Configurations of Graphs and the Master Equation

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Abstract of the Dissertation

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The "master equation package" of Sullivan is an algebraic structure where homotopy equivalences can be defined. It appears in many theories, for example, Donaldson theory, symplectic topology, string topology among others. This thesis provides an example of the "master equation package" in the setting of configuration spaces. We show that, under a canonical compactification, for any parallelizable manifold of dimension $d > 1$, the fundamental chains M of configuration spaces of certain decorated graphs in the manifold and the fundamental chains R of moduli spaces of configurations of these graphs provide a solution to the master equation system $\partial R + R * R = 0, \partial M + M * R = 0$, where the

product $*$ is the sum over all possible subgraph insertions. As an application, Kontsevich-Kuperberg-Thurston's construction of quantum invariants of 3-manifolds is discussed in this formalism.

To the memory of my grandmother

and

to my parents and Haiyan

Contents

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Chapter 1

Introduction

The purpose of this thesis is to describe, in the setting of configuration spaces, an algebraic structure called the master equation package [45], which also appears in many other theories, for example, Donaldson theory, symplectic topology, string topology among others. We focus on the setting of configuration spaces of graphs, where the Kontsevich-Kuperberg-Thuston's construction of quantum invariants of 3-manifolds lives.

1.1 Background

As classical objects in algebraic topology, configuration spaces have been studied from many different point of views. More recent interests in configuration spaces mainly came from two aspects:

One is from the algebraic aspect. Collections of configuration spaces form operads which parametrize certain algebras. For example, Getzler and Jones [21] observed the fact that the collection $\{\overline{\mathrm{conf}(n, \mathbb{R}^d)}\}_{n\geq 1}$ of moduli spaces of configurations of n distinct points in $\mathbb{R}^d$ has an operad structure. The $d = 1$ case gives the associahedra introduced by Stasheff [41].

The other is from the geometric/topological aspect. Configuration spaces, combined with ideas from perturbative Chern-Simons field theory [1, 4], are used to construct invariants of links and 3-manifolds (see, e.g. [2, 11, 12, 13, 28, 31, 47]).

In both aspects, a special compactification of configuration spaces is used, which is a real manifold analogue of the Fulton-Macpherson compactification [19] of configuration spaces of distinct points in complex varieties. One of our motivations is to understand the connections between the algebraic and topological approaches. In this thesis, we investigate configuration spaces of finite graphs, which are used by Kuperberg and Thurston [31] to give an improved, topological construction of Kontsevich's invariants [28] of rational homology 3-spheres.

A graph Γ is a pair consisting of a set of vertices and a set of edges. The configuration space $\mathrm{Conf}(\Gamma, M)$ of a graph Γ in a manifold M is the space of maps from the vertex set of Γ to the space M such that the images of any two vertices can not be equal whenever they are connected by an edge. The moduli space of configurations of graph Γ is the quotient space of the configuration space of graph Γ in the Euclidean space by the action of the group of translations and dilations.

There are various ways to compactify the configuration space $\mathrm{Conf}(\Gamma, M)$. The one we are interested in is obtained by iteratively blowing up the space $\mathrm{Map}(V(\Gamma), M)$ of maps from the vertex set of Γ to M along the set of diagonals corresponding to the set of connected full subgraphs of Γ. This compactification is called the canonical compactification in the thesis, due to its history.

The above described picture is a just small portion of a large picture. Starting from last eighties, based on ideas from quantum field theory in physics, many new theories emerged in mathematics. These theories include, e.g., finite type invariants of links and 3-manifolds (or Vassiliev theory), Donaldson's theory of 4-manifolds, symplectic topology, string topology. A common feature of these theories is the use of certain system of moduli spaces of some structures, and certain compactification of the moduli spaces. Based on this, an algebraic structure named "master equation package" was introduced by Sullivan [45, 46].

1.2 The master equation package

In this section, the notions of "master equation" and "master equation package" will be explained.

Some algebraic structures, e.g., groups, can be represented in the form of generators with some relations. Consider such an algebraic structure, and suppose, in addition, that there is a differential on it. To represent it, extra relations with differential are needed.

In its easiest form, a master equation is, symbolically, an equation of the following form:

$$\partial X = X * X,$$

where ∂ is a differential; $X = \{X_i\}$ is a linear basis indexed by some partially ordered set $\{i\}$ with all descending chains finite, such that the right hand side of the equation for ∂X_i has only terms with strictly lower index; and $X * X$ is a sum over a collection $*$ of binary operations.

There may have some initial terms X_i such that $\partial X_i = 0$.

The master equation is understood as a presentation of a free triangular differential graded algebra (or module over differential graded algebra) generated by a linear basis subject to the differential relation given by the master equation, where "free" means free over certain combination operad, i.e., there are no other algebraic relations besides those generated by the basic composition rules which define the combination operad. In our case, the master equation has the form

$$\partial\Gamma = \sum_{\gamma} \Gamma/\gamma \odot \gamma$$

where $\Gamma, \Gamma/\gamma, \gamma$ are configurations of graphs (Definition 3.2.1). The algebra it represents will be called configuration algebra (Definition 3.2.7), which is a differential graded many-sorted algebra (in the sense of universal algebra [14], see Remark 3.2.3). Also we will have a master equation system of the form

$$\begin{aligned} \partial R_\Gamma &= \sum R_{\Gamma/\gamma} \odot R_\gamma; \\ \partial M_\Gamma &= \sum M_{\Gamma/\gamma} \odot R_\gamma \end{aligned}$$

where R_Γ, M_Γ are configurations of graphs Γ coloured in colours R, M respectively. The differential graded many-sorted algebra represented by this master equation system will be called a right module over the configuration algebra.

A master equation package is a triple (F, S, A) where F and A are differential graded (many-sorted) algebras (or differential graded modules over differential graded algebras) with F free and triangular, and $S : F \to A$ is a map of differential graded (many-sorted) algebras (or differential graded modules over differential graded algebras). S is also called a solution map to the master equation (system). And if there is a master equation package (F, S, A), we may as well say that there is a solution in A to the master equation (system) representing F.

There is a homotopy theory for master equation packages [45, 46], which is a generalization of rational homotopy theory [44].

1.3 Outline and summary of results

The thesis has two parts and an appendix. The first part is from Chapter 2 to Chapter 6; Chapter 7 is the second part.

Chapters 2 and 3 are the algebraic/combinatorial basis of the thesis. In Chapter 2, after fixing the definition of graphs, the composition properties of insertions of graphs are studied. In Chapter 3, the notion of configurations of abstract graphs is defined, and a differential graded many-sorted algebra called configuration algebra is constructed for a class of decorated graphs named $\mathcal{A}$-labeled graphs (see Definition 2.2.2).

Chapters 4 to 6 have more flavors of geometry and topology. Chapter 4 is a detailed account of the canonical compactification of configuration spaces of graphs. In Chapter 5, the fibrations on the boundary strata of the canonically compactified moduli spaces of graphs and canonically compactified configuration spaces of graphs in manifolds are studied. Chapter 6 investigates the properties of geometric chains on moduli spaces and certain modified configuration spaces of graphs. In particular, the main result of the first part is given in this chapter.

Theorem 6.3.6 (Master equation package). *Let M be a smooth parallelizable manifold of dimension $d > 1$. Then the fundamental geometric chains $\{\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}\}$ of the canonically compactified moduli spaces of connected $\mathcal{A}$-labeled graphs Γ with two or more vertices, and the fundamental geometric chains $\{\widehat{\mathrm{Conf}(\Gamma, M)}\}$ of certain modified configuration spaces of connected $\mathcal{A}$-labeled graphs Γ with two or* *more vertices in M, provide a solution to the following master equation system*

$$\begin{aligned}
\partial R_\Gamma &= 0, \text{if } |V(\Gamma)| = 2;\\
\partial M_\Gamma &= 0, \text{if } |V(\Gamma)| = 2;\\
\partial R_\Gamma &= \sum_\gamma R_{\Gamma/\gamma} \odot_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} R_\gamma, \text{if } |V(\Gamma)| > 2;\\
\partial M_\Gamma &= \sum_\gamma M_{\Gamma/\gamma} \odot_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} R_\gamma, \text{if } |V(\Gamma)| > 2
\end{aligned}$$

in the sum of complexes of geometric chains of moduli spaces of Γ *and modified configuration spaces of* Γ *in M, with insertion products* $*$ *of graphs, where the sum is over all connected* $\mathcal{A}$*-labeled graphs* Γ *with two or more vertices.*

In the second part of the thesis, i.e., Chapter 7, the case when the manifold M^d is of odd dimensions is investigated. In particular, we show that Kontsevich-Kuperberg-Thurston's construction [28, 31] of quantum invariants of 3-manifolds fits the framework developed in the first part.

Due to dimension reason (see Proposition 7.2.6), for many $\mathcal{E}$-decorated graphs Γ (Definition 7.1.2), there exist orientation reversing involutions on $\mathrm{Conf}(\Gamma, \mathbb{R}^d)$. This allows us to introduce certain equivalence relations on the differential right module over configuration algebra. The quotient algebraic structure of the differential right module over configuration algebra of $\mathcal{E}$-decorated graphs modulo the ideal generated by certain involutions and IHX relations (§7.2.3) will be called the differential right module represented by the reduced master equation B. Sums of trivalent graphs provide non trivial cycles to the differential right module represented by the reduced master equation B.

The Kontsevich-Kuperberg-Thurston's construction is summarized in §7.3.1.

Theorem 7.3.4. *Kontsevich-Kuperberg-Thurston invariants can be obtained from geometric realization $(\bar{C}'_n, D)$ of the non trivial cycles of the differential right module represented by the reduced master equation B which are of the form of sums of trivalent graphs.*

The appendix is a supplement to Chapter 2. Here the relations between insertions of graphs (with external edges) and operads are discussed.

Part I

The general theory

Chapter 2

Graphs and insertions

The pre-Lie algebra and Hopf algebra of Feynman diagrams were studied by Connes and Kreimer [15, 16]. Inspired by their work, in this chapter, we study the properties of insertion operations on the set of connected graphs.

2.1 Graphs

Outside the scope of graph theory, various definitions and terminologies related to graphs are used in the literature. In this section, we will introduce and fix our definitions and notations about graphs.

Definition 2.1.1 (Pregraph)**.** A finite pregraph Γ is a pair $(V(\Gamma), E(\Gamma))$ of finite sets, where

- An element of $V(\Gamma)$ is called a vertex of Γ.
- An element of $E(\Gamma)$ is called an edge of Γ. Each edge is labeled by a two-element subset of $V(\Gamma)$. The labels of different edges may be the same.

If $\{u, v\}$ is the label of an edge $e \in E(\Gamma)$, then vertices u and v are called the endpoints of e.

Definition 2.1.2 (Graph)**.** Two pregraphs Γ and Γ' are isomorphic if $V(\Gamma) = V(\Gamma')$ and there exist a bijective map $f : (V(\Gamma), E(\Gamma)) \to (V(\Gamma'), E(\Gamma'))$ such that

- the first factor of f is the identity map, and
- the map induced by f from the multiset of labels of edges of Γ to that of Γ' is the identity map.

A finite graph is the isomorphism class of a finite pregraph.

The adjective *finite* before the term *graph(s)* will be omitted from now on, unless we want to stress it, since only finite graphs will be considered in this thesis.

Remark 2.1.1. Graphs can be visualized as 1-dimensional CW-complexes: vertices are 0-dimensional cells and edges are 1-dimensional cells.

Let v be a vertex of a graph Γ. The set of edges of Γ having v as an endpoint will be denoted by $E(v)$. The valence of the vertex v is the cardinality $|E(v)|$ of the set $E(v)$. A trivalent graph Γ is a graph with $|E(v)| = 3$ for all of its vertices v.

Definition 2.1.3 (Subgraph, full subgraph)**.** A graph γ is a subgraph of a graph Γ if γ satisfies the following:

- $V(\gamma) \subset V(\Gamma)$;
- $E(\gamma) \subset E(\Gamma)$ and the endpoints of elements of $E(\gamma)$ are the same as those of them as elements of $E(\Gamma)$;

Let γ be a subgraph of Γ. γ is called a full subgraph of Γ if every edge of Γ with both endpoints in $V(\gamma)$ is also an edge of γ.

A subgraph of a graph Γ is called non trivial if it is not a vertex or the graph Γ itself.

Definition 2.1.4 (Connected graph)**.** Two vertices u and v of a graph Γ are connected by an edge if they are the endpoints of an edge of Γ.

A graph Γ is connected if for any two of its vertices $p, q \in V(\Gamma)$, there exists a sequence of sets of endpoints of its edges

$$\{x_0, x_1\}, \{x_1, x_2\}, \cdots, \{x_{k-1}, x_k\}$$

such that $x_0 = p$ and $x_k = q$.

The graph with only one vertex and no edges is considered connected.

Definition 2.1.5 (Cut vertex)**.** Let Γ be a connected graph. A vertex $v \in V(\Gamma)$ is called a cut vertex if the graph obtained by removing v is not connected.

Definition 2.1.6 (Vertex-2-connected graph)**.** A connected graph Γ is vertex-2-connected if it has no cut vertices.

Definition 2.1.7 (Complete graph)**.** A complete graph is a graph Γ in which any two vertices are connected by a single edge.

Example 2.1.1. The connected graphs with two vertices, and complete graphs are vertex-2-connected.

Definition 2.1.8 (Quotient graph)**.** Let Γ be a graph and γ its connected full subgraph. The quotient graph Γ/γ is the graph defined by the following data:

- $V(\Gamma/\gamma) = (V(\Gamma) \setminus V(\gamma)) \cup \{v_\gamma\}$;

- $E(\Gamma/\gamma) = E(\Gamma) \setminus E(\gamma)$, and the endpoints of elements $e \in E(\Gamma/\gamma)$ are the same as those of them as elements of $E(\Gamma)$ if none of the endpoints is in $V(\gamma)$, otherwise, the endpoints of e are the endpoint of e which is not in $V(\gamma)$ and the vertex v_γ.

Let γ_1, γ_2 be subgraphs of a graph Γ. We will use $\gamma_1 \cup \gamma_2$ to denote the subgraph of Γ corresponding to the union of the geometric realizations (see Remark 2.1.1) of γ_1, γ_2 in the geometric realization of Γ. For example, if γ_1, γ_2 are disjoint, $\gamma_1 \cup \gamma_2$ is the subgraph defined by $V(\gamma_1 \cup \gamma_2) = V(\gamma_1) \cup V(\gamma_2)$ and $E(\gamma_1 \cup \gamma_2) = E(\gamma_1) \cup E(\gamma_2)$; if γ_1 is a subgraph of γ_2, then $\gamma_1 \cup \gamma_2 = \gamma_2$.

Let Γ be a graph, and γ_1, γ_2 its connected full subgraphs. Sometimes we will use the notation $\Gamma/(\gamma_1 \cup \gamma_2)$ to denote $(\Gamma/\gamma_1)/\gamma_2$ or $(\Gamma/\gamma_2)/\gamma_1$, even when $\gamma_1 \cup \gamma_2$ is not connected or not a full subgraph, if no ambiguities will appear.

We will use $\mathcal{G}$ to denote the set of finite graphs. For any $\Gamma_1, \Gamma_2 \in \mathcal{G}$, define $\Gamma_1 \prec \Gamma_2$ if and only if $|V(\Gamma_1)| < |V(\Gamma_2)|$. Then the following is immediate:

Proposition 2.1.1. *The pair $(\mathcal{G}, \prec)$ where $\prec$ is defined as above is a strict partially ordered set and all descending chains for any $\Gamma \in \mathcal{G}$ are finite.*

2.2 Insertions of labeled graphs

A graph can be inserted into a vertex of another graph to form a third graph. In this section, we focus on a special class of graphs with labeled vertices and study their properties under insertions.

Definition 2.2.1 (Inserting a graph into another)**.** Let Γ_1, Γ_2 be connected graphs and $v \in V(\Gamma_1)$. Let $f : E(v) \to V(\Gamma_2)$ be a map.

Define a graph Γ by the following data

- $V(\Gamma) = (V(\Gamma_1) \setminus \{v\}) \sqcup V(\Gamma_2)$;
- $E(\Gamma) = E(\Gamma_1) \sqcup E(\Gamma_2)$; an edge of Γ_1 with endpoint set $\{p, v\}$ corresponds to an edge of Γ with endpoint set $\{p, f(e)\}$; the endpoints of the other edges remain unchanged.

We will call Γ the graph obtained by inserting Γ_2 into Γ_1 at vertex v according to the map f. The graph Γ is denoted by $\Gamma_1 \circ_{(v,f)} \Gamma_2$.

Let $\mathcal{A}$ be a countably infinite discrete set. A prototypic example of $\mathcal{A}$ is the set $\mathbb{N}$ of natural numbers.

Definition 2.2.2 ($\mathcal{A}$-labeled graph)**.** A finite $\mathcal{A}$-labeled graph is a finite graph Γ which is labeled in $\mathcal{A}$ subjecting to the following rules:

- Each vertex of Γ is labeled by a non empty finite subset of $\mathcal{A}$.
- The intersection of each pair of subsets labeling vertices of Γ is empty.
- The label of the graph is the union of labels of its vertices.

The subsets of $\mathcal{A}$ labeling vertices of an $\mathcal{A}$-labeled graph or the graph itself will be called $\mathcal{A}$-labels.

The set of $\mathcal{A}$-labeled graphs will be denoted by $\mathcal{AG}$.

For $\mathcal{A}$-labeled graphs, Definition 2.2.1 can be made more specific.

Definition 2.2.3 (Insertion operation of $\mathcal{A}$-labeled graphs)**.** Let Γ_1, Γ_2 and Γ_3 be connected $\mathcal{A}$-labeled graphs. If there exist $v \in V(\Gamma_1)$ whose $\mathcal{A}$-label is equal to the $\mathcal{A}$-label of Γ_2 and a map $f : E(v) \to V(\Gamma_2)$ such that $\Gamma_3 = \Gamma_1 \circ_{(v,f)} \Gamma_2$, we say that there is an insertion operation $\circ_{((\Gamma_1,v),\Gamma_2,\Gamma_3)} : (\Gamma_1, \Gamma_2) \mapsto \Gamma_3$.

Remark 2.2.1. The introduction of the notation $\circ_{((\Gamma_1,v),\Gamma_2,\Gamma_3)}$ is, in a sense, not necessary, because it can be equivalently substituted by some other notations like $\circ_{((\Gamma_1,v),\Gamma_2,f)}$ in which the output Γ_3 does not appear in the subscript. But for the sake of symbol computations, the notation $\circ_{((\Gamma_1,v),\Gamma_2,\Gamma_3)}$ has obvious advantages in keeping track of data.

Similar remarks apply to Definitions 3.2.2, 6.2.1, 6.3.1, 6.3.2, 7.1.5, 7.1.6, etc.

Let Γ be an $\mathcal{A}$-labeled graph and γ a connected full subgraph of Γ. To be consistent with Definition 2.2.3, the vertex v_γ of Γ/γ will be labeled by the subset of $\mathcal{A}$ whose elements occur in the labels of the vertices of γ.

Sometimes when we want to refer insertion operations in general, or do not want to specify them, we will just use the notation $\circ$.

The following proposition gives the basic properties of insertion operations $\circ$.

Proposition 2.2.1. *Let Γ be a connected $\mathcal{A}$-labeled graph; γ_1 and γ_2 be connected full subgraphs of Γ with the induced labels from Γ. Then we have the following statements of graphs:*

1. *If $V(\gamma_1) \cap V(\gamma_2) = \phi$, then*

$$\Gamma/(\gamma_1 \cup \gamma_2) = (\Gamma/\gamma_1)/\gamma_2 = (\Gamma/\gamma_2)/\gamma_1,$$

$$\Gamma/(\gamma_1 \cup \gamma_2) \circ_{((\Gamma/(\gamma_1\cup\gamma_2),v_{\gamma_1}),\gamma_1,\Gamma/\gamma_2)} \gamma_1 = \Gamma/\gamma_2,$$

$$\Gamma/(\gamma_1 \cup \gamma_2) \circ_{((\Gamma/(\gamma_1\cup\gamma_2),v_{\gamma_2}),\gamma_2,\Gamma/\gamma_1)} \gamma_2 = \Gamma/\gamma_1,$$

and thus

$$\begin{aligned}
&(\Gamma/(\gamma_1 \cup \gamma_2) \circ_{((\Gamma/(\gamma_1\cup\gamma_2),v_{\gamma_2}),\gamma_2,\Gamma/\gamma_1)} \gamma_2) \circ_{((\Gamma/\gamma_1,v_{\gamma_1}),\gamma_1,\Gamma)} \gamma_1 \\
&= (\Gamma/(\gamma_1 \cup \gamma_2) \circ_{((\Gamma/(\gamma_1\cup\gamma_2),v_{\gamma_1}),\gamma_1,\Gamma/\gamma_2)} \gamma_1) \circ_{((\Gamma/\gamma_2,v_{\gamma_2}),\gamma_2,\Gamma)} \gamma_2. \qquad (2.2.1)
\end{aligned}$$

2. *If $V(\gamma_2) \subset V(\gamma_1)$, then*

$$\Gamma/\gamma_1 \circ_{((\Gamma/\gamma_1,v_{\gamma_1}),\gamma_1/\gamma_2,\Gamma/\gamma_2)} \gamma_1/\gamma_2 = \Gamma/\gamma_2,$$

and thus

$$\begin{aligned}
&(\Gamma/\gamma_1 \circ_{((\Gamma/\gamma_1,v_{\gamma_1}),\gamma_1/\gamma_2,\Gamma/\gamma_2} \gamma_1/\gamma_2) \circ_{((\Gamma_2/\gamma_2,v_{\gamma_2}),\gamma_2,\Gamma)} \gamma_2 \\
&= \Gamma/\gamma_1 \circ_{((\Gamma/\gamma_1,v_{\gamma_1}),\gamma_1,\Gamma)} (\gamma_1/\gamma_2 \circ_{((\gamma_1/\gamma_2,v_{\gamma_2}),\gamma_2,\gamma_1)} \gamma_2). \qquad (2.2.2)
\end{aligned}$$

Proof. The statements follows directly from definitions.

Intuitively, the first statement says that if $V(\gamma_1) \cap V(\gamma_2) = \phi$, then the resulted graphs from different orders of collapsing γ_1 and γ_2 are the same, and when inserting back, different orders of inserting γ_1 and γ_2 will both produce the original graphs. The second statement has similar idea. □

Remark 2.2.2. There are many non trivial identities on the insertion algebra of graphs. In particular (see Example 2.2.1), there are pairwise non isomorphic graphs A, B, C and D, such that $A \circ_1 B = C \circ_2 D$, where $\circ_1$ and $\circ_2$ are certain insertion operations. So we do not expect that the insertion algebra of graphs be free.

Example 2.2.1. Let Γ be a connected $\mathcal{A}$-labeled graph; γ_1 and γ_2 be disjoint connected full subgraphs of Γ with the induced labels from Γ. Then

$$\Gamma/\gamma_1 \circ_{((\Gamma/\gamma_1, v_{\gamma_1}),\gamma_1,\Gamma)} \gamma_1 = \Gamma = \Gamma/\gamma_2 \circ_{((\Gamma/\gamma_2, v_{\gamma_2}),\gamma_2,\Gamma)} \gamma_2.$$

Chapter 3

Graph configurations and the master equation

In this chapter, we will study the algebraic/combinatorial properties of configurations of sets and graphs under insertions and define (or construct) a differential graded algebra of configurations of $\mathcal{A}$-labeled graphs.

3.1 Configurations of finite sets

Before going into configurations of graphs, in this section we study configurations of sets. All sets considered in this section are finite.

Based on the work of Axelrod, Bott, Fulton, MacPherson, Singer and Taubes ([2, 13, 19]), we have the definition of a screen of a set.

