Representation Theory of Categorified Quantum $\mathfrak{sl}_2$

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Abstract of the Dissertation

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Quantum $\mathfrak{sl}_2$ gives rise to the Jones polynomial knot invariant. One of the insights of categorification is that this 3-dimensional picture is a shadow, the decategorification, of a 4-dimensional picture. Thus, the categorification of quantum $\mathfrak{sl}_2$ gives rise through its representation theory to Khovanov homology, the categorification of the Jones polynomial. In the 3-dimensional picture, the algebra of Temperley-Lieb diagrams, used in the construction of the Jones polynomial, gives a graphical calculus for intertwiners of the representations of quantum $\mathfrak{sl}_2$. We show that the algebra of Bar-Natan's dotted cobordisms, used in the construction of Khovanov homology, gives a graphical calculus for intertwiners of representations of categorified quantum $\mathfrak{sl}_2$.

To my parents, Raffy and Dvora.

Contents

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I am deeply indebted to:

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Chapter 1

Introduction

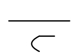
1.1 The decategorified picture

Quantum $\mathfrak{sl}_2$, denoted $U = U_q(\mathfrak{sl}_2)$ is a deformation of the universal enveloping algebra of $\mathfrak{sl}_2$ by a parameter q. U is the $\mathbb{Z}[q, q^{-1}]$-algebra generated by $\downarrow, \uparrow$ and an invertible element K with relations

$$K\uparrow = q^2 \uparrow K \qquad K\downarrow = q^{-2}\downarrow K \qquad \uparrow\downarrow - \downarrow\uparrow = \frac{K - K^{-1}}{q - q^{-1}}$$

U has irreducible representations $V_\lambda = U^- \underset{\lambda}{|}$ for $\lambda \in \mathbb{N}$ where U^- is the subalgebra of U generated by $\downarrow$, and, $\underset{\lambda}{|}$ is a vector in V_λ with $K\underset{\lambda}{|} = q^\lambda \underset{\lambda}{|}$ and $\uparrow \underset{\lambda}{|} = 0$. Tensor products of irreducible representations $V_{\underline{\lambda}} = V_{\lambda_N} \otimes \cdots \otimes V_{\lambda_1}$ are also representations of U.

The space of Temperley-Lieb diagrams $TL(m, n)$ is the $\mathbb{Z}[q, q^{-1}]$-module generated by isotopy classes of planar curves with endpoints, m to the left and n to the right, and elements in $TL(l, m)$ and $TL(m, n)$ may be glued together along their m endpoints, for instance,

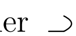
 $\in TL(3,1)$  $\in TL(1,3)$  $\in TL(3,3)$

subject to the Temperley-Lieb relation for a circle,

 $= q + q^{-1}$

Temperley-Lieb diagrams provide a graphical calculus for intertwiners of tensor powers of the fundamental representation $V_{1,\cdots,1}$ due to the following theorem.

Theorem. *There is a representation of the category TL of all Temperley-Lieb diagrams in the category of U-intertwiners of the representations $V_{1,\cdots,1}$ which sends q to $-q$.*

The key to constructing this representation is the invariant dual canonical basis vector $v^1 \heartsuit v^{-1} \in V_1 \otimes V_1$. One has that the intertwiner  $\in TL(2,0)$ is mutiplication of a scalar by $-qv^1 \heartsuit v^{-1}$ and that the intertwiner  $\in TL(0,2)$ is inner product with $v^1 \heartsuit v^{-1}$. See [6] for details. Since the category of Temperley-Lieb diagrams is a ribbon category it may used to define invariants of tangles, but even more profoundly, by applying the theory of Jones-Wenzl idempotents in TL, one obtains an invariant of 3-manifolds, the Reshetikhin-Turaev invariant.

The task of this dissertation is to prove a categorified version of the above theorem. First we will review some category theory and 2-category theory which we will then utilize to describe three categorifications, Lauda's categorification of quantum $\mathfrak{sl}_2$, Webster's categorification of tensor representations, and Bar-Natan's categorification of Temperley-Lieb diagrams. Then we will develop a theory of intertwiners and construct a representation of the 2-category of Bar-Natan cobordisms as intertwiners of the categorification of tensor powers of the fundamental representation of quantum $\mathfrak{sl}_2$. Finally, we will make note of some considerations about the categorification of Jones-Wenzl idempotents.

1.2 Previous work

The program of categorification was initiated by Louis Crane and Igor Frenkel in an attempt to lift 3-dimensional topological quantum field theory to 4-dimensions [5]. Mikhail Khovanov then showed that the Jones polynomial can be categorified resulting in the celebrated Khovanov homology [7]. Dror Bar-Natan then gave a new construction of Khovanov homology using a graphical calculus of surfaces which are cobordisms between resolutions of knot diagrams, categorifying the standard construction of the Jones' polynomial via the Kauffman bracket, essentially developing a categorification of Temperley-Lieb diagrams [1]. With Scott Morrison, he also noticed the importance of passing to the Karoubi envelope of a category [2].

Aaron Lauda then categorified quantum $\mathfrak{sl}_2$, or more precisely Lusztig's idempotented modification, by using a graphical calculus of string diagrams [11]. With Khovanov, he then generalized to categorifications of any Drinfeld-Jimbo quantum group [8, 9, 10]. Ben Webster then categorified tensor products of irreducible representations of quantum groups by modifying Khovanov and Lauda's graphical calculus of string diagrams [13, 14]. Webster showed that his categorification provided a knot invariant equivalent to Khovanov's in the case of $\mathfrak{sl}_2$ but did not yet extend it functorially to cobordisms.

Chapter 2

Algebraic Preliminary

2.1 Linear categories

We will work over a field $\mathbb{K}$. A linear category is a category $\mathcal{C}$ whose morphism sets $Hom_{\mathcal{C}}(a \to b)$ are $\mathbb{K}$-vector spaces such that composition is bilinear. Functors between linear categories are assumed to be linear. A graded, linear category has a strictly invertible endofunctor q. This gives rise to endofunctors q^n called grade shifts for $n \in \mathbb{Z}$, compositions of q or its inverse. Functors between graded, linear categories are assumed to be grade preserving, $Fq = qF$.

Given morphisms $a \xrightarrow{f} q^{n_1}b$ and $b \xrightarrow{g} q^{n_2}c$, define graded composition $a \xrightarrow{gf} q^{n_1+n_2}c$ to be $q^{n_1}(g)\,f$. As this abuse of notation implies, we will not distinguish between grade shifts of a given morphism. For $q^{n_0}a \xrightarrow{f} q^{n_1}b$ define q-degree, $deg^q(f) = n_1 - n_0$. Define graded morphism and endomorphism sets,

$$Hom^q_{\mathcal{C}}(a \to b) = \bigoplus_{n \in \mathbb{Z}} Hom(a \to q^n b)$$

$$End^q_{\mathcal{C}}(a) = Hom^q_{\mathcal{C}}(a \to a)$$

Then $End^q_{\mathcal{C}}(a)$ are graded rings and $Hom^q_{\mathcal{C}}(a \to b)$ are graded $End^q_{\mathcal{C}}(b)$, $End^q_{\mathcal{C}}(a)$-bimodules. We will assume all rings and modules are finite-dimensional $\mathbb{K}$-vector spaces.

Example. Given a graded ring R, the category $\mathcal{M}_R$ of graded, right modules and degree preserving homomorphisms over R is a graded, linear category.

An idempotent is a morphism $e \in End_{\mathcal{C}}(a)$ such that $e^2 = e$. Functors $\mathcal{F}$ preserve idempotents since

$$\mathcal{F}(e)^2 = \mathcal{F}\left(e^2\right) = \mathcal{F}(e)$$

In a linear category if e is an idempotent then $1_a - e$ is an idempotent since

$$(1_a - e)^2 = 1_a^2 - 2e + e^2 = 1_a - 2e + e = 1_a - e$$

A split idempotent is a morphism $e \in End_{\mathcal{C}}(a)$ such that there is an object $im(e)$, the image of e, with morphisms $im(e) \underset{i}{\overset{p}{\leftrightarrows}} a$, the projection and inclusion, such that $pi = 1_{im(e)}$ and $ip = e$. A split idempotent is an idempotent since

$$e^2 = (ip)^2 = ipip = i1_{im(e)}p = ip = e$$

The image of a split idempotent is uniquely determined up to canonical isomorphism.

A 0 object in a linear category is an object with identity $1_0 = 0$. A direct sum of a finite set of objects $a_1, \ldots, a_n$ in a linear category is an object $\bigoplus_{i=1}^n a_i = a_1 \oplus \cdots \oplus a_n$ with morphisms $a_k \underset{i_k}{\overset{p_k}{\leftrightarrows}} \bigoplus_{i=1}^n a_i$ such that the direct sum relations hold: $p_k i_k = 1_{a_k}$ and $p_k i_j = 0$ for $k \neq j$ and $\sum_{i=1}^n i_i p_i = 1_{\bigoplus_i a_i}$. Thus, an empty direct sum is a 0 object. Direct sums are uniquely determined up to canonical isomorphisms and linear functors preserve direct sums. Notice each $e_k = i_k p_k$ is a split idempotent with $a_k \cong im(e_k)$. Conversely,

Proposition. *If $e, 1_a - e \in End_{\mathcal{C}}(a)$ are split idempotents, then $a \cong im(e) \oplus im(1_a - e)$*

Proof. We must check the direct sum relations. We have that $p_e i_e = 1_{im(e)}$ and $p_{1-e} i_{1-e} = 1_{im(1-e)}$ by definition of split idempotents and,

$$\begin{aligned}
p_e i_{1-e} &= 1_{im(e)} p_e i_{1-e} 1_{im(1-e)} = p_e i_e p_e i_{1-e} p_{1-e} i_{1-e} \\
&= p_e e (1_a - e) i_{1-e} = p_e (e - e^2) i_{1-e} = 0 \\
p_{1-e} i_e &= 1_{im(1-e)} p_{1-e} i_e 1_{im(e)} = p_{1-e} i_{1-e} p_{1-e} i_e p_e i_e \\
&= p_{1-e} (1_a - e) e i_e = p_{1-e} \left(e - e^2\right) i_e = 0 \\
i_e p_e + i_{1-e} p_{1-e} &= e + 1_a - e = 1_a
\end{aligned}$$

□

Thus we can identify images of split idempotents as direct summands.

2.2 Pseudo-Abelian categories

A graded pseudo-Abelian category is a graded linear category which is closed under direct sums and direct summands, that is, it is additive and all idempotents are split.

Example. The category $\mathcal{P}_R$ of graded, projective, right modules and degree preserving homomorphisms over R is a graded, pseudo-Abelian category.

Define the pseudo-Abelian closure $\dot{\mathcal{C}}$ of a linear category $\mathcal{C}$ by formally adjoining direct sums and direct summands. That is $\dot{\mathcal{C}} = Add\,(Split\,(\mathcal{C}))$ where $Split\,(\mathcal{C})$, the Karoubi envelope of $\mathcal{C}$, is the category with:

- objects $im\,(e)$ are formal images of idempotents $e \in End_{\mathcal{C}}\,(a)$ from $\mathcal{C}$,
- morphisms $im\,(e_1) \xrightarrow{f} im\,(e_2)$ are morphisms $a_1 \xrightarrow{f} a_2$ from $\mathcal{C}$ such that $fe_1 = f = e_2 f$,
- composition $im\,(e_1) \xrightarrow{f} im\,(e_2) \xrightarrow{g} im\,(e_3)$ is just composition gf from $\mathcal{C}$,

- the identity on $im\,(e)$ is e,
- $Split\,(\mathcal{C})$ contains $\mathcal{C}$ as a full, faithful embedding with an object a in $\mathcal{C}$ identified with the object $im\,(1_a)$ in $Split\,(\mathcal{C})$,
- If $\mathcal{C}$ is graded, linear then $Split\,(\mathcal{C})$ is too with the same addition and $q\,(im\,(e)) = im\,(q\,(e))$;

and $Add\,(\mathcal{D})$, the additive closure of a linear category $\mathcal{D}$, is the category with:

- objects are formal finite direct sums of objects $\bigoplus_i a_i$ from $\mathcal{D}$,
- morphisms $\bigoplus_i a_i \xrightarrow{f} \bigoplus_j b_j$ are matrices of morphisms $a_i \xrightarrow{f_{ji}} b_j$ from $\mathcal{D}$,
- composition $\bigoplus_i a_i \xrightarrow{f} \bigoplus_j b_j \xrightarrow{g} \bigoplus_k c_k$ is given by matrix multiplication $(gf)_{ki} = \sum_j g_{kj} f_{ji}$,
- the identity on $\bigoplus_i a_i$ is the identity matrix $1_{jj} = 1_{a_j}$ and $1_{kj} = 0$ for $k \neq j$,
- $Add\,(\mathcal{D})$ contains $\mathcal{D}$ as a full, faithful embedding,
- If $\mathcal{D}$ is graded, linear then $Add\,(\mathcal{D})$ is too with matrix addition and $q\,(\bigoplus_i a_i) = \bigoplus_i q\,(a_i)$.

