Enumerative Geometry
via
Topological Computations

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Ritwik Mukherjee

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Ritwik Mukherjee

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Aleksey Zinger - Advisor
Associate Professor, Department of Mathematics

Dennis Sullivan - Committee Chair
Professor, Department of Mathematics

Jason Starr
Associate Professor, Department of Mathematics

Dusa McDuff
Professor of Mathematics, Columbia University (Barnard College)

This dissertation is accepted by the Graduate School.

Lawrence Martin
Dean of the Graduate School

Abstract of the Dissertation

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Enumerative geometry is a rich and fascinating subject that has been studied extensively by algebraic geometers. In our thesis however, we approach this subject using methods from differential topology. The method comprises of two parts. The first part involves computing the Euler class of a vector bundle and evaluating it on the fundamental class of a manifold. This is straightforward. The second part involves perturbing a section and computing its contribution near the boundary. This is usually difficult. We have used this method to compute how many degree d curves are there in $\mathbb{CP}^2$ that pass through $\frac{d(d+3)}{2}-(\delta+m)$ points having δ nodes and one singularity of codimension m provided $\delta+m \leq 7$. We also indicate how to extend this approach if $\delta+m$ is greater than 7.

Contents

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Chapter 1

Introduction

Enumerative geometry is a classical subject that dates back to over 150 years ago. The general goal of this subject is to count how many geometric objects are there that satisfy certain conditions. It has been an active field of research since the nineteenth century. In fact, Hilbert's fifteenth problem was to lay a rigorous foundation for enumerative Schubert calculus. While the problems in this field are typically easy to state, solutions to almost all of them require various deep concepts from mathematics.

This subject has been extensively studied by algebraic geometers. An example of a well-known enumerative problem is:

Question 1.1. *What is the number $\mathcal{N}_d(\delta)$ of degree d curves in $\mathbb{P}^2$ that pass through $\kappa(d)-\delta$ points and have δ simple nodes, where $\kappa(d) = \frac{d(d+3)}{2}$?*

Using methods of algebraic geometry, Vainsencher [18] and Kleiman-Piene [9] find explicit formulas for $N_d(\delta)$ with $\delta \leq 6$ and $\delta \leq 8$, respectively. Recursive formulas for $N_d(\delta)$ are derived by Caporaso-Harris [1] and Ran [13], [14]. They both use algebro-geometric methods.

In [25], the author uses a purely topological method to compute the number of degree d plane curves with up to 3 nodes passing through the appropriate number of points. In this thesis, we greatly extend this approach to enumerate curves with up to 7 nodes as well as curves with many other types of singularities. In fact, one of the main difficulties in extending this approach is the enumeration of curves with one highly singular point. Our ultimate aim is to enumerate curves with an arbitrary collection of singularities, provided the degree is sufficiently high.

We present our formulas for enumerating curves with singularities as recursions on the number and complexity of singular points. We have also created a mathematica program that uses these formulas to produce formulas expressing the number of degree d curves with specified singularities in terms of d.

1.1 Plane curves and their singularities

For each $d\in\mathbb{Z}^+$, let

$$\mathcal{D}_d \approx \mathbb{P}^{\kappa(d)}\,, \qquad \text{where} \qquad \kappa(d) := \binom{d+2}{2} - 1\,,$$

denote space of degree d curves in $\mathbb{P}^2$. For any non-negative integer r, denote by

$$\mathcal{D}_d(r) \approx \mathbb{P}^r \subset \mathcal{D}_d \approx \mathbb{P}^{\kappa(d)}$$

the subspace of curves passing through $\mathcal{D}_d - r$ general points. We write elements of $\mathcal{D}_d$ as $[s]$, with s denoting a non-zero degree d homogeneous polynomial on $\mathbb{C}^3$ or equivalently a non-zero element of $H^0(\mathbb{P}^2; \mathcal{O}(d))$, i.e. a non-zero holomorphic section of the holomorphic line bundle

$$\gamma_{\mathbb{P}^2}^{*d} \equiv (\gamma_{\mathbb{P}^2}^*)^{\otimes d} \longrightarrow \mathbb{P}^2,$$

where $\gamma_{\mathbb{P}^2} \longrightarrow \mathbb{P}^2$ is the tautological line bundle. Denote by $\pi_{\mathbb{P}T\mathbb{P}^2} : \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathbb{P}^2$ the bundle projection map. Let

$$\gamma_{\mathcal{D}} \longrightarrow \mathcal{D}_d \qquad \text{and} \qquad \tilde{\gamma} \longrightarrow \mathbb{P}(T\mathbb{P}^2)$$

be the tautological line bundles over $\mathcal{D}_d$ and $\mathbb{P}(T\mathbb{P}^2)$, respectively. We define

$$\lambda_{\mathcal{D}} = c_1(\gamma_{\mathcal{D}}^*), \qquad \lambda_{\mathbb{P}^2} = c_1(\gamma_{\mathbb{P}^2}^*), \qquad \text{and} \qquad \tilde{\lambda} = c_1(\tilde{\gamma}^*).$$

Definition 1.2. *Let $[s] \in \mathcal{D}_d$. A point $p \in s^{-1}(0)$ is of singularity type A_k with $k \geq 0$, D_k with $k \geq 4$, E_6, E_7, or E_8 if there exists a coordinate system $(x, y): (U, p) \longrightarrow (\mathbb{C}^2, 0)$ around p on $\mathbb{P}^2$ such that $s^{-1}(0) \cap U$ is given by the equation*

$$y^2 + x^{k+1} = 0, \quad y^2 x + x^{k-1} = 0, \quad y^3 + x^4 = 0, \quad y^3 + yx^3 = 0, \quad \text{or} \quad y^3 + x^5 = 0,$$

respectively.

We write $\chi_s(p)$ for the singularity type of $p \in s^{-1}(0)$. Thus, p is a smooth point of $s^{-1}(0)$ if $\chi_s(p) = A_0$, a simple node if $\chi_s(p) = A_1$, a cusp if $\chi_s(p) = A_2$, a tacnode if $\chi_s(p) = A_3$, and a triple point if $\chi_s(p) = D_4$. Let

$$c_{A_k} = c_{D_k} = c_{E_k} := k$$

be the codimension of the singularity.

Fix linear subspaces $\mathbb{P}^0 \subset \mathbb{P}^1 \subset \mathbb{P}^2$ in general position with respect to the $\kappa(d)$ points used to define $\mathcal{D}_d(r)$. If $r \in \mathbb{Z}^{\geq 0}$ and $\sigma \in \mathbb{Z}^{\geq 0}$, let

$$\mathbb{P}\mathcal{S}_r^d(0, A_1, \sigma) := \big\{([s], \ell) \in \mathcal{D}_d(r+k) \times \mathbb{P}T\mathbb{P}^2\big|_{\mathbb{P}^{2-\sigma}} :\ \chi_s(\pi_{\mathbb{P}T\mathbb{P}^2}(\ell)) = A_1,\ \nabla^2 s(v, v) = 0\ \forall\, v \in \ell\big\}.$$

In addition $\delta \in \mathbb{Z}^{\geq 0}$ and χ is a singularity type, let

$$\mathcal{S}_r^d(\delta, \chi, \sigma) := \big\{([s], p_1, \ldots, p_{\delta+1}) \in \mathcal{D}_d(r+\delta+c_\chi) \times (\mathbb{P}^2)^\delta \times \mathbb{P}^{2-\sigma} :\ p_i \neq p_j\ \forall\, i \neq j,$$
$$\chi_s(p_1), \chi_s(p_2), \ldots, \chi_s(p_\delta) = A_1,\ \chi_s(p_{\delta+1}) = \chi\big\}.$$

If $k \geq 2$, let

$$\mathbb{P}\mathcal{S}_r^d(0, A_k, \sigma) := \big\{([s], \ell) \in \mathcal{D}_d(r+k) \times \mathbb{P}T\mathbb{P}^2\big|_{\mathbb{P}^{2-\sigma}} :\ \chi_s(\pi_{\mathbb{P}T\mathbb{P}^2}(\ell)) = A_k,\ \nabla^2 s(v, \cdot) = 0\ \forall\, v \in \ell\big\}.$$

Similarly, for $\chi = D_k$ with $k \geq 4$ and $\chi = E_6, E_7, E_8$, let

$$\mathbb{P}\mathcal{S}_r^d(0, \chi, \sigma) := \big\{([s], \ell) \in \mathcal{D}_d(r+c_\chi) \times \mathbb{P}T\mathbb{P}^2\big|_{\mathbb{P}^{2-\sigma}} :\ \chi_s(\pi_{\mathbb{P}T\mathbb{P}^2}(\ell)) = \chi,\ \nabla^3 s(v, v, \cdot) = 0\ \forall\, v \in \ell\big\}.$$

If $\delta\in\mathbb{Z}^+$ and $\chi=A_k$ with $k\geq 2$, $\chi=D_k$ with $k\geq 4$, or $\chi=E_6,E_7,E_8$, let

$$\mathbb{P}\mathcal{S}_r^d(\delta,\chi,\sigma) := \big\{([s],p_1,\ldots,p_\delta,\ell)\in\mathcal{D}_d(r+\delta+c_\chi)\times(\mathbb{P}^2)^\delta\times\mathbb{P}T\mathbb{P}^2\colon\ ([s],\ell)\in\mathbb{P}\mathcal{S}_{r+\delta}^d(0,\chi,\sigma)\\ ([s],p_1,\ldots,p_\delta,\pi_{\mathbb{P}T\mathbb{P}^2}(\ell))\in\mathcal{S}_r^d(\delta,\chi,\sigma)\big\}.$$

The expected dimensions of $\mathcal{S}_r^d(\delta,\chi,\sigma)$ and $\mathbb{P}\mathcal{S}_r^d(\delta,\chi,\sigma)$ are r. Denote by

$$\overline{\mathcal{S}}_r^d(\delta,\chi,\sigma)\subset\mathcal{D}_d(r+\delta+c_\chi)\times(\mathbb{P}^2)^\delta\times\mathbb{P}^2 \qquad \text{and}$$
$$\overline{\mathbb{P}\mathcal{S}}_r^d(\delta,\chi,\sigma)\subset\mathcal{D}_d(r+\delta+c_\chi)\times(\mathbb{P}^2)^\delta\times\mathbb{P}T\mathbb{P}^2$$

the closures of $\mathcal{S}_r^d(\delta,\chi,\sigma)$ and $\mathbb{P}\mathcal{S}_r^d(\delta,\chi,\sigma)$, respectively. These are algebraic varieties of the expected dimension if d is sufficiently high.

If $\delta,\sigma_1,\sigma_2\in\mathbb{Z}^{\geq 0}$, $\chi=A_1,D_4$, and d is sufficiently high, let

$$\mathcal{N}^d(\delta,\chi,\sigma_1) := \big|\mathcal{S}_0^d(\delta,\chi,\sigma_1)\big|,$$
$$\mathcal{N}^d(\delta,\tilde{\chi},\sigma_1,) := \big\langle\tilde{\lambda}^{\sigma_2},[\overline{\mathbb{P}\mathcal{S}}_{\sigma_2}^d(\delta,\chi,\sigma_1)]\big\rangle.$$

If in addition $\chi=A_k$ with $k\geq 2$, $\chi=D_k$ with $k\geq 4$, or $\chi=E_6,E_7,E_8$, let

$$\mathcal{N}^d(\delta,\chi,\sigma_1,\sigma_2) := \langle\tilde{\lambda}^{\sigma_2},[\overline{\mathbb{P}\mathcal{S}}_{\sigma_2}^d(\delta,\chi,\sigma_1)]\rangle, \qquad \mathcal{N}^d(\delta,\chi,\sigma_1) := \mathcal{N}^d(\delta,\chi,\sigma_1,0).$$

Finally, for any singularity type χ as above, let

$$\mathcal{N}^d(\delta,\chi) := \mathcal{N}^d(\delta,\chi,0), \quad \mathcal{N}^d(\delta) := \mathcal{N}^d(\delta,A_1).$$

Thus, $\mathcal{N}^d(\delta,\chi,\sigma_1)$ is the number of degree d curves that pass through $\kappa(d)-\delta-c_\chi-\sigma_1$ general points and have δ ordered nodes and another singular point of type χ that lies on the intersection of σ_1 lines, while $\mathcal{N}^d(\delta)$ is the number of degree d curves that pass through $\kappa(d)-\delta-1$ general points and have $\delta+1$ ordered nodes.

1.2 Summary of results

Among the main results of this papers are the following theorems that provide recursion formulas for some expressing counts of curves with certain collections of singularities in terms of counts of curves with "simpler" collections (either fewer singular points or less complicated singularities).

Theorem 1. *If $1\leq\delta\leq 6$ and $d\geq 2\delta+1$, the number of degree d curves in $\mathbb{CP}^2$ with $\delta+1$ distinct ordered nodes with one of them lying on the intersection of σ generic lines is given by*

$$\mathcal{N}^d(\delta,A_1,\sigma)=\mathcal{N}^d(\delta-1,A_1)\cdot\mathcal{N}^d(0,A_1,\sigma)-\Bigg\{\delta\big(\mathcal{N}^d(\delta-1,A_1,\sigma)+d\mathcal{N}^d(\delta-1,A_1,\sigma+1)\big)\\ +3\binom{\delta}{1}\mathcal{N}^d(\delta-1,A_2,\sigma)+4\binom{\delta}{2}\mathcal{N}^d(\delta-2,A_3,\sigma)+18\binom{\delta}{3}\mathcal{N}^d(\delta-3,D_4,\sigma)\Bigg\}.$$

Theorem 2. *If $0\leq\delta\leq 5$ and $d\geq 2\delta+2$, the number of degree d curves in $\mathbb{CP}^2$ with a A_1-node with a marked direction lying on the intersection of σ_1 generic lines and one lambda class and δ other distinct ordered nodes is given by*

$$\mathcal{N}^d(\delta,\tilde{A}_1,\sigma,1)=\mathcal{N}^d(\delta,A_1,\sigma)+(d-6)\mathcal{N}^d(\delta,A_1,\sigma+1)-6\binom{\delta}{2}\mathcal{N}(\delta-2,D_4,\sigma).$$

Theorem 3. *If $0 \le \delta \le 6$ and $d \ge 2\delta+2$, the number of degree d curves in $\mathbb{CP}^2$ with a cusp lying on the intersection of σ generic lines and δ distinct ordered nodes is given by*

$$\mathcal{N}^d(\delta, A_2, \sigma) = 2\mathcal{N}^d(\delta, A_1, \sigma) + 2(d-3)\mathcal{N}^d(\delta, A_1, \sigma+1) - \left\{ 2\binom{\delta}{1}\mathcal{N}^d(\delta-1, A_3, \sigma) + 12\binom{\delta}{2}\mathcal{N}^d(\delta-2, D_4, \sigma) \right\}.$$

Theorem 4. *If $0 \le \delta \le 4$ and $d \ge 2\delta+2$, the number of degree d curves in $\mathbb{CP}^2$ with a cusp lying on the intersection of σ generic lines and one lambda class and δ distinct ordered nodes is given by*

$$\begin{aligned}\mathcal{N}^d(\delta, A_2, \sigma, 1) &= \mathcal{N}^d(\delta, \tilde{A}_1, \sigma, 1) + (d-3)\mathcal{N}^d(\delta, \tilde{A}_1, \sigma+1, 1) \\ &- \left\{ 2\binom{\delta}{1}\mathcal{N}^d(\delta-1, A_3, \sigma, 1) + 3\binom{\delta}{1}\mathcal{N}^d(\delta-1, D_4, \sigma) \right. \\ &\left. + 4\binom{\delta}{2}\left(\mathcal{N}^d(\delta-2, \tilde{D}_4, \sigma, 1) + \mathcal{N}^d(\delta-2, D_5, \sigma)\right) + 12\binom{\delta}{3}\mathcal{N}^d(\delta-3, D_6, \sigma) \right\}.\end{aligned}$$

Theorem 5. *If $0 \le \delta \le 4$ and $d \ge 2\delta+3$, the number of degree d curves in $\mathbb{CP}^2$ with a tacnode lying on the intersection of σ_1 generic lines and σ_2 lambda classes and δ distinct ordered nodes is given by*

$$\begin{aligned}\mathcal{N}^d(\delta, A_3, \sigma_1, \sigma_2) &= \mathcal{N}^d(\delta, A_2, \sigma_1, \sigma_2) + 3\mathcal{N}^d(\delta, A_2, \sigma_1, \sigma_2+1) + d\mathcal{N}^d(\delta, A_2, \sigma_1+1, \sigma_2) \\ &- \left\{ 2\binom{\delta}{1}\mathcal{N}^d(\delta-1, A_4, \sigma_1, \sigma_2) + 2\binom{\delta}{2}\mathcal{N}^d(\delta-2, D_5, \sigma_1, \sigma_2) \right\}.\end{aligned}$$

Theorem 6. *If $0 \le \delta \le 3$ and $d \ge 2\delta+3$, the number of degree d curves in $\mathbb{CP}^2$ with a D_4-node lying on the intersection of σ_1 generic lines and δ distinct ordered nodes is given by*

$$\begin{aligned}3\mathcal{N}^d(\delta, D_4, \sigma_1) &= \mathcal{N}^d(\delta, A_3, \sigma_1) - 2\mathcal{N}^d(\delta, A_3, \sigma_1, 1) + (d-6)\mathcal{N}^d(\delta, A_3, \sigma_1+1) \\ &- \left\{ 2\binom{\delta}{1}\mathcal{N}^d(\delta-1, D_5, \sigma_1) + 2\binom{\delta}{2}\left(\mathcal{N}^d(\delta-2, D_6, \sigma_1)\right) \right\}.\end{aligned}$$

Theorem 7. *If $0 \le \delta \le 2$ and $d \ge 2\delta+3$, the number of degree d curves in $\mathbb{CP}^2$ with a D_4-node with a marked direction lying on the intersection of σ_1 generic lines and one lambda classes and δ distinct ordered nodes is given by*

$$\mathcal{N}^d(\delta, \tilde{D}_4, \sigma, 1) = \mathcal{N}^d(\delta, D_4, \sigma) + (d-9)\mathcal{N}^d(\delta, D_4, \sigma+1).$$

Theorem 8. *If $0 \le \delta \le 3$ and $d \ge 2\delta+4$, the number of degree d curves in $\mathbb{CP}^2$ with an A_4-node lying on the intersection of σ_1 generic lines and $\sigma_2 \le 3-\delta$ lambda classes and δ distinct ordered nodes is given by*

$$\begin{aligned}\mathcal{N}^d(\delta, A_4, \sigma_1, \sigma_2) &= 2\mathcal{N}^d(\delta, A_3, \sigma_1, \sigma_2) + 2\mathcal{N}^d(\delta, A_3, \sigma_1, \sigma_2+1) + (2d-6)\mathcal{N}^d(\delta, A_3, \sigma_1+1, \sigma_2) \\ &- \left\{ 2\binom{\delta}{1}\mathcal{N}^d(\delta-1, A_5, \sigma_1, \sigma_2) + 4\binom{\delta}{2}\mathcal{N}^d(\delta-2, D_6, \sigma_1, \sigma_2) \right\}.\end{aligned}$$

Theorem 9. *If $0 \le \delta \le 2$ and $d \ge 2\delta+5$, the number of degree d curves in $\mathbb{CP}^2$ with an A_5-node lying on the intersection of σ_1 generic lines and $\sigma_2 \le 2-\delta$ lambda classes and δ distinct ordered nodes is given by*

$$\begin{aligned}\mathcal{N}^d(\delta, A_5, \sigma_1, \sigma_2) &= 2\mathcal{N}^d(\delta, A_4, \sigma_1, \sigma_2) + 3\mathcal{N}^d(\delta, A_4, \sigma_1, \sigma_2+1) + (2d-6)\mathcal{N}^d(\delta, A_4, \sigma_1+1, \sigma_2) \\ &- \left\{2\binom{\delta}{1}\mathcal{N}^d(\delta-1, A_6, \sigma_1, \sigma_2) + \binom{\delta}{1}\mathcal{N}^d(\delta-1, E_6, \sigma_1, \sigma_2) + 4\binom{\delta}{2}\big(\mathcal{N}^d(\delta-2, D_7, \sigma_1, \sigma_2)\right\}.\end{aligned}$$

Theorem 10. *If $0 \le \delta \le 2$ and $d \ge 2\delta+3$, the number of degree d curves in $\mathbb{CP}^2$ with a D_5-node lying on the intersection of σ_1 generic lines and $\sigma_2 \le 2-\delta$ lambda classes and δ distinct ordered nodes is given by*

$$\begin{aligned}\mathcal{N}^d(\delta, D_5, \sigma_1, \sigma_2) &= \mathcal{N}^d(\delta, \tilde{D}_4, \sigma_1, \sigma_2) + \mathcal{N}^d(\delta, \tilde{D}_4, \sigma_1, \sigma_2+1) + (d-3)\mathcal{N}^d(\delta, \tilde{D}_4, \sigma_1+1, \sigma_2) \\ &- 2\binom{\delta}{1}\mathcal{N}^d(\delta-1, D_6, \sigma_1, \sigma_2).\end{aligned}$$

Theorem 11. *If $\delta = 0, 1$ and $d \ge 2\delta+6$, the number of degree d curves in $\mathbb{CP}^2$ with an A_6-node lying on the intersection of σ_1 generic lines and $\sigma_2 \le 1-\delta$ lambda classes and δ nodes is given by*

$$\begin{aligned}\mathcal{N}^d(\delta, A_6, \sigma_1, \sigma_2) &= 3\mathcal{N}^d(\delta, A_5, \sigma_1, \sigma_2) + 2\mathcal{N}^d(\delta, A_5, \sigma_1, \sigma_2+1) + (3d-12)\mathcal{N}^d(\delta, A_5, \sigma_1+1, \sigma_2) \\ &- \Big\{2\mathcal{N}^d(\delta, D_6, \sigma_1, \sigma_2) + \mathcal{N}^d(\delta, E_6, \sigma_1, \sigma_2) \\ &+ 2\binom{\delta}{1}\mathcal{N}^d(\delta-1, A_7, \sigma_1, \sigma_2) + 3\binom{\delta}{1}\mathcal{N}^d(\delta-1, E_7, \sigma_1, \sigma_2)\Big\}.\end{aligned}$$

Theorem 12. *If $\delta = 0, 1$ and $d \ge 2\delta+4$, the number of degree d curves in $\mathbb{CP}^2$ with a D_6-node lying on the intersection of σ_1 generic lines and $\sigma_2 \le 1-\delta$ lambda classes and δ nodes is given by*

$$\begin{aligned}\mathcal{N}^d(\delta, D_6, \sigma_1, \sigma_2) &= \mathcal{N}^d(\delta, D_5, \sigma_1, \sigma_2) + 4\mathcal{N}^d(\delta, D_5, \sigma_1, \sigma_2+1) + d\mathcal{N}^d(\delta, D_5, \sigma_1+1, \sigma_2) \\ &- \left\{2\binom{\delta}{1}\mathcal{N}^d(\delta-1, D_7, \sigma_1, \sigma_2) + \binom{\delta}{1}\mathcal{N}^d(\delta-1, E_7, \sigma_1, \sigma_2)\right\}.\end{aligned}$$

Theorem 13. *If $\delta = 0, 1$ and $d \ge 2\delta+3$, the number of degree d curves in $\mathbb{CP}^2$ with an E_6-node lying on the intersection of σ_1 generic lines and $\sigma_2 \le 1-\delta$ lambda classes and δ distinct ordered nodes is given by*

$$\begin{aligned}\mathcal{N}^d(\delta, E_6, \sigma_1, \sigma_2) &= \mathcal{N}^d(\delta, D_5, \sigma_1, \sigma_2) - \mathcal{N}^d(\delta, D_5, \sigma_1, \sigma_2+1) + (d-6)\mathcal{N}^d(\delta, D_5, \sigma_1+1, \sigma_2) \\ &- \binom{\delta}{1}\mathcal{N}^d(\delta-1, E_7, \sigma_1, \sigma_2).\end{aligned}$$

Theorem 14. *If $d \ge 5$, the number of degree d curves in $\mathbb{CP}^2$ with an D_7-node lying on the intersection of σ generic lines is given by*

$$\mathcal{N}^d(0, D_7, \sigma) = 2\mathcal{N}^d(0, D_6, \sigma) + 4\mathcal{N}^d(0, D_6, \sigma, 1) + (2d-6)\mathcal{N}^d(0, D_6, 1).$$

Theorem 15. *If $d \ge 4$, the number of degree d curves in $\mathbb{CP}^2$ with an E_7-node lying on the intersection of σ generic lines is given by*

$$\mathcal{N}^d(0, E_7, \sigma) = \mathcal{N}^d(0, D_6, \sigma) - \mathcal{N}^d(0, D_6, \sigma, 1) + (d-6)\mathcal{N}^d(0, D_6, \sigma+1).$$

Theorem 16. *If $d \geq 7$, the number of degree d curves in $\mathbb{CP}^2$ with an A_7-node lying on the intersection of σ generic lines is given by*

$$\mathcal{N}^d(0, A_7, \sigma) = 5\mathcal{N}^d(0, A_6, \sigma) - \mathcal{N}^d(0, A_6, \sigma, 1) + (5d-24)\mathcal{N}^d(0, A_6, \sigma+1) \\ - \Big\{ 6\mathcal{N}^d(0, D_7, \sigma) + 7\mathcal{N}^d(0, E_7, \sigma) \Big\}.$$

The base case for the recursion is provided by the counts of one-nodal curves, obtained in Lemma 4.1:

$$\mathcal{N}^d(0, A_1, \sigma) = \begin{cases} 3(d-1)^{2-\sigma}, & \text{if } \sigma = 0, 1; \\ 1, & \text{if } \sigma = 2; \\ 0, & \text{otherwise.} \end{cases}$$

Since $\tilde{\lambda}^2 = -3\tilde{\lambda}\lambda_{\mathbb{P}^2} - 3\lambda_{\mathbb{P}^2}^2$, for every singularity type χ we have

$$\mathcal{N}^d(\delta, \chi, \sigma_1, \sigma_2) = -3\mathcal{N}^d(\delta, \chi, \sigma_1+1, \sigma_2-1) - 3\mathcal{N}^d(\delta, \chi, \sigma_1+2, \sigma_2-2) \qquad \forall\, \sigma_2 \geq 2.$$

All together these recursions allow us to obtain explicit formulas for the numbers $\mathcal{N}(\delta, \chi)$ with $\delta + c_\chi \leq 7$. These formulas agree with previous known results:

(1) the formulas for $\mathcal{N}^d(\delta, A_1)$ with $\delta+1 \leq 6$ nodes agree with [18, Example 5.1];

(2) the formulas for $\mathcal{N}^d(\delta, A_1)$ with $\delta+1 \leq 7$ nodes agree with [9, Theorem 1.1];

(3) the formulas for $\mathcal{N}^d(\delta, D_4)$ with $\delta \leq 3$ agree with [9, Theorem 1.2];

(4) the formulas for $\mathcal{N}^d(\delta, \chi)$ with $\delta + c_\chi \leq 7$ agree with [4];

(5) the numbers $\mathcal{N}^d(0, \chi)$ with $c_\chi \leq 7$ agree with [5, Proposition 1.2].

Moreover, our formulas pass all the lower degree checks we could think of; see Appendix A.

1.3 Outline of thesis

The main tool used in the thesis is the following lemma:

Lemma 1.3. *Let $M \subset \mathbb{P}^N$ be a compact algebraic variety and $V \longrightarrow \mathbb{P}^N$ a rank m holomorphic vector bundle. Assume that the dimension of M is also m. If s is a holomorphic section which is transverse to the zero set on every stratum of M, then the number of zeros of s is given by the Euler class of V evaluated on the fundamental class of M:*

$$|s^{-1}(0)| = \langle e(V), [M] \rangle.$$

However, the situation we are faced with is as follows:

Question 1.4. *Let ∂M be a (Zariski) closed subset of M. What is the number $\mathcal{N}$ of zeros of s that lie inside $M - \partial M$, if s is transverse to the zero set when restricted to $M - \partial M$?*

In order to answer this question, we have to look at the following problem:

Question 1.5. *Let ν be a generic section of $V \longrightarrow M$. What is the number $\mathcal{C}_{\partial M}$ of solutions (counted with a sign) for the equation*

$$s(m) + t\nu(m) = 0$$

for "small" t that lie "near" ∂M?

It can be shown that $\mathcal{C}_{\partial M}$ doesn't depend upon ν or t. The number $\mathcal{N}$ is therefore

$$\mathcal{N} = \langle e(V), [M] \rangle - \mathcal{C}_{\partial M}.$$

A global algebro-geometric excess intersection approach is described in [2]; in this thesis instead we follow the purely topological approach of [25]. In order to enumerate curves with just one node, we can take $M = \mathcal{D} \times \mathbb{P}^2$, where $\mathcal{D} \approx \mathbb{P}^1$ is a one-dimensional family of degree-d curves and

$$V = \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \oplus \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \otimes T^*\mathbb{P}^2$$

where $\gamma_{\mathcal{D}}$ and $\gamma_{\mathbb{P}^2}$ are the tautological line bundles over $\mathcal{D}$ and $\mathbb{P}^2$, respectively. A simple application of the splitting principle and Kunneth formula shows that

$$\langle e(V), [M] \rangle = 3(d-1)^2.$$

Hence the number of degree-d curves through $\kappa(d) - 1$ points and having one node is $3(d-1)^2$. However, to enumerate curves with two *distinct* nodes we need to count the number of zeros inside the space

$$\mathcal{D} \times (\mathbb{P}^2 \times \mathbb{P}^2 - \Delta)$$

where $\mathcal{D} \approx \mathbb{P}^2$ is a two-dimensional family of degree-d curves and $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the diagonal. This space is noncompact! Hence we have to use excess-intersection theory with

$$M = \mathcal{D} \times \mathbb{P}^2 \times \mathbb{P}^2, \qquad \partial M = \mathcal{D} \times \Delta.$$

In order to compute $\mathcal{C}_{\partial M}$, we have to have an understanding of a one-dimensional family of curves that have a simple node. This family can degenerate into a curve with a cusp (a cusp is locally given by the equation $y^2 + x^3 = 0$). The number $\mathcal{C}_{\partial M}$ consists of two parts: the contribution from a one-dimensional non-compact family of curves with a simple node and the contribution from a finite set of cuspidal curves. Hence we have to know how many curves are there through $\kappa(d) - 2$ points that have one cusp!

In general, to enumerate curves with k nodes we may have to first enumerate curves with other singularities of total codimension k, but with fewer singular points. As the number of node increases, the situation becomes more and more complicated, as more and more of them can sink together and effect the boundary contribution. However, we believe that the conclusion of Theorems 1-3, 6, and 7 and the $\sigma_2 = 0$ cases of Theorems 5, 8-10 holds for any number of nodes δ, as no new types of boundary strata occur.

In Chapter 2, we collect a number of preliminary results concerning local structure of holomorphic maps, which are then used to define the bundle sections that are central to this thesis. We also show that these bundle sections are generically transverse, even after cutting down by general point conditions. In Chapter 3, we study closures of spaces of curves with a singular point of certain types and some number of nodes; this is used to determine boundary contributions in Chapters 4-7. In Chapter 4, we focus on one-point singularities and in particular prove Theorems 14-16. In Chapter 5, we continue on to two-point singularities and complete proofs of Theorems 11-13. In Chapter 6, we finish proofs of Theorems 9 and 10, which involve up to 3 singular points. The remaining theorems are proved in Chapter 7. In Appendix A, we describe a number of low-degree checks, in cases when our numbers can be obtained by direct geometric arguments.

Chapter 2

Preliminaries

2.1 Local structure of holomorphic maps

If $f = f(x,y)$ is a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^2$ and i,j are non-negative integers, let

$$f_{ij} = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}\bigg|_{(x,y)=0}.$$

Lemma 2.1. *Let $f = f(x,y)$ be a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^2$ such that $f_{00} = 0$. If $f_{01} \neq 0$,*

$$(u,v) = (x,f)$$

is a coordinate chart around the origin.

Proof. This follows immediately from the Inverse Function Theorem. □

Proposition 2.2. *Let $f = f(x,y)$ be a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^2$ such that $f_{00}, f_{01} = 0$. If $f_{02} \neq 0$, there exist a coordinate chart $(x, v = v(x,y))$ centered at the origin in $\mathbb{C}^2$ and a holomorphic function g on a neighborhood of 0 in $\mathbb{C}$ such that*

$$f(x,y) = g(x) + v(x,y)^2. \tag{2.1}$$

Furthermore, the germ of $g = g_f$ at the origin is uniquely determined by f; if it is nonzero, there exists a coordinate $u = u(x)$ centered at the origin in $\mathbb{C}$ such that

$$g(x) = u(x)^{k+1}$$

for some $k = k_f \in \mathbb{Z}^{\geq 0}$ determined by f. Finally, if h is a holomorphic function defined around the origin in $\mathbb{C}^2$ such that $h(0) \neq 0$, then $k_{hf} = k_f \equiv k$ and

$$\frac{d^k g_{hf}}{dx^k} = h(0)\frac{d^k g_f}{dx^k}.$$

Proof. (1) Since $f_{01} = 0$ and $f_{02} \neq 0$, there exists a holomorphic function $B = B(x)$ on a neighborhood of the origin in $\mathbb{C}$ such that

$$B(0) = 0 \qquad \text{and} \qquad f_y\big(x, B(x)\big) = 0 \ \ \forall\, x.$$

With respect to the coordinate chart $(x, \hat{y} = y - B(x))$,

$$\frac{\partial f}{\partial \hat{y}}\bigg|_{(x,0)} = \frac{\partial f}{\partial y}\bigg|_{(x,B(x))}.$$

Thus, the function f is of the form

$$f(x,y) = g(x) + h(x,\hat{y})\hat{y}^2$$

for some holomorphic functions g and h defined on neighborhoods of the origin in $\mathbb{C}$ and $\mathbb{C}^2$, respectively. Since $f_{01} = 0$ and $f_{02} \neq 0$, $h(0,0) \neq 0$; thus, the function

$$v = \sqrt{h(x,\hat{y})}\hat{y}$$

has the desired properties.

(2) If v and g are as in (2.1), the derivatives of g at the origin in $\mathbb{C}$ are polynomials in the partial derivatives of f and of x-partials of $y = y(x,v)$ at the origin in $\mathbb{C}^2$. Since $f_v = f_y y_v = 2v$ vanishes along $v = 0$, $f_y|_{v=0} = 0$ and so

$$\big(f_{xy} + f_{yy}y_x\big)|_{v=0} = 0.$$

Since $f_{02} \neq 0$, this equation expresses the x-partials of $y = y(x,v)$ at $(x,v) = (0,0)$ as polynomial in the partial derivatives of f at the origin and in f_{02}^{-1}. Thus, the germ of the holomorphic function g is determined by f.

(3) If the germ of g at 0 is nonzero, there exist $k \in \mathbb{Z}^{\geq 0}$ and a holomorphic function h on a neighborhood of the origin such that

$$g(x) = h(x)x^{k+1}, \qquad h(0) \neq 0.$$

The function

$$u(x) = \sqrt[k+1]{h(x)}\,x$$

then has the desired properties.

(4) It is sufficient to prove the last statement for $f(x,y) = g(x) + y^2$. For hf, the function $B(x)$ in (1) above is of the form $g(x)b(x)$ with $b = b(x)$ determined by the Implicit Function Theorem from

$$b(0) = -\frac{h_y(0,0)}{2}, \quad h_y\big(x, g(x)b(x)\big) + 2b(x)h\big(x, g(x)b(x)\big) + g(x)b(x)^2 h_y\big(x, g(x)b(x)\big) = 0 \ \ \forall\, x.$$

Thus,

$$g_{hf}(x) \equiv f(x, B(x)) = h(x, g(x)b(x))g(x) + g(x)^2 h(x)^2$$

has the same first nonzero derivative at the origin as g. □

If f and g_f are as in Proposition 2.2 and $k \in \mathbb{Z}^+$, let

$$A_k^f = \frac{1}{k!}\frac{d^k g_f}{dx^k}\bigg|_{x=0}.$$

If in addition $f_{10}, f_{20}, f_{11} = 0$, we find that

$$A_3^f = f_{30}\,, \qquad A_4^f = f_{40} - \frac{3f_{21}^2}{f_{02}}\,, \qquad A_5^f = \frac{f_{50}}{24} - \frac{5f_{21}f_{31}}{12f_{02}} + \frac{5f_{12}f_{21}^2}{8f_{02}^2}$$

$$A_6^f = \frac{f_{60}}{120} - \frac{f_{21}f_{41}}{8f_{02}} + \frac{f_{31}^2}{12f_{02}} + \frac{f_{12}f_{21}f_{31}}{2f_{02}^2} + \frac{3f_{21}^2f_{22}}{8f_{02}^2} - \frac{f_{03}f_{21}^3}{8f_{02}^3} - \frac{3f_{12}^2f_{21}^2}{4f_{02}^3}$$

$$A_7^f = \frac{f_{70}}{720} - \frac{7f_{21}f_{51}}{240f_{02}} - \frac{7f_{31}f_{41}}{144f_{02}} + \frac{7f_{12}f_{21}f_{41}}{48f_{02}^2} + \frac{7f_{21}^2f_{32}}{48f_{02}^2} + \frac{7f_{12}f_{31}^2}{72f_{02}^2} + \frac{7f_{21}f_{22}f_{31}}{24f_{02}^2}$$
$$- \frac{7f_{03}f_{21}^2f_{31}}{48f_{02}^3} - \frac{7f_{12}^2f_{21}f_{31}}{12f_{02}^3} - \frac{7f_{12}f_{21}^2f_{22}}{8f_{02}^3} - \frac{7f_{13}f_{21}^3}{48f_{02}^3} + \frac{7f_{03}f_{12}f_{21}^3}{16f_{02}^4} + \frac{7f_{12}^3f_{21}^2}{8f_{02}^4}.$$

Thus, the curve $f^{-1}(0)$ has an A_k-node at the origin if and only if $A_i^f = 0$ for all $i \le k$ and $A_{k+1}^f \neq 0$. By the last statement of Proposition 2.2, the minimal integer k for which $A_k^f \neq 0$ depends only on the germ of f at the origin and for this value of k

$$A_k^{hf} = h(0)A_k^f \tag{2.2}$$

for any holomorphic function h around the origin in $\mathbb{C}^2$ such that $h(0) \neq 0$. Note that A_k^f is not defined if $f_{02} = 0$ and $k > 3$, but $f_{02}^{k-3}A_k^f$ is defined even if $f_{02} = 0$.

