Rigidity of Conformally Compact Manifolds

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Abstract of the Dissertation
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Rigidity problems for conformally compact manifolds have been studied by several mathematicians over the past. We have proved that a conformally compact manifold with a pole and with round sphere as the conformal infinity, under a lower bound on the Ricci curvature and a weak asymptotic bound on the scalar curvature has to be isometric to the hyperbolic space. We have also sharpened a result by Shi and Tian.

To my uncle, Late Professor Subhashis Nag

Contents

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Chapter 1

Introduction

The study of conformally compact manifolds has been a topic of great interest to both physicists and mathematicians. From the point of view of physics, conformally compact Einstein manifolds play a very important role in the ADS/CFT correspondence, first proposed by Maldacena. It asserts the existence of a correspondence between string theory on an asymptotically locally anti-de Sitter spacetime and an appropriate conformal field theory on the boundary at-infinity.

We will begin with the definition of *Conformally Compact* manifolds.

Definition 1.0.1. Let $\bar{M}$ be an oriented compact manifold with boundary. A $C^{m,\alpha}$ ($m \in \mathbb{Z}^+, \alpha \in (0,1)$) function $\rho : \bar{M} \to \mathbb{R}$ is said to be a *defining function* for $\partial\bar{M}$ if $\rho > 0$ in M, $\rho|_{\partial\bar{M}} = 0$, and $d\rho \neq 0$ on $\partial\bar{M}$.

Definition 1.0.2. A complete metric g on M is said to be $C^{m,\alpha}$ conformally compact if there exists a $C^{m,\alpha}$ defining function ρ for $\partial\bar{M}$ such that $\bar{g} = \rho^2 g$ extends to a $C^{m,\alpha}$ metric on $\bar{M}$.

We wish to point out here that defining functions are unique up to multiplication by positive functions on $\bar{M}$. In other words, if ρ is a defining function of the boundary then so is $f\rho$, where $f : \bar{M} \to \mathbb{R}$ is any smooth positive bounded function. Therefore, only the conformal class $[\bar{g}]$ is uniquely determined by g, same goes with the conformal class $[\gamma] = [\bar{g}|_{\partial\bar{M}}]$ of the induced boundary metric. The class $[\gamma]$ is called the *Conformal Infinity.* A formal definition is provided below.

Definition 1.0.3. Let (M, g) be a complete conformally compact manifold. The conformal class, $[\gamma]$ of boundary metrics determined uniquely by g is called the *conformal infinity* of (M, g).

Example 1.0.1. *As an example we consider the hyperbolic space* $\mathbb{H}^{n+1}$ *with the Poincare metric* g_{hyp} *and the unit ball* B^{n+1} *with the flat Euclidean metric* g_{euc}. *It is a well known fact that,*

$$g_{euc} = \left(\frac{1 - |x|^2}{2}\right)^2 g_{hyp},$$

where $|x|$ *is the Euclidean distance of a point* $x \in \bar{\mathrm{B}}$ *from the origin. Therefore,* $(\mathbb{H}^{n+1}, g_{hyp})$ *is conformally compact and a defining function is* $\frac{1-|x|^2}{2}$. *The compactification in this case is smooth and the conformal infinity is the conformal class of the round metric* $[\gamma_0]$ *on* S^n.

Even though defining functions in general are not unique, one can still prove uniqueness in the sense below.

Definition 1.0.4. A defining function ρ is called a *geodesic defining function* if $|\bar{\nabla}\rho|_{\bar{g}} = 1$ in a neighborhood of ∂M.

It has been proved that if the compactification is at least C^2, then for a given boundary metric, every such conformal compactification has a unique geodesic defining function.

It is easy to see that conformally compact manifolds have the following asymptotic curvature bounds:

$$|K_{ij} + 1| = O(\rho^2), \tag{1.0.1}$$

for every sectional curvature K_{ij} measured with respect to the complete metric g. Therefore, such manifolds are *asymptotically locally hyperbolic.*

Definition 1.0.5. A metric g on an open manifold M is called asymptotically hyperbolic if the sectional curvatures of 2-planes at $p \in M$ converge to -1 as $dist_g(p, p_0) \to \infty$, where $p_0 \in M$ is a fixed point.

This raises the following question:
When is an asymptotically hyperbolic manifold isometric to the hyperbolic space?
Rigidity of asymptotically Euclidean manifolds ties up with the positive mass theorem for Riemannian manifolds. It says that an asymptotically Euclidean manifold of dimension between 3 and 7 with everywhere non-negative scalar curvature and zero mass is isometric to the Euclidean space. This result was proved by Schoen and Yau ([16],[17]). Witten [20] showed that the conclusion is true in any dimension under the extra assumption that the manifold is spin. We recall here that a spin manifold (M^{n+1}, g) is an oriented Riemannian manifold together with a lift of the structure group $SO(n+1)$ of its principal bundle $SO(M^{n+1}, g)$ of all oriented orthonormal frames to its simply connected double cover $Spin(n+1)$. The hyperbolic analog of this result was proved by

Min O-O in the following theorem.

Definition 1.0.6. A smooth Riemannian manifold (M^{n+1}, g) is said to be *strongly asymptotically hyperbolic* if there exists a compact subset $B \subset M$ and a diffeomorphism $\phi : M - B \to \mathbb{H}^{n+1} - \bar{B}_{r_0}(0)$ for some $r_0 > 0$, such that the transformation $A : T(M - B) \to T(M - B)$ defined by the conditions,

$$g(Au, Av) = \phi^* g(u, v) = g(d\phi(u), d\phi(v)), \qquad g(Au, v) = g(u, Av)$$

for all $u, v \in T(M - B)$, satisfies the following properties: (a) There exists a uniform Lipschitz constant $C \geq 1$ such that for all $v \in T(M - B)$, $C^{-1} \leq \min_{|v|=1} |Av| \leq \max_{|v|=1} |Av| \leq C$ and (b) $\exp(\phi \circ r)(A - \mathrm{id}) \in L^{1,2}(T^*(M - B) \otimes T(M - B), g)$.

Theorem 1.0.1. *(Min O-O, [12]) A strongly asymptotic hyperbolic spin manifold of dimension $n \geq 3$, whose scalar curvature satisfies $R \geq -n(n-1)$ everywhere, is isometric to the hyperbolic space.*

This raises the next question:

Can one prove rigidity without having the spin structure?

The following theorem due to M.C.Leung gives a partial answer.

Definition 1.0.7. A smooth conformally compact metric g on a manifold M is said to be a *generalized Poincaré metric* if g is Einstein and $h = \rho^2 g$ is an even function of ρ, where ρ is a defining function of the boundary.

Theorem 1.0.2. *(Leung, [11]) For any odd integer $n \geq 2$, let g be a generalized Poincare metric on $\mathrm{B}^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| < 1\}$. Suppose the sectional* *curvatures of* (B^{n+1}, g) *approach -1 in order* $O(\rho^\tau)$ *for* $\tau > 2$, *then* (B^{n+1}, g) *is isometric to the standard hyperbolic space.*

The curvature constraints are stronger than eqn(1.0.1). As we know already, conformally compact manifolds are by definition asymptotically hyperbolic. They are the natural candidates for a generalization of Leung's result. More recently, J.Qing, cf [15] proved a rigidity result for conformally compact Einstein manifold (M^{n+1}, g).