The pseudo-Abelian closure $\dot{\mathcal{C}}$ is characterized by the universal property that $\dot{\mathcal{C}}$ is pseudo-Abelian and any functor from $\mathcal{C}$ to a pseudo-Abelian category factors through $\dot{\mathcal{C}}$ uniquely up to functorial isomorphism. Thus, if $\mathcal{C}$ is pseudo-Abelian, then $\dot{\mathcal{C}}$ is equivalent to $\mathcal{C}$.

An object a projectively generates $\mathcal{C}$ if and only if every object in $\mathcal{C}$ is isomorphic to an object generated by a under direct sums, direct summands and grade shifts. In that case $\dot{\mathcal{C}}$ is equivalent to the pseudo-Abelian closure of the subcategory of $\mathcal{C}$ with objects restricted to $q^n a$, $n \in \mathbb{Z}$.

Proposition. *If a projectively generates $\mathcal{C}$ and $R \cong End^q_{\mathcal{C}}\,(a)$, then $\dot{\mathcal{C}}$ is equivalent to $\mathcal{P}_R$.*

Proof. We will show that $Hom^q_{\dot{\mathcal{C}}}(a \to -)$ gives a functor from $\dot{\mathcal{C}}$ to $\mathcal{P}_R$. Any object b in $\dot{\mathcal{C}}$ is isomorphic to a direct sum of grade shifts of images of idempotents $b \cong \bigoplus_{i=1}^{N} q^{n_i} im(e_i)$, $e_i \in End_{\mathcal{C}}(a)$ so it is a direct summand of $a^{\oplus N}$; letting $b' = \bigoplus_{i=1}^{N} q^{-n_i} im(1_a - e_i)$,

$$b \oplus b' \cong \bigoplus_{i=1}^{N} im(e_i) \oplus im(1_a - e_i) \cong a^{\oplus N}$$

$$\text{So, } Hom^q_{\dot{\mathcal{C}}}(a \to b) \oplus Hom^q_{\dot{\mathcal{C}}}(a \to b') \cong End^q_{\mathcal{C}}(a)^{\oplus N} \cong R^{\oplus N}$$

since $Hom^q_{\dot{\mathcal{C}}}(a \to -)$ preserves direct sums and grade shifts. Thus, $Hom^q_{\dot{\mathcal{C}}}(a \to b)$ is a projective, right R-module.

We must show for any objects b, c in $\dot{\mathcal{C}}$ that $Hom^q_{\dot{\mathcal{C}}}(a \to -)$ induces an isomorphism of graded $End^q_{\dot{\mathcal{C}}}(c), End^q_{\dot{\mathcal{C}}}(b)$-bimodules between $Hom^q_{\dot{\mathcal{C}}}(b \to c)$ and

$$Hom_{\mathcal{P}_R}\left(Hom^q_{\dot{\mathcal{C}}}(a \to b) \to Hom^q_{\dot{\mathcal{C}}}(a \to c)\right)$$

In the case $a \cong b \cong c$, we have that $End^q_{\dot{\mathcal{C}}}(a) \cong End_{\mathcal{P}_R}\left(End^q_{\dot{\mathcal{C}}}(a)\right)$, with $f \in End^q_{\dot{\mathcal{C}}}(a)$ identified with left graded composition by f and $g \in End_{\mathcal{P}_R}\left(End^q_{\dot{\mathcal{C}}}(a)\right)$ identified with $g(1_a)$. The general case follows from this and the fact that $Hom^q_{\dot{\mathcal{C}}}(a \to -)$ preserves direct sums, direct summands and grade shifts. Thus, $Hom^q_{\dot{\mathcal{C}}}(a \to -)$ is full and faithful.

We must show that for any graded, projective, right R-module P, there is an object b in $\dot{\mathcal{C}}$ such that $P \cong Hom^q_{\dot{\mathcal{C}}}(a \to b)$. P is a direct summand of $R^{\oplus N} \cong End^q_{\mathcal{C}}(a)^{\oplus N}$ so let $End^q_{\mathcal{C}}(a)^{\oplus N} \xleftarrow{e} End^q_{\mathcal{C}}(a)^{\oplus N}$ be the idempotent with $P \cong im(e)$. Let $e' \in End^q_{\dot{\mathcal{C}}}\left(a^{\oplus N}\right)$ be the morphism induced by e. Then $e'^2 = 0$, so take $b = im(e')$. Thus, $Hom^q_{\dot{\mathcal{C}}}(a \to -)$ is essentially surjective and is an equivalence of categories. □

A finite set of objects $a_1, \ldots, a_N$ projectively generate $\mathcal{C}$ if and only if every object in $\mathcal{C}$ is isomorphic to an object generated by them under direct sums, direct summands and grade shifts.

Corollary. *If $a_1, \ldots, a_N$ projectively generate $\mathcal{C}$ and $R \cong \bigoplus_{i=1}^{N}\bigoplus_{j=1}^{N} Hom_{\mathcal{C}}^{q}(a_i \to a_j)$ then $\dot{\mathcal{C}}$ is equivalent to $\mathcal{P}_R$.*

Proof. This follows since $\bigoplus_{i=1}^{N} a_i$ projectively generates of $\dot{\mathcal{C}}$ and

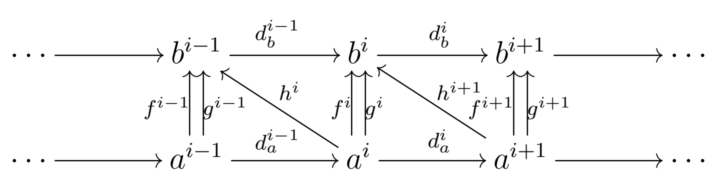
$$End_{\dot{\mathcal{C}}}^{q}\left(\bigoplus_{i=1}^{N} a_i\right) \cong \bigoplus_{i=1}^{N}\bigoplus_{j=1}^{N} Hom_{\mathcal{C}}^{q}(a_j \leftarrow a_i)$$

with multiplication given by graded composition where well-defined and 0 otherwise. □

2.3 Derived categories

For a linear category $\mathcal{C}$, let $\mathcal{K}(C)$ denote the homotopy category of complexes in $\mathcal{C}$ with:

- objects are formal cochain complexes a, sequences of objects a^i with $i \in \mathbb{Z}$ and differentials, morphisms $a^i \xrightarrow{d_a^i} a^{i+1}$, from $\mathcal{C}$ such that $d_a^i d_a^{i-1} = 0$,
- morphisms $a \xrightarrow{f} b$ are sequences of morphisms $a^i \xrightarrow{f^i} b^i$ from $\mathcal{C}$ with $d_b^i f^i = f^{i+1} d_a^i$. They are considered up to homotopies $f \sim g$, sequences $a^i \xrightarrow{h^i} b^{i-1}$ of morphisms such that $f^i - g^i = h^{i+1} d_a^i + d_b^{i-1} h^i$,



- morphisms in $\mathcal{K}(\mathcal{C})$ compose term-wise; the identity morphism 1_a is a sequence of identity morphisms 1_{a^i},
- $\mathcal{K}(\mathcal{C})$ is linear, morphisms are linearly combined term-wise; the 0 morphism is a sequence of 0 morphisms,

- $\mathcal{K}(\mathcal{C})$ has a grading t, called translation, with ta given by $(ta)^i = a^{i+1}$ and $d^i_{ta} = -d^{i+1}_a$ for objects and $t(f)^i = f^{i+1}$ for morphisms,

- $\mathcal{K}(\mathcal{C})$ contains $\mathcal{C}$ as a full, faithful embedding with an object a in $\mathcal{C}$ identified with the complex with $a^0 = a$ and $a^i = 0$ for $i \neq 0$,

- if $\mathcal{C}$ has a grading q, then $\mathcal{K}(\mathcal{C})$ inherits a grading q with qa given by $(qa)^i = q(a^i)$ and $d^i_{qa} = q(d^i_a)$ for objects and $q(f)^i = q(f^i)$ for morphisms, and $qt = tq$,

- if $\mathcal{C}$ is additive, then $\mathcal{K}(\mathcal{C})$ is too with $a \oplus b$ having $(a \oplus b)^i \cong a^i \oplus b^i$ and $d^i_{a \oplus b} = \begin{pmatrix} d^i_a & 0 \\ 0 & d^i_b \end{pmatrix}$,

- functors between linear categories canonically extend to functors between their homotopy categories of complexes.

If $\mathcal{C}$ has a 0 object, then let $\mathcal{K}^-(\mathcal{C})$, the homotopy category of bounded above complexes in $\mathcal{C}$, be the subcategory of $\mathcal{K}(C)$ with objects restricted to complexes a with $a^i \cong 0$ for all $i > n$ for some n.

Given a graded ring R, the derived category $\mathcal{D}_R$ is $\mathcal{K}(\mathcal{M}_R)$ localized at quasi-isomorphisms, that is, $\mathcal{D}_R$ includes formal inverses of those morphisms of complexes which induce isomorphisms on homology. Let $\mathcal{D}^-_R$ be its subcategory with objects restricted to bounded above complexes.

Corollary. *If* $a_1, \ldots, a_N$ *projectively generate* $\mathcal{C}$ *and* $R \cong \bigoplus_{i=1}^{N} \bigoplus_{j=1}^{N} Hom^q_{\mathcal{C}}(a_i \to a_j)$ *then* $\mathcal{K}^-\left(\dot{\mathcal{C}}\right)$ *is equivalent to* $\mathcal{D}^-_R$.

Proof. This follows from a theorem of homological algebra that the bounded-above derived category $\mathcal{D}^-_R$ is equivalent to $\mathcal{K}^-(\mathcal{P}_R)$. □

Given graded rings R_1, R_2, let ${}_{R_1}\mathcal{M}_{R_2}$ be the category of graded R_1, R_2-bimodules and degree preserving homomorphisms, let the derived category ${}_{R_1}\mathcal{D}_{R_2}$ be the localization of

$\mathcal{K}\left({}_{R_1}\mathcal{M}_{R_2}\right)$ at quasi-isomorphisms, and let ${}_{R_1}\mathcal{D}^-_{R_2}$ be its subcategory with objects restricted to bounded-above complexes. Given objects a in ${}_{R_1}\mathcal{D}^-_R$ and b in ${}_R\mathcal{D}^-_{R_2}$, there is an object, the left derived tensor product $a \overset{\mathcal{L}}{\underset{R}{\otimes}} b$, in ${}_{R_1}\mathcal{D}^-_{R_2}$ obtained by replacing a with its image in $\mathcal{K}^-\left(\mathcal{P}_R\right)$ and taking the standard tensor product, and $- \overset{\mathcal{L}}{\underset{R}{\otimes}} -$ extends to a bifunctor.

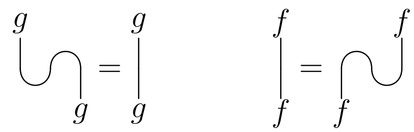
Let $\mathcal{D}^b_R$ be the subcategory of $\mathcal{D}_R$ with objects restricted to bounded complexes. Then by a theorem of Happel, if R has finite global dimension, then there is an invertible endofunctor of $\mathcal{D}^b_R$, the Serre functor, $\mathcal{S}_R = - \overset{\mathcal{L}}{\underset{R}{\otimes}} R^*$, where $-^*$ is vector space dual. See [3] for details.

2.4 2-categories

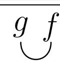
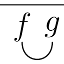
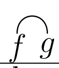
For a 2-category $\mathcal{C}$, we will follow the convention that horizontal sources are on the right and targets on the left, and that vertical sources are on the bottom and targets on the top. All 2-categories $\mathcal{C}$ will be enriched in graded, linear categories, that is, the morphism categories $Hom_{\mathcal{C}}\left(b \leftarrow a\right)$ will be graded, linear categories. The 2-categories $\dot{\mathcal{C}}$, and $\mathcal{K}^-\left(\dot{\mathcal{C}}\right)$ are obtained by taking pseudo-Abelian closures and homotopy cochain categories of morphism categories and extending horizontal composition. 2-categories will be assumed to be strict, either by definition or by strictification and 2-functors between them will be strict.

Example. There is a 2-category with graded linear categories as 0-morphisms, functors as 1-morphisms and functorial morphisms as 2-morphisms.