Proposition 2.3. *Let $f = f(x,y)$ be a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^2$ such that $f(0,0), \nabla f|_{(0,0)}, \nabla^2 f|_{(0,0)} = 0$. If either*

$$\nabla^3 f|_{(0,0)}(w,w,\cdot) \neq 0 \quad \forall\, w \in \mathbb{C}^2 - 0 \qquad \text{or} \qquad f_{30}, f_{21} = 0, \quad f_{12} \neq 0,$$

there exist a coordinate chart

$$(\hat{x},\hat{y}) = \big(x - C(y)y, \hat{y}\big)$$

centered at the origin in $\mathbb{C}^2$ and a holomorphic function g on a neighborhood of 0 in $\mathbb{C}$ such that

$$f(x,y) = \hat{x}\big(g(\hat{x}) + \hat{y}^2\big)\,. \tag{2.3}$$

Furthermore, the germ of $g = g_f$ at the origin is nonzero in the first case and is uniquely determined by f in the second case. In either case, if it is nonzero, there exists a coordinate chart (u,v) centered at the origin in $\mathbb{C}^2$ such that

$$f(x,y) = u(x,y)\big(u(x,y)^{k_f-2} + v(x,y)^2\big) \tag{2.4}$$

with $k_f = 4$ in the first case and for some $k_f \ge 5$ determined by f in the second case. Finally, if h is a holomorphic function defined around the origin in $\mathbb{C}^2$ such that $h(0) \neq 0$, then $k_{hf} = k_f \equiv k$ and in the second case

$$\frac{d^k g_{hf}}{dx^k} = h(0)\frac{d^k g_f}{dx^k}\,.$$

Proof. (1) We first show that there exists a holomorphic function $C = C(y)$ on a neighborhood of the origin so that

$$f\big(C(y)y, y\big) = 0 \qquad \forall\, y. \tag{2.5}$$

If $\nabla^3 f|_{(0,0)}(w,w,\cdot) \neq 0$ for all $w \in \mathbb{C}^2 - 0$, the cubic term in the Taylor expansion of f has no repeated factors. Thus,

$$f(x,y) = \tilde{x}\big(b(\tilde{x},y)y^2 + \tilde{x}yc(\tilde{x},\tilde{y}) + \tilde{x}^2 d(\tilde{x},\tilde{y})\big) + e(\tilde{y})\tilde{y}^4\,,$$

where $\tilde{x}=x+ay$ for some $a\in\mathbb{C}$ and $b=b(\tilde{x},y)$, $c=c(\tilde{x},y)$, $d=d(\tilde{x},y)$, and $e=e(y)$ are holomorphic around the origin in $\mathbb{C}^2$ and $\mathbb{C}$ such that $b(0,0)\neq 0$ (because the cubic in the Taylor expansion of $f(\tilde{x},y)$ is not a multiple of $\tilde{x}$). The condition (2.5) on $C(y)=-a+D(y)y$ is equivalent to

$$D(y)\big(b(D(y)y^2,y)+c(D(y)y^2,y)yD(y)+d(D(y)y^2,y)y^2D(y)^2\big)+e(y)=0\quad\forall\, y.$$

Since $b(0,0)\neq 0$, by the Implicit Function Theorem there exists a holomorphic function $D=D(y)$ on a neighborhood of the origin in $\mathbb{C}$ with $D(0)=-e(0)/b(0,0)$ solving this equation.

If $f_{30},f_{21}=0$ and $f_{12}\neq 0$,

$$f(x,y)=a(x,y)xy^2+b(y)y^3+c(x)x^4+d(x,y)x^3y$$

for some holomorphic functions a,b,c,d with $a(0,0)\neq 0$. The condition (2.5) on $C=C(y)$ is equivalent to

$$C(0)=-\frac{b(0)}{a(0,0)}\,,\quad a\big(C(y)y,y\big)C(y)+b(y)+c\big(C(y)\big)yC(y)^4+d\big(C(y)y,y\big)yC(y)^3=0\quad\forall\, y.$$

Since $a(0,0)\neq 0$, by the Implicit Function Theorem there exists a holomorphic function $C=C(y)$ on a neighborhood of the origin in $\mathbb{C}$ with $C(0)=-b(0)/a(0,0)$ solving this equation. In either case, by the Inverse Function Theorem

$$(\hat{x},y)=(x+C(y)y,y)$$

is a coordinate system centered at the origin.

(2) By (2.5),

$$f(x,y)=\hat{x}\hat{f}(\hat{x},y)$$

for some holomorphic function $\hat{f}$ on a neighborhood of the origin in $\mathbb{C}^2$. By the assumptions on f,

$$\hat{f}_{00},\hat{f}_{10},\hat{f}_{01}=0,\qquad \hat{f}_{02}\neq 0.$$

By Proposition 2.2, there exist a coordinate chart $(\hat{x},\hat{y})$ centered at the origin in $\mathbb{C}^2$ and a holomorphic function g on a neighborhood of 0 in $\mathbb{C}$ such that

$$f(x,y)=\hat{x}\hat{f}(\hat{x},y)=\hat{x}\big(g(\hat{x})+\hat{y}^2\big),\qquad g(0),g'(0)=0.$$

In the first case, $g''(0)\neq 0$, while in the second case $g''(0)=0$. Since the germ of $C=C(y)$ at the origin is uniquely determined by f, so is the germ of $\hat{f}$; Proposition 2.2 then implies that the germ of $g_f=g$ at the origin is also determined by f. It also implies the last claim in Proposition 2.3.

(3) If the germ of g at 0 is nonzero, there exist $k\geq 4$ and a holomorphic function h on a neighborhood of the origin such that

$$g(\hat{x})=h(\hat{x})\hat{x}^{k-2}\,,\qquad h(0)\neq 0.$$

By the Inverse Function Theorem,

$$u=\sqrt[k-1]{h(\hat{x})}\hat{x}\,,\qquad v=\frac{\hat{y}}{\sqrt[2(k-1)]{h(\hat{x})}},$$

is a coordinate chart centered at the origin so that (2.4) holds. □

If f and g_f are as in the second case of Proposition 2.3 and $k \geq 6$, let

$$D_k^f = \frac{1}{(k-3)!} \frac{d^{k-3} g_f}{d\hat{x}^{k-3}}\bigg|_{\hat{x}=0}.$$

In particular, we find that

$$D_6^f = f_{40}, \qquad D_7^f = -\frac{f_{31}^2}{24 f_{12}} + \frac{f_{50}}{40}.$$

By the last statement of Proposition 2.3, the minimal integer k for which $D_k^f \neq 0$ depends only on the germ of f at the origin and if this value of $k \geq 6$

$$D_k^{hf} = h(0) D_k^f \tag{2.6}$$

for any holomorphic function h around the origin in $\mathbb{C}^2$ such that $h(0) \neq 0$. Note that the curve $f^{-1}(0)$ has a D_k-node at the origin if and only if $D_i^f = 0$ for all $i < k$ and $D_{k+1}^f \neq 0$.

Proposition 2.4. *Let $f = f(x, y)$ be a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^2$ such that $f(0,0), \nabla f|_{(0,0)}, \nabla^2 f|_{(0,0)} = 0$. If $f_{30}, f_{21}, f_{12} = 0$ and $f_{03}, f_{40} \neq 0$, there exists a coordinate chart (u, v) centered at the origin in $\mathbb{C}^2$ such that*

$$f(x, y) = u(x, y)^4 + v(x, y)^3. \tag{2.7}$$

Proof. By the assumptions on f,

$$f(x, y) = a(x, y) y^3 + b(x, y) x^4 + \alpha x^3 y + \beta x^2 y^2 + \gamma x^3 y^2 \tag{2.8}$$

for some $\alpha, \beta, \gamma \in \mathbb{C}$ and holomorphic functions a and b on a neighborhood of the origin in $\mathbb{C}^2$ such that $a(0,0), b(0,0) \neq 0$. Let $A, B, C \in \mathbb{C}$ be given by

$$4b_{00}A + \alpha = 0, \qquad 3a_{00}B + 6b_{00}A^2 + 3\alpha A + \beta = 0,$$
$$3a_{00}C + 3a_{10}B + 10b_{10}A^2 + 4b_{01}A + 3\alpha A^2 B + 4\beta AB + \gamma = 0.$$

By the Inverse Function Theorem, the equations

$$x = \hat{x} + A\hat{y}, \qquad y = \hat{y} + B\hat{x}^2 + C\hat{x}^3$$

determine a coordinate chart $(\hat{x}, \hat{y})$ centered at the origin in $\mathbb{C}^2$. By (2.8),

$$f(x, y) = \hat{a}(\hat{x}, \hat{y}) \hat{y}^3 + \hat{b}(\hat{x}, \hat{y}) \hat{x}^4$$

for some holomorphic functions $\hat{a}$ and $\hat{b}$ around the origin in $\mathbb{C}^2$ such that $\hat{a}(0,0), \hat{b}(0,0) \neq 0$ (the three defining equations for A, B, C describe the coefficients of $\hat{x}^3\hat{y}, \hat{x}^2\hat{y}^2, \hat{x}^3\hat{y}^2$ in $f(x, y)$). Thus,

$$u = \sqrt[4]{\hat{b}(\hat{x}, \hat{y})}\,\hat{x}, \qquad v = \sqrt[3]{\hat{a}(\hat{x}, \hat{y})}\,\hat{y},$$

is a coordinate chart centered at the origin that satisfies (2.7). □

Proposition 2.5. *Let $f = f(x, y)$ be a holomorphic function defined on a neighborhood of the origin in $\mathbb{C}^2$ such that $f(0,0), \nabla f|_{(0,0)}, \nabla^2 f|_{(0,0)} = 0$. If $f_{30}, f_{21}, f_{12}, f_{40} = 0$ and $f_{03}, f_{31} \neq 0$, there exists a coordinate chart (u, v) centered at the origin in $\mathbb{C}^2$ such that*

$$f(x, y) = v(x, y)^3 + u(x, y)^3 v(x, y). \tag{2.9}$$

Proof. By the assumptions on f,

$$f(x,y)=a(x,y)y^3+b(x,y)x^3y+\alpha x^2y^2+\beta(x)x^5 \tag{2.10}$$

for some $\alpha\in\mathbb{C}$ and holomorphic functions a, b, and β on a neighborhood of the origin in $\mathbb{C}^2$ and $\mathbb{C}$ such that $a(0,0),b(0,0)\neq 0$. By the Implicit Function Theorem, there exists a holomorphic function $B=B(\hat{x})$ on a neighborhood of the origin in $\mathbb{C}$ such that

$$b(\hat{x},B(\hat{x})\hat{x}^2)B(\hat{x})+a(\hat{x},B(\hat{x})\hat{x}^2)\hat{x}B(\hat{x})+\hat{x}B(\hat{x})^2+\beta(\hat{x})=0, \qquad B(0)=-\frac{\beta(0)}{b_{00}}. \tag{2.11}$$

Let $A\in\mathbb{C}$ be given by

$$3b_{00}A+3a_{00}B(0)+\alpha=0. \tag{2.12}$$

By the Inverse Function Theorem, the equations

$$x=\hat{x}+A\hat{y}, \qquad y=\hat{y}+B(\hat{x})\hat{x}^2$$

determine a coordinate chart $(\hat{x},\hat{y})$ centered at the origin in $\mathbb{C}^2$. By (2.10),

$$f(x,y)=\hat{a}(\hat{x},\hat{y})\hat{y}^3+\hat{b}(\hat{x},\hat{y})\hat{x}^3\hat{y}$$

for some holomorphic functions $\hat{a}$ and $\hat{b}$ around the origin in $\mathbb{C}^2$ such that $\hat{a}(0,0),\hat{b}(0,0)\neq 0$ (the LHS of the first equation in (2.11) is $f|_{\hat{y}=0}/\hat{x}^5$, while the LHS of (2.12) is the coefficient of $\hat{x}^2\hat{y}^2$ in $f(x,y)$). Thus,

$$u=\sqrt[3]{\frac{\hat{b}(\hat{x},\hat{y})}{\sqrt[3]{\hat{a}(\hat{x},\hat{y})}}}\hat{x}, \qquad v=\sqrt[3]{\hat{a}(\hat{x},\hat{y})}\hat{y},$$

is a coordinate chart centered at the origin that satisfies (2.9). □

2.2 Transversality of sections

Let

$$\mathcal{P}_d\approx\mathbb{P}^{\kappa(d)+1}$$

denote the space of homogeneous polynomials of degree d on $\mathbb{C}^3$ or equivalently of polynomials of degree at most d on $\mathbb{C}^2$. Let

$$\mathcal{P}_d^*=\mathcal{P}_d-0$$

be the subspace of nonzero polynomials. If $V\longrightarrow M$ is any vector bundle over a smooth manifold, a section ψ of

$$\pi_1^*\gamma_{\mathcal{D}}^*\otimes\pi_2^*V\longrightarrow\mathcal{D}_d\times M$$

induces a section $\tilde{\psi}$ of $\pi_2^*V\longrightarrow\mathcal{P}_d^*\times M$ by

$$\tilde{\psi}(s,p)=\big\{\psi([s],p)\big\}(s).$$

We note that ψ is transverse to the zero set at $([s],p)$ if and only if $\tilde{\psi}$ is transverse to the zero set at (s,p).

Lemma 2.6. *The sections*

$$\begin{aligned}
&\psi_{A_0} \in \Gamma(\mathcal{D}_d \times \mathbb{P}^2, \pi_1^*\gamma_{\mathcal{D}}^* \otimes \pi_2^*\gamma_{\mathbb{P}^2}^{*d}), && \psi_{A_0}([s],p) = s(p)\,,\\
&\psi_{A_1} \in \Gamma(\psi_{A_0}^{-1}(0), \pi_1^*\gamma_{\mathcal{D}}^* \otimes \pi_2^*\gamma_{\mathbb{P}^2}^{*d} \otimes T^*\mathbb{P}^2), && \psi_{A_1}([s],p) = \nabla s|_p\,,\\
&\psi_{D_4} \in \Gamma\big(\psi_{A_1}^{-1}(0), \pi_1^*\gamma_{\mathcal{D}}^* \otimes \pi_2^*\gamma_{\mathbb{P}^2}^{*d} \otimes \mathrm{Sym}^2(T^*\mathbb{P}^2)\big), && \psi_{D_4}([s],p) = \nabla^2 s|_p\,,
\end{aligned}$$

are transverse to the zero set for all $[s]\in\mathcal{D}_d$*, provided* $d \geq 0,1,2$*, respectively.*

Proof. (1) Suppose $([s],p)\in\psi_{A_0}^{-1}(0)$. Choose homogeneous coordinates $[X_0,X_1,X_2]$ on $\mathbb{P}^2$ so that $p{=}[1,0,0]$ and let

$$U_0 = \big\{[X_0,X_1,X_2]\colon\ X_0\neq 0\big\}, \qquad x=\frac{X_1}{X_0}, \qquad y=\frac{X_2}{X_0}\,.$$

Viewing $\mathcal{P}_d$ as the space of polynomials in x,y of degree at most d, we show that the restriction of the induced section $\tilde{\psi}_{A_0}$ to $\mathcal{P}_d^*\times U_0$ is transverse to the zero set at $(s,0)$. With respect to the standard trivialization of $\gamma_{\mathbb{P}^2}^{*d}$ over U_0, $\tilde{\psi}_{A_0}$ is given by

$$\mathcal{P}_d^*\times U_0 \longrightarrow \mathbb{C}, \qquad (f,x,y)\longrightarrow f(x,y).$$

The differential of this map at $(s,0)$ is given by

$$\mathcal{P}_d\times\mathbb{C}^2 \longrightarrow \mathbb{C}, \qquad (f,x,y)\longrightarrow f_{00}+s_{10}x+s_{01}y.$$

The restriction of this linear map to the first component is surjective for any $d\in\mathbb{Z}^{\geq 0}$, and so $\tilde{\psi}_{A_0}$ is transverse to the zero set at $(s,0)$.

(2) Suppose $([s],p)\in\psi_{A_1}^{-1}(0)\subset\psi_{A_0}^{-1}(0)$; we continue with the setup of (1) above. Since the restriction of ∇s to $s^{-1}(0)$ is independent of the choice of ∇, the restriction of the induced section $\tilde{\psi}_{A_1}$ to $\tilde{\psi}_{A_0}^{-1}(0)\cap\mathcal{P}_d^*\times U_0$ with respect to the standard trivializations of $\gamma_{\mathbb{P}^2}^{*d}$ and $T^*\mathbb{P}^2$ over U_0 is given by

$$\tilde{\psi}_{A_0}^{-1}(0)\cap\mathcal{P}_d^*\times U_0 \longrightarrow \mathbb{C}^2, \qquad (f,x,y)\longrightarrow \big(f_x(x,y), f_y(x,y)\big).$$

Since the section $\tilde{\psi}_{A_0}$ is transverse to the zero set at $(s,0)$, the transversality of $\tilde{\psi}_{A_1}$ at $(s,0)$ is equivalent to the transversality of the map

$$\mathcal{P}_d^*\times U_0 \longrightarrow \mathbb{C}^3\,, \qquad (f,x,y)\longrightarrow \big(f(x,y), f_x(x,y), f_y(x,y)\big).$$

The differential of this map at $(s,0)$ is given by

$$\mathcal{P}_d\times\mathbb{C}^2 \longrightarrow \mathbb{C}, \qquad (f,x,y)\longrightarrow \big(f_{00}, f_{10}+s_{20}x+s_{11}y, f_{01}+s_{11}x+s_{02}y\big).$$

The restriction of this linear map to the first component is surjective for any $d\in\mathbb{Z}^+$, since f_{00},f_{10},f_{01} can be chosen arbitrarily then. Thus, $\tilde{\psi}_{A_1}$ is transverse to the zero set at $(s,0)$.

(3) Suppose $([s],p)\in\psi_{D_4}^{-1}(0)\subset\psi_{A_1}^{-1}(0)$; we continue with the setup above. Since the restriction of $\nabla^2 s$ to the zero set of s and ∇s is independent of the choice of ∇, the restriction of the induced section $\tilde{\psi}_{D_4}$ to $\tilde{\psi}_{A_1}^{-1}(0)\cap\mathcal{P}_d^*\times U_0$ with respect to the standard trivializations of $\gamma_{\mathbb{P}^2}^{*d}$ and $T^*\mathbb{P}^2$ over U_0 is given by

$$\tilde{\psi}_{A_1}^{-1}(0)\cap\mathcal{P}_d^*\times U_0 \longrightarrow \mathbb{C}^3, \qquad (f,x,y)\longrightarrow \big(f_{xx}(x,y), f_{xy}(x,y), f_{yy}\big).$$

Since the sections $\tilde{\psi}_{A_0}$ and $\tilde{\psi}_{A_0}$ are transverse to the zero set at $(s,0)$, the transversality of $\tilde{\psi}_{D_4}$ at $(s,0)$ is equivalent to the transversality of the map

$$\mathcal{P}_d^*\times U_0 \longrightarrow \mathbb{C}^6\,, \qquad (f,x,y)\longrightarrow \big(f(x,y), f_x(x,y), f_y(x,y), f_{xx}(x,y), f_{xy}(x,y), f_{yy}\big).$$

The restriction of the differential of this map at $(s,0)$ to the first component of the tangent bundle is given by

$$\mathcal{P}_d \longrightarrow \mathbb{C}^6, \qquad f\longrightarrow \big(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02}\big).$$

This map is surjective if $d\geq 2$. □

Let

$$V_2=\tilde{\gamma}^*\otimes\pi^*(\pi_1^*\gamma_{\mathcal{D}}^*\otimes\pi_2^*(\gamma_{\mathbb{P}^2}^{*d}\otimes T^*\mathbb{P}^2))\longrightarrow \mathcal{D}_d\times\mathbb{P}(T\mathbb{P}^2)\,,$$
$$V_3=\tilde{\gamma}^{*2}\otimes\pi^*(\pi_1^*\gamma_{\mathcal{D}}^*\otimes\pi_2^*(\gamma_{\mathbb{P}^2}^{*d}\otimes T^*\mathbb{P}^2))\longrightarrow \mathcal{D}_d\times\mathbb{P}(T\mathbb{P}^2),$$

where $\pi\colon \mathcal{D}_d\times\mathbb{P}(T\mathbb{P}^2)\longrightarrow \mathcal{D}_d\times\mathbb{P}^2$ is the projection map.

Lemma 2.7. *The sections*

$$\psi_{A_2}\in\Gamma\big((\mathcal{D}_d\times\mathbb{P}(T\mathbb{P}^2))|_{\psi_{A_1}^{-1}(0)}, V_2\big), \qquad \big\{\psi_{A_2}([s],p,\ell)\big\}(v)=\nabla^2 s|_p(v,\cdot),$$
$$\psi_{D_5}\in\Gamma\big((\mathcal{D}_d\times\mathbb{P}(T\mathbb{P}^2))|_{\psi_{D_4}^{-1}(0)}, V_3\big), \qquad \big\{\psi_{D_5}([s],p,\ell)\big\}(v^2)=\nabla^3 s|_p(v,v,\cdot),$$

are transverse to the zero set, provided $d\geq 2,3$, respectively.

Proof. (1) Suppose $([s],p,\ell)\in\psi_{A_2}^{-1}(0)$. We continue with the setup in the proof of Lemma 2.6 above, but choose the homogeneous coordinates so that ℓ is the span of the tangent vector $\frac{\partial}{\partial x}$ at p. Let

$$\tilde{U}_0=\big\{[w]\in\mathbb{P}(TP^2)|_{U_0}\colon\, dx(w)\neq 0\big\}.$$

Since the restriction of $\nabla^2 s$ to the zero set of s and ∇s is independent of the choice of ∇, the restriction of the induced section $\tilde{\psi}_{A_2}$ to $(\mathcal{P}_d^*\times\tilde{U}_0)|_{\tilde{\psi}_{A_1}^{-1}(0)}$ with respect to the standard trivializations of $\gamma_{\mathbb{P}^2}^{*d}$, $T^*\mathbb{P}^2$, and $\tilde{U}_0$ over U_0 and of $\tilde{\gamma}^*$ over $\tilde{U}_0$ is given by

$$(\mathcal{P}_d^*\times\tilde{U}_0)|_{\tilde{\psi}_{A_1}^{-1}(0)}\longrightarrow\mathbb{C}^2, \qquad (f,x,y,\eta)\longrightarrow\big(f_{xx}(x,y)+\eta f_{xy}(x,y), f_{xy}(x,y)+\eta f_{yy}(x,y)\big).$$

Since the sections $\tilde{\psi}_{A_0}$ and $\tilde{\psi}_{A_1}$ are transverse to the zero set at $(s,0)$, the transversality of $\tilde{\psi}_{A_2}$ at $(s,0,\ell)$ is equivalent to the transversality of the map

$$\mathcal{P}_d^*\times\tilde{U}_0\longrightarrow\mathbb{C}^5\,,$$
$$(f,x,y,\eta)\longrightarrow\big(f(x,y), f_x(x,y), f_y(x,y), f_{xx}(x,y)+\eta f_{xy}(x,y), f_{xy}(x,y)+\eta f_{yy}(x,y)\big).$$

The restriction of the differential of this map at $(s,0,0)$ to the first component of the tangent bundle is given by

$$\mathcal{P}_d\longrightarrow\mathbb{C}^5, \qquad f\longrightarrow\big(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}\big).$$

This map is surjective if $d\geq 2$.

(2) Suppose $([s],p,\ell)\in\psi_{D_5}^{-1}(0)$; we continue with the setup in (1) above. Since the restriction of $\nabla^3 s$ to the zero set of s, ∇s, and $\nabla^2 s$ is independent of the choice of ∇, the restriction of the

induced section $\tilde{\psi}_{D_5}$ to $(\mathcal{P}_d^*\times\tilde{U}_0)|_{\tilde{\psi}_{D_4}^{-1}(0)}$ with respect to the standard trivializations of $\gamma_{\mathbb{P}^2}^{*d}$, $T^*\mathbb{P}^2$, and $\tilde{U}_0$ over U_0 and of $\tilde{\gamma}^*$ over $\tilde{U}_0$ is given by

$$(\mathcal{P}_d^*\times\tilde{U}_0)|_{\tilde{\psi}_{D_4}^{-1}(0)} \longrightarrow \mathbb{C}^2,$$
$$(f,x,y,\eta) \longrightarrow \big(f_{xx}(x,y)+\eta f_{xy}(x,y)+\eta^2 f_{xyy}, f_{xy}(x,y)+\eta f_{yy}(x,y)+\eta^2 f_{yyy}\big).$$

Since the sections $\tilde{\psi}_{A_0}$, $\tilde{\psi}_{A_1}$, and $\tilde{\psi}_{D_4}$, are transverse to the zero set at $(s,0)$, the transversality of $\tilde{\psi}_{D_5}$ at $(s,0,\ell)$ is equivalent to the transversality of the map

$$\mathcal{P}_d^*\times\tilde{U}_0 \longrightarrow \mathbb{C}^8\,,$$
$$(f,x,y) \longrightarrow \big(f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}+2\eta f_{xxy}+\eta^2 f_{xyy}, f_{xxy}+2\eta f_{xyy}+\eta^2 f_{yyy}\big)_{(x,y)}.$$

The restriction of the differential of this map at $(s,0,0)$ to the first component of the tangent bundle is given by

$$\mathcal{P}_d \longrightarrow \mathbb{C}^8, \qquad f \longrightarrow \big(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02}, f_{30}, f_{21}\big).$$

This map is surjective if $d\ge 3$. □

Let

$$L_3' = \tilde{\gamma}^*\otimes\big(\pi^*\pi_2^*T\mathbb{P}^2/\tilde{\gamma}\big)^{*2}\otimes\pi^*(\pi_1^*\gamma_{\mathcal{D}}^*\otimes\pi_2^*\gamma_{\mathbb{P}^2}^{*d}) \longrightarrow \mathcal{D}_d\times\mathbb{P}(T\mathbb{P}^2)\,,$$
$$L_4' = \tilde{\gamma}^{*4}\otimes\pi^*(\pi_1^*\gamma_{\mathcal{D}}^*\otimes\pi_2^*\gamma_{\mathbb{P}^2}^{*d}) \longrightarrow \mathcal{D}_d\times\mathbb{P}(T\mathbb{P}^2).$$

Lemma 2.8. *The sections*

$$\psi_{E_6}\in\Gamma\big(\psi_{D_5}^{-1}(0), L_3'\big), \qquad \big\{\psi_{E_6}([s],p,\ell)\big\}(v\otimes w^2) = \nabla^3 s|_p(v,w,w),$$
$$\psi_{E_7}\in\Gamma\big(\psi_{E_6^{-1}(0)}, L_4'\big), \qquad \big\{\psi_{E_7}([s],p,\ell)\big\}(v^4) = \nabla^4 s|_p(v,v,v,v),$$

are transverse to the zero set, provided $d\ge 3,4$, respectively.

Proof. (1) Suppose $([s],p,\ell)\in\psi_{E_6}^{-1}(0)\subset\psi_{D_5}^{-1}(0)$; we continue with the setup in the proof of Lemma 2.7. Since the restriction of $\nabla^3 s$ to the zero set of s, ∇s, and $\nabla^2 s$ is independent of the choice of ∇ and $\nabla^3 s$ vanishes with two inputs from the distinguished tangent direction, the restriction of the induced section $\tilde{\psi}_{E_6}$ to $\tilde{\psi}_{D_5}^{-1}(0)\cap(\mathcal{P}_d^*\times\tilde{U}_0)$ with respect to the standard trivializations of $\gamma_{\mathbb{P}^2}^{*d}$, $T^*\mathbb{P}^2$, and $\tilde{U}_0$ over U_0 and of $\tilde{\gamma}^*$ over $\tilde{U}_0$ is given by

$$\tilde{\psi}_{D_5}^{-1}(0)\cap(\mathcal{P}_d^*\times\tilde{U}_0) \longrightarrow \mathbb{C}, \qquad (f,x,y,\eta) \longrightarrow f_{xyy}(x,y)+\eta f_{yyy}(x,y).$$

Since the sections $\tilde{\psi}_{A_0}$, $\tilde{\psi}_{A_1}$, $\tilde{\psi}_{D_4}$, and $\tilde{\psi}_{D_5}$ are transverse to the zero set at $(s,0,\ell)$, the transversality of $\tilde{\psi}_{E_6}$ at $(s,0,\ell)$ is equivalent to the transversality of the map

$$\mathcal{P}_d^*\times\tilde{U}_0 \longrightarrow \mathbb{C}^9\,,$$
$$(f,x,y,\eta) \longrightarrow \big(f, f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}+\eta f_{xxy}, f_{xxy}+\eta f_{xyy}, f_{xyy}+\eta f_{yyy}\big)_{(x,y)}.$$

The restriction of the differential of this map at $(s,0,0)$ to the first component of the tangent bundle is given by

$$\mathcal{P}_d \longrightarrow \mathbb{C}^9, \qquad f \longrightarrow \big(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02}, f_{30}, f_{21}, f_{12}\big).$$

This map is surjective if $d \geq 3$.

(2) Suppose $([s], p, \ell) \in \psi_{E_7}^{-1}(0) \subset \psi_{E_6}^{-1}(0)$; we continue with the setup as above. Since s vanishes on $\psi_{E_6}^{-1}(0)$ and ∇s, $\nabla^2 s$, and $\nabla^3 s$ vanish along the distinguished direction, the restriction of the induced section $\tilde{\psi}_{E_7}$ to $\tilde{\psi}_{E_6}^{-1}(0) \cap (\mathcal{P}_d^* \times \tilde{U}_0)$ with respect to the standard trivializations of $\gamma_{\mathbb{P}^2}^{*d}$, $T^*\mathbb{P}^2$, and $\tilde{U}_0$ over U_0 and of $\tilde{\gamma}^*$ over $\tilde{U}_0$ is given by

$$\tilde{\psi}_{E_6}^{-1}(0) \cap (\mathcal{P}_d^* \times \tilde{U}_0) \longrightarrow \mathbb{C},$$
$$(f, x, y, \eta) \longrightarrow f_{xxxx}(x, y) + 4\eta f_{xxxy}(x, y) + 6\eta^2 f_{xxyy}(x, y) + 4\eta^3 f_{xyyy}(x, y) + \eta^4 f_{yyyy}(x, y).$$

Since the sections $\tilde{\psi}_{A_0}$, $\tilde{\psi}_{A_1}$, $\tilde{\psi}_{D_4}$, $\tilde{\psi}_{D_5}$, and $\tilde{\psi}_{E_6}$ are transverse to the zero set at $(s, 0, \ell)$, the transversality of $\tilde{\psi}_{E_7}$ at $(s, 0, \ell)$ is equivalent to the transversality of the map

$$\tilde{\psi}_{A_0} \oplus \tilde{\psi}_{A_1} \oplus \tilde{\psi}_{D_4} \oplus \tilde{\psi}_{D_5} \oplus \tilde{\psi}_{E_6} \oplus \tilde{\psi}_{E_7} : \mathcal{P}_d^* \times \tilde{U}_0 \longrightarrow \mathbb{C}^{10}.$$

The restriction of the differential of this map at $(s, 0, 0)$ to the first component of the tangent bundle is given by

$$\mathcal{P}_d \longrightarrow \mathbb{C}^{10}, \qquad f \longrightarrow \big(f_{00}, f_{10}, f_{01}, f_{20}, f_{11}, f_{02}, f_{30}, f_{21}, f_{12}, f_{40}\big).$$

This map is surjective if $d \geq 4$. $\square$

For each $k \in \mathbb{Z}^+$, let

$$L_k = \tilde{\gamma}^{*^k} \otimes \big(\pi^*\pi_2^* T\mathbb{P}^2/\tilde{\gamma}\big)^{*(k-3)} \otimes \pi^*\big(\pi_1^*\gamma_{\mathcal{D}}^* \otimes \pi_2^*\gamma_{\mathbb{P}^2}^{*^d}\big)^{k-2} \longrightarrow \mathcal{D}_d \times \mathbb{P}(T\mathbb{P}^2)\,,$$
$$\mathbb{L}_k = \tilde{\gamma}^{*^{2(k-4)}} \otimes \big(\pi^*\pi_2^* T\mathbb{P}^2/\tilde{\gamma}\big)^{*2(k-6)} \otimes \pi^*\big(\pi_1^*\gamma_{\mathcal{D}}^* \otimes \pi_2^*\gamma_{\mathbb{P}^2}^{*^d}\big)^{k-5} \longrightarrow \mathcal{D}_d \times \mathbb{P}(T\mathbb{P}^2)\,.$$

For $k \geq 3$, the maps

$$\big\{f \in \mathcal{P}_d :\ f_{02} \neq 0\big\} \longrightarrow \mathbb{C}, \qquad f \longrightarrow f_{02}^{k-3} A_f^k\,, \tag{2.13}$$

of Section 2.1 are locally bounded on $\mathbb{P}^d$. Thus, by induction and (2.2), these maps induce sections

$$\psi_{A_k} \in \Gamma\big(\overline{\psi_{A_{k-1}}^{-1}(0) - (\mathcal{D}_d \times \mathbb{P}^2)|_{\psi_{D_4}^{-1}(0)}}, L_k\big).$$

For $k \geq 6$, the maps

$$\big\{f \in \mathcal{P}_d :\ f_{12} \neq 0\big\} \longrightarrow \mathbb{C}, \qquad f \longrightarrow f_{12}^{k-6} D_f^k\,, \tag{2.14}$$

are also are locally bounded on $\mathbb{P}^d$. Thus, by induction and (2.6), these maps induce sections

$$\psi_{D_k} \in \Gamma\big(\overline{\psi_{D_{k-1}}^{-1}(0) - \psi_{E_6}^{-1}(0)}, \mathbb{L}_k\big).$$

Lemma 2.9. *(1) For every $k \geq 3$, the section*

$$\psi_{A_k} \in \Gamma\big(\psi_{A_{k-1}}^{-1}(0) - (\mathcal{D}_d \times \mathbb{P}^2)|_{\psi_{D_4}^{-1}(0)}, L_k\big)$$

is transverse to the zero set, provided $d \geq k$; the section ψ_{A_3} is transverse over $\psi_{A_2}^{-1}(0)$.
(2) For every $k \geq 6$, the section

$$\psi_{D_k} \in \Gamma\big(\psi_{D_{k-1}}^{-1}(0) - \psi_{E_6}^{-1}(0), \mathbb{L}_k\big)$$

is transverse to the zero set, provided $d \geq k-2$; the section ψ_{D_6} is transverse over $\psi_{D_5}^{-1}(0)$.

Proof. (1) Suppose $([s],p,\ell)\in\psi_{A_k}^{-1}(0)$; we continue with the setup in the proof of Lemma 2.7. Since the map (2.13) is a polynomial in the derivatives of f at 0 with one of the directions being distinguished, the restrictions of the induced sections $\tilde\psi_{A_l}$ to $\tilde\psi_{A_{l-1}}^{-1}(0)\cap(\mathcal{P}_d^*\times\tilde U_0)$ naturally extend to a map

$$\tilde\psi_{A_l}\colon\mathcal{P}_d^*\times\tilde U_0\longrightarrow\mathbb{C}.$$

Since the sections $\tilde\psi_{A_0},\tilde\psi_{A_1},\ldots,\tilde\psi_{A_{k-1}}$ are transverse to the zero set at $(s,0,\ell)$, the transversality of $\tilde\psi_{A_k}$ at $(s,0,\ell)$ is equivalent to the transversality of the map

$$\tilde\psi_{A_0}\oplus\tilde\psi_{A_1}\oplus\ldots\oplus\tilde\psi_{A_k}\colon\mathcal{P}_d^*\times\tilde U_0\longrightarrow\mathbb{C}^{k+3}. \tag{2.15}$$

The restriction of the differential of $\tilde\psi_{A_0}\oplus\tilde\psi_{A_1}\oplus\tilde\psi_{A_2}$ at $(s,0,0)$ to the first component of the tangent bundle is given by

$$\mathcal{P}_d\longrightarrow\mathbb{C}^5,\qquad f\longrightarrow\big(f_{00},f_{10},f_{01},f_{20},f_{11},f_{02}\big).$$

The restriction of the differential of $\tilde\psi_{A_l}$ with $l\geq3$ is a linear combination of the maps

$$\mathfrak{D}_{ij}\colon\mathcal{P}_d\longrightarrow\mathbb{C},\qquad f\longrightarrow f_{ij}\,,$$

with $i+j\leq k$ and the coefficient of $\mathfrak{D}_{k0}$ is a nonzero multiple of s_{02}^{k-3}. Thus, the restriction of the differential of (2.15) at $(s,0,0)$ to the first component of the tangent bundle is surjective if $d\geq k$ and either $s_{02}\neq0$ or $k=3$.

(2) Suppose $([s],p,\ell)\in\psi_{D_k}^{-1}(0)$; we continue with the setup in (1) above. Since the map (2.14) is a polynomial in the derivatives of f at 0 with one of the directions being distinguished, the restrictions of the induced sections $\tilde\psi_{D_l}$ to $\tilde\psi_{D_{l-1}}^{-1}(0)\cap(\mathcal{P}_d^*\times\tilde U_0)$ naturally extend to a map

$$\tilde\psi_{D_l}\colon\mathcal{P}_d^*\times\tilde U_0\longrightarrow\mathbb{C}.$$

Since the sections $\tilde\psi_{A_0},\tilde\psi_{A_1},\tilde\psi_{D_4},\tilde\psi_{D_5}\ldots,\tilde\psi_{D_{k-1}}$ are transverse to the zero set at $(s,0,\ell)$, the transversality of $\tilde\psi_{D_k}$ at $(s,0,\ell)$ is equivalent to the transversality of the map

$$\tilde\psi_{A_0}\oplus\tilde\psi_{A_1}\oplus\tilde\psi_{D_4}\oplus\tilde\psi_{D_5}\oplus\ldots\oplus\tilde\psi_{D_k}\colon\mathcal{P}_d^*\times\tilde U_0\longrightarrow\mathbb{C}^{k+3}. \tag{2.16}$$

The restriction of the differential of $\tilde\psi_{A_0}\oplus\tilde\psi_{A_1}\oplus\tilde\psi_{D_4}\oplus\tilde\psi_{D_5}$ at $(s,0,0)$ to the first component of the tangent bundle is given by

$$\mathcal{P}_d\longrightarrow\mathbb{C}^8,\qquad f\longrightarrow\big(f_{00},f_{10},f_{01},f_{20},f_{11},f_{02},f_{30},f_{21}\big).$$

The restriction of the differential of $\tilde\psi_{D_l}$ with $l\geq6$ is a linear combination of the maps $\mathfrak{D}_{ij}$ above with $i+j\leq k-2$ and the coefficient of $\mathfrak{D}_{(k-2)0}$ is a nonzero multiple of s_{12}^{k-6}. Thus, the restriction of the differential of (2.16) at $(s,0,0)$ to the first component of the tangent bundle is surjective if $d\geq k-2$ and either $s_{12}\neq0$ or $k=6$. $\square$

Lemma 2.10. *The section*

$$\psi_{\tilde D_4}\in\Gamma\big(\psi_{A_3}^{-1}(0),(T\mathbb{P}^2/\tilde\gamma)^{*2}\otimes\gamma_{\mathcal{D}}^*\otimes\gamma_{\mathbb{P}^2}^{*d}\otimes T^*\mathbb{P}^2\big),\qquad\big\{\psi_{\tilde D_4}([s],p,\ell)\big\}(w^2)=\nabla^2s|_p(w,w),$$

is transverse to the zero set, provided $d\geq3$.