Theorem 1.0.3. *(Qing, [15]) Suppose that* (X^{n+1}, g) *is a conformally compact Einstein manifold with the round sphere as its conformal infinity, and* $3 \leq n \leq 6$. *Then* (X, g) *is the hyperbolic space.*

One limitation of this theorem is the restriction on the dimension of the manifold. Anderson, cf [4] generalized this result and showed that it is true in any dimension.

Theorem 1.0.4. *(Anderson, [4]) A* C^2 *conformally compact Einstein metric with conformal infinity given by the class of the round metric* g_+ *on the sphere* S^n *is necessarily isometric to the Poincaré metric on the ball* B^{n+1}.

A very recent theorem of Shi and Tian [18] proves the following.

Definition 1.0.8. A manifold (M, g) is said to have a pole $p \in M$ if

$$\exp_p : T_pM \to M$$

is a diffeomorphism.

Definition 1.0.9. A complete non-compact Riemannian manifold (M^{n+1}, g) is called asymptotically locally hyperbolic(ALH) of order α if $|K_{i,j} + 1| = O(e^{-\alpha\rho(x)})$, where $\rho(x) = dist_g(x, q)$, where q is some point in M.

Theorem 1.0.5. *(Shi-Tian, [18]) Suppose that (X^{n+1}, g), $n \geq 2$ and $n \neq 3$ is an ALH manifold of order α with a pole and there is a $\rho > 1$ such that the geodesic sphere with radius ρ and center at the pole is convex. If we further have $\alpha > 2$ and $Ric(g) \geq -ng$, then (X^{n+1}, g) is isometric to $\mathbb{H}^{n+1}$.*

As a remark, we wish to point out that the condition on α in the above theorem, is much stronger than equation(1.0.1). We now present the main theorem:

Theorem 1.0.6. *Let (M^{n+1}, g), $n \geq 2$ be a complete smoothly conformally compact manifold with a pole p with conformal infinity $(S^n, [\gamma_0])$ and $Ric \geq -ng$, where $[\gamma_0]$ is the conformal class of the round metric. Let $t(x) = dist_g(p, x)$ and $\frac{1}{Vol(\Sigma_t)} \int_{\Sigma_t} |R + n(n+1)| = o(e^{-2t})$, where Σ_t's are level sets of t and R is the scalar curvature of (M^{n+1}, g). Then (M^{n+1}, g) is isometric to (B^{n+1}, g_{-1}).*

It assumes a weak asymptotic bound on the scalar curvature and only a lower bound on the Ricci curvature. We make no additional assumption on the sectional curvatures other than what is inherited through conformally compactness. The scalar curvature of any conformally compact manifold satisfies $|R + n(n+1)| = O(e^{-2t})$, by virtue of equation(1.0.1). We require a slightly stronger decay rate. During the course of the proof we have also shown that Shi-Tian's result in ([18]) holds under a slightly weaker assumption on the sectional curvature bound.

Chapter 2

Background Material

Before getting into the proof of our main result, we will gather here some essential definitions and results that will be used in the subsequent chapter.

2.1 Definitions of Various Curvatures

Curvature Tensor

The curvature tensor $\mathfrak{R}$ is a $(1,3)$-tensor defined as

$$\mathfrak{R}(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \tag{2.1.1}$$

for any three vector fields $X, Y, Z \in TM$

Next we define three important curvature quantities:

Sectional Curvature

For any $p \in M$ and any $v, w \in T_pM$, the sectional curvature $K_{v,w}$ is defined as:

$$K_{v,w} = \frac{g(\mathfrak{R}(w,v)v, w)}{g(v,v)g(w,w) - (g(v,w))^2}$$

In terms of an orthonormal basis $\{e_i\}$ of T_pM:

$$K_{i,j} = K_{e_i,e_j} = g(\Re(e_j, e_i)e_i, e_j) \tag{2.1.2}$$

Ricci Curvature

For any point $p \in M$, let $\{e_i\}$ be an orthonormal basis of T_pM, then for any $v, w \in T_pM$, the Ricci curvature,

$$Ric(v, w) = \sum_{i=1}^{n} g(\Re(v, e_i)e_i, w) \tag{2.1.3}$$

Scalar Curvature

The scalar curvature at any point $p \in M$ is trace of the Ricci curvature, i.e.,

$$\begin{aligned} R &= \sum_{i=1}^{n} Ric(e_i, e_i) \\ &= \sum_{i,j=1}^{n} K_{ij}. \end{aligned} \tag{2.1.4}$$

Let (M^{n+1}, g) be a Riemannian manifold, under a conformal change of the metric g by a conformal factor e^{2f}, that is $\bar{g} = e^{2f}g$, one can relate the new curvatures of M with respect to $\bar{g}$ to those with respect to g. Here we will list the formulas without proof:

$$\begin{aligned}
\bar{Ric} &= Ric - (n-1)(Ddf - df \circ df) + (\Delta f - (n-1)|df|^2)g, && (2.1.5)\\
\bar{R} &= e^{2f}(R + 2n\Delta f - n(n-1)|dt|^2), && (2.1.6)\\
\bar{\nabla}_X Y &= \nabla_X Y + g(X,\nabla f)Y + g(Y,\nabla f)X - g(X,Y)\nabla f. && (2.1.7)
\end{aligned}$$

A convenient way to express the scalar curvature equation (2.1.6) is:

$$\bar{R} = u^{\frac{n+3}{1-n}}(n(n+1)u - \frac{4n}{n-1}\Delta_g u), \qquad (2.1.8)$$

where $\bar{g} = u^{\frac{4}{n-1}}g$. We have chosen the Laplacian as the trace of the second fundamental form.

For any distance function $f : U \subset (M^{n+1}, g) \to \mathbb{R}$, (for our purpose we would assume f to be smooth) the level sets, $f^{-1}(r) = U_r$ are smooth hypersurfaces in U, with induced metrics $\gamma_r = g|_{U_r}$. Clearly $N = \nabla f$ is a unit normal vector field to U_r. The Hessian of f denoted by $S = \nabla^2 f$ also called the *shape operator* or *second fundamental form* measures how the induced metric on U_r changes.

The following formula relates the curvature tensor, $\mathfrak{R}_r$ of (U_r, γ_r) to the curvature tensor $\mathfrak{R}$ of the ambient manifold M, and the shape operator $S = \nabla^2 f$.

Gauss-Codazzi Equations:

For any three vector fields X, Y, Z tangent to U_r, we have

$$(1) \quad \tan\mathfrak{R}(X,Y)Z = \mathfrak{R}_r(X,Y)Z - g(S(Y),Z)X + g(S(X),Z)Y, \tag{2.1.9}$$

$$(2) \quad \mathrm{nor}\mathfrak{R}(X,Y)Z = g((\nabla_Y S)(X) - (\nabla_X S)(Y), Z).N. \tag{2.1.10}$$

where $\tan(W) = W - g(W,N)N$ and $\mathrm{nor}(W) = g(W,N).N$

Let us now choose coordinates in U such that $\{\partial_1, ..\partial_n\}$ define coordinates on the level set U_r and $\partial_r = \nabla f$ is the unit normal. Let us introduce the following notations

$$g_{ij} = g(\partial_i, \partial_j), \tag{2.1.11}$$

$$R_{\partial_r}(\partial_i) = \mathfrak{R}(\partial_i, \partial_r)\partial_r = \sum R_i^j \partial_j, \tag{2.1.12}$$

$$S(\partial_i) = \nabla_{\partial_i}\partial_r = \sum S_i^j \partial_j, \tag{2.1.13}$$

$$m = \mathrm{tr}(S_i^j) = \Delta f. \tag{2.1.14}$$

We will now write out the following set of equations that we are going to use extensively in the next chapter. Proofs and further discussions can be found in [14]

$$\begin{aligned}
\partial_r(S_i^j) + (S_i^k)(S_k^j) &= -(R_i^j), & (2.1.15)\\
\partial_r(g_{ij}) &= 2(S_i^k).(g_{kj}), & (2.1.16)\\
\partial_r m + \frac{m^2}{n} &\leq -Ric(\partial_r, \partial_r), & (2.1.17)\\
\partial_r\sqrt{\det(g_{i,j})} &= m.\sqrt{\det(g_{i,j})}. & (2.1.18)
\end{aligned}$$