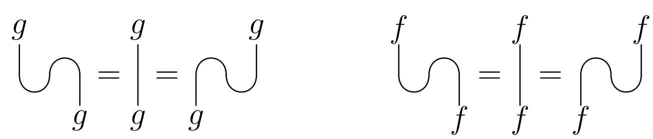
We will use the graphical calculus of string diagrams to denote morphisms of 2-categories. 1-morphisms $b \underset{g}{\overset{f}{\leftrightarrows}} a$ in a 2-category $\mathcal{C}$ are adjoint $f \dashv g$ if and only if there are 2-morphisms, the unit $\cup \in Hom_{End_{\mathcal{C}}(a)}\begin{pmatrix} gf \\ \Uparrow \\ 1_a \end{pmatrix}$ and counit $\cap \in Hom_{End_{\mathcal{C}}(b)}\begin{pmatrix} 1_b \\ \Uparrow \\ fg \end{pmatrix}$, which in the graphical calculus of string diagrams are denoted as  and  such that the unit-counit relations hold



In particular adjoint functors $F \dashv G$ are adjoint as 1-morphisms in the 2-category of categories, functors and functorial morphisms. Adjoints are determined up to 2-isomorphisms. If $\mathcal{C}$ is graded, linear-enriched, then say that f, g are graded biadjoint of degree n if and only if $q^{-n} f \dashv g \dashv q^n f$. Thus, f, g are graded biadjoint of degree n if and only if

there is a 2-morphism,	a morphism in,	with target,	and source.
	$End_{\mathcal{C}}(a)$	$q^{-n} g f$	1_a
	$End_{\mathcal{C}}(b)$	$q^n f g$	1_b
	$End_{\mathcal{C}}(a)$	$q^{-n} 1_a$	$g f$
	$End_{\mathcal{C}}(b)$	$q^n 1_b$	$f g$

such that the relations for a graded biadjunction hold



A semistrict monoidal structure on a 2-category is an associative product of 0-morphisms with an identity 0-morphism that extends 2-functorially to 1-morphisms and 2-morphisms.

Chapter 3

Categorifications

3.1 Lauda's categorification of quantum $\mathfrak{sl}_2$.

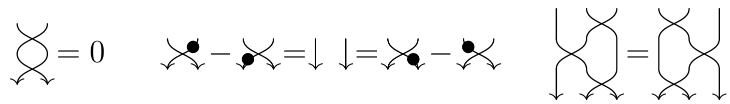
$\mathcal{U}^-$, the categorification of the negative half of quantum $\mathfrak{sl}_2$, is the graded, linear-enriched, strict 2-category consisting of:

- 0-morphisms are integers $n \in \mathbb{Z}$.
- 1-morphisms
  - For all $n \in \mathbb{Z}$, there is a 1-morphism $\downarrow$, an object in $Hom_{\mathcal{U}^-}(n-2 \leftarrow n)$.
  - 1-morphisms are generated by horizontal composition, and grade shifts of these and the horizontal identity 1-morphisms n.
- 2-morphisms
  - For all $n \in \mathbb{Z}$

there is a 2-morphism,	a morphism in,	with target,	and source.
	$Hom_{\mathcal{U}^-}(n-2 \leftarrow n)$	$q^2 \downarrow n$	$\downarrow n$
	$Hom_{\mathcal{U}^-}(n-4 \leftarrow n)$	$q^{-2} \downarrow\downarrow n$	$\downarrow\downarrow n$

– 2-morphisms are generated by horizontal and vertical composition, grade shifts, and linear combinations of these and the vertical identity 2-morphisms $\downarrow$, and n.

- 2-morphisms are subject to the nil-Hecke relations



The nil-Hecke rings are

$$R_a = End^q_{Hom_{\mathcal{U}^-}(n-2a \leftarrow n)} \left(\underset{a}{\downarrow} \, n \right)$$

where $\underset{a}{\downarrow} = \underbrace{\downarrow \cdots \downarrow}_{a}$. For different values of n, the R_a are all isomorphic. And, there are equivalences of categories,

$$Hom_{\dot{\mathcal{U}}^-}(n-2a \leftarrow n) \cong \mathcal{P}_{R_a} \qquad Hom_{\mathcal{K}^-(\dot{\mathcal{U}}^-)}(n-2a \leftarrow n) \cong \mathcal{D}^-_{R_a}$$

$\mathcal{U}$, Lauda's categorification of quantum $\mathfrak{sl}_2$, is the graded, linear-enriched, strict 2-category containing $\mathcal{U}^-$ consisting of:

- 0-morphisms are integers $n \in \mathbb{Z}$.

- 1-morphisms

  – For all $n \in \mathbb{Z}$, there is a 1-morphism $\uparrow$, an object in $Hom_{\mathcal{U}^-}(n+2 \leftarrow n)$.

  – 1-morphisms are generated by horizontal composition, and grade shifts of these, the 1-morphisms $\downarrow$ in $\mathcal{U}^-$, and the horizontal identity 1-morphisms n.

- 2-morphisms

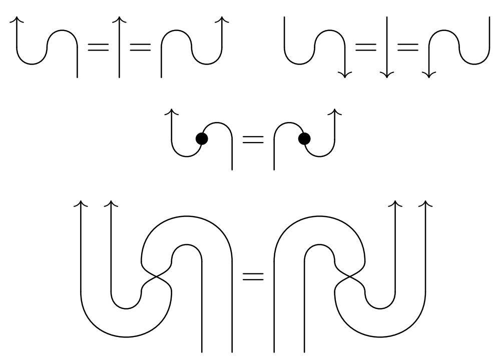
  – For all $n \in \mathbb{Z}$

there is a 2-morphism,	a morphism in,	with target,	and source.
	$End_{\mathcal{U}}(n)$	$q^{1-n}\uparrow\downarrow n$	n
	$End_{\mathcal{U}}(n)$	$q^{1+n}\downarrow\uparrow n$	n
	$End_{\mathcal{U}}(n)$	$q^{1-n}n$	$\uparrow\downarrow n$
	$End_{\mathcal{U}}(n)$	$q^{1+n}n$	$\downarrow\uparrow n$

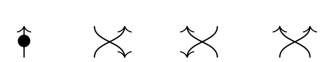
- 2-morphisms are generated by horizontal and vertical composition, grade shifts, and linear combinations of these, the 2-morphisms  and  in $\mathcal{U}^-$, and the vertical identity 2-morphisms $\uparrow$, $\downarrow$, and n.

- 2-morphisms are subject to the following relations.

  - nil-Hecke relations from $\mathcal{U}^-$
  - Duality relations



    - These relations guarantee that any 2-morphisms which are planar isotopic as oriented diagrams are equal. Thus, notation can be extended to planar isotopy classes of diagrams making sense of such diagrams as:



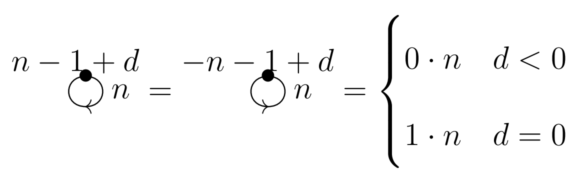
  - Bubble relations

* Define inductively by vertical composition

* The "dotted bubbles"

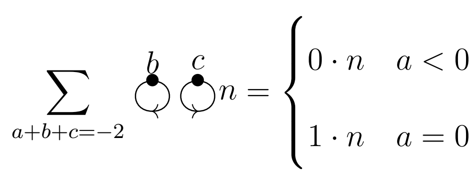
both have degree $2d$ in the graded unital ring $End^q_{End_{\mathcal{U}(n)}}(n)$. They make sense only when $d \geq 1 - n$ and $d \geq 1 + n$ respectively. However, we extend notation to include "fake" bubbles with $d < 1 - n$ and $d < 1 + n$ respectively.

* We require the relation

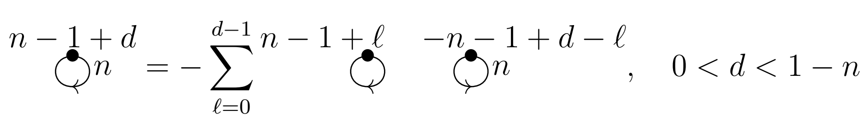


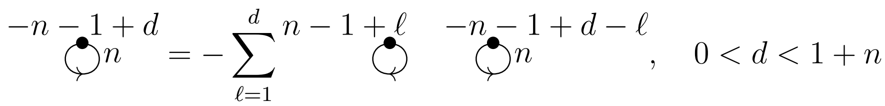
which defines fake bubbles when $d \leq 0$.

* We require the relation



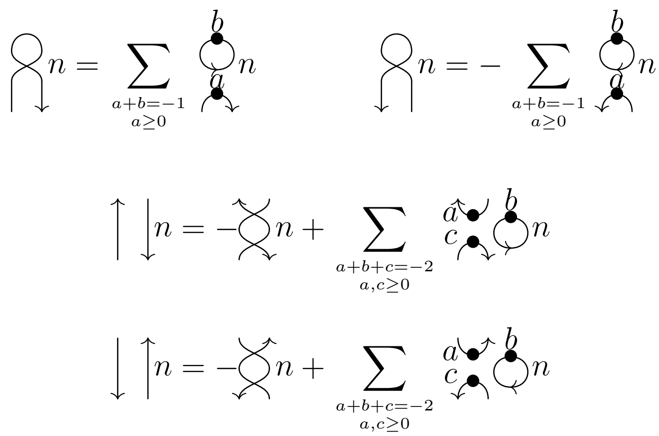
Setting $a = -d$ implies that





inductively defining fake bubbles when $d > 0$.

∗ Finally, we require the relations

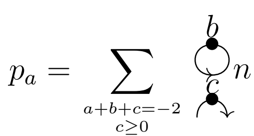


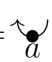
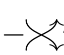
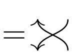
Proposition. *In $\mathcal{U}$, there are direct sum decompositions of 1-morphisms*

$$\uparrow\downarrow n \cong \bigoplus_{a=0}^{n-1} q^{n-1-2a} n \oplus \downarrow\uparrow n \text{ for } n \geq 0 \tag{3.1.1}$$

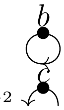
$$\downarrow\uparrow n \cong \bigoplus_{a=0}^{-n-1} q^{-n-1-2a} n \oplus \uparrow\downarrow n \text{ for } n \leq 0 \tag{3.1.2}$$

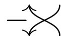
Proof. For $0 \leq a < n$, let



and $i_a =$  n and let $p_n = -$n and $i_n =$n. The relations for the direct sum decomposition (3.1.1); $\begin{matrix} p_a \\ i_a \end{matrix} = 1 \cdot n$ for $0 \leq a < n$, $\begin{matrix} p_n \\ i_n \end{matrix} = \downarrow\uparrow n$, $\begin{matrix} p_{a'} \\ i_a \end{matrix} = 0 \cdot n$ for $a \neq a'$ and $\sum_{a=0}^{n} \begin{matrix} i_a \\ p_a \end{matrix} = \uparrow\downarrow n$, all follow from bubble relations.

For $0 \leq a < -n$, let

$$p_a = \sum_{\substack{a+b+c=-2 \\ c \geq 0}}$$  n

and $i_a =$  n and let $p_{-n} = -$  n and $i_{-n} =$  n. The relations for the direct sum decomposition (3.1.2); $\begin{matrix} p_a \\ i_a \end{matrix} = 1 \cdot n$ for $0 \leq a < -n$, $\begin{matrix} p_{-n} \\ i_{-n} \end{matrix} = \uparrow\downarrow n$, $\begin{matrix} p_{a'} \\ i_a \end{matrix} = 0 \cdot n$ for $a \neq a'$ and $\sum_{a=0}^{-n} \begin{matrix} i_a \\ p_a \end{matrix} = \downarrow\uparrow n$, all follow from bubble relations. □

3.2 Webster's categorification of tensor representations

Let $\widetilde{\mathcal{U}}^-$ be the graded, linear-enriched, strict 2-category containing $\mathcal{U}^-$ consisting of:

- 0-morphisms are integers $n \in \mathbb{Z}$
- 1-morphisms
  - For all $n \in \mathbb{Z}$, $\lambda \in \mathbb{N}$, there is a 1-morphism $\underset{\lambda}{|}$, an object in $Hom_{\widetilde{\mathcal{U}}^-}(n + \lambda \leftarrow n)$;
  - 1-morphisms are generated by horizontal composition, and grade shifts of these, the 1-morphisms $\downarrow$ in $\mathcal{U}^-$, and the horizontal identity 1-morphisms n.
- 2-morphisms
  - For all $n \in \mathbb{Z}$, $\lambda \in \mathbb{N}$, there is

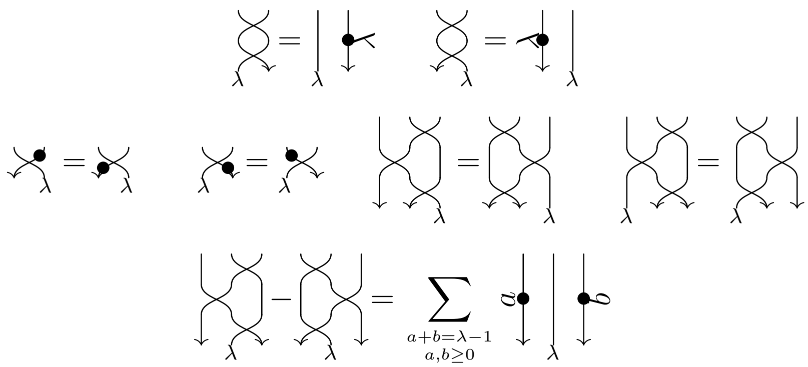
a 2-morphism,	a morphism in,	with target,	and source.
	$Hom_{\widetilde{\mathcal{U}}^-}(n + \lambda - 2 \leftarrow n)$	$q^\lambda \underset{\lambda}{|} \downarrow n$	$\downarrow \underset{\lambda}{|} n$
	$Hom_{\widetilde{\mathcal{U}}^-}(n + \lambda - 2 \leftarrow n)$	$q^\lambda \downarrow \underset{\lambda}{|} n$	$\underset{\lambda}{|} \downarrow n$

– 2-morphisms are generated by horizontal and vertical composition, grade shifts, and linear combinations of these, the 2-morphisms and in $\mathcal{U}^-$ and the vertical identity 2-morphisms $\downarrow$, $\underset{\lambda}{|}$ and n.