Definition 3.1.1 (Screen of a set)**.** A screen $\mathcal{S}$ of a set S is a set of subsets of S such that

- For any $A \in \mathcal{S}$, $|A| > 1$.
- If $A, B \in \mathcal{S}$ and $A \cap B \neq \phi$, then either $A \subseteq B$ or $B \subseteq A$.

A screen of a set can be expressed as a nested set, and this nested set presentation is unique.

Example 3.1.1. Let $S = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. Then the screen

$$\mathcal{S} = \{\{1, 2, 3\}, \{5, 6, 7, 8, 9\}, \{6, 7\}\}$$

has the nested set presentation

$$\{0, \{1, 2, 3\}, 4, \{5, \{6, 7\}, 8, 9\}\}.$$

Let $(S, \mathcal{S})$ be a configuation and N the nested set presentation of the screen $\mathcal{S}$. We will refer to S as the underlying set of N.

Definition 3.1.2 (Configuration)**.** A configuration X of a set S is a pair $(S, \mathcal{S})$, where $\mathcal{S}$ is a screen of the set S.

Definition 3.1.3 (Insertion product of configurations)**.** Let X_1, X_2, X_3 be configurations of sets S_1, S_2, S_3 respectively, and $s_1 \in S_1$. Define $X_3 = X_1 \boxdot_{((S_1, s_1), S_2, S_3)} X_2$ if and only if the following two conditions are satisfied:

- There exists $s_1 \in S_1$ such that $S_3 = (S_1 \setminus \{s_1\}) \sqcup S_2$.
- The nested set presentation of the screen of X_3 is equal to the nested set obtained by replacing s_1 in the nested set presentation of the screen of X_1 with the nested set presentation of the screen of X_2.

Definition 3.1.4 (Elementary configuration of a set)**.** The elementary configuration of a set S is a configuration of the set S where the screen is empty set.

Proposition 3.1.1. *A configuration of a finite set can be expressed as a monomial of elementary configurations of some sets, where the product is a collection of insertions as in Definition 3.1.3.*

Proof. Here we provide an algorithm whose input is the nested set presentation of a configuration, and whose output is a monomial presentation of the configuration.

Step 1). Let $X = (S, \mathcal{S})$ be a configuration of a finite set S and N be the nested set presentation of $\mathcal{S}$. There are two cases.

If $\mathcal{S} = \phi$, then done: X is an elementary configuration.

Otherwise there exists $x \in N$ such that x itself is a nested set, then

$$X = (S_1, \mathcal{S}_1) \boxdot_{((S_1,s_1),S_2,S)} (S_2, \mathcal{S}_2) \tag{3.1.1}$$

where

- S_1 is the underlying set of the nested set obtained by replacing in N the element x by the symbol s_1;
- $\mathcal{S}_1$ is the induced screen when replacing x by s_1;
- S_2 is the underlying set of the nested set x;
- $\mathcal{S}_2$ is the screen of the set S_2 whose nested set presentation is x.

Step 2). Repeat Step 1) for each argument configuration appearing in the right hand side of equation (3.1.1).

Since S is a finite set, the algorithm will end in finite steps. $\square$

Example 3.1.2. Let S and $\mathcal{S}$ be the same as in Example 3.1.1. Then the configuration $(S, \mathcal{S})$ can be expressed as a product of elementary configurations of sets:

$$((\{0, a, 4, b\}, \phi) \boxdot_{((\{0,a,4,b\},a),\{1,2,3\},A)} (\{1, 2, 3\}, \phi))$$

$$\boxdot_{((A,b),D,S)}((\{5, c, 8, 9\}, \phi) \boxdot_{((\{5,c,8,9\},c),\{6,7\},D)} (\{6, 7\}, \phi))$$

where $A = \{0, 1, 2, 3, 4, b\}$, $D = \{5, 6, 7, 8, 9\}$.

3.2 An algebra of configurations of labeled graphs

Definition 3.2.1 (Graph configuration)**.** Let Γ be a connected graph. A configuration X of the graph Γ is a pair $(\Gamma, \mathcal{S})$, where $\mathcal{S}$ is a screen of of the vertex set of Γ, such that the elements of the screen are the vertex sets of some connected subgraphs of the graph Γ.

The insertion product of configurations of $\mathcal{A}$-labeled graphs is a special case of the insertion product of configurations of sets (see Definition 3.1.3).

Definition 3.2.2 (Insertion product of configurations of labeled graphs)**.** Let Γ_i be a connected $\mathcal{A}$-labeled graph, and X_i be a configuration of Γ_i, $i = 1, 2, 3$. Define

$$X_3 = X_1 \odot_{((\Gamma_1,v),\Gamma_2,\Gamma_3)} X_2$$

if and only if the following two conditions are satisfied:

- There exists $v \in V(\Gamma_1)$ whose label is the same as the label of Γ_2 such that

$$\Gamma_3 = \Gamma_1 \circ_{((\Gamma_1,v),\Gamma_2,\Gamma_3)} \Gamma_2.$$

- The nested set presentation of the screen of X_3 is equal to the nested set obtained by replacing v in the nested set presentation of the screen of X_1 with the nested set presentation of the screen of X_2.

Remark 3.2.1. About the notation: sometimes the subscripts of the insertion products of configurations will be omitted, if no confusion will be caused and we do not want to emphasize them.

Definition 3.2.3 (Elementary configuration of a graph)**.** The elementary configuration of a connected graph Γ is Γ with the empty screen on its set of vertices.

Let Γ be a connected graph. Since the nested set presentation of the empty screen of the set $V(\Gamma)$ is canonically isomorphic to the set $V(\Gamma)$, we will interchangeably use the terms "an elementary configuration of a graph" and "a graph", and will sometimes also use the same notation Γ to denote the elementary configuration of the graph Γ. These will cause no confusion from the contexts.

Proposition 3.2.1. *A configuration of a connected $\mathcal{A}$-labeled graph can be expressed as a monomial of elementary configurations of $\mathcal{A}$-labeled graphs, where the products are insertion products of configurations.*

Proof. This follows from Proposition 3.1.1 and Definitions 3.1.3 and 3.2.2. $\square$

Proposition 3.2.2 (Basic identities of graph configurations)**.** *Let Γ be a connected $\mathcal{A}$-labeled graph and γ_1 and γ_2 connected full subgraphs of Γ. Then we have the following statements of graph configurations:*

1. *If $V(\gamma_1) \cap V(\gamma_2) = \phi$, then*

$$(\Gamma/(\gamma_1 \cup \gamma_2) \odot_{((\Gamma/(\gamma_1\cup\gamma_2),v_{\gamma_2}),\gamma_2,\Gamma/\gamma_1)} \gamma_2) \odot_{((\Gamma/\gamma_1,v_{\gamma_1}),\gamma_1,\Gamma)} \gamma_1$$

$$= (\Gamma/(\gamma_1 \cup \gamma_2) \odot_{((\Gamma/(\gamma_1\cup\gamma_2), v_{\gamma_1}), \gamma_1, \Gamma/\gamma_2)} \gamma_1) \odot_{((\Gamma/\gamma_2, v_{\gamma_2}), \gamma_2, \Gamma)} \gamma_2.$$

2. *If $V(\gamma_2) \subset V(\gamma_1)$, then*

$$(\Gamma/\gamma_1 \odot_{((\Gamma/\gamma_1, v_{\gamma_1}), \gamma_1/\gamma_2, \Gamma/\gamma_2)} \gamma_1/\gamma_2) \odot_{((\Gamma_2/\gamma_2, v_{\gamma_2}), \gamma_2, \Gamma)} \gamma_2$$

$$= \Gamma/\gamma_1 \odot_{((\Gamma/\gamma_1, v_{\gamma_1}), \gamma_1, \Gamma)} (\gamma_1/\gamma_2 \odot_{((\gamma_1/\gamma_2, v_{\gamma_2}), \gamma_2, \gamma_1)} \gamma_2).$$

Proof. This follows from Proposition 2.2.1 and Definition 3.2.2. □

Proposition 3.2.3. *Let Γ be a connected $\mathcal{A}$-labeled graph. The presentations of a configuration of graph Γ as monomials of elementary configurations of $\mathcal{A}$-labeled graphs are unique modulo the two basic identities in Proposition 3.2.2.*

Proof. This is because that when inserting two graphs Γ_1 and Γ_2 into a graph Γ, either Γ_1 or Γ_2 is inserted first. Suppose Γ_1 is inserted first, then when inserting Γ_2, it is inserted either inside or outside Γ_1. All these different cases are described by Proposition 3.2.2. When inserting more graphs, the relations between two different presentations of a configuration are generated by the two basic identities in Proposition 3.2.2. □

Proposition 3.2.4. *Let Γ be a connected $\mathcal{A}$-labeled graph and X a configuration of Γ. Then*

(1) The leftmost terms in all monomial presentations of X are the same.

(2) The numbers of $\odot$ operations in all monomial presentations of X are the same.

Proof. These follow from Propositions 3.2.2 and 3.2.3. □

Let n be the number of $\odot$ operations in a monomial presentation of X. We will call the number $n+1$ the length of X and denoted by $l(X)$.

Definition 3.2.4 (The boundary operator on configurations of $\mathcal{A}$-labeled graphs)**.** Let $\mathbf{V}$ be the module over a commutative unital ring $\mathbf{R}$ generated by the set of configurations of connected $\mathcal{A}$-labeled graphs with two or more vertices. Define an operator $\partial : \mathbf{V} \to \mathbf{V}$ by the following

1. ∂ (a connected $\mathcal{A}$-labeled graph with two vertices) $= 0$;

2. if Γ is a connected $\mathcal{A}$-labeled graph with more than three vertices, then

$$\partial\Gamma = \sum_{\gamma} \Gamma/\gamma \odot_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} \gamma \tag{3.2.1}$$

   where the sum is over all connected non trivial full sub $\mathcal{A}$-labeled graphs γ with the induced labels from the graph Γ.

Then extending (multi)linearly ∂ to all configurations of connected $\mathcal{A}$-labeled graphs with two or more vertices by derivation with respect to insertion product $\odot$.

Proposition 3.2.5. *Let X be a configuration of a connected $\mathcal{A}$-labeled graph Γ with $|V(\Gamma)| > 1$. Then $\partial^2 X = 0$ over $\mathbb{Z}_2$.*

Proof. Let Γ be the elementary configuration of a connected $\mathcal{A}$-labeled graph. By definition,

$$\partial\Gamma = \sum_{\alpha} \Gamma/\alpha \odot_{((\Gamma/\alpha, v_\alpha),\alpha,\Gamma)} \alpha$$

where α runs over the set of connected full sub $\mathcal{A}$-labeled graphs of Γ. Then

$$\partial^2\Gamma = \sum_{\alpha} (\partial(\Gamma/\alpha) \odot \alpha + \Gamma/\alpha \odot \partial\alpha)$$

$$= \sum_{\alpha} \sum_{\beta} ((\Gamma/\alpha)/\beta \odot_{(((\Gamma/\alpha)/\beta, v_\beta), \beta, \Gamma/\alpha)} \beta) \odot_{((\Gamma/\alpha, v_\alpha), \alpha, \Gamma)} \alpha$$

$$+ \sum_{\alpha} \sum_{\beta} \Gamma/\alpha \odot_{((\Gamma/\alpha, v_\alpha), \alpha, \Gamma)} (\alpha/\beta \odot_{((\alpha/\beta, v_\beta), \beta, \alpha)} \beta).$$

There are two cases for the first item in the right hand side of the above formula:

Case 1) if $V(\alpha) \cap V(\beta) = \phi$, then $(\Gamma/\alpha)/\beta = \Gamma/(\alpha \cup \beta)$, thus we have

$$\sum_{\alpha} \sum_{\beta} (\Gamma/(\alpha \cup \beta) \odot_{((\Gamma/(\alpha\cup\beta), v_\beta), \beta, \Gamma/\alpha)} \beta) \odot_{((\Gamma/\alpha, v_\alpha), \alpha, \Gamma)} \alpha.$$

This equals 0 by the basic identity (1) of Proposition 3.2.2.

Case 2) otherwise there exist a connected full subgraph $\bar{\beta}$ of Γ such that $\beta = \bar{\beta}/\alpha$. Then $V(\alpha) \subset V(\bar{\beta})$, we have

$$\sum_{\alpha} \sum_{\bar{\beta}} (\Gamma/\bar{\beta} \odot_{((\Gamma/\bar{\beta}, v_{\bar{\beta}}), \bar{\beta}/\alpha, \Gamma/\alpha)} \bar{\beta}/\alpha) \odot_{((\Gamma/\alpha, v_\alpha), \alpha, \Gamma)} \alpha.$$

This cancels with the second item in the above formula, by the basic identity (2) of Proposition 3.2.2.

So $\partial^2 = 0$ for elementary configurations. The general case can then be shown by induction and by the derivation property of ∂. □

Definition 3.2.5 (Oriented graph configuration)**.** Let X be a configuration of a connected graph Γ. An oriented graph configuration is a pair (X, or), where or is an orientation of X, which is a choice of the generators of an infinite group $\mathbb{Z}$ associated with the graph configuration.

Let R be a commutative unital ring. Then on the module generated by isomorphism classes of oriented graph configurations, we assume that there is an imposed relation

$$(X, -or) = -(X, or), \tag{3.2.2}$$

where $-or$ means the opposite orientation of the orientation or.

Remark 3.2.2. The situation here is like to assign orientations to simplexes to create a chain complex . All the graphs we considered are orientable.

Definition 3.2.6 (Orientation with diamond property)**.** An orientation or is called having the diamond property if it satisfies the following property of oriented graph configurations:

For any Γ, γ_1 and γ_2, where Γ is a connected $\mathcal{A}$-labeled graph; γ_1 and γ_2 are connected $\mathcal{A}$-labeled full subgraphs of Γ,

1. if $V(\gamma_1) \cap V(\gamma_2) = \phi$, then

$$((\Gamma/(\gamma_1 \cup \gamma_2), or) \odot_{((\Gamma/(\gamma_1\cup\gamma_2), v_{\gamma_2}), \gamma_2, \Gamma/\gamma_1)} (\gamma_2, or)) \odot_{((\Gamma/\gamma_1, v_{\gamma_1}), \gamma_1, \Gamma)} (\gamma_1, or)$$

$$= -((\Gamma/(\gamma_1 \cup \gamma_2), or) \odot_{((\Gamma/(\gamma_1\cup\gamma_2), v_{\gamma_1}), \gamma_1, \Gamma/\gamma_2)} (\gamma_1, or)) \odot_{((\Gamma/\gamma_2, v_{\gamma_2}), \gamma_2, \Gamma)} (\gamma_2, or);$$

2. if $V(\gamma_2) \subset V(\gamma_1)$, then

$$((\Gamma/\gamma_1, or) \odot_{((\Gamma/\gamma_1, v_{\gamma_1}), \gamma_1/\gamma_2, \Gamma/\gamma_2)} (\gamma_1/\gamma_2, or)) \odot_{((\Gamma_2/\gamma_2, v_{\gamma_2}), \gamma_2, \Gamma)} (\gamma_2, or)$$

$$= -(\Gamma/\gamma_1, or) \odot_{((\Gamma/\gamma_1, v_{\gamma_1}), \gamma_1, \Gamma)} ((\gamma_1/\gamma_2, or) \odot_{((\gamma_1/\gamma_2, v_{\gamma_2}), \gamma_2, \gamma_1)} (\gamma_2, or)).$$

Theorem 3.2.6. *Let (X, or) be an oriented configuration of a connected $\mathcal{A}$-labeled graph Γ, where or is an orientation with diamond property. Then $\partial^2 X = 0$ over any commutative unital ring $\mathbf{R}$.*

Proof. This follows immediately from Proposition 3.2.5 and the diamond property of the orientation or. □

Here we gave a name to the type of differential algebraic structures represented by the master equation of a certain class of decorated graphs (e.g., $\mathcal{A}$-labeled graphs) with subgraph insertion products $\odot$.

Definition 3.2.7 (Configuration algebra)**.** Let $\mathcal{B}$ be a class of decorated graphs, which is closed under subgraph insertions and taking quotient graphs. A configuration algebra of decorated graphs in $\mathcal{B}$ over a commutative unital ring $\mathbf{R}$ is the (many-sorted) algebra generated over $\mathbf{R}$ by the set of isomorphism classes of oriented elementary configurations (X, or) of connected decorated graphs in $\mathcal{B}$ with the set of bilinear operations $\odot$, where or is an orientation with diamond property, subject to the differential relations as in Definition 3.2.4.

Remark 3.2.3. In the language of universal algebra, a many-sorted algebra is a family of sets with a collection of operations among them. Here we list only the basic definitions of S-sorted algebras and homomorphism between S-sorted algebras of the same signature. For more, see, e.g. [8, 23, 40].

Let S be a set whose elements will be called sorts.

- An S-sorted signature Σ is a family $(\Sigma_{w,s})_{s\in S, w\in S^*}$ of sets, where S^* is the set of all finite words formed by elements of S. An element $F \in \Sigma_{w,s}$ is called an operation symbol of rank w, s, of arity w, and of sort s.

- Let Σ be an S-sorted signature. A Σ-algebra A is a family of sets $(A_s)_{s\in S}$, where A_s is called the carrier of A of sort s, and a function

$$\sigma_A : A_{s_1} \times A_{s_2} \times \cdots \times A_{s_n} \to A_s$$

for each $\sigma \in \Sigma_{w,s}$, where $w = s_1 s_2 \cdots s_n$. σ_A is called the operation of A named by σ.

- If A and B are both Σ-algebras, a Σ-homomorphism $h : A \to B$ is a family of functions $(h_s : A_s \to B_s)_{s \in S}$ which preserve the operations in the sense that if $\sigma \in \Sigma_{s_1 \cdots s_n, s}$ and $a_i \in A_{s_i}, i = 1, \cdots, n$, then

$$h_s[\sigma_A(a_1, \cdots, a_n)] = \sigma_B[h_{s_1}(a_1), \cdots, h_{s_n}(a_n)]. \tag{3.2.3}$$

Let Σ be an S-sorted signature.

Definition 3.2.8 (differential Σ-algebra, map of differential Σ-algebras)**.** A differential Σ-algebra is a pair (A, ∂) where A is a Σ-algebra and ∂ is a differential on A which is a derivation.

Let $(A, \partial_A), (B, \partial_B)$ be differential Σ-algebras. A map $f : A \to B$ is called a map of differential Σ-algebras if the following are satisfied:

- $f : A \to B$ is a Σ-homomorphism.
- f commutes with differentials, i.e., $f\partial_A = \partial_B f$.

Let $\mathcal{AG}_{\geq 2}$ be the set of connected $\mathcal{A}$-labeled graphs with two or more vertices.

Proposition 3.2.7. *Let F denote the configuration algebra of graphs in $\mathcal{AG}_{\geq 2}$. Then F is a differential graded $\mathcal{AG}_{\geq 2}$-sorted algebra.*

Proof. Let $\gamma \in \mathcal{AG}_{\geq 2}$ and F_γ be the module spanned by configurations of γ.

Given a sequence of $\Gamma_1, \cdots, \Gamma_n, \Gamma$, there are operations

$$\sigma : F_{\Gamma_1} \times F_{\Gamma_2} \cdots \times F_{\Gamma_n} \to F_\Gamma$$

if and only if there are ways to combine $\Gamma_1, \cdots, \Gamma_n$ using the operations $\circ$ in Definition 2.2.3 to obtain Γ.

The binary operations are exactly the $\odot$'s. In general, an operation σ is a combination of a collection of $\odot$'s, in the order prescribed by a binary tree. $\square$

In this thesis, we will mostly use the term S-sorted algebra while keeping its signature implicit. In fact, it is not possible to explicitly give the signature of a configuration algebra, since there are infinitely many finite graphs and infinitely many ways to combine them. But the lucky thing is that sometimes it is obvious to recognize whether the signatures of two insertion algebras are the same or not.

By Proposition 3.2.3, the configuration algebra of $\mathcal{A}$-labeled graphs is a free differential S-sorted algebra, where "free" means that there are no algebraic relations besides those implied by the defining relations in Definition 3.2.6. Or, it can be understood to be free over a combination operad [46] defined by the composition rules as in Proposition 2.2.1.

Later on, we will have the case that the class $\mathcal{B}$ is coloured in two colours M and R, and have a master equation system of the form

$$\partial R_\Gamma = \sum R_{\Gamma/\gamma} \odot_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} R_\gamma; \tag{3.2.4}$$

$$\partial M_\Gamma = \sum M_{\Gamma/\gamma} \odot_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} R_\gamma \tag{3.2.5}$$

Since equations (3.2.4) and (3.2.5) has the same underlying master equation (3.2.1), and the operations $\odot_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)}$ satisfy the same set of composition rules as in Proposition 3.2.2, the corresponding colored configuration algebra of decorated graphs in $\mathcal{B}$ over the commutative unital ring $\mathbf{R}$ will be called the right module over the configuration algebra of decorated graphs in $\mathcal{B}$. This colored configuration algebra can also be understood as a many-sorted algebra, thus the definition of maps $f : A \to B$ of right modules over the configuration algebra can be given if A, B are of the same signature as many-sorted algebras.

Proposition 3.2.8. *Let Γ be a connected $\mathcal{A}$-labeled graphs with two vertices. Then Γ is a non trivial cycle of the configuration algebra of $\mathcal{A}$-labeled graphs.*

Proof. By Definition 3.2.4,

- $\partial\Gamma = 0$.
- Γ is not in the image of ∂, because any monomial configuration in the image of ∂ has to be of length > 1.

$\square$

Chapter 4

Compactifications of configuration spaces of graphs

This chapter contains a detailed account of a compactification of the configuration space of maps from a finite graph to a manifold, which is a variant of the canonical compactification (see Axelrod and Singer [2], Fulton and Macpherson [19], and Kontsevich [28]) of the configuration space of maps from a finite set to a manifold. The construction here is due to Kuperberg and Thurston [31].

4.1 Defintions and conventions

This section contains some basic definitions and conventions for the rest of the thesis.

4.1.1 Configuration spaces

Definition 4.1.1 (Configuration space)**.** Let V be a finite set of labeled points and X a topological space. The configuration space $\mathrm{Conf}(V, X)$ of V in X is the space of the maps

$$\{f : V \to X | f(u) \neq f(v), \forall u, v \in V\}.$$

This concept can be generalized to

Definition 4.1.2 (Configuration space of a graph)**.** Let Γ be a connected graph and X a topological space. The space

$$\{f : V(\Gamma) \to X | f(u) \neq f(v), \text{if } \{u, v\} \in E(\Gamma)\}$$

is called the configuration space of Γ in X, and is denoted by $\mathrm{Conf}(\Gamma, X)$.

Proposition 4.1.1. *Given a finite set V and a topological space X, the configuration space of V in X is exactly the configuration space* $\mathrm{Conf}(\Gamma, X)$ *of a complete graph* Γ *with* $V(\Gamma) = V$.

Proof. Compare Definition 4.1.1 and Definition 4.1.2. □

4.1.2 Tangent cones

In this thesis, we will use the term *smooth* as a synonym of C^∞, or infinitely differentiable. In most of the cases, C^k (for some integer k from the contexts) is sufficient for our needs. But we will use the term *smooth* just for notational simplicity.

Let M be a smooth manifold and $p \in M$. Let us first recall the definitions of a tangent vector at p and the tangent space T_pM:

Let $\varphi_1 : [0, \epsilon_1) \to M$, $\varphi_2 : [0, \epsilon_2) \to M$, where $\epsilon_1, \epsilon_2 > 0$, be two smooth curves such that $\varphi_1(0) = \varphi_2(0) = p$. Define $\varphi_1 \sim \varphi_2$ if and only if for every real-valued function f defined in a neighborhood of p, $(f \circ \varphi_1)'(0) = (f \circ \varphi_2)'(0)$. One can check that $\sim$ is an equivalence relation. A tangent vector at p is just an equivalence class (under $\sim$) of smooth curves starting at p. The set of the tangent vectors at p, which has a linear space structure, is the tangent space T_pM.