Notice that taking the trace of equation (2.1.15), gives:

$$\partial_r m + \operatorname{tr}(S_i^k)(S_k^j) = -Ric(\partial_r, \partial_r). \tag{2.1.19}$$

To get equation (2.1.17) from equation (2.1.19) one simply needs to apply Cauchy-Schwartz inequality for matrices. Having set the notations, we would now state the following comparison theorem, the proof of which can also be found in [14]. Let us suppose that (M^{n+1}, g) is a Riemannian manifold. Let (r, θ) be polar coordinates around a point $p \in M$, Then the following holds:

Theorem 2.1.1. *Assume that* (M^{n+1}, g) *satisfies* $k \leq \operatorname{Sec} \leq K$. *If* (g_{ij}) *represents the metric in the polar coordinates and* (S_i^j) *the Hessian of* f, *then we have*

$$\begin{aligned}
\operatorname{sn}_K^2(r) &\leq (g_{ij}(r,\theta))_{2\leq i,j\leq n+1} \leq \operatorname{sn}_k^2(r), & (2.1.20)\\
\sqrt{K}\frac{\operatorname{sn}_K'(r)}{\operatorname{sn}_K(r)} &\leq (S_i^j(r,\theta))_{2\leq i,j\leq n+1} \leq \sqrt{k}\frac{\operatorname{sn}_k'(r)}{\operatorname{sn}_k(r)}, & (2.1.21)
\end{aligned}$$

where,

$$\mathrm{sn}_k(r) = \begin{cases} \frac{1}{\sqrt{k}} \sin(\sqrt{k}t) & \textit{if } k > 0 \\ t & \textit{if } k = 0 \\ \frac{1}{\sqrt{k}} \sinh(\sqrt{k}t) & \textit{if } k < 0 \end{cases}$$

In particular, if $\lambda_{max}, \lambda_{min}$ are the maximum and minimum eigenvalues of S_i^j respectively, then

$$\sqrt{K}\frac{\mathrm{sn}_K'(r)}{\mathrm{sn}_K(r)} \leq \lambda_{max}(r,\theta), \lambda_{min}(r,\theta) \leq \sqrt{k}\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}. \tag{2.1.22}$$

2.2 Comparison Theorems

We will use Bishop-Gromov volume comparison theorem on several occasions. Therefore, we will record the theorem here.

Theorem 2.2.1. *(Relative Volume Comparison, Bishop-Cheeger-Gromov) Suppose (M^{n+1}, g) is a complete Riemannian manifold with Ric $\geq nk$, $k \in \mathbb{R}$. Then*

$$r \to \frac{\mathrm{vol}B(p,r)}{v(n+1,k,r)} \tag{2.2.1}$$

is a nondecreasing function whose limit as $r \to 0$ is 1.

Here $v(n+1,k,r)$ is the volume of an (n+1)-ball of radius r in a space-form of constant sectional curvature $k \in \mathbb{R}$ and $p \in M$. The other useful result is the Laplacian comparison lemma (9.1.1)in [14].

Lemma 2.2.2. *Suppose (M^{n+1}, g) has Ric $\geq nk$ for some $k \in \mathbb{R}$. Then*

$$m(r,\theta) \leq n\frac{\mathrm{sn}_k'(r)}{\mathrm{sn}_k(r)}. \tag{2.2.2}$$

We will now discuss the function spaces briefly. We are going to use an embedding lemma towards the end of our proof, as a step towards boosting up the regularity of a limit function. Here we will state the lemma without proof.

2.3 Function Spaces

Definition 2.3.1 (Hölder Spaces). The (m, α) Hölder norm of a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$, is defined as:

$$[f]_{m,\alpha} = \sum_{|i| \leq m} \sup |\partial^i f| + \sum_{|i|=m} \left\{ \sup_{p,q \in \Omega} \frac{|\partial^i f(p) - \partial^i f(q)|}{|p-q|^\alpha} \right\} \tag{2.3.1}$$

The $C^{m,\alpha}(\Omega)$ Hölder space is the space of functions on Ω, that have bounded $C^{m,\alpha}$ norm.

Definition 2.3.2. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ be locally integrable and $\alpha = \{\alpha_1, ..., \alpha_n\}$ any multi-index. Then a locally integrable function v is called the α^{th} *weak derivative* of u if it satisfies

$$\int_\Omega \phi v dx = (-1)^{|\alpha|} \int_\Omega u D^\alpha \phi dx \quad for\, all\ \phi \in C_0^{|\alpha|}(\Omega).$$

Definition 2.3.3 (Sobolev Spaces). The (k,p) Sobolev norm if a function $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ is defined as:

$$||f||_{k,p;\Omega} = \sum_{|\alpha| \leq k} \left(\int_\Omega |D^\alpha f|^p dV \right)^{1/p}, \tag{2.3.2}$$

where D^α is the α weak derivative of the function f and dV is the volume form. The Sobolev space $W^{k,p}(\Omega)$ is the space of all functions f on Ω with bounded (k,p) Sobolev norm.

Under certain conditions on the exponents $W^{k,p}$ spaces embed into $C^{m,\alpha}$ Hölder spaces. The following embedding lemma is of particular interest to us.

Definition 2.3.4. Let $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}$ and let e_i be the unit coordinate vector in the x_i direction. We define the difference quotient in the direction e_i by

$$\Delta^h u(x) = \Delta_i^h(x) = \frac{u(x+he_i) - u(x)}{h}, \quad h \neq 0.$$

Now, we will state the lemma. The proof can be found in [8](lemma 7.24)

Lemma 2.3.1. *Let $u \in L^p(\Omega)$, $1 < p < \infty$, and suppose there exists a constant K such that $\Delta^h u \in L^p(\Omega')$ and $||\Delta^h u||_{L^p(\Omega')} \leq K$ for all $h > 0$ and $\Omega' \subset\subset \Omega$ satisfying $h < \operatorname{dist}(\Omega', \partial\Omega)$. Then the weak derivative $D_i u$ exists and satisfies $||D_i u||_{L^p(\Omega)} \leq K$.*

In particular if Ω is bounded and u has a uniform $C^{0,1}$ bound on Ω then $u \in W^{1,p}(\Omega)$ for all $p > 1$.
We will state Arzela-Ascoli theorem, that we will use to prove convergence of a sequence of functions.

Theorem 2.3.2 (Arzela-Ascoli). *Let X be a compact Hausdorff space and $C(X)$ the Banach space of the continuous functions on X with the norm of uniform convergence. A subset $S \subset C(X)$ is precompact, i.e. $\bar{S}$ is compact, if and only if it is bounded and equicontinuous.*

2.4 Yamabe Quotient

Let us assume that (M^n, g) is a compact Riemannian manifold. Under a conformal change of metric $\bar{g} = \phi^{4/(n-2)}g$, the scalar curvature $\bar{R}$ with respect to metric $\bar{g}$ satisfies the following P.D.E. (as in equation (2.1.8))

$$\bar{R}\phi^{2n/(n-2)} = R\phi^2 - \frac{4(n-1)}{(n-2)}\phi\Delta\phi, \tag{2.4.1}$$

where R is the scalar curvature with respect to the metric g.