- 2-morphisms are subject to the following relations.

  – nil-Hecke relations from $\mathcal{U}^-$.

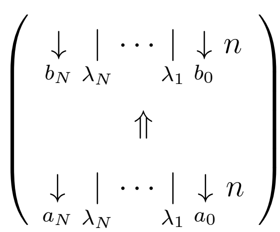
  – Webster relations



Any 2-morphism in $\widetilde{\mathcal{U}}^-$ has the same ordering of labels $\underline{\lambda} = \lambda_N, \dots, \lambda_1$ on its target and source 1-morphisms, so let $\widetilde{\mathcal{U}}^-_{\underline{\lambda}}$ be the sub-2-category with diagrams in only that ordering.

Let $\left[\begin{smallmatrix}\lambda \\ a\end{smallmatrix}\right]$ be the set of $\underline{a} = a_N, \dots, a_0$ with $a_i \in \mathbb{N}$ and $a = \sum_{i=0}^{N} a_i$ and let $|\underline{\lambda}| = \sum_{i=1}^{N} \lambda_i$. The Webster rings (for $\mathfrak{sl}_2$) are

$$\widetilde{R}^{\underline{\lambda}}_{a} = \bigoplus_{\underline{a},\underline{b} \in \left[\begin{smallmatrix}\lambda \\ a\end{smallmatrix}\right]} Hom^q_{Hom_{\widetilde{\mathcal{U}}^-_{\underline{\lambda}}}(n+|\underline{\lambda}|-2a \leftarrow n)}$$

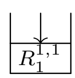
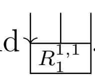


For different values of n the $\widetilde{R}^{\underline{\lambda}}_{a}$ are all isomorphic. Let $R^{\underline{\lambda}}_{a}$, the cyclotomic Webster ring, be the quotient of $\widetilde{R}^{\underline{\lambda}}_{a}$ (with $n = 0$) by the relation that a diagram with $\downarrow 0$ on the right is 0. Let ${}_m\mathcal{V}_{\underline{\lambda}}$ be the quotient of $Hom_{\widetilde{\mathcal{U}}^-_{\underline{\lambda}}}(m \leftarrow 0)$ by the relation that a diagram with $\downarrow 0$ on the right is 0. Then there are equivalences of categories

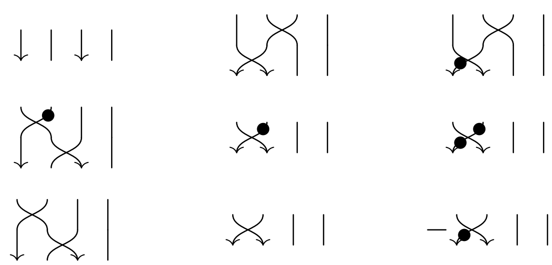
$$ {}_{|\underline{\lambda}|-2a}\dot{\mathcal{V}}_{\underline{\lambda}} \cong \mathcal{P}_{R_a^{\underline{\lambda}}} \qquad \mathcal{K}^-\left({}_{|\underline{\lambda}|-2a}\dot{\mathcal{V}}_{\underline{\lambda}}\right) \cong \mathcal{D}^-_{R_a^{\underline{\lambda}}} $$

Example. The cyclotomic Webster rings $R_a^{1,1}$, suppressing the labels $1, 1$, are

- $R_0^{1,1} = \mathbb{K}|\quad| \cong \mathbb{K}$
- $R_1^{1,1}$ has basis

  and has indecomposable projective generators  and .

- $R_2^{1,1}$ has basis



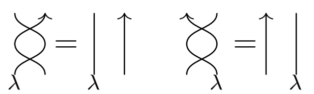
  and as a ring there is an isomorphism with the matrix ring $R_2^{1,1} \cong Mat^{3\times 3}(\mathbb{K})$ sending these generators to the elementary generators. Thus $R_2^{1,1}$ is Morita equivalent to $\mathbb{K}$.

- $R_a^{1,1}$ with $a > 2$ is trivial and is trivially trivial for $a < 0$.

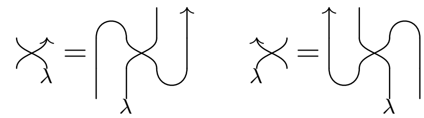
By a representation of $\dot{\mathcal{U}}$ we mean a 2-functor from $\dot{\mathcal{U}}$.

Proposition. *There is a representation $\dot{\mathcal{V}}_{\underline{\lambda}}$ of $\dot{\mathcal{U}}$ for which the 0-morphism n is sent to the category ${}_n\dot{\mathcal{V}}_{\underline{\lambda}}$.*

Proof. Let $\widetilde{\mathcal{U}}$ be the 2-category containing both $\mathcal{U}$ and $\widetilde{\mathcal{U}}^{-}$, with all 0, 1, and 2-morphisms from both, subject to all relations from both and the additional relations



where



Thus, in $\widetilde{\mathcal{U}}$,

$$\uparrow \underset{\lambda}{|} \cong \underset{\lambda}{|} \uparrow \tag{3.2.1}$$

Let $\widetilde{\mathcal{U}}_{\underline{\lambda}}$ be the sub-2-category of $\widetilde{\mathcal{U}}$ with the ordering of labels given by $\underline{\lambda}$, and let ${}_n\mathcal{W}_{\underline{\lambda}} = Hom_{\widetilde{\mathcal{U}}_{\underline{\lambda}}}(n \leftarrow 0) / \sim$ where the relation $\sim$ is that a diagram with $\downarrow 0$ or $\uparrow 0$ on the right is 0. Define rings

$$\widetilde{T}_a^{\underline{\lambda}} = \bigoplus_{\underline{a},\underline{b} \in \left[\begin{smallmatrix}\lambda \\ a\end{smallmatrix}\right]} Hom^q_{Hom_{\widetilde{\mathcal{U}}_{\underline{\lambda}}}(|\underline{\lambda}|-2a \leftarrow 0)} \begin{pmatrix} \underset{b_N}{\downarrow} \ \underset{\lambda_N}{|} \cdots \underset{\lambda_1}{|} \ \underset{b_0}{\downarrow} \ 0 \\ \Uparrow \\ \underset{a_N}{\downarrow} \ \underset{\lambda_N}{|} \cdots \underset{\lambda_1}{|} \ \underset{a_0}{\downarrow} \ 0 \end{pmatrix}$$

$$T_a^{\underline{\lambda}} = \bigoplus_{\underline{a},\underline{b} \in \left[\begin{smallmatrix}\lambda \\ a\end{smallmatrix}\right]} Hom^q_{{}_{|\underline{\lambda}|-2a}\mathcal{W}_{\underline{\lambda}}} \begin{pmatrix} \underset{b_N}{\downarrow} \ \underset{\lambda_N}{|} \cdots \underset{\lambda_1}{|} \ \underset{b_0}{\downarrow} \ 0 \\ \Uparrow \\ \underset{a_N}{\downarrow} \ \underset{\lambda_N}{|} \cdots \underset{\lambda_1}{|} \ \underset{a_0}{\downarrow} \ 0 \end{pmatrix}$$

Using (3.1.1), (3.1.2) and (3.2.1) we see that any 1-morphism in $\widetilde{\mathcal{U}}$ is isomorphic to a direct summand of a direct sum of grade shifts of 1-morphisms for which all $\uparrow$ are on the right. Applying the relation $\sim$ we then see that the objects $\underset{a_N}{\downarrow} \ \underset{\lambda_N}{|} \cdots \underset{\lambda_1}{|} \ \underset{a_0}{\downarrow} \ 0$, $\underline{a} \in \left[\begin{smallmatrix}\lambda \\ a\end{smallmatrix}\right]$ projectively generate ${}_{|\underline{\lambda}|-2a}\mathcal{W}_{\underline{\lambda}}$. Thus, there is an equivalence of categories, ${}_{|\underline{\lambda}|-2a}\dot{\mathcal{W}}_{\underline{\lambda}} \cong \mathcal{P}\left(T_a^{\underline{\lambda}}\right)$.

Consider the 2-functor $Hom_{\widetilde{\mathcal{U}}_{\underline{\lambda}}}(- \leftarrow 0) / \sim$ applied to the image of $\mathcal{U}$ in $\widetilde{\mathcal{U}}$. It sends n to ${}_n\mathcal{W}_{\underline{\lambda}}$, and is the category of functors and functorial morphisms to ${}_{n'}\mathcal{W}_{\underline{\lambda}}$ from ${}_n\mathcal{W}_{\underline{\lambda}}$ given by left horizontal composition by $Hom_{\mathcal{U}}(n' \leftarrow n)$, which is well-defined.

There is an inclusion of rings given by horizontal composition

$$\widetilde{R}^{\underline{\lambda}}_{\underline{a}} \underset{\mathbb{K}}{\otimes} End^q_{End_{\mathcal{U}}(0)}(0) \to \widetilde{T}^{\underline{\lambda}}_{\underline{a}}$$

Using relations in $\widetilde{\mathcal{U}}$, we can combinatorially simplify diagrams in $\widetilde{T}^{\underline{\lambda}}_{\underline{a}}$ to show that this is an isomorphism. Modding out both sides by the relation $\sim$ shows that $T^{\underline{\lambda}}_{\underline{a}} \cong R^{\underline{\lambda}}_{\underline{a}}$. Thus, ${}_n\dot{\mathcal{W}}_{\underline{\lambda}} \cong {}_n\dot{\mathcal{V}}_{\underline{\lambda}}$, so that applying the pseudo-Abelian envelope induces a representation of $\dot{\mathcal{U}}$ that sends n to ${}_n\dot{\mathcal{V}}_{\underline{\lambda}}$. Let $\dot{\mathcal{V}}_{\underline{\lambda}}$ be the image of this representation of $\dot{\mathcal{U}}$. $\square$

Corollary. *In $\dot{\mathcal{V}}_{\underline{\lambda}}$, $\downarrow n+1$ and $\uparrow n-1$ are equivalent to Ind and $q^n Res$, the extension and shifted restriction of scalars functions induced by the inclusion of rings $R^{\underline{\lambda}}_{a} \xrightarrow{\downarrow\cdot} R^{\underline{\lambda}}_{a+1}$, $n = |\lambda| - 2a - 1$, given by left horizontal composition by $\downarrow$. And, $\downarrow n+1$ is also equivalent to $q^{-2n}coInd$, the shifted coinduction functor.*

Proof. The equivalence of categories ${}_{n+1}\dot{\mathcal{V}}_{\underline{\lambda}} \cong \mathcal{P}_{R^{\underline{\lambda}}_a}$ is given by the functor

$$Hom^q_{{}_{n+1}\dot{\mathcal{V}}_{\underline{\lambda}}}\left(\begin{array}{c} - \\ \Uparrow \\ \bigoplus_{\underline{a}\in\left[\begin{smallmatrix}\underline{\lambda}\\ a\end{smallmatrix}\right]} \underset{a_N}{\downarrow} \underset{\lambda_N}{|} \cdots \underset{\lambda_1}{|} \underset{a_0}{\downarrow} 0 \end{array}\right)$$

so that left horizontal composition by $\downarrow$ as a functor with target ${}_{n-1}\dot{\mathcal{V}}_{\underline{\lambda}}$ and source ${}_{n+1}\dot{\mathcal{V}}_{\underline{\lambda}}$ will be equivalent to the extension of scalars functor with target $\mathcal{P}_{R^{\underline{\lambda}}_{a+1}}$ and source $\mathcal{P}_{R^{\underline{\lambda}}_a}$ induced by the homomorphism $R^{\underline{\lambda}}_{a} \xrightarrow{\downarrow\cdot} R^{\underline{\lambda}}_{a+1}$, i.e. the following diagram commutes.