Now we are ready to give the definition of tangent cones:

Definition 4.1.3 (Tangent cone)**.** Let M be a smooth manifold and $X \subset M$ a subset of M. The tangent cone TC_pX of a point $p \in X$ is the cone formed by those tangent vectors $v \in T_pM$ such that there exists a smooth curve $\varphi : [0, \epsilon) \to M$ so that $\varphi(0) = p, \varphi([0, \epsilon]) \subset X$ and φ is a representative of v.

Example 4.1.1. Let M be a smooth manifold and $X \subset M$ an open subset of M. Then the tangent cone TC_pX of a point $p \in X$ is the entire tangent space T_pM.

Definition 4.1.4 (Cone-like space)**.** Let M be a smooth manifold and $X \subset M$ a subset of M. The set X is called cone-like if each point $p \in X$ has a neighborhood in X which is diffeomorphic to its tangent cone TC_pX.

4.1.3 Manifolds with corners

Here we introduce some definitions on manifolds with corners.

Let $(\mathbb{R}_{\geq 0})^n = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_i \geq 0, i = 1, \cdots, n\}$ and $x \in (\mathbb{R}_{\geq 0})^n$. Define $d(x)$=the number of zeros in $(x_1, \cdots, x_n)$. We call $d(x)$ the depth of the point x.

The following proposition will lead us to the notion of manifolds with corners.

Proposition 4.1.2. *Let U, V be two (relative) open subsets of $(\mathbb{R}_{\geq 0})^n$, and let $f : U \to V$ be a diffeomorphism. Then $d(x) = d(f(x))$, for all $x \in U$.*

Proof. The proof is similar to the proof of Lemma 14.17 (invariance of corner points) in Lee's book [33]. □

Because of Proposition 4.1.2, we have the following definitions:

Definition 4.1.5 (Manifold with corners)**.** Suppose M is a topological n-manifold with boundary. A *chart with corners* for M is a pair (U, φ), where U is an open subset of M and φ is a homeomorphism from U to a (relative) open set $\widetilde{U} \subset (\mathbb{R}_{\geq 0})^n$. Two charts with corners $(U, \varphi), (V, \psi)$ are said to be smoothly compatible if the composite map $\varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$ is a diffeomorphism.

A smooth structure with corners on a topological manifold with boundary M is a maximal collection of smoothly compatible charts with corners whose domains cover M.

A topological manifold with boundary together with a smooth structure with corners is called a (smooth) manifold with corners.

Definition 4.1.6 (Depth-k stratum, codimension-k stratum)**.** Let M be a manifold with corners. Denote

$$\partial_k M = \{x \in M | d(x) = k\}.$$

A depth-k stratum of M is defined to be a connected component of $\partial_k M$.

Remark 4.1.1. Let N be a depth k stratum of M. It is easy to see that the codimension in M of N is equals to k. So sometimes we refer N as a codimension-k stratum of M.

Definition 4.1.7 (Transversal submanifold)**.** Let M be a manifold with corners. A closed subset $N \subset M$ is called a transversal submanifold with corners of M if it satisfies the following conditions

- N with the smooth structure with corners induced from M is a manifold with corners.

- N is transversal with each stratum of M.
- Each depth k stratum of N is contained in a depth k stratum of M.

4.1.4 Whitney-stratified immersion

Definition 4.1.8 (Mutually transverse)**.** Let M be a smooth manifold (with corners) and $X_1, X_2, \cdots, X_k \subset M$ submanifolds (with corners) of M. Then $X_1, X_2, \cdots, X_k$ are called mutually transverse if for any collection of submanifolds $X_{i_1}, \cdots, X_{i_m}$ ($m \leqslant k$), either

$$X_{i_1} \cap \cdots \cap X_{i_m} = \phi$$

or

$$\operatorname{codim} X_{i_1} \cap \cdots \cap X_{i_m} = \operatorname{codim} X_{i_1} + \cdots + \operatorname{codim} X_{i_m}.$$

Definition 4.1.9 (Whitney-stratified immersion)**.** Let M be a smooth manifold (with corners) and $X \subset M$ a closed subset. X is called a Whitney-stratified immersion if it can be decomposed into a locally finite, strict partially ordered set $(I, \prec)$ of strata X_i,

$$X = \bigcup_{i \in I} X_i$$

such that

- Each stratum $X_i \subset M$ is a smoothly embedded manifold.
- Each $\overline{X}_i$, the closure of the stratum X_i in M, is a union of strata.
- For any $i, j \in I$, $i \prec j \iff X_i \subset \overline{X}_j$.
- If $i_1, \cdots, i_n$ are pairwise incomparable, then the corresponding strata $X_{i_1}, \cdots,$ X_{i_n} are mutually transverse.

4.2 Blowups along a Whitney-stratified immersion

Let M be a smooth manifold and $X \subset M$ a cone-like Whitney-stratified immersion. This section will provide an algorithm to blow up M along X.

4.2.1 The blowup of a manifold along its smooth submanifold

The blowup of a complex manifold along its complex submanifold is well known to algebraic geometers (see, for example, [26]). Our situation differs from that in two aspects: first, we work with smooth real manifolds; second, instead of replacing a point in the blowup locus with the set of lines passing through the point in its normal space, we replace it with the set of rays in its normal space emanating from this point.

Let us begin with the definition of the blowup of an open disk along a coordinate plane.

Let $S^{n-1} = \mathbb{R}^n - \{0\}/\sim$, where $x \sim x'$ if and only if there exists some number $\lambda > 0$ such that $x = \lambda x'$ for any $x, x' \in \mathbb{R}^n - \{0\}$.

Definition 4.2.1 (Blowup)**.** Let $\mathbb{D}$ be an n-dimensional open disc with smooth coordinates $x_1, \cdots, x_n$, and let $H \subset \mathbb{D}$ be the plane $x_{m+1} = \cdots = x_n = 0$. Define a smooth manifold with boundary

$$\widetilde{\mathbb{D}} = \{(x_1, \cdots, x_n, [y_{m+1}, \cdots, y_n]) \in \mathbb{D} \times S^{n-m-1} | \exists \lambda \geq 0$$

$$\text{s.t. } (x_{m+1}, \cdots, x_n) = \lambda(y_{m+1}, \cdots, y_n)\}.$$

The space $\widetilde{\mathbb{D}}$, together with the map $\pi : \widetilde{\mathbb{D}} \to \mathbb{D}$, which is induced by the projection

$$\mathbb{D} \times S^{n-m-1} \to \mathbb{D}$$

on the first factor, is called the (ray) blowup of $\mathbb{D}$ along H. Later we will denote $\widetilde{\mathbb{D}}$ by $\mathrm{Bl}(\mathbb{D}, H)$.

The blowup map $\pi : \widetilde{\mathbb{D}} \to \mathbb{D}$ is an isomorphism away from H, and the inverse image of a point $x \in H$ is a sphere.

Let

$$\mathbb{R}^n_{j+} = \{(x_1, \cdots, x_n) | x_i \in \mathbb{R}^n, i = 1, \cdots, n; x_j \geq 0\};$$

$$\mathbb{R}^n_{j-} = \{(x_1, \cdots, x_n) | x_i \in \mathbb{R}^n, i = 1, \cdots, n; x_j \leq 0\}$$

and

$$\widetilde{U}_{j+} = \{(x_1, \cdots, x_n, [y_{m+1}, \cdots, y_n]) \in \widetilde{\mathbb{D}} | y_j > 0\};$$

$$\widetilde{U}_{j-} = \{(x_1, \cdots, x_n, [y_{m+1}, \cdots, y_n]) \in \widetilde{\mathbb{D}} | y_j < 0\}$$

for $j = m+1, \cdots, n$. The manifold with boundary $\widetilde{\mathbb{D}}$ can be covered by coordinate charts

$$(\widetilde{U}_{j+}, \phi_{j+}), (\widetilde{U}_{j-}, \phi_{j-})$$

where

$$\phi_{j+} : \widetilde{U}_{j+} \to \mathbb{R}^n_{j+}, (x_1, \cdots, x_n, [y_{m+1}, \cdots, y_n]) \mapsto (x_1,$$

$$\cdots, x_m, \frac{x_{m+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j});$$

$$\phi_{j-} : \widetilde{U}_{j-} \to \mathbb{R}^n_{j-}, (x_1, \cdots, x_n, [y_{m+1}, \cdots, y_n]) \mapsto (x_1,$$

$$\cdots, x_m, \frac{x_{m+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j}),$$

(when $x_j = 0$, replace $\frac{x_i}{x_j}$ with $\frac{y_i}{y_j}$) are smooth coordinates for $j = m+1, \cdots, n$. Later we will refer these coordinate charts as the standard covering coordinate charts of $\widetilde{\mathbb{D}}$.

Proposition 4.2.1. *The blowup $\pi : \widetilde{\mathbb{D}} \to \mathbb{D}$ does not depend on the coordinate system used in $\mathbb{D}$.*

Proof. Let $x'_i = g_i(x), i = 1, \cdots, n$ be another smooth coordinate system in $\mathbb{D}$, $H \subset \mathbb{D}$ the plane $x'_{m+1} = \cdots = x'_n = 0$, and

$$\widetilde{\mathbb{D}}' = \{(x'_1, \cdots, x'_n, [y'_{m+1}, \cdots, y'_n]) \in \mathbb{D} \times S^{n-m-1} | \exists \lambda \geq 0$$

$$\text{s.t. } (x'_{m+1}, \cdots, x'_n) = \lambda(y'_{m+1}, \cdots, y'_n)\}$$

the blowup of $\mathbb{D}$ in this new coordinate system, then we can extend the isomorphism

$$\widetilde{g} : \widetilde{\mathbb{D}} - \pi^{-1}(H) \to \widetilde{\mathbb{D}}' - \pi'^{-1}(H), x \mapsto g(x)$$

over $\pi^{-1}(H)$ by sending a point $(x, [y_{m+1}, \cdots, y_n])$ with $x_{m+1} = \cdots = x_n = 0$ to the point $(g(x), [y'_{m+1}, \cdots, y'_n])$, where

$$y'_j = \sum_{i=m+1}^{n} y_i \frac{\partial g_j}{\partial x_i}(x). \tag{4.2.1}$$

One can easily check that the extension (4.2.1) does not depend on the coordinates chosen. $\square$

Remark 4.2.1. The map

$$(x, y) \mapsto [\sum_{i=m+1}^{n} y_i \frac{\partial}{\partial x_i}\Big|_x]$$

gives an identification of the fiber of $\pi : \pi^{-1}(H) \to H$ over a point $x = (x_1, \cdots, x_m, 0, \cdots, 0)$ with its normal sphere S^{n-m-1}. This identification is independent of the choice of the coordinates.

The blowup construction can be carried out globally. Let M be an n-dimensional smooth manifold, and $N \subset M$ an m-dimensional smooth submanifold. Let $\{\mathbb{D}_\alpha\}_{\alpha \in I}$ be a collection of discs in M covering N such that $N \cap \mathbb{D}_\alpha$ is given by $x_{m+1} = \cdots = x_n = 0$, and let $\pi_\alpha : \widetilde{\mathbb{D}}_\alpha \to \mathbb{D}_\alpha$ be the blowup of $\mathbb{D}_\alpha$ along $N \cap \mathbb{D}_\alpha$, for each $\alpha \in I$. Then we have a family of isomorphisms

$$\pi_{\alpha\beta} : \pi_\alpha^{-1}(\mathbb{D}_\alpha \cap \mathbb{D}_\beta) \to \pi_\beta^{-1}(\mathbb{D}_\alpha \cap \mathbb{D}_\beta).$$

The local blowups $\widetilde{\mathbb{D}}_\alpha$ with π_α can be glued together, by these isomorphisms, to form a manifold with boundary

$$L = \cup_{\pi_{\alpha\beta}} \widetilde{\mathbb{D}}_\alpha$$

with a map

$$\pi : L \to \cup \mathbb{D}_\alpha.$$

We still denote by π the map extending π on L and the identity on $M - N$, and let

$$\mathrm{Bl}(M, N) = L \cup_\pi (M - N).$$

Definition 4.2.2 (Blowup along a submanifold)**.** Let M be an n-dimensional smooth manifold, and $N \subset M$ an m-dimensional smooth submanifold. The manifold with boundary $\mathrm{Bl}(M, N)$, together with the projection map $\pi : \mathrm{Bl}(M, N) \to M$, is called the blowup of M along N.

By the construction, the blowup has the following immediate properties:

Proposition 4.2.2. *Let M be an n-dimensional smooth manifold, and $N \subset M$ an m-dimensional smooth submanifold.*

1. *The map π is an isomorphism away from $N \subset M$ and $\pi^{-1}(N) \subset \mathrm{Bl}(M, N)$.*

2. *(Locality property) Locally the blowup is isomorphic to the blowup of a disc along a coordinate plane as given in Definition 4.2.1.*

3. *$\pi^{-1}(N) \to N$ is a fibre bundle over N with fibre S^{n-m-1}. It can be naturally identified with the normal sphere bundle of N in M.*

4. *The same blowup construction can be carried out if M is a manifold with corners and $N \subset M$ is a transversal submanifold with corners, provided that N has a well-defined normal bundle in M.*

Proof. Statements 1) and 2) follow directly from definitons. Statement 3) follows from Remark 4.2.1. Statement 4) follows from 2) and 3). $\square$

4.2.2 Iterated blowups

Based on the study of the blowup of a smooth manifold along a smooth submanifold in § 4.2.1, this subsection studies the more general case: iterated blowups. We will first look at the local pictures of blowups, and then go to the global picture.

Lemma 4.2.3. *Let V_1 and V_2 be linear subspaces of $\mathbb{R}^n$ of dimensions k and m respectively, where $k < m < n$. If V_1 is a subspace of V_2, then $\overline{\pi^{-1}(V_2 - V_1)}$, the closure of $\pi^{-1}(V_2 - V_1)$ in* $\mathrm{Bl}(\mathbb{R}^n, V_1)$*, has a well-defined normal bundle.*

Proof. Without loss of generality, we can assume

$$V_1 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_{k+1} = \cdots = x_n = 0\};$$

$$V_2 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_{m+1} = \cdots = x_n = 0\}.$$

Let $\{(\widetilde{U}_{j+}, \phi_+), (\widetilde{U}_{j-}, \phi_-)\}_{j=k+1}^n$ be the standard covering coordinate charts of $\mathrm{Bl}(\mathbb{R}^n, V_1)$. We claim that $\{\widetilde{U}_{j+}, \widetilde{U}_{j-}\}_{j=k+1}^m$ is a covering of $\overline{\pi^{-1}(V_2 - V_1)}$. In fact, for all $p \in \overline{\pi^{-1}(V_2 - V_1)}$, there exist $j(k + 1 \leq j \leq n)$, a neighborhood $U(p)$ of p such that $U(p) \subset \widetilde{U}_{j+}$ (or $\widetilde{U}_{j+}$), and a point $\widetilde{x} \in \pi^{-1}(V_2 - V_1) \cap U(p)$ with the coordinates $(x_1, \cdots, x_k, \frac{x_{k+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j})$. But since $x_{m+1} = \cdots = x_n = 0$, and $\lambda \neq 0$, we know $y_{m+1} = \cdots = y_n = 0$, which implies $j \leq m$.

Now since $j \leq m$, each point in $\overline{\pi^{-1}(V_2 - V_1)} \cap \widetilde{U}_{j\pm}$ has coordinates

$$(x_1, \cdots, x_k, \frac{x_{k+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j})$$

$$= (x_1, \cdots, x_k, \frac{x_{k+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_m}{x_j}, 0, \cdots, 0)$$

So each point has a normal space spanned by

$$\{\frac{\partial}{\partial x_{m+1}}, \cdots, \frac{\partial}{\partial x_n}\},$$

which is diffeomorphic to $\mathbb{R}^{n-m}$. □

Lemma 4.2.4. *Let V_1 and V_2 be linear subspaces of $\mathbb{R}^n$ of dimensions k and m respectively, where $k \geq n-m$. If V_1 and V_2 intersect transversely, then $\overline{\pi^{-1}(V_2 - V_1)}$, the closure of $\pi^{-1}(V_2 - V_1)$ in $\mathrm{Bl}(\mathbb{R}^n, V_1)$, has a well-defined normal bundle.*

Proof. Without loss of generality, we can assume

$$V_1 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_{k+1} = \cdots = x_n = 0\};$$

$$V_2 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_1 = \cdots = x_{n-m} = 0\}.$$

Let $\{(\widetilde{U}_{j+}, \phi_+), (\widetilde{U}_{j-}, \phi_-)\}_{j=k+1}^{n}$ be the standard covering coordinate charts of $Bl(\mathbb{R}^n, V_1)$. Each point in $\overline{\pi^{-1}(V_2 - V_1)} \cap \widetilde{U}_{j\pm}$ has the coordinates

$$(x_1, \cdots, x_k, \frac{x_{k+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j})$$

$$= (0, \cdots, 0, x_{n-m+1}, \cdots, x_k, \frac{x_{k+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j})$$

So each point has a normal space spanned by

$$\{\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{n-m}}\},$$

which is diffeomorphic to $\mathbb{R}^{n-m}$. □

Remark 4.2.2. It follows that $\overline{\pi^{-1}(V_2 - V_1)}$ is transverse to the boundary stratum of $\mathrm{Bl}(\mathbb{R}^n, V_1)$.

Lemma 4.2.5. *Let V_1 and V_2 be linear subspaces of $\mathbb{R}^n$ of dimensions k and m respectively, where $k \geq n - m$. If V_1 and V_2 intersect transversely, then*

1. *$\mathbb{R}^n$ can be blown up iteratedly along V_1 and V_2.*

2. *The two manifolds with corners*

$$\mathrm{Bl}(\mathrm{Bl}(\mathbb{R}^n, V_1), \overline{\pi^{-1}(V_2 - V_1)})$$

and

$$\mathrm{Bl}(\mathrm{Bl}(\mathbb{R}^n, V_2), \overline{\pi^{-1}(V_1 - V_2)})$$

are diffeomorphic, where $\overline{\pi^{-1}(V_2 - V_1)}$ *is the closure of* $\pi^{-1}(V_2 - V_1)$ *in* $\mathrm{Bl}(\mathbb{R}^n, V_1)$*, and* $\overline{\pi^{-1}(V_1 - V_2)}$ *is the closure of* $\pi^{-1}(V_1 - V_2)$ *in* $\mathrm{Bl}(\mathbb{R}^n, V_2)$*.*

Proof. The first statement follows from Lemma 4.2.4 and Proposition 4.2.2.

For the second statement, as in the proof of Lemma 4.2.4, we can assume

$$V_1 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_{k+1} = \cdots = x_n = 0\};$$

$$V_2 = \{(x_1, \cdots, x_n) \in \mathbb{R}^n | x_1 = \cdots = x_{n-m} = 0\}.$$

Let $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$. Without loss of generality, we may assume that $x_i, x_j \geq 0$, where i and j are two fixed indices: $i \leq n - m$, $j \geq k$.

In $\widetilde{U}_{j+}$ (or $\widetilde{U}_{j-}$) of $\mathrm{Bl}(\mathbb{R}^n, V_1)$, $\tilde{x} = (x_1, \cdots, x_k, \frac{x_{k+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j})$. In $\widetilde{\widetilde{U}}_{i+}$ (or $\widetilde{\widetilde{U}}_{i-}$) of $\mathrm{Bl}(\mathrm{Bl}(\mathbb{R}^n, V_1), \overline{\pi^{-1}(V_2 - V_1)})$, $\tilde{\tilde{x}} = (\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n-m}}{x_i}, x_{n-m+1}, \cdots, x_k, \frac{x_{k+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j})$.

Similarly, in $\widetilde{\widetilde{U}}_{i+}$ (or $\widetilde{\widetilde{U}}_{i-}$) of $\mathrm{Bl}(\mathrm{Bl}(\mathbb{R}^n, V_2), \overline{\pi^{-1}(V_1 - V_2)})$, $\tilde{\tilde{x}} = (\frac{x_1}{x_i}, \cdots, \frac{x_{i-1}}{x_i}, x_i, \frac{x_{i+1}}{x_i}, \cdots, \frac{x_{n-m}}{x_i}, x_{n-m+1}, \cdots, x_k, \frac{x_{k+1}}{x_j}, \cdots, \frac{x_{j-1}}{x_j}, x_j, \frac{x_{j+1}}{x_j}, \cdots, \frac{x_n}{x_j})$.

So, we get a diffeomorphism between

$$\mathrm{Bl}(\mathrm{Bl}(\mathbb{R}^n, V_1), \overline{\pi^{-1}(V_2 - V_1)})$$

and

$$\mathrm{Bl}(\mathrm{Bl}(\mathbb{R}^n, V_2), \overline{\pi^{-1}(V_1 - V_2)}).$$

□

By induction on the number of subspaces of $\mathbb{R}^n$, we can generalize the above lemma as

Lemma 4.2.6. *Let $V_1, \cdots, V_k$ be linear subspaces of $\mathbb{R}^n$, which are mutually transverse. Then successive blowups along the V_i give a result independent of the order, up to diffeomorphism.*

Definition 4.2.3 (Minimal sequence)**.** Let $(I, \prec)$ be a finite or countably infinite partially ordered set. A sequence s:

$$i_1, i_2, \cdots, i_k, \cdots$$

is called a minimal sequence of I if the following conditions are satisfied:

- i_1 is a minimal element of $(I, \prec)$, i_2 is a minimal element of $(I \setminus \{i_1\}, \prec)$, $\cdots$, i_k is a minimal element of $(I \setminus \{i_1, \cdots, i_{k-1}\}, \prec)$ and so on;
- $I = \{i_1, i_2, \cdots, i_k, \cdots\}$.

Remark 4.2.3. A minimal sequence s of I turns the partially ordered set I into a well-ordered set. The transfinite induction (see, for example, [17]), an extension of mathematical induction, is valid on a well-ordered set.

Example 4.2.1. Let Γ be a connected graph with k vertices. Let I be the set of connected full subgraphs of Γ. Define a partial order on I by

$$\Gamma_i \prec \Gamma_j \Leftrightarrow V(\Gamma_i) \supset V(\Gamma_j).$$

Here is a minimal sequence of the poset $(I, \prec)$: beginning with the graph Γ itself, then the connected full subgraphs with $(k-1)$ vertices in an arbitrary order, then the connected full subgraphs with $(k-2)$ vertices in an arbitrary order, and so on.

Theorem 4.2.7. *Let M be a smooth manifold and $X = \bigcup_{i \in I} X_i$ a cone-like, Whitney-stratified immersion.*

1. *Let s be a minimal sequence of I, then M can be blown up successively along strata X_i of X in the order specified by s. Denote the final resulted manifold with corners by* $\mathrm{Bl}(M, (X, s))$.

2. *Let s, s' be minimal sequences of I, then* $\mathrm{Bl}(M, (X, s))$ *is diffeomorphic to* $\mathrm{Bl}(M, (X, s'))$.

Proof. The second statement follows from the first statement, Lemma 4.2.6 and the locality property of blowups.

Here is the proof of the first statement:

Let the sequence s:

$$i_1, i_2, \cdots, i_k, \cdots$$

be a minimal sequence of the partially ordered set I. As given in Definiton 4.2.2, the smooth manifold M can be blown up along the stratum X_{i_1}, resulting in a manifold with corners $\mathrm{Bl}(M, X_{i_1})$.

We notice that the partial order are preserved under the blowup process:

1. If $i, j \neq i_1 \in I$ and $X_i \subset \overline{X_j}$, then it is easy to see that

$$\overline{\pi^{-1}(X_i - X_{i_1})} \subset \overline{\pi^{-1}(\overline{X_j} - X_{i_1})}.$$

   It follows that if $i \prec j \neq i_1$ in I, then we have $i \prec j$ in $I \setminus \{i_1\}$ after the blowup.