The volume forms $d\eta$ and $d\bar{\eta}$ satisfy the following equation:

$$d\bar{\eta} = \phi^{2n/(n-2)}d\eta \tag{2.4.2}$$

Yamabe Quotient $Q(g)$ is defined as follows:

$$Q(g) = \frac{\int\limits_M \bar{R}d\bar{\eta}}{\left(\int\limits_M d\bar{\eta}\right)^{\frac{n-2}{n}}} = \frac{\int\limits_M \left(n(n-1)\phi^2 + \frac{4(n-1)}{n-2}|\nabla_g\phi|^2\right)d\eta}{\left(\int\limits_M \phi^{\frac{2n}{n-2}}d\eta\right)^{\frac{n-2}{n}}}. \tag{2.4.3}$$

Yamabe Invariant, μ_g is the defined as the infimum of $Q(g)$ over the conformal class of g, i.e.

$$\mu_g = \inf\{Q(\bar{g})\,|\bar{g} \in [g]\}$$

From the definition it is trivial to see that μ_g is a conformal invariant. It is also known due to Aubin that for any compact Riemannian manifold (M^n, γ), $\mu_\gamma \leq \mu_{\gamma_0}(S^n)$, where γ_0 is the standard round metric on S^n. In particular, if

$(M^n, [\gamma]) = (S^n, [\gamma_0])$, then

$$\mu_\gamma = n(n-1)(\omega_n)^{\frac{2}{n}}, \tag{2.4.4}$$

where ω_n is the volume of (S^n, γ_0).

We will use the following Proposition of Obata [13] to argue that a certain limit metric is isometric to the round metric γ_0 on S^n.

2.5 Obata's Theorem

Proposition 2.5.1 (Obata [13]). *Let (S^n, g) be a Euclidean n-sphere of radius 1, and g^* another Riemannian metric on S^n conformal to g. Then g^* is of constant scalar curvature $n(n-1)$ if and only if it is of constant sectional curvature 1.*

2.6 Notations

Starting with a complete, conformally compact manifold (M^{n+1}, g) with other assumptions as in the theorem 1.0.6, we consider two compactifications namely, $\bar{g} = e^{-2r}g$ and $\bar{g}' = e^{-2t}g$. Recall, the function r is defined as $e^{-r(x)} = \rho(x)$, where ρ is the defining function of the boundary, and $t(x) = dist_g(x, p)$ is the distance from the pole.

• The barred and primed, barred and unbarred, unprimed quantites are computed with respect to the metrics $\bar{g}'$, $\bar{g}$ and g respectively.

• $\Sigma_r = \bar{\Sigma}_r$ are the level sets of r, $\bar{\Sigma}_t{}' = \Sigma_t$ are the level sets of t.

• The function $c(x) = t(x) - r(x)$ and c_t (or c_r) is the restriction of c on Σ_t (or Σ_r)

• $\bar{\gamma}'_t$ (or $\bar{\gamma}_r$) are the restrictions of the metric $\bar{g}'$ (or $\bar{g}$) to the level sets $\bar{\Sigma}'_t$ (or $\bar{\Sigma}_r$). Same goes with γ_t.

• The barred and primed quantities with subscript t are computed on the level sets $\bar{\Sigma}'_t$ with respect to $\bar{\gamma}'_t$. Similar notations hold for barred with subscript r or unbarred, unprimed with subscript r.

Chapter 3

Main Result

We start off with a geodesic defining function $\rho : \bar{M} \to \mathbb{R}$ of the boundary that is $|\bar{\nabla}\rho|_{\bar{g}} \equiv 1$ in a neighborhood of the boundary of M. Let us set $\rho = 2e^{-r}$. It is easy to show that $|\nabla r|_g = 1$ if and only if $|\bar{\nabla}\rho|_{\bar{g}} = 1$. Therefore, outside of a compact set $K \subset M$, $|\nabla r|_g = 1$, and therefore r is a distance function. Choose r_0, such that $M_{r_0} = \{x \in M | \rho(x) > \rho_0 = 2e^{-r_0}\}$ contains the cut locus of ρ, the set K and the pole p. Let x be any point in M and outside of M_{r_0}, and let $t(x) = dist_g(p, x)$. Recall, p is the pole and therefore t is a smooth distance function on M. Clearly the function $c(x) = t(x) - r(x)$ is a smooth bounded function in $M - M_{r_0}$. We will show that assuming $(\bar{M}, \bar{g})$ is a smooth compactification, the function c extends to a $C^{0,1}$ function on the boundary. If we assume that the conformal infinity is the round sphere as in the theorem, then c extends to a smooth function on the boundary.

We begin with the two compactifications $\bar{g} = 4e^{-2r}g$ and $\bar{g}' = 4e^{-2t}g$. By our assumption, $(\bar{M}, \bar{g})$ is the smooth compactification of (M, g), with the round sphere as its boundary.

3.1 Asymptotic Bounds on the Shape Operators

Here we will find asymptotic bounds on the shape operators $\nabla^2 r$ and $\nabla^2 t$ with respect to the complete metric g.

Lemma 3.1.1. *Let (M, g) be a complete conformally compact manifold. Let r and t be smooth distance functions as defined earlier. For all r and t sufficiently large, the eigenvalues $\{\lambda_i(r)\}$ and $\{\mu_i(r)\}$ of the shape operators $\nabla^2 r$ and $\nabla^2 t$, respectively are non-negative and bounded from above.*

Proof. Since (M, g) is conformally compact, we have $|K_{ij} + 1| = O(e^{-2r})$. So outside of a compact set K, we can assume that $-\alpha^2 \leq K_{ij} \leq -\frac{1}{\alpha^2}$, for $\alpha > 1$ but close to 1. Without loss of generality, we can assume K to be the same compact set as above. Choose r_0' such that K lies inside $M_{r_0'}$. For any $r > r_0'$ we have the following differential inequality for the each eigenvalue, $\lambda_i(r)$ of the shape operator $\nabla^2 r$, (cf equation(2.1.21)).

$$\frac{1}{\alpha^2} \leq \lambda_i'(r) + \lambda_i^2(r) \leq \alpha^2.$$

From the second inequality, we get,

$$\frac{\lambda_i'(r)}{\alpha^2 - \lambda_i^2(r)} \leq 1.$$

Integrating from r_0' to r, we get,

$$\ln\Big(\frac{\alpha + \lambda_i(r)}{\alpha - \lambda_i(r)} \frac{1}{\hat{C}_0(r_0')}\Big) \leq 2\alpha(r - r_0'),$$

where $\hat{C}_0(r'_0) = \ln(\frac{\alpha+\lambda_i(r'0)}{\alpha-\lambda_i(r'_0)})$

or,

$$\lambda_i(r) \leq \alpha\Big(\frac{C_0(r'_0)e^{2\alpha r} - 1}{C_0(r'_0)e^{2\alpha r} + 1}\Big),$$

where $C_0(r'_0) = \hat{C}_0(r'_0)e^{-2\alpha r'_0}$. The first inequality,

$$\frac{1}{\alpha^2} \leq \lambda'_i(r) + \lambda_i^2(r)$$

gives,

$$\lambda_i(r) \geq \frac{1}{\alpha}\Big(\frac{C'_0(r'_0)e^{\frac{2}{\alpha}r} - 1}{C'_0(r'_0)e^{\frac{2}{\alpha}r} + 1}\Big),$$

where $C'_0(r'_0) = \ln(\frac{\frac{1}{\alpha}+\lambda_i(r'0)}{\frac{1}{\alpha}-\lambda_i(r'_0)})e^{-\frac{2}{\alpha}r'_0}$ Without loss of generality, we can assume $r'_0 = r_0$. Therefore, we see that $0 \leq \lambda_i(r) \leq C_1(r_0)$ for all $r > r_0$. Since for any point $x \in M$, $|t(x) - r(x)|$ is bounded, we also have, $|K_{ij} + 1| = O(e^{-2t})$. Therefore, a similar analysis for the eigenvalues $\mu_i(t)$ of $\nabla^2 t$, would give us the following inequality

$$0 \leq \mu_i(t) \leq E_1(t_0)$$

for every $\mu_i(t)$ and for all $t > t_0$.