Proof. Suppose $([s],p,\ell)\in\psi_{\tilde{D}_4}^{-1}(0)\subset\psi_{A_3}^{-1}(0)$; we continue with the setup in the proof of Lemma 2.7. Since the restriction of $\nabla^2 s$ to the zero set of s and ∇s is independent of the choice of ∇ and $\nabla^2 s$ vanishes with either input from the distinguished tangent direction, the restriction of the induced section $\tilde{\psi}_{\tilde{D}_4}$ to $\tilde{\psi}_{A_3}^{-1}(0)\cap(\mathcal{P}_d^*\times\tilde{U}_0)$ with respect to the standard trivializations of $\gamma_{\mathbb{P}^2}^{*d}$, $T^*\mathbb{P}^2$, and $\tilde{U}_0$ over U_0 and of $\tilde{\gamma}^*$ over $\tilde{U}_0$ is given by

$$\tilde{\psi}_{A_3}^{-1}(0)\cap(\mathcal{P}_d^*\times\tilde{U}_0)\longrightarrow\mathbb{C},\qquad (f,x,y,\eta)\longrightarrow f_{yy}(x,y).$$

Since the sections $\tilde{\psi}_{A_0}$, $\tilde{\psi}_{A_1}$, $\tilde{\psi}_{A_2}$, and $\tilde{\psi}_{A_3}$, are transverse to the zero set at $(s,0,\ell)$, the transversality of $\tilde{\psi}_{\tilde{D}_4}$ at $(s,0,\ell)$ is equivalent to the transversality of the map

$$\mathcal{P}_d^*\times\tilde{U}_0\longrightarrow\mathbb{C}^7,$$
$$(f,x,y,\eta)\longrightarrow\big(f,f_x,f_y,f_{xx}+\eta f_{xy},f_{xy}+\eta f_{yy},f_{xxx}+3\eta f_{xxy}+3\eta^2 f_{xyy}+\eta^3 f_{yyy},f_{yy}\big)_{(x,y)}.$$

The restriction of the differential of this map at $(s,0,0)$ to the first component of the tangent bundle is given by

$$\mathcal{P}_d\longrightarrow\mathbb{C}^7,\qquad f\longrightarrow\big(f_{00},f_{10},f_{01},f_{20},f_{11},f_{30},f_{02}\big).$$

This map is surjective if $d\geq 3$. □

2.3 General position arguments

We first start with the following important lemma

Lemma 2.11. *Let $\mathcal{A}\subset\mathbb{P}^N$ be a smooth variety (not necessarily closed). Then $\overline{\mathcal{A}}$ and $\overline{\mathcal{A}}-\mathcal{A}$ are both algebraic varieties and*

$$\dim(\overline{\mathcal{A}}-\mathcal{A})<\dim(\mathcal{A}),$$

where the closure is taken inside $\mathbb{P}^N$.

Lemma 2.12. *Let $\mathcal{A}\subset\mathcal{D}_d\times(\mathbb{P}^2)^\delta$ be a smooth variety of dimension k, not necessarily closed. Then there exists a Zariski open $U\subset(\mathbb{P}^2)^{k+1}$ such that*

$$\overline{\mathcal{A}}\cap H_{p_1}\dots H_{p_{k+1}}=\emptyset,\qquad\forall\ \ (p_1,\dots,p_{k+1})\in U,$$

where

$$H_p:=\{([s],p_1,\dots,p_\delta)\in\mathcal{D}_d\times(\mathbb{P}^2)^\delta : s(p)=0\}.$$

Proof. We use induction on k. By induction assumption there exists an open set U' such that for all $(p_1,\dots,p_k)\in U'$

$$\partial\overline{\mathcal{A}}\cap H_{p_1}\dots H_{p_k}=\emptyset.$$

This is because the boundary $\partial\overline{\mathcal{A}}$ is a variety of dimension less than or equal to $k-1$ and is stratified into smooth varieties of dimension $k-1$ or less. We can apply the result to each stratum and the result follows, since a finite intersection of Zariski open sets is again Zariski open. Hence

$$\mathcal{A}\cap H_{p_1}\dots H_{p_k}=\overline{\mathcal{A}}\cap H_{p_1}\dots H_{p_k}.$$

Choose p_{k+1} such that $\mathcal{A}$ is transverse to $H_{p_{k+1}}^*$ and $\overline{\mathcal{A}}$ is not a subset $H_{p_{k+1}}$. Hence

$$\dim(\overline{\mathcal{A}}\cap H_{p_{k+1}})<\dim\overline{\mathcal{A}}.$$

Hence by the induction hypothesis

$$\overline{\mathcal{A}}\cap H_{p_{k+1}}\cap H_{p_1}\cap H_{p_2}\dots H_{p_k}=\emptyset.$$

Lemma 2.13. *Let $\mathcal{A} \subset \mathcal{D}_d \times (\mathbb{P}^2)^\delta$ be a smooth variety of dimension k, not necessarily closed. Then there exists a Zariski open set $U \subset \mathbb{P}^2$ such that for all $p \in U$,*

$$\dim(\mathcal{A} \cap (H_p - H_p^*)) \leq k - 3$$

provided a generic element of $\mathcal{A}$ has only finitely many singular points.

Lemma 2.14. *Let $\mathcal{A} \subset \mathcal{D}_d \times (\mathbb{P}^2)^\delta$ be a smooth variety of dimension k, not necessarily closed. Then there exists a Zariski open $U \subset (\mathbb{P}^2)^{k+1}$ such that*

$$\overline{\mathcal{A}} \cap H_{p_1} \dots H_{p_k} = \mathcal{A} \cap H_{p_1}^* \dots H_{p_k}^*, \qquad \forall \ \ (p_1, \dots, p_{k+1}) \in U,$$

and every intersection is transverse where

$$H_p^* := \{([s], p_1, \dots, p_\delta) \in \mathcal{D}_d \times (\mathbb{P}^2)^\delta : s(p) = 0, \nabla s|_p \neq 0\},$$

provided a generic element of every stratum of $\overline{\mathcal{A}}$ has finitely many singular points.

Proof. This follows from lemma 2.11, 2.12 and 2.13 by applying it to every stratum of $\overline{\mathcal{A}}$.

2.4 Transversality for multiple points

Lemma 2.15. *The section*

$$\psi_{A_1}^\delta \in \Gamma(\mathcal{D}_d \times (\mathbb{P}^2)^\delta, \bigoplus_{i=1}^{\delta} \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \oplus \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \otimes T^*\mathbb{P}^2)$$
$$\psi_{A_1}^\delta([s], p_1, \dots, p_\delta) = s(p_1), \nabla s|_{p_1}, \dots, s(p_\delta), \nabla s|_{p_\delta}$$

is transverse to the zero set for all $[s] \in \mathcal{D}_d$, provided $d \geq 2\delta + 1$.

Proof. We will show transversality at p_1, $p_2, \dots, p_\delta$. Let us assume $p_i = (x_i, y_i)$ and consider 3δ vectors in the space of polynomials given by

$$f_{00}^i = 1 + 0(x - x_i) + 0(y - y_i) + \dots$$
$$f_{10}^i = 0 + 1(x - x_i) + 0(y - y_i) + \dots$$
$$f_{01}^i = 0 + 0(x - x_i) + 1(y - y_i) + \dots$$

for $i = 1$ to δ. We now define the following vectors

$$g_{\alpha\beta}^i = (f_{10}^{i-1})^2 (f_{10}^{i-2})^2 \cdot \dots \cdot (f_{10}^1)^2 f_{\alpha\beta}^i$$

where α and β are 0 or 1. We choose $p_2, p_3, \dots, p_\delta$ so that $f_{10}^j(p_m)$ is not zero for any j or m. We now evaluate the polynomials $g_{\alpha\beta}^i$ at p_i and get 3δ vectors (evaluating a polynomial at a point gives us a vector by looking at the coefficients). Using the fact hat $f_{10}^j(p_m)$ is not zero, we get that these 3δ vectors are linearly independent.

Lemma 2.16. *The section*

$$\psi_{A_1^\delta D_4} \in \Gamma\big(\psi_{A_1}^{-1}(0) \times (\mathbb{P}^2)^\delta, \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \otimes \mathrm{Sym}^2(T^*\mathbb{P}^2 \otimes T^*\mathbb{P}^2)\big)$$
$$\psi_{A_1^\delta D_4}([s], p_1, \dots, p_\delta, p_{\delta+1}) = s(p_1), \nabla s|_{p_1}, \dots, s(p_\delta), \nabla s|_{p_\delta}, \nabla^2 s|_{p_{\delta+1}}$$

is transverse to the zero set, provided $d \geq 2\delta + 4$.

Proof. The proof is similar to the previous Lemma. We consider the vectors

$$g^i_{\alpha\beta} = (f^{i-1}_{10})^3 (f^{i-2}_{10})^3 \ldots . (f^1_{10})^3 f^i_{\alpha\beta}$$

instead.

Lemma 2.17. *Let*

$$V_2 = \tilde{\gamma}^* \otimes \pi^*(\pi_1^*\gamma_{\mathcal{D}}^* \otimes \pi_2^*(\gamma_{\mathbb{P}^2}^{*d} \otimes T^*\mathbb{P}^2)) \longrightarrow \mathcal{D}_d \times \mathbb{P}(T\mathbb{P}^2)\,,$$
$$V_3 = \tilde{\gamma}^{*2} \otimes \pi^*(\pi_1^*\gamma_{\mathcal{D}}^* \otimes \pi_2^*(\gamma_{\mathbb{P}^2}^{*d} \otimes T^*\mathbb{P}^2)) \longrightarrow \mathcal{D}_d \times \mathbb{P}(T\mathbb{P}^2),$$

where $\pi\colon \mathcal{D}_d \times \mathbb{P}(T\mathbb{P}^2) \longrightarrow \mathcal{D}_d \times \mathbb{P}^2$ *is the projection map. The section*

$$\psi_{A_1^\delta A_2} \in \Gamma\big((\mathbb{P}(T\mathbb{P}^2))|_{\psi_{A_1}^{-1}(0)} \times (\mathbb{P}^2)^\delta,\ V_2\big)$$
$$\big\{\psi_{A_1^\delta A_2}([s], p_1, \ldots, p_\delta, p_{\delta+1}, \ell)\big\}(v) = s(p_1), \nabla s|_{p_1}, \ldots, s(p_\delta), \nabla s|_{p_\delta}, \nabla^2 s|_{p_{\delta+1}}(v, \cdot)$$

and

$$\psi_{A_1^\delta D_5} \in \Gamma\big(\mathbb{P}(T\mathbb{P}^2)|_{\psi_{A_1}^{-1}(0)} \times (\mathbb{P}^2)^\delta,\ V_3\big)$$
$$\big\{\psi_{A_1^\delta D_5}([s], p_1, \ldots, p_\delta, p_{\delta+1}, \ell)\big\}(v, v) = s(p_1), \nabla s|_{p_1}, \ldots, s(p_\delta), \nabla s|_{p_\delta}, \nabla^2 s|_{p_{\delta+1}}(v, v, \cdot)$$

are transverse to the zero set, provided $d \geq 2\delta + 4, 2\delta + 5$*, respectively.*

Proof. The proof is similar to the previous lemma. We consider the vectors

$$g^i_{\alpha\beta} = (f^{i-1}_{10})^3 (f^{i-2}_{10})^3 \ldots . (f^1_{10})^3 f^i_{\alpha\beta}$$

instead for the first case and the vectors

$$g^i_{\alpha\beta} = (f^{i-1}_{10})^4 (f^{i-2}_{10})^4 \ldots . (f^1_{10})^4 f^i_{\alpha\beta}$$

in the second case.

Lemma 2.18. *Let*

$$L_k = \tilde{\gamma}^{*^k} \otimes \big(\pi^*\pi_2^* T\mathbb{P}^2/\tilde{\gamma}\big)^{*(k-3)} \otimes \pi^*\big(\pi_1^*\gamma_{\mathcal{D}}^* \otimes \pi_2^*\gamma_{\mathbb{P}^2}^{*^d}\big)^{k-2} \longrightarrow \mathcal{D}_d \times \mathbb{P}(T\mathbb{P}^2)\,,$$
$$\mathbb{L}_k = \tilde{\gamma}^{*^{2(k-4)}} \otimes \big(\pi^*\pi_2^* T\mathbb{P}^2/\tilde{\gamma}\big)^{*2(k-6)} \otimes \pi^*\big(\pi_1^*\gamma_{\mathcal{D}}^* \otimes \pi_2^*\gamma_{\mathbb{P}^2}^{*^d}\big)^{k-5} \longrightarrow \mathcal{D}_d \times \mathbb{P}(T\mathbb{P}^2)\,.$$

(1) For every $k \geq 3$*, the section*

$$\psi_{A_1^\delta A_k} \in \Gamma\big((\psi_{A_{k-1}}^{-1}(0) - (\mathcal{D}_d \times \mathbb{P}^2)|_{\psi_{D_4}^{-1}(0)}) \times (\mathbb{P}^2)^\delta, L_k\big)$$

is transverse to the zero set, provided $d \geq 2\delta + 2 + k$
(2) For every $k \geq 6$*, the section*

$$\psi_{A_1^\delta D_k} \in \Gamma\big((\psi_{D_{k-1}}^{-1}(0) - \psi_{E_6}^{-1}(0)) \times (\mathbb{P}^2)^\delta, \mathbb{L}_k\big)$$

is transverse to the zero set, provided $d \geq 2\delta + k$*.*

Proof. We consider the vectors

$$g^i_{\alpha\beta} = (f^{i-1}_{10})^{k+1} (f^{i-2}_{10})^{k+1} \ldots . (f^1_{10})^{k+1} f^i_{\alpha\beta}$$

for the first case and the vectors

$$g^i_{\alpha\beta} = (f^{i-1}_{10})^{k-2} (f^{i-2}_{10})^{k-2} \ldots . (f^1_{10})^{k-2} f^i_{\alpha\beta}$$

in the second case.

Chapter 3

Closure of spaces

Recall that we have defined the following sections

$$\begin{aligned}
A_3^f &= f_{30}\\
A_4^f &= f_{40} - \frac{3f_{21}^2}{f_{02}}\\
A_5^f &= \frac{f_{50}}{24} - \frac{5f_{21}f_{31}}{12f_{02}} + \frac{5f_{12}f_{21}^2}{8f_{02}^2}\\
A_6^f &= \frac{f_{60}}{120} - \frac{f_{21}f_{41}}{8f_{02}} + \frac{f_{31}^2}{12f_{02}} + \frac{f_{12}f_{21}f_{31}}{2f_{02}^2} + \frac{3f_{21}^2f_{22}}{8f_{02}^2} - \frac{f_{03}f_{21}^3}{8f_{02}^3} - \frac{3f_{12}^2f_{21}^2}{4f_{02}^3}\\
A_7^f &= \frac{f_{70}}{720} - \frac{7f_{21}f_{51}}{240f_{02}} - \frac{7f_{31}f_{41}}{144f_{02}} + \frac{7f_{12}f_{21}f_{41}}{48f_{02}^2} + \frac{7f_{21}^2f_{32}}{48f_{02}^2} + \frac{7f_{12}f_{31}^2}{72f_{02}^2} + \frac{7f_{21}f_{22}f_{31}}{24f_{02}^2}\\
&\quad - \frac{7f_{03}f_{21}^2f_{31}}{48f_{02}^3} - \frac{7f_{12}^2f_{21}f_{31}}{12f_{02}^3} - \frac{7f_{12}f_{21}^2f_{22}}{8f_{02}^3} - \frac{7f_{13}f_{21}^3}{48f_{02}^3}\\
&\quad + \frac{7f_{03}f_{12}f_{21}^3}{16f_{02}^4} + \frac{7f_{12}^3f_{21}^2}{8f_{02}^4}.
\end{aligned}$$

Note that A_k^f is not defined if $f_{02} = 0$. We now define the following quantities

$$\begin{aligned}
\alpha_{A_3} &= f_{30}\\
\alpha_{A_4} &= f_{02}f_{40} - 3f_{21}^2\\
\alpha_{A_5} &= \frac{f_{02}f_{50}}{24} - \frac{5f_{21}f_{31}}{12} + \frac{5f_{12}f_{40}}{24}\\
\alpha_{A_6} &= f_{02}\big(-\frac{f_{31}^2}{24} + \frac{f_{60}f_{02}}{240} + \frac{f_{50}f_{12}}{40}\big) - \frac{f_{21}}{2}\big(\frac{f_{40}f_{03}}{144} - \frac{f_{21}f_{22}}{16}\big)\\
\alpha_{A_7} &= \frac{f_{02}^4f_{70}}{720} - \frac{7f_{02}^3f_{21}f_{51}}{240} - \frac{7f_{02}^3f_{31}f_{41}}{144} + \frac{7f_{02}^2f_{12}f_{21}f_{41}}{48} + \frac{7f_{02}^2f_{21}^2f_{32}}{48} + \frac{7f_{02}^2f_{21}f_{22}f_{31}}{24}\\
&\quad - \frac{7f_{02}f_{03}f_{21}^2f_{31}}{48} - \frac{7f_{02}f_{12}f_{21}^2f_{22}}{8} - \frac{7f_{02}f_{13}f_{21}^3}{48} + \frac{7f_{03}f_{12}f_{21}^3}{16}\\
&\quad - \frac{7f_{02}^2f_{12}}{3}\big(-\frac{f_{31}^2}{24} + \frac{f_{50}f_{12}}{40}\big).
\end{aligned}$$

Note that if $f_{02} \neq 0$ then

$$\alpha_{A_i} = 0 \qquad \forall i \leq k \qquad \text{iff} \qquad A_i^f = 0 \qquad \forall i \leq k.$$

Unlike A_k^f, α_{A_k} is defined even when $f_{02} = 0$.

3.1 Closure

3.2 One point singularity

Let us define

$$\mathcal{S}^d(k,\chi_m) := \mathcal{S}^d_{\kappa(d)-(m+k)}(k,\chi_m), \qquad \mathbb{P}\mathcal{S}^d(k,\chi_m) := \mathbb{P}\mathcal{S}^d_{\kappa(d)-(m+k)}(k,\chi_m).$$

Lemma 3.1. *The element* $([f],p) \in \partial\overline{\mathcal{S}^d(0,A_1)}$ *if and only if* $([f],p) \in \overline{\mathcal{S}^d(0,A_2)}$

Proof. We write the section s in local coordinates and fix the marked point to be $(0,0)$. The Taylor expansion of a function f vanishing at the origin is

$$f = f_{10}x + f_{01}y + \frac{f_{20}}{2}x^2 + f_{11}xy + \frac{f_{02}}{2}y^2 + \dots$$

We can think of $f \in \mathbb{C}^{M_d}$. We claim that

$$\overline{\{f \in \mathbb{C}^{M_d} : f_{10} = 0, f_{01} = 0, f_{11}^2 - f_{20}f_{02} \neq 0\}} = \{f \in \mathbb{C}^{M_d} : f_{10} = 0, f_{01} = 0\}$$

To prove this statement, it suffices to show that if there is a function f such that

$$f_{11}(0)^2 - f_{20}(0)f_{02}(0) = 0,$$

then there exists a sequence (or curve) $f_{ij}(t)$ such that

$$\begin{aligned} f_{10}(t) &= 0 \\ f_{01}(t) &= 0 \\ f_{11}(t)^2 - f_{20}(t)f_{02}(t) &\neq 0 \qquad \forall\ \ t \neq 0 \end{aligned}$$

There are three possible cases. For the first case, let us assume that

$$f_{20}(0) \neq 0$$

Then we take the sequence

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \ \ (i,j) \neq (2,0) \\ f_{02}(t) &= \frac{f_{11}(t)^2}{f_{20}(t)} + t \end{aligned}$$

We can construct a similar curve if we assume

$$f_{02}(0) \neq 0$$

The remaining case is if $f_{02}(0) = f_{20}(0) = f_{11}(0) = 0$. Then the curve

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \ \ (i,j) \neq (2,0),(1,1),(0,2) \\ f_{20} &= t \\ f_{02} &= t \\ f_{11} &= 2t \end{aligned}$$

This proves the claim.

Lemma 3.2. *The element* $([f], p) \in \partial\overline{\mathcal{S}_r^d(0, A_1)}$ *if and only if* $([f], p) \in \overline{\mathcal{S}_r^d(0, A_2)}$

Proof. This is less obvious, i.e. that after the curve f passes through a certain number of points the statement will still be true. More precisely consider a subspace

$$\mathbb{C}^{M_d - r} \subset \mathbb{C}^{M_d}$$

that arises after passing through r generic points. We wish to claim that

$$\overline{\{f \in \mathbb{C}^{M_d-r} : f_{10} = 0, f_{01} = 0, f_{11}^2 - f_{20}f_{02} \neq 0\}} = \{f \in \mathbb{C}^{M_d-r} : f_{10} = 0, f_{01} = 0\}$$

The place where our previous proof will break down is that the sequence we construct $f_{ij}(t)$ may not lie in $\mathbb{C}^{M_d-r}$, even though $f_{ij}(0)$ does. To fix this, we claim that we can *perturb* the original sequence $f_{ij}(t)$ to a new sequence $\tilde{f}_{ij}(t)$ such that it does lie in in $\mathbb{C}^{M_d-r}$. More precisely let the r points be $(x_1, y_1), \ldots (x_r, y_r)$. This gives us r linear equations in the coefficients f_{ij}. More explicitly let the equations be

$$\begin{aligned} L_1(f(t)) &:= \epsilon_1(t) \\ L_2(f(t)) &:= \epsilon_2(t) \ldots \\ L_r(f(t)) &:= \epsilon_r(t) \end{aligned}$$

Note that $\epsilon(0) = 0$. If $\epsilon(t) = 0$ then we would be done. Hence we now modify the coefficients using each of the equations one by one so that

$$L(\tilde{f}(t)) = 0$$

and $\tilde{f}(0) = f(0)$. To see why this is so, we explicitly show the procedure. Let us assume that the first equation L_1 is given by

$$A_{20}^1 \frac{f_{20}(t)}{2} + A_{11}^1 f_{11}(t) + A_{02}^1 \frac{f_{02}(t)}{2} + \ldots + A_{mn}^1 \frac{f_{mn}(t)}{m!n!} + \ldots = \epsilon_1(t)$$

Here A_{mn}^1 are the coefficients we get when we plug in the first point (x_1, y_1). More precisely

$$A_{mn}^r = x_r^m y_r^n.$$

To avoid confusion with the notation, the author emphasizes that

$$x_r^m$$

is the number x_r raised to the power m. Let us choose any of the terms we like, say f_{mn}, such that $A_{mn}^1 \neq 0$ and define a new quantity

$$\frac{\tilde{f}_{mn}(t)}{m!n!} := \frac{f_{mn}(t)}{m!n!} - \frac{\epsilon_1(t)}{A_{mn}^1}$$

This of course only makes sense when $A_{mn}^1 \neq 0$. Since

$$A_{mn}^r = x_r^m y_r^n$$

we can do this provided $x_r \neq 0$ and $y_r \neq 0$. This has full measure in the space of all possible points (more precisely the complement is a variety of strictly smaller dimension). To fit the next equation we can modify another term $\tilde{f}_{m'n'}$. Hence if there are r points through which the curve should pass, then we modify r of the coefficients and get a new function $\tilde{f}(t)$ that agrees with $f(t)$ when $t = 0$ and also passes through those points when $t \neq 0$. In particular if this is a k parameter family of curves, then we can choose any k of the $f_{ij}(t)$ we like so that in the end

$$\tilde{f}_{ij}(t) = f_{ij}(t)$$

This will be important when we compute multiplicities. We will not want to change the important $f_{ij}(t)$ that affect the multiplicity. In this example for instance, if it was a one parameter family of curves with a node degenerating to a cusp, then we could ensure that

$$\tilde{f}_{02}(t) = f_{02}(t)$$

Hence, if there was a multiplicity computation, then this new curve would not effect it, unless it was a triple point, which is to be expected.

Lemma 3.3. *The element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0, A_2)}$ *if and only if* $([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, A_3)}$

Proof. Consider a path given by

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (3,0) \\ f_{30}(t) &= t \end{aligned}$$

Lemma 3.4. *The element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0, A_k)}$ *and* $f_{02} \neq 0$ *if and only if* $([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, A_{k+1})}$*, provided* $k \geq 3$.

Proof. Just consider the path given by

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (k+1,0) \\ A^f_{k+1}(t) &= t \end{aligned}$$

Lemma 3.5. *The element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0, A_3)}$ *and* $f_{02} = 0$ *if and only if* $([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_4)}$.

Proof. Consider a path given by

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (0,2) \\ f_{20}(t) &= t \end{aligned}$$

Lemma 3.6. *The element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0, A_4)}$ *and* $f_{02} = 0$ *if and only if* $([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_5)}$.

Proof. Let us consider two cases. First assume that $f_{40}(0) \neq 0$. Then the path given by Consider a path given by

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (0,2),(2,1) \\ f_{21} &= t \\ f_{20}(t) &= \frac{3t^2}{f_{40}(0)} \end{aligned}$$

If $f_{40}(0) = 0$ then consider the path given by

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (0,2), (2,1), (4,0) \\ f_{40}(t) &= t \\ f_{21} &= t \\ f_{20}(t) &= 3t \end{aligned}$$

Lemma 3.7. *The element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0, A_5)}$ *and* $f_{02} = 0$ *if and only if*

$$([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_6)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, E_6)}.$$

Proof. It is easy to see that $\overline{\mathbb{P}\mathcal{S}^d(0, A_5)}$ is given by

$$\begin{aligned} \alpha_{A_3} &= f_{30} = 0 \\ \alpha_{A_4} &= f_{02}f_{40} - 3f_{21}^2 = 0 \\ \alpha_{A_5} &= \frac{f_{02}f_{50}}{24} - \frac{5f_{21}f_{31}}{12} + \frac{5f_{12}f_{40}}{24} = 0 \\ f_{02} &= 0 \end{aligned}$$

It is easy to see that if $f_{02} = 0$ then $f_{21} = 0$ and since $\alpha_{A_6} = 0$ we get that either $f_{40} = 0$ or $f_{12} = 0$ which corresponds to either a D_6 node or E_6 node (at least). To prove the other direction we can construct a path in a way similar to the previous lemma.

Lemma 3.8. *The element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0, A_6)}$ *and* $f_{02} = 0$ *if and only if*

$$([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_7)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, E_7)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, X_8)},$$

where X_8 *is a quadruple point.*

Proof. If $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0, A_6)}$, then $\exists$ a sequence $f_{ij_n} \in \mathcal{A}_6$ such that $f_{ij_n} \longrightarrow f_{ij}$.

Case 1: Let us assume $f_{02} = 0$ and $f_{40} = 0$. After passing to a subsequence there are two possibilities

Case 1a): The limit

$$\lim_{n \longrightarrow \infty} \frac{f_{02_n}}{f_{21_n} f_{40_n}} = L$$

exists. Since $\alpha_{A_4} = 0$, that implies that

$$\lim_{n \longrightarrow \infty} \frac{f_{02_n}}{f_{40_n}} = 0, \tag{3.1}$$

$$\lim_{n \longrightarrow \infty} \frac{f_{21_n}}{f_{40_n}} = 0. \tag{3.2}$$

Since $\alpha_{A_5} = 0$, using equations (3.2) we get

$$\begin{aligned} \lim_{n \longrightarrow \infty} f_{12_n} &= \lim_{n \longrightarrow \infty} -\frac{f_{02_n} f_{50_n}}{24 f_{40_n}} + \frac{5 f_{21_n} f_{31_n}}{12 f_{40_n}} \\ &= 0. \end{aligned}$$

This corresponds to being an E_7-node
Case 1b): The limit

$$\lim_{n\longrightarrow\infty}\frac{f_{21_n}f_{40_n}}{f_{02_n}}=0.$$

Since $\alpha_{A_6}=0$, we get

$$\begin{aligned}\lim_{n\longrightarrow\infty}-\frac{f_{31_n}^2}{24}+\frac{f_{50_n}f_{12_n}}{40}&=-\frac{f_{60_n}f_{02_n}}{240}+\frac{f_{21_n}f_{40_n}f_{03_n}}{288f_{02_n}}-\frac{f_{21_n}^2f_{22_n}}{32f_{02_n}}\\&=0\end{aligned}$$

since

$$\lim_{n\longrightarrow\infty}\frac{f_{21_n}^2}{f_{02_n}}=0.$$

which is a consequence of $\alpha_{A_4}=0$. This corresponds to being a D_7-node.

Case 2): Let us assume $f_{02}=0$ and $f_{40}\neq 0$. Since $\alpha_{A_4}=0$, that implies

$$\begin{aligned}\lim_{n\longrightarrow\infty}\frac{f_{02_n}}{f_{21_n}}&=\lim_{n\longrightarrow\infty}\frac{3f_{21_n}}{f_{40_n}}\\&=0.\end{aligned}\tag{3.3}$$

Using equation (3.3) and $\alpha_{A_6}=0$, we get that

$$\begin{aligned}\lim_{n\longrightarrow\infty}f_{03_n}&=\frac{288f_{02_n}A_n}{f_{21_n}f_{40_n}}-\frac{9f_{21_n}^2f_{22_n}}{f_{40_n}}\\&=0\end{aligned}$$

where

$$A_n=-\frac{f_{31_n}^2}{24}+\frac{f_{60_n}f_{02_n}}{240}+\frac{f_{50_n}f_{12_n}}{40}$$

this corresponds to being a quadruple point. To prove the other direction we can construct a path in a way similar to the previous lemmas.

Lemma 3.9. *The element* $([f],p)\in\partial\overline{\mathcal{S}^d(0,D_4)}$ *if and only if* $([f],p)\in\overline{\mathcal{S}^d(0,D_5)}$.

Proof. Consider a path given by

$$\begin{aligned}f_{ij}(t)&=f_{ij}(0)\qquad\text{if}\qquad (i,j)\neq(2,1)\\f_{21}(t)&=t\end{aligned}$$

Lemma 3.10. *The element* $([f],p)\in\partial\overline{\mathbb{P}\mathcal{S}^d(0,D_k)}$ *and* $f_{12}\neq 0$ *if and only if* $([f],p)\in\overline{\mathbb{P}\mathcal{S}^d(0,D_{k+1})}$, *provided* $k\geq 5$.

Proof. Consider a path given by

$$\begin{aligned}f_{ij}(t)&=f_{ij}(0)\qquad\text{if}\qquad (i,j)\neq(k-1,0)\\D_{k+1}^f(t)&=t\end{aligned}$$

Lemma 3.11. *The element* $([f],p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0,D_5)}$ *and* $f_{12}=0$ *if and only if* $([f],p) \in \overline{\mathbb{P}\mathcal{S}^d(0,E_6)}$.

Proof. Consider a path given by

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (1,2) \\ f_{12}(t) &= t \end{aligned}$$

Lemma 3.12. *The element* $([f],p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0,D_6)}$ *and* $f_{12}=0$ *if and only if* $([f],p) \in \overline{\mathbb{P}\mathcal{S}^d(0,E_7)}$.

Proof. Consider a path given by

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (1,2) \\ f_{12}(t) &= t \end{aligned}$$

Lemma 3.13. *The element* $([f],p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0,E_6)}$ *if and only if*

$$([f],p) \in \overline{\mathbb{P}\mathcal{S}^d(0,E_7)} \cup \overline{\mathbb{P}\mathcal{S}^d(0,X_8)}.$$

Proof. Consider a path given by

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (4,0) \\ f_{40}(t) &= t \end{aligned}$$

or consider the path given by

$$\begin{aligned} f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (3,0) \\ f_{30}(t) &= t \end{aligned}$$

3.3 Two point singularities

Lemma 3.14. *The element* $([f],p) \in \partial\overline{\mathcal{S}^d(1,A_1)}$ *if and only if* $([f],p) \in \overline{\mathcal{S}^d(0,A_3)}$.

Proof. The space $\mathcal{S}^d(1,A_1)$ is given by

$$\begin{aligned} f &= \frac{f_{20}}{2}x^2 + f_{11}xy + \frac{f_{02}}{2}y^2 + \frac{f_{30}}{6}x^3 + \frac{f_{21}}{2}x^2y + \frac{f_{12}}{2}xy^2 + \frac{f_{03}}{6}y^3 + G(x,y) = 0 \\ f_x &= f_{20}x + f_{11}y + \frac{f_{30}}{2}x^2 + f_{21}xy + \frac{f_{12}}{2}y^2 + G_x = 0 \\ f_y &= f_{11}x + f_{02}y + \frac{f_{21}}{2}x^2 + f_{12}xy + \frac{f_{03}}{2}y^2 + G_y = 0 \\ (x,y) &\neq (0,0) \end{aligned}$$

If $(f,0,0) \in \partial\overline{\mathcal{S}^d(1,A_1)}$ then there exists a sequence $(f_n,x_n,y_n) \in \mathcal{S}^d(1,A_1)$ that converges to $(f,0,0)$. Let us assume that after passing to a subsequence, the limit

$$\lim_{n\longrightarrow\infty} \frac{x_n}{y_n} = L$$

exists. The equations $f_x, f_y = 0$ imply that in the limit

$$\begin{aligned} f_{20}L + f_{11} &= 0 \\ f_{11}L + f_{02} &= 0 \end{aligned}$$

Finally, the equation

$$f - \frac{xf_x}{2} - \frac{yf_y}{2} = 0$$

implies that in the limit

$$f_{30}L^3 + 3f_{21}L^2 + 3f_{12}L + f_{03} = 0$$

A similar statement holds if

$$\lim_{n\longrightarrow\infty} \frac{y_n}{x_n} = L$$

exists. Hence $(f, 0, 0) \in \overline{\mathcal{S}^d(0, A_3)}$ To show the other direction we need to construct a path $(f(t), x(t), y(t)) \in \mathcal{S}^d(1, A_1)$ that converges to $\overline{\mathcal{S}^d(0, A_3)}$.

$$\begin{aligned}
x(t) &= Lt \\
y(t) &= t \\
f_{ij}(t) &= f_{ij} \qquad \text{if } (i,j) \neq (1,1),\ (0,3) \text{ or } (0,2). \\
\frac{f_{03}(t)}{6} &= -(\frac{f_{12}}{2}L + \frac{f_{21}}{2}L^2 + \frac{f_{30}}{6}L^3) + 2G - xG_x - yG_y \\
f_{11}(t) &= -(\frac{f_{30}}{2}L^2 + f_{21}L + \frac{f_{12}}{2})y - \frac{G_x}{y} - f_{20}L \\
f_{02}(t) &= -(\frac{f_{21}}{2}L^2 + f_{12}L + \frac{f_{03}}{2})y - \frac{G_y}{y} - f_{11}L
\end{aligned}$$

is such a path if

$$\lim_{n\longrightarrow\infty} \frac{x_n}{y_n} = L$$

exists. We can construct a similar path if

$$\lim_{n\longrightarrow\infty} \frac{y_n}{x_n} = L.$$

Lemma 3.15. *The element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(1, A_k)}$ *and* $f_{02} \neq 0$ *if and only if* $([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, A_{k+2})}$, *provided* $k \geq 2$.

Proof. Since $f_{02} \neq 0$, we can find coordinates (u, v) so that the curve is given by

$$f = v^2 + A^f_{k+1}u^{k+1} + A^f_{k+2}u^{k+2} + \ldots$$

The set of equations we are solving for are

$$\begin{aligned}
f &= v^2 + A^f_{k+1}u^{k+1} + A^f_{k+2}u^{k+2} + \ldots = 0 \\
f_u &= (k+1)A^f_{k+1}u^k + (k+2)A^f_{k+2}u^{k+1} + \ldots = 0 \\
f_v &= 2v = 0
\end{aligned}$$

Solving these three equations we get

$$\begin{aligned}
A^f_{k+2} &= \frac{k+1}{k+2}A^f_{k+3}u + O(u^2) \\
A^f_{k+1} &= \frac{2k+4}{k+1}A^f_{k+3}u^2 + O(u^3)
\end{aligned}$$

Hence in the limit A^f_{k+1} and A^f_{k+2} vanish. The above equations also prove the only if part, i.e if a $(f,0,0) \in \overline{\partial \mathbb{P}\mathcal{S}^d(0, A_{k+2})}$, and $f_{02} \neq 0$ then there exists a curve $(f(t), u(t), v(t)) \in \mathcal{S}^d(1, A_k)$ converging to $(f,0,0)$.

Lemma 3.16. *The element* $([f], p) \in \partial\overline{\mathcal{S}^d(1, D_4)}$ *if and only if* $([f], p) \in \overline{\mathcal{S}^d(0, D_6)}$.

Proof. This is completely analogous to the proof of Lemma 3.14. Consider the equations

$$\begin{aligned}
f &= \frac{f_{30}}{6}x^3 + \frac{f_{21}}{2}x^2y + \frac{f_{12}}{2}xy^2 + \frac{f_{03}}{6}y^3 + \frac{f_{40}}{24}x^4 + \frac{f_{31}}{6}x^3y + \frac{f_{22}}{4}x^2y^2 + \frac{f_{13}}{6}xy^3 + \frac{f_{04}}{24}y^4 + G(x,y) = 0\\
f_x &= \frac{f_{30}}{2}x^2 + f_{21}xy + \frac{f_{12}}{2}y^2 + \frac{f_{40}}{6}x^3 + \frac{f_{31}}{2}x^2y + \frac{f_{22}}{2}xy^2 + \frac{f_{13}}{6}y^3 + G_x = 0\\
f_y &= \frac{f_{21}}{2}x^2 + f_{12}xy + \frac{f_{03}}{2}y^2 + \frac{f_{31}}{6}x^3 + \frac{f_{22}}{2}x^2y + \frac{f_{13}}{2}xy^2 + \frac{f_{04}}{6}y^3 + G_y = 0
\end{aligned}$$

$(x,y) \neq (0,0)$

If $(f,0,0) \in \partial\overline{\mathcal{S}^d(1, D_4)}$ then there exists a sequence $(f_n, x_n, y_n) \in \mathcal{S}^d(1, D_4)$ that converges to $(f,0,0)$. Let us assume that after passing to a subsequence, the limit

$$\lim_{n\longrightarrow\infty} \frac{x_n}{y_n} = L$$

exists. The equations $f_x, f_y = 0$ imply that in the limit

$$\begin{aligned}
f_{30}L^2 + 2f_{21}L + f_{12} = 0\\
f_{21}L^2 + 2f_{12}L + f_{03} = 0
\end{aligned}$$

Finally, the equation

$$f - \frac{xf_x}{3} - \frac{yf_y}{3} = 0$$

implies that in the limit

$$f_{40}L^4 + 4f_{31}L^3 + 6f_{22}L^2 + 4f_{13}L + f_{04} = 0$$

A similar statement holds if

$$\lim_{n\longrightarrow\infty} \frac{y_n}{x_n} = L$$

exists. Hence $(f,0,0) \in \overline{\mathcal{S}^d(0, D_6)}$ To show the other direction we need to construct a path $(f(t), x(t), y(t)) \in \mathcal{S}^d(1, D_4)$ that converges to $\overline{\mathcal{S}^d(0, D_6)}$, which we can do in an analogous way as in Lemma 3.14.

