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On Conformal Geometry of Kähler Surfaces

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Caner Koca

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Abstract of the Dissertation

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In this thesis, we study several problems related to conformal geometry of Kähler and Einstein metrics on compact 4-manifolds, by using the conformally invariant Weyl functional.

We first study a coupled system of equations on oriented compact 4-manifolds which we call the Bach–Merkulov equations. These equations can be thought of as the conformally invariant version of the classical Einstein–Maxwell equations. Inspired by the work of C. LeBrun on Einstein–Maxwell equations on compact Kähler surfaces, we give a variational characterization of solutions to Bach–Merkulov equations as critical points of the Weyl functional. We also show that extremal Kähler metrics are solutions to these equa-

tions, although, contrary to the Einstein–Maxwell analogue, they are not necessarily minimizers of the Weyl functional. We illustrate this phenomenon by studying the Calabi action on Hirzebruch surfaces.

Next we prove that the only compact 4-manifold M with an Einstein metric of positive sectional curvature which is also hermitian with respect to some complex structure on M, is $\mathbb{CP}_2$, with its Fubini–Study metric.

Finally we present an alternative proof of existence of conformally compact Einstein metrics on some complex ruled surfaces fibered over Riemann surfaces of genus at least 2. This result was first proved by C. Tønnesen-Friedman. We prove the existence by finding the critical points of the Weyl functional on space of all extremal Kähler metrics on these ruled surfaces.

Contents

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Chapter 1

Bach–Merkulov Equations and Extremal Kähler Metrics

1.1 Introduction

Let M be a smooth oriented n-manifold. A Riemannian metric g on M is said to satisfy the *Einstein–Maxwell equations* if

$$\begin{aligned} [r + F \circ F]_\circ = 0 \\ dF = 0, \quad d * F = 0 \end{aligned} \tag{1.1.1}$$

for some 2-form F on M. Here, r is the Ricci tensor of g; $(F \circ F)_{ij} = F_i{}^s F_{sj}$ is the composition of F with itself as an endomorphism of the tangent bundle TM; $[\cdot]_\circ$ denotes the trace-free part of a $(2, 0)$-tensor, and $*$ is the Hodge operator with respect to the metric g. When M is *compact*, the second line of (1.1.1), which is called *Maxwell equations*, is equivalent to saying that F is *harmonic* with respect to g, i.e. $\Delta F = 0$.

By Hodge theory we know that any harmonic form F minimizes the L^2 norm $F \mapsto \int_M |F|_g^2 d\mu_g$ among the forms cohomologous to F, namely on $[F] \in H^2_{\mathrm{dR}}(M, \mathbb{R})$. If, in addition, M has dimension 4, the integral $\int_M |F|_g d\mu_g$ is unchanged if g is replaced by any conformally related metric $\tilde{g} := ug$, for a positive smooth function u on M. Therefore, if F is harmonic with respect to g, it will be harmonic with respect to $\tilde{g}$. By contrast, the first line of (1.1.1) is certainly not conformally invariant in any dimension. There is, however, an interesting conformally invariant counterpart of these equations introduced by Merkulov in [28]:

$$\begin{gathered} B + [F \circ F]_\circ = 0 \\ dF = 0, \quad d * F = 0 \end{gathered} \tag{1.1.2}$$

where $B_{ij} = (\nabla^s \nabla^t + \frac{1}{2} r^{st}) W_{isjt}$ is the *Bach tensor* [3]. When M is compact, this tensor arises as the Euler–Lagrange equations for the *Weyl energy functional* $g \mapsto \int_M |W|^2 d\mu_g$ over the space of all metrics. That is, if we vary the metric $g_t = g_o + th + o(t^2)$, then [7]

$$\left.\frac{d}{dt}\right|_{t=0} \mathcal{W}(g_t) = \int_M \langle h, B \rangle d\mu = \int_M g^{is} g^{tj} h_{st} B_{ij} d\mu. \tag{1.1.3}$$

Note that in 4 dimensions, $\mathcal{W}$ is indeed conformally invariant since the conformal change $\tilde{g} = ug$ of the metric implies

$$d\tilde{\mu} = u^2 d\mu \quad \text{and} \quad \tilde{W}_{ijk}{}^l = W_{ijk}{}^l.$$

Bach tensor, too, behaves well under conformal change: $\tilde{B}_{ij} = \frac{1}{u}B_{ij}$. To see this note that for the rescaled variation $\tilde{g}_t = ug_0 + tuh + o(t^2)$ we have

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{W}(\tilde{g}_t) = \int_M \langle uh, \tilde{B}\rangle d\tilde{\mu} = \int_M \tilde{g}^{is}\tilde{g}^{tj}uh_{st}\tilde{B}_{ij}d\tilde{\mu}$$

and comparing it to (1.1.3) we deduce that $\tilde{B}_{ij} = \frac{1}{u}B_{ij}$. Also B is symmetric, trace-free and divergence-free. Note also that $[F \circ F]_\circ$, the other term in (1.1.2), rescales similar to B_{ij} under conformal rescaling. Clearing out the $\frac{1}{u}$ factors, we see that, when M is a compact manifold of dimension 4, the coupled system of equations (1.1.2) is conformally invariant in the sense that if (g, F) is a solution, so is (ug, F) for any positive smooth function u.

Both Einstein–Maxwell and Bach–Merkulov equations stem from a variational origin. For any given de Rham class $\Omega \in H^2_{\mathrm{d}R}(M, \mathbb{R})$, solutions (g, F) of Einstein-Maxwell solutions with $[F] = \Omega$ are in fact the critical point of the coupled action

$$\begin{aligned} \mathcal{G}_1 \times \Omega &\longrightarrow \mathbb{R} \\ (g, F) &\longmapsto \int_M s_g + |F|_g^2 d\mu_g \end{aligned}$$

where $\mathcal{G}_1$ stands for the space of unit volume metrics [25]. Similarly [3],[28], Bach–Merkulov solutions are the critical point of the action

$$\begin{aligned} \mathcal{G}_1 \times \Omega &\longrightarrow \mathbb{R} \\ (g, F) &\longmapsto \int_M |W|_g^2 + |F|_g^2 d\mu_g. \end{aligned}$$

In [25], C. LeBrun studied Einstein–Maxwell equations (1.1.1) on compact smooth 4-manifolds, and discovered some fascinating properties of these equations in relation to Kähler geometry. He showed that *constant scalar curvature Kähler* metrics satisfy (1.1.1); all solutions to (1.1.1) are critical points of L^2-norm of scalar curvature on $\mathcal{G}_\Omega$, the space of metrics for which a fixed cohomology class Ω is represented by a self-dual harmonic form Ω_g; and on complex surfaces constant scalar curvature Kähler metrics are *global minimizers* of that action if $c_1 \cdot \Omega \leq 0$. These results are summarized in Section 2. The aim of this chapter is to state and prove the relevant properties of the Bach–Merkulov equations. The first main results is:

Theorem 1.1.1. *Let M be a compact complex surface, and let g be a metric conformal to an extremal Kähler metric on M. Then g solves the Bach–Merkulov equations for some F. As a consequence, on any compact complex surface Kähler type we can solve* (1.1.2).

In other words, extremal Kähler metrics are standard solutions of Bach–Merkulov equations on a compact complex surface. On a more general compact oriented 4–manifold the Bach–Merkulov equations naturally become critical points of the Weyl energy functional $g \mapsto \int_M |W|^2 d\mu$:

Theorem 1.1.2. *Let M be a smooth compact oriented 4–manifold, and $\Omega \in H^2_{\mathrm{d}R}(M, \mathbb{R})$ be any de Rham class. A metric $g \in \mathcal{G}_\Omega$ is a critical point of the restriction of Weyl functional to $\mathcal{G}_\Omega$ iff g is a solution of Bach–Merkulov equations in conjunction with a unique harmonic form F whose self-dual part is $F^+ = \Omega_g$.*

On a compact Kähler surface, one could therefore ask analogously if extremal Kähler metrics are *absolute minimizers* of the Weyl functional on $\mathcal{G}_\Omega$ where Ω is the Kähler class represented by the extremal Kähler metric. It turns out that this is not the case:

Theorem 1.1.3. *For any given $\Omega \in H^2_{\mathrm{d}R}(M,\mathbb{R})$ on Kähler-type smooth 4-manifolds $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$ or $\mathbb{CP}_1 \times \mathbb{CP}_1$ the extremal Kähler metrics in $\mathcal{G}_\Omega$ (with respect to some complex structure) are not necessarily minimizers of the Weyl functional restricted to $\mathcal{G}_\Omega$.*

Theorem 1.1.1 and 1.1.2 are proved in Section 3 in Propositions 1.3.1, 1.3.2, 1.3.3. Theorem 1.1.3 is a consequence of the discussion in Section 4.