$$\begin{array}{ccc} \mathcal{P}_{R^{\underline{\lambda}}_{a+1}} & \xleftarrow{Ind} & \mathcal{P}_{R^{\underline{\lambda}}_{a}} \\ \cong\uparrow & & \uparrow\cong \\ {}_{n-1}\dot{\mathcal{V}}_{\underline{\lambda}} & \underset{\downarrow}{\longleftarrow} & {}_{n+1}\dot{\mathcal{V}}_{\underline{\lambda}} \end{array}$$

By the duality relations in $\mathcal{U}$, $\downarrow n+1$ and $\uparrow n-1$ are graded biadjoint of degree n,

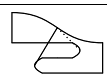
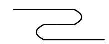
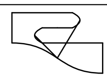
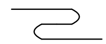
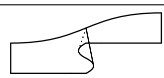
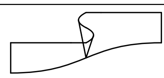
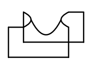
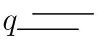
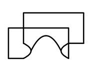
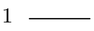
$$q^{-n} \downarrow n+1 \quad \dashv \quad \uparrow n-1 \quad \dashv \quad q^{n} \downarrow n+1$$

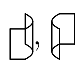
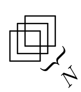
But $Ind \dashv Res \dashv coInd$, so $q^{-n}Ind \dashv q^{n}Res \dashv q^{-n}coInd$. Thus, $\uparrow n-1$ is equivalent to $q^{n}Res$ and $\downarrow n+1$ is equivalent to $q^{-2n}coInd$. $\square$

3.3 Bar-Natan's categorification of Temperley-Lieb diagrams

Let $\sqsupset$ be the graded, linear-enriched, semistrict, monoidal 2-category consisting of:

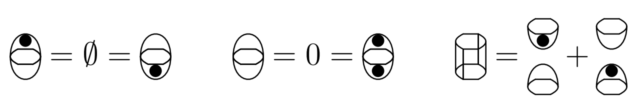
- 0-morphisms
  - There is a 0-morphism $\cdot$ the monoidal generator.
  - 0-morphisms are generated by monoidal products of these $\underbrace{\cdot\;\cdot\;\cdot}_N$ and the monoidal identity is $\underbrace{\cdot\;\cdot\;\cdot}_0 = \emptyset$.
- 1-morphisms
  - There is a 1-morphism $\supset$ an object in $Hom_{\sqsupset}(.\,^{\cdot} \leftarrow \emptyset)$.
  - There is a 1-morphism $\subset$ an object in $Hom_{\sqsupset}(\emptyset \leftarrow .\,^{\cdot})$.
  - 1-morphisms are generated by horizontal composition, monoidal products and grade shifts of these and the horizontal identity 1-morphisms $\underbrace{\equiv}_N$.
- 2-morphisms

There is a 2-morphism,	a morphism in,	with target,	and source.
	$End_{\sqsupseteq}(\emptyset)$		$q^{-1}\emptyset$
	$End_{\sqsupseteq}(\emptyset)$	$q\emptyset$	
	$End_{\sqsupseteq}(\emptyset)$		$q\emptyset$
	$End_{\sqsupseteq}(\emptyset)$	$q^{-1}\emptyset$	
	$End_{\sqsupseteq}(\cdot)$		
	$End_{\sqsupseteq}(\cdot)$		
	$End_{\sqsupseteq}(\cdot)$		
	$End_{\sqsupseteq}(\cdot)$		
	$End_{\sqsupseteq}(.\,^{\cdot})$		q 
	$End_{\sqsupseteq}(.\,^{\cdot})$	q^{-1} 	

– 2-morphisms are generated by horizontal and vertical composition, monoidal products, grade shifts, and linear combinations of these and the vertical identity 2-morphisms  and  .

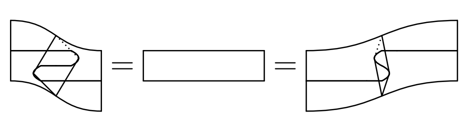
– 2-morphisms are subject to the following relations:

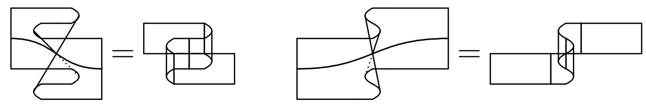
  * Bar-Natan relations



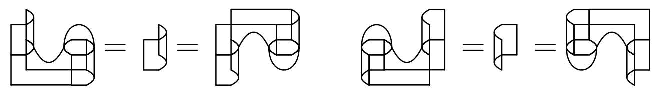
  * Duality relations:

    Cusp cancellation relations

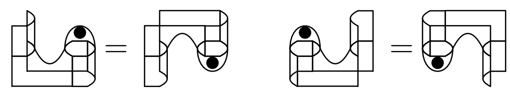




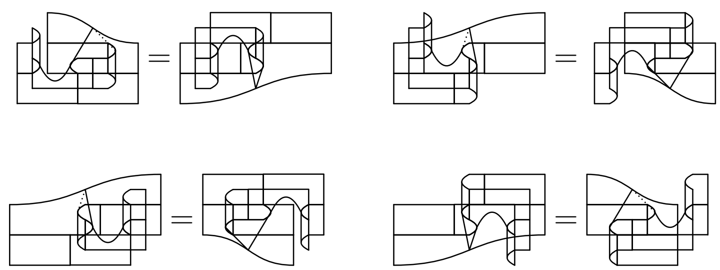
Morse cancellation relations



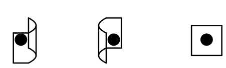
Dot slide relations



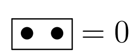
Saddle slide relations



∗ The duality relations guarantee that any 2-morphisms which are spatial isotopic as dotted surfaces with corners are equal. See [12] for details. Thus, notation can be extended to spatial isotopy classes of such dotted surfaces with corners making sense of such dotted surfaces as:



and we require the nilpotency relation for dotted surfaces



Proposition. *In $\beth$, there is a direct sum decomposition of 1-morphisms*

$$\bigcirc \cong \left(q \oplus q^{-1}\right) 1_{\emptyset}$$

Proof. The Bar-Natan relations are the relations for the direct sum decomposition. □

Given a dotted surface Σ, let $\chi(\Sigma)$ be its Euler characteristic, $h(\Sigma)$ be its number of horizontal boundary components and $d(\Sigma)$ be its number of dots $\bullet$.

Proposition. *The degree of a dotted surface Σ in $\beth$ is*

$$deg^{q}(\Sigma) = \mathcal{X}(\Sigma) - \frac{h(\Sigma)}{2} - 2d(\Sigma)$$

Proof. Degrees of monoidal products of dotted surfaces are additive since each of χ, h, d is additive under disjoint union. Degrees of horizontal and of vertical compositions of dotted surfaces are additive since $\chi - \frac{h}{2}$ is additive, which follows from the inclusion-exclusion principle for χ and comparing Euler characteristics of shared boundary curves with numbers of horizontal boundary components, and also d is additive. Thus, in order to be the formula for degree, it suffices that deg^{q} is the degree of the generating 2-morphisms, which, upon inspection, it is. □

Chapter 4

Intertwiner theory

4.1 Intertwiner 2-category

The 2-category *Twine* of intertwiners of categorified tensor representations consists of:

- 0-morphisms are sequences of categories $\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda}}\right)$, $n \in \mathbb{Z}$.

- 1-morphisms are intertwiners, sequences of functors

$$\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\mu}}\right) \xleftarrow{{}_nF} \mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda}}\right)$$

  such that for any object u in $hom_{\mathcal{K}^-(\dot{\mathcal{U}})}(n' \leftarrow n)$, the following diagram commutes up to functorial isomorphism.

$$\begin{array}{ccc} \mathcal{K}^-\left({}_{n'}\dot{\mathcal{V}}_{\underline{\mu}}\right) & \xleftarrow{{}_{n'}F} & \mathcal{K}^-\left({}_{n'}\dot{\mathcal{V}}_{\underline{\lambda}}\right) \\ {\scriptstyle u}\uparrow & \Leftrightarrow & \uparrow{\scriptstyle u} \\ \mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\mu}}\right) & \xleftarrow[{}_nF]{} & \mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda}}\right) \end{array}$$

  – Horizontal composition of intertwiners is the sequence of compositions of functors

${}_n(GF) = {}_nG_nF$ and horizontal identities are sequences of identity functors on $\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda}}\right)$.

- A 2-morphism is a sequence of functorial morphisms between intertwiners. Horizontal and vertical compositions are sequences of horizontal and vertical compositions of functorial morphisms and vertical identities are sequences of identity functorial morphisms.

- The morphism categories of $Twine$ inherit gradings q, t and a linear structure from the categories $\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda}}\right)$.

Proposition. *$Twine$ has a monoidal structure.*

Proof. Define the monoidal product of the 0-morphisms to be the concatenation,

$$\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda_1}}\right) \sqcup \mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda_2}}\right) = \mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda_1\lambda_2}}\right)$$

The categories $\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda}}\right)$ are equivalent to the derived categories $\mathcal{D}^-_{R^{\underline{\lambda}}_a}$ where $n = |\underline{\lambda}| - 2a$. Under this equivalence, a 1-morphism $\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\mu}}\right) \xleftarrow{{}_nF} \mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda}}\right)$ in $Twine$ gets sent to a sequence of functors $\mathcal{D}^-_{R^{\underline{\mu}}_b} \xleftarrow{{}_nF} \mathcal{D}^-_{R^{\underline{\lambda}}_a}$ which are functorially isomorphic to the derived tensor products $- \overset{\mathcal{L}}{\underset{R^{\underline{\lambda}}_a}{\otimes}} {}_nX$ where ${}_nX = {}_nF\left(R^{\underline{\lambda}}_a\right)$ is an object in the derived category ${}_{R^{\underline{\lambda}}_a}\mathcal{D}^-_{R^{\underline{\mu}}_b}$.

Given X_1, a right $R^{\underline{\lambda_1}}_{a_1}$-module, and X_2, a right $R^{\underline{\lambda_2}}_{a_2}$-module, X_1 is also a right $\widetilde{R}^{\underline{\lambda_1}}_{a_1}$-module and $X_1 \underset{\mathbb{K}}{\otimes} X_2$ is a right $\widetilde{R}^{\underline{\lambda_1}}_{a_1} \underset{\mathbb{K}}{\otimes} R^{\underline{\lambda_2}}_{a_2}$-module, so there is a right $R^{\underline{\lambda_1\lambda_2}}_{a_1+a_2}$-module $Ind_{a_1,a_2}\left(X_1 \underset{\mathbb{K}}{\otimes} X_2\right)$ where Ind_{a_1,a_2} is the extension of scalars functor induced by the inclusion of rings $\widetilde{R}^{\underline{\lambda_1}}_{a_1} \underset{\mathbb{K}}{\otimes} R^{\underline{\lambda_2}}_{a_2} \to R^{\underline{\lambda_1\lambda_2}}_{a_1+a_2}$ given by horizontal composition. Extension of scalars is exact so given objects X_1 in $\mathcal{D}^-_{R^{\underline{\lambda_1}}_{a_1}}$ and X_2 in $\mathcal{D}^-_{R^{\underline{\lambda_2}}_{a_2}}$, the definition extends to give an object $Ind_{a_1,a_2}\left(X_1 \underset{\mathbb{K}}{\otimes} X_2\right)$ in $\mathcal{D}^-_{R^{\underline{\lambda_1,\lambda_2}}_{a_1+a_2}}$. If X_1 is an object in ${}_{R^{\underline{\lambda_1}}_{a_1}}\mathcal{D}^-_{R^{\underline{\mu_1}}_{b_1}}$ and X_2 is an object in ${}_{R^{\underline{\lambda_2}}_{a_2}}\mathcal{D}^-_{R^{\underline{\mu_2}}_{b_2}}$, then define the object $Ind^{a_1,a_2}_{b_1,b_2}\left(X_1 \underset{\mathbb{K}}{\otimes} X_2\right)$ by extension of scalars for both the left and right actions.