Lemma 3.17. *The element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(1, D_k)}$ *and* $f_{12} \neq 0$ *if and only if* $([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_{k+2})}$, *provided* $k \geq 5$.

Proof. This is analogous to the proof of Lemma 3.15. Since $f_{12} \neq 0$, we can find coordinates (u, v) so that the curve is given by

$$f = v^2u + D^f_{k+1}u^{k-1} + D^f_{k+2}u^k + \ldots$$

The set of equations we are solving for are

$$\begin{aligned} f &= v^2u + D^f_{k+1}u^{k-1} + D^f_{k+2}u^k + \ldots = 0 \\ f_u &= v^2 + (k-1)D^f_{k+1}u^{k-2} + (k)D^f_{k+2}u^{k-1} + \ldots = 0 \\ f_v &= 2vu = 0 \end{aligned}$$

The last equation implies that either v or u is zero. But if u is zero then $f_u = 0$ implies that v is also zero. Hence v is zero. Solving these three equations we get

$$\begin{aligned} D^f_{k+2} &= O(u) \\ D^f_{k+1} &= O(u^2) \end{aligned}$$

Hence in the limit D^f_{k+1} and D^f_{k+2} vanish. The above equations also prove the only if part, i.e if a $(f,0,0) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0, D_{k+2})}$, and $f_{12} \neq 0$ then there exists a curve $(f(t), u(t), v(t)) \in \mathcal{S}^d(1, D_k)$ converging to $(f,0,0)$.

Lemma 3.18. *If the element* $([f],p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(1, A_2)}$ *and* $f_{30}, f_{02} = 0$*, then at least one of the following holds*

$$4f_{21}f_{03} - 3f_{12}^2 = 0, \qquad \text{or} \qquad f_{21} = 0,$$

i.e $([f],p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(0, D_5)}$

Proof. Consider the set of equations we are solving

$$\begin{aligned} f &= \frac{f_{02}}{2}y^2 + \frac{f_{30}}{6}x^3 + \frac{f_{21}}{2}x^2y + \frac{f_{12}}{2}xy^2 + \frac{f_{03}}{6}y^3 + \frac{f_{40}}{24}x^4 + \ldots = 0 \\ f_x &= \frac{f_{30}}{2}x^2 + f_{21}xy + \frac{f_{12}}{2}y^2 + \frac{f_{40}}{6}x^3 + \ldots = 0 \\ f_y &= f_{02}y + \frac{f_{21}}{2}x^2 + f_{12}xy + \frac{f_{03}}{2}y^2 + \ldots = 0 \end{aligned}$$

Since $([f],0,0)$ is in the closure, there exists a sequence $(f_n, x_n, y_n) \in \mathcal{S}^d(1, A_2)$ that converges to $([f],0,0)$. Furthermore we are assuming that in the limit, f_{02} and f_{30} are 0. We now consider three cases.

Case 1: Let us assume that after passing to a subsequence

$$\lim_{n\longrightarrow\infty} \frac{x_n}{y_n} = L$$

exists. Then the equation $f_x = 0$ will imply that in the limit

$$f_{30}L^2 + 2f_{21}L + f_{12} = 0 \tag{3.4}$$

The equation

$$f - \frac{yf_y}{2} - \frac{xf_x}{3} = 0$$

will imply that in the limit

$$f_{21}L^2 + 2f_{12}L + f_{03} = 0$$

Let us assume $f_{21} \neq 0$. Combining the fact that $f_{30} = 0$, $f_{21} \neq 0$ and eliminating L we get that

$$4f_{21}f_{03} - 3f_{12}^2 = 0$$

If $f_{21} = 0$ then $f_{12} = 0$ by equation (3.4), which still satisfies the equation.

Case 2 a): After passing to a subsequence

$$\lim_{n \longrightarrow \infty} \frac{y_n}{x_n} = 0$$

but

$$\lim_{n \longrightarrow \infty} \frac{x_n^2}{y_n} = 0.$$

Recall that we are assuming in the limit f_{02}, f_{30} vanish. The equation

$$f - \frac{yf_y}{2} - \frac{xf_x}{4} = 0$$

will imply that in the limit f_{21} vanishes. Note that here the condition

$$\lim_{n \longrightarrow \infty} \frac{x_n^2}{y_n} = 0$$

is crucial.

Case 2 b): After passing to a subsequence

$$\lim_{n \longrightarrow \infty} \frac{y_n}{x_n} = 0$$

but

$$\lim_{n \longrightarrow \infty} \frac{y_n}{x_n^2} = L$$

exists. The condition $f_y = 0$ implies that in the limit f_{21} vanishes (since f_{02} vanishes).

Remark: Both the conditions

$$4f_{21}f_{03} - 3f_{12}^2 = 0, \qquad \text{or} \qquad f_{21} = 0.$$

refer to at least a D_5 node. To see that, recall that a D_5 node is given by the condition that there exists a non zero vector $v = L_1\partial_x + L_2\partial_y$ such that

$$\nabla^3 f(v, v, \cdot) = 0.$$

This is equivalent to the condition that

$$L_1^2 f_{30} + 2L_1L_2f_{21} + L_2^2 f_{12} = 0$$
$$L_1^2 f_{21} + 2L_1L_2f_{12} + L_2^2 f_{03} = 0$$

Since $f_{30} = 0$, we get that

$$2L_1 L_2 f_{21} + L_2^2 f_{12} = 0.$$

If $L_2 = 0$ then $f_{21} = 0$. That is a particular D_5 node where the preferred direction is ∂_x. If $f_{21} \neq 0$ and $L_2 \neq 0$ then we can eliminate $\frac{L_1}{L_2}$ and get

$$4f_{21}^2 f_{03} - 3f_{12}^2 f_{21} = 0.$$

Since $f_{21} \neq 0$, this implies

$$4f_{21} f_{03} - 3f_{12}^2 = 0.$$

More precisely this corresponds to a D_5 node, where the preferred direction is

$$f_{12}\partial_x - 2f_{21}\partial_y.$$

Lemma 3.19. *If the element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(1, A_3)}$ *and* $f_{02} = 0$ *then* $([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_5)}$. *Furthermore if*

$$4f_{21} f_{03} - 3f_{12}^2 = 0$$

then $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(1, A_3)}$.

Remark: The condition $4f_{21}f_{03} - 3f_{12}^2 = 0$ corresponds to a D_5 node with the preferred direction

$$f_{12}\partial_x - 2f_{21}\partial_y.$$

There is another D_5 node with the preferred direction ∂_x, which corresponds to $f_{21} = 0$. This D_5 node is not necessarily in the closure (unless some other quantity vanishes).

Proof. The proof is similar to the previous Lemma. First we write down the equations

$$\begin{aligned} f &= \frac{f_{02}}{2}y^2 + \frac{f_{21}}{2}x^2 y + \frac{f_{12}}{2}xy^2 + \frac{f_{40}}{24}x^4 + \ldots = 0 \\ f_x &= f_{21}xy + \frac{f_{12}}{2}y^2 + \frac{f_{40}}{6}x^3 + \ldots = 0 \\ f_y &= f_{02}y + \frac{f_{21}}{2}x^2 + f_{12}xy + \frac{f_{03}}{2}y^2 + \ldots = 0 \end{aligned}$$

Since $([f], 0, 0)$ is in the closure, there exists a sequence $(f_n, x_n, y_n) \in \mathcal{S}^d(1, A_3)$ that converges to $([f], 0, 0)$. Furthermore we are assuming that in the limit, f_{02} is 0. We now consider three cases.

Case 1: Let us assume that after passing to a subsequence

$$\lim_{n \longrightarrow \infty} \frac{x_n}{y_n} = L$$

exists. Then the equation $f_x = 0$ will imply that in the limit

$$2f_{21}L + f_{12} = 0$$

The equation

$$f - \frac{yf_y}{2} - \frac{xf_x}{3} = 0$$

will imply that in the limit

$$f_{21}L^2 + 2f_{12}L + f_{03} = 0$$

Eliminating L we get that

$$f_{21}(4f_{21}f_{03} - 3f_{12}^2) = 0$$

Hence we get that, either

$$f_{21}, f_{12}, f_{30} = 0 \qquad \text{or} \qquad 4f_{21}f_{03} - 3f_{12}^2.$$

Both of these are *at least* D_5 nodes (but with different preferred directions).
Case 2 a): After passing to a subsequence

$$\lim_{n \longrightarrow \infty} \frac{y_n}{x_n} = 0$$

but

$$\lim_{n \longrightarrow \infty} \frac{x_n^2}{y_n} = 0.$$

Recall that we are assuming in the limit f_{02} vanishes. The equation

$$f_x = 0$$

will imply that in the limit f_{21} vanishes. The equation

$$f - \frac{yf_y}{2} - \frac{xf_x}{4} = 0$$

implies that in the limit f_{12} vanishes. Note that the coefficient of x^4 gets canceled, which is crucial. This is at least an E_6 node, which lies in the closure of D_5 nodes.

Case 2 b): After passing to a subsequence

$$\lim_{n \longrightarrow \infty} \frac{y_n}{x_n} = 0$$

but

$$\lim_{n \longrightarrow \infty} \frac{y_n}{x_n^2} = L$$

exists. The condition $f_y = 0$ implies that in the limit f_{21} vanishes (since f_{02} vanishes). Next, the equation $f_x = 0$ implies that in the limit f_{40} vanishes, since we have shown that f_{21} vanishes. This is at least a D_6 node, which lies in the closure of D_5 nodes.

We now prove the converse. We need to show that if

$$4f_{21}f_{03} - 3f_{12}^2 = 0$$

then there exists a path $(f(t), x(t), y(t)) \in \mathbb{P}\mathcal{S}^d(1, A_3)$ that converges to this D_5 node. Let

$$\begin{aligned} y(t) &= t \\ x(t) &= Lt \\ f_{ij}(t) &= f_{ij}(0) \qquad \text{if} \qquad (i,j) \neq (1,2), (0,3), (0,2) \\ -\frac{f_{12}(t)}{2} &= Lf_{21} + \dots \qquad \text{using} \;\; f_x = 0 \\ -\frac{f_{03}(t)}{6} &= \qquad \text{using} \;\; f - \frac{yf_y}{2} - \frac{xf_x}{3} = 0 \\ f_{02}(t) &= \qquad \text{using} \;\; f_y = 0 \end{aligned}$$

Note that when we are defining $f_{12}(t)$ there is no f_{03} or f_{02} involved and when we are defining $f_{03}(t)$ there is no f_{02} involved.

Lemma 3.20. *If the element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(1, A_4)}$ *and* $f_{02} = 0$*, then either*

$$f_{21}, f_{12} = 0 \qquad \text{or} \qquad f_{21}, f_{40}, -\frac{f_{31}^2}{24} + \frac{f_{50}f_{12}}{40} = 0,$$

i.e.

$$([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, E_6)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, D_7)}$$

Proof. Most of the proposition follows from the previous Lemma, since an A_4 node is at least an A_3 node. We need to simply consider the case that after passing to a subsequence

$$\lim_{n \longrightarrow \infty} \frac{y_n}{x_n} = 0$$

but

$$\lim_{n \longrightarrow \infty} \frac{y_n}{x_n^2} = L$$

exists. As before, in the limit f_{21} and f_{40} will vanish. We need to show that in addition,

$$-\frac{f_{31}^2}{24} + \frac{f_{50}f_{12}}{40} = 0.$$

It has been shown by Dmitry Kerner in his paper [7], that the closure of one node and one A_4 node can not be a strict D_6 node which proves the claim.

Lemma 3.21. *The element* $([f], p) \in \partial\overline{\mathcal{S}^d(1, A_5)}$ *and* $f_{02} = 0$ *if and only if*

$$([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, E_7)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, D_8)}.$$

Proof. If $f_{12} = 0$, then by the previous Lemma, it is at least an E_6 node. Since the delta invariant of an E_6 node is 3, it has to be at least an E_7 node. If $f_{12} \neq 0$ then the curve has to have at least a D_7 node using delta invariants. Dmitry Kerner has shown in his paper [7] that the closure of one node and one A_5 node can not be a strict D_7-node, which proves the claim.

Lemma 3.22. *The element* $([f], p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(1, D_5)}$ *and* $f_{12} = 0$*, then*

$$([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, E_7)}.$$

Proof. We have shown in Lemma 3.16 that the curve has at least a D_6 node. Since $f_{12} = 0$, we get at least an E_7 node.

3.4 Three point singularities

Lemma 3.23. *The element* $([f], p, p, p) \in \partial\overline{\mathcal{S}^d(2, A_1)}$ *if and only if*

$$([f], p) \in \overline{\mathcal{S}^d(0, D_4)} \cup \overline{\mathcal{S}^d(0, A_5)}.$$

Proof. We have shown that it has to be at least a A_3-node. The delta invariant of an A_3 is 2. Hence it has to be more singular. If the Hessian is not zero, then it has to be at least a A_4 node. But the delta invariant of that is again 2. Hence it has to be at least an A_5 node. If the Hessian is zero, then it has to be at least a D_4 node.

We can also show the only if direction by constructing a path.

Lemma 3.24. *If the element* $([f], p, p, p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(2, A_2)}$ *and* $f_{30} = 0$ *then*

$$([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_5)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, A_6)}.$$

Proof. By Lemma (3.18) we know that it has to be at least a D_5-node if the Hessian vanishes and by Lemma (3.23) if the Hessian is not zero it has to be at least an A_5-node. We can show that if the Hessian does not vanish, then it has to be at least an A_6-node using sequences. That completes the proof. We can also prove the converse by constructing a path.

Lemma 3.25. *If the element* $([f], p, p, p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(2, A_3)}$ *then*

$$([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_6)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, A_7)}.$$

Proof. Using Lemma (3.23), this is easy. We know that it has to be at least an A_5 or D_4 node. The total delta invariant has to be 4. Hence if the Hessian is not zero, then it has to be at least an A_7 node. If the Hessian is zero, then it has to be at least a D_6 node. It can not be a strict E_6, since the delta invariant of that is 3.

Lemma 3.26. *If the element* $([f], p, p, p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(2, A_4)}$ *then*

$$([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_7)} \cup \partial\overline{\mathbb{P}\mathcal{S}^d(0, E_7)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, A_8)}.$$

Proof. If the Hessian is zero, then by Lemma 3.20 we know that it is at least an E_6 node or a D_7 node. Since the delta invariant of an E_6 node is 3, it has to be at least an E_7 node. I haven't been able to show it can not be a strict E_7 node. If the Hessian is not zero then we can show that it has to at least an A_8 node using sequences.

3.5 Four and more point singularities

Lemma 3.27. *The element* $([f], p, p, p, p) \in \partial\overline{\mathcal{S}^d(3, A_1)}$ *if and only if*

$$([f], p) \in \overline{\mathcal{S}^d(0, D_6)} \cup \overline{\mathcal{S}^d(0, A_7)}.$$

Proof. Follows easily using delta invariants. If the Hessian is not zero, it has to be at least A_7. If the Hessian is zero, it has to be at least a D_6. We can also show the other direction, by constructing a path.

Lemma 3.28. *If* $([f], p, p, p, p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(3, A_2)}$ *then*

$$([f], p) \in \overline{\mathbb{P}\mathcal{S}^d(0, D_7)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, E_7)} \cup \overline{\mathbb{P}\mathcal{S}^d(0, A_7)}.$$

Proof. Since four nodes can sink to a strict D_6-node, three nodes and one cusp can not sink to a strict D_6 node.

Lemma 3.29. *If* $([f], p, p, p, p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(3, A_3)}$ *then*

$$([f], p) \in \partial\overline{\mathcal{S}^d(0, D_7)} \cup \overline{\mathcal{S}^d(0, A_9)}.$$

Proof. This one is trivial using delta invariants.

The remaining results are trivially true. In particular all the elements of the closure are singularities of codimension 8 or more. Hence they will not occur in the enumeration of curves with up to 7 nodes.

Lemma 3.30. *If* $([f], p, p, p, p, p) \in \partial\overline{\mathcal{S}^d(4, A_1)}$ *then*

$$([f], p) \in \partial\overline{\mathcal{S}^d(0, D_7)} \cup \partial\overline{\mathcal{S}^d(0, E_7)} \cup \overline{\mathcal{S}^d(0, A_9)}.$$

Proof. Follows from lemma 3.29.

Lemma 3.31. *If* $([f], p, p, p, p, p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(4, A_2)}$ *then*

$$([f], p) \in \partial\overline{\mathcal{S}^d(0, D_7)} \cup \partial\overline{\mathcal{S}^d(0, E_7)} \cup \overline{\mathcal{S}^d(0, A_9)}.$$

Proof. Follows from lemma 3.29.

Lemma 3.32. *If* $([f], p, p, p, p, p, p) \in \partial\overline{\mathbb{P}\mathcal{S}^d(5, A_1)}$ *then*

$$([f], p) \in \partial\overline{\mathcal{S}^d(0, D_7)} \cup \partial\overline{\mathcal{S}^d(0, E_7)} \cup \overline{\mathcal{S}^d(0, A_9)}.$$

Proof. Follows from lemma 3.29.

Chapter 4

Enumeration of curves with one singular point

For each $d\in\mathbb{Z}^+$, let

$$\mathcal{D}_d \approx \mathbb{P}^{\kappa(d)}\,, \qquad \text{where} \qquad \kappa(d) := \binom{d+2}{2} - 1\,,$$

denote space of degree d curves in $\mathbb{P}^2$. For any non-negative integer r, denote by

$$\mathcal{D}_d(r)\approx\mathbb{P}^r \subset \mathcal{D}_d\approx\mathbb{P}^{\kappa(d)}$$

the subspace of curves passing through $\mathcal{D}_d - r$ general points. We write elements of $\mathcal{D}_d$ as $[s]$, with s denoting a non-zero degree d homogeneous polynomial on $\mathbb{C}^3$ or equivalently a non-zero element of $H^0(\mathbb{P}^2;\mathcal{O}(d))$, i.e. a non-zero holomorphic section of the holomorphic line bundle

$$\gamma_{\mathbb{P}^2}^{*d} \equiv (\gamma_{\mathbb{P}^2}^*)^{\otimes d} \longrightarrow \mathbb{P}^2,$$

where $\gamma_{\mathbb{P}^2} \longrightarrow \mathbb{P}^2$ is the tautological line bundle. Denote by $\pi : \mathbb{P}T\mathbb{P}^2 \longrightarrow \mathbb{P}^2$ the bundle projection map. Let

$$\gamma_{\mathcal{D}} \longrightarrow \mathcal{D}_d \qquad \text{and} \qquad \tilde{\gamma} \longrightarrow \mathbb{P}(T\mathbb{P}^2)$$

be the tautological line bundles over $\mathcal{D}_d$ and $\mathbb{P}(T\mathbb{P}^2)$, respectively. We define

$$\lambda_{\mathcal{D}} = c_1(\gamma_{\mathcal{D}}^*), \qquad \lambda_{\mathbb{P}^2} = c_1(\gamma_{\mathbb{P}^2}^*), \qquad \text{and} \qquad \lambda = c_1(\tilde{\gamma}^*).$$

4.1 Curves with one A_1-node

Lemma 4.1. *The number of degree d curves passing through $\kappa(d) - 1$ generic points and having one simple node (A_1-node) is*

$$\mathcal{N}^d(1) = 3(d-1)^2.$$

Proof. Recall that we have defined the space $\mathcal{S}_r^d(0, A_1) \subset \mathcal{D}_d(r+1) \times \mathbb{P}^2$, given by

$$\mathcal{S}_r^d(0, A_1) := \big\{([s], p) \in \mathcal{D}_d(r+1) \times \mathbb{P}^2 : \psi_{A_1} = 0,\ \psi_{A_2} \neq 0\big\}$$

where

$$\psi_{A_1} = s(p),\ \nabla s$$
$$\psi_{A_2} = \det(\nabla^2 s)$$

It is the space of degree d curves $[s] \in \mathcal{D}_d(r+1)$ passing through $\kappa(d)-(r+1)$ generic points and a marked point $p \in \mathbb{P}^2$, such that the curve has a *strict* node at the point p. The expected dimension of this space is r. By lemma 3.1 and 3.2

$$\overline{\mathcal{S}_r^d(0, A_1)} := \big\{([s], p) \in \mathcal{D}_d(r+1) \times \mathbb{P}^2 : \psi_{A_1} = 0\big\}.$$

A similar fact will later on turn out to be false, i.e. the closure of a space will *not* be given by the zero set of a section. The quantity ψ_{A_1} is a section of the rank 3 vector bundle

$$V = \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \oplus \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \otimes T^*\mathbb{P}^2.$$

We need to compute the cardinality of the set $\overline{\mathcal{S}_0^d(0, A_1)}$. By lemma 2.6, the section ψ_{A_1} is transverse to the zero set. Since the points are in general position all the elements of $\overline{\mathcal{S}_0^d(0, A_1)}$ are *strict* nodes which follows from lemma 2.14. Assuming the claims we made about closure and transversality, the desired number $\mathcal{N}^d(1)$ is given by

$$\begin{aligned}
\mathcal{N}^d(1) &= |\overline{\mathcal{S}_0^d(0, A_1)}| \\
&= \langle e(V), [\mathcal{D}_d(1) \times \mathbb{P}^2] \rangle \\
&= 3(d-1)^2
\end{aligned}$$

which can be seen from the splitting principle and Kunneth formula.

Remark: This formula is trivially true for $d = 1$ and $d = 2$. It can also be seen to be true for $d = 3$ (recall that the number of degree three rational curves through 8 points is 12).

Lemma 4.2. *The number of degree d curves passing through $D_d - 2$ points with a simple node (A_1-node) on a fixed line is*

$$\mathcal{N}^d(0, A_1, 1) = 3(d-1).$$

Proof. The proof is similar to that of lemma 4.1. Let $l \in \mathbb{P}^2$ be a generic line in $\mathbb{P}^2$. As before we consider the space $\mathcal{S}_r^d(0, A_1)$. Notice that a line in $\mathbb{P}^2$ is the zero set of a section of $\gamma_{\mathbb{P}^2}^* \longrightarrow \mathbb{P}^2$. Hence the desired number is

$$\mathcal{N}^d(0, A_1, 1) = \langle e(\gamma_{\mathbb{P}^2}^*), [\overline{\mathcal{S}_1^d(0, A_1)}] \rangle$$

Note that although the space $\overline{\mathcal{S}_1^d(0, A_1)}$ will contain curves with singularities worse than a simple node, those singular points will not lie on a generic line l. Also note that since

$$\overline{\mathcal{S}_1^d(0, A_1)} = \psi_{A_1}^{-1}(0)$$

the Poincare dual of the homology class $[\overline{\mathcal{S}_1^d(0, A_1)}]$ in $\mathcal{D}(2) \times \mathbb{P}^2$ is in fact the Euler class of V, where

$$V = \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \oplus \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \otimes T^*\mathbb{P}^2.$$

Hence

$$\begin{aligned}
\mathcal{N}^d(0, A_1, 1) &= \langle e(\gamma_{\mathbb{P}^2}^*),\ [\overline{\mathcal{S}_1^d(0, A_1)}] \rangle \\
&= \langle e(\gamma_{\mathbb{P}^2}^*) e(V),\ [\mathcal{D}_d(2) \times \mathbb{P}^2] \rangle \\
&= 3(d-1)
\end{aligned}$$

Lemma 4.3. *The number of degree d curves passing through $D_d - 3$ points with a simple node (A_1-node) on a fixed point is*

$$\mathcal{N}^d(0, A_1, 2) = 1.$$

Proof. As before we consider the space $\mathcal{S}_r^d(0, A_1)$. Notice that a point in $\mathbb{P}^2$ is the intersection of two generic lines. In other words, it is the zero set of a section of $\gamma_{_{\mathbb{P}^2}}^* \oplus \gamma_{_{\mathbb{P}^2}}^* \longrightarrow \mathbb{P}^2$. Hence the desired number is

$$\mathcal{N}^d(0, A_1, 2) = \langle e(\gamma_{_{\mathbb{P}^2}}^*)^2, [\overline{\mathcal{S}_2^d(0, A_1)}]\rangle$$

Note that although the space $\overline{\mathcal{S}_2^d(0, A_1)}$ will contain curves with singularities worse than a simple node, those singular points will not lie on a generic point. As before, since

$$\overline{\mathcal{S}_2^d(0, A_1)} = \psi_{A_1}^{-1}(0)$$

the Poincare dual of the homology class $[\overline{\mathcal{S}_2^d(0, A_1)}]$ in $\mathcal{D}(3) \times \mathbb{P}^2$ is in fact the Euler class of V, where

$$V = \gamma_{\mathcal{D}}^* \otimes \gamma_{_{\mathbb{P}^2}}^{*d} \oplus \gamma_{\mathcal{D}}^* \otimes \gamma_{_{\mathbb{P}^2}}^{*d} \otimes T^*\mathbb{P}^2.$$

Hence

$$\begin{aligned}\mathcal{N}^d(0, A_1, 2) &= \langle e(\gamma_{_{\mathbb{P}^2}}^*)^2,\ [\overline{\mathcal{S}_2^d(0, A_1)}]\rangle \\ &= \langle e(\gamma_{_{\mathbb{P}^2}}^*)^2 e(V),\ [\mathcal{D}_d(2) \times \mathbb{P}^2]\rangle \\ &= 1\end{aligned}$$

Remark: What we have done here is not simply compute three different numbers. We have actually been able to "compute the homology class" $[\overline{\mathcal{S}_2^d(0, A_1)}]$. The three results stated above can be rephrased as follows

$$\begin{aligned}\langle e(\gamma_{\mathcal{D}}^*)^2,\ [\overline{\mathcal{S}_2^d(0, A_1)}]\rangle &= 3(d-1)^2 && \text{the number of curves with a node} \\ \langle e(\gamma_{\mathcal{D}}^*)e(\gamma_{_{\mathbb{P}^2}}^*),\ [\overline{\mathcal{S}_2^d(0, A_1)}]\rangle &= 3(d-1) && \text{the number of curves with a node on a line} \\ \langle e(\gamma_{_{\mathbb{P}^2}}^*)^2,\ [\overline{\mathcal{S}_2^d(0, A_1)}]\rangle &= 1 && \text{the number of curves with a node on a fixed point}\end{aligned}$$

4.2 Curves with one A_2-node

Lemma 4.4. *The number of degree d curves passing through $\kappa(d) - 2$ points with a cusp (A_2-node) is*

$$\mathcal{N}^d(0, A_2) = 2\mathcal{N}^d(0, A_1) + 2(d-3)\mathcal{N}^d(0, A_1, 1).$$

Proof. We will do this problem in two different ways.

Method 1: A cusp occurs when the determinant of the Hessian is 0. Recall that we have defined

$$\mathcal{S}_{r+1}^d(0, A_1) \subset \mathcal{D}_d(r+2) \times \mathbb{P}^2$$

to be the $r+1$ dimensional space of curves with a node and

$$\mathcal{S}_r^d(0, A_2) := \big\{([s], p) \in \mathcal{S}_{r+1}^d(0, A_1) :\ \psi_{A_2} = 0,\ \psi_{A_3} \neq 0\big\}$$

where

$$\psi_{A_2} = \det(\nabla^2 s)$$
$$\psi_{A_3} = \text{third derivative along kernel of Hessian}$$

It is the space of degree d curves $[s] \in \mathcal{D}_d(r+2)$ passing through $D_d - (r+2)$ generic points and a marked point $p \in \mathbb{P}^2$, such that the curve has a *strict* cusp at the point p. The expected dimension of this space is r. Note that

$$\overline{\mathcal{S}_r^d(0, A_2)} := \big\{([s], p) \in \overline{\mathcal{S}_{r+1}^d(0, A_1)} : \psi_{A_2} = 0\big\}$$

where the closure is taken inside $\overline{\mathcal{S}_{r+1}^d(0, A_1)}$. A reminder to the attentive reader that soon a similar fact will be false! The quantity ψ_{A_2} is a section of the line bundle

$$L = (\gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d} \otimes \Lambda^2 T^*\mathbb{P}^2)^{\otimes 2}$$

We need to compute the cardinality of the set $\overline{\mathcal{S}_0^d(0, A_2)}$. We can show that ψ_{A_2} restricted to $\mathcal{S}_0^d(0, A_1)$ is transverse to the zero set. Since the points are in general position all the elements of $\overline{\mathcal{S}_0^d(0, A_2)}$ are *strict* cusps, i.e. $\psi_{A_3} \neq 0$. Hence the desired number $\mathcal{N}^d(0, A_2)$ is given by

$$\begin{aligned}
\mathcal{N}^d(0, A_2) &= |\overline{\mathcal{S}_0^d(0, A_2)}| \\
&= \langle e(L), \quad [\overline{\mathcal{S}_1^d(0, A_1)}]\rangle \\
&= 2\mathcal{N}^d(0, A_1) + 2(d-3)\mathcal{N}^d(0, A_1, 1)
\end{aligned}$$

which can be seen from the splitting principle and Kunneth formula.

Method 2: A cusp occurs when the Hessian is degenerate, i.e. there exists a non zero vector v such that $\nabla^2 s(v, \cdot) = 0$ and the third derivative along v is non zero. Recall that we have defined

$$\mathcal{S}_{r+1}^d(0, A_1) \subset \mathcal{D}_d(r+2) \times \mathbb{P}^2$$

to be the $r+1$ dimensional space of curves with a node. Let

$$\widehat{\mathcal{S}_{r+1}^d(0, A_1)} \subset \mathcal{D}_d(r+2) \times \mathbb{P}T\mathbb{P}^2$$

be the space of curves with a node and a tangent vector on top of it. Note that this is a $r+2$ dimensional space. We further define

$$\mathbb{P}\mathcal{S}_r^d(0, A_2) := \big\{([s], p) \in \widehat{\mathcal{S}_{r+1}^d(0, A_1)} : \ \hat{\psi}_{A_2} = 0, \ \hat{\psi}_{A_3} \neq 0\big\}$$

where

$$\hat{\psi}_{A_2} = \nabla^2 s(v, \cdot)$$
$$\hat{\psi}_{A_3} = \nabla^3 s(v, v, v)$$

It is the space of degree d curves $[s] \in \mathcal{D}_d(r+2)$ passing through $D_d - (r+2)$ generic points and a marked point $p \in \mathbb{P}^2$ and *marked direction* v , such that the Hessian at the point p evaluated on

v is zero and the third derivative along v is not. The expected dimension of this space is r. Note that

$$\overline{\mathbb{P}\mathcal{S}_r^d(0,A_2)} := \big\{([s],p) \in \overline{\widehat{\mathcal{S}_{r+1}^d}(0,A_1)} : \hat{\psi}_{A_2} = 0\big\}$$

where the closure is taken inside $\overline{\widehat{\mathcal{S}_{r+1}^d}(0,A_1)}$. The quantity $\hat{\psi}_{A_2}$ is a section of a rank 2 vector bundle

$$W = \tilde{\gamma}^* \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{_{\mathbb{P}^2}}^{*^d} \otimes T^*\mathbb{P}^2$$

We need to compute the cardinality of the set $\overline{\mathbb{P}\mathcal{S}_0^d(0,A_2)}$. We can show that $\hat{\psi}_{A_2}$ restricted to $\widehat{\mathcal{S}^d}_0(0,A_1)$ is transverse to the zero set. Since the points are in general position all the elements of $\overline{\mathbb{P}\mathcal{S}_0^d(0,A_2)}$ are *strict* cusps, i.e. $\hat{\psi}_{A_3} \neq 0$. Hence the desired number $\mathcal{N}^d(0,A_2)$ is given by

$$\mathcal{N}^d(0,A_2) = |\overline{\mathbb{P}\mathcal{S}_0^d(0,A_2)}| = \langle e(W), \quad [\overline{\widehat{\mathcal{S}^d}_1(0,A_1)}]\rangle.$$

Note that the dimension of $\overline{\widehat{\mathcal{S}^d}_1(0,A_1)}$ is two not one.

Remark 1: For this computation it is essential to know the ring structure of $H^*(\mathbb{P}(T\mathbb{P}^2))$ in addition to using the splitting principle and Kunneth formula.

Remark 2: Note that $\overline{\mathcal{S}_0^d(0;A_2)}$ and $\overline{\mathbb{P}\mathcal{S}_0^d(0;A_2)}$ are subsets of two different spaces. As sets they are different. They also count two different things. The first one counts the number of degree d curves with a marked point where the determinant of the Hessian at the marked point vanishes. The second quantity counts the number of degree d curves with a with a marked point and a marked direction such that the Hessian at the point evaluated on the marked direction vanishes. For a strict cusp, there is a unique direction along which the Hessian is degenerate. Hence these two sets happen to have the same cardinality.

Remark 3: The second method seems to be unnecessarily complicated. In fact for this problem alone it is. However, it will be necessary to think of a cusp in the second way to proceed to the next problem, i.e. enumerating curves with a tacnode (A_3-node).

Remark 4: Let us consider the number

$$\langle e(\tilde{\gamma}^*), \quad [\overline{\mathbb{P}\mathcal{S}_1^d(0,A_2)}]\rangle$$

It is worth pointing out that this number is a *signed* cardinality of a set and is often negative. The next question is how do we compute this number? As before, we "expect" it to be

$$\begin{aligned}
\mathcal{N}^d(0,A_2,0,1) &= \langle e(\tilde{\gamma}^*), \quad [\overline{\mathbb{P}\mathcal{S}_1^d(0,A_2)}]\rangle \\
&= \langle e(\tilde{\gamma}^*)e(W), \quad [\overline{\widehat{\mathcal{S}_2^d}(0,A_1)}] \\
&= \langle e(\tilde{\gamma}^*)e(W), \quad [\overline{\widehat{\mathcal{S}_2^d}(0,A_1)}]
\end{aligned}$$

This is true, but it requires some care to justify. A quick reminder that the dimension of $\overline{\widehat{\mathcal{S}_2^d}(0,A_1)}$ is *three* not two. Let us consider the space $\overline{\widehat{\mathcal{S}_2^d}(0,A_1)}$. The boundary of $\overline{\widehat{\mathcal{S}_2^d}(0,A_1)}$ comprises of curves with a cusp which can further degenerate to curves with a tacnode. Now consider a representative of the space

$$\overline{\widehat{\mathcal{S}_2^d}(0, A_1)} \cap e(\tilde{\gamma}^*)$$

which we get by taking a generic section of the line bundle $\tilde{\gamma}^*$. The boundary of this space comprises of curve with a tacnode with a marked direction that is generic. The section $\hat{\psi}_{A_2}$ will not vanish on that tacnode, since the direction is generic. Let us skip a step ahead and look at the computation of one node and one cusp on a lambda. The computation of $\mathcal{N}^d(1, A_2, 0, 1)$ is more involved than the computation of $\mathcal{N}^d(1, A_2, 0, 0)$. This is because in the closure of two nodes we get a tacnode, which is already dealt with in the computation of $\mathcal{N}^d(1, A_2)$. However in the computation of $\mathcal{N}^d(1, A_2, 0, 1)$ we have to realize that a tacnode could degenerate to either an A_4-node or a triple point. The section $\hat{\psi}_{A_2}$ will not vanish along a A_4 node because the A_4-node is assigned a generic direction. However the section $\hat{\psi}_{A_2}$ will vanish along a triple point, because the section will vanish *no matter what the assigned direction is.* We call the number

$$\mathcal{N}^d(0, A_2, 0, 1)$$

"number of degree d-curves with a cusp on a lambda class."

4.3 Conditions for A_k-node

If $f = f(x, y)$ is a holomorphic function defined on a neighborhood of the origin and i, j are non-negative integers, let

$$f_{ij} = \frac{\partial^{i+j} f}{\partial x^i \partial y^j}\bigg|_{(x,y)=0}.$$

Theorem 4.5. *Let $f(x, y) = 0$ be a curve such that $f_{00}, f_{10}, f_{01}, f_{11}, f_{20}$=0. Then the curve has a singularity of type A_k (i.e. it can be expressed after a change of coordinates in the form $\hat{y}^2+\hat{x}^{k+1} = 0$ if and only if*

$$\begin{aligned} \alpha_{A_3}, \alpha_{A_4}, \ldots, \alpha_{A_k} &= 0 \\ \alpha_{A_{k+1}} &\neq 0 \\ f_{20} &\neq 0 \end{aligned}$$

where the α_{A_k} are described below.

First we define the following quantities:

$$\alpha_{A_3} = f_{30}, \qquad \alpha_{A_4} = f_{02}f_{40} - 3f_{21}^2, \qquad \alpha_{A_5} = \frac{f_{02}f_{50}}{24} - \frac{5f_{21}f_{31}}{12} + \frac{5f_{12}f_{40}}{24}$$

$$\alpha_{A_6} = f_{02}\big(-\frac{f_{31}^2}{24} + \frac{f_{60}f_{02}}{240} + \frac{f_{50}f_{12}}{40}\big) - \frac{f_{21}}{2}\big(\frac{f_{40}f_{03}}{144} - \frac{f_{21}f_{22}}{16}\big)$$

$$\begin{aligned}\alpha_{A_7} = &\frac{f_{02}^4 f_{70}}{720} - \frac{7f_{02}^3 f_{21} f_{51}}{240} - \frac{7f_{02}^3 f_{31} f_{41}}{144} + \frac{7f_{02}^2 f_{12} f_{21} f_{41}}{48} + \frac{7f_{02}^2 f_{21}^2 f_{32}}{48} + \frac{7f_{02}^2 f_{21} f_{22} f_{31}}{24} \\ &- \frac{7f_{02} f_{03} f_{21}^2 f_{31}}{48} - \frac{7f_{02} f_{12} f_{21}^2 f_{22}}{8} - \frac{7f_{02} f_{13} f_{21}^3}{48} + \frac{7f_{03} f_{12} f_{21}^3}{16} - \frac{7f_{02}^2 f_{12}}{3}\big(-\frac{f_{31}^2}{24} + \frac{f_{50}f_{12}}{40}\big).\end{aligned}$$

Note that if $f_{02} \neq 0$ then

$$\alpha_{A_i} = 0 \qquad \forall i \leq k \qquad \text{iff} \qquad A_i^f = 0 \qquad \forall i \leq k.$$

Unlike A_k^f, α_{A_k} is defined even when $f_{02} = 0$.