2. If $k_1, \cdots, k_m \neq i_1 \in I$ and $X_{k_1}, \cdots, X_{k_m}$ are mutually transverse, then in $\mathrm{Bl}(M, X_{i_1})$, $\overline{\pi^{-1}(X_{k_1} - X_{i_1})}, \cdots, \overline{\pi^{-1}(X_{k_m} - X_{i_1})}$ are mutually transverse. To ease of notation, let us consider only their complete intersection,

$$\begin{aligned}
& \operatorname{codim} \overline{\pi^{-1}(X_{k_1} - X_{i_1})} + \cdots + \operatorname{codim} \overline{\pi^{-1}(X_{k_m} - X_{i_1})} \\
= \;& \operatorname{codim} X_{k_1} + \cdots + \operatorname{codim} X_{k_m} \\
= \;& \operatorname{codim} X_{k_1} \cap \cdots \cap X_{k_m} \\
= \;& \operatorname{codim} (\pi^{-1}(X_{k_1} \cap \cdots \cap X_{k_m} - X_i)) \\
= \;& \operatorname{codim} \overline{\pi^{-1}(X_{k_1} \cap \cdots \cap X_{k_m} - X_{i_1})} \\
= \;& \operatorname{codim} \overline{\pi^{-1}(X_{k_1} - X_{i_1})} \cap \cdots \cap \overline{\pi^{-1}(X_{k_m} - X_{i_1})}.
\end{aligned}$$

   It follows that if $k_1, \cdots, k_m$ are pairwise incomparable in I, then we have $k_1, \cdots, k_m$ are pairwise incomparable in $I \setminus \{i_1\}$ after the blowup.

Now let us consider the blowup preimages of the other strata X_j of X in $\mathrm{Bl}(M, X_{i_1})$:

- For those indices j such that i_1, j are incomparable, we have X_{i_1}, X_j are transverse in M. By Lemma 4.2.4, and the locality property of blowups, $\overline{\pi^{-1}(X_j - X_{i_1})}$ has a well-defined normal bundle in $\mathrm{Bl}(M, X_{i_1})$.

- For those indices j such that $i_1 \prec j$, we have $X_{i_1} \subset \overline{X_j}$, where the closure is taken in M. So if $i_1 \prec j_1, j_2, \cdots, j_k$, then $X_{i_1} \subset \overline{X_{j_1}} \cap \overline{X_{j_2}} \cap \cdots \cap \overline{X_{j_k}}$. Since X

is locally cone-like, for any $p \in X_{i_1}$, there exist a neighborhood $U(p) \subset X$ of p which is diffeomorphic to the tangent cone T_pX with p the tip of the cone. If in addition we assume that $j_1, j_2, \cdots, j_k$ are pairwise incomparable, then it follows that in $\mathrm{Bl}(M, X_{i_1})$, $\overline{\pi^{-1}(X_{j_1} - X_{i_1})}, \overline{\pi^{-1}(X_{j_2} - X_{i_1})}, \cdots, \overline{\pi^{-1}(X_{j_k} - X_{i_1})}$ are disjoint in a sufficiently small neighborhood of $\pi^{-1}(X_{i_1})$. By Lemma 4.2.3 and the locality property of blowups, each $\overline{\pi^{-1}(X_{j_s} - X_{i_1})}$, $s \in \{1, \cdots, k\}$, has a well-defined normal bundle in $\mathrm{Bl}(M, X_{i_1})$.

Thus $\overline{\pi^{-1}(X_{i_2} - X_{i_1})}$ has a well-defined normal bundle in $\mathrm{Bl}(M, X_{i_1})$. So we can blow up $\mathrm{Bl}(M, X_{i_1})$ along $\overline{\pi^{-1}(X_{i_2} - X_{i_1})}$.

Suppose that we already finished the blowups along the strata corresponding to indices $i_1, i_2, \cdots, i_k$, and have the following:

- For any $i, j \in I \backslash \{i_1, i_2, \cdots, i_k\}$, $i \prec j$ if and only if the preimage in the resulted partially blown-up space Y of the stratum X_i is a subset of the closure in Y of the preimage of X_j.

- If $k_1, \cdots, k_m \in I \backslash \{i_1, i_2, \cdots, i_k\}$ are pairwise incomparable, then the preimage in Y of the corresponding strata $X_{k_1}, \cdots, X_{k_m}$ are mutually transverse.

- The preimage in Y of each stratum of X with index in $I \backslash \{i_1, i_2, \cdots, i_k\}$ has well-defined normal bundle.

So we can continue blowing up the partially blown up space Y along the preimage of the next stratum $X_{i_{k+1}}$.

By the principal of induction, the iterated blowup procedure can proceed along every stratum of X. □

4.3 Compactifications of configuration spaces of graphs

Let Γ be a connected graph and M a d-dimensional smooth manifold. Let $\mathrm{Map}(V(\Gamma), M)$ be the space of maps $f : V(\Gamma) \to M$. It is easy to see that $\mathrm{Map}(V(\Gamma), M)$ is isomorphic to the Cartesian product $M^{\times|V(\Gamma)|}$.

If Γ' is a connected full subgraph of Γ, denote by $\Delta_{\Gamma'}$ the diagonal of $\mathrm{Map}(V(\Gamma), M)$ in which all vertices of Γ' are mapped to the same point.

If one could blow up $\mathrm{Map}(V(\Gamma), M)$ along

$$\{\Delta_e \subset \mathrm{Map}(V(\Gamma), M) | e \in E(\Gamma)\},$$

then a compactification of the configuration space of the graph Γ in the manifold M would be obtained. But unfortunately, there are some technical difficulties in doing so. Instead, there is a compactification of the configuration space of the graph, which is obtained by iteratively blowing up all connected full subgraphs of Γ. To see this, one needs the following theorem

Theorem 4.3.1. *Let Γ be a connected graph and M a smooth manifold. Let*

$$X = \bigcup_{\Gamma'} \{\Delta_{\Gamma'}\} \subset \mathrm{Map}(V(\Gamma), M),$$

where Γ' varies over the set of connected full subgraphs of Γ. Then X admits a cone-like, Whitney-stratified immersion structure in $\mathrm{Map}(V(\Gamma), M)$.

Proof. The subset $X \subset \mathrm{Map}(V(\Gamma), M)$ is cone-like since it is the union of a set of diagonals.

Now we check that X admits a Whitney-stratified immersion structure whose strata are corresponding to diagonals in certain way which we now describe.

Let Γ be a connected graph and I the set of connected full subgraphs of Γ. We say a filtration

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k,$$

where $\Gamma_1 = \Gamma$, Γ_k is a two-vertex connected full subgraph of Γ, $\Gamma_i \in I$ and the vertex set $V(\Gamma_{i-1}) \supsetneq V(\Gamma_i)$, is *good* if there is no $\Gamma' \in I$ such that $V(\Gamma_{i-1}) \supsetneq V(\Gamma') \supsetneq V(\Gamma_i)$.

Now let

$$\Gamma_1 \supset \Gamma_2 \supset \cdots \supset \Gamma_k$$

be a good filtration, then we say

$$\Delta_{\Gamma_1}, \Delta_{\Gamma_2} \backslash \Delta_{\Gamma_1}, \cdots, \Delta_{\Gamma_k} \backslash \Delta_{\Gamma_{k-1}}$$

are strata coming from the filtration.

Consider all the strata coming from good filtrations on I. For any connected full subgraph Γ_i, let

$$\Delta_{\Gamma_i} \backslash \Delta_{\Gamma_{i-1,1}}, \cdots, \Delta_{\Gamma_i} \backslash \Delta_{\Gamma_{i-1,m}}$$

be all the strata which have the form $\Delta_{\Gamma_i} \backslash \Delta_{\Gamma_{i-1}}$. Define

$$X_{\Gamma_i} = (\Delta_{\Gamma_i} \backslash \Delta_{\Gamma_{i-1,1}}) \cap \cdots \cap (\Delta_{\Gamma_i} \backslash \Delta_{\Gamma_{i-1,m}}),$$

and $X_\Gamma = \Delta_\Gamma$. Thus we obtain a decomposition

$$X = \bigcup_{\Gamma_i \in I} X_{\Gamma_i}.$$

We call X_{Γ_i} the stratum of X corresponding to the subgraph Γ_i.

On I, we define

$$\Gamma_i \prec \Gamma_j \Leftrightarrow X_{\Gamma_i} \subset \overline{X_{\Gamma_j}}.$$

Since by our construction, $\overline{X_{\Gamma_i}} = \Delta_{\Gamma_i}$, for some $\Gamma_i \in I$, $\overline{X_{\Gamma_i}}$ is a union of strata. If $\overline{X_{\Gamma_i}} = \Delta_{\Gamma_i}$, $\overline{X_{\Gamma_j}} = \Delta_{\Gamma_j}$, then

$$\Gamma_i \prec \Gamma_j \Leftrightarrow X_{\Gamma_i} \subset \overline{X_{\Gamma_j}} \Leftrightarrow V(\Gamma_i) \supset V(\Gamma_j).$$

It remains to check that if $\Gamma_{i_1}, \cdots, \Gamma_{i_k}$ are pairwise incomparable, then the corresponding strata $X_{\Gamma_{i_1}}, \cdots, X_{\Gamma_{i_k}}$ are mutually transverse. We do this by induction on k.

When $k = 2$. Let Γ_1, Γ_2 be connected full subgraphs of Γ. We have $\overline{X_{\Gamma_1}} = \Delta_{\Gamma_1}$, $\overline{X_{\Gamma_2}} = \Delta_{\Gamma_2}$.

i) If $V(\Gamma_1) \cap V(\Gamma_2) \neq \phi$ and $\Gamma_1 \cup \Gamma_2$ is a full subgraph, then $\Delta_{\Gamma_1} \cap \Delta_{\Gamma_2} = \Delta_{\Gamma_1 \cup \Gamma_2}$. So $X_{\Gamma_1} \cap X_{\Gamma_2} = \phi$.

ii) If $V(\Gamma_1) \cap V(\Gamma_2) \neq \phi$ and $\Gamma_1 \cup \Gamma_2$ is not a full subgraph, let Γ' be the connected full subgraph of Γ with vertex set $V(\Gamma_1 \cup \Gamma_2)$. Then we have

$$\Delta_{\Gamma_1} \cap \Delta_{\Gamma_2} = \Delta_{\Gamma_1 \cup \Gamma_2} = \Delta_{\Gamma'}.$$

So $X_{\Gamma_1} \cap X_{\Gamma_2} = \phi$.

iii) Otherwise $V(\Gamma_1) \cap V(\Gamma_2) = \phi$. Then a point on $\Delta_{\Gamma_1} \cap \Delta_{\Gamma_2}$ can be viewed as the graph obtained from Γ by contracting Γ_1 to a vertex and Γ_2 to another vertex. Thus

$$\operatorname{codim} X_{\Gamma_1} \cap X_{\Gamma_2} = \operatorname{codim} \Delta_{\Gamma_1} \cap \Delta_{\Gamma_2} = (|V(\Gamma_1)| + |V(\Gamma_2)| - 2)d.$$

Now since

$$\operatorname{codim} X_{\Gamma_1} = \operatorname{codim} \Delta_{\Gamma_1} = (|V(\Gamma_1)| - 1)d;$$

$$\operatorname{codim} X_{\Gamma_2} = \operatorname{codim} \Delta_{\Gamma_2} = (|V(\Gamma_2)| - 1)d,$$

it follows that

$$\operatorname{codim} X_{\Gamma_1} \cap X_{\Gamma_2} = \operatorname{codim} X_{\Gamma_1} + \operatorname{codim} X_{\Gamma_2}.$$

When $k > 2$. From definition we have $\overline{X_{\Gamma_{i_1}}} = \Delta_{\Gamma_{i_1}}, \overline{X_{\Gamma_{i_2}}} = \Delta_{\Gamma_{i_2}}, \cdots, \overline{X_{\Gamma_{i_k}}} = \Delta_{\Gamma_{i_k}}$. By induction, it's enough to consider their complete intersection $X_{\Gamma_{i_1}} \cap \cdots \cap X_{\Gamma_{i_k}}$ only.

i) If there exist $s, t \in \{i_1, i_2, \cdots, i_k\}$ such that $V(\Gamma_s) \cap V(\Gamma_t) \neq \phi$ and $\Gamma_s \cup \Gamma_t$ is a full subgraph of Γ, then $X_{\Gamma_s} \cap X_{\Gamma_t} = \phi$, which implies that

$$X_{\Gamma_{i_1}} \cap \cdots \cap X_{\Gamma_{i_k}} = \phi.$$

ii) If there exist $s, t \in \{i_1, i_2, \cdots, i_k\}$ such that $V(\Gamma_s) \cap V(\Gamma_t) \neq \phi$ and $\Gamma_s \cup \Gamma_t$ is not a full subgraph of Γ, let Γ' be the full subgraph with vertex set $V(\Gamma_s \cup \Gamma_t)$. Then

$$\Delta_{\Gamma_s} \cap \Delta_{\Gamma_t} = \Delta_{\Gamma_s \cup \Gamma_t} = \Delta_{\Gamma'}.$$

So $X_{\Gamma_s} \cap X_{\Gamma_t} = \phi$. This implies that

$$X_{\Gamma_{i_1}} \cap \cdots \cap X_{\Gamma_{i_k}} = \phi.$$

iii) Otherwise, $V(\Gamma_{i_1}), V(\Gamma_{i_2}), \cdots, V(\Gamma_{i_k})$ are pairwise disjoint. On the one hand, a point on $\Delta_{\Gamma_{i_1}} \cap \cdots \cap \Delta_{\Gamma_{i_k}}$ can be viewed as a graph obtained from Γ by collapsing each $\Gamma_{i_j} (j = 1, \cdots, k)$ to a vertex. Thus

$$\begin{aligned} \text{codim } X_{\Gamma_{i_1}} \cap \cdots \cap X_{\Gamma_{i_k}} &= \text{codim } \Delta_{\Gamma_{i_1}} \cap \cdots \cap \Delta_{\Gamma_{i_k}} \\ &= (|V(\Gamma_{i_1})| + \cdots + |V(\Gamma_{i_k})| - k)d. \end{aligned}$$

On the other hand,

$$\text{codim } X_{\Gamma_{i_j}} = \text{codim } \Delta_{\Gamma_{i_j}} = (|V(\Gamma_{i_j})| - 1)d, \, j = 1, \cdots, k.$$

So we have,

$$\text{codim } X_{\Gamma_1} \cap \cdots \cap X_{\Gamma_k} = \text{codim } X_{\Gamma_1} + \cdots + \text{codim } X_{\Gamma_k}.$$

□

Corollary 4.3.2. *The space* $\mathrm{Map}(V(\Gamma), M)$ *can be blown up along* $X = \cup\{\Delta_{\Gamma'}\}$*, the set of diagonals corresponding to all connected full subgraphs.*

Proof. It follows immediately from Theorems 4.2.7 and 4.3.1. □

The resulted space in the above corollary will be denoted by $\overline{\mathrm{Conf}(\Gamma, M)}$.

Corollary 4.3.3. *Let* Γ *be a connected finite graph and M a smooth compact manifold. Then* $\overline{\mathrm{Conf}(\Gamma, M)}$ *is compact.*

Proof. It follows immediately from Theorems 4.2.7 and 4.3.1. □

Let Γ be a connected finite graph. If the space M is compact, by the above Corollary 4.3.3, $\overline{\mathrm{Conf}(\Gamma, M)}$ is a compact space; if M is not compact, then in general $\overline{\mathrm{Conf}(\Gamma, M)}$ is not compact. For example, the space $\overline{\mathrm{Conf}(\Gamma, \mathbb{R}^d)}$ is not compact, because a sequence of configurations going to the spatial infinity has no limit. But in both cases, we will follow the history and call the iterated blowup procedure from $\mathrm{Map}(V(\Gamma), M)$ to $\overline{\mathrm{Conf}(\Gamma, M)}$ a compactification of the configuration space $\mathrm{Conf}(\Gamma, M)$, and call $\overline{\mathrm{Conf}(\Gamma, M)}$ the compactified configuration space of the graph Γ in the space M.

Remark 4.3.1. There are different compactifications of configuration spaces. But the compactification constructed as above, i.e., the compactification obtained by iteratively blown up diagonals corresponding to all connected subgraphs in the order specified by a minimal sequence as in Example 4.2.1, is the one we are most interested of. It will be referred to as the canonical compactification of the configuration space of a graph in a manifold.

In one of the examples of other compactifications, the space $\mathrm{Map}(V(\Gamma), M)$ are blown up along the set of its diagonals which correspond to all vertex-2-connected diagonals only, due to the following proposition

Proposition 4.3.4. *Let Γ be a connected graph and M a smooth manifold. Let*

$$X = \bigcup_{\Gamma'} \{\Delta_{\Gamma'}\} \subset \mathrm{Map}(V(\Gamma), M),$$

where Γ' varies over the set of vertex-2-connected full subgraphs of Γ. Then X admits a cone-like, Whitney-stratified immersion structure in $\mathrm{Map}(V(\Gamma), M)$.

Proof. Similar to the proof of Theorem 4.3.1, see Kuperberg-Thurston [31]. □

The points in $\overline{\mathrm{Conf}(\Gamma, M)}$, or other spaces that we will form from it, will be called configurations of the graph Γ in the space M.

Corollary 4.3.5. *Let M be a smooth manifold and Γ a connected graph. The compactified configuration space $\overline{\mathrm{Conf}(\Gamma, M)}$ is a manifold with corners.*

Proof. This follows immediately from the fact that $\overline{\mathrm{Conf}(\Gamma, M)}$ can be constructed by the iterated ray blowups from the manifold $\mathrm{Map}(V(\Gamma), M)$ (see Theorem 4.2.7). □

Proposition 4.3.6. *Let M be a smooth manifold and Γ a connected graph. The two spaces $\overline{\mathrm{Conf}(\Gamma, M)}$ and $\mathrm{Conf}(\Gamma, M)$ are homotopy equivalent.*

Proof. This is obvious. □

Chapter 5

Boundary fibrations of $\overline{\mathrm{Conf}(\Gamma, M)}$ and $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$

This chapter is devoted to the study of the fibration structures on the boundary strata of the canonical compactified configuration spaces of graphs in d-dimensional smooth closed manifolds, and of the compactified moduli spaces of configurations of graphs in the d-dimensional Euclidean space $\mathbb{R}^d$, where $d > 1$. In particular, in these cases, the bases and fibers of these bundles are also configuration spaces.

5.1 Boundary fibrations of $\overline{\mathrm{Conf}(\Gamma, M)}$

Let Γ be a connected graph and M a smooth manifold. Let $\overline{\mathrm{Conf}(\Gamma, M)}$ be the canonical compactified configuration space of Γ in M, obtained by iteratively blowing up the space $\mathrm{Map}(V(\Gamma), M)$ along its set of diagonals corresponding to connected full subgraphs of Γ (see Chapter 4). By Corollary 4.3.5, $\overline{\mathrm{Conf}(\Gamma, M)}$ is a smooth manifold with corners. According to Theorem 4.3.1, each codimension one stratum of this manifold with corners corresponds to a connected full subgraph γ of the graph Γ. Let F_γ denote the stratum corresponding to γ.

Proposition 5.1.1. *Let Γ be a connected graph and M a smooth manifold. The* *stratum F_γ of $\overline{\mathrm{Conf}(\Gamma, M)}$ admits a fibre bundle* [42] *structure where the base is isomorphic to the space* $\mathrm{Conf}(\Gamma/\gamma, M)$.

Proof. By the construction in Theorems 4.2.7 and 4.3.1, the stratum F_γ is obtained from the normal sphere bundle of the diagonal corresponding to γ, with its intersections with all diagonals corresponding to connected full subgraphs of Γ/γ blown up. So F_γ is a fibre bundle over a base which is isomorphic to $\mathrm{Conf}(\Gamma/\gamma, M)$. □

Remark 5.1.1. There exist examples (see Example 5.1.1) of graphs Γ and a version (see Proposition 4.3.4) of compactification of configuration spaces $\mathrm{Conf}(\Gamma, M)$ of graphs Γ, in which the space $\mathrm{Map}(V(\Gamma), M)$ is blown up along the set of diagonals corresponding to vertex-2-connected full subgraphs of Γ, such that the compactified space of $\mathrm{Conf}(\Gamma, M)$ does not satisfy the property in Proposition 5.1.1.

Example 5.1.1. Let Γ be the graph defined by

$$V(\Gamma) = \{1, 2, 3, 4, 5, 6, 7\},$$

$$E(\Gamma) = \{\{1, 2\}, \{1, 3\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{4, 6\}, \{4, 7\}, \{5, 6\}, \{6, 7\}\},$$

and Γ' be the edge with endpoints $\{4, 6\}$. In the vertex-2-connected subgraph version of compactification (see Proposition 4.3.4), the base of the fibration whose total space is the diagonal corresponding to Γ' is not $\mathrm{Conf}(\Gamma/\Gamma', M)$, but the space obtained by blowing up $\mathrm{Conf}(\Gamma/\Gamma', M)$ along the diagonal corresponding to the connected full subgraph γ with $V(\gamma) = \{\{46\}, 5, 7\}$, which is not a vertex-2-connected subgraph of the graph Γ/Γ'.

Recall that a smooth manifold M is parallelizable if it admits a smooth global frame, or equivalently (see, for example [33]), if its tangent bundle TM is a product bundle. Examples of parallelizable manifolds include $\mathbb{R}^d$, Lie groups and orientable 3-manifolds [43].

Theorem 5.1.2. *If M is a d-dimensional parallelizable smooth manifold, then each boundary fibration F_γ of $\overline{\mathrm{Conf}(\Gamma, M)}$ is a product bundle with base $\mathrm{Conf}(\Gamma/\gamma, M)$, where γ is a connected full subgraph of Γ.*

Proof. By Proposition 5.1.1, we only need to check that the boundary fibration is a product bundle.

Recall the construction of the compactification of configuration space. The boundary stratum F_γ is obtained by blowup the diagonal Δ_γ in $\mathrm{Map}(V(\Gamma), M)$, which is isomorphic to the cartesian product of several copies of M. Since M is parallelizable, there exist global framings on M. Fix a such framing f, then f induces global framings on Δ_γ and $\mathrm{Map}(V(\Gamma), M)$. It follows that the normal bundle of each diagonal $\Delta_\gamma \subset \mathrm{Map}(V(\Gamma), M)$ is trivial; so is the normal sphere bundle of Δ_γ. Thus F_γ is a product bundle. □

5.2 Boundary fibrations of $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$

Let

$$G = \{g : \mathbb{R}^d \to \mathbb{R}^d | g(x) = ax + b, a > 0, b \in \mathbb{R}^d\}$$

be the group of translations and dilations of $\mathbb{R}^d$.

The group G has a left action on the space $\mathrm{Conf}(\Gamma, \mathbb{R}^d)$.

Proposition 5.2.1. *Let Γ be a connected graph with $V(\Gamma) > 1$, then the left group action of G on $\mathrm{Conf}(\Gamma, \mathbb{R}^d)$ is free.*

Proof. Let $g \in G$ be a non identity element of the group G. Then the equation $gx = x$ has at most one solution in $\mathbb{R}^d$, since it is a linear equation. So if $c \in \mathrm{Conf}(\Gamma, \mathbb{R}^d)$ is a configuration with at least two vertices, it is impossible to have $gc = c$. □

Definition 5.2.1 (Moduli space of configurations)**.** Let Γ be a connected graph with $V(\Gamma) > 1$. The space

$$\mathrm{conf}(\Gamma, \mathbb{R}^d) = \mathrm{Conf}(\Gamma, \mathbb{R}^d)/G$$

will be called the moduli space of configurations of the graph Γ in $\mathbb{R}^d$.

