha_{i1}, \cdots, \alpha_{ij}\}$. The corresponding weight α_{ij+1} the minimum in the later set of weights. The previous one is obviously a finite process, and by construction it follows that $F^{j+1}_{z_i} \subset F^j_{z_i}$.

Definition C.2.1. *A parabolic bundle E_* is* ***parabolic decomposable*** *if there exist two subbundles E', E'' such that $E = E' \oplus E''$ and the parabolic structures of E_* and $E'_* \oplus E''_*$ coincide.*

Proposition C.2.2. *Given a representation ρ, the parabolic bundle $(E_\rho)_*$ is parabolic indecomposable if and only if ρ is irreducible.*

Proof. This is a straightforward generalization of the fact, proved in proposition B.2.3, that the function Υ asociated to ρ is block diagonal if and only if the representation ρ is block diagonal. We just have to emphasize that the induced parabolic structures in the subbundles E_{ρ_1}, E_{ρ_2} are complementary since the corresponding flags are given in terms of the columns of the blocks, and the weights in terms of the eigenvalues of the blocks. □

Remark 18. It should be observed that *the dual of a parabolic bundle E_* of parabolic degree 0 cannot be a parabolic bundle of parabolic degree 0.* This should be contrasted with the fact that duality is a well-defined operation in the category of unitary representations. One could keep in mind Proposition 1.4.4 (at least if the system of weights is generic), and define the *parabolic dual* of a parabolic vector bundle E_* to have $\mathcal{O}(-n) \otimes E^*$ as its underlying vector bundle, with parabolic structure induced by replacing the flags in the following way: at the point $\{z_i\}$ set $E_{z_i} = F^1_{z_i} \supset F^1_{z_i}/F^{l_i}_{z_i} \supset \cdots \supset F^1_{z_i}/F^2_{z_i} \supset \{0\}$ with weights $1 - \alpha_{il_i} \leq 1 - \alpha_{il_i - 1} \leq \cdots \leq 1 - \alpha_{i1} < 1$.

C.3 Parabolic stability

Definition C.3.1. *A parabolic bundle E_* is called **parabolic semistable** (resp.* parabolic stable*) if for any parabolic subbundle $E'_* \subset E_*$ we have*

$$\frac{\text{Par deg}(E'_*)}{\text{rk}(E')} \leq \frac{\text{Par deg}(E_*)}{\text{rk}(E)} \qquad (\textit{resp.} <).$$

A major result of the theory of parabolic bundles is the next theorem, which relates the parabolic stability with a suitable notion of uniformization.

Theorem C.3.2 (Mehta & Seshadri)**.** *A parabolic bundle E_* of parabolic degree 0 (with rational weights) is semistable if and only if $E_* \cong (E_\rho)_*$ for a unitary representation $\rho : \pi_1(S) \to \mathrm{U}(r)$. E_* is stable if and only if ρ is irreducible. Two parabolic semistable bundles E_*, E'_* are equivalent if and only if the corresponding representations ρ, ρ' are equivalent.*

The isomorphism is realized through the branched covering extending the uniformization map $J : \mathbb{H}^+ \to S$ as

$$\begin{tikzcd} (E_\rho)_* \arrow[r, leftrightarrow, "J^*"] \arrow[d, "\mathrm{pr}"'] & E_* \arrow[d, "\pi"] \\ \Gamma \setminus \mathbb{H}^+ \arrow[r, leftrightarrow, "J"] & S \end{tikzcd}$$

It is important to emphasize that the proof of the theorem of Mehta and Seshadri is only valid for systems of weights consisting of rational numbers. In the spirit of S. Donaldson, O. Biquard [9] proved that in the category of parabolic bundles, the notion of parabolic stability of a parabolic vector bundle E_* of parabolic degree 0 is equivalent to the existence of a meromorphic metric connection on E with logarithmic singularities at the parabolic locus of S. Moreover, his proof works for arbitrary real weights. In their work [12], H. Boden and Y. Hu proved that if a system of weights is generic, then every parabolic semistable bundle with such system of weights is parabolic stable.

S. Bauer proved in [6] that over $\mathbb{P}^1$, for generic weights the moduli space of stable parabolic bundles of rank 2 is birational to a projective space. This result was generalized to arbitrary rank in [12].

C.4 The infinitesimal deformation map

It is a standard fact of Kodaira's deformation theory that for any smooth or holomorphic vector bundle E, the infinitesimal deformations of it are parametrized by the Čech cohomology group $H^1(\mathrm{End}(E))$, and given any smooth or holomorphic family $\mathcal{F} \to \mathscr{M} \times S$ of vector bundles $E_x \to S$, $x \in \mathscr{M}$ such that for for some $x_0 \in \mathcal{F}$, $E_{x_0} \cong E$, there is a canonical *infinitesimal deformation map*,

$$\mathscr{T}_{x_0} : T_{x_0}\mathscr{M} \to H^1(\mathrm{End}(E))$$

A classical result ([40], cf. Lemma 15.5 Atiyah-Bott, [5]) states that in fact any holomorphic vector bundle over a Riemann surface possesses "sufficiently big" families, that is, for every holomorphic vector bundle E, there exist a family $\mathscr{M}$ of vector bundles so that for some $x_0 \in \mathscr{M}$, $E_{x_0} \cong E$, and moreover, the infinitesimal deformation map $T_{x_0}\mathscr{M} \to H^1(\mathrm{End}(E))$ is an isomorphism.

Proceeding by analogy, it is natural to expect that the infinitesimal deformations of a given parabolic bundle E_* are parametrized by the Čech cohomology group $H^1(\mathrm{Par\ End}(E_*))$ instead. It is not hard to prove [41] that any smooth or holomorphic family $\mathcal{F} \to \mathscr{M} \times S$ of parabolic bundles induces an infinitesimal deformation map

$$\mathscr{T}_{x_0} : T_{x_0}\mathscr{M} \to H^1(\mathrm{Par\ End}(E_*)) \tag{C.3}$$

and that every parabolic bundle E_* is contained in a family whose infinitesimal deformation map is an isomorphism.

For the family of parabolic bundles arising from irreducible unitary representations of $\pi_1(S)$ (with a fixed set of weights at the parabolic generators), this isomorphism can be constructed explicitly through the character variety. the particular case of $S = \mathbb{P}^1$ is discussed explicitly in section 3.2.3.

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