Then, given intertwiners $\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\mu_1}}\right) \xleftarrow{{}_nF_1} \mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda_1}}\right)$ and $\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\mu_2}}\right) \xleftarrow{{}_nF_2} \mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{\underline{\lambda_2}}\right)$, they are equivalent to derived tensor products $- \overset{\mathcal{L}}{\underset{R^{\underline{\lambda_1}}_{a_1}}{\otimes}} {}_nX_1$ and $- \overset{\mathcal{L}}{\underset{R^{\underline{\lambda_1}}_{a_1}}{\otimes}} {}_nX_2$ so define their monoidal product to be the derived tensor products ${}_n(F_1 \sqcup F_2) = - \overset{\mathcal{L}}{\underset{R^{\underline{\lambda}}_{a}}{\otimes}} {}_n(X_1 \sqcup X_2)$ where

$$ {}_n(X_1 \sqcup X_2) = \bigoplus_{n_1+n_2=n} Ind^{a_1,a_2}_{b_1,b_2}\left({}_{n_1}X_1 \underset{\mathbb{K}}{\otimes} {}_{n_2}X_2\right) $$

Then ${}_n(F_1 \sqcup F_2)$ is an intertwiner for the action of $\mathcal{K}^-\left(\dot{\mathcal{U}}\right)$ since $\uparrow$ and $\downarrow$ commute with extension of scalars,

$$ \downarrow Ind^{a_1,a_2}_{b_1,b_2}\left(X_1 \underset{\mathbb{K}}{\otimes} X_2\right) \cong Ind^{a_1+1,a_2}_{b_1+1,b_2}\left(\downarrow X_1 \underset{\mathbb{K}}{\otimes} X_2\right) $$

$$ \uparrow Ind^{a_1,a_2}_{b_1,b_2}\left(X_1 \underset{\mathbb{K}}{\otimes} X_2\right) \cong Ind^{a_1-1,a_2}_{b_1-1,b_2}\left(\uparrow X_1 \underset{\mathbb{K}}{\otimes} X_2\right) $$

Also, $\sqcup$ is trivially unital and is associative since

$$ Ind^{a_1+a_2,a_3}_{b_1+b_2,b_3}\left(Ind^{a_1,a_2}_{b_1,b_2}\left(X_1 \underset{\mathbb{K}}{\otimes} X_2\right) \underset{\mathbb{K}}{\otimes} X_3\right) \cong Ind^{a_1,a_2+a_3}_{b_1,b_2+b_3}\left(X_1 \underset{\mathbb{K}}{\otimes} Ind^{a_2,a_3}_{b_2,b_3}\left(X_2 \underset{\mathbb{K}}{\otimes} X_3\right)\right) $$

Finally, $\sqcup$ extends the monoidal product defined on 0-morphisms since

$$ \bigoplus_{a_1+a_2=a} Ind^{a_1,a_2}_{a_1,a_2}\left(R^{\underline{\lambda_1}}_{a_1} \underset{\mathbb{K}}{\otimes} R^{\underline{\lambda_2}}_{a_2}\right) \cong R^{\underline{\lambda_1\lambda_2}}_{a} $$

The monoidal product extends functorially to morphisms between intertwiners and thus gives a monoidal structure on $Twine$. □

Let $Twine_1$ be the sub-2-category of $Twine$ with 0-morphisms restricted to $\mathcal{K}^-\left({}_n\dot{\mathcal{V}}_{1,\ldots,1}\right)$.

Theorem. *There is a representation of* $\beth$ *in* $Twine_1$ *which sends* q *to* $\frac{q}{t}$.

This theorem will be proved in a number of steps, but the critical ingrediant will be a module L which is the categorified analogue of the invariant dual canonical basis vector $v^1 \heartsuit v^{-1}$.

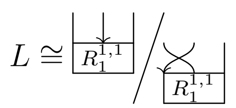
4.2 Graphical calculus of intertwiners

Define 0-morphisms in $Twine$

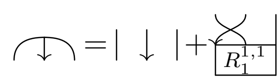
$$ {}_n\dot{\wr}_N = \mathcal{K}^- \left({}_n\dot{\mathcal{V}}_{\underbrace{1,\ldots,1}_{N}} \right) \cong \mathcal{D}^-_{R^{1,\ldots,1}_{\frac{N-n}{2}}} $$

thus representing the 0-morphisms $\dot{\wr}_N$ from $\sqsupset$ as the sequence of categories ${}_n\dot{\wr}_N$.

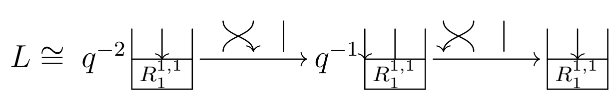
Let L be the irreducible right $R_1^{1,1}$-module



Then $L \cong \mathbb{K}$ with generator



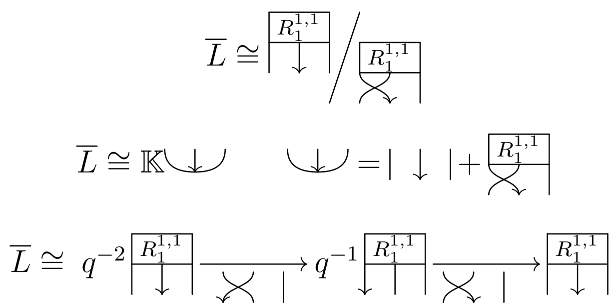
and L is quasi-isomorphic to its projective resolution



with coboundaries given by graded top composition by the indicated diagrams.

For a diagram D let $\overline{D}$ be the diagram obtained from D by reflecting it vertically and reversing all orientations. We must be careful to keep track of the grade shifts of targets and sources. If D has target $q^m b$ and source $q^n a$, then $\overline{D}$ has target $q^{-n} a$ and source $q^{-m} b$. The operation is an involution $\overline{\overline{D}} = D$. If D_1 and D_2 are graded composable, then as diagrams $\overline{\frac{D_2}{D_1}} = \frac{\overline{D_1}}{\overline{D_2}}$, but keeping track of the grade shifts, equality only holds if we redefine graded composition on the right hand side. Given two morphisms $q^{-n_1}a \xrightarrow{f} b$ and $q^{-n_2}b \xrightarrow{g} c$ in a graded category, define a second graded composition $q^{-n_1-n_2}a \xrightarrow{gf} c$ by $gq^{-n_2}(f)$. Now the equality holds using the first graded composition on the left hand side and the second graded composition on the right hand side.

Applying the bar involution to all diagrams in L and its projective resolution defines a left $R_1^{1,1}$-module $\overline{L}$,

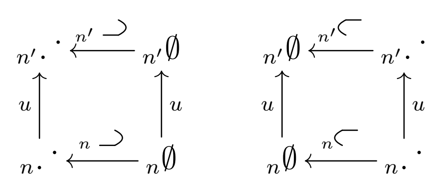


with coboundaries in the projective resolution given by second graded bottom composition by the indicated diagrams.

Define functors $_{n}\cdot^{\cdot} \xleftarrow{n\supset} {}_{n}\emptyset$ and $_{n}\emptyset \xleftarrow{n\subset} {}_{n}\cdot^{\cdot}$ which are both necessarily 0 for $n \neq 0$ and let $_{0}\supset = \frac{q}{t}\left(- \underset{\mathbb{K}}{\otimes} L\right)$ and $_{0}\subset = - \overset{\mathcal{L}}{\underset{R_1^{1,1}}{\otimes}} \overline{L}$.

Proposition. *$_{n}\supset$ and $_{n}\subset$ are intertwiners, 1-morphisms in Twine.*

Proof. We must show that given an object u in $hom_{\mathcal{K}^-(\dot{\mathcal{U}})}(n' \leftarrow n)$, the following diagrams commutes up to functorial isomorphism.



All such diagrams trivially commute except for the diagrams

$$
\begin{array}{ccc}
{}_{-2}\cdot^{\cdot} & \xleftarrow{{}_{-2}\supset} & {}_{-2}\emptyset \\
{\scriptstyle\downarrow}\big\uparrow & & \big\uparrow{\scriptstyle\downarrow} \\
{}_{0}\cdot^{\cdot} & \xleftarrow{{}_{0}\supset} & {}_{0}\emptyset
\end{array}
\qquad
\begin{array}{ccc}
{}_{2}\cdot^{\cdot} & \xleftarrow{{}_{2}\supset} & {}_{2}\emptyset \\
{\scriptstyle\uparrow}\big\uparrow & & \big\uparrow{\scriptstyle\uparrow} \\
{}_{0}\cdot^{\cdot} & \xleftarrow{{}_{0}\supset} & {}_{0}\emptyset
\end{array}
\qquad
\begin{array}{ccc}
{}_{0}\emptyset & \xleftarrow{{}_{0}\subset} & {}_{0}\cdot^{\cdot} \\
{\scriptstyle\downarrow}\big\uparrow & & \big\uparrow{\scriptstyle\downarrow} \\
{}_{2}\emptyset & \xleftarrow{{}_{2}\subset} & {}_{2}\cdot^{\cdot}
\end{array}
\qquad
\begin{array}{ccc}
{}_{0}\emptyset & \xleftarrow{{}_{0}\subset} & {}_{0}\cdot^{\cdot} \\
{\scriptstyle\uparrow}\big\uparrow & & \big\uparrow{\scriptstyle\uparrow} \\
{}_{-2}\emptyset & \xleftarrow{{}_{-2}\subset} & {}_{-2}\cdot^{\cdot}
\end{array}
$$

It suffices to check that the first two diagrams commute starting with $\mathbb{K}$ in the lower right, the third diagram commutes starting with $R_0^{1,1}$ in the lower right and the last diagram commutes starting with $R_2^{1,1}$ in the lower right since these are projective generators.

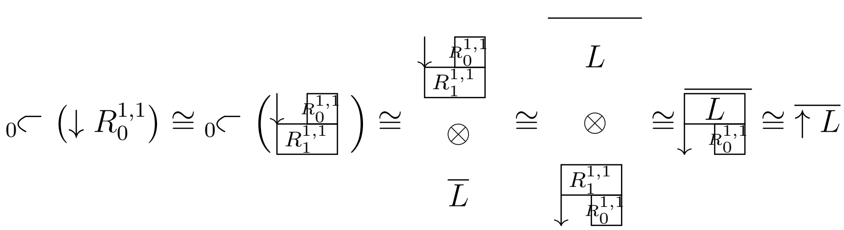
The compositions

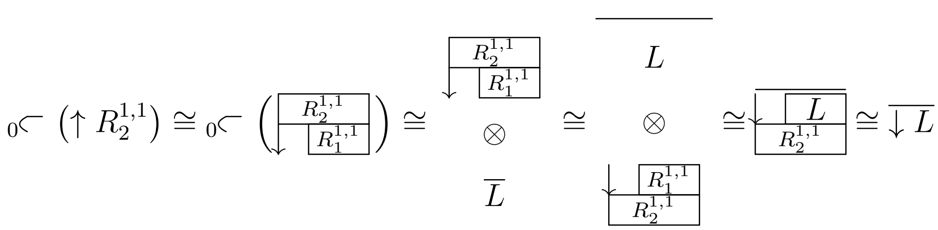
$$
{}_{-2}\supset(\downarrow(\mathbb{K})) \qquad {}_{2}\supset(\uparrow(\mathbb{K})) \qquad \downarrow\left({}_{2}\subset\left(R_0^{1,1}\right)\right) \qquad \uparrow\left({}_{-2}\subset\left(R_2^{1,1}\right)\right)
$$

are 0 since ${}_n\supset$ and ${}_n\subset$ are 0 for $n \neq 0$. We have that

$$
\downarrow {}_0\supset(\mathbb{K}) \cong \frac{q}{t}\downarrow L \qquad \uparrow {}_0\supset(\mathbb{K}) \cong \frac{q}{t}\uparrow L \qquad {}_0\subset\left(\downarrow R_0^{1,1}\right) \cong \overline{\uparrow L} \qquad {}_0\subset\left(\uparrow R_2^{1,1}\right) \cong \overline{\downarrow L}
$$

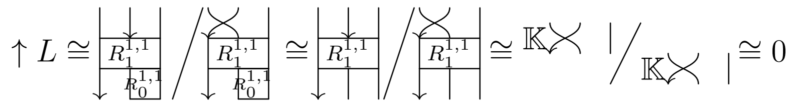
The latter two isomorphisms follow from considering the bar involution on diagrams,



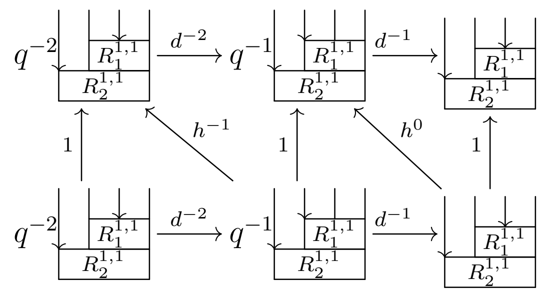


Thus, it suffices to show that $\uparrow L$ and $\downarrow L$ are isomorphic to 0 since this implies that the diagrams commute.