□

The next lemma shows that the shape operators actually tend to 1 at a certain rate. Under a stronger assumption on the Ricci curvature, we will show that the Laplacian of t satisfies certain asymptotic bound. We wish to emphasize here that this extra Ricci curvature assumption is not necessary to prove our result and is not used anywhere in the proof.

Lemma 3.1.2. *Under the above assumptions, for $r > r_0$ and $t > t_0$, we have:*

$$\nabla^2 r(e_i, e_j) = \delta_i^j + o(e^{-\frac{3}{2}\beta r}) \tag{3.1.1}$$

$$\nabla^2 t(e_i', e_j') = \delta_i^j + o(e^{-\frac{3}{2}\beta t}) \tag{3.1.2}$$

where $\beta < 1$, $\{e_i\}_{i=1}^n$ are tangent to the level sets Σ_r and $\{e_i'\}_{i=1}^n$ are tangent to the level sets Σ_t. If, in addition $-(n - \psi(t)e^{-2t})g \leq Ric \leq -ng$, where $\psi(t) \geq 0$ and $\psi(t) \in L^1(\mathbf{R}^+)$, then

$$\Delta t = n + O(e^{-2t}) \tag{3.1.3}$$

Proof. For $r > r_0$, we have the following inequality for $\{\lambda_i(r)\}_{i=1}^n$ from the proof of lemma 3.1.1:

$$0 \leq \lambda_i(r) \leq C_1(r_0).$$

We also have:

$$-1 - Ce^{-2r} \leq K_{ij} \leq -1 + Ce^{-2r}.$$

This gives us the following differential inequality:

$$1 - Ce^{-2r} \leq \lambda_i'(r) + \lambda_i^2(r) \leq 1 + Ce^{-2r}. \tag{3.1.4}$$

Let $u_i(r) = \lambda_i(r) - 1$, therefore $-1 \leq u_i(r) \leq C_1(r_0)$. Thus,

$$\begin{aligned}(u_i^2(r))' + 2u_i^2(r) &\leq (u_i^2(r))' + (4 + 2u_i(r))u_i^2(r) \\ &\leq 2Cu_i(r)e^{-2r}.\end{aligned}$$

That is,

$$(u_i^2(r))' + 2u_i^2(r) \leq 2Cu_i(r)e^{-2r}.$$

Choose $0 < \beta' < 1$, and write the above inequality in the following manner,

$$\begin{aligned}(u_i^2(r))' + 2\beta' u_i^2(r) &\leq (u_i^2(r))' + 2u_i^2(r) \\ &\leq 2Cu_i(r)e^{-2\beta' r}e^{-2(1-\beta')r}.\end{aligned}$$

Therefore, bringing $e^{-2\beta' r}$ over to the other side,

$$(u_i^2(r)e^{2\beta' r})' \leq 2Cu_i(r)e^{-2(1-\beta')r}.$$

Integrating from r_0 to r,

$$u_i^2(r)e^{2\beta' r} \leq C_2^2(r_0),$$

or,

$$|u_i(r)| \leq C_2(r_0)e^{-\beta' r}. \tag{3.1.5}$$

This estimate is far weaker than what we had set out to show. To improve this we use the following differential inequality,

$$\begin{aligned}(u_i^2(r))' + (4 - 2|u_i(r)|)u_i^2(r) &\leq (u_i^2(r))' + (4 + 2u_i(r))u_i^2(r) \\ &\leq 2C|u_i(r)|e^{-2r}. \qquad (3.1.6)\end{aligned}$$

Putting back the estimate on $|u_i(r)|$ obtained from equation(3.1.5) to equation(3.1.6)

$$(u_i^2(r))' + 4u_i^2(r) \leq 2C.C_2(r_0)e^{-(2+\beta')r} + 2C_2^3(r_0)e^{-3\beta' r}.$$

Choose $0 < \beta < \beta' < \frac{(2+\beta')}{3}$, and split the right hand side of the inequality in the following way,

$$(u_i^2(r))' + 4u_i^2(r) \leq e^{-3\beta r}[2C.C_2(r_0)e^{-(2+\beta'-3\beta)r} + 2C_2^3(r_0)e^{-3(\beta'-\beta)r}]$$

or,

$$\begin{aligned}(u_i^2(r))' + 3\beta u_i^2(r) &\leq (u_i^2(r))' + 4u_i^2(r) \\ &\leq e^{-3\beta r}[2C.C_2(r_0)e^{-(2+\beta'-3\beta)r} + 2C_2^3(r_0)e^{-3(\beta'-\beta)r}]\end{aligned}$$

or,

$$(u_i^2(r)e^{3\beta r})' \leq [2C.C_2(r_0)e^{-(2+\beta'-3\beta)r} + 2C_2^3(r_0)e^{-3(\beta'-\beta)r}]$$

Integrating from r_0 to r,

$$u_i^2(r)e^{3\beta r} \leq C_3^2(r_0)$$

or,

$$\lambda_i(r) = 1 + o(e^{-\frac{3}{2}\beta r}). \tag{3.1.7}$$

Similar analysis with the eigenvalues $\mu_i(t)$ proves the other claim.

Here we prove our last claim. As before, let $\mu_i(t)$'s be the eigenvalues of the Hessian of t, therefore, $\Delta t = \Sigma_{i=1}^n \mu_i(t)$. Let us assume $\mu_i(t) = 1 + T_i(t)e^{-2t}$, we would show that $|\Sigma_{i=1}^n T_i(t)| = O(1)$. From equation (3.1.2) we already know that for each i, $|T_i(t)| \leq O(e^{(2-\frac{3}{2}\beta)t})$. Taking the trace of equation (2.1.18), we get,

$$n - \psi(t)e^{-2t} \leq (\Sigma_{i=1}^n \mu_i(t))' + \Sigma_{i=1}^n \mu_i^2(t) \leq n.$$

Replacing $\mu_i(t)$ by $1 + T_i(t)e^{-2t}$, we get,

$$-\psi(t) \leq (\Sigma_{i=1}^n T_i)' + (\Sigma_{i=1}^n T_i^2)e^{-2t} \leq 0.$$

Now, the second inequality, that is $(\Sigma_{i=1}^n T_i)' + (\Sigma_{i=1}^n T_i^2)e^{-2t} \leq 0$ implies, $(\Sigma_{i=1}^n T_i)' \leq 0$, that is $\Sigma_{i=1}^n T_i(t)$ is non-increasing. Therefore it can only go to $-\infty$. Whereas the first inequality stops that from happening.

$$\begin{aligned} -\psi(t) &\leq (\Sigma_{i=1}^n T_i)' + (\Sigma_{i=1}^n T_i^2)e^{-2t}, \\ -\psi(t) - (\Sigma_{i=1}^n T_i^2)e^{-2t} &\leq (\Sigma_{i=1}^n T_i)'. \end{aligned} \tag{3.1.8}$$

Since we already know that, for each i, $|T_i(t)| \leq O(e^{(2-\frac{3}{2}\beta)t})$, and $2-\frac{3}{2}\beta \leq 1-\epsilon$, for some $\epsilon > 0$. We have the left hand side in $L^1(\mathbf{R}^+)$, and therefore $\Sigma_{i=1}^n T_i(t)$ is greater than some finite number. Hence we conclude that $|\Sigma_{i=1}^n T_i(t)| = O(1)$.