Remark 1: The first three conditions f_{00}, f_{10}, f_{01} implies that the curve has at least a node. The other two conditions imply that the Hessian is degenerate and the vector $(1, 0)$ is in the kernel of the Hessian.

Remark 2: Note that while $\alpha_{A_3} = f_{30}$, naively we would expect $\alpha_{A_4} = f_{40}$ which is not the case! The expressions for α_{A_k} soon become very complicated.

Remark 3: The condition $f_{02} \neq 0$ is identical to saying that the Hessian is not identically zero. This is the condition that is in some sense makes the entire problem so hard!

Theorem 4.6. *Let $f(x, y) = 0$ be a curve such that $f_{00}, f_{10}, f_{01}, f_{11}, f_{20}, f_{02}, f_{30}, f_{21}$=0. Then the curve has a singularity of type D_k (i.e. it can be expressed after a change of coordinates in the form $\hat{y}^2 x + \hat{x}^{k-1} = 0$ if and only if*

$$\alpha_{D_6}, \alpha_{D_7}, \ldots, \alpha_{D_k} = 0, \quad \alpha_{D_{k+1}} \neq 0, \quad f_{12} \neq 0, \quad f_{30} \neq 0.$$

The first few values of α_{D_k} are

$$\alpha_{D_5} = f_{21}, \quad \alpha_{D_6} = f_{40}, \quad \alpha_{D_7} = -\frac{f_{31}^2}{24} + \frac{f_{50} f_{12}}{40}.$$

Remark 1: Here we are assuming the Hessian is identically zero. The last two conditions imply that there exists a non zero vector v such that $\nabla^3 f(v, v, \cdot)$ is zero and we have fixed that vector to be $(1, 0)$.

Theorem 4.7. *Let $f(x, y) = 0$ be a curve such that $f_{00}, f_{10}, f_{01}, f_{11}, f_{20}, f_{02}, f_{30}, f_{21}$=0. Then the curve has a singularity of type E_6 (i.e. it can be expressed after a change of coordinates in the form $\hat{y}^3 + \hat{x}^4 = 0$) if and only if*

$$\alpha_{E_6} = 0, \quad \alpha_{E_7} \neq 0, \quad f_{30} \neq 0.$$

where

$$\alpha_{E_6} = f_{12}, \qquad \alpha_{E_7} = f_{40}.$$

Theorem 4.8. *Let $f(x, y) = 0$ be a curve such that $f_{00}, f_{10}, f_{01}, f_{11}, f_{20}, f_{02}, f_{30}, f_{21}$=0, $\alpha_{E_6} = 0$. Then the curve has a singularity of type E_7 (i.e. it can be expressed after a change of coordinates in the form $\hat{y}^3 + \hat{y}\hat{x}^3 = 0$ if and only if*

$$\alpha_{E_7} = 0, \quad \alpha_{E_8} \neq 0, \quad f_{30} \neq 0.$$

where

$$\alpha_{E_7} = f_{40}$$

4.4 Curves with one A_3-node

Lemma 4.9. *The number of degree d curves passing through $\kappa(d)-3$ points with a tacnode (A_3-node) is*

$$\mathcal{N}^d(0;A_3) = \mathcal{N}^d(0,A_2) + d\mathcal{N}^d(0,A_2,1) + 3\mathcal{N}^d(0,A_2,0,1).$$

Proof. Recall that we have defined

$$\mathbb{P}\mathcal{S}^d_{r+1}(0,A_2) \subset \mathcal{D}_d(r+2) \times \mathbb{P}T\mathbb{P}^2$$

to be the $r+1$ dimensional space of curves with a cusp and a preferred direction along which the Hessian vanishes and

$$\mathbb{P}\mathcal{S}^d_r(0,A_3) := \big\{([s],p,v) \in \overline{\mathbb{P}\mathcal{S}^d_{r+1}(0,A_2)} : \alpha_{A_3} = 0, f_{02} \neq 0, \alpha_{A_4} \neq 0\big\}$$

The expected dimension of this space is r. Note that

$$\overline{\mathbb{P}\mathcal{S}^d_r(0,A_3)} := \big\{([s],p) \in \overline{\mathbb{P}\mathcal{S}^d_{r+1}(0,A_2)} : \alpha_{A_3} = 0\big\}$$

where the closure is taken inside $\overline{\mathbb{P}\mathcal{S}^d_{r+1}(0,A_2)}$. The quantity α_{A_3} is a section of the line bundle

$$L = \tilde{\gamma}^{*^3} \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^{*^d}_{\mathbb{P}^2}$$

We need to compute the cardinality of the set $\overline{\mathbb{P}\mathcal{S}^d_0(0,A_3)}$. We can show that α_{A_3} restricted to $\overline{\mathbb{P}\mathcal{S}^d_1(0,A_2)}$ is transverse to the zero set. Since the points are in general position all the elements of $\overline{\mathbb{P}\mathcal{S}^d_0(0,A_3)}$ are *strict* tacnodes. Hence the desired number $\mathcal{N}^d(0,A_3)$ is given by

$$\begin{aligned}\mathcal{N}^d(0,A_3) &= |\overline{\mathbb{P}\mathcal{S}^d_0(0,A_3)}| = \langle e(L), \quad [\overline{\mathbb{P}\mathcal{S}^d_1(0,A_2)}]\rangle \\ &= 3\mathcal{N}^d(0,A_2,0,1) + \mathcal{N}^d(0,A_2) + d\mathcal{N}^d(0,A_2,1).\end{aligned}$$

Remark: Note that although the number $\mathcal{N}^d(0,A_2,0,1)$ may not be a genuine number, it arises naturally while computing a perfectly genuine quantity i.e. $\mathcal{N}^d(0,A_3)$.

Lemma 4.10. *The number of degree d curves passing through $\kappa(d)-4$ points with a tacnode (A_3-node) on a lambda class is*

$$\mathcal{N}^d(0,A_3,0,1) \quad = \mathcal{N}^d(0,A_2,0,1) + d\mathcal{N}^d(0,A_2,1,1) + 3\mathcal{N}^d(0,A_2,0,2) \tag{4.1}$$

Proof. This number is what we expect it to be. It requires some justification. We note that a tacnode can degenerate to a A_4-node or a triple point with a generic direction. The section α_{A_3} will not vanish along such a point because the third derivative along a generic direction does not vanish for either a triple point or a A_4-node. This same argument will hold when we compute $\mathcal{N}^d(0,A_4,0,1)$, $\mathcal{N}^d(0,A_5,0,1)$, $\mathcal{N}^d(0,A_6,0,1)$. They are all what we expect them to be from the computation of $\mathcal{N}^d(0,A_k)$. That is because the section the section α_{A_k} will not vanish on the more degenerate points that arise, because they are assigned a generic direction.

4.5 Curves with one A_4-node

Theorem 4.11. *The number of degree d curves through $\kappa(d)-4$ points with a A_4-node is*

$$\mathcal{N}^d(0,A_4) = 2\mathcal{N}^d(0,A_3,0,1) + \mathcal{N}^d(0,A_3) + 2(d-3)\mathcal{N}^d(0,A_3,1).$$

Proof. Recall that we have defined

$$\mathbb{P}\mathcal{S}^d_{r+1}(0,A_3) \subset \mathcal{D}_d(r+2) \times \mathbb{P}T\mathbb{P}^2$$

to be the $r+1$ dimensional space of curves with a *strict* tacnode and a preferred direction along which the Hessian vanishes and the third derivative vanishes. Also note that

$$\mathbb{P}\mathcal{S}^d_r(0,A_4) := \big\{([s],p,v) \in \overline{\mathbb{P}\mathcal{S}^d_{r+1}(0,A_3)} : \alpha_{A_4} = 0, \alpha_{A_5} \neq 0, f_{02} \neq 0\big\}$$

The expected dimension of this space is r. Note that

$$\overline{\mathbb{P}\mathcal{S}^d_r(0,A_4)} := \big\{([s],p) \in \overline{\mathbb{P}\mathcal{S}^d_{r+1}(0,A_3)} : \alpha_{A_4} = 0\big\}$$

where the closure is taken inside $\overline{\mathbb{P}\mathcal{S}^d_{r+1}(0,A_3)}$. This is the last time this kind of a statement will be true. Note that, the quantity α_{A_4} is a section of the line bundle

$$L = \tilde{\gamma}^{*^4} \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^{*^d}_{\mathbb{P}^2} \otimes \big(T\mathbb{P}^2/\tilde{\gamma}\big)^{*2} \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^{*^d}_{\mathbb{P}^2}$$

We need to compute the cardinality of the set $\overline{\mathbb{P}\mathcal{S}^d_0(0,A_4)}$. We can show that α_{A_4} restricted to $\overline{\mathbb{P}\mathcal{S}^d_1(0,A_3)}$ is transverse to the zero set. Since the points are in general position all the elements of $\overline{\mathbb{P}\mathcal{S}^d_0(0,A_4)}$ are *strict* A_4-node. Hence the desired number $\mathcal{N}^d(0,A_4)$ is given by

$$\mathcal{N}^d(0,A_4) = |\overline{\mathbb{P}\mathcal{S}^d_0(0,A_4)}| = \langle e(L), \quad [\overline{\mathbb{P}\mathcal{S}^d_1(0,A_4)}]\rangle.$$

Lemma 4.12. *The number of degree d curves passing through $\kappa(d)-5$ points with a A_4-node on a lambda class is*

$$\mathcal{N}^d(0,A_4,0,1) \quad = 2\mathcal{N}^d(0,A_3,0,2) + \mathcal{N}^d(0,A_3,0,1) + 2(d-3)\mathcal{N}^d(0,A_3,1,1). \tag{4.2}$$

Proof. This number is what we expect it to be. It requires some justification. We note that a A_4-node can degenerate to an A_5-node or a D_5-node with a generic direction. We claim that the section α_{A_4} will not vanish along such a point. Let us consider the behavior of α_{A_4} along a D_5 node. It is true that the quantity f_{02} will vanish. We claim that f_{21} will not vanish. Let us consider more carefully what we are claiming. We are claiming that for a generic $v \in \tilde{\gamma}*$ and a $w \in T\mathbb{P}^2/\tilde{\gamma}^*$, the quantity

$$\nabla^2 s(v,v,w) \neq 0.$$

This quantity will vanish only for a very specific choice of $v \in T\mathbb{P}^2$. The v has to be in the kernel of the Hessian. The same argument holds for the other sections α_{A_k}. Hence, the number

$$\mathcal{N}^d(0,A_k,0,1)$$

is what we "expect" it to be.

4.6 Curves with one A_5-node

Theorem 4.13. *The number of degree d curves with a A_5-node is*

$$\mathcal{N}^d(0, A_5) = 3\mathcal{N}^d(0, A_4, 0, 1) + 2\mathcal{N}^d(0, A_4) + 2(d-3)\mathcal{N}^d(0, A_4, 1).$$

Proof. **Method 1:** This is the first place where the condition $f_{02} \neq 0$ creates a problem. But in this case we have an alternative method as seen in method 2.

Recall that we have defined

$$\mathbb{P}\mathcal{S}^d_{r+1}(0, A_4) \subset \mathcal{D}_d(r+2) \times \mathbb{P}T\mathbb{P}^2$$

to be the $r+1$ dimensional space of curves with a *strict* A_4-node. Also note that

$$\mathbb{P}\mathcal{S}^d_r(0, A_5) := \big\{([s], p, v) \in \overline{\mathbb{P}\mathcal{S}^d_{r+1}(0, A_4)} : f_{02}^2 A^f_5 = 0, A^f_6 \neq 0, f_{02} \neq 0\big\}$$

The expected dimension of this space is r. Although A^f_5 is not defined when $f_{02} = 0$, $f_{02}^2 A^f_5$ is well defined. However

$$\overline{\mathbb{P}\mathcal{S}^d_r(0, A_5)} \neq \big\{([s], p) \in \overline{\mathbb{P}\mathcal{S}^d_{r+1}(0, A_4)} : f_{02}^2 A^f_5 = 0\big\}.$$

In other words, the closure of the space

$$\begin{aligned} A^f_3 &= f_{30} = 0 \\ f_{02}A^f_4 &= f_{02}f_{40} - 3f_{21}^2 = 0 \\ f_{02}^2 A^f_5 &= \frac{f_{50}f_{02}}{24} - \frac{5f_{21}f_{31}f_{02}}{12} + \frac{5f_{12}f_{21}^2}{8} = 0 \\ &f_{02}^3 A^f_6 \neq 0 \\ &f_{02} \neq 0 \end{aligned}$$

is *not* the same as

$$\begin{aligned} A^f_3 &= f_{30} = 0 \\ f_{02}A^f_4 &= f_{02}f_{40} - 3f_{21}^2 = 0 \\ f_{02}^2 A^f_5 &= \frac{f_{50}f_{02}}{24} - \frac{5f_{21}f_{31}f_{02}}{12} + \frac{5f_{12}f_{21}^2}{8} = 0 \end{aligned}$$

Hence, although $f_{02}^2 A^f_5$ is a section of the bundle

$$L = \tilde{\gamma}^{*^5} \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^{*^d}_{\mathbb{P}^2} \otimes \big(\big(T\mathbb{P}^2/\tilde{\gamma}\big)^{*2} \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^{*^d}_{\mathbb{P}^2}\big)^{\otimes 2}$$

the desired number $\mathcal{N}^d(0, A_5)$ is *not* the Euler class of L, i.e.

$$\mathcal{N}^d(0, A_5) \neq \langle e(L), [\overline{\mathbb{P}\mathcal{S}^d_1(0, A_4)}]\rangle$$

The reason is that the section $f_{02}^2 A_5^f$ vanishes on points that are at the boundary of $\overline{\mathbb{P}\mathcal{S}_1^d(0, A_4)}$, that have $f_{02} = 0$. Hence the the desired number is

$$\mathcal{N}^d(0, A_5) = \langle e(L), [\overline{\mathbb{P}\mathcal{S}_1^d(0, A_4)}]\rangle - \mathcal{C}_{\partial \overline{M}}$$

This raises three questions.

Question 1: First of all what singularities are there in $\overline{\mathbb{P}\mathcal{S}_1^d(0; A_4)}$ when $f_{02} = 0$?

Question 2: How many of these are there (an enumerative question)?

Question 3: And finally what is the multiplicity with which the section $f_{02}^2 A_5^f$ vanishes around these points?

The answer to the question 1 is that if a curve is in the closure of $\overline{\mathbb{P}\mathcal{S}_1^d(0; A_4)}$ and $f_{02} = 0$, then the curve has a singularity of type at least D_5, i.e.

$$f_{30} = 0, \quad f_{20} = 0, \quad f_{21} = 0.$$

Since the points are in general position, it is a strict D_5-node.

The second question has been answered later on in the thesis. The question we need to answer is "How many degree d-curves are there that pass through $\kappa(d) - 5$ points and have a D_5-node?" Let us denote this number by

$$\mathcal{N}^d(0, D_5)$$

As seen from the above three equations, this number is merely the Euler class of a rank three bundle defined on top of $\overline{\mathbb{P}\mathcal{S}_3^d(0, A_2)}$.

Finally we need to compute the multiplicity of the section $f_{02}^2 A_5^f$ around a D_5-node. To do that let us construct a path $f_{ij}(t) \in \mathbb{P}\mathcal{S}_1^d(0; A_4)$ that converges to a a D_5-node. Let us assume $f_{40}(0) \neq 0$, which will be the case since the points are in general position. Then the curve

$$\begin{aligned} f_{21}(t) &= t \\ f_{02}(t) &= \frac{3t^2}{f_{40}(0)} \\ f_{ij}(t) &= f_{ij}(0) \qquad \text{otherwise} \end{aligned}$$

does lie in $\mathbb{P}\mathcal{S}_1^d(0; A_4)$ for $t \neq 0$ and $f_{ij}(0)$ is a D_5-node. The equation $f_{02}^2 A_5^f = \nu$ can be rewritten as

$$\frac{5f_{12}t^2}{8} - \frac{5f_{31}t^3}{4f_{40}} + \frac{3f_{50}t^4}{8f_{40}^2} = \nu$$

where ν is a "small perturbation". If $f_{12} \neq 0$ then for a generic ν this has 2 "small" solutions. The multiplicity is therefore 2, provided f_{40} and $f_{12} \neq 0$. This we can assume since the points are in general position. Hence the desired number is

$$\mathcal{N}^d(0; A_5) = \langle e(L), [\overline{\mathbb{P}\mathcal{S}_1^d(0; A_4)}]\rangle - 2\mathcal{N}^d(0; D_5).$$

The problem with this computation is that the sequence (or curve) $f(t)$ we constructed may not pass through the $\kappa(d)-5$ points in general position although $f(0)$ does. Using the same argument as in the proof of lemma 3.2 we can modify the sequence to $\tilde{f}(t)$ such that it does pass through the points in general position and $\tilde{f}(0) = f(0)$. Since this is a one parameter, family of curves, we can choose one of the $f_{ij}(t)$ we like and keep it the same. Let us choose our perturbation so that

$$\begin{aligned}\tilde{f}_{21}(t) &= f_{21}(t)\\ &= t\end{aligned}$$

Since the curve has to at least an A_4-node we get that

$$\tilde{f}_{02}(t) = \frac{3t^2}{\tilde{f}_{40}(t)}$$

Furthermore

$$\begin{aligned}\tilde{f}_{40}(t) &= f_{40}(t) + \epsilon(t)\\ &= f_{40}(0) + \epsilon(t)\end{aligned}$$

where $\epsilon(0) = 0$. It is easy to see that this perturbation does not affect the multiplicity computation.

Method 2: In this case a simpler solution is available. Recall that the problem we faced was

$$\overline{\mathbb{P}\mathcal{S}_r^d(0, A_5)} \neq \big\{([s], p) \in \overline{\mathbb{P}\mathcal{S}_{r+1}^d(0, A_4)} : f_{02}^2 A_5^f = 0\big\}$$

which was because the section $f_{02}^2 A_5^f$ vanishes on a D_5-node. But notice that using the fact that $A_4^f = 0$, we can rewrite $f_{02}^2 A_5^f$ as follows

$$\begin{aligned}f_{02}^2 A_5^f &= \frac{f_{50} f_{02}^2}{24} - \frac{5 f_{21} f_{31} f_{02}}{12} + \frac{5 f_{12} f_{21}^2}{8}\\ &= f_{02}\alpha_{A_5} \qquad \text{(using that } \alpha_{A_4} = 0)\end{aligned}$$

Hence an equivalent condition to have a A_5-node is that

$$\alpha_{A_3} = 0, \quad \alpha_{A_4} = 0, \quad \alpha_{A_5} = 0, \quad \alpha_{A_6} \neq 0, \quad f_{02} \neq 0$$

But now

$$\overline{\mathbb{P}\mathcal{S}_r^d(0, A_5)} = \big\{([s], p) \in \overline{\mathbb{P}\mathcal{S}_{r+1}^d(0, A_4)} : \alpha_{A_5} = 0\big\}!$$

The difference is that α_{A_5} is a section of different bundle

$$L = \tilde{\gamma}^{*^5} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{_{\mathbb{P}^2}}^{*^d} \otimes \big(T\mathbb{P}^2/\tilde{\gamma}\big)^{*2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{_{\mathbb{P}^2}}^{*^d}$$

The desired number is therefore

$$\mathcal{N}^d(0, A_5) = \langle e(L), \ \ [\overline{\mathbb{P}\mathcal{S}_1^d(0, A_4)}]\rangle$$

4.7 Curves with one A_6-node

Theorem 4.14. *The number of degree d curves with an A_6 node is*

$$\mathcal{N}^d(0, A_6) = 2\mathcal{N}^d(0, A_5, 0, 1) + 3\mathcal{N}^d(0, A_5) + (3d-12)\mathcal{N}^d(0, A_5, 1) - 2\mathcal{N}^d(0, D_6) - \mathcal{N}^d(0, E_6)$$

Proof.
Recall that we have defined

$$\mathbb{P}\mathcal{S}^d_{r+1}(0, A_5) \subset \mathcal{D}_d(r+2) \times \mathbb{P}T\mathbb{P}^2$$

to be the $r+1$ dimensional space of curves with a *strict* A_5-node. Also note that

$$\mathbb{P}\mathcal{S}^d_r(0, A_6) := \left\{([s], p, v) \in \overline{\mathbb{P}\mathcal{S}^d_{r+1}(0, A_5)} : \alpha_{A_6} = 0, \alpha_{A_7} \neq 0, f_{02} \neq 0\right\}$$

The expected dimension of this space is r. Again

$$\overline{\mathbb{P}\mathcal{S}^d_r(0, A_6)} \neq \left\{([s], p) \in \overline{\mathbb{P}\mathcal{S}^d_{r+1}(0, A_5)} : \alpha_{A_6} = 0\right\}$$

Hence, although α_{A_6} is a section of the bundle

$$L = \tilde{\gamma}^{*^6} \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^{*^d}_{\mathbb{P}^2} \otimes \left(\left(T\mathbb{P}^2/\tilde{\gamma}\right)^{*2} \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^{*^d}_{\mathbb{P}^2}\right)\right)^{\otimes 2}$$

the desired number $\mathcal{N}^d(0, A_6)$ is *not* the Euler class of L, i.e.

$$\mathcal{N}^d(0, A_6) \neq \langle e(L), [\overline{\mathbb{P}\mathcal{S}^d_1(0, A_5)}]\rangle$$

The reason is that the section α_{A_6} vanishes on points that are at the boundary of $\overline{\mathbb{P}\mathcal{S}^d_1(0, A_5)}$, that have $f_{02} = 0$. Hence the the desired number is

$$\mathcal{N}^d(0, A_6) = \langle e(L), [\overline{\mathbb{P}\mathcal{S}^d_1(0, A_5)}]\rangle - \mathcal{C}_{\partial \overline{M}}$$

Again we need to answer three questions.

Question 1: What singularities are there in $\overline{\mathbb{P}\mathcal{S}^d_1(0; A_5)}$ when $f_{02} = 0$?

Question 2: How many of these are there (an enumerative question)?

Question 3: What is the multiplicity with which the section α_{A_6} vanishes around these points?

The answer to the question 1 is that if a curve is in the closure of $\overline{\mathbb{P}\mathcal{S}^d_1(0; A_5)}$ and $f_{02} = 0$, then the curve has a singularity of type at least D_6 or E_6. This is not hard to see. Since the points are in general position, it is a strict D_6-node or a strict E_6 node. The fact that there can not be any other type of singularity can be seen directly or we can simply use the classification of singularities in dimension 6. The last argument is essentially "cheating" because beyond dimension 7 there isn't a complete classification of singularities available.

Recall that a singularity is at least of type D_6 if

$$f_{30} = 0, \quad f_{20} = 0, \quad f_{21} = 0, \quad f_{40} = 0$$

and it is of type at least E_6 if

$$f_{30} = 0, \quad f_{20} = 0, \quad f_{21} = 0, \quad f_{12} = 0.$$

Since the points are in general position, it is a strict D_6-node or a strict E_6-node.

The second question has been answered later on in the thesis. The question we need to answer is "How many degree d-curves are there that pass through $\kappa(d) - 6$ points and have a D_6/E_6-node ?" Let us denote the numbers by

$$\mathcal{N}^d(0, D_6), \qquad \mathcal{N}^d(0, E_6)$$

As seen from the above equations, both of these numbers are merely the Euler class of a line bundle defined on top of $\overline{\mathbb{P}\mathcal{S}_1^d(0, D_5)}$.

Finally we need to compute the multiplicity of the section around a D_6-node or a E_6-node.

Let us start with E_6-node. To do that let us construct a path $f_{ij}(t) \in \mathbb{P}\mathcal{S}_1^d(0; A_5)$ that converges to an E_6-node. Let us assume that $f_{40}(0) \neq 0$. Note that although there will be points in the closure where $f_{40} = 0$, they won't be near points that have an E_6-node. The curve

$$\begin{aligned}
f_{21}(t) &= t \\
f_{02}(t) &= \frac{3t^2}{f_{40}(0)} \\
f_{12}(t) &= \frac{-f_{02}(t)f_{50} + 10f_{21}(t)f_{31}}{5f_{40}(0)} \\
f_{ij}(t) &= f_{ij}(0) \qquad \text{otherwise}
\end{aligned}$$

does lie in $\mathbb{P}\mathcal{S}_1^d(0; A_5)$ for $t \neq 0$ and $f_{ij}(0)$ is an E_6-node. The equation $\alpha_{A_6} = \nu$ can be rewritten as

$$-\frac{f_{40}f_{03}}{288}t + O(t^2) = \nu$$

where ν is a "small perturbation". If $f_{40}f_{03} \neq 0$ then for a generic ν this has 1 "small" solutions. The multiplicity is therefore 1, provided f_{40} and $f_{03} \neq 0$. This we can assume since the points are in general position.

Next we need to compute the multiplicity near a D_6-node. To do that let us construct a path $f_{ij}(t) \in \mathbb{P}\mathcal{S}_1^d(0; A_5)$ that converges to a a D_6-node. We will not able to do this explicitly. We will do this implicitly by "solving" a quadratic equation. Let $x = f_{02}(t)$, $y = f_{40}(t)$ and $f_{21}(t) = t$. Then x and y can be implicitly written as

$$\begin{aligned}
xy &= 3t^2 \\
\frac{f_{50}}{24}x + \frac{5f_{12}}{24}y &= \frac{5f_{31}}{12}t
\end{aligned}$$

One value of t corresponds to 2 solutions for (x, y) (both of which are first order in t). The equation $\alpha_{A_6} = \nu$ can be rewritten as

$$At + O(t^2) = \nu$$

where the coefficient A will be non zero generically. Hence it has one small solution for t. Which means it has two possible solutions for (f_{02}, f_{40}). The desired number is therefore

$$\begin{aligned}\mathcal{N}^d(0; A_6) = \langle e(L), [\overline{\mathbb{P}\mathcal{S}_1^d(0; A_5)}]\rangle \\ - 2\mathcal{N}^d(0; D_6) - \mathcal{N}^d(0; E_6)\end{aligned}$$

Remark: One must pause to consider the validity of this computation. Let us look at the multiplicity computation around a D_6-node. We chose the parametrization

$$\begin{aligned}xy &= 3t^2 \\ \frac{f_{50}}{24}x + \frac{5f_{12}}{24}y &= \frac{5f_{31}}{12}t \\ f_{21} &= t \\ f_{ij}(t) &= f_{ij}(0) \qquad \text{otherwise}\end{aligned}$$

where $x = f_{02}(t)$, $y = f_{40}(t)$. Why could we simply not take the parametrization

$$\begin{aligned}f_{02} &= t \\ f_{21} &= t \\ f_{40} &= 3t \\ f_{ij}(t) &= f_{ij}(0) \qquad \text{otherwise}\end{aligned}$$

The reason is that with this parametrization the $f_{ij}(0)$ would not be arbitrary! The condition that $\alpha_{A_5} = 0$ would imply that in the limit

$$\begin{aligned}\frac{f_{02}f_{50}}{24} - \frac{5f_{21}f_{31}}{12} + \frac{5f_{12}f_{40}}{24} &= 0 \\ \implies \frac{f_{50}}{24} - \frac{5f_{31}}{12} + \frac{5f_{12}}{8} &= 0\end{aligned}$$

This is an extra condition on the $f_{ij}(0)$ which generically will not happen. The parametrization we chose doesn't impose an extra condition on the coefficients (aside from being a D_6-node).

4.8 Proof of Theorem 16

Recall that we have defined

$$\mathbb{P}\mathcal{S}_{r+1}^d(0, A_6) \subset \mathcal{D}_d(r+2) \times \mathbb{P}T\mathbb{P}^2$$

to be the $r+1$ dimensional space of curves with a *strict* A_6-node. Also note that

$$\mathbb{P}\mathcal{S}_r^d(0, A_7) := \left\{([s], p, v) \in \overline{\mathbb{P}\mathcal{S}_{r+1}^d(0, A_6)} : \alpha_{A_7} = 0, \alpha_{A_8} \neq 0, f_{02} \neq 0\right\}$$

The expected dimension of this space is r. Again

$$\overline{\mathbb{P}\mathcal{S}_r^d(0, A_7)} \neq \left\{([s], p) \in \overline{\mathbb{P}\mathcal{S}_{r+1}^d(0, A_6)} : \alpha_{A_7} = 0\right\}$$

Hence, although α_{A_7} is a section of the bundle

$$L = \tilde{\gamma}^{*^7} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d} \otimes ((T\mathbb{P}^2/\tilde{\gamma})^{*2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}))^{\otimes 4}$$

the desired number $\mathcal{N}^d(0, A_7)$ is *not* the Euler class of L, i.e.

$$\mathcal{N}^d(0, A_7) \neq \langle e(L), [\overline{\mathbb{P}\mathcal{S}_1^d(0, A_6)}]\rangle$$

The reason is that the section α_{A_7} vanishes on points that are at the boundary of $\overline{\mathbb{P}\mathcal{S}_1^d(0, A_6)}$, that have $f_{02} = 0$. Hence the the desired number is

$$\mathcal{N}^d(0, A_6) = \langle e(L), [\overline{\mathbb{P}\mathcal{S}_1^d(0, A_6)}]\rangle - \mathcal{C}_{\partial \overline{M}}$$

Again we need to answer three questions.

Question 1: What singularities are there in $\overline{\mathbb{P}\mathcal{S}_1^d(0; A_6)}$ when $f_{02} = 0$?

Question 2: How many of these are there (an enumerative question)?

Question 3: What is the multiplicity with which the section α_{A_7} vanishes around these points?

The answer to the question 1 is that if a curve is in the closure of $\overline{\mathbb{P}\mathcal{S}_1^d(0; A_6)}$ and $f_{02} = 0$, then the curve has a singularity of type at least D_7 or E_7. This is not hard to see. Since the points are in general position, it is a strict D_7-node or a strict E_7 node. The fact that there can not be any other type of singularity is in fact hard to see, but we can show it directly. We can also use the classification of singularities in dimension 7. That argument is essentially "cheating" because beyond dimension 7 there isn't a complete classification of singularities available.

Recall that a singularity is at least of type D_7 if

$$f_{30} = 0, \quad f_{20} = 0, \quad f_{21} = 0, \quad f_{40} = 0, \quad \alpha_{D_8} = -\frac{f_{31}^2}{24} + \frac{f_{50} f_{12}}{40} = 0,$$

and it is of type at least E_7 if

$$f_{30} = 0, \quad f_{20} = 0, \quad f_{21} = 0, \quad f_{12} = 0, \quad f_{40} = 0,$$

Since the points are in general position, it is a strict D_7-node or a strict E_7-node.

The second question has been answered later on in the thesis. The question we need to answer is "How many degree d-curves are there that pass through $\kappa(d) - 7$ points and have a D_7/E_7-node ?" Let us denote the numbers by

$$\mathcal{N}^d(0, D_7), \qquad \mathcal{N}^d(0, E_7)$$

Both of these numbers are merely the Euler class of a line bundle defined on top of $\overline{\mathbb{P}\mathcal{S}_1^d(0, D_6)}$ or $\overline{\mathbb{P}\mathcal{S}_1^d(0, E_6)}$ respectively.

Finally we need to compute the multiplicity of the section around a D_7-node or an E_7-node.

Let us start with E_7-node. To do that let us construct a path $f_{ij}(t) \in \mathbb{P}\mathcal{S}_1^d(0; A_6)$ that converges to an E_7-node. The curve

$$\begin{aligned} f_{21}(t) &= t^2, \quad f_{40}(t) = t, \quad f_{02}(t) = 3t^3 \\ f_{12}(t) &= \frac{-f_{02}(t)f_{50} + 10f_{21}(t)f_{31}}{5f_{40}(t)} = O(t) \\ f_{ij}(t) &= f_{ij}(0) \qquad \text{otherwise} \end{aligned}$$

does lie in $\mathbb{P}\mathcal{S}_1^d(0; A_6)$ for $t \neq 0$ and $f_{ij}(0)$ is an E_7-node. The equation $\alpha_{A_7} = \nu$ can be rewritten as

$$At^7 + O(t^8) = \nu$$

where ν is a "small perturbation". For generic values of $f_{ij}(0)$, the coefficient A will be non zero. The multiplicity is therefore 7, provided $A \neq 0$. This we can assume since the points are in general position.

We can also compute the multiplicity near a D_7-node by constructing a path $f_{ij}(t) \in \mathbb{P}\mathcal{S}_1^d(0; A_6)$ that converges to a a D_7-node similar to the proof of lemma 4.6. The multiplicity in terms of t is 3 and hence the total multiplicity is 6. The desired number is therefore

$$\mathcal{N}^d(0; A_6) = \langle e(L), [\overline{\mathbb{P}\mathcal{S}_1^d(0; A_6)}] \rangle - 6\mathcal{N}^d(0; D_7) - 7\mathcal{N}^d(0; E_7).$$

4.9 Curves with one D_4-node

Lemma 4.15. *The number of degree d curves with a $(3,3)$ node is*

$$\begin{aligned} \mathcal{N}^d(0, D_4) &= \mathcal{N}^d(0, A_1) + (-9 + 3d)\mathcal{N}^d(0, A_1, 1) + (30 - 18d + 3d^2)\mathcal{N}^d(0, A_1, 2) \\ &= -2\mathcal{N}^d(0, A_3, 0, 1) + \mathcal{N}^d(0, A_3) + (d - 6)\mathcal{N}^d(0, A_3, 1) \end{aligned}$$

Proof.

Method 1: Recall that we have defined

$$\mathcal{S}_3^d(0; A_1) \subset \mathcal{D}_d(4) \times \mathbb{P}^2$$

to be the three dimensional space of curves with a node and

$$\mathcal{S}_0^d(0; D_4) \subset \overline{\mathcal{S}_3^d(0; A_1)}$$

is the set of zero dimensional set of curves with a strict triple point (D_4-node). Since the points are all in general position

$$\mathcal{S}_0^d(0; D_4) = \overline{\mathcal{S}_0^d(0; D_4)}$$

and the desired number is

$$\mathcal{N}^d(0; D_4) = |\overline{\mathcal{S}_0^d(0; D_4)}| = \langle e(\mathrm{Sym}^2\mathrm{Hom}(T\mathbb{P}^2, T\mathbb{P}^2)), \quad [\overline{\mathcal{S}_3^d(0; A_1)}] \rangle.$$

Method 2: This method is conceptually involved. It is similar to the second method of enumerating curves with a cusp.

Recall that we have defined the space

$$\mathbb{P}\mathcal{S}_1^d(0; A_3) \subset \mathcal{D}_d(4) \times \mathbb{P}T\mathbb{P}^2$$

to be the one dimensional space of curves with a tacnode (A_3-node), i.e the third derivative along the kernel of the Hessian is zero (and the "next" thing is not zero). Let us define $\tilde{\gamma}$ to be the direction along which the third derivative vanishes. We now define

$$\mathbb{P}\mathcal{S}_0^d(0; D_4) \subset \overline{\mathbb{P}\mathcal{S}_1^d(0; A_3)}$$

to be the zero dimensional space of curves such that the second derivative along the quotient space $T\mathbb{P}^2/\tilde{\gamma}$ vanishes. In local coordinates we are looking at the zero set of the equations

$$f_{00} = 0, \quad f_{10} = 0, \quad f_{01} = 0, \quad f_{20} = 0, \quad f_{11} = 0, \quad f_{30} = 0, \quad f_{02} = 0.$$

Note that with the fourth and fifth conditions, we are not merely looking at a cusp ($\det(\nabla^2 s) = 0$), but we are looking at a cusp with a marked direction, which we have fixed here to be $(1, 0)$. Note also that the last condition $f_{02} = 0$ is well defined on the quotient space. The cardinality of the set $\overline{\mathbb{P}\mathcal{S}_0^d(0; D_4)}$ is therefore

$$\begin{aligned} |\overline{\mathbb{P}\mathcal{S}_0^d(0; D_4)}| &= \langle e((T\mathbb{P}^2/\tilde{\gamma})^{*^2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}), [\overline{\mathbb{P}\mathcal{S}_1^d(0; A_3)}]\rangle \\ &= 45(d-2)^2 \end{aligned}$$

This number is three times what we would expect. Let us see what we are counting more carefully. As explained in the remark 2 of enumerating cusps, what we are counting is not a curve with a singular point, but a curve, with a singular point and a *marked direction*. Notice that we are counting on top of a tacnode. Hence what we are counting is the set of curves with a point along which the Hessian is zero and a marked direction along which the third derivative is zero (tacnode). But for a given triple point there will be *three* distinct directions in which the third derivative vanishes. Hence the cardinality of $|\overline{\mathbb{P}\mathcal{S}_0^d(0; D_4)}|$ is indeed three times the cardinality of $\mathcal{N}^d(0; D_4)$. Hence

$$\begin{aligned} \mathcal{N}^d(0; D_4) &= \frac{|\overline{\mathbb{P}\mathcal{S}_0^d(0; D_4)}|}{3} \\ &= \frac{1}{3}\langle e((T\mathbb{P}^2/\tilde{\gamma})^{*^2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}), [\overline{\mathbb{P}\mathcal{S}_1^d(0; A_3)}]\rangle. \end{aligned}$$

Theorem 4.16. *The number of degree d curves with a triple point on a lambda class is*

$$\begin{aligned} \mathcal{N}^d(0, D_4, 0, 1) &= -2\mathcal{N}^d(0, A_3, 0, 2) + \mathcal{N}^d(0A_3, 0, 1) + (d-6)\mathcal{N}^d(0, A_3, 1, 1) \\ &= \mathcal{N}^d(0, D_4) + (d-9)\mathcal{N}^d(0, D_4, 1) \end{aligned}$$

Proof. This can be done in two ways. The author feels that the second method is the better one.