Recall that, given a compact complex manifold (M, J) with a Kähler class Ω (i.e. Ω is represented by a Kähler form), an *extremal Kähler metric* is, by definition, the critical point of the action

$$\begin{aligned} \Omega^+ &\longrightarrow \mathbb{R} \\ \omega &\longmapsto \int_M s_\omega^2 d\mu_\omega \end{aligned} \tag{1.1.4}$$

where Ω^+ stands for the space of *Kähler* forms in the de Rham class Ω. This notion of extremal metrics was introduced by Calabi [9] in an attempt to show existence of constant scalar curvature Kähler metrics on compact complex manifolds. The Euler-Lagrange equations of this action are given by $\frac{\partial^2 s}{\partial \bar{z}_i \partial \bar{z}_j} = 0$. In particular, every constant scalar curvature Kähler metric is extremal. The converse, however, is not true: For any given Kähler class on Hirzebruch srufaces $\mathbb{F}_k = \mathbb{P}(\mathcal{O}(-k) \oplus \mathcal{O})$, Calabi constructed explicit extremal Kähler metrics in that class. However, the first Hirzebruch surface $\mathbb{F}_1 \approx \mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$ cannot admit constant scalar curvature Kähler metric by Matsushima–Lichnerowicz theorem because the maximal compact subgroup Lie group of automorphisms is not reductive [9].

By computing the second variation of this action at a critical metric, Calabi was able to show that extremal Kähler metrics are local minimizers [9]. Indeed, they turn out to be *global* minimizers as proven recently by Donaldson and Chen [14], [11]. As we will discuss in Section 2, on compact complex surfaces, LeBrun showed that the constant scalar curvature Kähler metrics remain to be global minimizers of the action (1.1.4) if we extend the domain from Ω^+ to $\mathcal{G}_\Omega$, provided $c_1 \cdot \Omega \leq 0$. However, we will show in Section 4 that the extremal Kähler metrics are *not* necessarily global minimizers of the Weyl energy functional.

1.2 Einstein–Maxwell Equations

This section summarizes some of the results in [25].

Recall that the Euler–Lagrange equations of the action $g \mapsto \int_M s_g d\mu_g$, where g is allowed to vary over all *unit volume* metrics are precisely $\mathring{r} = 0$ (i.e. Einstein metrics). Also, from Hodge theory, the Euler–Lagrange equations of the action $F \mapsto \int_M |F|_g^2 d\mu_g$, where g is fixed but F is varying over all closed 2-forms in a fixed de Rham class $[F] \in H^2_{dR}(M, \mathbb{R})$ are the Laplace equation $\Delta F = 0$. Therefore, the Einstein–Maxwell equations are precisely the Euler-Lagrange equations of the joint action $(g, F) \mapsto \int_M s_g + |F|_g^2 d\mu_g$ where g is varying over unit volume Riemannian metrics and F is varying over a fixed de Rham class.

If we restrict the first action to the conformal class of a critical metric, we get the Einstein–Hilbert action whose critical points are well known to have constant scalar curvature [36]. Thus, any Einstein–Maxwell metric is of constant scalar curvature. Conversely, C. LeBrun observed the following remarkable fact:

Proposition 1.2.1 (LeBrun)**.** *Suppose that* (M^4, g, J) *is a Kähler surface with Kähler form* $\omega = g(J\cdot, \cdot)$ *and Ricci form* $\rho = r(J\cdot, \cdot)$*. If* g *is constant scalar curvature Kähler, then* g *satisfies Einstein–Maxwell equations with* $F = \omega + \frac{1}{2}\mathring{\rho}$*, where* $\mathring{\rho} = \mathring{r}(J\cdot, \cdot)$ *is the primitive part of the Ricci form* ρ *of* g*.*

Recall that constant scalar curvature Kähler metrics are in particular extremal Kähler, which are critical points of L^2-norm of scalar curvature $g \mapsto \int_M s_g^2 d\mu_g$, where g is varying over Kähler metrics on a fixed Kähler class $\Omega \in H^2_{dR}(M, \mathbb{R})$. C. LeBrun generalized the notion of a Kähler class and Calabi problem for a Kähler surface to the Riemannian setting, where, a priori, there may not be a complex structure at all. The generalization is as follows:

Let M be a smooth 4-manifold; and let Ω be a de Rham class as above. By Hodge theory, we know that any Riemannian metric g gives a unique harmonic representative Ω_g of Ω. If Ω_g is self-dual, g is called an Ω-adapted metric. The space of all Ω-adapted metrics is denoted by $\mathcal{G}_\Omega$; i.e. $\mathcal{G}_\Omega = \{g : *\Omega_g = \Omega_g\}$.

Observe that if M is a complex surface and Ω is a Kähler class, then $\mathcal{G}_\Omega$ contains all Kähler metrics in Ω, because any Kähler form is self-dual. In this sense, $\mathcal{G}_\Omega$ is a Riemannian generalization of a Kähler class. Also note that if $g \in \mathcal{G}_\Omega$, so is $\tilde{g} = ug \in \mathcal{G}_\Omega$ since Hodge $*$-operator is unchanged under conformal changes of the metric.

Now, as in the Calabi problem, C. LeBrun considers the action

$$g \mapsto \int_M s_g^2 d\mu_g \tag{*}$$

on $\mathcal{G}_\Omega$, and sees which metrics are critical points of this action:

Proposition 1.2.2 (LeBrun)**.** *Critical points of (*) are either*

(1) scalar-flat metrics (i.e. $s \equiv 0$*), or*

(2) Einstein–Maxwell metrics g *with* $F^+ = \Omega_g$*.*

Thus, in particular, constant scalar curvature Kähler metrics are critical points of (*). Moreover, they are actually *minimizers* if $c_1 \cdot \Omega \leq 0$.

Theorem 1.2.3 (LeBrun)**.** *Let* (M^4, J) *be a compact complex surface and* Ω *is a Kähler class with* $c_1 \cdot \Omega \leq 0$*. Then any metric* g *in* $\mathcal{G}_\Omega$ *satisfies* $\int_M s^2 d\mu \geq 32\pi^2 \frac{(c_1 \cdot \Omega)^2}{\Omega \cdot \Omega}$*, and equality holds iff* g *is constant scalar curvature Kähler.*

Another observation of C. LeBrun is that any compact smooth 4-manifold of Kähler type admits a solution of (1.1.1). This follows from Shu's result [30], which says that such 4-manifolds admit a constant scalar curvature Kähler metrics unless they are diffeomorphic to $\mathbb{C}P_2 \sharp \overline{\mathbb{C}P}_2$ or $\mathbb{C}P_2 \sharp 2\overline{\mathbb{C}P}_2$. However, both of these manifolds admit Einstein metrics (Page metric [29] and Chen–LeBrun–Weber metric [12]) which are automatically Einstein–Maxwell with $F = 0$.

1.3 Bach–Merkulov Equations

In this section we will state and prove analogues of LeBrun's results stated in section 1.2 for Bach–Merkulov equations.

First we start by observing the following proposition which shows that Bach–Merkulov equations possess an interesting family of solutions.