We get that

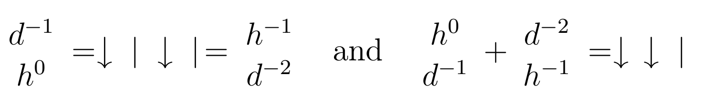


Also, the projective resolution of $\downarrow L$ has a contraction given by



where the coboundaries and homotopies are given by graded top compositions

h is a null-homotopy of the identity since

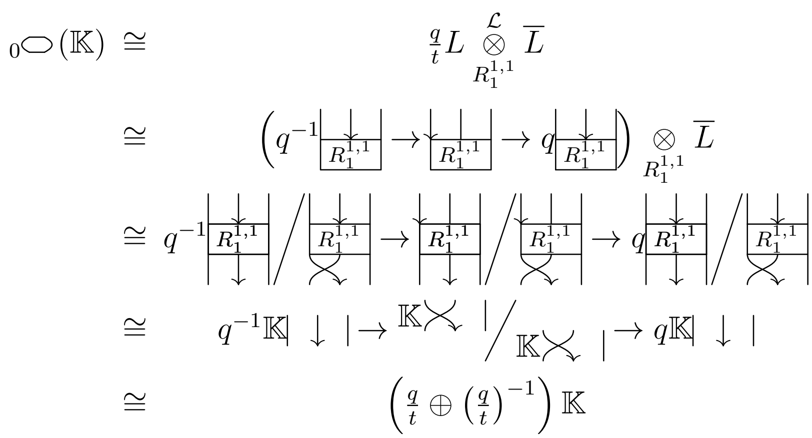


by the Webster and nil-Hecke relations. Thus, $\downarrow L \cong 0$. □

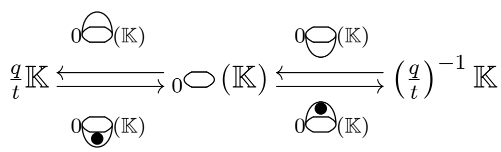
Thus, we have represented the generating 1-morphisms $\supset$ and $\subset$ of ℶ as intertwiners $_n\supset$ and $_n\subset$. Using the monoidal structures in ℶ and $Twine$, this extends to a representation of all 1-morphisms of ℶ.

4.3 Graphical calculus of bowls and dotted bowls

There is a direct sum decomposition

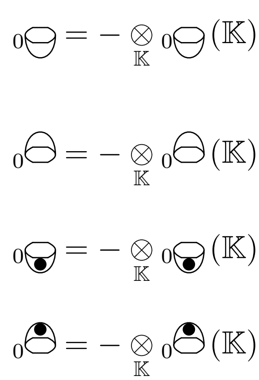


Denote its inclusions and projections as

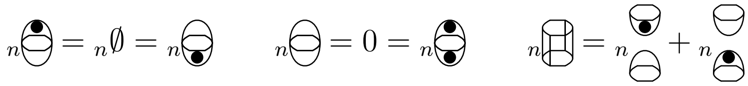


Define 2-morphisms

in $Twine$ which are necessarily 0 for $n \neq 0$ and let

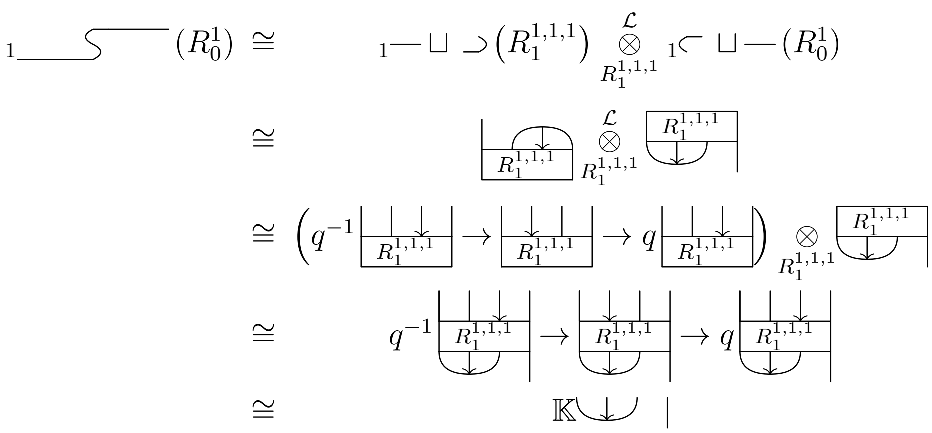


Then it follows that the Bar-Natan relations hold in *Twine*

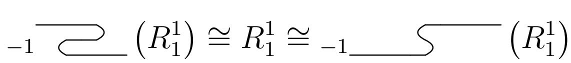


4.4 Graphical calculus of cusp surfaces

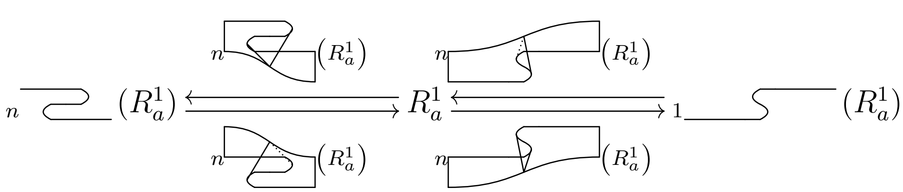
Calculate that



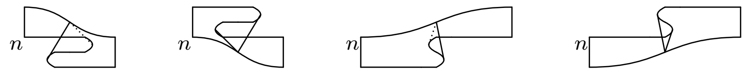
In the last step, plugging in generators, the first and third term vanish and we find that as R_0^1, R_0^1-bimodules  $(R_0^1) \cong R_0^1$ where  $|$ is identified with $|$. Similarly, we have that  $(R_0^1) \cong R_0^1$ and



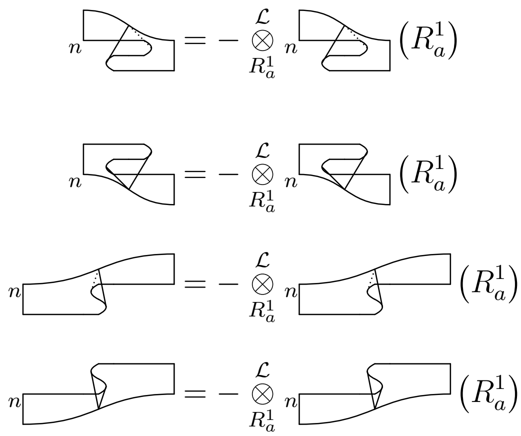
Denote these isomorphisms as



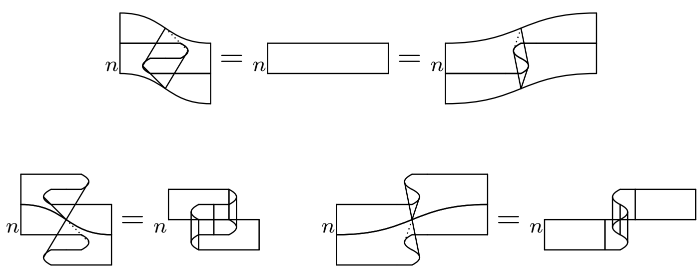
Define 2-morphisms



in $Twine$ which are necessarily 0 for $n \neq \pm 1$ and for $n = \pm 1$, let

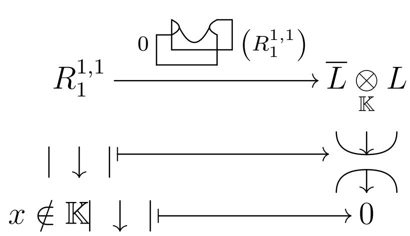


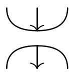
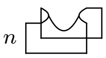
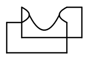
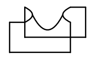
Then it follows that the cusp cancellation relations hold in $Twine$

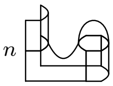
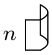


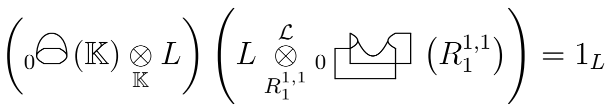
4.5 Graphical calculus of saddles

Define a homomorphism of $R_1^{1,1}, R_1^{1,1}$-bimodules,

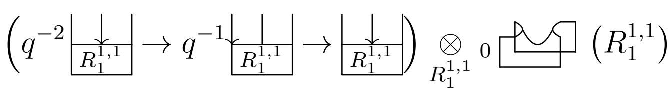


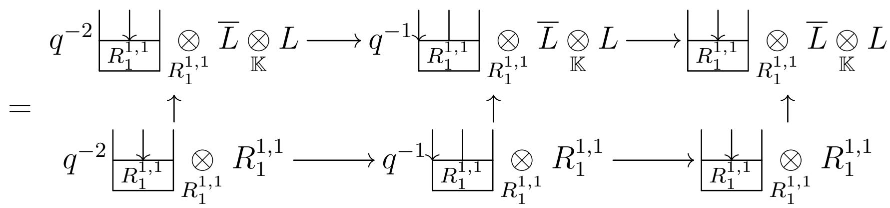
where  =  $\underset{\mathbb{K}}{\otimes}$  and define a 2-morphism $_n$  in *Twine* which is necessarily 0 for $n \neq 0$ and let $_0$  $= - \overset{\mathcal{L}}{\underset{R_1^{1,1}}{\otimes}}$ $_0$  $(R_1^{1,1})$.

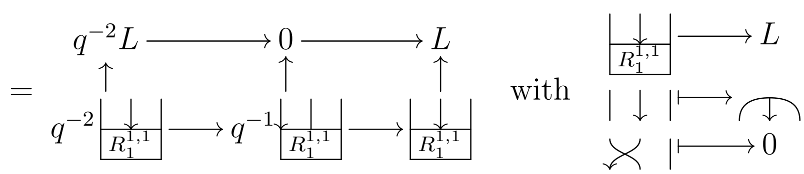
Then, to show that the Morse cancellation relation $_n$  $=$ $_n$  holds in *Twine* it suffices to show that



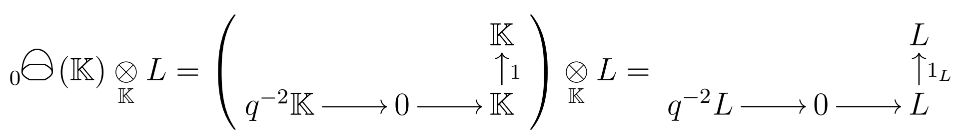
Replacing L with its projective resolution, calculate



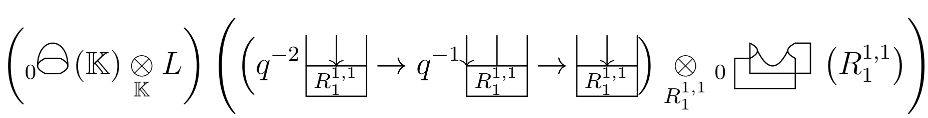




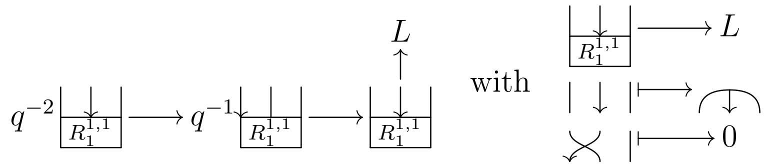
On the other hand,



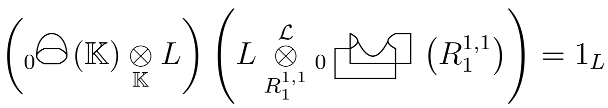
so that the composition

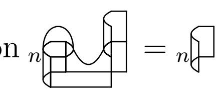


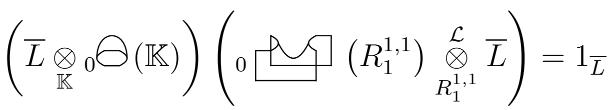
is equal to the quasi-isomorphism of L and its projective resolution,



so composing with the formal inverse of the quasi-isomorphism gives

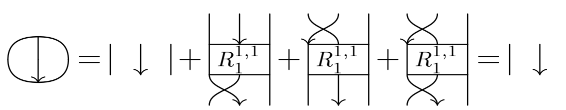


The Morse cancellation relation  follows from a similar calculation which shows that

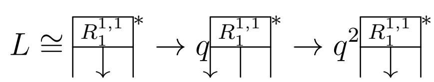


by replacing $\overline{L}$ with its projective resolution.