Method 1: This follows the second method of the previous problem. The desired number is

$$\mathcal{N}^d(0, D_4, 0, 1) = \langle e(\tilde{\gamma}^* \oplus (T\mathbb{P}^2/\tilde{\gamma})^{*^2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}), [\overline{\mathbb{P}\mathcal{S}_2^d(0, A_3)}]\rangle.$$

Method 2: For this method we have to understand what the number really means. Recall that we are looking at the space which is the space of curves with a tripe pointed and a marked direction along which the third derivative vanishes. In the first method we looked at the zero set of the section f_{02} on top of a tacnode. However there is a much simpler way to look at this three to one cover. Consider the space

$$\hat{\mathcal{S}}_r^d(0, D_4) \subset \mathcal{D}_d(r+4) \times \mathbb{P}T\mathbb{P}^2$$

which is the r dimensional space of curves with a D_4-node and an equivalence class of tangent vector on top of it. A reminder to the reader that the dimension of this space is $r+1$, not r. Notice that

$$\langle e(\tilde{\gamma}^{*3} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d}), \quad [\overline{\hat{\mathcal{S}}_0^d(0, D_4)}]\rangle = 3\mathcal{N}^d(0, D_4)$$

because

$$\langle e(\gamma_{\mathcal{D}}^*), \quad [\overline{\hat{\mathcal{S}}_0^d(0, D_4)}]\rangle = 0$$
$$\langle e(\gamma_{\mathbb{P}^2}^*), \quad [\overline{\hat{\mathcal{S}}_0^d(0, D_4)}]\rangle = 0$$
$$\langle e(\tilde{\gamma}^*), \quad [\overline{\hat{\mathcal{S}}_0^d(0, D_4)}]\rangle = \mathcal{N}^d(0, D_4)$$

The answer is of course not surprising because a generic section of the line bundle

$$\tilde{\gamma}^{*3} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d}$$

is the third derivative along a direction. Hence what we are saying is that

$$\begin{aligned}\langle e(\tilde{\gamma}^{*3} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d}), \quad [\overline{\hat{\mathcal{S}}_0^d(0, D_4)}]\rangle &= \langle e((T\mathbb{P}^2/\tilde{\gamma})^{*2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d}), [\overline{\mathbb{P}\mathcal{S}_1^d(0; A_3)}]\rangle \\ &= 3\mathcal{N}^d(0, D_4)\end{aligned}$$

This shows another way to think of the three to one cover of the space of curves with a triple point. This of course is of no use if we want to find the number

$$\mathcal{N}^d(0, D_4) \qquad \text{or} \qquad \mathcal{N}^d(0, D_4, 0, 1).$$

But once we do know these two numbers, we can use this fact to find the number $\mathcal{N}^d(0, D_4, 0, 1)$. In other words

$$\begin{aligned}\mathcal{N}^d(0, D_4, 0, 1) &= \langle e(\tilde{\gamma} \oplus \tilde{\gamma}^{*3} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d}), [\overline{\hat{\mathcal{S}}_1^d(0, D_4)}]\rangle \\ &= \mathcal{N}^d(0, D_4) + (d-9)\mathcal{N}^d(0, D_4, 1)\end{aligned}$$

which of course agrees with the previous answer.

Remark 1: Let us skip one step ahead and consider the computation of one node and one triple point on a lambda, $\mathcal{N}^d(1, D_4, 0, 1)$. We could of course compute the contribution from the main stratum using the first method. That would involve finding the closure of one node and one tacnode and a multiplicity computation. The second method gives us the answer for free. We claim that the contribution from the main stratum is in fact the desired number. The price to pay is that we have to know $\mathcal{N}^d(1, D_4)$ using a different method (using the first method in fact). To see why there is no contribution from the boundary we need to see what is going on. We claim that the desired number is

$$\begin{aligned}\mathcal{N}^d(1, D_4, 0, 1) &= \langle e(\tilde{\gamma} \oplus \tilde{\gamma}^{*3} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d}), [\overline{\hat{\mathcal{S}}_1^d(1, D_4)}] \rangle \\ &= \mathcal{N}^d(1, D_4) + (d-9)\mathcal{N}^d(1, D_4, 1)\end{aligned}$$

We need to see what happens in the closure of one node and one D_4-node. We get at least a D_6 node. After we cap with a lambda class, we get a D_6-node with a generic direction assigned. The third derivative along a generic direction will not vanish. Hence the section does not vanish along a D_6 node. Hence the desired number is simply the Euler class of a bundle. This number therefore must be correct, assuming that the numbers

$$\mathcal{N}^d(1, D_4) \qquad \text{and} \qquad \mathcal{N}^d(1, D_4, 1)$$

are correct. There is no doubt that these numbers are correct. They both pass low degree checks for $d = 4$. And the first number agrees with the computation of Kazarian and Kleiman and Piene.

4.10 Curves with one D_5-node

Theorem 4.17. *The number of degree d curves with a D_5 node is*

$$\mathcal{N}^d(0, D_5) = 4\mathcal{N}^d(0, D_4) + (-18 + 4d)\mathcal{N}^d(0, D_4, 1)$$

Proof. **Method 1:** This computation is almost the "same" as the computing A_2-node. A D_5-node occurs when there exists a non zero v such that $\nabla^2 s(v, v, \cdot)$ is zero. Recall that we have defined

$$\mathcal{S}_1^d(0; D_4) \subset \mathcal{D}_d(5) \times \mathbb{P}^2$$

to be the one dimensional space of curves with a D_4 and

$$\hat{\mathcal{S}}_1^d(0, D_4) \subset \mathcal{D}_d(5) \times \mathbb{P}T\mathbb{P}^2$$

is the one dimensional space of curves with a D_4-node and an equivalence class of tangent vector on top of it. Note that this is a two dimensional space. Further recall that

$$\mathbb{P}\mathcal{S}_0^d(0; D_5) \subset \overline{\mathbb{P}\mathcal{S}_1^d(0; D_4)}$$

is the zero dimensional space of curves with a strict D_5-node with a marked direction in along which the $\nabla^2 s(v, v, \cdot)$ vanishes and along which the fourth derivative does not. Since the points are in general position, the last condition is automatically satisfied and hence

$$\overline{\mathbb{P}\mathcal{S}_0^d(0; D_5)} = \mathbb{P}\mathcal{S}_0^d(0; D_5).$$

The desired number is therefore given by

$$\mathcal{N}^d(0; D_5) = |\overline{\mathbb{P}\mathcal{S}_0^d(0; D_5)}| = \langle e(\tilde{\gamma}^{*^2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*d} \otimes T^*\mathbb{P}^2), \quad [\overline{\hat{\mathcal{S}}^d{}_1(0; D_4)}] \rangle.$$

Method 2: Recall that we have defined the space

$$\mathbb{P}\widetilde{\mathcal{S}_1^d(0; D_4)} \subset \mathcal{D}_d(5) \times \mathbb{P}T\mathbb{P}^2$$

to be the one dimensional space of curves with a triple point (D_4-node) with a marked direction. Further recall that

$$\mathbb{P}\mathcal{S}_0^d(0;D_5) \subset \overline{\mathbb{P}\widetilde{\mathcal{S}_1^d}(0;D_4)}$$

is the zero dimensional space of curves with a strict D_5-node i.e.

$$\alpha_{D_5} = f_{21} = 0$$

but the "next" quantity $\alpha_{D_6} \neq 0$ and the third derivative tensor is not identically zero. Since the points are in general position the last two conditions are automatically satisfied. Since α_{D_5} is a section of the bundle

$$\tilde{\gamma}^{*^2} \otimes (T\mathbb{P}^2/\tilde{\gamma}^*)^* \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}$$

the desired number is given by

$$\mathcal{N}^d(0;D_5) = |\overline{\mathbb{P}\mathcal{S}_0^d(0;D_5)}| = \langle e(\tilde{\gamma}^{*^2} \otimes (T\mathbb{P}^2/\tilde{\gamma}^*)^* \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}), \quad [\overline{\mathbb{P}\widetilde{\mathcal{S}_1^d}(0;D_4)}]\rangle.$$

4.11 Curves with one D_6-node

Theorem 4.18. *The number of degree d curves with a D_6-node is*

$$\mathcal{N}^d(0,D_6) = 4\mathcal{N}^d(0,D_5,0,1) + \mathcal{N}^d(0,D_5) + d\mathcal{N}^d(0,D_5,1).$$

Proof. Recall that we have defined the space

$$\mathbb{P}\mathcal{S}_1^d(0;D_5) \subset \mathcal{D}_d(6) \times \mathbb{P}T\mathbb{P}^2$$

is the one dimensional space of curves with a D_5-node and marked direction v along which $\nabla^2 s(v,s,\cdot)$. Note that this is a one dimensional space. Further recall that

$$\mathbb{P}\mathcal{S}_0^d(0;D_6) \subset \overline{\mathbb{P}\mathcal{S}_1^d(0;D_5)}$$

is the zero dimensional space of curves with a strict D_6-node i.e. $f_{40} = 0$, but the "next" thing is not zero. Since the points are in general position,

$$\mathbb{P}\mathcal{S}_0^d(0;D_6) = \overline{\mathbb{P}\mathcal{S}_0^d(0;D_6)}$$

The desired number is therefore given by

$$\mathcal{N}^d(0;D_6) = |\overline{\mathbb{P}\mathcal{S}_0^d(0;D_6)}| = \langle e(\tilde{\gamma}^{*^4} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}), \quad [\overline{\mathbb{P}\mathcal{S}_1^d(0;D_5)}]\rangle.$$

4.12 Proof of Theorem 14

Proof. Recall that we have defined the space

$$\mathbb{P}\mathcal{S}_1^d(0;D_6) \subset \mathcal{D}_d(7) \times \mathbb{P}T\mathbb{P}^2$$

to be the one dimensional space of curves with a D_6-node. Further recall that

$$\mathbb{P}\mathcal{S}_0^d(0; D_7) \subset \overline{\mathbb{P}\mathcal{S}_1^d(0; D_6)}$$

is the zero dimensional space of curves with a strict D_7-node, i.e.

$$\alpha_{D_7} = 0$$

but the "next" quantity $\alpha_{D_8} \neq 0$ and $\nabla^3 f$ is not identically zero. Since the points are in general position the last two conditions are automatically satisfied. Since α_{D_7} is a section of the bundle

$$L \quad = \tilde{\gamma}^{*^5} \otimes \tilde{\gamma}^* \otimes (T\mathbb{P}^2/\tilde{\gamma})^{*^2} \otimes (\gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d})^{\otimes 2}$$

the desired number is given by

$$\mathcal{N}^d(0; D_7) = |\overline{\mathbb{P}\mathcal{S}_0^d(0; D_7)}| = \langle e(L), [\overline{\mathbb{P}\mathcal{S}_1^d(0; D_7)}] \rangle$$

4.13 Curves with one E_6-node

Theorem 4.19. *The number of degree d curves with a E_6-node is*

$$\mathcal{N}^d(0, E_6) = -\mathcal{N}^d(0, D_5, 0, 1) + \mathcal{N}^d(0, D_5) + (d-6)\mathcal{N}^d(0, D_5, 1)$$

Proof. Recall that we have defined the space

$$\mathbb{P}\mathcal{S}_1^d(0; D_5) \subset \mathcal{D}_d(6) \times \mathbb{P}T\mathbb{P}^2$$

is the one dimensional space of curves with a D_5-node and marked direction v along which $\nabla^2 s(v, s, \cdot)$. Note that this is a one dimensional space. Further recall that

$$\mathbb{P}\mathcal{S}_0^d(0; E_6) \subset \overline{\mathbb{P}\mathcal{S}_1^d(0; D_5)}$$

is the zero dimensional space of curves with a strict E_6-node i.e. $f_{12} = 0$, but the "next" thing is not zero. Since the points are in general position,

$$\mathbb{P}\mathcal{S}_0^d(0; E_6) = \overline{\mathbb{P}\mathcal{S}_0^d(0; E_6)}$$

The quantity f_{12} is a section of the line bundle

$$L = \tilde{\gamma}^* \otimes (T\mathbb{P}^2/\tilde{\gamma})^{*^2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}$$

The desired number is therefore given by

$$\mathcal{N}^d(0, E_6) = |\overline{\mathbb{P}\mathcal{S}_0^d(0; E_6)}| = \langle e(L), [\overline{\mathbb{P}\mathcal{S}_1^d(0; D_5)}] \rangle.$$

4.14 Proof of Theorem 15

Proof. Recall that we have defined the space

$$\mathbb{P}\mathcal{S}_1^d(0; E_6) \subset \mathcal{D}_d(7) \times \mathbb{P}T\mathbb{P}^2$$

to be the one dimensional space of curves with a E_6-node. Further recall that

$$\mathbb{P}\mathcal{S}_0^d(0; E_7) \subset \overline{\mathbb{P}\mathcal{S}_1^d(0; E_6)}$$

is the zero dimensional space of curves with a strict E_7-node i.e.

$$\alpha_{E_7} = f_{40} = 0$$

but the "next" quantity $\alpha_{E_8} \neq 0$ and $\nabla^3 f$ is not identically zero. Since the points are in general position the last two conditions are automatically satisfied. Since α_{E_7} is a section of the bundle

$$L \quad = \tilde{\gamma}^{*^4} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}$$

the desired number is given by

$$\mathcal{N}^d(0, E_7) = |\overline{\mathbb{P}\mathcal{S}_0^d(0, E_7)}| = \langle e(L), [\overline{\mathbb{P}\mathcal{S}_1^d(0, E_6)}]\rangle.$$

Chapter 5

Enumeration of curves with two singular points

Recall lemma 3.15 which states that if a curve is in the closure of one node and one A_k node and the Hessian is not zero, then the curve is at least as singular as an A_{k+2} node. We now prove the following lemma

Lemma 5.1. *The multiplicity of the section A^f_{k+1} around an A_{k+2} node, arising as the closure of one node and one A_k node is 2, generically.*

Proof. Since $f_{02} \neq 0$, we can find coordinates (u, v) so that the curve is given by

$$f = v^2 + A^f_{k+1}u^{k+1} + A^f_{k+2}u^{k+2} + \dots$$

The set of equations we are solving for are

$$\begin{aligned} f &= v^2 + A^f_{k+1}u^{k+1} + A^f_{k+2}u^{k+2} + \dots = 0 \\ f_u &= (k+1)A^f_{k+1}u^k + (k+2)A^f_{k+2}u^{k+1} + \dots = 0 \\ f_v &= 2v = 0 \end{aligned}$$

Solving these three equations we get

$$A^f_{k+1} = \frac{2k+4}{k+1}A^f_{k+3}u^2 + O(u^3)$$

Generically, A^f_{k+3} will not vanish. Hence

$$A^f_{k+1}(u) = O(u^2)$$

which is two to one.

Similarly, recall lemma 3.17 which states that if a curve is in the closure of one node and one D_k node and f_{12} is not zero, then the curve is at least as singular as a D_{k+2} node. We now prove the following lemma analogously

Lemma 5.2. *The multiplicity of the section D^f_{k+1} around an D_{k+2} node, arising as the closure of one node and one D_k node is 2, generically.*

Proof. Since $f_{12} \neq 0$, we can find coordinates (u, v) so that the curve is given by

$$f = v^2 u + D^f_{k+1} u^{k-1} + D^f_{k+2} u^k + \dots$$

The set of equations we are solving for are

$$\begin{aligned} f &= v^2 u + D^f_{k+1} u^{k-1} + D^f_{k+2} u^k + \dots = 0 \\ f_u &= (k-1) D^f_{k+1} u^k + (k) D^f_{k+2} u^{k-1} + \dots = 0 \\ f_v &= 2vu = 0 \end{aligned}$$

Solving these three equations we get

$$D^f_{k+1} = \frac{2k}{k+1} D^f_{k+3} u^2 + O(u^3)$$

Generically, D^f_{k+3} will not vanish. Hence

$$D^f_{k+1}(u) = O(u^2)$$

which is two to one.

5.1 Curves with one A_1-node and one A_3-node

Theorem 5.3. *The number of degree d curves with one node and one A_3-node is*

$$\mathcal{N}^d(1, A_3) = 3\mathcal{N}^d(1, A_2, 0, 1) + \mathcal{N}^d(1, A_1) + d\mathcal{N}^d(1, A_1, 1, 0) - 2\mathcal{N}^d(0, A_4).$$

Proof. This one follows from lemma 5.1. Let us see care fully what is going on. Compare with the computation of $\mathcal{N}^d(0, A_3)$. The contribution from the main stratum is

$$3\mathcal{N}^d(1, A_2, 0, 1) + \mathcal{N}^d(1, A_2) + d\mathcal{N}^d(1, A_2, 1)$$

By lemma 5.1 the contribution from the A_4-nodes is

$$2\mathcal{N}^d(0, A_4)$$

The non trivial fact is that the only thing in the closure of $\overline{\mathbb{P}\mathcal{S}^d_1(1, A_2)}$ is a A_4-node. By lemma 3.18, the singularities that can be in the closure of one node and one cusp when the Hessian is zero and when $f_{30} = 0$ is of codimension 5 or higher. Hence they will not occur (since the points are in general position). Hence the final number is

$$\mathcal{N}^d(1, A_3) = 3\mathcal{N}^d(1, A_2, 0, 1) + \mathcal{N}^d(1, A_2) + d\mathcal{N}^d(1, A_2, 1) - 2\mathcal{N}^d(0, A_4)$$

Theorem 5.4. *The number of degree d curves with one node and one tacnode with the tacnode on a lambda is*

$$\mathcal{N}^d(1, A_3, 0, 1) = 3\mathcal{N}^d(1, A_2, 0, 2) + 2\mathcal{N}^d(0, A_2, 0, 1) + 2(d-3)\mathcal{N}^d(1, A_1, 1, 1) - 2\mathcal{N}^d(0, A_4, 0, 1).$$

Proof. In analogy with the computation of $\mathcal{N}^d(1, A_2, 0, 1)$ we expect this computation to be more involved. However it is not. The answer is in fact what we "expect" it to be i.e.

$$\mathcal{N}^d(1, A_3, 0, 1) = 3\mathcal{N}^d(1, A_2, 0, 2) + \mathcal{N}^d(1, A_2, 0, 1) + d\mathcal{N}^d(1, A_2, 1, 1) - 2\mathcal{N}^d(0, A_4, 0, 1)$$

This is because the third derivative section will not vanish along a generic direction for either a A_4 node or a D_4 node.

5.2 Curves with one A_1-node and one A_4-node

Theorem 5.5. *The number of degree d curves with one node and one A_4-node is*

$$\mathcal{N}^d(1, A_4) = 2\mathcal{N}^d(1, A_3, 0, 1) + 2\mathcal{N}^d(1, A_3) + 2(d-3)\mathcal{N}^d(1, A_3, 1) - 2\mathcal{N}^d(0, A_5)$$

Proof. This one follows from lemma 5.1. The non trivial fact is that the only thing in the closure of $\overline{\mathbb{P}\mathcal{S}_1^d(A_1, A_3)}$ is a A_5-node which follows from lemma 3.19.

Theorem 5.6. *The number of degree d curves with one node and one A_4-node on a fixed lambda is*

$$\mathcal{N}^d(1, A_4, 0, 1) \quad = 2\mathcal{N}^d(1, A_3, 0, 2) + 2\mathcal{N}^d(1, A_3, 0, 1) + 2(d-3)\mathcal{N}^d(1, A_3, 1, 1) - 2\mathcal{N}^d(0, A_5, 0, 1)$$

Proof. Again this number is what we expect it to be. At the boundary, we get a one dimensional family of curves with a A_5-node. A A_5-node can degenerate into a A_6-node or a D_6-node. When we hit a one dimensional family of A_5-node with a generic lambda class, we only get D_6. However the section φ_{A_5} will not vanish on a D_6-node. That gives us the desired result.

5.3 Curves with one A_1-node and one A_5-node

Theorem 5.7. *The number of degree d curves with one A_1-node and one A_5-node is*

$$\mathcal{N}^d(1, A_5) = 3\mathcal{N}^d(1, A_4, 0, 1) + 2\mathcal{N}^d(1, A_4) + 2(d-3)\mathcal{N}^d(1, A_4, 1) - 2\mathcal{N}^d(0, A_6) - \mathcal{N}^d(0, E_6).$$

Proof. This one requires more care. The new thing that happens is that the closure of $\overline{\mathbb{P}\mathcal{S}_1^d(1, A_4)}$ contains E_6-nodes in addition to the obvious A_6-nodes. We claim that the section α_{A_5} vanishes on the E_6-nodes with a multiplicity of 1. To prove this claim we first construct a sequence in $\mathbb{P}\mathcal{S}_r^d(1, A_4)$ that converges to an E_6-node. We will merely describe the procedure to construct the sequence. We write down the Taylor expansion of f that has a A_5-node at $(0,0)$ and that also has a node at a point distinct from $(0,0)$. Hence we get three equations

$$f = 0, \quad f_x = 0, \quad f_y = 0.$$

Let us say the second node is at the point (Lt^3, t^4). Solve the equation

$$f - \frac{y f_y}{2} - \frac{x f_x}{4} = 0$$

and get

$$f_{12} = O(t)$$

Remark: Notice the way the $f_{03}y^3$ and $f_{40}x^4$ gets canceled, which is crucial.

Next solve

$$f_x = 0$$

and get

$$f_{21} = O(t^2)$$

And finally, solve

$$f_y = 0$$

and get

$$f_{02}(t) = O(t^4)$$

Since the curve already has an A_4-node we get that

$$\begin{aligned} f_{40} &= \frac{3f_{21}^2}{f_{02}} \\ &= O(1) \\ &= \text{Arbitrary, since } L \text{ is arbitrary.} \end{aligned}$$

The rest of the f_{ij} are arbitrary. Hence the equation

$$\alpha_{A_5} = O(t) = \nu$$

is one to one for small t.

5.4 Curves with one A_1-node and one A_5-node on a lambda

Theorem 5.8. *The number of degree d curves with one A_1-node and one A_5-node is*

$$\mathcal{N}^d(1, A_5, 0, 1) \quad = 3\mathcal{N}^d(1, A_4, 0, 2) + 2\mathcal{N}^d(1, A_4, 0, 1) + 2(d-3)\mathcal{N}^d(1, A_4, 1, 1) - 2\mathcal{N}^d(1, A_6, 0, 1).$$

Proof. This number is again what we "expect" it to be because the section α_{A_5} will not vanish on a A_7, D_7 or E_7 node with a generic direction assigned.

5.5 Curves with one A_1-node and one A_6-node

Theorem 5.9. *The number of degree d curves with one A_1-node and one A_6-node is*

$$\begin{aligned} \mathcal{N}^d(1, A_6) \quad &= 2\mathcal{N}^d(1, A_5, 0, 1) + 3\mathcal{N}^d(1, A_5) + (3d-12)\mathcal{N}^d(1, A_5, 1) \\ &-2\mathcal{N}^d(1, D_6) - \mathcal{N}^d(1, E_6) - 2\mathcal{N}^d(0, A_7) - 3\mathcal{N}^d(0, E_7) \end{aligned}$$

Proof. This one requires more care. The new thing that happens is that the closure of $\overline{\mathbb{P}\mathcal{S}_1^d(1, A_5)}$ contains E_7-nodes in addition to the obvious A_7-nodes. The section α_{A_6} vanishes on the E_7 nodes with a multiplicity of 3. To prove this claim we first construct a sequence in $\mathbb{P}\mathcal{S}_r^d(1, A_5)$ that converges to an E_7-node. We will merely describe the procedure to construct the sequence. We write down the Taylor expansion of f that has an A_5-node at $(0,0)$ and that also has a node at a point distinct from $(0,0)$. Hence we get three equations

$$f = 0, \quad f_x = 0, \quad f_y = 0$$

Let us say the second node is at the point $(L_1 t^2, L_2 t^3)$, where L_1 and L_2 are constants to be determined from the $f_{ij}(0)$. Let us say $f_{40}(t) = t$. Solve the equation

$$f - \frac{y f_y}{2} - \frac{x f_x}{4} = 0$$

and get

$$f_{12} = O(t)$$

Remark: Notice the way the $f_{03}y^3$ and $f_{40}x^4$ gets canceled, which is crucial.

Next solve

$$f_x = 0$$

and get

$$f_{21} = O(t^2)$$

And finally, solve

$$f_y = 0$$

and get

$$f_{02}(t) = O(t^4)$$

This gives us

$$\begin{aligned} f_{02} &= O(t^3) \qquad \text{since} \\ \alpha_{A_4} &= 0 \end{aligned}$$

Using the equations

$$\begin{aligned} f_y &= 0 \\ \alpha_{A_5} &= 0 \end{aligned}$$

we can determine the L_1 and L_2 in terms of the remaining *arbitrary* f_{ij}.
Hence the equation

$$\begin{aligned} \alpha_{A_6} &= O(t^3) \\ &= \nu \end{aligned}$$

is three to one for small t.

5.6 Curves with one A_1-node and one D_4-node

Theorem 5.10. *The number of degree d curves with one node and one D_4-node is*

$$\mathcal{N}^d(1, D_4) = \frac{1}{3}\big\{(d-6)\mathcal{N}^d(1, A_3, 1) - 2\mathcal{N}^d(1, A_3, 0, 1) + \mathcal{N}^d(1, A_3) - 2\mathcal{N}^d(0, D_5)\big\}.$$

Proof. The reader is urged to refresh his memory by going over the computation for $\mathcal{N}^d(0, D_4)$. As before we use the trick that we look at the problem in $\mathbb{P}T\mathbb{P}^2_2$ and divide by three. We consider the closure of space $\overline{\mathbb{P}\mathcal{S}_1^d(1, A_3)}$ of curves with one node and one tacnode. The condition for a triple point is $f_{02} = 0$ which is a section of the line bundle

$$L = (T\mathbb{P}^2/\tilde{\gamma})^{*^2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d}$$

Hence the contribution from the main stratum is

$$\langle e(L), \quad [\overline{\mathbb{P}\mathcal{S}_1^d(1, A_3)}] \rangle$$

To compute the contribution from the boundary, we need to first see what is in the closure of one node and one tacnode i.e. we need to know the space $\overline{\mathcal{S}_1^d(1, A_3)}$. We already know that there is an A_5-node in the closure. But the section f_{02} will not vanish there. We also have D_5-nodes in the closure where the section vanishes with a multiplicity of 2.

Theorem 5.11. *The number of degree d curves with one node and one D_4-node with a fixed lambda is*

$$\mathcal{N}^d(1, D_4, 0, 1) = \mathcal{N}^d(1, D_4, 0) + (d-9)\mathcal{N}^d(1, D_4, 1)$$

Proof. This follows from the second proof of lemma 4.15. A priori there could be contributions from a D_6 node since the closure of one node and one D_4 node is a D_6 node. However the third derivative along a generic direction will not vanish on a D_6 node.

5.7 Curves with one A_1-node and one D_5-node

Theorem 5.12. *The number of degree d curves with one node and one D_5-node is*

$$\mathcal{N}^d(1, D_5) = 4\mathcal{N}^d(1, D_4) + (4d-18)\mathcal{N}^d(1, D_4, 1, 0) - 2\mathcal{N}^d(0, D_6, 0, 0).$$

Proof. The reader is urged to refresh his memory by going over the computation for $\mathcal{N}^d(0, D_5)$. We will interpret a D_5-node as a section of

$$\tilde{\gamma}^{*2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d} \otimes T^*\mathbb{P}_2^2$$

i.e. we will be doing the computation on top of $\overline{\mathcal{S}_1^d(1, \widehat{D_4})}$. The contribution form the main stratum is

$$\langle e(\tilde{\gamma}^{*2} \otimes \gamma_{\mathcal{D}}^* \otimes \gamma_{\mathbb{P}^2}^{*^d} \otimes T^*\mathbb{P}_2^2), \quad [\overline{\mathcal{S}_1^d(1, \widehat{D_4})}] \rangle$$

The closure comprises of curves with a D_6-node which will contribute with a multiplicity of 2.

5.8 Curves with one A_1-node and one D_6-node

Theorem 5.13. *The number of degree d curves with one node and one D_6-node is*

$$\begin{aligned} \mathcal{N}^d(1, D_6) &= 4\mathcal{N}^d(1, D_5, 0, 1) + \mathcal{N}^d(1, D_5) + d\mathcal{N}^d(1, D_5, 1) \\ &\quad -2\mathcal{N}^d(0, D_7) - \mathcal{N}^d(0, E_7) \end{aligned}$$

Proof. The reader is urged to refresh his memory by going over the computation for $\mathcal{N}^d(0, D_6)$. A D_6-node is a section of the bundle

$$\tilde{\gamma}^{*4} \otimes \gamma_{\mathcal{D}}^{*} \otimes \gamma_{_{\mathbb{P}^2}}^{*^d}$$

on top of $\overline{\mathcal{S}_1^d(1, D_5)}$. The contribution form the main stratum is

$$\langle e(\tilde{\gamma}^{*4} \otimes \gamma_{\mathcal{D}}^{*} \otimes \gamma_{_{\mathbb{P}^2}}^{*^d}), \quad [\overline{\mathcal{S}_1^d(1, D_5)}] \rangle$$

The closure comprises of curves with a D_7-node and E_7-node. We claim they will contribute with a multiplicity of 2 and 1 respectively. Let us start with a D_7-node. We first write down the Taylor expansion of a function f that has a D_5-node at $(0, 0)$. We will construct a sequence $f_{ij}(t), x(t), y(t) \in \mathcal{S}_1^d(1, D_5)$ that converges to a D_7-node. The sequence is

$$x = Lt, \quad y = t^2, \quad f_{ij}(t) = \text{to be determined}$$

where L is a constant to be determined. We also have three equations

$$f = 0, \quad f_x = 0, \quad f_y = 0.$$

Using $f_x = 0$ we get that

$$f_{40}(t) = O(t^2)$$

Using that

$$\alpha_{D_7} = O(t) \qquad \text{provided } f_{12} \neq 0.$$

Finally using

$$f_y = 0$$

we get L in terms of $f_{12}(0)$ and $f_{03}(0)$. Hence the equation

$$\alpha_{D_6} = f_{40} = O(t^2) = \nu$$

is 2 to 1.

Next we consider E_7-nodes. The sequence we consider is

$$x = Lt^2, \quad y = t^3$$

The equation

$$f_y = 0$$

gives us

$$f_{12} = O(t)$$

and the equation

$$f_x = 0$$

gives us

$$f_{40} = O(t)$$

The last equation $f = 0$ gives the value of L. Hence the equation

$$\alpha_{D_6} = f_{40} = O(t) = \nu.$$

is 1 to 1.

5.9 Curves with one A_1-node and one E_6-node

Theorem 5.14. *The number of degree d curves with one node and one D_6-node is*

$$\mathcal{N}^d(1, E_6) \quad = -\mathcal{N}^d(1, D_5, 0, 1) + \mathcal{N}^d(1, D_5) + (d-6)\mathcal{N}^d(1, D_5, 1) - \mathcal{N}^d(0, E_7)$$

Proof. The reader is urged to refresh his memory by going over the computation for $\mathcal{N}^d(0, D_6)$. An E_6-node is a section of the bundle

$$\tilde{\gamma}^{*2} \otimes (T\mathbb{P}^2_2/\tilde{\gamma})^{*2} \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^{*^d}_{_{\mathbb{P}^2}}$$

on top of $\overline{\mathcal{S}^d_1(1, D_5)}$. The contribution form the main stratum is

$$\langle e(\tilde{\gamma}^{*2} \otimes (T\mathbb{P}^2_2/\tilde{\gamma})^{*2} \otimes \gamma^*_{\mathcal{D}} \otimes \gamma^{*^d}_{_{\mathbb{P}^2}}), \quad [\overline{\mathcal{S}^d_1(1, D_5)}]\rangle$$

The closure comprises of curves with a E_7-node. We claim they will contribute with a multiplicity of one. We construct the same sequence as we constructed in the computation of $\mathcal{N}^d(1, D_6)$ while finding the multiplicity around an E_7-node. Since

$$f_{12} = O(t),$$

the equation

$$\alpha_{E_6} = f_{12} = O(t) = \nu$$

is 1 to 1.

Chapter 6

Enumeration of curves with three singular points

6.1 Curves with two A_1-nodes and one A_3-node

Theorem 6.1. *The number of degree d curves with two nodes and one tacnode is*

$$\mathcal{N}^d(2, A_3) = 3\mathcal{N}^d(2, A_2, 0, 1) + \mathcal{N}^d(2, A_1) + d\mathcal{N}^d(2, A_1, 1, 0) - 4\mathcal{N}^d(1, A_4) - 2\mathcal{N}^d(0, D_5).$$

Proof. Similar to the computation of $\mathcal{N}^d(1, A_3)$, the contribution from the main stratum is

$$3\mathcal{N}^d(2, A_2, 0, 1) + \mathcal{N}^d(2, A_1) + d\mathcal{N}^d(2, A_1, 1, 0).$$

Furthermore, when one node and one cusp sink together, we get a A_4 node and the contribution to the boundary is 2 as shown in the computation of $\mathcal{N}^d(1, A_3)$. But there are two nodes that can collapse with a cusp to produce a A_4-node. hence the total contribution from $\mathcal{N}^d(1, A_4)$ is 4. The new thing that happens here is that two nodes and one cusp can collapse to a D_5-node. The multiplicity of the section around that point is 2. To see that let us consider a curve that is a union of a straight line and a curve with a cusp given by

$$\frac{f_{02}(t)}{2}y^2 + \frac{f_{30}}{6} + \ldots = (\frac{\widetilde{f_{02}}}{2}y^2 + \frac{\widetilde{f_{30}}}{6}x^3 + \ldots)(x - t)$$

Comparing coefficients we get that

$$f_{30}(t) = \widetilde{f_{30}}t$$

Hence the equation

$$\begin{aligned}\alpha_{A_3} &= \widetilde{f_{30}}t + O(t^2)\\ &= O(t)\end{aligned}$$

is generically one to one in t. But there are two nodes that can permute. Hence the total multiplicity is 2 to 1 around a D_5 node. Hence the final answer is

$$\mathcal{N}^d(2, A_3) = 3\mathcal{N}^d(2, A_2, 0, 1) + \mathcal{N}^d(2, A_1) + d\mathcal{N}^d(1, A_1, 1, 0) - 4\mathcal{N}^d(1, A_4) - 2\mathcal{N}^d(0, D_5).$$

6.2 Curves with two A_1-nodes and one A_4-node

Theorem 6.2. *The number of degree d curves with two nodes and one A_4-node is*

$$\mathcal{N}^d(2, A_4) = 2\mathcal{N}^d(2, A_3, 0, 1) + 2\mathcal{N}^d(2, A_3) + 2(d-3)\mathcal{N}^d(2, A_3, 1) - 4\mathcal{N}^d(1, A_5) - 4\mathcal{N}^d(0, D_6).$$

Proof. The new thing that happens here is that two nodes and one tacnode can collapse to a D_6-node. The multiplicity of the section around that point is 4. To see that let us consider a curve that is a union of a straight line and a curve with a tacnode given by

$$\frac{f_{02}(t)}{2}y^2 + + \ldots = (\frac{\widetilde{f_{02}}}{2}y^2 + \frac{\widetilde{f_{21}}}{2}x^2y + \frac{\widetilde{f_{12}}}{2}xy^2 + \frac{\widetilde{f_{03}}}{6}y^3 + \frac{\widetilde{f_{40}}}{24}x^4 + \ldots)(x-t)$$

Comparing coefficients we get that

$$f_{40}(t) = -t\widetilde{f_{40}}$$
$$f_{21}(t) = -t\widetilde{f_{21}}$$
$$f_{02}(t) = t\widetilde{f_{02}}$$

Hence the equation

$$\alpha_{A_4} = f_{02}f_{40} - 3f_{21}^2 = O(t^2) = \nu$$

is two to one in t. But there are two nodes that can permute. Hence the total multiplicity is 4 to one.

6.3 Curves with two A_1-nodes and one A_5-node

Theorem 6.3. *The number of degree d curves with two nodes and one A_5-node is*

$$\mathcal{N}^d(2, A_5) = 3\mathcal{N}^d(2, A_4, 0, 1) + 2\mathcal{N}^d(2, A_4) + 2(d-3)\mathcal{N}^d(2, A_4, 1)$$
$$- 4\mathcal{N}^d(1, A_6) - 2\mathcal{N}^d(1, E_6) - 4\mathcal{N}^d(0, D_7)$$

Proof. The new thing that happens here is that two nodes and one A_4-node can collapse to a D_7-node. The multiplicity of the section around that point is 4. To see that let us consider a curve that is a union of a straight line and a curve with a A_4-node given by

$$\frac{f_{02}(t)}{2}y^2 + + \ldots = (\frac{\widetilde{f_{02}}}{2}y^2 + \frac{\widetilde{f_{21}}}{2}x^2y + \frac{\widetilde{f_{12}}}{2}xy^2 + \frac{\widetilde{f_{03}}}{6}y^3 + \frac{\widetilde{f_{40}}}{24}x^4 + \ldots)(x-t)$$

Comparing coefficients we get that

$$f_{02}(t) = -t\widetilde{f_{02}}$$
$$f_{50}(t) = 5\widetilde{f_{40}} - t\widetilde{f_{50}}$$
$$f_{21}(t) = -t\widetilde{f_{21}}$$
$$f_{31}(t) = 3\widetilde{f_{21}} - t\widetilde{f_{31}}$$
$$f_{12}(t) = \widetilde{f_{02}} - t\widetilde{f_{12}}$$
$$f_{40}(t) = -t\widetilde{f_{40}}$$

Hence the equation

$$\alpha_{A_5} = O(t^2) = \nu$$

is 2 to one in t. But there are two nodes that can permute. Hence the total multiplicity is 4 to 1.

6.4 Curves with two A_1-nodes and one D_4-node

Theorem 6.4. *The number of degree d curves with two nodes and one D_4-node is*

$$\mathcal{N}^d(2, D_4) = \frac{1}{3}\big\{(d-6)\mathcal{N}^d(2, A_3, 1) - 2\mathcal{N}^d(2, A_3, 0, 1) + \mathcal{N}^d(2, A_3) - 4\mathcal{N}^d(1, D_5) - 2\mathcal{N}^d(0, D_6)\big\}$$

Proof. The new thing that happens here is that two nodes and one tacnode can collapse to a D_6-node. The multiplicity of the section around that point is 4. To see that let us consider a curve that is a union of a straight line and a curve with a tacnode given by

$$\frac{f_{02}(t)}{2}y^2 + + \ldots = (\frac{\widetilde{f_{02}}}{2}y^2 + \frac{\widetilde{f_{21}}}{2}x^2y + \frac{\widetilde{f_{12}}}{2}xy^2 + \frac{\widetilde{f_{03}}}{6}y^3 + \frac{\widetilde{f_{40}}}{24}x^4 + \ldots)(x-t)$$

Comparing coefficients we get that

$$f_{40}(t) = -t\widetilde{f_{40}}$$
$$f_{21}(t) = -t\widetilde{f_{21}}$$
$$f_{02}(t) = t\widetilde{f_{02}}$$

Hence the equation

$$\alpha_{D_4} = f_{02}(t) = O(t) = \nu$$

is one to one in t. But there are two nodes that can permute. Hence the total multiplicity is 2 to 1.

6.5 Curves with two A_1-nodes and one D_5-node

Theorem 6.5. *The number of degree d curves with two nodes and one D_5-node is*

$$\mathcal{N}^d(2, D_5) = 4\mathcal{N}^d(2, D_4) + (4d-18)\mathcal{N}^d(2, D_4, 1, 0) - 4\mathcal{N}^d(1, D_6)$$

Proof. Nothing new happens here. One dimensional family of three nodes and one triple point can not sink together.