Proposition 1.3.1. *Let g be an extremal Kähler metric on a compact complex surface (M, J). Then (g, F) satisfies the Bach–Merkulov equations with $F = \omega + \frac{1}{2}\psi$ where $\psi = B(J\cdot, \cdot)$. Hence any metric conformal to an extremal Kähler metric is a solution of* (1.1.2).

Proof. The proof is similar to the one of Proposition 1.2.1. First, observe that $[F \circ F]_\circ = 2F^+ \circ F^-$ where F^+ and F^- are the self-dual and anti-self-dual part of F, respectively. Since g is Kähler, ω is a self-dual harmonic 2-form. Moreover, since g is *extremal*, $\psi = B(J\cdot, \cdot)$ is an anti-self-dual harmonic 2-form (see [12]). Thus, setting $F^+ = \omega$ and $F^- = \frac{\psi}{2}$, we see that $2F^+ \circ F^- = \omega_i{}^s \psi_{sj} = \psi(J\cdot, \cdot) = -B$. Thus we get $B + [F \circ F]_\circ = 0$. Moreover, F is harmonic since both F^+ and F^- are so. Therefore, (g, F) is a solution of Bach–Merkulov equations. □

More explicitly, if g is extremal Kähler, then the Bach tensor can be re-written in the form

$$B = \frac{1}{12}(s\mathring{r} + 2\,\mathrm{Hess}_\circ(s))$$

and therefore $\psi = \frac{1}{12}[s\rho + i\partial\bar{\partial}s]_\circ$ where $[\,\cdot\,]_\circ$ stands for the primitive part of a (1,1)-form (see [12]). In particular, if the extremal Kähler metric turns out to have *non-zero constant scalar curvature*, then ψ simplifies to $\frac{s}{12}\mathring{\rho}$. So we see that the solution of Proposition 1.3.1 becomes $(g, F = \omega + \frac{s}{24}\mathring{\rho})$ which is quite similar to LeBrun's solution to Einstein–Maxwell equations $(g, F = \omega + \frac{\mathring{\rho}}{2})$.

Proposition 1.3.1 together with Shu's result implies the following:

Proposition 1.3.2. *Let M be the underlying 4-manifold of any compact complex surface of Kähler type. Then M admits a solution (g, F) of Bach–Merkulov equations.*

Next, we will prove the analogue of Proposition 1.2.2 for Bach–Merkulov equations:

Proposition 1.3.3. *An Ω-adapted metric g is a critical point of the restriction of Weyl functional to $\mathcal{G}_\Omega$ iff g is a solution of Bach–Merkulov equations in conjunction with a unique harmonic form F with $F^+ = \Omega_g$.*

Proof. The proof is similar to the one of Proposition 1.2.2. Let $g_t = g + th + O(t^2)$ be a variation of a metric g in $\mathcal{G}_\Omega$. Donaldson showed that the tangent space $T_g\mathcal{G}_\Omega$ is precisely the L^2-orthogonal complement of $\{\Omega_g \circ \varphi : \varphi \in \mathcal{H}_g^-\}$ in $\Gamma(\bigodot^2 T^*M)$. Thus, in our case, h can be taken such that $\int_M \langle h, \Omega_g \circ \varphi\rangle d\mu_g = 0$ for all $\varphi \in \mathcal{H}_g$.

The first variation of the Weyl functional is given by ([3], [7])

$$\frac{d}{dt}\int_M \|W\|^2 d\mu_{g_t}\bigg|_{t=0} = \int_M h^{ij}B_{ij}d\mu_g = \int_M \langle h, B\rangle d\mu_g.$$

Thus, g is a critical point iff h is L^2-orthogonal to B. By Donaldson's result, this implies that $B = \Omega_g \circ \varphi$ for some $\varphi \in \mathcal{H}_g^-$. So, taking $F^+ = \Omega_g$ and $F^- = -\frac{\varphi}{2}$, we see that g satisfies (1.1.2).

Conversely, if g is an Ω-adapted solution of (1.1.2) with $F^+ = \Omega_g$, then $\int \langle h, B \rangle d\mu = 2 \int \langle h, F^+ \circ F^- \rangle d\mu = 0$ for any variation h as above. Thus, by Donaldson, g is a critical point. □

In particular, extremal Kähler metrics are also critical points of this functional. The natural question to ask is whether they are global minimizers in $\mathcal{G}_\Omega$. In the next section, we will show that the answer to this question is negative: the analogue of Theorem 1.2.3 does not hold for Bach–Merkulov equations.

1.4 Example: Hirzebruch Surfaces

In this section we will show that extremal Kähler metrics adapted to a fixed cohomology class Ω do not necessarily have the same Weyl energy. We will illustrate this fact on Hirzebruch surfaces, by showing existence of two closed forms in Ω which are extremal Kähler with respect to different complex structures. Using the formula in [21] it will turn out that the Weyl energy of the corresponding extremal Kähler metrics are different.

Recall that the k-th Hirzebruch surface $\mathbb{F}_k$ is defined as the projectivization of the rank-2 complex vector bundle $\mathcal{O}(-k) \oplus \mathcal{O}$ over $\mathbb{CP}_1$ (see [5] and [4] for details). $\mathbb{F}_k$ is diffeomorphic to $S^2 \times S^2$ if k is even, and to $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}_2}$ if k is odd [19]. They are, however, all biholomorphically distinct as complex surfaces (see [19]). They are simply connected; they have second Betti number $b_2(\mathbb{F}_k) = 2$ and Euler characteristic $\chi(\mathbb{F}_k) = 4$.

For the generators of the homology of $\mathbb{F}_k$ we will take the fiber F and the image of the section $\{z \mapsto [0 : z]\} : \mathbb{CP}_1 \to \mathbb{P}(\mathcal{O}(-k) \oplus \mathcal{O}) = \mathbb{F}_k$, which we will denote by C_k. Note that $F \cdot F = 0$, $F \cdot C_k = 1$ and $C_k \cdot C_k = -k$ so that in this basis the intersection pairing becomes

$$\begin{pmatrix} 0 & 1 \\ 1 & -k \end{pmatrix}.$$

Let the Poincaré dual of C_k and F be $\mathfrak{c}_k$ and $\mathfrak{f}$ respectively. Then any de Rham class $\Omega \in H^2_{dR}(\mathbb{F}_k, \mathbb{R})$ can be written as $\Omega = p\mathfrak{c}_k + q\mathfrak{f}$ for some $p, q \in \mathbb{R}$. If k and n are two positive integers of same parity, then $\mathbb{F}_k$ and $\mathbb{F}_n$ are diffeomorphic; so we can represent Ω with respect to the basis $\{\mathfrak{c}_n, \mathfrak{f}\}$. The following lemma gives the change of basis formula:

Lemma 1.4.1. *We have*

$$\mathfrak{c}_k = \mathfrak{c}_n + \frac{n-k}{2}\mathfrak{f}.$$

Therefore,

$$\Omega = p\mathfrak{c}_k + q\mathfrak{f} = p\mathfrak{c}_n + \tilde{q}\mathfrak{f}$$

where $\tilde{q} = p\frac{n-k}{2} + q$

Proof. Let

$$C_k = sC_n + tF \qquad (1.4.1)$$

for some constants s, t. Take the intersection of both sides with F:

$$C_k \cdot F = sC_n \cdot F + tF \cdot F.$$

Since $C_k \cdot F = C_n \cdot F = 1$ and $F \cdot F = 0$, we have $s = 1$. On the other hand, take the self intersection of both sides:

$$\begin{aligned} C_k \cdot C_k &= (C_n + tF) \cdot (C_n + tF) \\ -k &= -n + 2t. \end{aligned}$$

Therefore $t = \frac{n-k}{2}$. Taking the Poincaré dual of (1.4.1) proves the first equality. The second equality follows immediately from this. $\square$

Note also that $\Omega = p\mathfrak{c}_k + q\mathfrak{f} \in H^2_{dR}(\mathbb{F}_k, \mathbb{R})$ is a Kähler class iff $\Omega \cdot C_k > 0$ and $\Omega \cdot F > 0$, that is, iff $p > 0$ and $q > kp$. Now we can deduce when the same de Rham class Ω is a Kähler class in $\mathbb{F}_n$, where n and k have the same parity. Let J_k denote the complex structure of the complex surface $\mathbb{F}_k$.