Proposition 5.2.2. *Let Γ be a connected graph with $V(\Gamma) > 1$, then the group G acts on $\overline{\mathrm{Conf}(\Gamma, \mathbb{R}^d)}$ freely.*

Proof. The elements of any orbit of the group G belong to one stratum of the manifold with corners $\overline{\mathrm{Conf}(\Gamma, \mathbb{R}^d)}$. And by Proposition 5.2.1 and Theorem 5.1.2, the group G acts on each stratum of $\overline{\mathrm{Conf}(\Gamma, \mathbb{R}^d)}$ freely. □

Definition 5.2.2 (Compactified moduli space)**.** Let Γ be a connected graph with $V(\Gamma) > 1$. The space

$$\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)} = \overline{\mathrm{Conf}(\Gamma, \mathbb{R}^d)}/G$$

will be called the compactified moduli space of configurations of the graph Γ in $\mathbb{R}^d$.

Example 5.2.1. Let Γ be a connected graph with two vertices. The space $\mathrm{conf}(\Gamma, \mathbb{R}^d)$, which is diffeomorphic to the $(d-1)$-dimensional sphere S^{d-1}, is already compact and thus is isomorphic to $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$.

Proposition 5.2.3. *Let Γ be a connected graph and γ a connected full subgraph with $V(\gamma) > 1$. Let M be a smooth manifold of dimension $d > 1$, then the fiber of the boundary fibration F_γ of $\overline{\mathrm{Conf}(\Gamma, M)}$ is isomorphic to $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$.*

Proof. This is immediate by the construction of $\overline{\mathrm{Conf}(\Gamma, M)}$ (see Chapter 4) and the definition of $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$. $\square$

Proposition 5.2.4. *Let Γ be a connected graph. Then the space $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ is a manifold with corners.*

Proof. The space $\overline{\mathrm{Conf}(\Gamma, \mathbb{R}^d)}$ is a manifold with corners (see Corollary 4.3.5), and it allows a free left G-action which preserves the strata. So $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ is a manifold with corners, whose strata are induced from those of $\overline{\mathrm{Conf}(\Gamma, \mathbb{R}^d)}$. $\square$

In Theorem 5.1.2, let $M = \mathbb{R}^d$. Let $E_\gamma = F_\gamma / G$.

Proposition 5.2.5. *Let Γ be a connected graph and γ a connected full subgraph with $V(\gamma) > 1$. Then the boundary stratum E_γ of $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ has a product bundle structure, where the base is isomorphic to $\mathrm{conf}(\Gamma/\gamma, \mathbb{R}^d)$ and the fiber is isomorphic to $\overline{\mathrm{conf}(\gamma, \mathbb{R}^d)}$.*

Proof. By Theorem 5.1.2, each codimension one stratum of $\overline{\mathrm{Conf}(\Gamma, \mathbb{R}^d)}$ is a trivial fiber bundle. The group G acts trivially on the fibre direction of each codimension one stratum E_γ of $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$. So the transition group of the bundle has only one element. Thus the fibre bundle is trivial. $\square$

Proposition 5.2.6. *Let Γ be a finite connected graph. Then the space $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ is compact.*

Proof. Since Γ is a finite graph, the number of strata of $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ is finite. Then the proposition follows from the fact that every sequence in $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ has a convergent subsequence, the limit point of which belongs to $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$. $\square$

Proposition 5.2.7. *The two spaces $\mathrm{Conf}(\Gamma, \mathbb{R}^d)$ and $\mathrm{conf}(\Gamma, \mathbb{R}^d)$ are homotopy equivalent.*

Proof. Let G be the Lie group of translations and dilations in $\mathbb{R}^d$. Since the G-action on $\mathrm{Conf}(\Gamma, \mathbb{R}^d)$ is free, we have a fibration

$$\begin{tikzcd} G \arrow[r] & \mathrm{Conf}(\Gamma, \mathbb{R}^d) \arrow[d, "p"] \\ & \mathrm{conf}(\Gamma, \mathbb{R}^d). \end{tikzcd}$$

Let $p_* : \pi_k(\mathrm{Conf}(\Gamma, \mathbb{R}^d)) \to \pi_k(\mathrm{conf}(\Gamma, \mathbb{R}^d))$ be the map induced by p. By the long homotopy exact sequence associated to the above fibration, p_* is an isomorphism for any k, since the Lie group G is contractible.

Now the statement follows by applying Whitehead's theorem (see, for example, p. 346 in [27]). $\square$

Chapter 6

Insertion products on chains of configuration spaces of graphs

On the moduli spaces of configurations of graphs, and the compactified configuration spaces of graphs in parallelizable manifolds, chain level insertion operations are induced by inserting a graph into another. In this chapter, we will show that the fundamental chains of the canonically compactified moduli spaces of configurations of graphs in the Euclidean space $\mathbb{R}^d$, and certain modified compactified configuration spaces of graphs in the parallelizable manifolds provide a solution to a master equation system of the form $\partial R + R * R = 0, \partial M + M * R = 0$.

6.1 Geometric chains of Whitney-stratified spaces

Although essentially the results in this thesis do not depend on the models of chains used, as long as certain out (or inner) normal direction can be made sense in the chain models, appropriate models of chains are helpful in understanding. A nice discussion of some ideas on chains can be found at the appendix of Wilson's Ph.D. thesis [48].

We will take a geometric chain model, as given in Goresky [25]. Here we list some definitions.

Definition 6.1.1 (Whitney object)**.** Let X be a closed subset of a smooth manifold M.

A Whitney stratification of X is a filtration

$$X_0 \subset X_1 \subset ... \subset X_n = X$$

where each X_i is closed and $X_i - X_{i-1}$ is a locally finite union of i-dimensional submanifolds of M, such that each pair of such submanifolds satisfies Whitney's condition B[1]. X_i is called the i-skeleton. The components of X_i are called i-dimensional strata of X.

The closed subset X with its Whitney stratification is called a Whitney object.

A closed subset $W \subset X$ is called a Whitney substratified object if it has a Whitney stratification such that each of its strata is contained in a single stratum of X.

Example 6.1.1. Manifolds with corners are Whitney stratified objects. Let M be a d-dimensional manifold with corners, its Whitney stratification is given by

$$X_{d-k} = \bigcup_{k \leq i \leq d} \overline{\partial_i M}, k = 0, 1, \cdots, d.$$

Definition 6.1.2 (Geometric chain)**.** Let X be a Whitney stratified object. A geometric k-chain α in X is a triple $(|\alpha|, o, m)$, where $|\alpha| \subset X$ is a compact k-dimensional Whitney substratified object, called the support of α; o is an orientation of $|\alpha|$; m is the multiplicity of each k-dimensional stratum.

[1] see [24]

The reduction of a geometric k-chain α is the geometric chain whose support is the closure of the union of all components of $|\alpha| - |\alpha|_{k-1}$ that have been assigned a nonzero multiplicity.

The boundary $\partial\alpha$ of a geometric k-chain α is the (reduction of the) geometric $(k-1)$-chain whose support is α_{k-1} and with the induced orientation.

A geometric k-cycle is a geometric k-chain α such that $\partial\alpha = 0$.

Remark 6.1.1.

1. A chain with a given orientation and multiplicity will be identified with the chain with the opposite orientation and negative multiplicity, just as in Equation (3.2.2).
2. As in [25], one can identify a geometric chain with its reduction and no confusion will be caused.

In this thesis, we will take the induced orientation on the boundary of a chain to be the so called "the out normal first orientation". That is, if α is a geometric chain, β is a connected boundary component of α, and v is an outward pointing normal vector of $\beta \subset \alpha$, then

$$v \oplus o_\beta = o_\alpha.$$

Let X be a Whitney stratified object. We will use $C_*(X)$ to denote the complex of geometric chains of X over a unital commutative ring $\mathbf{R}$, for example $\mathbb{Z}$.

Proposition 6.1.1. *Let Γ be a finite connected graph and M a smooth compact manifold, then*

- $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ *is a geometric chain.*
- $\overline{\mathrm{Conf}(\Gamma, M)}$ *is a geometric chain.*

Proof. By Propositions 5.2.4 and 5.2.6, the space $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ is a compact manifold with corners. The Whitney stratification on $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ is given by its manifold with corners structure.

Similarly for $\overline{\mathrm{Conf}(\Gamma, M)}$. By Corollary 4.3.3 it is compact, and by Corollary 4.3.5, it is a manifold with corners. □

6.2 Insertion products on chains of $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$

The insertion operations of configurations of graphs induce insertion operations on the chains of moduli spaces of configurations of graphs. We will study their properties in this section.

Definition 6.2.1. Let Γ_1, Γ_2 be connected $\mathcal{A}$-labeled graphs and Γ an $\mathcal{A}$-labeled graph which can be obtained by inserting Γ_2 into a vertex v of Γ_1. For any integers $i, j \geq 0$, define a bilinear operation

$$*_{((\Gamma_1,v),\Gamma_2,\Gamma)} : C_i(\overline{\mathrm{conf}(\Gamma_1, \mathbb{R}^d)}) \otimes C_j(\overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)}) \to C_{i+j}(\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}),$$

$$x \otimes y \mapsto z$$

where the support $|z|$ of chain z is a fibration with base $|x|$ and fiber $|y|$ so that this fibration is the restriction of the codimension one boundary fibration of $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$.

Proposition 6.2.1. *Let Γ_1, Γ_2 be connected $\mathcal{A}$-labeled graphs and Γ an $\mathcal{A}$-labeled graph which can be obtained by inserting Γ_2 into a vertex v of Γ_1. If $\alpha \in C_*(\overline{\mathrm{conf}(\Gamma_1, \mathbb{R}^d)})$, $\beta \in C_*(\overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)})$, then*

$$\partial(\alpha *_{((\Gamma_1,v),\Gamma_2,\Gamma)} \beta) = (\partial\alpha) *_{((\Gamma_1,v),\Gamma_2,\Gamma)} \beta + \alpha *_{((\Gamma_1,v),\Gamma_2,\Gamma)} \partial\beta$$

up to signs, where ∂ is the boundary operator of geometric chains. In other words, $*_{((\Gamma_1,v),\Gamma_2,\Gamma)}$ commutes with ∂.

Proof. Firstly, as topological spaces (the support of chains), the formula is correct: the boundary of the product consists of two parts: the boundary of the first factor times the second factor, and the first factor times the boundary of the second factor. Secondly, the differential ∂ is a linear map, so multiplicities are preserved. Lastly, both sides are with the induced orientation from $\alpha *_{((\Gamma_1,v),\Gamma_2,\Gamma)} \beta$. □

Proposition 6.2.2. *The sum over $\mathcal{AG}_{\geq 2}$ of the complexes of geometric chains of the canonically compactified moduli spaces of graphs is a differential graded $\mathcal{AG}_{\geq 2}$-sorted algebra with the same signature as the configuration algebra in Proposition 3.2.7, where the product is a collection of the insertions $*$.*

Proof. The proof that it is an $\mathcal{AG}_{\geq 2}$-sorted algebra is the same as Proposition 3.2.7. It is obvious that these two algebras are of the same signature.

That it is a differential graded $\mathcal{AG}_{\geq 2}$-sorted algebra then follows from Proposition 6.2.1. □

Proposition 6.2.3. *Let $\Gamma \in \mathcal{AG}_{\geq 2}$. As geometric chains,*

$$\partial\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)} = \sum_{\gamma} \overline{\mathrm{conf}(\Gamma/\gamma, \mathbb{R}^d)} *_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} \overline{\mathrm{conf}(\gamma, \mathbb{R}^d)}$$

where ∂ is the differential of the complex $C_*(\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)})$ of geometric chains and γ runs over the set of connected non trivial full sub $\mathcal{A}$-labeled graphs of Γ.

Proof. The space $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ is a manifold with corners, whose codimension one boundary strata are disjoint open manifolds. Some of the closures of these open manifolds may intersect, but the intersections are of much lower dimensions. So as chains, we have the above formula, where the product structure on the right hand side of the formula is due to the Proposition 5.2.5. □

Theorem 6.2.4 (Master equation package)**.** *The fundamental chains* $\{\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}\}_{\Gamma \in \mathcal{AG}_{\geq 2}}$ *provide a solution to the master equation*

$$\partial\Gamma = \begin{cases} 0 & \text{if } |V(\Gamma)| = 2; \\ \sum_\gamma \Gamma/\gamma \odot_{((\Gamma/\gamma, v_\gamma), \gamma, \Gamma)} \gamma & \text{if } |V(\Gamma)| > 2 \end{cases}$$

in the sum over $\mathcal{AG}_{\geq 2}$ *of complexes* $C_*(\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)})$ *of geometric chains with insertion products* $*$.

Proof. Let F be the configuration algebra of $\mathcal{A}$-labeled graphs and F' be the differential S-sorted algebra in Proposition 6.2.2. The map which sends $\Gamma \in \mathcal{AG}_{\geq 2}$ to the fundamental chain of its moduli space can be extended to a map $f : F \to F'$ of differential $\mathcal{AG}_{\geq 2}$-sorted algebras, following Propositions 6.2.1 and 6.2.3.

If $|V(\Gamma)| = 2$, then $\partial\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)} = 0$ as geometric chain, since its support is isomorphic to S^{d-1}. □

6.3 Insertion products on chains of modified configuration spaces

Let Γ be a connected graph and M a smooth parallelizable manifold of dimension $d > 1$. In Chapter 4, the canonically compactified space $\overline{\mathrm{Conf}(\Gamma, M)}$ of the configuration space $\mathrm{Conf}(\Gamma, M)$ is constructed by iteratively blowing up $\mathrm{Map}(V(\Gamma), M)$

along diagonals corresponding to connected full subgraphs γ of Γ with $|V(\gamma)| > 1$.

The insertion operations of configurations of graphs induce an action of the chains of moduli spaces of configurations of graphs on the chains of compactified configuration spaces $\overline{\mathrm{Conf}(\Gamma, M)}$ of graphs in the manifold M.

Definition 6.3.1. Let Γ_1, Γ_2 be connected $\mathcal{A}$-labeled graphs and Γ an $\mathcal{A}$-labeled graph which can be obtained by inserting Γ_2 to a vertex v of Γ_1. For any integers $i, j \geq 0$, define a bilinear operation

$$*_{((\Gamma_1,v),\Gamma_2,\Gamma)} : C_i(\overline{\mathrm{Conf}(\Gamma_1, M)}) \otimes C_j(\overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)}) \to C_{i+j}(\overline{\mathrm{Conf}(\Gamma, M)})$$

$$x \otimes y \mapsto z$$

where the support $|z|$ of chain z is a fibration with base $|x|$ and fiber $|y|$ so that this fibration is the restriction of the codimension one boundary fibration of $\overline{\mathrm{Conf}(\Gamma, M)}$.

Proposition 6.3.1. *Let Γ be a connected $\mathcal{A}$-labeled graph. As geometric chains,*

$$\partial\overline{\mathrm{Conf}(\Gamma, M)} = \sum_{\gamma} \overline{\mathrm{Conf}(\Gamma/\gamma, M)} *_{((\Gamma/\gamma,v_\gamma),\gamma,\Gamma)} \overline{\mathrm{conf}(\gamma, \mathbb{R}^d)}$$

$$+\overline{\mathrm{Conf}(\Gamma/\Gamma, M)} *_{((\Gamma/\Gamma,v_\Gamma),\Gamma,\Gamma)} \overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$$

where ∂ is the differential of the chain complex $C_*(\overline{\mathrm{Conf}(\Gamma, M)})$ and γ runs over the set of connected non trivial full sub $\mathcal{A}$-labeled graphs of Γ.

Proof. The proof is similar to that of Proposition 6.2.3. □

Let

$$\mathcal{M} = \{\overline{\mathrm{Conf}(\Gamma, M)} | \Gamma \text{ is a connected } \mathcal{A}\text{-labeled graph}\},$$

$$\mathcal{R} = \{\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)} | \Gamma \text{ is a connected } \mathcal{A}\text{-labeled graph}\}.$$

Then the boundary strata of the space $\overline{\mathrm{Conf}(\Gamma, M)}$ correspond to chains of the following forms:

$$\begin{gathered} m * r \\ (m * r_1) * r_2, m * (r_1 * r_2), \\ ((m * r_1) * r_2) * r_3, (m * (r_1 * r_2)) * r_3, (m * r_1) * (r_2 * r_3), \\ \cdots \end{gathered} \tag{6.3.1}$$

where $m \in \mathcal{M}, r, r_i \in \mathcal{R}$, and $*$ is the insertion product as in Definition 6.3.1.

Since M is a parallelizable manifold, we can fix a global frame f of M. For each stratum of $\overline{\mathrm{Conf}(\Gamma, M)}$, if the corresponding chain when expressed in the above forms has the support $|m|$ of its leftmost argument chain m isomorphic to M, we identify the fibers of the iterated fibration using the frame f. That is, intuitively, we collapse these strata of $\overline{\mathrm{Conf}(\Gamma, M)}$ along the M direction. Let $\widehat{\mathrm{Conf}(\Gamma}, M)$ denote the resulted modified compactified configuration space.

Proposition 6.3.2. *Let Γ be a connected finite graph and M a compact parallelizable manifold. Then $\widehat{\mathrm{Conf}(\Gamma}, M)$ is a Whitney stratified object, and thus a geometric chain.*

Proof. By Corollary 4.3.3, $\overline{\mathrm{Conf}(\Gamma, M)}$ is compact. Then by its construction, the space $\widehat{\mathrm{Conf}(\Gamma}, M)$ is compact and has a Whitney stratification induced from that of $\overline{\mathrm{Conf}(\Gamma, M)}$. □

A definition similar to Definition 6.3.1 can be made for chains of spaces $\widehat{\mathrm{Conf}(\Gamma}, M)$. We will use the same notation $*$.

Definition 6.3.2. Let Γ_1, Γ_2 be connected $\mathcal{A}$-labeled graphs and Γ an $\mathcal{A}$-labeled graph which can be obtained by inserting Γ_2 into a vertex v of Γ_1. For any integers $i, j \geq 0$, define a bilinear operation

$$*_{((\Gamma_1,v),\Gamma_2,\Gamma)} : C_i(\widehat{\mathrm{Conf}(\Gamma_1}, M)) \otimes C_j(\overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)}) \to C_{i+j}(\widehat{\mathrm{Conf}(\Gamma}, M))$$

$$x \otimes y \mapsto z$$

where the support $|z|$ of chain z is a fibration with base $|x|$ and fiber $|y|$ so that this fibration is the restriction of the codimension one boundary fibration of $\widehat{\mathrm{Conf}(\Gamma}, M)$.

Proposition 6.3.3. *Let Γ_1, Γ_2 be connected $\mathcal{A}$-labeled graphs and Γ an $\mathcal{A}$-labeled graph which can be obtained by inserting Γ_2 into a vertex v of Γ_1. If $\alpha \in C_*(\overline{\mathrm{conf}(\Gamma_1, M)})$, $\beta \in C_*(\overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)})$, then*

$$\partial(\alpha *_{((\Gamma_1,v),\Gamma_2,\Gamma)} \beta) = (\partial\alpha) *_{((\Gamma_1,v),\Gamma_2,\Gamma)} \beta + \alpha *_{((\Gamma_1,v),\Gamma_2,\Gamma)} \partial\beta$$

up to sign, where ∂ is the boundary operator of geometric chains. In other words, $*_{((\Gamma_1,v),\Gamma_2,\Gamma)}$ commutes with ∂.

Proof. Similar to Proposition 6.2.1. □

Proposition 6.3.4. *Let M be a parallelizable manifold of dimension $d > 1$. The sum over $\mathcal{AG}_{\geq 2}$ of complexes of geometric chains of modified configuration spaces $\widehat{\mathrm{Conf}(\Gamma}, M)$ of $\Gamma \in \mathcal{AG}_{\geq 2}$ in M is a right differential graded module over the differential graded $\mathcal{AG}_{\geq 2}$-sorted algebra of the sum over $\mathcal{AG}_{\geq 2}$ of the complexes of geometric chains of compactified moduli spaces of graphs.*

Proof. This follows from Definitions 6.2.1, 6.3.2 and Proposition 6.2.1, 6.3.3. The check is routine, similar to the proof of Proposition 6.2.2. □

Proposition 6.3.5. *Let Γ be a connected $\mathcal{A}$-labeled graph with $|V(\Gamma)| > 2$. As geometric chains,*

$$\partial \widehat{\mathrm{Conf}(\Gamma, M)} = \sum_{\gamma} \widehat{\mathrm{Conf}(\Gamma/\gamma, M)} *_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} \overline{\mathrm{conf}(\gamma, \mathbb{R}^d)},$$

where ∂ is the differential of the complex $C_*(\widehat{\mathrm{Conf}(\Gamma, M)})$ of geometric chains and γ runs over the set of connected non trivial full sub $\mathcal{A}$-labeled graphs of Γ.

Proof. By the construction of $\widehat{\mathrm{Conf}(\Gamma, M)}$, the codimension one face of $\overline{\mathrm{Conf}(\Gamma, M)}$ corresponding to the diagonal where the images of all the vertices of the graph Γ coincide is collapsed along the base M to a fibre $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ over a point, according to a framing of M. The dimension of $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ is $(d+1)$ less than that of $\overline{\mathrm{Conf}(\Gamma, M)}$ and $\widehat{\mathrm{Conf}(\Gamma, M)}$. So, as a chain, this term is not part of the boundary of $\widehat{\mathrm{Conf}(\Gamma, M)}$. The statement then follows from Proposition 6.3.1. □

Theorem 6.3.6 (Master equation package)**.** *Let M be a parallelizable manifold of dimension $d > 1$. Then the fundamental chains $\{\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}\}_{\Gamma \in \mathcal{AG}_{\geq 2}}$ and $\{\widehat{\mathrm{Conf}(\Gamma, M)}\}_{\Gamma \in \mathcal{AG}_{\geq 2}}$ provide a solution to the master equation system*

$$\begin{aligned}
\partial R_\Gamma &= 0, \text{if } |V(\Gamma)| = 2; \\
\partial M_\Gamma &= 0, \text{if } |V(\Gamma)| = 2; \\
\partial R_\Gamma &= \sum_{\gamma} R_{\Gamma/\gamma} \odot_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} R_\gamma, \text{if } |V(\Gamma)| > 2;
\end{aligned}$$

$$\partial M_\Gamma = \sum_{\gamma} M_{\Gamma/\gamma} \odot_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} R_\gamma, \text{if } |V(\Gamma)| > 2. \tag{6.3.2}$$

in the sum over $\mathcal{AG}_{\geq 2}$ of complexes of geometric chains of moduli spaces of Γ and modified configuration spaces of Γ in M, with insertion products $*$.

Proof. Let $\Gamma \in \mathcal{AG}_{\geq 2}$.

If $V(\Gamma) = 2$, then as topological spaces, $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$ is isomorphic to S^{d-1}; $\widehat{\mathrm{Conf}(\Gamma}, M)$ is obtained from $\overline{\mathrm{Conf}(\Gamma, M)}$ by collapsing its boundary, which is isomorphic to $S^{d-1} \times M$, along the M direction, to a point. Thus as geometric chains,

$$\partial \overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)} = 0,$$

$$\partial \widehat{\mathrm{Conf}(\Gamma}, M = 0.$$

The case when $V(\Gamma) > 2$ follows from Propostions 6.2.3 and 6.3.5 .

Now the statement follows from the fact that the map which sends R_Γ to $\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}$, and M_Γ to $\widehat{\mathrm{Conf}(\Gamma}, M)$ can be extended to a map of differential many sorted algebras. The check will not be given here since it is routine. $\square$

Part II

The odd dimensional case

Chapter 7

Reduced master equations in odd dimensions

Configuration spaces in odd dimensional manifolds will be investigated in this chapter. This is interesting mainly due to the fact that configuration spaces of many graphs in odd dimensional Euclidean spaces allow certain orientation reversing involutions. The Kontsevich-Kuperberg-Thurston's construction [28, 31] of quantum invariants of 3-manifolds, is discussed in the framework of the master equation package. Many of the proofs about the construction are following Kuperberg and Thurston [31, 47].