Chapter 7

Enumeration of curves with multiple singular points

7.1 Proof of Theorem 1

Let

$$\bar{X}=\overline{\mathcal{S}}_1^d(\delta-1,A_1)\times\mathbb{P}^{2-\sigma}\,,\qquad X=\big\{([s],p_1,\ldots,p_{\delta+1})\in\bar{X}:\ p_{\delta+1}\neq p_1,p_2,\ldots,p_\delta\big\}.$$

For any subset $I\subset\{1,\ldots,\delta\}$ and $i=1,2,\ldots,\delta$, let

$$\begin{aligned}\bar{X}_I&=\big\{([s],p_1,\ldots,p_{\delta+1})\in\bar{X}:\ p_{\delta+1}=p_i\ \forall\, i\in I\big\},\quad X_I=\bar{X}_I-\bigsqcup_{I\subsetneq I'\subset\{1,\ldots,\delta\}}\bar{X}_{I'}\,,\\ \bar{X}_i&=\bar{X}_{\{i\}}\,,\qquad X_i=X_{\{i\}}\,,\qquad X^*_{\{i\}}=\big\{([s],p_1,\ldots,p_{\delta+1})\in X_i:\ \chi_s(p_i)=A_1\big\}.\end{aligned}\tag{7.1}$$

For example, $X_\emptyset=X$. By lemma 3.1, 3.14 and 3.23,

$$\begin{aligned}([s],p_1,\ldots,p_{\delta+1})&\in X_i-X^*_i &&\Longrightarrow\quad \chi_s(p_i)=A_2\,;\\ ([s],p_1,\ldots,p_{\delta+1})&\in X_I,\ i\in I,\quad |I|=2 &&\Longrightarrow\quad \chi_s(p_i)=A_3\,;\\ ([s],p_1,\ldots,p_{\delta+1})&\in X_I,\ i\in I,\quad |I|=3 &&\Longrightarrow\quad \chi_s(p_i)=D_4\end{aligned}$$

respectively. In all of the above cases, the remaining points p_i are all distinct simple nodes of $s^{-1}(0)$. Furthermore, $X_I=\emptyset$ if $|I|>3$. Let

$$\pi_0,\pi_1,\ldots,\pi_{\delta+1}:\bar{X}\longrightarrow\mathcal{D}_d,\mathbb{P}^2,\mathbb{P}^2,\ldots,\mathbb{P}^2\tag{7.2}$$

be the projection maps.

We need to determine the cardinality of the set

$$\begin{aligned}\mathcal{S}_0^d(\delta,A_1,\sigma)&=\big\{([s],p_1,\ldots,p_{\delta+1})\in\mathcal{S}_1^d(\delta-1,A_1)\times\mathbb{P}^{2-\sigma}:\ p_{\delta+1}\neq p_1,p_2,\ldots,p_\delta,\ \chi_s(p_{\delta+1})=A_1\big\}\\ &=\big\{([s],p_1,\ldots,p_{\delta+1})\in X:\ s(p_{\delta+1})=0,\ \nabla s|_{p_{\delta+1}}=0\big\};\end{aligned}$$

the last equality is a special case of Proposition 3.1. By Lemma 2.15 the restriction of the sections

$$\begin{aligned}\psi_{\delta+1;A_0}&\in\Gamma\big(\bar{X},\pi_0^*\gamma_{\mathcal{D}}^*\otimes\pi_{\delta+1}^*\gamma_{\mathbb{P}^2}^{*d}\big), &\psi_{\delta+1;A_0}\big([s],p_1,\ldots,p_{\delta+1}\big)&=s(p_{\delta+1})\,,\\ \psi_{\delta+1;A_1}&\in\Gamma\big(\psi_{\delta+1;A_0}^{-1}(0),\pi_0^*\gamma_{\mathcal{D}}^*\otimes\pi_{\delta+1}^*(\gamma_{\mathbb{P}^2}^{*d}\otimes T^*\mathbb{P}^2)\big) &\psi_{\delta+1;A_1}\big([s],p_1,\ldots,p_{\delta+1}\big)&=\nabla s|_{p_{\delta+1}}\,,\end{aligned}$$

to X are transverse to the zero set. Thus,

$$\begin{aligned}\mathcal{N}^d(\delta,A_1,\sigma) &= \big|\mathcal{S}_0^d(\delta,A_1,\sigma)\big| \\ &= \big\langle e(\pi_0^*\gamma_{\mathcal{D}}^*\otimes\pi_{\delta+1}^*\gamma_{\mathbb{P}^2}^{*d})e(\pi_0^*\gamma_{\mathcal{D}}^*\otimes\pi_{\delta+1}^*(\gamma_{\mathbb{P}^2}^{*d}\otimes T^*\mathbb{P}^2)),[\bar{X}]\big\rangle - \mathcal{C}_{\partial\bar{X}}\big(\psi_{\delta+1;A_0}\oplus\psi_{\delta+1;A_1}\big),\end{aligned} \tag{7.3}$$

where the last term is the $\psi_{\delta+1;A_0}\oplus\psi_{\delta+1;A_1}$-contribution to the Euler class from

$$\partial X = \bar{X}-X = \bigcup_{i=1}^{\delta}\bar{X}_i = \bigsqcup_{\emptyset\neq I\subset\{1,\ldots,\delta\}} X_I\,.$$

The first term on the right-side side of (7.3) gives the first term on the right-hand side of the expression in Theorem 1.

For each $i=1,2,\ldots,\delta$, a neighborhood of $\bar{X}_i$ in $\bar{X}$ can be identified via the exponential with a neighborhood of the zero section in $\pi_i^*T\mathbb{P}^2$. For any identification of the bundles,

$$\big|\{\psi_{\delta+1;A_0}\oplus\psi_{\delta+1;A_1}\}\big([s],p_1,\ldots,p_\delta,p_i;v\big) - (0,\nabla^2 s|_{p_i}(v,\cdot)\big| \le C|v|^2$$

for some $C>0$, dependent only on the identification. Since the bundle map

$$\begin{aligned}\alpha\colon \pi_i^*T\mathbb{P}^2 &\longrightarrow \pi_0^*\gamma_{\mathcal{D}}^*\otimes\pi_{\delta+1}^*\gamma_{\mathbb{P}^2}^{*d}\oplus\pi_0^*\gamma_{\mathcal{D}}^*\otimes\pi_{\delta+1}^*(\gamma_{\mathbb{P}^2}^{*d}\otimes T^*\mathbb{P}^2),\\ \big([s],p_1,\ldots,p_\delta,p_i;v\big) &\longrightarrow \big(0,\nabla s|_{p_i}(v,\cdot)\big),\end{aligned}$$

over $\bar{X}_i$ is injective over X_i^*, by [23, Proposition 2.18B] the contribution of X_i^* is the number of zeros of a generic affine bundle map with linear map α,

$$\mathcal{C}_{X_i^*}\big(\psi_{\delta+1;A_0}\oplus\psi_{\delta+1;A_1}\big) = \mathcal{N}(\alpha).$$

Since α maps to the first component,

$$\begin{aligned}\mathcal{C}_{X_i^*}\big(\psi_{\delta+1;A_0}\oplus\psi_{\delta+1;A_1}\big) = \mathcal{N}(\alpha) &= \big\langle e(\pi_0^*\gamma_{\mathcal{D}}^*\otimes\pi_{\delta+1}^*\gamma_{\mathbb{P}^2}^{*d}),[\bar{X}_i]\big\rangle \\ &= \mathcal{N}^d(\delta-1,A_1,\sigma)+d\mathcal{N}^d(\delta,A_1,\sigma+1).\end{aligned} \tag{7.4}$$

It remains to compute the contribution from the finite sets $X_i-X_i^*$ and X_I with $|I|=2,3$.

If $([s],p_1,\ldots,p_\delta,p_i)\in X_i-X_i^*$, $s^{-1}(0)$ has a cusp at p_i. Thus, we can choose coordinates (x,y) centered at p_i so that $s=y^2+x^3$. Since the section $\psi_{i;A_2}$ is transverse over X_i by Lemma 2.17, a neighborhood of $([s],p_1,\ldots,p_\delta,p_i)$ in X_i can be parametrized by $t\in\mathbb{C}$,

$$t\longrightarrow([s^t],p_1^t,\ldots,p_\delta^t,p_i^t) \qquad \text{so that} \qquad s_{xx}^t(p_i^t)=2t.$$

Thus,

$$\begin{aligned}\Big|s^t(x,y)-\sum_{i+j=2,3}f_{ij}(t)x^iy^j\Big| &\le C|t|\big(|x|^4+|y|^4\big),\\ \big|s_y^t(x,y)-\big(2f_{02}(t)y+f_{11}(t)x+f_{21}(t)x^2+2f_{12}(t)xy+3f_{03}(t)y^2\big)\big| &\le C|t|\big(|x|^3+|y|^3\big),\end{aligned} \tag{7.5}$$

for some $C\in\mathbb{R}^+$ and holomorphic functions f_{ij} on a neighborhood of the origin in $\mathbb{C}$ such that

$$f_{20}(t)=t, \qquad f_{02}(0),f_{30}(0)=1, \qquad f_{ij}(0)=0 \ \text{ otherwise}.$$

Similarly to the proof of Proposition 2.2, by (7.5) and the Implicit Function Theorem there exists a holomorphic function $B=B(t,x)$ on a neighborhood of the origin in $\mathbb{C}^2$ so that

$$B(t,0)=0, \qquad s^t_y\big(x,B(t,x)\big)=0 \ \ \forall\, x.$$

This function satisfies

$$\big|B(t,x)\big|\le C|t||x|$$

for some $C\in\mathbb{R}^+$. Let

$$\hat{y}=y-B(t,x).$$

By the definition of $\hat{y}$,

$$s^t(x,y)=h\big(t,x,\hat{y}\big)\hat{y}^2+g(t,x)$$

for some holomorphic functions h and g on neighborhoods of the origin in $\mathbb{C}^3$ and $\mathbb{C}^2$, respectively. These functions satisfy

$$h\big(0,0,0\big)=1, \qquad \big|g(t,x)-(tg_2(t)x^2+g_3(t)x^3)\big|\le C|t||x|^4$$

for some $C\in\mathbb{R}^+$ and some holomorphic functions g_2,g_3 on a neighborhood of the origin in $\mathbb{C}$ such that $g_2(0),g_3(0)=1$. Thus, after the change of variables

$$(t,x,y)\longrightarrow\big(g_2(t)g_3(t)^{-2/3}t,\ \sqrt[3]{g_3(t)}x,\ \sqrt{h(t,x,\hat{y})}\hat{y}\big),$$

we can assume that a neighborhood of $([s],p_1,\ldots,p_\delta,p_i)$ in X_i can be parametrized by $t\in\mathbb{C}$,

$$t\longrightarrow([s^t],p_1^t,\ldots,p_\delta^t,p_i^t) \qquad \text{s.t.} \qquad \big|s^t(p_i^t)-(y^2+x^3+tx^2)\big|\le C|t||x|^4 \tag{7.6}$$

for some $C\in\mathbb{R}^+$.

Let $p_i^t(u,v)=p_i^t+(u,v)\in\mathbb{C}^2$. The contribution of each point of $X_i-X_i^*$ to the Euler class in (7.3) is the number of small solutions (t,u,v) of the system

$$s^t(p_1(u,v))=\tau\nu_0, \qquad s^t_x(p_1(u,v))=\tau\nu_{10}, \qquad s^t_y(p_1(u,v))=\tau\nu_{01}, \tag{7.7}$$

for a generic choice of $(\nu_0,\nu_{10},\nu_{01})\in\mathbb{C}^3$ and $\tau\in\mathbb{R}^+$ sufficiently small. By (7.6), the last equation in (7.7) is just

$$2v=\tau\nu_{01}\,;$$

it has a unique solution. By (7.6),

$$\begin{aligned}\Big|\big(s^t(p_1(u,v))-2us^t_x(p_1(u,v))-2vs^t_y(p_1(u,v)),us^t_x(p_1(u,v))\big)-(u^3,3u^3+tu^3)\Big|\\ \le C|t|\big|(u^3,3u^3+tu^3)\big|.\end{aligned}$$

Thus, by a rescaling and cobordism argument as in [21, Section 3.1], the number of small solutions (t,u) of the first two equations in (7.7) with $2v=\tau\nu_{01}$ is the number of solutions (t,u) of the system

$$u^3=\nu_0, \qquad 2tu^2+3u^3=0,$$

for a generic choice of $\nu_0\in\mathbb{C}$. Since this number is clearly 3,

$$\mathcal{C}_{X_i-X_i^*}\big(\psi_{\delta+1;A_0}\oplus\psi_{\delta+1;A_1}\big)=3\big|X_i-X_i^*\big|=3\mathcal{N}^d(\delta-1,A_2,\sigma). \tag{7.8}$$

We next compute the contribution of each element $([s],p_1,\ldots,p_\delta,p_{\delta+1})\in X_I$ with $|I|=2$. We can assume that $I=\{1,2\}$. By lemma 3.14, $p_1=p_2=p_{\delta+1}$ is a tacnode of $s^{-1}(0)$. Thus, we can choose coordinates (x,y) centered at p_i so that $s=y^2+x^4$. We will first describe a neighborhood of

$$\tilde{p}\equiv([s],p_1,\ldots,p_\delta,p_1,\mathbb{C}\oplus 0)\in\mathbb{P}(\pi_1^*T\mathbb{P}^2)$$

inside of the one-dimensional space $\tilde{X}$ of curves with δ nodes and a choice of a branch at the first node, which we can take to be the x-axis.

Since the sections $\psi_{1;A_2}$ and $\psi_{1;A_3}$ are transverse over X_I by lemma 2.17 and 2.14, a neighborhood of $\tilde{p}$ in $\tilde{X}_I$ can be parametrized by $t=(t_1,t_2)\in\mathbb{C}^2$,

$$t\longrightarrow([s^t],p_1^t,\ldots,p_\delta^t,p_i^t)\qquad\text{s.t.}\qquad s^t_{xx}(p_1^t)=0,\quad s^t_{xy}(p_1^t)=2t_1,\quad s^t_{xxx}(p_1^t)=6t_2.$$

Thus,

$$\begin{aligned}\Big|s^t(x,y)-\sum_{i+j=2,3,4}f_{ij}(t)x^iy^j\Big|&\le C|t|\big(|x|^5+|y|^5\big),\\ \Big|s^t_y(x,y)-\sum_{i+j=2,3,4}f_{ij}(t)jx^iy^{j-1}\Big|&\le C|t|\big(|x|^4+|y|^4\big),\end{aligned}\tag{7.9}$$

for some $C\in\mathbb{R}^+$ and holomorphic functions f_{ij} on a neighborhood of the origin in $\mathbb{C}$ such that

$$f_{20}=0,\quad f_{11}(t)=2t_1,\quad f_{30}(t)=t_2,\quad f_{02}(0),f_{40}(0)=1,\qquad f_{ij}(0)=0\ \text{ otherwise.}$$

By replacing $\sqrt{f_{02}(t)}y$ with y above, we can assume that $f_{02}=1$ above. Similarly to the proof of Proposition 2.2, by (7.9) and the Implicit Function Theorem there exists a holomorphic function $B=B(t,x)$ on a neighborhood of the origin in $\mathbb{C}^2$ so that

$$B(t,0)=0,\qquad s^t_y\big(x,B(t,x)\big)=0\ \ \forall\,x.$$

This function satisfies

$$\big|B(t,x)+t_1x\big|\le C|t||x|^2$$

for some $C\in\mathbb{R}^+$. Let

$$\hat{y}=y-B(t,x).$$

By the definition of $\hat{y}$,

$$s^t(x,y)=h\big(t,x,\hat{y}\big)\hat{y}^2+g(t,x)$$

for some holomorphic functions h and g on neighborhoods of the origin in $\mathbb{C}^3$ and $\mathbb{C}^2$, respectively. These functions satisfy

$$h\big(0,0,0\big)=1,\qquad \big|g(t,x)-(-t_1^2x^2+g_3(t)x^3+g_4(t)x^4)\big|\le C|t||x|^5$$

for some $C\in\mathbb{R}^+$ and some holomorphic functions g_3,g_4 on a neighborhood of the origin in $\mathbb{C}$ such that

$$g_4(0)=1,\qquad \big|g_3(t)-t_2\big|\le C|t|^2.$$

Thus, after the change of variables

$$(t_1,t_2,x,y) \longrightarrow \big(-\mathfrak{i}t_1g_4(t)^{-1/4}, t_2g_3(t)g_4(t)^{-3/4}, \sqrt[4]{g_4(t)}x, \sqrt{h(t,x,\hat{y})}\hat{y}\big),$$

we can assume that a neighborhood of $([s],p_1,\ldots,p_\delta,p_i)$ in X_i can be parametrized by $(t_1,t_2)\in\mathbb{C}$,

$$\begin{aligned} t=(t_1,t_2) \longrightarrow \big([s^t],p_1^t,\ldots,p_\delta^t,p_1^t,\{y=\mathfrak{i}t_1x\}\big) \qquad \text{s.t.} \\ \big|s^t(p_i^t)-(y^2+x^4+t_1^2x^2+t_2x^3)\big| \le C|t||x|^5 \end{aligned} \tag{7.10}$$

for some $C\in\mathbb{R}^+$.

Let $p_1^t(u,v)=p_1+u+v\in\mathbb{C}^2$. The complement of $\tilde{p}$ in a small neighnorhood in $\tilde{X}$ is the set of small solutions (t,x,y) of the system

$$s^t(p_1^t(x,y))=0, \qquad s_x^t(p_1^t(x,y))=0, \qquad s_y^t(p_1^t(x,y))=0 \tag{7.11}$$

with $(x,y)\neq 0$. By (7.10), the last equation is equivalent to $y=0$. Since

$$\big|x^{-3}\big(2xs_x^t(p_1^t(x,0))-s^t(p_1^t(x,y))-(2x^4+t_2x^3)\big)\big| \le C|t||x|^2,$$

by the Implicit Function Theorem the equation

$$x^{-3}\big(2xs_x^t(p_1^t(x,0))-s^t(p_1^t(x,y)\big)=0$$

has a unique small solution $x=x(t)$; it satisfies

$$|2x+t_2| \le C|t||t_2|^2.$$

The first equation in (7.11) is then equivalent to

$$t_1^2-\frac{t_2^2}{4}h(t)=0$$

for some holomorphic function h on a neighborhood of the origin in $\mathbb{C}$ such that

$$\big|h(t)-1\big| \le C|t||t_2|.$$

In particular, the first equation in (7.11) has two families of solutions, each parameterized by t_1. We conclude that a neighborhood of $\tilde{p}$ in $\tilde{X}$ consists of two copies of $\mathbb{C}$, each parametrized by $t\in\mathbb{C}$,

$$\begin{aligned} t \longrightarrow \big([s^t],p_1^t,\ldots,p_\delta^t,p_1^t,\{y=2\mathfrak{i}tx\}\big) \qquad \text{s.t.} \\ \big|s^t(p_i^t)-(y^2+x^4+4t^2x^2\pm tx^3)\big| \le C\big(|t||x|^5+|t|^3|x|^3\big) \end{aligned} \tag{7.12}$$

for some $C\in\mathbb{R}^+$. A neighborhood of p in $\bar{X}$ is parametrized by either of the two copies of $\mathbb{C}$.

The contribution of each point of X_I to the Euler class in (7.3) is the number of small solutions (t,u,v) of the system

$$s^t(p_1^t(u,v))=\tau\nu_0, \qquad s_x^t(p_1^t(u,v))=\tau\nu_{10}, \qquad s_y^t(p_1^t(u,v))=\tau\nu_{01}. \tag{7.13}$$

for a generic choice of $(\nu_0,\nu_{10},\nu_{01})\in\mathbb{C}^3$ and $\tau\in\mathbb{R}^+$ sufficiently small. By (7.12), the last equation in (7.13) is just

$$2v=\tau\nu_{01}\,;$$

it has a unique solution. Let

$$\alpha_0(t,u)=u^2(u^2+4t^2+tu), \qquad \alpha_{10}(t,u)=u^2(4u^2+8t^2+3tu).$$

Since the second factors in the two expressions have no common factors,

$$|u|^2+|t|^2\le C\big|(\alpha_0(t,u),\alpha_{10}(t,u))\big|.$$

Thus, by (7.10),

$$\big|\big(s^t(p_1(u,0)),us^t_x(p_1(u,0))\big)-(\alpha_0(t,u),\alpha_{10}(t,u))\big|\le C|t|\big|(\alpha_0(t,u),\alpha_{10}(t,u))\big|.$$

Thus, by a rescaling and cobordism argument as in [21, Section 3.1], the number of small solutions (t,u) of the first two equations in (7.13) with $2v=\tau\nu_{01}$ is the number of solutions (t,u) of the system

$$\alpha_0(t,u)=\nu_0, \qquad \alpha_{10}(t,u)=0,$$

for a generic choice of $\nu_0\in\mathbb{C}$. Dividing the second equation by u^2 and then factoring it, we find that it has two solutions $u=u(t_2)$ for each value of t. Thus, the total number of solutions of this system and the system (7.13) is 4. We conclude that

$$\mathcal{C}_{X_I}\big(\psi_{\delta+1;A_0}\oplus\psi_{\delta+1;A_1}\big)=4\big|X_I\big|=4\,\mathcal{N}^d(\delta-2,A_3,\sigma). \tag{7.14}$$

Finally, we compute the contribution of each element $([s],p_1,\ldots,p_\delta,p_{\delta+1})\in X_I$ with $|I|=3$. We can assume that $I=\{1,2,3\}$. By lemma 3.23, $p_1=p_2=p_3=p_{\delta+1}$ is a D_4-node of $s^{-1}(0)$. Thus, we can choose coordinates (x,y) centered at p_1 so that $s=x^2y+xy^2$. We will first describe a neighborhood of

$$\tilde{p}\equiv([s],p_1,\ldots,p_\delta,p_{\delta+1},\mathbb{C}\oplus 0)\in\mathbb{P}(\pi_1^*T\mathbb{P}^2)$$

inside of the one-dimensional space $\tilde{X}$ of curves with δ nodes and a choice of a branch at the first node, which we can take to be the x-axis.

Since the sections $\psi_{1;A_2}$, $\psi_{1;A_3}$, and $\psi_{1;\tilde{D}_4}$, are transverse over X_I by Lemma **??**, a neighborhood of $\tilde{p}$ in $\tilde{X}_I$ can be parametrized by $t=(t_1,t_2,t_3)\in\mathbb{C}^3$,

$$t\longrightarrow([s^t],p_1^t,\ldots,p_\delta^t,p_1^t) \qquad \text{so that}$$
$$s^t_{xy}(p_1^t)=t_1, \qquad s^t_{yy}(p_1^t)=2t_2, \qquad s^t_{xxx}(p_1^t)=6t_3.$$

Thus,

$$\big|s^t(x,y)-\big(t_1xy+t_2y^2+t_3x^3+f_{21}(t)x^2y+f_{12}(t)xy^2+f_{03}(t)y^3\big)\big|\le C|t|\big(|x|^4+|y|^4\big)$$

for some $C\in\mathbb{C}$ and holomorphic functions f_{21},f_{12},f_{03} on a neighborhood of the origin such that

$$f_{21}(0),f_{12}(0)=1, \qquad f_{03}(0)=0.$$

We first change the coordinates in order to turn these three functions into constants. Since the polynomial x^2y+xy^2 has three distinct factors, there exist functions g_{21},g_{12},g_{03} on a neighborhood of the origin in $\mathbb{C}^3$ so that

$$g_{21}(0),g_{12}(0)=1, \qquad g_{03}(0)=0,$$
$$t_3x^3+f_{21}(t)x^2y+f_{12}(t)xy^2+f_{03}(t)y^3$$
$$=\big(x+g_{03}(t)y\big)\big(t_3(x+g_{03}(t)y)^2+(x+g_{03}(t)y+g_{12}(t)y)g_{21}(t)y\big).$$

We make the change of variables

$$\tau_1 = \frac{g_{12}(t)}{g_{21}(t)}t_1, \quad \tau_2 = \frac{g_{12}(t)}{g_{21}(t)}t_2, \quad \tau_3 = t_3, \quad u = x + g_{03}(t)y, \quad v = g_{12}(t)y, \quad s^\tau = \frac{g_{12}(t)}{g_{21}(t)}s^t.$$

By the above, a neighborhood of $\tilde{p}$ in $\tilde{X}_I$ can be parametrized by $t=(t_1,t_2,t_3)\in\mathbb{C}^3$,

$$t \longrightarrow ([s^t], p_1^t, \ldots, p_\delta^t, p_i^t) \qquad \text{so that}$$
$$s_{xy}^t(p_1^t) = t_1, \qquad s_{yy}^t(p_1^t) = 2t_2, \qquad s_{xxx}^t(p_1^t) = 6t_3, \qquad s_{xxy}^t, s_{xyy}^t = 2, \; s_{yyy}^t = 0.$$

Let

$$p_1^t(u,v) = p_1^t + (u,v) \in \mathbb{C}^2, \qquad \alpha_0(t,u) = t_1uv + t_2v^2,$$
$$\alpha_{10}(t,u,v) = t_1v + 3t_3u^2 + 2uv + v^2, \qquad \alpha_{01}(t,u,v) = t_1u + 2t_2v + u^2 + 2uv.$$

In particular,

$$\big|\big(3s^t(p_1^t(u,v)) - us_u^t(p_1^t(u,v)) - vs_v^t(p_1^t(u,v))\big) - \alpha_0(t,u,v)\big| \le C|t|\big(|u|^4 + |v|^4\big), \tag{7.15}$$
$$\big|s_u^t(p_1^t(u,v)) - \alpha_{10}(t,u,v)\big| \le C|t|\big(|u|^3 + |v|^3\big), \tag{7.16}$$
$$\big|s_v^t(p_1^t(u,v)) - \alpha_{01}(t,u,v)\big| \le C|t|\big(|u|^3 + |v|^3\big) \tag{7.17}$$

for some $C\in\mathbb{R}^+$. The intersection of a neighborhood of $\tilde{p}$ with the main stratum of $\tilde{X}$ (where all the nodes are distinct) is the set of small solutions (t, x_2, y_2, x_3, y_3) of the system of 6 equations,

$$s^t(p_1^t(x_i,y_i)) = 0, \quad s_u^t(p_1^t(x_i,y_i)) = 0, \quad s_v^t(p_1^t(x_i,y_i)) = 0, \qquad i = 2,3, \tag{7.18}$$

with $(x_i, y_i) \neq (0,0)$ and $(x_2, y_2) \neq (x_3, y_3)$.

We first show neither of the two triples of equations has such a solution with $t_1 = 0$. Suppose (t, x_i, y_i) is such a solution. If $y_i = z_ix_i$ with $|z_i| \le 1$ and thus $x_i \neq 0$, (7.16) gives

$$\big|(z_i+2)z_i\big| \le C|t| \qquad \Longrightarrow \qquad |z_i| \le C|t|.$$

Combining this with (7.17) and then with (7.15), we obtain

$$|x_i| \le C|t_2||z_i| \qquad \Longrightarrow \qquad |t_2z_i^2| \le C|t||x_i|^2 \le C|t||t_2| \cdot |t_2z_i^2|$$
$$\Longrightarrow \qquad t_2z_i = 0 \qquad \Longrightarrow \qquad x_i = 0,$$

which is impossible. If $x_i = z_iy_i$ with $|z_i| \le 1$ and thus $y_i \neq 0$, (7.15) and (7.17) give

$$|t_2| \le C|t||y_i|^2 \qquad \Longrightarrow \qquad |z_i| \le C|t||y_i|.$$

However, by (7.16),

$$|2z_i + 1| \le C|t|,$$

which contradicts the previous conclusion if t is sufficiently small.

Suppose next that (t, x_i, y_i) is a solution of one of the triples of equations with $y_i = z_i x_i$ and $|z_i| \le 4$. Since $x_i \neq 0$ in this case, (7.15)-(7.17) give

$$\big|t_1 z_i + t_2 z_i^2\big| \le C|t||x_i|^2, \tag{7.19}$$

$$\big|t_1 z_i + 3t_3 x_i + 2x_i z_i + x_i z_i^2\big| \le C|t||x_i|^2, \tag{7.20}$$

$$\big|t_1 + 2t_2 z_i + x_i + 2x_i z_i\big| \le C|t||x_i|^2\,. \tag{7.21}$$

If in addition $4|z_i| \ge 1$, these inequalities give

$$\left|\begin{pmatrix} 1 & z_i & 0 \\ 1 & 0 & 2{+}z_i \\ 1 & 2z_i & 1{+}2z_i \end{pmatrix}\begin{pmatrix} t_1 \\ t_2 \\ x_i \end{pmatrix}\right| \le C|t||x_i|.$$

Computing the determinant of the above matrix, we find that this implies that $|1{+}z_i| \le C|t|$, provided t is sufficiently small. By (7.19) and (7.21),

$$\big|-t_1 + (1{+}2z_i)x_i\big| \le C|t||x_i|^2. \tag{7.22}$$

Thus, by (7.15), (7.17), and the Implicit Function Theorem, the equation

$$x_i^{-1} s_y^t(p_1^t(x_i, z_i x_i)) - 2z_i^{-1} x_i^{-2}\big(3s^t(p_1^t(x_i, z_i x_i)) - x_i s_x^t(p_1^t(x_i, z_i x_i)) - z_i x_i s_y^t(p_1^t(x_i, z_i x_i))\big) = 0$$

with $|1{+}z_i| \le C|t|$ has a unique small solution $x_i = x_i(t, z_i)$; it satisfies $|x_i| \le C|t_1|$. By (7.20) and (7.22),

$$\big|3t_3 + 3(1{+}z_i)z_i\big| \le C|t||x_i|.$$

Thus, by (7.15)-(7.17) and the Implicit Function Theorem, the equation

$$x_i^{-2}\big(x_i s_x^t(p_1^t(x_i, z_i x_i)) + z_i x_i s_y^t(p_1^t(x_i, z_i x_i)) - 2s^t(p_1^t(x_i, z_i x_i))\big) = 0$$

with $x_i = x_i(t, z_i)$ has a unique solution $z_i = z_i(t)$ with $1{+}z_i$ small; it satisfies

$$\big|(1{+}z_i) - t_3\big| \le C|t|^2.$$

Finally, by (7.15) and the Implicit Function Theorem, the equation

$$z_i^{-1} x_i^{-2}\big(3s^t(p_1^t(x_i, z_i x_i)) - x_i s_x^t(p_1^t(x_i, z_i x_i)) - z_i x_i s_y^t(p_1^t(x_i, z_i x_i))\big) = 0$$

with $x_i = x_i(t, z_i)$ and $z_i = z_i(t)$ has a unique small solution $t_2 = t_2(t_1, t_3)$; it satisfies

$$|t_2 - t_1| \le C\big(|t_1|^2 + |t_3|\big)|t_1|. \tag{7.23}$$

We next consider the case with $y_i = z_i x_i$, $4|z_i| \le 1$, $z_i = u_i x_i$, and $|u_i| \le 1$. Since $x_i \neq 0$,

$$\big|t_1 u_i + t_2 u_i^2 x_i\big| \le C|t||x_i|, \tag{7.24}$$

$$\big|t_1 u_i + 3t_3 + 2u_i x_i + u_i^2 x_i^2\big| \le C|t||x_i|, \tag{7.25}$$

$$\big|t_1 + (1{+}2t_2 u_1{+}2u_i x_i)x_i\big| \le C|t||x_i|^2\,. \tag{7.26}$$

By (7.17), (7.26), and the Implicit Function Theorem, the equation

$$x_i^{-1}s_y^t(p_1^t(x_i,u_ix_i^2))=0$$

has a unique small solution $x_i=x_i(t,u_i)$; it satisfies $|x_i|\le C|t_1|$. Thus, (7.15), (7.24), and the Implicit Function Theorem, the equation

$$t_1^{-1}x_i^{-3}\big(3s^t(p_1^t(x_i,z_ix_i))-x_is_x^t(p_1^t(x_i,u_ix_i^2))-u_ix_i^2s_y^t(p_1^t(x_i,u_ix_i^2))\big)=0$$

with $x_i\!=\!x_i(t,u_i)$ has a unique small solution $u_i\!=\!u_i(t)$; it satisfies $|u_i|\le C|t|$. Finally, by (7.17), (7.26), and the Implicit Function Theorem, the equation

$$x_2^{-1}s_x^t(p_1^t(x_i,u_ix_i^2))=0$$

with $x_i\!=\!x_i(t,u_i)$ and $u_i\!=\!u_i(t)$ has a unique small solution $t_3\!=\!t_3(t_1,t_2)$; it satisfies

$$|t_3|\le C\big(|t_1|+|t_2|\big)|t_1|. \tag{7.27}$$

Suppose next that $y_i\!=\!z_ix_i$, $4|z_i|\!\le\!1$, $x_i\!=\!u_iz_i$, and $|u_i|\!\le\!1$. Since $x_i\!\neq\!0$, by (7.19)-(7.21)

$$\big|t_1+t_2z_i\big|\le C|t||u_i|^2|z_i|, \tag{7.28}$$
$$\big|t_1+3t_3u_i+2u_iz_i+u_iz_i^2\big|\le C|t||u_i|^2|z_i|, \tag{7.29}$$
$$\big|t_1+2t_2z_i+u_iz_i+2u_iz_i^2\big|\le C|t||u_i|^2|z_i|\,. \tag{7.30}$$

Since

$$u_i^{-1}x_i^{-2}\big(x_is_x^t(p_1^t(x_i,z_ix_i))+z_ix_is_y^t(p_1^t(x_i,z_ix_i))-2s^t(p_1^t(x_i,z_ix_i))\big)=0,$$

(7.15)-(7.17) and (7.28)-(7.30) give

$$\big|1+4z_i+z_i^2\big|\le C|t|.$$

Since $4|z_i|\!\le\!1$, this is impossible if t is sufficiently small.

Finally, suppose $x_i\!=\!z_iy_i$ with $4|z_i|\!\le\!1$. Since $y_i\!\neq\!0$, (7.15)-(7.17) give

$$\big|t_1z_i+t_2\big|\le C|t||y_i|^2, \tag{7.31}$$
$$\big|t_1+3t_3z_i^2y_i+2z_iy_i+y_i\big|\le C|t||y_i|^2, \tag{7.32}$$
$$\big|t_1z_i+2t_2+z_i^2y_i+2z_iy_i\big|\le C|t||y_i|^2\,. \tag{7.33}$$

By (7.16), (7.32), and the Implicit Function Theorem, the equation

$$y_i^{-1}s_x^t(p_1^t(u_iy_i,y_i))=0$$

has a unique small solution $y_i=y_i(t,u_i)$; it satisfies $|y_i|\le C|t_1|$. Thus, by (7.15)-(7.17), (7.31)-(7.33), and the Implicit Function Theorem, the equation

$$y_i^{-3}\big(z_iy_is_x^t(p_1^t(z_iy_i,y_i))+y_is_y^t(p_1^t(z_iy_i,y_i))-2s^t(p_1^t(z_iy_i,y_i))\big)=0$$

with $y_i = y_i(t, z_i)$ has a unique solution $z_i = z_i(t)$ with $4|z_i| \le 1$; it satisfies $|z_i| \le C|t||t_1|$. Finally, by (7.15), (7.31), and the Implicit Function Theorem, the equation

$$y_i^{-2}\big(3s^t(p_1^t(z_iy_i, y_i)) - z_iy_is_x^t(p_1^t(z_iy_i, y_i)) - y_is_y^t(p_1^t(z_iy_i, y_i))\big) = 0$$

with $y_i = y_i(t, z_i)$ and $z_i = z_i(t)$ has a unique solution $t_2 = t_2(t_1, t_3)$; it satisfies

$$|t_2| \le C\big(|t_1| + |t_3|\big)|t_1|^2. \tag{7.34}$$

In summary, there are 3 possible types of solutions (t, x_i, y_i) for each of the two triples of equations (7.18). In each case, (x_i, y_i) is determined by the values of t. Since $(x_2, y_2) \ne (x_3, y_3)$, (x_2, y_2) and (x_3, y_3) are of two different types. By (7.23) and (7.34), the two corresponding types of pairs are not compatible. Thus, (x_2, y_2) must be of the type corresponding to (7.27), while (x_3, y_3) of the type corresponding to (7.23) or (7.34), or vice versa. If $t_3 = t_3(t_1, t_2)$ as in (7.27), it remains to solve the equation

$$t_2 = t_2\big(t_1, t_3(t_1, t_2)\big),$$

where $t_2 = t_2(t_1, t_3)$ is as in (7.23) and (7.34), respectively. By the Implicit Function Theorem, this has equation has a unique small solution $t_2 = t_2(t_1)$ in either case; it satisfies

$$\big|t_2 - t_1\big| \le C|t_1|^3 \qquad \text{and} \qquad \big|t_2\big| \le C|t_1|^3 \tag{7.35}$$

in the two respective cases. We conclude that the intersection of a neighborhood of $\tilde{p}$ with the main stratum of $\tilde{X}$ is 4 copies of a punctured disk inside of a neighborhood of $\tilde{p}$ in $\tilde{X}$ (which is isomorphic to $\mathbb{C}^3$). Since there are 3 choices of a branch at a D_4-node and 2 at a simple node, the intersection of a neighborhood of a D_4-node with X is 6 copies of a punctured disk.

It remains to determine the number of solutions of the system of equations

$$\begin{gathered} \tilde{s}^t(p_1^t(u,v)) \equiv 3s^t(p_1^t(u,v)) - s_x^t(p_1^t(u,v)) - ys_y^t(p_1^t(u,v)) = \tau\nu_0, \\ s_x^t(p_1^t(u,v)) = \tau\nu_{10}, \qquad s_y^t(p_1^t(u,v)) = \tau\nu_{01} \end{gathered} \tag{7.36}$$

for a generic choice of $\nu_0, \nu_{10}, \nu_{01} \in \mathbb{C}$, $\tau \in \mathbb{R}^+$ sufficiently small,

$$t = \big(t_1, t_2(t_1), t_3(t_1, t_2(t_1))\big),$$

and for each of the copies of a punctured disk around p. Let

$$\psi(t_1, u, v) = \big(\tilde{s}^t(p_1^t(u,v)), s_x^t(p_1^t(u,v)), s_y^t(p_1^t(u,v))\big).$$

For a punctured disk for which the second inequality in (7.35) holds, let

$$\alpha(t_1, u, v) = \big(t_1uv, v(t_1 + 2u + v), u(t_1 + u + 2v)\big),$$
$$Y_\alpha = \alpha^{-1}(0), Z_\alpha = \big\{\tau\psi(t_1, u, v) \colon (t_1, u, v) \in Y_\alpha,\ \tau \in \mathbb{C}\big\} \cup 0 \times \mathbb{C}^2.$$

Since Y_α consists of four lines through the origin, the closure $\bar{Z}_\alpha \subset \mathbb{C}^3$ of Z_α is an algebraic subvariety of dimension at most 2. We show below that for a generic choice of

$$\nu \equiv (\nu_0, \nu_{01}, \nu_{10}) \in \mathbb{C}^3 - \bar{Z}_\alpha,$$

there exists δ_ν such that the number of solutions of the system (7.36) with $|t_1|, |u|, |v|, \tau < \delta_\nu$ is the cardinality of the set $\alpha^{-1}(\nu_0, 0, 0)$; the latter is easily seen to be 3.