Lemma 1.4.2. *A Kähler class $\Omega = p\mathfrak{c}_k + q\mathfrak{f}$ in $\mathbb{F}_k$ is a Kähler class in $\mathbb{F}_n$ iff $n < 2\frac{q}{p} - k$. In particular, Ω is Kähler with respect to only* finitely many *J_n's.*

Proof. $\Omega = p\mathfrak{c}_k + q\mathfrak{f}$ is Kähler with respect to J_k iff $p > 0$ and $q > kp$. By Lemma 1.4.1, $\Omega = p\mathfrak{c}_n + \left(p\frac{n-k}{2} + q\right)$. Now, by the previous paragraph, this class is Kähler with respect to J_n iff $p > 0$ and $p\frac{n-k}{2} + q > np$. The second inequality is the same as $n < 2\frac{q}{p} - k$. Notice that $2\frac{q}{p} - k$ is positive since $\Omega = p\mathfrak{c}_k + q\mathfrak{f}$ is assumed to be Kähler since $q > pk$. Hence there are only finitely many possibilities for n so that Ω remains Kähler with respect to J_n. $\square$

So there are de Rham classes on smooth 4-manifolds $S^2 \times S^2$ or $\mathbb{CP}_2\sharp\overline{\mathbb{CP}_2}$ which are Kähler with respect to different complex structures. However, Calabi [9] showed that every Kähler class on a Hirzebruch surface is represented by an *extremal* Kähler metric.

So, with our previous notation all Riemannian metrics g whose Kähler form $\omega = g(J_k\cdot, \cdot)$ with respect to any of the complex structures J_k are in $\mathcal{G}_\Omega$. Thus, we have essentially distinct extremal Kähler metrics in $\mathcal{G}_\Omega$. Each of those metrics are critical points of the restriction of the Weyl functional to $\mathcal{G}_\Omega$.

Next, we will show that this the Weyl energy levels of those metrics are different. First note that [13] for Kähler metrics we have $|W^+|^2 = \frac{s^2}{24}$. By signature formula, $\int |W^+|^2 d\mu = \frac{1}{2}\int |W|^2 d\mu + 6\pi^2\tau$. Since the signature τ of the Hirzebruch surfaces is 0, we see that the Weyl energy of a Kähler metric is equal to its Calabi energy up to an overall multiplicative constant. Hwang&Simanca [21] gave the following formula for the Calabi energy of an extremal metric in a Kähler class on the Hirzebruch surface $\mathbb{F}_k$.

Proposition 1.4.3 (Hwang&Simanca)**.** *The Calabi energy of the extremal Kähler metric in the class* $\Omega = 4\pi\mathfrak{c}_k + 2\pi(a+k)\mathfrak{f}$ *in* $\mathbb{F}_k$ *is given as:*

$$\tilde{\mathcal{C}}(a,k) := 12\pi\frac{a^3 + 4a^2 + (4+k^2)a - 4k^2}{3a^2 - k^2}. \tag{1.4.2}$$

Note that the Calabi energy and the Weyl energy are scale-invariant in dimension four. Therefore by appropriate scaling we see that the Calabi energy of the extremal Kähler metric in $\Omega = p\mathfrak{c}_k + q\mathfrak{f}$ is given by

$$\mathcal{C}(p,q,\mathbb{F}_k) := \tilde{\mathcal{C}}(2\frac{q}{p}-k, k) = 12\pi\frac{(2\frac{q}{p}-k)^3 + 4(2\frac{q}{p}-k)^2 + (4+k^2)(2\frac{q}{p}-k) - 4k^2}{3(2\frac{q}{p}-k)^2 - k^2}. \tag{1.4.3}$$

We therefore see that the extremal Kähler metrics with respect to different complex structures in Ω have different energy, i.e. $\mathcal{C}(p,q,\mathbb{F}_k) \neq \mathcal{C}(p, p\frac{n-k}{2} +$

$q, \mathbb{F}_n$) in general. Thus, they also have different Weyl energy.

This shows that the analogue of of Theorem 1.2.3 cannot hold for the Bach–Merkulov equations.

1.5 Minimizers of Weyl functional

We are concluding this chapter with a discussion on critical points of the restriction of the Weyl functional to various subspaces of the space of all metrics $\mathcal{G}$.

As discussed earlier, the critical points of $\mathcal{W}$ as a functional on $\mathcal{G}$ are precisely the Bach-flat metrics. Locally conformally flat metrics and half-conformally flat metrics are minimizers of this action. Einstein metrics are Bach-flat by the second Bianchi identity, but they are not necessarily of minimal energy.

On a compact complex manifold (M, J) with a fixed Kähler class Ω, the critical points of $\mathcal{W}$ restricted to Ω^+ are extremal Kähler metrics. They are global minimizers of this action, and such a minimizer is unique (up to biholomorphism) if it exists ([11],[14]). Constant scalar curvature Kähler metrics (in particular Kähler-Einstein metrics) are examples of extremal Kähler metrics. Hirzebruch surfaces are examples of compact complex surfaces for which all Kähler classes are represented by an extremal Kähler metric which is not of constant scalar curvature. Tønnesen-Friedman [34] gave examples of complex surfaces for which some classes admit extremal Kähler representatives and some do not. Burns and de Bartolomeis [8] gave examples of surfaces for which no Kähler class has an extremal Kähler representative.

If, on a compact complex surface (M, J), we regard $\mathcal{W}$ as a functional on the extremal cone (i.e. the space of all extremal Kähler metrics), the critical points are surprisingly Bach-flat (see [12]), that is, they are critical points of the functional on the whole space $\mathcal{G}$. If g is such a Bach-flat Kähler metric, then the conformally related metric $h := s^{-2}g$ is Einstein wherever $s \neq 0$ (see [13]). This is how the (globally) Einstein metrics known as the Page metric and Chen–LeBrun–Weber metric are found. In the last chapter of this thesis, using the same method, we will show existence of Poincaré–Einstein metrics in some ruled surfaces which blow up on the three dimensional submanifold $\{s = 0\}$. In all these examples the critical points turn out to be minimizers, but no such general statement is known.

Gursky [16] studied the action $\mathcal{W}^+ := \int |W^+|^2 d\mu$ as a functional on Yamabe-positive metrics $\mathcal{Y}_+$ on compact oriented 4-manifolds with $b^+ > 0$. Here $\mathcal{Y}_+$ stands for the space of conformal classes of metrics with constant positive scalar curvature. He gave the following general topological lower bound for the functional:

$$\mathcal{W}^+[g] \geq \frac{4\pi^2}{3}(2\chi + 3\tau)$$

where χ is the Euler characteristic and τ is the signature of the manifold. Furthermore, he showed that equality is achieved only for Kähler–Einstein metrics (in which case they are automatically minimizers). This happens only when M is either

(i) $\mathbb{CP}_2 \sharp k \overline{\mathbb{CP}}_2$, $k = 3, 4, \ldots, 8$ with Tian's Kähler–Einstein metrics

(ii) $\mathbb{CP}_2$ with Fubini–Study metric

(iii) $\mathbb{CP}_1 \times \mathbb{CP}_1$ with the product Einstein metric.

It is interesting that the cases $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2$ and $\mathbb{CP}_2 \sharp 2\overline{\mathbb{CP}}_2$ are excluded in Gursky's results. These manifolds are known not to admit a Kähler–Einstein metric [33], so the lower bounds $\frac{4\pi^2}{3}(2\chi + 3\tau)$, which in our case is equal to $\frac{32\pi^2}{3}$ and $\frac{28\pi^2}{3}$, respectively, are certainly not attained in $\mathcal{Y}_+$. However, there are canonical Einstein metrics on these surfaces, namely the Page metric and the Chen–LeBrun–Weber metric, which are potential candidates for the minimizers. As a numerical evidence, note that the gap between $\mathcal{W}^+[g_{\mathrm{Page}}] \approx 107.63$ (see [23]) and the lower bound $\frac{32\pi^2}{3} \approx 105.28$ is very small. In passing, one should also keep in mind that the Kähler–Ricci solitons discovered by Koiso–Cao on $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2$ and Wang–Zhu on $\mathbb{CP}_2 \sharp 2\overline{\mathbb{CP}}_2$ are other candidates for the minimizer (if it exists).