7.1 $\mathcal{E}$-decorated graphs and the master equation

After introducing a class of decorated graphs, main definitions and propositions parallel to those of the usual $\mathcal{A}$-labeled graphs in previous chapters will be listed.

Definition 7.1.1. A linear order on a finite set S is a bijective map from the set of consecutive natural numbers $\{1, \cdots, |S|\}$ to S.

Definition 7.1.2 ($\mathcal{E}$-decorated graph)**.** An $\mathcal{E}$-decorated graph is an $\mathcal{A}$-labeled graph with the following data:

- a linear order σ on the set of edges;
- an orientation (direction) for each edge.

An isomorphism of $\mathcal{E}$-decorated graphs is an isomorphism of $\mathcal{A}$-labeled graphs which preserves the extra decoration. Let $\mathcal{E}$ be the set of isomorphism classes of connected $\mathcal{A}$-labeled graphs with two or more vertices.

Definition 7.1.3 (Shuffle)**.** Let S and T be two finite sets and s, t be linear orders on S and T respectively. A shuffle of s and t is a linear order h on the set $S \sqcup T$ such that

$$h^{-1}(s(1)) < h^{-1}(s(2)) < \cdots < h^{-1}(s(|S|))$$

and

$$h^{-1}(t(1)) < h^{-1}(t(2)) < \cdots < h^{-1}(t(|T|)).$$

Previous notions like "insertion of graph", "screen of graph", "configuration algebra of graphs" etc, can be moved to the case of $\mathcal{E}$-decorated graphs without difficulties. Some of them are listed here.

Definition 7.1.4 (Inserting an $\mathcal{E}$-decorated graph into another)**.** Let $\Gamma_1, \Gamma_2 \in \mathcal{E}$ such that there exists $v \in V(\Gamma_1)$ whose $\mathcal{A}$-label is equal to the $\mathcal{A}$-label of Γ_2. Let σ_1, σ_2 be the linear orders on $E(\Gamma_1)$ and $E(\Gamma_2)$, and h is a shuffle of σ_1, σ_2. Let $f : E(v) \to V(\Gamma_2)$ be a map.

Define an $\mathcal{E}$-decorated graph Γ by the following data

- $V(\Gamma) = (V(\Gamma_1) \setminus \{v\}) \sqcup V(\Gamma_2)$;
- $E(\Gamma) = E(\Gamma_1) \sqcup E(\Gamma_2)$ as sets of oriented edges; an edge of Γ_1 with endpoint set $\{p, v\}$ corresponds to an edge of Γ with endpoint set $\{p, f(e)\}$; the endpoints

of the other edges remain unchanged;

- the linear order on $E(\Gamma)$ is given by h.

This $\Gamma \in \mathcal{E}$ will be called the graph obtained by inserting Γ_2 into Γ_1 at vertex v according to the map f and shuffle h, and be denoted by $\Gamma_1 \circ_{(v,f,h)} \Gamma_2$.

Definition 7.1.5 (Insertion operation of $\mathcal{E}$-decorated graphs)**.** Let $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{E}$. If there exist $v \in V(\Gamma_1)$ whose $\mathcal{A}$-label is equal to the $\mathcal{A}$-label of Γ_2, a map $f : E(v) \to V(\Gamma_2)$ and a shuffle h of linear orders on $E(\Gamma_1), E(\Gamma_2)$ such that $\Gamma_3 = \Gamma_1 \circ_{(v,f,h)} \Gamma_2$, we say that there is an insertion operation $\circ_{((\Gamma_1,v),\Gamma_2,\Gamma_3)} : (\Gamma_1, \Gamma_2) \mapsto \Gamma_3$.

Definition 7.1.6 (Insertion of configurations of $\mathcal{E}$-decorated graphs)**.** Let $\Gamma_i \in \mathcal{E}$, and X_i be a configuration of Γ_i, $i = 1, 2, 3$. Define

$$X_3 = X_1 \odot_{((\Gamma_1,v),\Gamma_2,\Gamma_3)} X_2$$

if and only if the following two conditions are satisfied:

- There exists $v \in V(\Gamma_1)$ whose $\mathcal{A}$-label is the same as the $\mathcal{A}$-label of Γ_2 such that as $\mathcal{E}$-decorated graphs,

$$\Gamma_3 = \Gamma_1 \circ_{((\Gamma_1,v),\Gamma_2,\Gamma_3)} \Gamma_2.$$

- The nested set presentation of the screen of X_3 is equal to the nested set obtained by replacing v in the nested set presentation of the screen of X_1 with the nested set presentation of the screen of X_2.

Remark 7.1.1. Here and in the following, the same notation as in Chapters 2 - 6 will be used, but with a different meaning. This will not cause confusion from the contexts.

Given an $\mathcal{E}$-decorated graph Γ and a smooth manifold M, the configuration space $\mathrm{Conf}(\Gamma, M)$ is isomorphic to the configuration space of the underlying graph of Γ without extra decoration. Because of the canonical compactification of configuration spaces of graphs, one can define insertion products $*$ for geometric chains of moduli spaces and configuration spaces of $\mathcal{E}$-decorated graphs, similar to Definitions 6.2.1 and 6.3.2, and thus we have the following proposition:

Proposition 7.1.1 (Master equation package)**.** *Let M be a parallelizable manifold of dimension $d > 1$. Then the fundamental chains $\{\overline{\mathrm{conf}(\Gamma, \mathbb{R}^d)}\}_{\Gamma\in\mathcal{E}}$ and $\{\widehat{\mathrm{Conf}(\Gamma, M)}\}_{\Gamma\in\mathcal{E}}$ provide a solution to the master equation system (6.3.2) in the sum over $\mathcal{E}$ of complexes of geometric chains of moduli spaces of Γ and modified configuration spaces of Γ in M, with insertion products $*$.*

Proof. Similar to Theorem 6.3.6. □

7.2 Involutions and reduced master equations

The idea of involutions on configuration spaces was used by Axelrod, Bott, Cattaneo, Kontsevich, Kuperberg, Singer, Taubes, and Thurston ([1, 2, 11, 12, 13, 28, 31, 47]) et al., in their constructions of invariants of links and 3-manifolds. We give a review of involutions in this section: first on abstract graphs and then their geometric realization on configuration spaces.

7.2.1 Involutions on graphs

Definition 7.2.1 (separating edge)**.** An edge e of a graph Γ is called a separating edge if the removal of e will disconnect Γ into two components Γ_1 and Γ_2, in the sense that

- there are no other edges connecting Γ_1 and Γ_2;
- one of the endpoints of e belongs to Γ_1 and the other belongs to Γ_2.

Definition 7.2.2 (separating pair of edges)**.** Two edges e_1 and e_2 of a graph Γ are called a separating pair if the removal of e_1 and e_2 will disconnect Γ into two components Γ_1 and Γ_2 in the sense that

- there are no other edges connecting Γ_1 and Γ_2;
- both e_1 and e_2 have the property that one endpoint belongs to Γ_1 and the other belongs to Γ_2.

Remark 7.2.1. In Definitions 7.2.1 and 7.2.2, the graphs Γ, Γ_1 and Γ_2 themselves do not have to be connected. And in Definition 7.2.2 the endpoints of the two edges e_1 and e_2 are not required to be different.

Proposition 7.2.1 ([31, 47])**.** *Let Γ be a trivalent graph and γ a connected non trivial full subgraph of Γ which is not an edge. Then γ has at least a separating pair of edges.*

Proof. Since γ is a subgraph of a connected graph Γ, there exist $u \in V(\Gamma) \setminus V(\gamma)$, $v \in V(\gamma)$ such that u and v are connected by an edge. Thus the valence of u in γ is 1 or 2.

If there are two edges $e_1, e_2 \in E(\gamma)$ with endpoint v. Then e_1, e_2 are a separating pair: v is a component, and the subgraph of γ spanned by $V(\gamma) \setminus \{v\}$ is another component.

If there is only one edge $e \in E(\gamma)$ with endpoint v. Let u be the other endpoint of e. Since γ is not an edge, the valence of u in γ is 2 or 3. If it is 2, done. Otherwise there are two other edges $e_1, e_2 \in E(\gamma)$ having u as an endpoint. Then e_1, e_2 are a separating pair: the subgraph of γ spanned by u, v is a component, and the subgraph of γ spanned by $V(\gamma) \setminus \{u, v\}$ is another. $\square$

Definition 7.2.3 (involution)**.** A type I involution on an $\mathcal{E}$-decorated graph coloured in R is a reversing of the orientation of a separating edge.

A type II involution on an $\mathcal{E}$-decorated graph coloured in R is a permutation of the linear order of a separating pair of edges whose removal separate the graph into two components Γ_1 and Γ_2, with a possible change of orientations (directions) of these two edges as described in the following

- if both the two edges have orientations away from (or going into) Γ_1, then both of their orientations are reversed.
- otherwise, both orientations are unchanged.

An involution on an $\mathcal{E}$-decorated graph colored in R does not change its underlying $\mathcal{A}$-labeled graph. And there may exist none or many involutions on a given $\mathcal{E}$-decorated graph.

Proposition 7.2.2. *1. Given $\Gamma \in \mathcal{E}$, then the set of type I involutions on the set of $\mathcal{E}$-decorated graphs with the same underlying $\mathcal{A}$-labeled graph as Γ's generates a finite group.*

2. *Given $\Gamma \in \mathcal{E}$, then the set of type II involutions on the set of $\mathcal{E}$-decorated graphs with the same underlying $\mathcal{A}$-labeled graph as Γ's generates a finite group.*

Proof. Here we give the proof for type II involutions; type I case is similar.

A type II involution only changes the $\mathcal{E}$-decoration, not the underlying $\mathcal{A}$-labeled graph: permuting the order of a separating pair of edges and possibly reversing their orientations (if the orientation of one edge in the pair is reversed, so is the other). Since there are only finitely many $\mathcal{E}$-decorated graphs with the same underlying $\mathcal{A}$-labeled graph as Γ, the group generated by involutions is a finite group. □

Corollary 7.2.3. *The order of the groups in Proposition 7.2.2 are even.*

Proof. This is due to Lagrange's theorem in group theory and the fact that an involution is an element of order two. □

The involutions can be generalized to the configuration algebra F of R-coloured $\mathcal{E}$-decorated graphs. A type I (or type II) involution on a monomial of R-colored $\mathcal{E}$-decorated graphs is a type I (or type II) involution on an argument $\mathcal{E}$-decorated graph. This introduces on configuration algebra F an equivalent relation of the following form:

$$\text{(an involution)}m = -m,$$

where m is a monomial of R-colored $\mathcal{E}$-decorated graphs (i.e., elementary configurations).

Proposition 7.2.4. *The configuration algebra of $\mathcal{E}$-decorated graphs modulo equivalences generated by involution relations is a triangular differential right module* *over a differential graded algebra.*

Proof. This is immediate. □

7.2.2 Involutions on configuration spaces

This subsection discusses the geometric realization of involutions defined in §7.2.1.

Let $\Gamma \in \mathcal{E}$. Let $\mathrm{Map}(V(\Gamma), \mathbb{R}^d)$ denote the space of maps from $V(\Gamma)$ to $\mathbb{R}^d$. This space is isomorphic to $\mathbb{R}^{d\times|V(\Gamma)|}$.

Type I involution: Suppose Γ has a separating edge e, i.e., the removal of e disconnects Γ into two components Γ_1 and Γ_2. Let $v_1 \in V(\Gamma_1), v_2 \in V(\Gamma_2)$ be the endpoints of e. Define a map

$$f : \mathrm{Map}(V(\Gamma), \mathbb{R}^d) \to \mathrm{Map}(V(\Gamma), \mathbb{R}^d), \varphi \mapsto f(\varphi)$$

by the following:

$f(\varphi)(v_1) = 2\varphi(v_2) - \varphi(v_1)$;

$f(\varphi)(v) = \varphi(v) + 2\varphi(v_2) - 2\varphi(v_1)$ if $v \in V(\Gamma_1)$ and $v \neq v_1$;

$f(\varphi)(v) = \varphi(v)$ if $v \in V(\Gamma_2)$.

Type II involution: Suppose Γ has a separating pair of edges e_1, e_2: the removal of e_1 and e_2 disconnects Γ into two components Γ_1 and Γ_2. Let $v_{11} \in V(\Gamma_1), v_{12} \in V(\Gamma_2)$ be the endpoints of e_1 and $v_{21} \in V(\Gamma_1), v_{22} \in V(\Gamma_2)$ be the endpoints of e_2.

Define a map

$$f : \mathrm{Map}(V(\Gamma), \mathbb{R}^d) \to \mathrm{Map}(V(\Gamma), \mathbb{R}^d), \varphi \mapsto f(\varphi)$$

by the following:

1. If $v_{11} \neq v_{21}$ and $v_{12} \neq v_{22}$, then

   $f(\varphi)(v_{11}) = \varphi(v_{12}) + \varphi(v_{22}) - \varphi(v_{21})$;

   $f(\varphi)(v_{21}) = \varphi(v_{12}) + \varphi(v_{22}) - \varphi(v_{11})$;

   $f(\varphi)(v) = \varphi(v) + \varphi(v_{12}) + \varphi(v_{22}) - \varphi(v_{11}) - \varphi(v_{21})$ if $v \in V(\Gamma_1)$ and $v \neq v_{11}, v_{21}$;

   $f(\varphi)(v) = \varphi(v)$ if $v \in V(\Gamma_2)$.

2. If $v_{11} = v_{21} = v_1$ and $v_{12} \neq v_{22}$, then

   $f(\varphi)(v_1) = \varphi(v_{12}) + \varphi(v_{22}) - \varphi(v_1)$;

   $f(\varphi)(v) = \varphi(v) + \varphi(v_{12}) + \varphi(v_{22}) - 2\varphi(v_1)$ if $v \in V(\Gamma_1)$ and $v \neq v_1$;

   $f(\varphi)(v) = \varphi(v)$ if $v \in V(\Gamma_2)$.

3. If $v_{11} \neq v_{21}$ and $v_{12} = v_{22} = v_2$, then

   $f(\varphi)(v_{11}) = 2\varphi(v_2) - \varphi(v_{21})$;

   $f(\varphi)(v_{21}) = 2\varphi(v_2) - \varphi(v_{11})$;

   $f(\varphi)(v) = \varphi(v) + 2\varphi(v_2) - \varphi(v_{11}) - \varphi(v_{21})$ if $v \in V(\Gamma_1)$ and $v \neq v_{11}, v_{21}$;

   $f(\varphi)(v) = \varphi(v)$ if $v \in V(\Gamma_2)$.

4. If $v_{11} = v_{21} = v_1$ and $v_{12} = v_{22} = v_2$, then

   $f(\varphi)(v_1) = 2\varphi(v_2) - \varphi(v_1)$;

$f(\varphi)(v) = \varphi(v) + 2\varphi(v_2) - 2\varphi(v_1)$ if $v \in V(\Gamma_1)$ and $v \neq v_1$;

$f(\varphi)(v) = \varphi(v)$ if $v \in V(\Gamma_2)$.

Proposition 7.2.5. *The map* $f : \mathrm{Map}(V(\Gamma), \mathbb{R}^d) \to \mathrm{Map}(V(\Gamma), \mathbb{R}^d)$ *has the property that if* $\varphi(u) \neq \varphi(v)$*, then* $f(\varphi)(u) \neq f(\varphi)(v)$*, for any* $u, v \in V(\Gamma_i)$ *and* $\varphi \in \mathrm{Map}(V(\Gamma), \mathbb{R}^d)$*,* $i = 1, 2$*.*

Proof. This is immediate. □

Recall that $\mathrm{Conf}(\Gamma, \mathbb{R}^d)$ is the subspace of $\mathrm{Map}(V(\Gamma), \mathbb{R}^d)$ such that the images of two vertices of Γ can not coincide if there is an edge connecting these two vertices. Because of Proposition 7.2.5, $f(\varphi) \in \mathrm{Conf}(\Gamma, \mathbb{R}^d)$ for any $\varphi \in \mathrm{Conf}(\Gamma, \mathbb{R}^d)$. We will use the same symbol f to denote the map when restricted to $\mathrm{Conf}(\Gamma, \mathbb{R}^d)$.

Proposition 7.2.6. *The map* $f : \mathrm{Conf}(\Gamma, \mathbb{R}^d) \to \mathrm{Conf}(\Gamma, \mathbb{R}^d)$ *is an orientation reversing isomorphism when d is odd.*

Proof. The map f is an isomorphism: f^2 is the identity map on $\mathrm{Conf}(\Gamma, \mathbb{R}^d)$.

Since the space $\mathrm{Map}(V(\Gamma), \mathbb{R}^d)$ is isomorphic to $\mathbb{R}^{d \times |V(\Gamma)|}$, the map f can be viewed as a linear map from $\mathbb{R}^{d \times |V(\Gamma)|}$ to $\mathbb{R}^{d \times |V(\Gamma)|}$. Using linear algebra, one can check that in all the cases, the determinant of the map f is equal to -1 when d is odd. Thus f is an orientation reversing map. □

Example 7.2.1. Let Γ be the graph of a single edge with labeled endpoints p, q. When $d = 3$, the type I involution

$$f : \mathrm{Map}(\{p, q\}, \mathbb{R}^3) \to \mathrm{Map}(\{p, q\}, \mathbb{R}^3), \varphi \mapsto f(\varphi),$$

is given by

$$\begin{aligned} f(\varphi(p)) &= f(\varphi)(p) = 2\varphi(q) - \varphi(p); \\ f(\varphi(q)) &= f(\varphi)(q) = \varphi(q). \end{aligned}$$

So, the determinant of the map f is

$$\begin{vmatrix} -1 & 0 & 0 & 2 & 0 & 0 \\ 0 & -1 & 0 & 0 & 2 & 0 \\ 0 & 0 & -1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{vmatrix} = -1.$$

The map f commutes with translations and dilations in $\mathbb{R}^d$. Let $[f] : \mathrm{conf}(\Gamma, \mathbb{R}^d) \to \mathrm{conf}(\Gamma, \mathbb{R}^d)$ be the map induced by f.

Corollary 7.2.7. *The map $[f]$ is an orientation reversing isomorphism when d is odd.*

Proof. This follows from Proposition 7.2.6 and the definition of $\mathrm{conf}(\Gamma, \mathbb{R}^d)$. □

Now let M be a smooth parallelizable manifold of odd dimension $d > 1$ and $\Gamma_i, \overline{\Gamma}_{i_0} \in \mathcal{E}$ such that $\overline{\Gamma}_{i_0} = f(\Gamma_{i_0})$, where f is an involution, $i = 1, ..., i_0, .., k$ and $1 < i_0 \leq k$. Let c_1 be a monomial chain of the form in (6.3.1), more explicitly, say, of the form $\overline{\mathrm{Conf}(\Gamma_1, M)} * \overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_k, \mathbb{R}^d)}$ with parentheses inserted in appropriate places; c_2 be a monomial chain which differs from c_1 in that the graph Γ_{i_0} is replaced by $\overline{\Gamma}_{i_0}$. Then on the algebra of chains F, impose equivalent relations of the following form:

$$(\overline{\mathrm{Conf}(\Gamma_1, M)} * \overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\overline{\Gamma}_{i_0}, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_k, \mathbb{R}^d)}, or)$$

$$= -(\overline{\mathrm{Conf}(\Gamma_1, M)} * \overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_{i_0}, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_k, \mathbb{R}^d)}, or)$$

This is reasonable since the involution f is orientation reversing,

$$\begin{aligned}
& (\overline{\mathrm{Conf}(\Gamma_1, M)} * \overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\overline{\Gamma}_{i_0}, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_k, \mathbb{R}^d)}, or) \\
= \quad & (\overline{\mathrm{Conf}(\Gamma_1, M)} * \overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)} * \cdots * (-\overline{\mathrm{conf}(\Gamma_{i_0}, \mathbb{R}^d)}) * \cdots * \overline{\mathrm{conf}(\Gamma_k, \mathbb{R}^d)}, or) \\
= \quad & (\overline{\mathrm{Conf}(\Gamma_1, M)} * \overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_{i_0}, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_k, \mathbb{R}^d)}, -or) \\
= \quad & -(\overline{\mathrm{Conf}(\Gamma_1, M)} * \overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_{i_0}, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_k, \mathbb{R}^d)}, or).
\end{aligned}$$

Involutions also induces relations on the chains whose supports are inside the above monomial chains. Let $[f] : \mathrm{conf}(\Gamma_{i_0}, \mathbb{R}^d) \to \mathrm{conf}(\overline{\Gamma}_{i_0}, \mathbb{R}^d)$ be a involution. Then $[f]$ induces an involution on the level of geometric chains

$$[f]^\sharp : C_*(\overline{\mathrm{conf}(\Gamma_{i_0}, \mathbb{R}^d)}) \to C_*(\overline{\mathrm{conf}(\overline{\Gamma}_{i_0}, \mathbb{R}^d)}).$$

And then the map $[f]^\sharp$ can be extend to $C_*(\overline{\mathrm{Conf}(\Gamma_1, M)} * \overline{\mathrm{conf}(\Gamma_2, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_{i_0}, \mathbb{R}^d)} * \cdots * \overline{\mathrm{conf}(\Gamma_k, \mathbb{R}^d)})$.

7.2.3 Cycles of reduced master equations

The quotient differential algebraic structure of the differential right module over the configuration algebra represented by the master equation system of $\mathcal{E}$-decorated graphs (see Poposition 7.1.1) modulo the ideal generated by certain equivalences relations will be called, by abuse of language, a reduced master equation of $\mathcal{E}$-decorated graphs. Two versions of reduced master equations will be considered: the reduced master equation A, in which the equivalences relations are generated by involutions of both type I and type II, and the reduced master equation B, which is obtained by modulo relations generated by involutions of type II and generalized IHX relations.

In the study of finite type invariants (see [5] for an almost complete bibliography on the theory of finite type invariants), a relation called IHX relation is often used. It is a relation among connected trivalent graphs with more than three vertices which only differ at the neighborhood of a single edge. There are three different ways to divide the four edges connecting to the endpoints of the single edge into two pairs, which correspond to three trivalent graphs. The IHX relation is defined to be the identities of the form claiming that the formal sum of these three graphs is 0.

For $\mathcal{E}$-decorated graphs, since edges are oriented and vertices are labeled, there are more graphs which differ only at a neighborhood of a single edge. Let γ be the $\mathcal{E}$-decorated graph consisting of an oriented single edge e from endpoint u to endpoint v. There are six ways to insert γ into a 4-valent vertex v of an $\mathcal{E}$-decorated graph so that in the new graph obtained both u and v are trivalent. Let $I, H, X, \underline{I}, \underline{H}$ and $\underline{X}$ be six $\mathcal{E}$-labeled graphs which can be obtained this way. A relation of the form

$$I/\gamma \odot_{(I/\gamma, v_\gamma), \gamma, I)} \gamma + H/\gamma \odot_{(H/\gamma, v_\gamma), \gamma, H)} \gamma + X/\gamma \odot_{(X/\gamma, v_\gamma), \gamma, X)} \gamma +$$

$$+ \underline{I}/\gamma \odot_{(\underline{I}/\gamma, v_\gamma), \gamma, \underline{I})} \gamma + \underline{H}/\gamma \odot_{(\underline{H}/\gamma, v_\gamma), \gamma, \underline{H})} \gamma + \underline{X}/\gamma \odot_{(X/\gamma, v_\gamma), \gamma, \underline{X})} \gamma = 0 \tag{7.2.1}$$

will be referred as an IHX relation on configurations of $\mathcal{E}$-decorated graphs. Relations induced on the right module over the configuration algebra of $\mathcal{E}$-decorated graphs by relations (7.2.1) will be called generalized IHX relations. Note that in the case of length two configurations of trivalent $\mathcal{E}$-decorated graphs, the only generalized IHX relations are exactly of the form given in (7.2.1).