□

Remark 3.1.1. Since the compactification by the function r is smooth, all curvature quantities, in particular $\bar{Ric}$, are bounded. A stronger asymptotic condition holds for $\nabla^2 r$, i.e. $\nabla^2 r(e_i, e_j) = \delta_i^j + O(e^{-2r})$. A straightforward calculation using equation (2.1.5) proves it.

The following corollary is a direct consequence of lemma 3.1.2.

Corollary 3.1.3. *The second fundamental form of the level-sets* $(\bar{\Sigma}_t{}', \bar{\gamma}_t')$, $\bar{\nabla}'^2 t(\bar{e}_i', \bar{e}_j') = \bar{S}'(\bar{e}_i', \bar{e}_j') \to 0$ *as* $t \to \infty$, *where* $\{\bar{e}_i'\}_{i=1}^n$ *are unit tangent vectors to* $(\bar{\Sigma}_t{}', \bar{\gamma}_t')$.

Proof. Since we are considering the metric $\bar{g}' = e^{-2t}g$, where $t(x) = dist_g(x,p)$, the unit normal $\bar{N}'$ to the level set$(\bar{\Sigma}_t{}', \bar{\gamma}_t')$, in $\bar{g}'$ metric can be written as,

$$\bar{N}' = \bar{\nabla}' e^{-t} = -e^{-t}\bar{\nabla}' t. \tag{3.1.9}$$

Therefore, the second fundamental form of the level set $(\bar{\Sigma}_t', \bar{\gamma}_t')$ is as follows,

$$
\begin{aligned}
\bar{S}'(\bar{e}'_i, \bar{e}'_j) &= \bar{g}'(\bar{\nabla}'_{\bar{e}'_i}\bar{N}', \bar{e}'_j) \\
&= g(\bar{\nabla}'_{e'_i}\bar{N}', e'_j) \\
&= g(\nabla_{e'_i}\bar{N}', e'_j) - g(\nabla t, e'_i)g(\bar{N}', e'_j) - g(\nabla t, e'_j)g(\bar{N}', e'_i) \\
&\quad + g(\nabla t, \bar{N}')g(e'_i, e'_j).
\end{aligned}
$$

We used equation (2.1.7) to get the last equality. Since, $\{e'_i\}_{i=1}^n$ are tangent to the level-sets and ∇t is normal, the second and the third terms above are zero. Hence we have,

$$
\begin{aligned}
\bar{S}'(\bar{e}'_i, \bar{e}'_j) &= g(\nabla_{e'_i}\bar{N}', e'_j) + g(\nabla t, \bar{N}')\delta_{ij} \\
&= -g(\nabla_{e'_i}(e^{-t}\bar{\nabla}'t), e'_j) + e^{-t}\delta_{ij}g(\nabla t, \bar{\nabla}'t) \\
&= e^{-t}(g(\bar{\nabla}'t, \nabla_{e'_i}e'_j)) - e'_i(g(e^{-t}\bar{\nabla}'t, e'_j)) + e^{t}\delta_{ij}g(\nabla t, \nabla t).
\end{aligned}
$$

The second term, $g(\bar{\nabla}'t, e'_j) = 0$, since $\bar{\nabla}'t$ is normal to the level-sets. To simplify the first term we use the following,

$$
\begin{aligned}
g(\bar{\nabla}'t, X) &= e^{2t}\bar{g}'(\bar{\nabla}'t, X) \\
&= e^{2t}g(\nabla t, X).
\end{aligned}
\tag{3.1.10}
$$

Thus we have,

$$
\begin{aligned}
e^t(g(\nabla t, \nabla_{e'_i} e'_j)) + e^t\delta_{ij} g(\nabla t, \nabla t) &= e^t(e'_i(g(\nabla t, e'_j)) - g(\nabla_{e'_i}\nabla t, e'_j) + \delta_{ij}) \\
&= -e^t(g(\nabla_{e'_i} N, e'_j) - \delta_{ij}) \\
&= -e^t(\nabla^2 t(e'_i, e'_j) - \delta_{ij}).
\end{aligned}
$$

Hence, replacing $\nabla^2 t(e'_i, e'_j)$ by $\delta_{ij} + O(e^{-\frac{3}{2}\beta t})$ proved in lemma (3.1.2), we get,

$$
\bar{S}'(\bar{e}'_i, \bar{e}'_j) = O((e^{-(\frac{3}{2}\beta - 1)t}). \tag{3.1.11}
$$

□

We now prove the following lemma, which is one of the key estimates for the result in [18]. We show here in this lemma that the main theorem in [18] stays valid even under a weaker assumption on the asymptotic behavior of the sectional curvatures. One readily finds that assuming the following lemma is true, the theorem of [18] follows immediately.

Lemma 3.1.4. *Let (M^{n+1}, g), $n \geq 2$ be a complete manifold with a pole, p. If the sectional curvatures, K_{ij} satisfy*

$$
-1 - \phi(t)e^{-2t} \leq \mathrm{K}_{ij} \leq -1 + \phi(t)e^{-2t} \tag{3.1.12}
$$

where, $t(x) = dist_g(x, p)$ $\phi \in \mathrm{L}^1(\mathbb{R}^+)$ and $\phi \geq 0$. and $Ric(g) \geq -ng$, then, for $t > t_0$, we have:

$$
\nabla^2 t(e'_i, e'_j) = g(S(e'_i), e'_j) = \delta_{ij} + O(e^{-2t}). \tag{3.1.13}
$$

Proof. Our goal here is to improve the estimates derived in lemma (3.1.2), by using the extra assumption on the sectional curvatures. We begin with equation (3.1.4) and under the stronger assumption on the sectional curvatures, we have:

$$1 - \phi(t)e^{-2t} \leq \mu_i'(t) + \mu_i^2(t) \leq 1 + \phi(t)e^{-2t}. \tag{3.1.14}$$

Let,

$$v_i(t) = (\mu_i(t) - 1)e^{2t}.$$

In terms of $v_i(t)$, equation (3.1.14) becomes:

$$v_i^2(t) \leq (\phi(t) - v_i'(t))e^{2t}.$$

Which implies that $\phi(t) \geq v_i'(t)$. Integrating this from t_0 to t and using the fact that $\phi \in \mathrm{L}^1(\mathbb{R}^+)$, we get $v_i(t) \leq C_1$, and hence

$$\mu_i(t) \leq 1 + C_2 e^{-2t} \tag{3.1.15}$$

To get a lower bound, we write

$$\mu_i(t) = 1 + T_i(t)e^{-2t}$$

We already know from equation (3.1.14) and the last part of lemma (3.1.2) with extra Ricci curvature assumption that obviously holds here, that for each i and $t \geq t_0$,

$$-C_3 e^{(2-\frac{3}{2}\beta)t} \leq T_i(t) \leq C_2. \qquad (*)$$

Taking the trace of equation (2.1.15), we get:

$$(\Sigma\mu_i(t))' + \Sigma\mu_i^2(t) = -Ric(N, N).$$

Since $|K_{ij} + 1| \leq \phi(t)e^{-2t}$, we have a bound on $Ric(N, N)$

$$|Ric(N, N) + n| \leq n\phi(t)e^{-2t}.$$

Therefore,

$$n - n\phi e^{-2t} \leq (\Sigma\mu_i)' + \Sigma\mu_i^2 \leq n + n\phi e^{-2t}.$$