We conjecture that the Page metric and the Chen–LeBrun–Weber metric are indeed minimizers on the space of all metrics $\mathcal{G}$. More weakly, we suspect that the Page metric (which is given explicitly in coordinates) is indeed a local minimizer. One way to show this would be to use O. Kobayashi's formula [22] for the second variation of the Weyl functional at an Einstein metric: If $g = g_0 + th + O(t^2)$ is a variation of an Einstein metric g_0, then

$$\frac{d^2}{dt^2}\mathcal{W}(g_t)\bigg|_{t=0} = \int \langle (\Delta_L + \frac{s}{2})h, (\Delta_L + \frac{s}{3})h)\rangle d\mu$$

where $(\Delta_L h)_{ij} = (\Delta h)_{ij} + h_{ik;kj} - h_{ik;jk} + h_{jk;ki} - h_{jk;ik}$ is the Lichnerowicz Laplacian and Δ is the rough Laplacian. This complicated calculation may illuminate the answer of our problem.

Chapter 2

Positively Curved Einstein Hermitian Metrics

2.1 Introduction

Let M be a smooth n-manifold. A Riemannian metric g on M is called *Einstein* if the Ricci tensor is a constant multiple of the metric tensor, i.e.

$$r = \lambda g$$

for some constant $\lambda \in \mathbb{R}$, called the *Einstein constant* [7]. If $\lambda > 0$ and g is complete, M is compact by Myer's Theorem. Since the Ricci tensor is, by definition, the pointwise average of all sectional curvatures, for an Einstein metric the positivity of λ is assured if all sectional curvatures are positive.

In this chapter, we are dealing with compact smooth manifolds in dimension 4 which admit Einstein metrics with $\lambda > 0$. Examples of such manifolds are 4-sphere S^4 with its standard round metric (which has all sectional curvatures $K \equiv 1$), and the complex projective plane $\mathbb{CP}_2$ with the Fubini-Study metric g_{FS} (which has $1 \le K \le 4$ everywhere). Notice that these examples are of strictly *positive* sectional curvature. The product metric on $S^2 \times S^2$ is Einstein with $\lambda > 0$, too, its sectional curvatures, however, are *non-negative.* They are actually 0 for transverse planes (i.e. $K_\Pi = 0$ if the plane Π is not tangent to each of the factors). In fact, the famous Hopf Conjecture asks whether or not there are any metrics on $S^2 \times S^2$ of positive sectional curvature.

The most fruitful resource of Einstein metrics is the Kähler geometry. For a compact complex surface M,

a. there is a unique Kähler-Einstein metric with $\lambda < 0$ if $c_1(M) < 0$ (see [38], [2]),

b. there is a unique Kähler-Einstein metric with $\lambda = 0$ (i.e. Ricci-flat) in each Kähler class if $c_1(M) = 0$ (see [37]).

c. In $\lambda > 0$ case, Tian [33] showed that M admits a Kähler-Einstein metric with $\lambda > 0$ iff M has $c_1(M) > 0$ and its automorphism group $\mathrm{Aut}(M)$ is a reductive Lie group. The diffeomorphism types of such complex surfaces are $\mathbb{CP}_2$, $\mathbb{CP}_1 \times \mathbb{CP}_1$ and $\mathbb{CP}_2 \sharp k \overline{\mathbb{CP}}_2$ with $k = 3, 4, \ldots, 8$.

Among those Kähler-Einstein metrics, only the ones on $\mathbb{CP}_2$ are of *positive* sectional curvature. This follows, for example, by Andreotti's Theorem [1], which says that any compact complex surface M with a Kähler metric of positive sectional curvature must be $\mathbb{CP}_2$. Andreotti's theorem is a special case of Frankel's conjecture [15], which was later proved by Siu and Yau [32], which asserts that the generalization of the statement is true for all complex dimensions n.

As a consequence of the following theorem, Gursky and LeBrun [17] reached at the same conclusion that $\mathbb{CP}_2$ is the unique compact complex surface with a positively curved Kähler-Einstein metric:

Theorem 2.1.1 (Gursky-LeBrun)**.** *Let (M^4, g) be an compact oriented Einstein 4-manifold of non-negative sectional curvature.*

(i) if M has positive intersection form, then $(M, g) = (\mathbb{CP}_2, g_{FS})$, up to rescaling and isometry;

(ii) if g is neither self-dual nor anti-self-dual, then $\frac{15}{4}|\tau| < \chi \leq 9$ where τ is the signature and χ is the Euler characteristic of M.

Note that $\mathbb{CP}_2 \sharp k\overline{\mathbb{CP}}_2$ have $\tau = 1 - k$ and $\chi = 3 + k$, so the inequality is not satisfied if $k = 3, 4, \ldots, 8$. The Kähler-Einstein metrics on those surfaces cannot be self-dual (that is, $W_- \equiv 0$) or anti-self-dual (that is, $W_+ \equiv 0$) as a consequence of

$$12\pi^2\tau = \int_M |W_+|^2 - |W_-|^2 d\mu \qquad \text{(Signature Formula)}$$
$$8\pi^2\chi = \int_M |W_+|^2 + |W_-^2| + \frac{s^2}{24} - \frac{|\mathring{r}|}{2} d\mu \qquad \text{(Gauss-Bonnet)}$$

and the fact that $|W_+|^2 = \frac{s^2}{24}$ for Kähler metrics. Indeed, since $\tau < 0$, the signature formula implies that g cannot be self-dual. If, on the other hand, g were anti-self-dual, then $s = 0$ since g is Kähler, and also $\mathring{r} = 0$ since g is Einstein. So, the two formulas would give different values for the integral

$\int_M |W_-|^2 d\mu$ unless $k = 9$; but this case is excluded from the range for k.

Observe, in passing, that part (ii) has a similar taste with Hitchin's theorem [20], which says that a compact oriented Einstein 4-manifold of positive sectional curvature should satisfy $\chi \geq (\frac{3}{2})^{3/2}|\tau|$; and the Hitchin-Thorpe inequality $\chi \geq \frac{3}{2}|\tau|$ which holds for all compact orientable Einstein 4-manifolds.

If we relax the Kähler condition on the Einstein metric g, and merely assume that g is *hermitian*, that is $g(J\cdot, J\cdot) = g(\cdot, \cdot)$ for a complex structure J on the manifold M, interesting enough, we get only two more exceptional metrics:

Theorem 2.1.2 (LeBrun [26])**.** *Let (M^4, J) be a compact complex surface. If g is Einstein and Hermitian, then only one of the following holds:*

(1) g is Kähler-Einstein with $\lambda > 0$.

(2) (M, J) is biholomorphic to $\mathbb{CP}_2 \sharp \overline{\mathbb{CP}}_2$ and g is the Page metric g_{Page} (up to rescaling and isometry).

(3) (M, J) is biholomorphic to $\mathbb{CP}_2 \sharp 2\overline{\mathbb{CP}}_2$ and g is the Chen-LeBrun-Weber metric g_{CLW} (up to rescaling and isometry).

Thus, if we in addition assume that g is of positive sectional curvature, then the first case of the above theorem is possible only when M is $\mathbb{CP}_2$, by Andreotti's theorem. Moreover, by a theorem of Berger [6], the Kähler-Einstein metric g on $\mathbb{CP}_2$, is the Fubini-Study metric (up to rescaling and isometry) since it has positive holomorphic bisectional curvature.