Now let us consider some examples of non trivial cycles of reduced master equations.

Let $\Gamma \in \mathcal{E}$. If $|V(\Gamma)| = 2$, then by Proposition 3.2.8, Γ is a non trivial cycle of the reduced master equations A and B.

Proposition 7.2.8. *For each set of $\mathcal{A}$-labels on vertices, the sum of all the $\mathcal{E}$-decorated trivalent graphs with $2k$ vertices and with the given $\mathcal{A}$-labels is a non trivial cylce of the reduced master equation A.*

Proof. 1) The sum is a cycle. Given an $\mathcal{E}$-decorated graph Γ, we have

$$\partial\Gamma = \sum_{\gamma} \Gamma/\gamma \odot_{((\Gamma/\gamma, v_\gamma),\gamma,\Gamma)} \gamma.$$

By Proposition 7.2.1, for each term where γ is not an edge, there exists at least one type II involutions. By Corollary 7.2.3, there are even number of elements in the group generated by involutions on the underlying $\mathcal{A}$-decorated graphs, thus the sum of terms where γ is non edge is 0.

Similarly, terms in which γ is an edge are canceled by type I involutions.

2) The cycle is non trivial, because any monomial configuration in the image of ∂ has to be of length > 1. $\square$

Proposition 7.2.9. *For each set of $\mathcal{A}$-labels on vertices, the sum of all the $\mathcal{E}$-decorated trivalent graphs with $2k$ vertices and with the given $\mathcal{A}$-labels is a non trivial cylce of the reduced master equation B.*

Proof. Similar to the proof of Proposition 7.2.8. The only difference is that those terms in which γ is an edge are canceled by IHX relations. □

Question: Do there exist non trivial cycles of the reduced master equations which are of the form of sums of some non trivalent graphs?

Invariants of manifolds may be obtained from non trivial cycles of reduced master equations. One way is by the Kontsevich-Kuperberg-Thurston's construction, which will be discussed in the next section.

7.3 Finite type invariants and the master equation package

A family of quantum invariants of links and 3-manifolds were defined by Kontsevich [28], based on ideas from perturbative Chern-Simons field theory. Kuperberg and Thurston [31] gave a purely topological construction of these invariants and showed that they are universally finite type in certain sense. In this section we will show that their construction fits the framework of master equation package.

In this section, let M be a rational homology 3-sphere.

7.3.1 Kuperberg-Thurston's construction

For the needs of next subsection, we include here a brief review of Kuperberg and Thurston's construction [31] of quantum invariants of rational homology spheres.

For cohomological reason (Lemma 7.3.2), these invariants are given in terms of $M_{\text{fin}} = M \setminus \{\infty\}$ with a fixed asymptotically constant framing, where $\infty \in M$ is a marked point.

Let $\widehat{J_n}$ be the set of isomorphism classes of $\mathcal{E}$-decorated trivalent graphs Γ with $2n$ vertices, for a fixed set of $\mathscr{A}$-labels on vertices. Each of these graphs has $3n$ edges. Let $\overline{\mathrm{Conf}_\infty(\Gamma, M)}$ denote the space obtained by blowing up $\mathrm{Map}(V(\Gamma), M)$ along infinity locus where the images of vertices of some connected full subgraph of Γ coincide at ∞ and diagonals corresponding to connected full subgraphs.

The key point of their construction is a generalized gauss map

$$\Phi : (\bar{C}_n(M), D) \to (\bar{P}(M)^{\times 3n}, Q),$$

for each n, where

- $\bar{C}_n(M)$ is obtained from $C_n(M) = \coprod_{\Gamma \in \widehat{J_n}} \overline{\mathrm{Conf}_\infty(\Gamma, M)}$ by the following gluings or modifications of boundary strata:
  - Boundary strata corresponding to single edges are glued together according to IHX relation.
  - Boundary strata corresponding to non single edges are glued together according to type II involutions.
  - The particular boundary strata where the images of all $2n$ vertices of Γ coincide are collapsed using the framing of M_{fin}.
  - Points in boundary strata corresponding to infinite locus are identified according their directions according to framing, and form the degenerate locus D.
- $\bar{P}(M)$ is obtained from $\overline{\mathrm{Conf}_\infty(e, M)}$, the compactified configuration space of

a single edge e, with points in boundary identified according to their directions in the framing.

- Q is certain degenerate locus of $\bar{P}(M)$.

To obtain invariants, Kuperberg and Thurston proved the following two lemmas:

Lemma 7.3.1. *The top cohomology of $\bar{C}_n(M)$ is independent of M and there is a surjective map*

$$f : H^{6n}(\bar{C}_n(M), D) \twoheadrightarrow V_n,$$

where V_n is the vector space over $\mathbb{Q}$ spanned by isomorphism classes of connected Lie-oriented trivalent graphs with $2n$ vertices, modulo IHX relation.

Lemma 7.3.2. *The space $\bar{P}(M)$ has a generating class $\alpha \in H^2(\bar{P}(M), \mathbb{Q})$, and there is a well-defined cohomology class $\alpha^{\otimes 3n} \in H^{6n}(\bar{P}^{\times 3n}(M), Q)$.*

Now, by Lemma 7.3.1, there is an injection $f^* : V_n^* \hookrightarrow H_{6n}(\bar{C}_n(M), D)$. If $w \in V_n^*$, then there is an invariant of M associated to w by the pairing of homology classes with cohomology classes:

$$I_w(M) =< w, \Phi^*(\alpha^{\otimes 3n}) > .$$

By linear duality, an invariant $I_n(M) \in V_n$ of M is defined.

Kuperberg and Thurston then showed that $I_n(M)$ is a finite-type invariant of degree n in both the algebraically split and Torelli senses for framed rational homology 3-spheres M, and it is universal for integer homology spheres; and there is an invariant $\delta_n(M) \in V_n$ such that $I_n(M) - \delta_n(M)$ does not depend on the framing of M.

7.3.2 Finite type invariants and the master equation package

Here we construct a geometric realization $(\bar{C}'_n, D)$ of the non trivial cycle in Proposition 7.2.9 of the reduced master equation B, in the sum over $\mathcal{E}$ of complexes of geometric chains of the modified configuration spaces.

Let

$$C'_n = \coprod_{\Gamma \in \widehat{\mathcal{J}_n}} \widehat{\mathrm{Conf}_\infty(\Gamma}, M).$$

$\bar{C}'_n$ is obtained from C'_n with the following gluings and modifications:

- The codimensional one strata corresponding to single edges are glued together according to IHX relation.

- The codimensional one strata corresponding to non single edges are glued together according to type II involutions.

- Points in boundary strata corresponding to infinite locus are identified according their directions according to framing, and form the degenerate locus D.

- Lower dimensional strata, i.e., those of codimensional two or more, are also identified according to relations implied by the master equation B.

The pair $(\bar{C}'_n, D)$ is a geometric realization of the non trivial cycle in Proposition 7.2.9. The difference from $(\bar{C}_n, D)$ is that there are more gluings and modifications in $(\bar{C}'_n, D)$:

- There are more collapses, namely the collapses used in modifying $\overline{\mathrm{Conf}_\infty(\gamma, M)}$ into $\widehat{\mathrm{Conf}_\infty(\gamma}, M)$, where γ is a non trivial connected full subgraph of a trivalent graph Γ with $2n$ vertices. These collapses are modifications on strata of codimensional two or more, relative to $\bar{C}'_n$.

- There are extra identification of strata of codimensional two or more, according to relations implied by the master equation B.

Proposition 7.3.3. *Let X, Y be n-dimensional CW complexes such that Y is obtained from X by adding more relations on generating cells of dimension $n-2$ and lower. Then $H^n(X, \mathbb{Q}) \cong H^n(Y, \mathbb{Q})$.*

Proof. By definition of homology, $H_n(X, \mathbb{Q}) \cong H_n(Y, \mathbb{Q})$. Then the statement follows from the universal coefficient theorem. □

Theorem 7.3.4. *Kontsevich-Kuperberg-Thurston invariants can be obtained from geometric realization $(\bar{C}'_n, D)$ of the non trivial cycles of the differential right module represented by the reduced master equation B which are of the form of sums of trivalent graphs.*

Proof. By the construction of $\bar{C}_n$ and $\bar{C}'_n$, $(\bar{C}'_n, D)$ differs from $(\bar{C}_n, D)$ only in that there are more gluings in strata of codimension two or more. In the gluings of codimensional one strata of compactified and modified configuration spaces of a trivalent graph, a generalized IHX relation is exactly the usual IHX relation used in Kuperberg and Thurston's construction. The collapses from $\overline{\mathrm{Conf}_\infty(\Gamma, M)}$ to $\widehat{\mathrm{Conf}_\infty(\Gamma}, M)$ only reduce the dimension of anomalous faces and do not introduce any new relations. Thus by Proposition 7.3.3, the top cohomology with rational coefficient of $(\bar{C}'_n, D)$ is isomorphic to that of $(\bar{C}_n, D)$:

$$H^{6n}(\bar{C}_n(M), D) \cong H^{6n}(\bar{C}'_n(M), D).$$

Then since the Kontsevich-Kuperberg-Thurston invariants is the degree of the generalized gauss map, if $\bar{C}$ is replaced by $\bar{C}'$, we will have the same invariants. $\square$

Bibliography

[1] Axelrod, S. and Singer, I. M. *Chern-Simons perturbation theory*. In: Proceedings of the XXth International Conference on Differential Geometric Methods in Theoretical Physics, Vol. 1, 2 (New York, 1991), pp. 3–45. World Sci. Publ., River Edge, NJ, 1992.

[2] ——. *Chern-Simons perturbation theory. II*. J. Differential Geom., vol. 39(1) (1994) pp. 173–213.

[3] Baez, J. C. and Dolan, J. *Categorification*. In: Higher category theory (Evanston, IL, 1997), *Contemp. Math.*, vol. 230, pp. 1–36. Amer. Math. Soc., Providence, RI, 1998.

[4] Bar-Natan, D. *Perturbative Chern-Simons theory*. J. Knot Theory Ramifications, vol. 4(4) (1995) pp. 503–547.

[5] Bar-Natan, D. and Duzhin, S. *Bibliography of vassiliev invariants*. `http://www.pdmi.ras.ru/~duzhin/VasBib/`.

[6] Beilinson, A. and Drinfeld, V. Chiral algebras, *American Mathematical Society Colloquium Publications*, vol. 51. American Mathematical Society, Providence, RI, 2004.

[7] Berger, C. and Moerdijk, I. *Resolution of coloured operads and rectification of homotopy algebras*. In: Categories in algebra, geometry and mathematical physics, *Contemp. Math.*, vol. 431, pp. 31–58. Amer. Math. Soc., Providence, RI, 2007.

[8] Birkhoff, G. and Lipson, J. D. *Heterogeneous algebras*. J. Combinatorial Theory, vol. 8 (1970) pp. 115–133.

[9] Boardman, J. M. and Vogt, R. M. Homotopy invariant algebraic structures on topological spaces. Springer-Verlag, Berlin, 1973. Lecture Notes in Mathematics, Vol. 347.

[10] Borcherds, R. E. *Vertex algebras*. In: Topological field theory, primitive forms and related topics (Kyoto, 1996), *Progr. Math.*, vol. 160, pp. 35–77. Birkhäuser Boston, Boston, MA, 1998. `arXiv:q-alg/9706008`.

[11] Bott, R. and Cattaneo, A. S. *Integral invariants of* 3*-manifolds*. J. Differential Geom., vol. 48(1) (1998) pp. 91–133.

[12] ——. *Integral invariants of 3-manifolds. II*. J. Differential Geom., vol. 53(1) (1999) pp. 1–13.

[13] Bott, R. and Taubes, C. *On the self-linking of knots*. J. Math. Phys., vol. 35(10) (1994) pp. 5247–5287.

[14] Cohn, P. M. Universal algebra, *Mathematics and its Applications*, vol. 6. D. Reidel Publishing Co., Dordrecht, second edn., 1981.

[15] Connes, A. and Kreimer, D. *Renormalization in quantum field theory and the Riemann-Hilbert problem. I. The Hopf algebra structure of graphs and the main theorem*. Comm. Math. Phys., vol. 210(1) (2000) pp. 249–273.

[16] ——. *Insertion and elimination: the doubly infinite Lie algebra of Feynman graphs*. Ann. Henri Poincaré, vol. 3(3) (2002) pp. 411–433.

[17] Dugundji, J. Topology. Allyn and Bacon Inc., Boston, Mass., 1966.

[18] Elmendorf, A. D. and Mandell, M. A. *Rings, modules, and algebras in infinite loop space theory*. Adv. Math., vol. 205(1) (2006) pp. 163–228.

[19] Fulton, W. and MacPherson, R. *A compactification of configuration spaces*. Ann. of Math. (2), vol. 139(1) (1994) pp. 183–225.

[20] Gerstenhaber, M. *The cohomology structure of an associative ring*. Ann. of Math. (2), vol. 78 (1963) pp. 267–288.

[21] Getzler, E. and Jones, J. D. S. *Operads, homotopy algebra and iterated integrals for double loop spaces*. `arXiv:hep-th/9403055`.

[22] Ginzburg, V. and Kapranov, M. *Koszul duality for operads*. Duke Math. J., vol. 76(1) (1994) pp. 203–272.

[23] Goguen, J. A., Thatcher, J. W. and Wagner, E. G. *An initial algebra approach to the specification, correctness and implementation of abstract data types*. In: Current Trends in Programming Methodology, vol. IV, pp. 80–149. Prentice-Hall, Inc., Englewood Cliffs, NJ, 1978.

[24] Goresky, M. and MacPherson, R. Stratified Morse theory, *Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)]*, vol. 14. Springer-Verlag, Berlin, 1988.

[25] Goresky, R. M. *Whitney stratified chains and cochains*. Trans. Amer. Math. Soc., vol. 267(1) (1981) pp. 175–196.

[26] Griffiths, P. and Harris, J. Principles of algebraic geometry. Wiley-Interscience [John Wiley & Sons], New York, 1978. Pure and Applied Mathematics.

[27] Hatcher, A. Algebraic topology. Cambridge University Press, Cambridge, 2002. `http://www.math.cornell.edu/~hatcher/AT/ATpage.html`.

[28] Kontsevich, M. *Feynman diagrams and low-dimensional topology*. In: First European Congress of Mathematics, Vol. II (Paris, 1992), *Progr. Math.*, vol. 120, pp. 97–121. Birkhäuser, Basel, 1994.

[29] ——. *Operads and motives in deformation quantization*. Lett. Math. Phys., vol. 48(1) (1999) pp. 35–72. Moshé Flato (1937–1998).

[30] Kontsevich, M. and Soibelman, Y. *Deformations of algebras over operads and the Deligne conjecture*. In: Conférence Moshé Flato 1999, Vol. I (Dijon), *Math. Phys. Stud.*, vol. 21, pp. 255–307. Kluwer Acad. Publ., Dordrecht, 2000.

[31] Kuperberg, G. and Thurston, D. P. *Perturbative 3-manifold invariants by cut-and-paste topology*. UC Davis Math 1999-36. `arXiv:math.GT/9912167`.

[32] Lambek, J. *Deductive systems and categories. II. Standard constructions and closed categories*. In: Category Theory, Homology Theory and their Appli-

cations, I (Battelle Institute Conference, Seattle, Wash., 1968, Vol. One), pp. 76–122. Springer, Berlin, 1969.

[33] Lee, J. M. Introduction to smooth manifolds, *Graduate Texts in Mathematics*, vol. 218. Springer-Verlag, New York, 2003.

[34] Leinster, T. Higher operads, higher categories, *London Mathematical Society Lecture Note Series*, vol. 298. Cambridge University Press, Cambridge, 2004.

[35] Mac Lane, S. Categories for the working mathematician, *Graduate Texts in Mathematics*, vol. 5. Springer-Verlag, New York, second edn., 1998.

[36] Manin, Y. I. Frobenius manifolds, quantum cohomology, and moduli spaces, *American Mathematical Society Colloquium Publications*, vol. 47. American Mathematical Society, Providence, RI, 1999.

[37] Markl, M. *Homotopy algebras are homotopy algebras*. Forum Math., vol. 16(1) (2004) pp. 129–160.

[38] Markl, M., Shnider, S. and Stasheff, J. Operads in algebra, topology and physics, *Mathematical Surveys and Monographs*, vol. 96. American Mathematical Society, Providence, RI, 2002.

[39] May, J. P. The geometry of iterated loop spaces. Springer-Verlag, Berlin, 1972. Lectures Notes in Mathematics, Vol. 271.

[40] Plotkin, B. Universal algebra, algebraic logic, and databases, *Mathematics and its Applications*, vol. 272. Kluwer Academic Publishers Group, Dordrecht, 1994. Translated from the 1991 Russian original by J. Cīrulis, A. Nenashev and V. Pototsky and revised by the author.

[41] Stasheff, J. D. *Homotopy associativity of H-spaces. I, II.* Trans. Amer. Math. Soc. 108 (1963), 275-292; ibid., vol. 108 (1963) pp. 293–312.

[42] Steenrod, N. The topology of fibre bundles. Princeton Mathematical Series, vol. 14. Princeton University Press, Princeton, N. J., 1951.

[43] Stiefel, E. *Richtungsfelder und Fernparallelismus in Mannigfaltigkeiten*. Comm. Math. Helv., vol. 8 (1936) pp. 3–51.

[44] Sullivan, D. *Infinitesimal computations in topology*. Inst. Hautes Études Sci. Publ. Math., (47) (1977) pp. 269–331 (1978).

[45] ——. *String topology: Background and present state*. to appear in Current Developments in Mathematics 2005, International Press of Boston, 2007. arXiv:0710.4141v1[math.GT].

[46] ——. *Homotopy theory of the master equation package applied to algebra and geometry: a sketch of two interlocking programs*. to appear in Algebraic Topology: Old and New, M. M. Postnikov Memorial Conference, 2008. arXiv:0803.0588v1[math.QA].

[47] Thurston, D. P. *Integral expressions for the Vassiliev knot invariants*. A.B. thesis, Harvard University, 1995. arXiv:math/9901110[math.QA].

[48] Wilson, S. *On the algebra and geometry of a manifold's chains and cochains*. Ph.D. thesis, Stony Brook University, Stony Brook, NY, 2005.

Appendix A

Graphs, insertions and operads

We investigate in this appendix the properties of insertion operations on some classes of finite connected decorated graphs with external or half edges, and construct coloured operads using these graphs and insertions. This can be seen as a supplement to Chapter 2, but is not used in later chapters.

A.1 Graphs with external edges

In this section, we introduce and fix our definitions and notations about graphs with external or half edges. Since this appendix can be read independently, we will just refer them as graphs.

Definition A.1.1 (Pregraph)**.** A finite pregraph Γ is a triple $(V(\Gamma), E(\Gamma), H(\Gamma))$ of finite sets, where

- An element of $V(\Gamma)$ is called a vertex of Γ.
- An element of $E(\Gamma)$ is called an internal edge of Γ. Each internal edge is labeled by a two-element subset of $V(\Gamma)$. The labels of different internal edges may be the same.

If $\{u, v\}$ is the label of an internal edge $e \in E(\Gamma)$, then vertices u and v are called the endpoints of e.

- An element of $H(\Gamma)$ is called an external edge or half edge of Γ. Each external edge is labeled by a singleton subset of $V(\Gamma)$. The labels of different external edges may be the same.

  If $\{v\}$ is the label of an external edge $h \in H(\Gamma)$, then v is called the endpoint of h.

Definition A.1.2 (Graph)**.** Two pregraphs Γ and Γ' are isomorphic if $V(\Gamma) = V(\Gamma')$ and there exist a bijective map $f : (V(\Gamma), E(\Gamma), H(\Gamma)) \to (V(\Gamma'), E(\Gamma'), H(\Gamma'))$ such that

- the first factor of f is the identity map, and
- the map induced by f from the multiset of labels of internal and external edges of Γ to that of Γ' is the identity map.

A finite graph is the isomorphism class of a finite pregraph.

The adjective *finite* before the term *graph(s)* will be omitted from now on, unless we want to stress it, since only finite graphs will be considered in this thesis.

Remark A.1.1. A graph can be visualized as a 1-dimensional topological space: a vertex corresponds to a 0-dimensional cell, an internal edge corresponds to an open 1-dimensional cell $(0, 1)$ with its ends glued to different vertices, and an external edge corresponds to an open 1-dimensional cell $(0, 1/2)$ with only one end glued to a vertex.

Let v be a vertex of a graph Γ. The set of internal edges of Γ having v as an endpoint will be denoted by $E(v)$. The set of internal and external edges of Γ having v as an endpoint will be denoted by $H(v)$; an element $h \in H(v)$ will be called a half edge of v and labeled by $\{v\}$, and we will say v is the only endpoint of h. For example, if Γ is a graph with a single vertex v, then $H(\Gamma) = H(v)$.

Definition A.1.3 (Connected graph)**.** Two vertices u and v of a graph Γ are connected by an internal edge if they are the endpoints of an internal edge of Γ.

A graph Γ is connected if for any two of its vertices $p, q \in V(\Gamma)$, there exists a sequence of sets of endpoints of its internal edges

$$\{x_0, x_1\}, \{x_1, x_2\}, \cdots, \{x_{k-1}, x_k\}$$

such that $x_0 = p$ and $x_k = q$.

The graph with only one vertex and no internal edges is considered connected.

We will use $\mathcal{G}$ to denote the set of finite graphs. For any $\Gamma_1, \Gamma_2 \in \mathcal{G}$, define $\Gamma_1 \prec \Gamma_2$ if and only if $|V(\Gamma_1)| < |V(\Gamma_2)|$. Then it is immediate that we have the following

Proposition A.1.1. *The pair $(\mathcal{G}, \prec)$ where $\prec$ is defined as above is a strict partially ordered set and all descending chains for any $\Gamma \in \mathcal{G}$ are finite.*

There are some combinatorial constraints on the existence of graphs and connected graphs.

Proposition A.1.2. *Let V be a finite set, and $c_v, c \in \mathbb{Z}_{\geq 0}$, for all $v \in V$. Then*

(1) there exists a graph Γ with vertex set $V(\Gamma) = V$, $|H(v)| = c_v$ and $|H(\Gamma)| = c$ if and only if the following are satisfied:

- $\sum_{v \in V} c_v \geq c$, *where the equality holds if and only if there are no internal edges; and*

- $\sum_{v \in V} c_v \equiv c \mod 2$.

(2) if $|V| > 1$, *there exists a connected graph* Γ *with vertex set* $V(\Gamma) = V$, $|H(v)| = c_v$ *and* $|H(\Gamma)| = c$ *if and only if the following are satisfied:*

- $c_v > 0$ *for all* $v \in V$;

- $\sum_{v \in V} c_v \geq c + 2(|V| - 1)$, *where the equality holds if and only if there are no internal edges; and*

- $\sum_{v \in V} c_v \equiv c \mod 2$.

Proof. (1) is easy. (2) is because in order that Γ be connected, at least $|V(\Gamma)| - 1$ internal edges are needed. □

A.2 Coloured operads

Many algebraic structures can be described in the language of ordinary or generalized operads. For standard facts on ordinary operads, the reader is referred to [21, 22, 29, 36, 38, 39]. In this thesis we will use the notion of a generalized operad called coloured operad [7, 9, 30, 37], which is also known as many-sorted operad or typed operad [3], or multicategory [18, 32, 34], or relaxed multilinear category [10], or pseudo-tensor category [6]. In this section, for convenience, we introduce the definitions of and list certain basic facts about this generalized operad.