Replacing $\mu_i(t)$'s by $1 + T_i(t)e^{-2t}$'s, we get:

$$-n\phi(t)e^{-2t} \leq (\Sigma T_i(t))'e^{-2t} + (\Sigma T_i^2(t))e^{-4t} \leq n\phi(t)e^{-2t}.$$

or,

$$-n\phi(t) \leq (\Sigma T_i(t))' + (\Sigma T_i^2(t))e^{-2t} \leq n\phi(t). \qquad (3.1.16)$$

Now, since $T_i^2(t) \leq max\{C_2^2, C_3^2 e^{(4-3\beta)t}\}$. We have

$$T_i^2(t)e^{-2t} \leq max\{C_2^2 e^{-2t}, C_3^2 e^{2-3\beta t}\}.$$

And since $2 - 3\beta < 0$. That implies that $(\Sigma T_i^2(t)e^{-2t}) \in L^1(\mathbf{R}^+)$.

By assumption $\phi(t) \in L^1(\mathbf{R}^+)$ that forces $(\Sigma T_i(t))' \in L^1(\mathbf{R}^+)$. Which means $\Sigma T_i(t)$ is bounded. Again since each $T_i(t)$ is bounded from above (equation (*)), we conclude that the $|T_i(t)|$'s are uniformly bounded for all t large.

We have therefore established the inequalities for S_i^j.

□

3.2 Scalar Curvature and Volume Bounds on Level Sets

Next, we prove the following lemma,

Lemma 3.2.1. *Let (M^{n+1}, g), $n \geq 3$ be a complete conformally compact manifold with a pole p and $Ric \geq -ng$. Let $t(x) = dist_g(p, x)$ and $\frac{1}{Vol(\Sigma_t)} \int_{\Sigma_t} |R + n(n+1)| = o(e^{-2t})$, where for each t, Σ_t is the level set of t, and R, the scalar curvature of (M^{n+1}, g). The scalar curvatures and volumes of $(\bar{\Sigma}_t', \bar{\gamma}_t')$ of t, satisfy the following inequalities,*

$$\int_{\bar{\Sigma}_t'} \bar{R}_t' \leq n(n-1)Vol(\bar{\Sigma}_t') + o(1) \tag{3.2.1}$$

$$Vol(\bar{\Sigma}_t') \leq \omega_n \tag{3.2.2}$$

where ω_n is the volume of (S^n, γ_0).

In particular, if $|R + n(n+1)| = o(e^{-2t})$ then one has $\bar{R}_t' \leq n(n-1) + o(1)$.

Proof. We will show that for the compactification $\bar{g}' = e^{-2t}\mathrm{g}$,

$$\int_{\bar{\Sigma}'_t} \bar{R}'_t \leq 4n(n-1)Vol(\bar{\Sigma}'_t) + o(1).$$

We refer to equations (2.1.5) and (2.1.6), namely,

$$\bar{Ric}' = Ric - (n-1)(Ddt - dt \circ dt) + (\Delta t - (n-1)|dt|^2)g, \quad (3.2.3)$$

$$\bar{R}' = e^{2t}(R + 2n\Delta t - n(n-1)|dt|^2). \quad (3.2.4)$$

From corollary (3.1.3), we know that the second fundamental form $|\bar{S}'| \to 0$ as $t \to \infty$. By Gauss - Codazzi equation and corollary (3.1.3), we get:

$$\bar{R}'_t = \bar{R}' - 2\bar{Ric}'(\bar{N}', \bar{N}') + o(1). \quad (3.2.5)$$

Our assumption on the lower bound of the Ricci curvature gives the following upper bound on the Laplacian of t, $\Delta t \leq n \coth(t)$ (lemma (2.2.2)). Since t is the distance function, we also have the following $g(\nabla t, \nabla t) = |dt|^2 = 1$ equation (3.2.4) becomes,

$$\bar{R}' = e^{2t}(R + 2n\Delta t - n(n-1)). \quad (3.2.6)$$

Let $\bar{N}'$ be the unit normal to the level set $\bar{\Sigma}'_t$ in $\bar{g}'$ metric. Then from equation (3.2.3) we get:

$$
\begin{aligned}
\bar{Ric}'(\bar{N}', \bar{N}') &= Ric(\bar{N}', \bar{N}') - (n-1)(Ddt - dt \circ dt)(\bar{N}', \bar{N}') \\
&\quad +(\Delta t - (n-1))g(\bar{N}', \bar{N}') \\
&= Ric(e^t N, e^t N) - (n-1)(Ddt - dt \circ dt)(e^t N, e^t N) \\
&\quad +(\Delta t - (n-1))g(e^t N, e^t N) \\
&= e^{2t}[Ric(N,N) - (n-1)(Ddt - dt \circ dt)(N,N) \\
&\quad +(\Delta t - (n-1))]
\end{aligned}
$$

Since $Ddt(N,N) = g(\nabla_N^2 t, N) = 0$ and $dt \circ dt(N,N) = |dt|^2 = 1$, we get:

$$
\bar{Ric}'(\bar{N}', \bar{N}') = e^{2t}(Ric(N,N) + \Delta t) \tag{3.2.7}
$$

Substituting equation (3.2.6) and equation (3.2.7) in equation (3.2.5), we get:

$$
\bar{R}_t' = e^{2t}[R + 2(n-1)\Delta t - n(n-1) - 2Ric(N,N)] + o(1).
$$

Replacing Δt by $n(\coth(t) - 1) + n$ on the right hand side, we get the following inequality:

$$
\begin{aligned}
\bar{R}_t' &\leq e^{2t}[R + 2(n-1)(\frac{2ne^{-2t}}{1-e^{-2t}} + n) - n(n-1) - 2Ric(N,N)] + o(1) \\
&\leq e^{2t}[R + \frac{4n(n-1)e^{-2t}}{1-e^{-2t}} + n(n-1) - 2Ric(N,N)] + o(1) \\
&\leq e^{2t}[R + n(n+1) + \frac{4n(n-1)e^{-2t}}{1-e^{-2t}} - 2(Ric(N,N) + n)] + o(1)
\end{aligned}
$$

Since $Ric(N,N) + n \geq 0$, we have,

$$\bar{R}'_t \leq e^{2t}[R + n(n+1) + \frac{4n(n-1)e^{-2t}}{1-e^{-2t}}] + o(1) \tag{3.2.8}$$

Let $d\bar{\eta}'_t$ and $d\eta_t$ be the volume elements on $(\bar{\Sigma}'_t, \bar{g}'|_{\Sigma_t})$ and $(\Sigma_t, g|_{\Sigma_t})$ respectively. So,

$$d\bar{\eta}'_t = e^{-nt} d\eta_t \tag{3.2.9}$$

Then,

$$\begin{aligned}\int_{\Sigma_t} \bar{R}'_t d\bar{\eta}'_t &\leq e^{2t} \int_{\Sigma_t} [R + n(n+1) + \frac{n(n-1)e^{-2t}}{1-e^{-2t}}] d\bar{\eta}'_t + o(1) \\ &\leq e^{-(n-2)t} \Big(\int_{\Sigma_t} |R + n(n+1)| d\eta_t \Big) + \frac{n(n-1)}{1-e^{-2t}} Vol(\bar{\Sigma}'_t) + o(1)\end{aligned} \tag{3.2.10}$$