In the next two sections we will prove that the two exceptional metrics g_{Page} and g_{CLW} are not of positive sectional curvature either. This will conclude the proof of our main theorem:

Theorem 2.1.3. *Let M^4 be a compact smooth 4-manifold, and let g be an Einstein metric of positive sectional curvature. If g is hermitian with respect to some complex structure J on M, then (M, J) is biholomorphic to $\mathbb{CP}_2$, and g is the Fubini-Study metric (up to rescaling and isometry).*

One of the key facts in the proof of this theorem is Frankel's Theorem [15] which says that totally geodesic submanifolds of complementary dimensions on positively curved manifolds necessarily intersect. Since the Page metric has an explicit form, we are also able to give a computational proof of the failure of positivity. Note that, on the contrary, Chen-LeBrun-Weber metric does not possess such an explicit formula.

2.2 Page metric

The Page metric was first introduced by D. Page in 1978 as a limiting metric of Kerr-de Sitter solution (see [29]). To define it formally, we first think of the following metric on the product $S^3 \times I$ where I is the closed interval $[0, \pi]$:

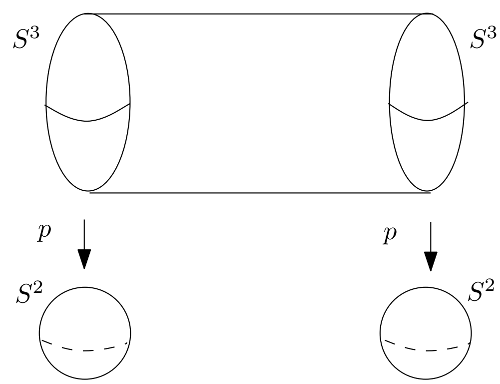
$$g = V(r)dr^2 + f(r)(\sigma_1^2 + \sigma_1^2) + \frac{C\sin^2 r}{V(r)}\sigma_3^2$$

where the coefficient functions are given as

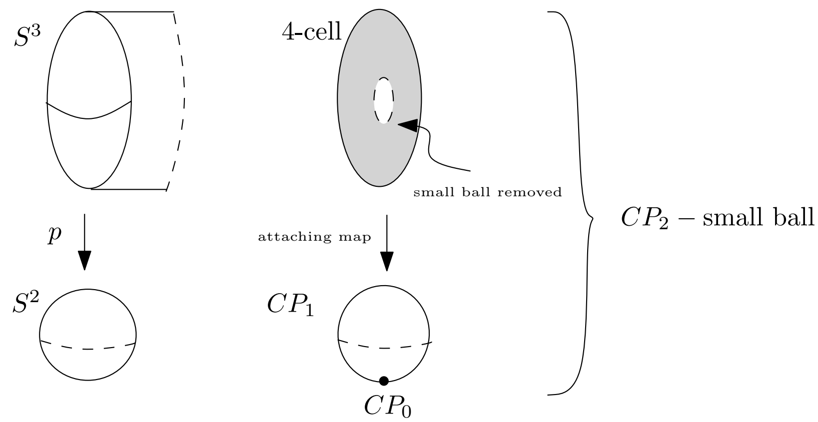
$$V(r) = \frac{1 - a^2\cos^2 r}{3 - a^2 - a^2(1+a^2)\cos^2 r}$$
$$f(r) = \frac{4}{3 + 6a^2 - a^4}(1 - a^2 cos^2 r)$$
$$C = \left(\frac{2}{3+a^2}\right)^2$$

and a is the unique positive root of $a^4 + 4a^3 - 6a^2 + 12a - 3 = 0$. Here, $\sigma_1, \sigma_2, \sigma_3$ is the standard left invariant 1-forms on the Lie group $SU(2) \approx S^3$.

When $r = 0$ or π, we see from the formula that the metric reduces to a round metric on S^2. Thus, g descends to a metric, denoted by g_{Page}, on the quotient $(S^3 \times I)/\sim$ where $\sim$ identifies the fibers of the Hopf fibration $p : S^3 \to S^2$ on the two ends $S^3 \times \{0\}$ and $S^3 \times \{\pi\}$ of the cylinder $S^3 \times I$.



The resulting manifold is indeed the connected sum $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$. To see this, recall that in the cell decomposition of $\mathbb{CP}_2$, the attaching map from the boundary of the 4-cell (which is S^3) to the 2-skeleton (which is $\mathbb{CP}_1 \approx S^2$) is given by the Hopf map [18]. So, if we cut the cylinder $S^3 \times I$ in two halves and identify the Hopf fibers of S^3 at each end, we get $\mathbb{CP}_2 - \{\text{small ball}\}$. Since the right and left halves have different orientations, we obtain $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$ in the quotient.



Now, we will prove that the Page metric is not of positive sectional curvature. We will use the following classical theorem by Frankel:

Theorem 2.2.1 (Frankel [15])**.** *Let M be a smooth n-manifold, and let g be a complete Riemannian metric of positive sectional curvature. If X and Y are two compact totally geodesic submanifolds of dimensions d_1 and d_2 such that $d_1 + d_2 \geq n$, then X and Y intersect.*

In our case, the two 2-spheres on each end of the above quotient will play the role of X and Y. They are compact and the dimensions add up to 4. So it remains to show that those two submanifolds are totally geodesic with respect to g_{Page}. Since they are obviously disjoint, this will imply that g_{Page} cannot have positive sectional curvature.

There is a very well-known lemma to detect totally geodesic submanifolds:

Lemma 2.2.2. *Let (M, g) be a Riemannian manifold. If f is an isometry, then each connected component of the fix point set* $\mathrm{Fix}(f)$ *of f is a totally geodesic submanifold of M.*

So, below we will show that there is an isometry of the Page metric whose fix point set is precisely the two end spheres.

What are the isometries of the Page metric? Derdziński [13] showed that the Page metric is indeed conformal to one of Calabi's extremal Kähler metrics on $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$. On the other hand, the identity component of the isometry group of extremal Kähler metrics is a maximal compact subgroup of the identity component of the automorphism group [10]. In the case of $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$, this implies that the identity component of the isometry group of the Page metric is $U(2) = (SU(2) \times S^1)/\mathbb{Z}_2$. By the formula of the metric, we see that the isometries in the $SU(2)$ component are precisely given by the left multiplication action of $SU(2)$ on the first factor of $S^3 \times I$. Note that the forms σ_i, $i = 1, 2, 3$ are invariant under the action, but the action on the 3-spheres $S^3 \times \{r\}$, $r \in (0, \pi)$ is fixed-point-free! The metric is invariant under this action as the coefficients of the metric only depend on the parameter r.

Now, let us see what happens at the endpoints $r = 0$ and $r = \pi$: It is well-known that the action of $U \in SU(2)$ on the 2-sphere S^2 (after the quotient) is given by the conjugation $A \mapsto UAU^{-1}$, where we regard the 2×2 complex matrix $A = x\sigma_1 + y\sigma_2 + z\sigma_3$ with $x^2 + y^2 + z^2 = 1$ as a point of S^2. It is now straightforward to see that the action of $-I \in SU(2)$ is trivial on S^2 (since $(-I)A(-I)^{-1} = A$); thus, it fixes every point on S^2. Therefore, we conclude that the fixed point set of the isometry given by the "antipodal map" $-I \in SU(2)$ consists of the two 2-spheres at each end of the quotient $((S^3 \times I)/\sim) \approx \mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$. Note that, indeed, there is an S^1-family of isometries generated by rotation in direction of σ_3 having the exact same fixed point set.

So we showed that there are two disjoint compact totally geodesic submanifolds of $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$. Therefore, by Frankel's theorem, we conclude that g_{Page} is not of positive sectional curvature.