The notion of coloured operads can be defined in any symmetric monoidal category [35] $(\mathcal{E}, \otimes)$. Here we require that $\mathcal{E}$ has internal Hom and finite products and coproducts. We will only give the definition when $\mathcal{E}$ is the category of sets (modules, vector spaces, chain complexes, ...) or its subcategory. Let Σ_n be the symmetric group on n symbols.

Definition A.2.1 (C-coloured operad)**.** A coloured operad O in the category $\mathcal{E}$ consists of the following data:

(1) a set C, whose elements will be called colours;

(2) for any $n \in \mathbb{N}$, and any $(n+1)$-tuple $(c_1, ..., c_n; c)$ of colours, there is an object $O(c_1, ..., c_n; c)$ in $\mathcal{E}$, whose elements are operations from n inputs of colours $c_1, ..., c_n$ to an output of colour c;

(3) for any colour c, there is an element $1_c \in O(c; c)$, called the identity of colour c;

(4) for any $(n+1)$-tuple $(c_1, ..., c_n; c)$ of colours and n other tuples of colours

$$(c_{1,1}, ..., c_{1,k_1}), ..., (c_{n,1}, ..., c_{n,k_n}),$$

of lengths $k_1, ..., k_n$ respectively, there is a composition product

$$O(c_1, ..., c_n; c) \otimes O(c_{1,1}, ..., c_{1,k_1}; c_1) \otimes \cdots \otimes O(c_{n,1}, ..., c_{n,k_n}; c_n)$$

$$\xrightarrow{\theta} O(c_{1,1}, ..., c_{n,k_n}; c);$$

(5) for any $\sigma \in \Sigma_n$, there is a map $\sigma^* : O(c_1, ..., c_n; c) \to O(c_{\sigma(1)}, ..., c_{\sigma(n)}; c)$.

These data are required to satisfy the following properties:

- each 1_c is a 2-sided unit for the composition product θ, i.e., we have the following identity
$$1_c \circ \theta = \theta = \theta \circ (1_{c_1}, ..., 1_{c_n});$$

- the maps in (5) define a right Σ_n-action on $O(n) = \oplus_{c_1,...,c_n,c\in C} O(c_1, ..., c_n; c)$, i.e., $\sigma^*\tau^* = (\tau\sigma)^*$ for any $\sigma, \tau \in \Sigma_n$;

- the composition product θ is associative and Σ_n-equivariant in some natural sense (see, e.g., [18]).

Remark A.2.1. When the set C contains only one colour, a C-coloured operad is the same as an ordinary operad.

By removing from Definition A.2.1 all references to the symmetric group actions, one obtains the notion of non symmetric C-coloured operads.

Example A.2.1 (C-coloured endomorphism operad)**.** Let $A = (A(c))_{c\in C}$ be a family of objects of $\mathcal{E}$. The endomorphism operad $\mathrm{End}(A)$ of A is given by

$$\mathrm{End}(A)(c_1, ..., c_n; c) = \mathrm{Hom}(A(c_1) \otimes \cdots \otimes A(c_n), A(c))$$

for $c_1, ..., c_n, c \in C$, where the composition product and Σ_n-action are the evident substitution and permutation.

Definition A.2.2 (Map of coloured operads)**.** Let O be a C-coloured operad and O' a C'-coloured operad. A map of coloured operads from O to O' consists of a function $f : C \to C'$, and for each $c_1, ..., c_n, c \in C$, a function

$$O(c_1, ...c_n; c) \to O'(f(c_1), ..., f(c_n); f(c))$$

which preserves the symmetric group Σ_n-action, the identities and the composition.

Definition A.2.3 (Algebra over a coloured operad)**.** Let O be a C-coloured operad. An O-algebra structure on a family $A = (A(c))_{c\in C}$ of objects of $\mathcal{E}$ is a map of C-coloured operads

$$\alpha : O \to \mathrm{End}(A)$$

which preserves the colours, i.e., is a family of maps

$$\alpha_{c_1,\dots,c_n;c} : O(c_1, \dots c_n; c) \otimes A(c_1) \otimes \dots \otimes A(c_n) \to A(c)$$

satisfying obvious axioms for associativity, unit and equivariance.

Let $\underline{A} = \oplus_{c\in C} A(c)$ for a fixed order of C. We also say $\underline{A}$ is an O-algebra.

If $\mathcal{P}$ is an ordinary operad and $\underline{\mathcal{P}} = \oplus_{j\geq 1}\mathcal{P}(j)$, then $\underline{\mathcal{P}}$ has a canonical structure of $\mathcal{P}$-algebra (see, for example Manin [36], p.177). Similarly, we have the following

Proposition A.2.1. *Let O be a C-coloured operad. Let $O(c) = \oplus_{n\in\mathbb{N};c_1,\dots c_n\in C} O(c_1, \dots c_n; c)$ and $\underline{O} = \oplus_{c\in C} O(c)$ for a fixed order of C. Then $\underline{O}$ has a canonical structure of O-algebra.*

Proof. The O-algebra structure on $\underline{O}$ is given by the maps

$$\alpha_{c_1,\dots,c_n;c} : O(c_1, \dots c_n; c) \otimes O(c_1) \otimes \dots \otimes O(c_n) \to O(c).$$

□

Definition A.2.4 (Map of O-algebras)**.** Let O be a coloured operad and A, A' be O-algebras. A map of O-algebras $f : A \to A'$ is a family of maps $(f_c : A(c) \to A'(c))_{c\in C}$ which commute with the O-algebra structure maps of A, A'.

The above definition (Definition A.2.1) of coloured operads makes use of the canonical order on the set $[n] = \{1, 2, ..., n\}$. In many discussions, instead of $[n]$, we need to work with an arbitrary finite set, then it is useful and natural to have a version of the definition which does not depend on a particular order.

Definition A.2.5 (C-coloured operad; unordered version)**.** A coloured operad O in the category $\mathcal{E}$ consists of the following data:

(1) a set C, whose elements will be called colours;

(2) for any finite set I, family of colours $(c_i)_{i\in I}$, and colour c, there is an object $O((c_i)_{i\in I}, c)$ in $\mathcal{E}$, whose elements are operations $\varphi : (c_i)_{i\in I} \to c$;

(3) for any surjective map between finite sets $p : I \to J$, families of colours $(a_i)_{i\in I}, (b_j)_{j\in J}$, and colour c, there is a composition product

$$O((b_j)_{j\in J}, c) \otimes \prod_{j\in J} O((a_i)_{i\in p^{-1}\{j\}}, b_j) \xrightarrow{\theta} O((a_i)_{i\in I}, c), \tag{A.2.1}$$

$$(\varphi, (\psi_j)_{j\in J}) \mapsto \varphi \circ (\psi_j)_{j\in J} = \varphi((\psi_j)_{j\in J});$$

(4) for any singleton I, colour c, there is an identity element $1_c^I \in O((c_i)_{i\in I}; c)$;

These data are required to satisfy the following associativity and unit axioms:

- If $p : I \to J, q : J \to K$ are maps of finite sets, and $\phi_j : (a_i)_{i\in p^{-1}\{j\}} \to b_j, \psi_k : (b_j)_{j\in q^{-1}\{k\}} \to c_k, \varphi : (c_k)_{k\in K} \to d$ are operations, where a_i, b_j, c_k, d are colours, then

$$(\varphi \circ (\psi_k)_{k\in K}) \circ (\phi_j)_{j\in J} = \varphi \circ (\psi_k \circ (\phi_j)_{j\in q^{-1}\{k\}})_{k\in K};$$

- If I is a finite set, J is a singleton, and $\varphi : (c_i)_{i \in I} \to c$ is an operation, then

$$1_c^J \circ (\varphi)_{j \in J} = \varphi = \varphi \circ (1_{c_i}^{\{i\}})_{i \in I}.$$

The definitions of maps of coloured operads, algebras over a coloured operads etc for the unordered version can be given similarly as in the usual version.

There is an alternative way (Definition A.2.7) of viewing operads, in which instead of a family of n-ary operations, an operad is viewed as a family of binary operations with certain constraints.

Definition A.2.6. Let X, Y be finite sets and $x \in X$, define

$$X \sqcup_x Y = (X \setminus \{x\}) \sqcup Y$$

where the "$\sqcup$" on the right hand side is the disjoint union, i.e, the coproduct in the category of finite sets and maps between them.

Proposition A.2.2 ([38])**.** *Let X, Y, Z be finite sets, $x, x_1, x_2 \in X$ and $y \in Y$, then*

$$(X \sqcup_x Y) \sqcup_y Z = X \sqcup_x (Y \sqcup_y Z),$$

$$(X \sqcup_{x_1} Y) \sqcup_{x_2} Z = (X \sqcup_{x_2} Z) \sqcup_{x_1} Y.$$

Proof. It follows directly from definition. $\square$

Remark A.2.2. In category theory, the coproduct of a family of objects in a category is defined via universal property, and thus is unique only up to canonical isomorphisms [14, 35]. So when we use the phrase "the disjoint union" of two finite sets, we understand that it is up to these canonical isomorphisms. Similarly, the "="s in the above preposition are actually canonical isomorphisms.

The above proposition is used implicitly to make sense of the associativity conditions in the following definition of coloured pseudo-operad.

Definition A.2.7 (C-coloured pseudo-operad; unordered version)**.** A coloured pseudo-operad O in the category $\mathcal{E}$ consists of the following data:

(1) a set C, whose elements will be called colours;

(2) for any finite set I, family of colours $(c_i)_{i\in I}$, and colour c, there is an object $O((c_i)_{i\in I}, c)$ in $\mathcal{E}$, whose elements are operations from $(c_i)_{i\in I}$ to c;

(3) for any constant map between finite sets $p : I \to J$, families of colours $(a_i)_{i\in I}, (b_j)_{j\in J}$, and colour c, there is a composition product

$$O((b_j)_{j\in J}, c) \otimes O((a_i)_{i\in I}, b_{j_0}) \xrightarrow{\circ_{((J,j_0),I)}} O((c_k)_{k\in J\sqcup_{j_0} I}, c),$$

where $j_0 \in p(I)$, and

$$c_k = \begin{cases} a_k, & \text{if } k \in I; \\ b_k, & \text{if } k \in J \setminus \{j_0\}. \end{cases}$$

These data are required to satisfy the following associativity axioms:

- if I, J, K are finite sets, $j_0 \in J, k_0 \in K$, and $\phi : (a_i)_{i\in I} \to b_{j_0}, \psi : (b_j)_{j\in J} \to c_{k_0}, \varphi : (c_k)_{k\in K} \to d$ are operations, where a_i, b_j, c_k, d are colours, then

$$\varphi \circ_{((K,k_0),J\sqcup_{j_0} I)} (\psi \circ_{((J,j_0),I)} \phi) = (\varphi \circ_{((K,k_0),J)} \psi) \circ_{((K\sqcup_{k_0} J, j_0),I)} \phi;$$

- if I, J, K are finite sets, $k_1, k_2 \in K$, and $\phi : (a_i)_{i\in I} \to c_{k_1}, \psi : (b_j)_{j\in J} \to c_{k_2}, \varphi : (c_k)_{k\in K} \to d$ are operations, where a_i, b_j, c_k, d are colours, then

$$(\varphi \circ_{((K,k_2),J)} \psi) \circ_{((K\sqcup_{k_2} J,k_1),I)} \phi = (\varphi \circ_{((K,k_1),I)} \phi) \circ_{((K\sqcup_{k_1} I,k_2),J)} \psi.$$

Proposition A.2.3. *A coloured operad is a coloured pseudo-operad.*

Proof. Let O be a coloured operad. Let I, J be finite sets, $\phi : (a_i)_{i\in I} \to b_{j_0}, \psi : (b_j)_{j\in J} \to c$ are operations, where a_i, b_j are colours of O. Define composition product $\circ_{((J,j_0),I)}$ as follows:

$$\circ_{((J,j_0),I)}(\psi, \phi) = \theta(\psi, (\varphi_j)_{j\in J}),$$

where θ is the composition product of the coloured operad O (see equation (A.2.1)), and

$$\varphi_j = \begin{cases} \phi, & \text{if } j = j_0; \\ 1^{\{j\}}_{b_j}, & \text{if } j \in J \setminus \{j_0\}. \end{cases}$$

One can check that a coloured pseudo-operad is obtained in this way. □

A.3 Insertions of decorated graphs and operads

A graph can be inserted into a vertex of another graph to form a third graph. In this section, we focus on a special class of graphs with labeled vertices and study their properties under insertions.

Definition A.3.1 (Inserting a graph into another)**.** Let Γ_1, Γ_2 be connected graphs and $v \in V(\Gamma_1)$ such that there is a bijective map $f : H(v) \to H(\Gamma_2)$.

Define a graph Γ by the following data

- $V(\Gamma) = (V(\Gamma_1) \setminus \{v\}) \sqcup V(\Gamma_2)$;
- $E(\Gamma) = E(\Gamma_1) \sqcup E(\Gamma_2)$; an internal edge of Γ_1 with endpoint set $\{p, v\}$ corresponds to an internal edge of Γ with endpoint set $\{p, q\}$, where $p \in V(\Gamma_1) \setminus \{v\}$ and $q \in V(\Gamma_2)$ is the endpoint of $f(v)$;
- $H(\Gamma) = H(\Gamma_1)$; an external edge of Γ_1 with endpoint v corresponds to an external edge of Γ with endpoint the endpoint of $f(v)$.

We will call Γ the graph obtained by inserting Γ_2 into Γ_1 at vertex v according to the map f. The graph Γ is denoted by $\Gamma_1 \circ_{(v,f)} \Gamma_2$.

A.3.1 *L*-graphs

Definition A.3.2 (*L*-graph)**.** An *L*-graph Γ is a a graph along with the following data:

- there is a linear order on the set $H(\Gamma)$ of external edges.
- for each $v \in V(\Gamma)$, there is a linear order on the set $H(v)$ of half edges of v.

For *L*-graphs, Definition A.3.1 can be made more specific.

Definition A.3.3 (Insertion operation of *L*-graphs)**.** Let Γ_1, Γ_2, and Γ_3 be connected *L*-graphs. If there exist $v \in V(\Gamma_1)$ and a linear order preserving bijective map $f : H(v) \to H(\Gamma_2)$ such that $\Gamma_3 = \Gamma_1 \circ_{(v,f)} \Gamma_2$, we say that there is an insertion operation $\circ_{((\Gamma_1,v),\Gamma_2,\Gamma_3)} : (\Gamma_1, \Gamma_2) \mapsto \Gamma_3$.

Remark A.3.1. In the notation $\circ_{((\Gamma_1,v),\Gamma_2,\Gamma_3)}$ above, the Γ_3 in the subscript is redundant, i.e., one can safely use $\circ_{((\Gamma_1,v),\Gamma_2)}$ instead and there will be no confusion. But we prefer the notation adopted, since it is more convenient in certain symbolic proofs.

Theorem A.3.1. *The set $\mathcal{L}_0$ of connected labeled L-graphs with two or more vertices forms a $\mathbb{Z}_{\geq 0}$-coloured pseudo-operad in the category of sets.*

Proof. Here we will check Definition A.2.7.

- Take the set $\mathbb{Z}_{\geq 0}$ of non negative integers as the set C of colours.
- For any finite set I, family of colours $(c_i)_{i\in I}$, and colour c, there is a set $O((c_i)_{i\in I}, c)$ of connected L-graphs Γ: each $\Gamma \in O((c_i)_{i\in I}, c)$ is an L-graph with $V(\Gamma) = I$, $|H(\Gamma)| = c$ and $|H(i)| = c_i$ for all $i \in I$. Because of combinatorial constraints (see Proposition A.1.2), some $O((c_i)_{i\in I}, c)$s may be empty.
- For any constant map between finite sets $p : I \to J$, families of colours $(a_i)_{i\in I}, (b_j)_{j\in J}$, and colour c, there is a composition product

  $$O((b_j)_{j\in J}, c) \otimes O((a_i)_{i\in I}, b_{j_0}) \xrightarrow{\circ_{((J,j_0),I)}} O((c_k)_{k\in J\sqcup_{j_0} I}, c),$$

  where $j_0 \in p(I)$, and

  $$c_k = \begin{cases} a_k, & \text{if } k \in I; \\ b_k, & \text{if } k \in J \setminus \{j_0\}. \end{cases}$$

  In our situation, if some $O((c_i)_{i\in I}, c)$s are empty, the composition product is trivial; otherwise, given graph $\Gamma \in O((b_j)_{j\in J}, c)$, $\gamma \in O((a_i)_{i\in I}, b_{j_0})$, one has a graph $\Gamma \circ_{((J,j_0),I)} \gamma \in O((c_k)_{k\in J\sqcup_{j_0} I}, c)$.

Now one can readily see that the associativity axioms in Definition A.2.7 are satisfied. □

Corollary A.3.2. *The set $\mathcal{L}_0$ of connected labeled L-graphs with two or more vertices forms an algebra over the above defined coloured pseudo-operad of L-graphs.*

Proof. This follows from Theorem A.3.1 and Proposition A.2.1. □

A.3.2 $\mathcal{A}$-labeled L-graphs

Let $\mathcal{A}$ be a countably infinite discrete set. A prototypal example of $\mathcal{A}$ is the set $\mathbb{N}$ of natural numbers.

Definition A.3.4 ($\mathcal{A}$-labeled graph)**.** A finite $\mathcal{A}$-labeled L-graph is a finite L-graph Γ which is labeled in $\mathcal{A}$ subjecting to the following rules:

- Each vertex of Γ is labeled by a non empty finite subset of $\mathcal{A}$.
- Each element of $\mathcal{A}$ can occur at most once for each $\mathcal{A}$-labeled graph.
- The label of the graph is the union of labels of its vertices.

An isomorphism of $\mathcal{A}$-labeled L-graphs is a graph isomorphism which preserves the $\mathcal{A}$-labeling and all linear orders. Let $\mathcal{AL}$ denote the set of isomorphism classes of connected $\mathcal{A}$-labeled L-graphs.

Definition A.3.5 (Insertion operation of $\mathcal{A}$-labeled L-graphs)**.** Let $\Gamma_1, \Gamma_2, \Gamma_3 \in \mathcal{AL}$. If there exist $v \in V(\Gamma_1)$ whose label is equal to the label of Γ_2 and a linear order preserving bijective map $f : H(v) \to H(\Gamma_2)$ such that $\Gamma_3 = \Gamma_1 \circ_{(v,f)} \Gamma_2$, we say that there is an insertion operation $\circ_{((\Gamma_1,v),\Gamma_2,\Gamma_3)} : (\Gamma_1, \Gamma_2) \mapsto \Gamma_3$.

Theorem A.3.3. *The set $\mathcal{AL}$ of connected $\mathcal{A}$-labeled L-graphs forms a coloured operad in the category of sets.*

Proof. Here we check that the set $\mathcal{AL}$ satisfies Definition A.2.5.

- Define a colour c to be a pair (n, A), where n is a non negative integer and A is a non empty finite subset of $\mathcal{A}$.

- For any finite set I, family of colours $(c_i)_{i\in I}$, where $c_i = (n_i, A_i)$, and colour $c = (n, A)$, there is a set $O((c_i)_{i\in I}, c)$ of connected $\mathcal{A}$-labeled L-graphs Γ: each $\Gamma \in O((c_i)_{i\in I}, c)$ is an $\mathcal{A}$-labeled L-graph satisfies the following conditions:

  - $V(\Gamma) = I$, $|H(\Gamma)| = n$ and $|H(i)| = n_i$ for all $i \in I$;
  - vertex i is labeled by A_i, for all $i \in I$, and the graph Γ is labeled by A.

  Because of the combinatorial constraints (see Proposition A.1.2) and $\mathcal{A}$-labeling restriction, some $O((c_i)_{i\in I}, c)$s are empty sets.

- For any surjective map between finite sets $p : I \to J$, families of colours $(a_i)_{i\in I}, (b_j)_{j\in J}$, and colour c, there is a composition product

$$O((b_j)_{j\in J}, c) \otimes \prod_{j\in J} O((a_i)_{i\in p^{-1}\{j\}}, b_j) \xrightarrow{\theta} O((a_i)_{i\in I}, c),$$

$$(\varphi, (\psi_j)_{j\in J}) \mapsto \varphi \circ (\psi_j)_{j\in J} = \varphi((\psi_j)_{j\in J}).$$

  More explicitly, if all $O((c_i)_{i\in I}, c)$s in the domain are not empty, the composition means that after inserting each vertex j of a graph $\Gamma \in O((b_j)_{j\in J}, c)$ a graph $\gamma \in O((a_i)_{i\in p^{-1}\{j\}}, b_j)$, a new graph $\Gamma' \in O((a_i)_{i\in I}, c)$ is obtained.

  In case that some $O((c_i)_{i\in I}, c)$ in the domain is empty, the composition product is trivial.

- For any singleton $I = \{i\}$, colour $c = (n, A) \in \mathbb{Z}_{\geq 0}$, the graph with only one vertex i labeled by A, no internal edges and n external edges is the unit $1_c^{\{i\}}$.

One can easily check that the above data satisfy the associativity and unit axioms. □

Remark A.3.2. Strictly speaking, the definition of pseudo-tensor category of Beilinson Drinfeld [6] is slightly different from the definition of coloured operad we adopted (Definition A.2.5). But by the same construction, one can check that the set $\mathcal{AL}$ of connected $\mathcal{A}$-labeled L-graphs forms a pseudo-tensor category. Similar remarks applies to all the related statements.

A.4 Relation to pre-Lie algebra

This section discusses the relations between the above algebras of graphs under insertions and pre-Lie algebra.

Definition A.4.1 (Pre-Lie algebra)**.** Let k be a field of characteristic 0 and W be a vector space. A pre-Lie algebra is W with a bilinear operation $\cdot : W \times W \to W$ which satisfies

$$(\alpha \cdot \beta) \cdot \gamma - \alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \gamma) \cdot \beta - \alpha \cdot (\gamma \cdot \beta)$$

for any $\alpha, \beta, \gamma \in W$.

Let $\alpha, \beta \in W$, define a bilinear operation $[,] : W \times W \to W$ by

$$[\alpha, \beta] = \alpha \cdot \beta - \beta \cdot \alpha.$$

Proposition A.4.1. *Let* $(W, \cdot)$ *be a pre-Lie algebra and* $[,]$ *the binary operation defined as above. Then* $(W, [,])$ *is a Lie algebra. More explicitly, if* $\alpha, \beta, \gamma \in W$*, then*

1. *(anti-symmetry)* $[\alpha, \beta] = -[\beta, \alpha]$.

2. *(Jacobi identity)* $[\alpha, [\beta, \gamma]] + [\beta, [\gamma, \alpha]] + [\gamma, [\alpha, \beta]] = 0$.

Proof. It is a direct check, or see Gerstenhaber [20]. □

There is a notion of the sum of all ways of insertions of graphs which will lead to a pre-Lie algebra. We give it in the case of L-graphs; other cases are similar.

Definition A.4.2 (All possible insertions operation on graphs)**.** Let $\mathcal{W}$ be the linear space spanned by the set of finite L-graphs. Define a bilinear operator $* : \mathcal{W} \otimes \mathcal{W} \to \mathcal{W}$, by

$$\Gamma_1 * \Gamma_2 = \sum_{v \in V(\Gamma_1), \Gamma_3} \Gamma_1 \circ_{((\Gamma_1, v), \Gamma_2, \Gamma_3)} \Gamma_2 \tag{A.4.1}$$

for any L-graphs Γ_1, Γ_2 and then extend bilinearly to $\mathcal{W}$.

Remark A.4.1. It is easy to see that the sum at the right hand side of equation (A.4.1) has only finitely many nonzero terms. Thus the above operation is well defined.

Proposition A.4.2. *The pair* $(\mathcal{W}, *)$ *is a pre-Lie algebra.*

Proof. See Connes-Kreimer [16]. □