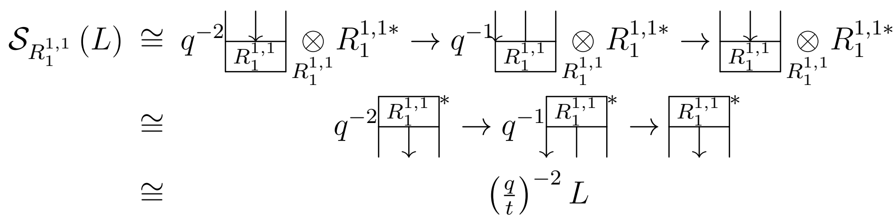
L and $\overline{L}$ are vector space duals of each other, $L \cong \overline{L}^*$, since vertically composing their generators is non-degenerate,



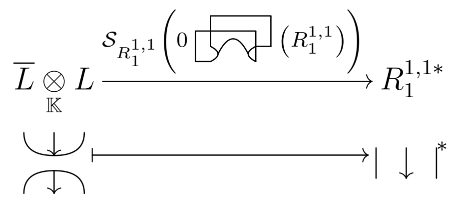
It follows that L has an injective resolution which is the vector space dual of the projective resolution of $\overline{L}$,



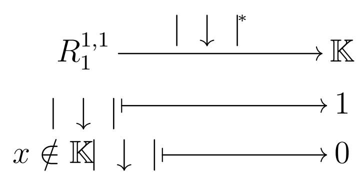
Thus, applying the Serre functor, $\mathcal{S}_{R_1^{1,1}} = - \overset{\mathcal{L}}{\underset{R_1^{1,1}}{\otimes}} R_1^{1,1*}$, to the projective resolution of L gives



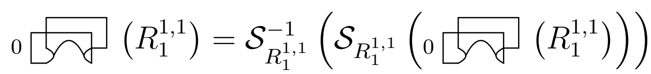
Define a homomorphism of $R_1^{1,1}, R_1^{1,1}$-bimodules,

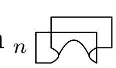
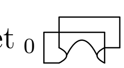
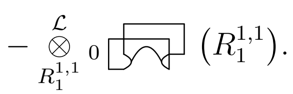


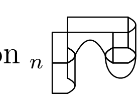
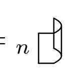
where

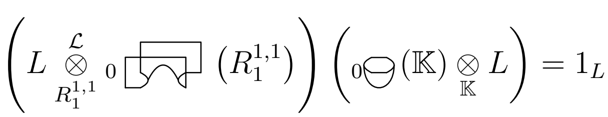


and let

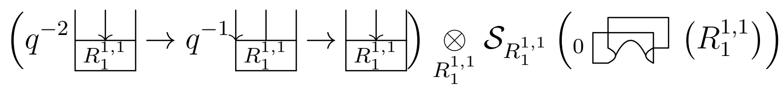


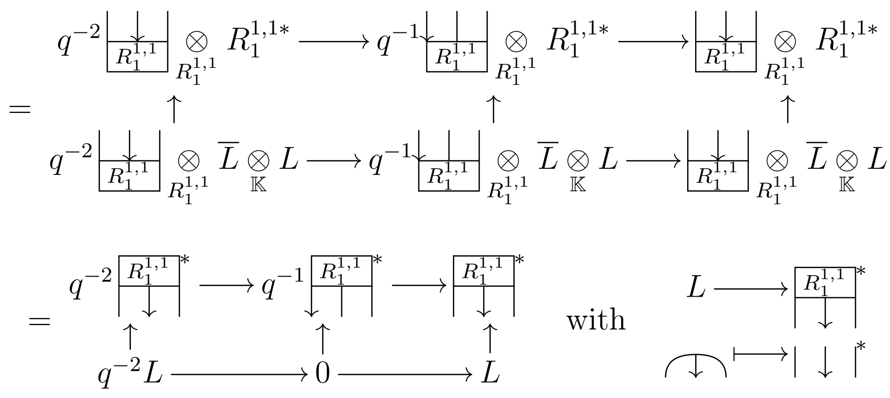
and define a 2-morphism  in $Twine$ which is necessarily 0 for $n \neq 0$ and let  $=$ .

To show that the Morse cancellation relation  $=$  holds in $Twine$ it suffices to show that

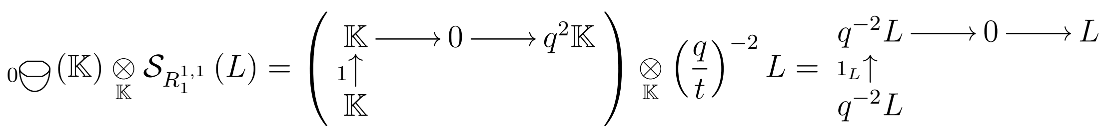


Replacing L with its projective resolution, calculate

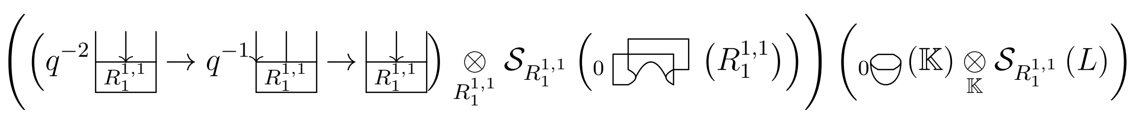




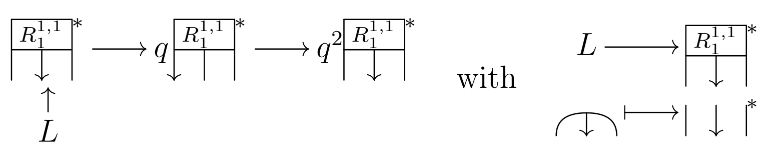
On the other hand,



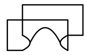
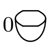
so that the composition

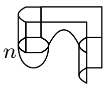
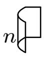


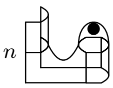
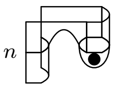
is equal to $\left(\frac{q}{t}\right)^{-2}$ applied to the quasi-isomorphism of L and its injective resolution,

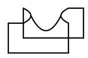
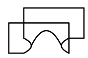


so composing with the formal inverse of the quasi-isomorphism and applying $\mathcal{S}^{-1}_{R_1^{1,1}}$ gives

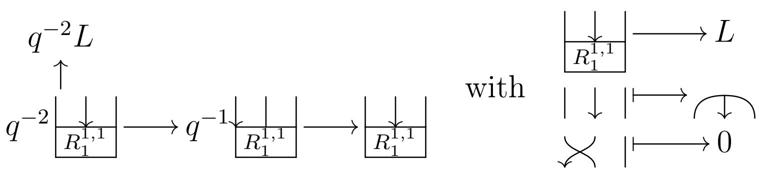
$\left(L \overset{\mathcal{L}}{\underset{R_1^{1,1}}{\otimes}} {}_0$  $\left(R_1^{1,1}\right) \right) \left({}_0$  $(\mathbb{K}) \underset{\mathbb{K}}{\otimes} L \right) = 1_L$

The Morse cancellation relation $_n$  $=$ $_n$  follows from a similar calculation.

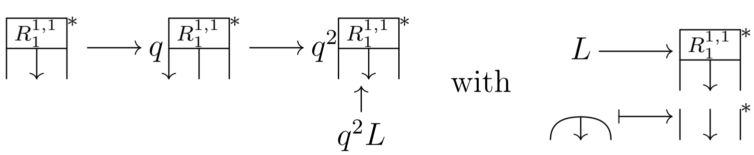
To show that the dot slide relation $_n$  $=$ $_n$  holds in *Twine* it suffices to show that

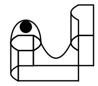
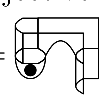
$\left({}_0$  $(\mathbb{K}) \underset{\mathbb{K}}{\otimes} L \right) \left(L \overset{\mathcal{L}}{\underset{R_1^{1,1}}{\otimes}} {}_0$  $\left(R_1^{1,1}\right) \right) = \left(L \overset{\mathcal{L}}{\underset{R_1^{1,1}}{\otimes}} {}_0$  $\left(R_1^{1,1}\right) \right) \left({}_0$  $(\mathbb{K}) \underset{\mathbb{K}}{\otimes} L \right)$

which follows since the left side equals the composition of

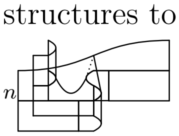
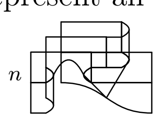


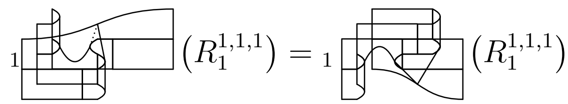
with the formal inverse of the quasi-isomorphism of L and its projective resolution and the right side equals the composition of



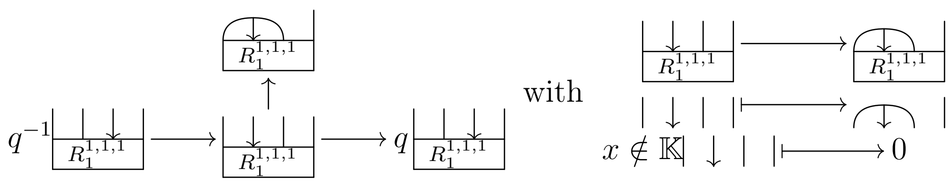
with the formal inverse of the quasi-isomorphism of L and its injective resolution and the two are equal up to a grade shift. The dot slide relation  $=$  similarly holds.

We can use the monoidal structures to represent all 2-morphisms of $\sqsupset$.

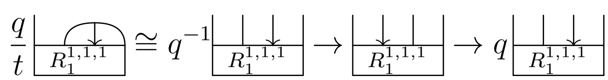
The saddle slide relation $_n$  $=$ $_n$  holds in *Twine* since it can be shown that both sides of



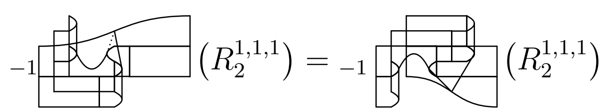
equal the composition of



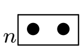
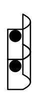
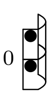
and the formal inverse of the quasi-isomorphism

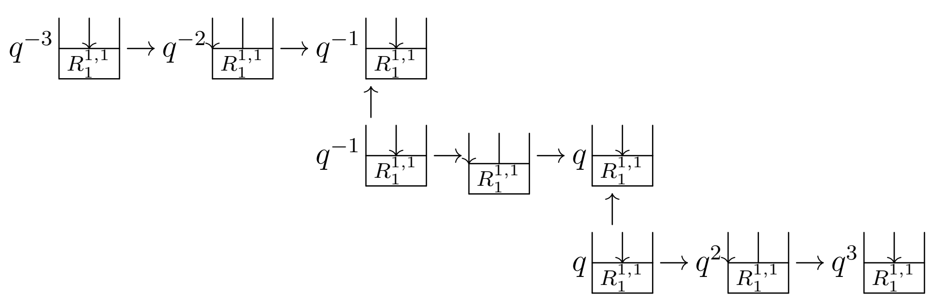


and similarly for $n = -1$,



while for $n \neq \pm 1$, both sides are 0. All the saddle slide relations hold similarly in *Twine*.

To show that the nilpotency relation $_n$ $= 0$ holds in *Twine* it suffices by spatial isotopy to show that $_n$ $= 0$. This is trivial for $n \neq 0$ and $_0$$\left(R_1^{1,1}\right)$ factors through the composition,



which is 0.

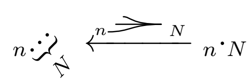
This proves the theorem that there is a representation of $\beth$ in $Twine_1$. As a corollary, we immediately get that there is a representation of $\mathcal{K}^-\left(\dot{\beth}\right)$ in $Twine_1$. I conjecture that $\mathcal{K}^-\left(\dot{\beth}\right)$ and $Twine_1$ are equivalent as monoidal 2-categories.

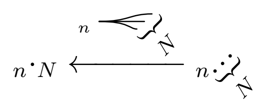
4.6 Categorification of Jones-Wenzl idempotents

Define 0-morphisms in $Twine$,

$$ {}_n\cdot_N = \mathcal{K}^-\left({}_n\dot{\mathcal{V}}_N\right) \cong \mathcal{D}^-_{R_a^N} $$

with $n = N - 2a$. Then define intertwiners,

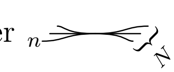


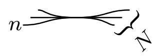


which are extensions of scalars Ind and restrictions of scalars Res induced by the inclusions of rings,

$$ R_a^N \longrightarrow R_a^{\overbrace{1,\cdots,1}^{N}} $$

$$ x\,\underset{N}{|} \longmapsto x\,\underbrace{|\cdots|}_{N} $$

Then  is functorially isomorphic to ${}_n\text{—}_N$, the identity on ${}_n\cdot_N$ since extending and then restricting scalars leaves a module unchanged. Thus, the intertwiner  is an idempotent,

The Jones-Wenzl idempotent has been categorified by Cooper and Krushkal using the graphical calculus of Bar-Natan's dotted surfaces [4]. I conjecture that their categorification corresponds with the intertwiner 

, although theirs is a bounded below complex in $\beth$, so it may be necessary to work with the categories $\mathcal{D}^{+}_{R^{\lambda}_{a}} \cong \mathcal{K}^{+}\left(\mathcal{I}_{R^{\lambda}_{a}}\right)$ of bounded below complexes of injective modules instead of the categories $\mathcal{D}^{-}_{R^{\lambda}_{a}} \cong \mathcal{K}^{-}\left(\mathcal{P}_{R^{\lambda}_{a}}\right)$ we used to define $Twine$.

Bibliography

[1] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *arXiv*, math.GT, October 2004.

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