Finally, we note that we can actually show the failure of positivity *directly by brute-force* using tensor calculus: Introduce a new coordinate function $x := \cos(r)$, so that the metric becomes

$$g = W^2(x)dx^2 + G^2(x)(\sigma_1^2 + \sigma_1^2) + \frac{D^2}{W^2(x)}\sigma_3^2$$

where the coefficient functions are given as

$$\begin{aligned} W(x) &= \sqrt{\frac{1 - a^2x^2}{(3 - a^2 - a^2(1 + a^2)x^2)(1 - x^2)}} \\ G(x) &= 2\sqrt{\frac{1 - a^2x^2}{3 + 6a^2 - a^4}} \\ D &= \frac{2}{3 + a^2} \end{aligned}$$

and choose the following vierbein: $\{W dx, G\sigma_1, G\sigma_2, \frac{D}{W}\sigma_3\} =: \{e^0, e^1, e^2, e^3\}$. Then by a standard tensor calculus, we see that the sectional curvature of the plane generated by e_0 and e_1 is given by

$$K_{01} = 2\frac{G'W' - G''W}{GW^3}.$$

Using a computer program like *Maple*, one can easily verify that this function $K_{01}(x)$ can take both positive and negative values for $x \in (-1, 1)$.

2.3 Chen-LeBrun-Weber metric

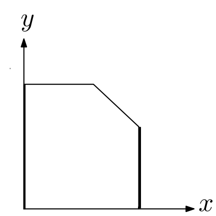
After the discovery of the Kähler-Einstein metrics on $\mathbb{CP}_2\sharp k\overline{\mathbb{CP}}_2$ for $k = 3, \ldots, 8$, and the Einstein metric (namely the Page metric) on $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$, it was speculated that whether $\mathbb{CP}_2\sharp 2\overline{\mathbb{CP}}_2$ admits an Einstein metric. Derdziński [13] had discovered in early '80s that even though Page metric is not Kähler, it is actually confomally related to a Kähler metric on $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$; indeed to one of Calabi's extremal Kähler metrics in [9]. Inspired by this result, LeBrun showed in 1995 that an Einstein hermitian metric h on $\mathbb{CP}_2\sharp 2\overline{\mathbb{CP}}_2$ has to be conformally related to an extremal Kähler metric [23], in such a way that $h = s^{-2}g$, where s is the scalar curvature of g, which turns out to be necessarily positive in this setting. Conversely, it was proved that for an extremal Kähler metric g, the metric $h := s^{-2}g$ is Einstein (defined wherever $s \neq 0$) if g is the critical point of the Calabi functional regarded as an action on extremal Kähler cone [31], [12].

Unlike the $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$ case, not every Kähler class on $\mathbb{CP}_2\sharp 2\overline{\mathbb{CP}}_2$ is represented by an extremal Kähler metric. Nevertheless, using the computations of Futaki invariant in [27], Chen, LeBrun and Weber [12] showed that this action has a critical point, and this critical class is indeed represented by an *extremal* Kähler metric g of positive scalar curvature! Thus, the conformally related metric $s^{-2}g$ is an Einstein metric on $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$, denoted by g_{CLW}. More recenty, LeBrun [26] showed that this is the *unique* Einstein hermitian metric on $\mathbb{CP}_2\sharp\overline{\mathbb{CP}}_2$, and it can be obtained also as an appropriate deformation of the Kähler-Einstein metric in $\mathbb{CP}_2\sharp 3\overline{\mathbb{CP}}_2$ representing the first Chern class c_1 (see [24]).

The identity component of the isometry group of g_{CLW} lies in the identity component of the group of biholomorphisms of $\mathbb{CP}_2\sharp 2\overline{\mathbb{CP}}_2$ as a maximal compact subgroup. Indeed, there is a natural torus action [12] of $S^1 \times S^1$ on $\mathbb{CP}_2\sharp 2\overline{\mathbb{CP}}_2$ obtained by lifting the torus action

$$(e^{i\theta}, e^{i\xi}) \longmapsto ([u_1 : e^{i\theta}u_2], [v_1 : e^{i\xi}v_2])$$

on $\mathbb{CP}_1 \times \mathbb{CP}_1$ to its blowup at $([0 : 1], [0 : 1])$. Note that $\mathbb{CP}_2\sharp 2\overline{\mathbb{CP}}_2$ is isomorphic to the blowup of $\mathbb{CP}_1 \times \mathbb{CP}_1$ at one point. This torus action has the following moment map profile [26]:



Here, x and y are Hamiltonians. In particular, the fix point set of the vector field of the Hamiltonian x contains the two vertical edges of the pentagon, which indeed correspond to two disjoint holomorphic $\mathbb{CP}_1$'s lying in $\mathbb{CP}_2\sharp 2\overline{\mathbb{CP}}_2$. Again, by Lemma 2.2.2, they are totally geodesic. Therefore, by Frankel's theorem, it follows that g_{CLW} is not of positive sectional curvature. This completes the proof of Theorem 2.1.3.

Chapter 3

Calabi Energy of Tønnesen-Friedman Metrics

Let (M^4, J) be a compact complex surface, and let g be an extremal Kähler metric. Denote the Kähler form of g by ω and its Kähler class by Ω. Then its *Calabi Energy* is given by

$$\mathcal{C}(g) = \int_M s^2 d\mu = s_0 \int_M d\mu + \int_M (s - s_0)^2 d\mu = 32\pi^2 \frac{(c_1 \cdot \Omega)^2}{\Omega^2} - \mathcal{F}(\xi, \Omega)$$

where $\xi = \nabla^{1,0} s = (\bar{\partial} s)^\sharp$ is the extremal vector field, s_0 is the average scalar curvature, and $\mathcal{F}(\cdot, \Omega)$ is the Futaki character.

Consider the ruled surface $M = \mathbb{P}(\mathcal{O} \oplus L)$ over a Riemann surface $\Sigma_{\mathfrak{g}}$ of genus $\mathfrak{g} \geq 2$ where L is a holomorphic line bundle of degree $l > 0$ over $\Sigma_{\mathfrak{g}}$. The homology of M is generated by two holomorphic curves: the fiber C with $C \cdot C = 0$, and the section E_0 with $E_0 \cdot E_0 = l = \deg L$.

In [34], based on the computations of LeBrun&Simanca in [27], Tønnesen-

Friedman showed that for the Kähler class $\Omega = m_1C + m_2E_0$ (with $m_1, m_2 > 0$) the Futaki invariant in direction of the Euler vector field X (of the natural $\mathbb{C}^*$ action) is given by

$$\mathcal{F}(X, \Omega) = m_2^2 l \left(1 - \frac{1}{3}\frac{2m_1 + m_2(l - 2(g-1))}{2m_1 + m_2 l}\right).$$

The Euler vector field is a constant multiple of the extremal vector field ξ:

$$\xi = s_t X$$

where s_t is the derivative of the scalar curvature s with respect to the moment map coordinate t. Note that s_t is indeed *constant* because in these coordinates the extremal condition $s_{,\bar{j}\bar{k}}$ becomes $s_{tt} = 0$.

Now assume $\Omega_k = 4\pi\left(\frac{l}{k-1}C + E_0\right)$. If we assume that this class is extremal, Tønnesen-Friedman showed that its extremal Kähler representative has the property that

$$s_t = -cq$$

where $q = l/(\mathfrak{g} - 1)$ and

$$c = \frac{6[(q+1)k + (q-1)](k-1)}{(k^2 + 4k + 1)q^2}.$$

Thus, keeping in mind that $c_1 \cdot \Omega = 2m_1 + m_2(l - 2(\mathfrak{g} - 1))$, the $\mathcal{A}$-energy of the Kähler class Ω_k is:

$$\begin{aligned}\mathcal{A}(\Omega_k) &= \frac{(c_1\cdot\Omega_k)^2}{\Omega_k^2} - \frac{1}{32\pi^2}\mathcal{F}(\xi,\Omega_k)\\ &= \frac{(2m_1+m_2(l-2(\mathfrak{g}-1)))^2}{2m_1m_2+m_2^2 l}\\ &+ \frac{1}{32\pi^2}\frac{6[(q+1)k+(q-1)](k-1)}{(k^2+4k+1)q}m_2^2 l\left(1-\frac{1}{3}\frac{2m_1+m_2(l-2(\mathfrak{g}-1))}{2m_1+m_2 l}\right).\end{aligned}$$

where $m_1 = 4\pi l/(k-1)$ and $m_2 = 4\pi$. More explicitly:

$$\frac{1}{l}\mathcal{A}(\Omega_k) = \tfrac{((1-2/q)k+(1+2/q))^2}{k^2-1} + \tfrac{((q+1)k+q-1)(k-1)}{((k^2+4k+1))q}\left(3-\tfrac{(1-2/q)k+(2/q+1)}{k+1}\right)$$

We see that the right hand side depends only on k, and the ratio $q = l/g-1$, rather than degree and genus individually. Therefore critical points of this functional depend only on q.