For any $r \in \mathbb{R}^+$, let

$$B_r = \big\{(t_1, u, v) \in \mathbb{C}^3 \colon\ |(t_1, u, v)| \le r\big\}.$$

It can be assumed that $\alpha^{-1}(\nu_0, 0, 0) \subset B_{1/2}$. Choose a precompact neighborhood

$$K_\nu \subset (\mathbb{C}^3 - Y_\alpha) \cap B_{3/4}$$

of $\alpha^{-1}(\nu_0, 0, 0)$ and let

$$m_\nu = \min\big\{\big|\alpha(t_1, u, v) - (\nu_0, 0, 0)\big| \colon\ (t_1, u, v) \in B_1 - K_\nu\big\} > 0. \tag{7.37}$$

If $\tau \in \mathbb{R}^+$, let

$$\begin{aligned}
t_\tau &= \big(\tau^{1/3} t_1, t_2(\tau^{1/3} t_1), t_3(\tau^{1/3} t_1, t_2(\tau^{1/3} t_1))\big),\\
\psi_\tau(t_1, u, v) &= \big(\tau^{-1} \tilde{s}^{t_\tau}(p_1^{t_\tau}(\tau^{1/3} u, \tau^{1/3} v)), \tau^{-2/3} s_x^{t_\tau}(p_1^{t_\tau}(\tau^{1/3} u, \tau^{1/3} v)), \tau^{-2/3} s_y^{t_\tau}(p_1^{t_\tau}(\tau^{1/3} u, \tau^{1/3} v))\big),\\
\varepsilon_\tau(t_1, u, v) &= \psi_\tau(t_1, u, v) - \alpha(t_1, u, v), \qquad \nu_\tau = \big(\nu_0, \tau^{1/3} \nu_{10}, \tau^{1/3} \nu_{01}\big).
\end{aligned}$$

By (7.15)-(7.15), (7.27), and (7.35),

$$\begin{aligned}
\big|\psi(t_1, u, v) - \alpha(t_1, u, v)\big| &\le C_\psi |t| \big(|u|^3 + |v|^3 + |t||u|^2 + |t|^2 |v|\big)\\
&\le 8 C_\psi \big(|t|^4 + |u|^4 + |v|^4\big) \le 8 C_\psi \big|(t_1, u, v)\big|^4.
\end{aligned} \tag{7.38}$$

For any precompact open subset $\mathcal{K} \subset \mathbb{C}^3 - \bar{Z}_\alpha$, $\mathbb{C}^* \bar{\mathcal{K}} \subset \mathbb{C}^3 - \bar{Z}_\alpha$ is closed in $\mathbb{C}^3 - 0$. Since the proper transform of $\psi^{-1}(\mathbb{C}^* \bar{\mathcal{K}})$ in the blowup of $\mathbb{C}^3$ at the origin is disjoint from the proper transform of Y_α, the closure of

$$S_{\mathcal{K}} \equiv \left\{ \frac{(t_1, u, v)}{|(t_1, u, v)|} \colon\ (t_1, u, v) \in \bar{K}_\nu \cup \psi^{-1}(\mathbb{C}^* \bar{\mathcal{K}}) \cap B_1 \right\}$$

is disjoint from Y_α. Thus, there exists $C_{\mathcal{K}} \in \mathbb{R}^+$ such that

$$|(t_1, u, v)|^3 \le C_{\mathcal{K}} \big|\alpha(t_1, u, v)\big| \qquad \forall\ (t_1, u, v) \in \mathbb{C}^* S_{\mathcal{K}} \cap B_1. \tag{7.39}$$

Since $C_{\mathcal{K}}$ depends only on $\mathbb{C}^* \mathcal{K}$, it can be assumed that $3(C_{\mathcal{K}} |\nu|)^{1/3} \le 1$. Let

$$\delta_{\mathcal{K}} = \frac{1}{16 C_\psi C_{\mathcal{K}}}.$$

By (7.38) and (7.39),

$$|(t_1, u, v)|^3 \le C_{\mathcal{K}} \big|\alpha(t_1, u, v)\big| \le 2 C_{\mathcal{K}} |\nu| \tau \quad \forall\ (t_1, u, v) \in \psi^{-1}(\tau\nu) \cap B_{\delta_{\mathcal{K}}}$$

for any $\nu \in \mathcal{K}$ and $\tau \in \mathbb{R}^+$. Since

$$\psi(t_1, u, v) = \tau\nu \qquad \Longleftrightarrow \qquad \psi_\tau\big(\tau^{-1/3} t_1, \tau^{-1/3} u, \tau^{-1/3} v\big) = \big(\nu_0, \tau^{1/3} \nu_{10}, \tau^{1/3} \nu_{01}\big),$$

it follows that

$$\big\{(t_1, u, v) \in B_{\tau^{-1/3} \delta_{\mathcal{K}}} \colon\ \psi_\tau(t_1, u, v) = \nu_\tau\big\} \subset B_{2(C_{\mathcal{K},\nu} |\nu|)^{1/3}} \subset B_{2/3}.$$

By (7.38),

$$\big|\varepsilon_\tau(t_1,u,v)\big| \le 8C_\psi \tau^{1/3}\big|(t_1,u,v)\big|^4 \qquad \forall\ (t_1,u,v)\in B_{\tau^{-1/3}}\,. \tag{7.40}$$

For any smooth map $\phi\colon I\times B_1\longrightarrow\mathbb{C}^3$, where $I=[0,1]$, and $\tau\in\mathbb{R}$, define

$$\Psi_{\tau,\phi}\colon I\times B_1\longrightarrow\mathbb{C}^3 \qquad \text{by}$$

$$\Psi_{\tau,\phi}(\eta,t_1,u,v)=\alpha(t_1,u,v)+\eta\varepsilon_\tau(t_1,u,v)-\big(\nu_0,\eta\tau^{1/3}\nu_{10},\eta\tau^{1/3}\nu_{01}\big)-\phi(\eta,t_1,u,v).$$

If $(\max\phi), |\nu|\tau^{1/3}, 8C_\psi|\tau|^{1/3} < m_\nu/3$,

$$\Psi_{\tau,\phi}^{-1}(0)\subset I\times K_\nu$$

by (7.40) and (7.37). Thus, if ν_τ is a regular value for ψ_τ, for a generic choice of small ϕ vanishing on $\{0,1\}\times B_1$, $\Psi_{\tau,\phi}^{-1}(0)$ is a cobordism between $\psi_\tau^{-1}(\nu_\tau)$ and $\alpha^{-1}(\nu_0,0,0)$. It follows that the signed cardinalities of the sets $\psi^{-1}(\tau\nu)$ and $\alpha^{-1}(\nu_0,0,0)$ are the same for all $\tau\in\mathbb{R}^+$ sufficiently small, as claimed.

For a punctured disk for which the first inequality in (7.35) holds, we replace the variables (u,v) by $(x,y)=(u+v,-v)$ and the (s,s_x,s_y)-equations by the $(-s,s_x,s_y-s_x)$ equations. This reduces the problem of finding the number of solutions of (7.36) to the case just considered, and so the number of solutions is again 3. Since every point of X_I, with $|I|=3$, is a 6-fold point in $\bar{X}$, we conclude that

$$\mathcal{C}_{X_I}\big(\psi_{\delta+1;A_0}\oplus\psi_{\delta+1;A_1}\big)=18\big|X_I\big|=18\,\mathcal{N}^d(\delta-3,D_4,\sigma). \tag{7.41}$$

Taking into account the number of possibilities for i and I in (7.4), (7.8), (7.14), and (7.41) and plugging these equations in (7.3), we obtain Theorem 1.

7.2 Proof of Theorem 3

Let

$$\bar{X}=\overline{\mathcal{S}}_1^d(\delta,A_1,\sigma)\,, \qquad X=\big\{([s],p_1,\ldots,p_{\delta+1})\in\bar{X}\colon\ p_{\delta+1}\neq p_1,p_2,\ldots,p_\delta\big\}$$

and define the spaces X_I and X_i as in (7.1) and the maps π_i as in (7.2). By lemma 3.14, 3.23,

$$\begin{aligned} ([s],p_1,\ldots,p_{\delta+1})\in X_i \qquad &\Longrightarrow \qquad \chi_s(p_{\delta+1})=A_3\,;\\ ([s],p_1,\ldots,p_{\delta+1})\in X_I,\quad |I|=2 \qquad &\Longrightarrow \qquad \chi_s(p_{\delta+1})=D_4. \end{aligned}$$

In both cases, the remaining points p_i are all distinct simple nodes of $s^{-1}(0)$. Furthermore, $X_I=\emptyset$ if $|I|>2$.

We need to determine the cardinality of the set

$$\begin{aligned}\mathcal{S}_0^d(\delta,A_2,\sigma)&=\big\{([s],p_1,\ldots,p_{\delta+1})\in X\colon\ \chi_s(p_{\delta+1})=A_2\big\}\\ &=\big\{([s],p_1,\ldots,p_{\delta+1})\in X\colon\ \det(\nabla^2 s|_{p_{\delta+1}})=0\big\};\end{aligned}$$

the first equality is a special case of Proposition 3.1. By Lemma 2.17, the restriction of the section

$$\psi_{\delta+1;A_2}\in\Gamma\big(\bar{X},\pi_0^*\gamma_{\mathcal{D}}^{*2}\otimes\pi_{\delta+1}^*(\gamma_{\mathbb{P}^2}^{*2d}\otimes\Lambda^2T^*\mathbb{P}^2)\big), \qquad \psi_{\delta+1;A_2}\big([s],p_1,\ldots,p_{\delta+1}\big)=\det(\nabla^2 s|_{p_{\delta+1}})\,,$$

to X is transverse to the zero set. Thus,

$$\begin{aligned}\mathcal{N}^d(\delta, A_2, \sigma) &= \big|\mathcal{S}_0^d(\delta, A_2, \sigma)\big| \\ &= \big\langle e(\pi_0^*\gamma_{\mathcal{D}}^{*2} \otimes \pi_{\delta+1}^*(\gamma_{\mathbb{P}^2}^{*2d}\otimes\Lambda^2 T^*\mathbb{P}^2)), [\bar{X}]\big\rangle - \mathcal{C}_{\partial\bar{X}}\big(\psi_{\delta+1;A_2}\big),\end{aligned} \tag{7.42}$$

where the last term is the $\psi_{\delta+1;A_2}$-contribution to the Euler class from

$$\partial X = \bar{X} - X = \bigsqcup_{\emptyset\neq I\subset\{1,\ldots,\delta\}} X_I\,.$$

The first term on the right-side side of (7.42) gives the first two terms on the right-hand side of the expression in Theorem 3.

If $i=1,2,\ldots,\delta$ and $([s],p_1,\ldots,p_\delta,p_{\delta+1})\in X_i$, $p_i=p_{\delta+1}$ is a tacnode of $s^{-1}(0)$ by Proposition 3.14. By the proof of Theorem 1, a neighborhood of this point in $\bar{X}$ can be parametrized by $t\in\mathbb{C}$,

$$t \longrightarrow ([s^t],p_1^t,\ldots,p_\delta^t,p_{\delta+1}^t) \qquad \text{s.t.} \qquad s_{xx}^t(p_{\delta+1}^t) = 8t^2, \quad s_{xy}^t(p_{\delta+1}^t) = 0, \quad s_{yy}^t(p_{\delta+1}^t) = 2;$$

see (7.12). Thus, it is immediate that the equation

$$\det\big(\nabla^2 s^t|_{p_{\delta+1}^t}\big) \equiv 2s_{xx}^t(p_{\delta+1}^t) = \tau\nu$$

has 2 small solutions for all $\nu\in\mathbb{C}^*$ and all $\tau\in\mathbb{R}^+$ sufficiently small. We conclude that

$$\mathcal{C}_{X_i}\big(\psi_{\delta+1;A_2}\big) = 2\big|X_i\big| = 2\mathcal{N}^d(\delta-1, A_3, \sigma). \tag{7.43}$$

Finally, we compute the contribution of each element $([s],p_1,\ldots,p_\delta,p_{\delta+1})\in X_I$ with $|I|=2$. We can assume that $I=\{1,2\}$. By lemma 3.23, $p_1=p_2=p_{\delta+1}$ is a D_4-node of $s^{-1}(0)$. By the proof of Theorem 1, a neighborhood of this point in $\bar{X}$ consists of 6 copies of $\mathbb{C}$, each of which can be parametrized by $t\in\mathbb{C}$,

$$t \longrightarrow ([s^t],p_1^t,\ldots,p_\delta^t,p_{\delta+1}^t) \qquad \text{s.t.} \qquad s_{xx}^t(p_{\delta+1}^t) = 0, \quad s_{xy}^t(p_{\delta+1}^t) = t.$$

Thus, it is immediate that the equation

$$\det\big(\nabla^2 s^t|_{p_{\delta+1}^t}\big) \equiv s_{xy}^t(p_{\delta+1}^t)^2 = \tau\nu$$

has 2 small solutions for all $\nu\in\mathbb{C}^*$ and all $\tau\in\mathbb{R}^+$ sufficiently small. We conclude that

$$\mathcal{C}_{X_I}\big(\psi_{\delta+1;A_2}\big) = 6\cdot 2\big|X_I\big| = 12\,\mathcal{N}^d(\delta-1, A_4, \sigma). \tag{7.44}$$

Taking into account the number of possibilities for i and I in (7.43) and (7.44) and plugging these equations into (7.42), we obtain Theorem 3.

7.3 Proof of Theorem 2

Let

$$\bar{X} = \overline{\mathcal{S}}_1^d(\delta, A_1, \sigma), \qquad X = \big\{([s], p_1, \ldots, p_{\delta+1}) \in \bar{X}:\ p_{\delta+1} \neq p_1, p_2, \ldots, p_\delta\big\},$$

define the spaces X_I and X_i as in (7.1) and the maps π_i as in (7.2), and let

$$\pi: \mathbb{P}(\pi_{\delta+1}^* T\mathbb{P}^2) \longrightarrow \bar{X} \tag{7.45}$$

be the bundle projection map. By lemma 3.14 and 3.23,

$$\begin{aligned} ([s], p_1, \ldots, p_{\delta+1}) \in X_i \qquad & \Longrightarrow \qquad \chi_s(p_{\delta+1}) = A_3\,;\\ ([s], p_1, \ldots, p_{\delta+1}) \in X_I, \quad |I| = 2 \qquad & \Longrightarrow \qquad \chi_s(p_{\delta+1}) = D_4. \end{aligned}$$

In both cases, the remaining points p_i are all distinct simple nodes of $s^{-1}(0)$. Furthermore, $X_I = \emptyset$ if $|I| > 2$. Choose a generic section ϕ of the hyperplane line bundle

$$\tilde{\gamma} \longrightarrow \widehat{\mathbb{P}\mathcal{S}}_1^d(\delta, A_1, \sigma) \equiv \mathbb{P}(\pi_{\delta+1}^* T\mathbb{P}^2),$$

and let $\hat{X}_\phi = \phi^{-1}(0)$.

We need to determine the cardinality of the set

$$\mathbb{P}\mathcal{S}_0^d(\delta, A_1, \sigma, 1) = \big\{([s], p_1, \ldots, p_{\delta+1}, \ell) \in \hat{X}_\phi|_X:\ \nabla^2 s|_{p_{\delta+1}}(v, v) = 0\ \forall\, v \in \ell\big\}$$

this equality is a special case of Proposition **??**. By Lemma 3.1 and 2.14, the restriction of the section

$$\begin{gathered} \psi_{\delta+1;\tilde{A}_1} \in \Gamma\big(\bar{X}, \tilde{\gamma}^{*2} \otimes \pi^*(\pi_0^* \gamma_{\mathcal{D}}^* \otimes \pi_{\delta+1}^* \gamma_{\mathbb{P}^2}^{*d})\big),\\ \big\{\psi_{\delta+1;\tilde{A}_1}\big([s], p_1, \ldots, p_{\delta+1}, \ell\big)\big\}(v^{\otimes 2}) = \nabla^2 s|_{p_{\delta+1}}(v, v)\,, \end{gathered}$$

to $\hat{X}_\phi|_X$ is transverse to the zero set. Thus,

$$\begin{aligned} \mathcal{N}^d(\delta, \tilde{A}_1, \sigma, 1) &= \big|\mathbb{P}\mathcal{S}_0^d(\delta, A_1, \sigma, 1)\big|\\ &= \big\langle e(\tilde{\gamma}^{*2} \otimes \pi^*(\pi_0^* \gamma_{\mathcal{D}}^* \otimes \pi_{\delta+1}^* \gamma_{\mathbb{P}^2}^{*d})), [\hat{X}_\phi]\big\rangle - \mathcal{C}_{\partial \hat{X}_\phi}\big(\psi_{\delta+1;\tilde{A}_1}\big),\\ &= \big\langle \tilde{\lambda}(2\tilde{\lambda} + \pi^*(\pi_0^* \lambda_{\mathcal{D}} + d\pi_{\delta+1}^* \lambda_{\mathbb{P}^2})), [\widehat{\mathbb{P}\mathcal{S}}_1^d(\delta, A_1, \sigma)]\big\rangle - \mathcal{C}_{\partial \hat{X}_\phi}\big(\psi_{\delta+1;\tilde{A}_1}\big), \end{aligned} \tag{7.46}$$

where the last term is the $\psi_{\delta+1;\tilde{A}_1}$-contribution to the Euler class from

$$\partial \hat{X}_\phi = \hat{X}_\phi - \hat{X}_\phi|_X = \bigsqcup_{\emptyset \neq I \subset \{1, \ldots, \delta\}} \hat{X}_\phi|_{X_I}\,.$$

The first term on the right-side side of (7.46) gives the first two terms on the right-hand side of the expression in Theorem 3.

If $i = 1, 2, \ldots, \delta$ and

$$\tilde{p} = ([s], p_1, \ldots, p_\delta, p_{\delta+1}, \ell) \in \hat{X}_\phi|_{X_i}\,,$$

$p_i = p_{\delta+1}$ is a tacnode of $s^{-1}(0)$ by lemma 3.14, while $\ell \subset T_{p_{\delta+1}}\mathbb{P}^2$ is a line determined by ϕ. Since $X_i \subset X$ is a finite collection of points, ℓ is not tangent to $s^{-1}(0)$ at $p_{\delta+1}$ and thus $\psi_{\delta+1;\tilde{A}_1}$ does not vanish at $\tilde{p}$. We conclude that

$$\mathcal{C}_{\hat{X}_\phi|_{X_i}}\big(\psi_{\delta+1;\tilde{A}_1}\big) = 0. \tag{7.47}$$

If $|I|=2$ and

$$\tilde{p} = ([s], p_1, \ldots, p_\delta, p_{\delta+1}, \ell) \in \hat{X}_\phi|_{X_I},$$

$p_{\delta+1}$ is a D_4-node of $s^{-1}(0)$ by lemma 3.23, while $\ell \subset T_{p_{\delta+1}}\mathbb{P}^2$ is a line determined by ϕ. By the proof of Theorem 1, a neighborhood of this point in $\hat{X}_\phi$ consists of 6 copies of $\mathbb{C}$, each of which can be parametrized by $t \in \mathbb{C}$,

$$t \longrightarrow ([s^t], p_1^t, \ldots, p_\delta^t, p_{\delta+1}^t) \qquad \text{s.t.} \qquad s^t_{xx}(p^t_{\delta+1}) = 0, \quad s^t_{xy}(p^t_{\delta+1}) = t, \quad \big|s^t_{yy}(p^t_{\delta+1})\big| \leq Ct.$$

Thus, it is immediate that for a generic choice of $\eta \in \mathbb{C}$ (corresponding to a generic choice of ϕ) the equation

$$\nabla^2 s^t|_{p^t_{\delta+1}}\big(\eta e_1 + e_2, \eta e_1 + e_2\big) = \tau\nu,$$

where $e_1, e_2 \in \mathbb{C}^2$ are the two standard coordinate vectors, has 1 small solution for all $\nu \in \mathbb{C}^*$ and all $\tau \in \mathbb{R}^+$ sufficiently small. We conclude that

$$\mathcal{C}_{X_I}\big(\psi_{\delta+1;\tilde{A}_1}\big) = 6 \cdot \big|X_I\big| = 6\mathcal{N}^d(\delta-1, A_4, \sigma). \tag{7.48}$$

Taking into account the number of possibilities for I in (7.48) and plugging these equations into (7.46), we obtain Theorem 3.

7.4 Proof of Theorem 4

Let π be as in (7.45). Choose a generic section ϕ of the hyperplane line bundle

$$\tilde{\gamma} \longrightarrow \overline{\mathbb{P}\mathcal{S}}_2^d(\delta, A_1, \sigma) \subset \mathbb{P}(\pi^*_{\delta+1}T\mathbb{P}^2).$$

Let

$$\bar{X} = \phi^{-1}(0), \qquad X = \big\{([s], p_1, \ldots, p_{\delta+1}, \ell) \in \bar{X} \colon \; p_{\delta+1} \neq p_1, p_2, \ldots, p_\delta\big\}$$

and define the spaces X_I and X_i as in (7.1). By lemma 3.14, 3.23 and 3.27,

$$\begin{aligned}
([s], p_1, \ldots, p_{\delta+1}, \ell) \in X_i \qquad\qquad\quad & \Longrightarrow \quad \chi_s(p_{\delta+1}) = A_3, A_4, D_4\,; \\
([s], p_1, \ldots, p_{\delta+1}, \ell) \in X_I, \quad |I| = 2 \quad & \Longrightarrow \quad \chi_s(p_{\delta+1}) = D_4, A_5, D_5; \\
([s], p_1, \ldots, p_{\delta+1}, \ell) \in X_I, \quad |I| = 3 \quad & \Longrightarrow \quad \chi_s(p_{\delta+1}) = D_6.
\end{aligned}$$

In all cases, the remaining points p_i are all distinct simple nodes of $s^{-1}(0)$. Furthermore, $X_I = \emptyset$ if $|I| > 3$.

We need to determine the cardinality of the set

$$\mathbb{P}\mathcal{S}_0^d(\delta, A_2, \sigma, 1) = \big\{([s], p_1, \ldots, p_{\delta+1}, \ell) \in X \colon \; \nabla^2 s|_{p_{\delta+1}}(v, w) = 0 \;\forall\, v \in \ell, \; w \in T_{p_{\delta+1}}\mathbb{P}^2/\ell\big\};$$

this equality is a special case of lemma 3.1. By Lemma 2.17, the restriction of the section

$$\psi_{\delta+1;A_2} \in \Gamma\big(\bar{X}, \tilde{\gamma}^*\otimes(\pi^*\pi_{\delta+1}^*T\mathbb{P}^2/\tilde{\gamma})^* \otimes \pi^*(\pi_0^*\gamma_{\mathcal{D}}^*\otimes\pi_{\delta+1}^*\gamma_{\mathbb{P}^2}^{*d})\big),$$
$$\big\{\psi_{\delta+1;A_2}\big([s], p_1, \ldots, p_{\delta+1}, \ell\big)\big\}(v,w) = \nabla^2 s|_{p_{\delta+1}}(v,w)\,,$$

to X is transverse to the zero set. Thus,

$$\begin{aligned}\mathcal{N}^d(\delta, A_2, \sigma, 1) &= \big|\mathbb{P}\mathcal{S}_0^d(\delta, A_2, \sigma, 1)\big| \\ &= \big\langle e\big(\tilde{\gamma}^*\otimes(\pi^*\pi_{\delta+1}^*T\mathbb{P}^2/\tilde{\gamma})^* \otimes \pi^*(\pi_0^*\gamma_{\mathcal{D}}^*\otimes\pi_{\delta+1}^*\gamma_{\mathbb{P}^2}^{*d})\big), [\bar{X}]\big\rangle - \mathcal{C}_{\partial\hat{X}_\phi}\big(\psi_{\delta+1;A_2}\big),\end{aligned} \tag{7.49}$$

where the last term is the $\psi_{\delta+1;A_2}$-contribution to the Euler class from

$$\partial\bar{X} = \bar{X} - X = \bigsqcup_{\emptyset\neq I\subset\{1,\ldots,\delta\}} X_I\,.$$

The first term on the right-side side of (7.49) gives the first two terms on the right-hand side of the expression in Theorem 4.

If $i=1,2,\ldots,\delta$ and

$$\tilde{p} = ([s], p_1, \ldots, p_\delta, p_{\delta+1}, \ell) \in X_i\,,$$

$p_i = p_{\delta+1}$ is a tacnode, A_4, or D_4-node of $s^{-1}(0)$ by lemma 3.14, while $\ell\subset T_{p_{\delta+1}}\mathbb{P}^2$ is in the zero set of ϕ. If $\chi_s(p_{\delta+1})=A_4$, $\psi_{\delta+1;A_2}$ does not vanish at $\tilde{p}$ for a generic choice of ϕ, and so $\tilde{p}$ does not contribute to (7.49). If $\chi_s(p_{\delta+1})=A_3$, $\ell\subset T_{p_{\delta+1}}s^{-1}(0)$. Thus, the set of points of this type is isomorphic to $\mathbb{P}\mathcal{S}_0^d(\delta-1, A_3, \sigma, 1)$. By the proof of Theorem 1, a neighborhood of $\tilde{p}$ in $\tilde{X}$ can be parametrized by $t\in\mathbb{C}$,

$$t \longrightarrow ([s^t], p_1^t, \ldots, p_\delta^t, p_{\delta+1}^t, \{y=2\mathrm{i}tx\}) \quad \text{s.t.} \quad s_{xx}^t(p_{\delta+1}^t) = 8t^2, \quad s_{xy}^t(p_{\delta+1}^t) = 0, \quad s_{yy}^t(p_{\delta+1}^t) = 2;$$

see (7.12). Thus, it is immediate that the equation

$$\nabla^2 s^t|_{p_{\delta+1}^t}\big((1, 2\mathrm{i}t), (0,1)\big) = \tau\nu$$

has 1 small solution for all $\nu\in\mathbb{C}^*$ and all $\tau\in\mathbb{R}^+$ sufficiently small. We conclude that the contribution of the subset of points of type A_3 in X_i is

$$\mathcal{C}_{X_i,A_3}\big(\psi_{\delta+1;A_2}\big) = 2\,\mathcal{N}^d(\delta-1, A_3, \sigma). \tag{7.50}$$

Suppose next that $\chi_s(p_{\delta+1})=D_4$, $\mathbb{C}\times 0$ is one of the tangent directions of $s^{-1}(0)$ at $p_{\delta+1}$, $\ell=[a,1]$ for some generic $a\in\mathbb{C}^*$. By the proof of Theorem 1, a neighborhood of $\tilde{p}$ in $\bar{X}$ is isomorphic a hypersurface in $\mathbb{C}^3$ with coordinates $t=(t_1,t_2,t_3)$ such that

$$t \longrightarrow \big([s^t], p_1^t, \ldots, p_\delta^t, p_{\delta+1}^t, [a^t, 1]\big) \quad \text{with} \quad s_{xx}^t(p_{\delta+1}^t) = 0, \quad s_{xy}^t(p_{\delta+1}^t) = t_1, \quad s_{yy}^t(p_{\delta+1}^t) = 2t_2.$$

Furthermore, either $t_3=t_3(t_1,t_2)$ and (7.27) holds or $t_2=t_2(t_1,t_3)$ and (7.23) or (7.34) holds. Since $[a^t,1]$ is a tangent direction to $s^{-1}(0)$ at $p_{\delta+1}^t=0$,

$$2t_1a^t + 2t_2 = 0. \tag{7.51}$$

Unless $a \equiv a^0 = 1$, this equation and (7.23) imply that $t_1 = 0$; unless $a = 0$, this equation and (7.34) imply that $t_1 = 0$. However, by the proof of Theorem 1, the triple of equations (7.18) has no small solutions (t, x_i, y_i) with $t_1 = 0$ and $(x_i, y_i) \neq 0$; thus, neither of these two cases occurs. On the other hand, if $t_3 = t_3(t_1, t_2)$ is as in the case corresponding to (7.27), by the Implicit Function Theorem the equation (7.51) with $t = (t_1, t_2, t_3(t_1, t_2))$ has a unique small solution $t_2 = t_2(t_1)$. In particular, a neighborhood of $\tilde{p}$ inside of X is parametrized by 3 copies of $\mathbb{C}$ (one copy for each choice of a branch of $s^{-1}(0)$ at $p_{\delta+1}$ to be identified with $\mathbb{C} \times 0$, not for each of the 3 cases just considered). It is immediate that the equation

$$s^t_{xy}(p^t_{\delta+1}) - \big(2t_1 a^t + 2t_2\big) \equiv -a^t t_1 = \tau\nu,$$

with $t = (t_1, t_2(t_1), t_3(t_1, t_2(t_1)))$ has 1 small solution for all $\nu \in \mathbb{C}^*$ and all $\tau \in \mathbb{R}^+$ sufficiently small. We conclude that the contribution of the subset of points of type D_4 in X_i is

$$\mathcal{C}_{X_i, D_4}\big(\psi_{\delta+1; A_2}\big) = 3\mathcal{N}^d(\delta-1, D_4, \sigma).$$

Combining this with (7.50), we find that

$$\mathcal{C}_{X_I}\big(\psi_{\delta+1; \tilde{A}_2}\big) = 2\mathcal{N}^d(\delta-1, A_3, \sigma) + 3\mathcal{N}^d(\delta-1, D_4, \sigma). \tag{7.52}$$

If $|I| = 2$ and

$$\tilde{p} = ([s], p_1, \ldots, p_\delta, p_{\delta+1}, \ell) \in X_I\,,$$

$p_{\delta+1}$ is a D_4, A_5, or D_5-node of $s^{-1}(0)$ by lemma 3.23, while $\ell \subset T_{p_{\delta+1}}\mathbb{P}^2$ is in the zero set of ϕ If $\chi_s(p_{\delta+1}) = A_5$, $\psi_{\delta+1; A_2}$ does not vanish at $\tilde{p}$ for a generic choice of ϕ, and so $\tilde{p}$ does not contribute to (7.49). If $\chi_s(p_{\delta+1}) = D_4$, $\ell \subset T_{p_{\delta+1}} s^{-1}(0)$; so, the set of such points is isomorphic to $\mathbb{P}\mathcal{S}^d_0(\delta-2, D_4, \sigma, 1)$. By the proof of Theorem 1, a neighborhood of such a point in $\bar{X}$ consists of 4 copies of $\mathbb{C}$, each of which can be parametrized $t \in \mathbb{C}$,

$$t \longrightarrow ([s^t], p^t_1, \ldots, p^t_\delta, p^t_{\delta+1}, \mathbb{C} \times 0) \qquad \text{s.t.} \qquad s^t_{xx}(p^t_{\delta+1}) = 0, \qquad s^t_{xy}(p^t_{\delta+1}) = t.$$

Since the equation

$$s^t_{xy}(p^t_{\delta+1}) = \tau\nu$$

has 1 small solution for all $\nu \in \mathbb{C}^*$ and all $\tau \in \mathbb{R}^+$ sufficiently small, the contribution of the subset of points of type D_4 in X_I is

$$\mathcal{C}_{X_I, D_4}\big(\psi_{\delta+1; A_2}\big) = 4\mathcal{N}^d(\delta-2, \tilde{D}_4, \sigma, 1).$$

The contributions from the other two types of boundaries are computed similarly. They do not arise when enumerating curves with up to 5 nodes.

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Appendix A

Low-degree checks

A.1 Curves with one singular point

1. Curves with 1 node

$d=1$: there are no nodal lines;

$d=2$: the number of line pairs that pass through 4 general points is $\frac{1}{2}\binom{4}{2}=3$;

$d=3$: the nodal cubics passing through 8 general points are the rational cubics passing through these points; their number, 12, can also be computed through Kontsevich's recursion [16, Theorem 10.4].

- Curves with a node on a fixed line

$d=1$: there are no nodal lines;

$d=2$: the number of line pairs that pass through 3 general points and meet on a general line is $\binom{4}{2}=3$.

These are all special cases of theorem 1 with $\delta = 0$.

2. Curves with a cusp.

$d=1$: There are no lines with a cusp.

$d=2$: The only way a conic can have a cusp is if its a double line. There are no double lines through three generic points

$d=4$: The number of quartics with a cusp is 72. This is same as the number of genus two curves with a cusp. This is given to be 72 in page 19 of [19].

These are all special cases of theorem 3 with $\delta = 0$.

3. Curves with a tacnode.

$d=3$: The number of conics that pass through four points and tangent to a fixed line is 2. The number of lines that pass through a given point and tangent to a given conic is 2. Hence the number of cubics passing through 6 points having a tacnode is

$$\binom{6}{2} \times 2 + \binom{6}{1} \times 2 = 42.$$

These are all special cases of theorem 5 with $\delta = 0$.

4. Curves with an A_4-node.

$d=3$: There are no cubics with a A_4-node.

These are all special cases of theorem 8 with $\delta = 0$.

5. Curves with a D_4-node.

$d=2$: There are no conics with a $(3,3)$ node.

$d=3$: The only way a cubic can have a $(3,3)$ node is, if it breaks into three distinct lines intersecting at a common point. The number of such configurations passing through 5 points is

$$\frac{1}{3} \times \binom{5}{2} \times \binom{3}{2} = 15.$$

6. Curves with D_4-node on a line

$d=2$: There are no conics with a D_4 node on a line.

$d=3$: The number of triple lines, having a common point at a given line and passing through four points is

$$\binom{4}{2} = 6.$$

That verifies the claim.

These are all special cases of theorem 6 with $\delta = 0$.

A.2 Curves with two singular points

1. Curves with 2 nodes.

$d=2$: The only way a conic can have 2 nodes is if it is a double line. There are no double lines through 3 generic points.

$d=3$: The only way a cubic can have 2 nodes is if it breaks into a line and a conic. Hence the number of cubics with 2 unordered nodes is

$$\binom{7}{2} = 21$$

2. Curves with two nodes, one on a line.

3. Curves with one node and one cusp.

$d=3$: There are no cubics with one node and one cusp.

4. Curves with one node and one tacnode.

$d=3$: There are no cubics with one node and one tacnode.

$d=4$: There are two possibilities here. The curve could break into a line and a cubic. The number of lines through a given point and tangent to a fixed cubic is 6. The number of cubics through through 8 points, tangent to a given line is 4. Hence the total number of quartics with one node and one tacnode that breaks into a line and a cubic is

$$\binom{10}{1} \times 6 + \binom{10}{2} \times 4 = 240.$$

It is also known that the number of genus zero quartics with one node and one tacnode is 1296. Hence the total number of quartics with one node and one tacnode is 1536.

These are all special cases of theorem 1 with $\delta = 1$.

A.3 Curves with three singularities

1. Curves with three nodes.

$d=3$: The only way a cubic can have three nodes is if it breaks into three distinct lines. The number of such configurations through six points is

$$\frac{1}{6} \times \binom{6}{2} \times \binom{4}{2} \times \binom{2}{2} = 15.$$

Hence the number of cubics with three unordered nodes is 15.

These are all special cases of theorem 1 with $\delta = 2$.

2. Curves with two nodes and one cusp.

$d=3$: There are no cubics with two nodes and one cusp.

$d=4$: It is known that the number of genus zero quartics with two unordered nodes and one cusp is 2304. That verifies the claim.

3. Curves with two nodes and one tacnode.

$d = 4$: This passes through 9 points. A smooth quartic has genus 3. Since a tacnode contributes 2 to the genus, the curve has to break. There are three possibilities here.

- It could break into two conics tangent to each other. The number of conics through 4 points, tangent to a given conic is 6. Hence the total number of ways is

$$\binom{9}{5} \times 6 = 756.$$

- A nodal cubic could go through 8 points and a line through the remaining point tangent to the cubic. The number of lines through a given point tangent to a nodal cubic is 4. The number of nodal cubics through 8 points is 12. Hence the total number is

$$\binom{9}{8} \times 4 \times 12 = 432.$$

- A line could pass through two points and a nodal cubic through the remaining 7 points, tangent to the given line. The number of nodal cubics through 7 points tangent to a given line is 36. Hence the total number is

$$\binom{9}{2} \times 36 = 1296.$$

Hence the total number is

$$756 + 432 + 1296 = 2484.$$

These are all special cases of theorem 3 with $\delta = 2$.

A.4 Curves with four singularities

1. Curves with four nodes.

$d=3$: There are no cubics with four nodes.

$d=4$: There are two possibilities here. The curve could break into two conics. The possible configurations for that are

$$\frac{1}{2} \times \binom{10}{5} = 126.$$

It could also break into a nodal cubic and a line. The possible configurations for that are

$$\binom{10}{8} \times 12 = 540.$$

Hence the total number of quartics with four unordered nodes is

$$126 + 540 = 666.$$

2. Curves with four nodes, one on a line.

$d=4$: There are four possibilities here. First of all we observe that there are nine points.

- The curve could break into two conics. One of the conic passes through 5 points, the remaining one has two choices. Hence the total possibilities are

$$\binom{9}{5} \times 2 = 252.$$

The curve could break into a nodal cubic and a line. There are three ways this could happen.

- The nodal cubic goes through eight points. The line has three choices. Hence the total possibilities are

$$\binom{9}{8} \times 12 \times 3 = 324.$$

- The line goes through two points. The cubic goes through seven points, with the node on the given line. Hence the total possibilities are

$$\binom{9}{2} \times 6 = 216.$$

- The line goes through two points. It meets the given line at some point p. The nodal cubic goes through the remaining seven points and p. Hence the total possibilities are

$$\binom{9}{2} \times 12 = 432.$$

Hence the total number of quartics with four nodes one of them on a line is

$$252 + 324 + 216 + 432 = 1224.$$

These are all special cases of theorem 1 with $\delta = 3$.

3. Curves with three nodes and one cusp.

$d=4$: The curve has to break. This goes through nine points. A cubic with a cusp goes through 7 points and a line passes through the remaining point. The number of cubics with a cusp through 7 points is 24. Hence the total number is

$$\binom{9}{7} \times 24 = 864.$$

$d=5$: The number of degree five, genus two curves with one cusp is given to be 239400 in page 19 of [19]. This agrees with our computation.

These are all special cases of theorem 3 with $\delta = 3$.

A.5 Curves with five singular points

- Curves with five nodes.

$d=5$: There are two possibilities here. The curve could be genus one. By the theorem in page 212 of [3], the number of degree 5 genus one curves is 87192. The other possibility is that the curve could be nodal quartic and a line. The number of such curves is

$$\binom{15}{13} \times 27 = 2835.$$

Hence, the total number is

$$87192 + 2835 = 90027.$$

This also agrees with the number stated in page 5 of [1].

These are all special cases of theorem 1 with $\delta = 4$.