If a critical point k of this functional lies in the *extremal range*, then the extremal representative g_k of Ω_k will be Bach-flat (see [12]), and hence by Derdzinski [13], it will be conformally equivalent to a Poincaré–Einstein metric $s^{-2}g_k$.

For example, if $q = 4$, that is, if $l = 4(g-1)$, one can check on computer that the derivative of $\mathcal{A}$-functional with respect to the parameter k has a zero at $k \approx 3.59$. The extremal range when $q = 4$ is $(1, \widetilde{k}_q)$ where $\widetilde{k}_q > q+\sqrt{1+q} \approx 6.23$ (see [34]). Therefore there is a critical Kähler class which is represented by an *extremal* Kähler metric. So we have the following conclusion:

Proposition 3.0.1. *Over any compact Riemann surface $\Sigma_{\mathfrak{g}}$ of genus $\mathfrak{g} \geq 2$, there is a ruled surface $\mathbb{P}(\mathcal{O}\oplus L)$ which carries a Bach-flat Kähler metric, and* *hence a Poincaré–Einstein Metric.*

This proposition was first proved by Tønnesen-Friedman (see [35]) using Derdziński's equation [13] for Bach-flat metrics.

Bibliography

[1] A. Andreotti. On the complex structures of a class of simply-connected manifolds. In *Algebraic geometry and topology. A symposium in honor of S. Lefschetz*, pages 53–77. Princeton University Press, Princeton, N. J., 1957.

[2] T. Aubin. *Nonlinear analysis on manifolds. Monge-Ampère equations*, volume 252 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, New York, 1982.

[3] R. Bach. Zur Weylschen Relativitätstheorie und der Weylschen Erweiterung des Krümmungstensorbegriffs. *Math. Z.*, 9(1-2):110–135, 1921.

[4] W. Barth, C. Peters, and A. Van de Ven. *Compact complex surfaces*, volume 4 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1984.

[5] A. Beauville. *Complex algebraic surfaces*, volume 34 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, second edition, 1996. Translated from the 1978 French original by R. Barlow, with assistance from N. I. Shepherd-Barron and M. Reid.

[6] M. Berger. Sur les variétés d'Einstein compactes. In *Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (Namur, 1965)*, pages 35–55. Librairie Universitaire, Louvain, 1966.

[7] A. L. Besse. *Einstein manifolds*, volume 10 of *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*. Springer-Verlag, Berlin, 1987.

[8] D. Burns and P. De Bartolomeis. Stability of vector bundles and extremal metrics. *Invent. Math.*, 92(2):403–407, 1988.

[9] E. Calabi. Extremal Kähler metrics. In *Seminar on Differential Geometry*, volume 102 of *Ann. of Math. Stud.*, pages 259–290. Princeton Univ. Press, Princeton, N.J., 1982.

[10] E. Calabi. Extremal Kähler metrics. II. In *Differential geometry and complex analysis*, pages 95–114. Springer, Berlin, 1985.

[11] X. Chen. Space of Kähler metrics. III. On the lower bound of the Calabi energy and geodesic distance. *Invent. Math.*, 175(3):453–503, 2009.

[12] X. Chen, C. LeBrun, and B. Weber. On conformally Kähler, Einstein manifolds. *J. Amer. Math. Soc.*, 21(4):1137–1168, 2008.

[13] A. Derdziński. Self-dual Kähler manifolds and Einstein manifolds of dimension four. *Compositio Math.*, 49(3):405–433, 1983.

[14] S. K. Donaldson. Lower bounds on the Calabi functional. *J. Differential Geom.*, 70(3):453–472, 2005.

[15] T. Frankel. Manifolds with positive curvature. *Pacific J. Math.*, 11:165–174, 1961.

[16] M. J. Gursky. The Weyl functional, de Rham cohomology, and Kähler-Einstein metrics. *Ann. of Math. (2)*, 148(1):315–337, 1998.

[17] M. J. Gursky and C. LeBrun. On Einstein manifolds of positive sectional curvature. *Ann. Global Anal. Geom.*, 17(4):315–328, 1999.

[18] A. Hatcher. *Algebraic topology*. Cambridge University Press, Cambridge, 2002.

[19] F. Hirzebruch. Über eine Klasse von einfachzusammenhängenden komplexen Mannigfaltigkeiten. *Math. Ann.*, 124:77–86, 1951.

[20] N. Hitchin. Compact four-dimensional Einstein manifolds. *J. Differential Geometry*, 9:435–441, 1974.

[21] A. D. Hwang and S. R. Simanca. Extremal Kähler metrics on Hirzebruch surfaces which are locally conformally equivalent to Einstein metrics. *Math. Ann.*, 309(1):97–106, 1997.

[22] O. Kobayashi. On a conformally invariant functional of the space of Riemannian metrics. *J. Math. Soc. Japan*, 37(3):373–389, 1985.

[23] C. LeBrun. Einstein metrics on complex surfaces. In *Geometry and physics (Aarhus, 1995)*, volume 184 of *Lecture Notes in Pure and Appl. Math.*, pages 167–176. Dekker, New York, 1997.

[24] C. LeBrun. Einstein Manifolds and Extremal Kahler Metrics. *ArXiv e-prints*, Sept. 2010.

[25] C. LeBrun. The Einstein-Maxwell equations, extremal Kähler metrics, and Seiberg-Witten theory. In *The many facets of geometry*, pages 17–33. Oxford Univ. Press, Oxford, 2010.

[26] C. LeBrun. On Einstein, Hermitian 4-Manifolds. *ArXiv e-prints*, Oct. 2010.

[27] C. LeBrun and S. R. Simanca. Extremal Kähler metrics and complex deformation theory. *Geom. Funct. Anal.*, 4(3):298–336, 1994.

[28] S. A. Merkulov. A conformally invariant theory of gravitation and electromagnetism. *Classical Quantum Gravity*, 1(4):349–354, 1984.

[29] D. Page. A Compact Rotating Gravitational Instanton. *Phys. Let.*, 79B(3):235–238, 1978.

[30] Y. Shu. Compact complex surfaces and constant scalar curvature Kähler metrics. *Geom. Dedicata*, 138:151–172, 2009.

[31] S. R. Simanca. Strongly extremal Kähler metrics. *Ann. Global Anal. Geom.*, 18(1):29–46, 2000.

[32] Y. T. Siu and S. T. Yau. Compact Kähler manifolds of positive bisectional curvature. *Invent. Math.*, 59(2):189–204, 1980.

[33] G. Tian. On Calabi's conjecture for complex surfaces with positive first Chern class. *Invent. Math.*, 101(1):101–172, 1990.

[34] C. W. Tønnesen-Friedman. Extremal Kähler metrics on minimal ruled surfaces. *J. Reine Angew. Math.*, 502:175–197, 1998.

[35] C. W. Tønnesen-Friedman. Extremal Kähler metrics and Hamiltonian functions. II. *Glasg. Math. J.*, 44(2):241–253, 2002.

[36] H. Yamabe. On a deformation of Riemannian structures on compact manifolds. *Osaka Math. J.*, 12:21–37, 1960.

[37] S. T. Yau. Calabi's conjecture and some new results in algebraic geometry. *Proc. Nat. Acad. Sci. U.S.A.*, 74(5):1798–1799, 1977.

[38] S. T. Yau. On the Ricci curvature of a compact Kähler manifold and the complex Monge-Ampère equation. I. *Comm. Pure Appl. Math.*, 31(3):339–411, 1978.