By our assumption on the integral of the scalar curvature, we have:

$$\begin{aligned}e^{-(n-2)t} \Big(\int_{\Sigma_t} |R + n(n+1)| d\eta_t \Big) &= e^{-(n-2)t} Vol(\Sigma_t).o(e^{-2t}) \\ &= e^{-nt} Vol(\Sigma_t).o(1) \\ &= Vol(\bar{\Sigma}'_t).o(1) \\ &= o(1).\end{aligned} \tag{3.2.11}$$

Substituting equation (3.2.11) in equation (3.2.10) we get:

$$\int_{\Sigma_t} \bar{R}'_t d\bar{\eta}'_t \leq \frac{n(n-1)}{1-e^{-2t}} Vol(\bar{\Sigma}'_t) + o(1).$$

Expanding the denominator in Taylor series, we see that:

$$\int_{\Sigma_t} \bar{R}'_t d\bar{\eta}'_t \leq n(n-1) Vol(\bar{\Sigma}'_t) + o(1). \tag{3.2.12}$$

In particular, if $|R + n(n+1)| = o(e^{-2t})$, from equation (3.2.8) we have:

$$\bar{R}'_t \leq 4n(n-1) + o(1)$$

For the compactification $\bar{g}' = 4e^{-2t}g$, we have:

$$\bar{R}'_t \leq n(n-1) + o(1) \tag{3.2.13}$$

Proof of the volume bound

From Gromov's relative volume comparison theorem (2.2.1) we have,

$$t \rightarrow \frac{Vol(B(p,t)}{Vol(B_0(0,t))}$$

is a non-increasing function of t, where $B_0(0,t)$ is the ball of radius t centered at the origin of the hyperbolic space, (H^{n+1}, g_{-1}). It also gives an upper bound on the ratio namely,

$$\frac{Vol(B(p,t)}{Vol(B_0(0,t))} \leq 1.$$

The volume elements of (M,g) and (H, g_{-1}) in polar coordinates are $\lambda(r,\theta)drd\theta$ and $\lambda_{-1}(r,\theta)drd\theta$, respectively. Therefore

$$
\begin{aligned}
Vol(B(p,t)) &= \int_0^t \int_{S^{n-1}} \lambda(r,\theta) dr d\theta, \\
Vol(B_0(0,t)) &= \int_0^t \int_{S^{n-1}} \lambda_{-1}(r,\theta) dr d\theta.
\end{aligned}
$$

So,

$$
\frac{Vol(B(p,t)}{Vol(B_0(0,t))} = \frac{\int_0^t \int_{S^{n-1}} \lambda(r,\theta) dr d\theta}{\int_0^t \int_{S^{n-1}} \lambda_{-1}(r,\theta) dr d\theta}.
$$

Since this is a non-increasing function, its derivative is non-positive, so the following inequality must hold

$$
\frac{\int_{S^{n-1}} \lambda(t,\theta) d\theta}{\int_{S^{n-1}} \lambda_{-1}(t,\theta) d\theta} \leq \frac{\int_0^t \int_{S^{n-1}} \lambda(r,\theta) dr d\theta}{\int_0^t \int_{S^{n-1}} \lambda_{-1}(r,\theta) dr d\theta}.
$$

Therefore,

$$
\frac{Vol(\Sigma_t)}{Vol(S_t)} \leq \frac{Vol(B(p,t)}{Vol(B_0(0,t))} \leq 1,
$$

where S_t is the level set of the distance function from the origin at t in the hyperbolic space. So,

$$
Vol(\Sigma_t) \leq Vol(S_t).
$$

Or, equivalently

$$
Vol(\bar{\Sigma}_t') = \frac{Vol(\Sigma_t)}{\frac{e^{nt}}{2^n}} \leq \frac{Vol(\Sigma_t)}{\sinh^n t} \leq \frac{Vol(S_t)}{\sinh^n t} = \omega_n \qquad (3.2.14)
$$

□

3.3 Lipschitz Bound on Function c

On one hand, per our assumption $(\bar{M}, \bar{g})$ is a smooth compact manifold with $\bar{g}$ extending to a smooth metric up to the boundary ∂M. While on the other, $(\bar{M}, \bar{g}')$ is another compactification with $\bar{g}'$ smooth in M. We do not know yet, whether $\bar{g}'$ has a smooth extension on ∂M. In the next few lemmas we show that in general the extension can only be $C^{0,1}$. But, if the conformal infinity is (S^n, γ_0) as in the theorem, then the extension is C^∞. As a first step we choose local coordinates in a neighborhood of the boundary as follows.

Since ∂M is compact,it can be covered by finitely many coordinate n-balls $S_k(2R)$, such that $S_k(R)$ also cover ∂M. Each $S_k(2R)$ can be extended to $U_k(2R) = S_k(2R) \times [0, \epsilon_0) \subset \bar{M}$, such that $\cup U_k(2R)$ and $\cup U_k(R)$ both cover a tubular neighborhood of ∂M and there exist smooth coordinates in $U_k(2R)$, $\{\theta_1, ..., \theta_n, \theta_{n+1}\}$ such that at any point $x \in M \cap U_k(2R)$, $\bar{g}' = ds^2 + \bar{g}'_{ij} d\theta^i d\theta^j$ where $s(x) = e^{-t(x)}$ and, $g = dt^2 + e^{2t}\bar{g}'_{ij} d\theta^i d\theta^j = dt^2 + g_{ij} d\theta^i d\theta^j$. Let us define: $S_{k;t}(2R) = U_k(2R) \cap \Sigma_t$ and $c_t = c|_{\Sigma_t}$, and $|\bar{\nabla}'^t c_t|^2 = \bar{g}'^{ij} \partial_i c \partial_j c$

Lemma 3.3.1. *We have the following asymptotic bound on the $C^{0,1}$ norm of the function c:*

(1) $|\nabla c(x)|^2 = O(e^{-2t})$

(2) $|\bar{\nabla}' c(x)|^2 = e^{2t} |\nabla c(x)|^2 = O(1)$

Therefore, $|\bar{\nabla}'^t c_t| < \Lambda_1$ *for constant* Λ_1 *and on every* $S_{k;t}(2R)$ *for* t *large.*

Proof. We first observe that $g(\nabla c, \nabla c) = |\nabla r|^2 + |\nabla t|^2 - 2g(\nabla t, \nabla r) = 2(1 - g(\nabla r, \nabla t))$.

Define $\phi_1(t) = e^t(1 - g(\nabla r, \nabla t))$. Clearly $\phi_1(t) \geq 0$.

$$\begin{aligned}
\partial_t \phi_1(t) &= e^t(1 - g(\nabla r, \nabla t)) - e^t \partial_t g(\nabla r, \nabla t) \\
&= \phi_1(t) - e^t[g(\nabla_{\partial_t} \nabla r, \nabla t) + g(\nabla_{\partial_t} \nabla t, \nabla r)] \\
&= \phi_1(t) - e^t g(\nabla_{\partial_t} \nabla r, \nabla t) \\
&= \phi_1(t) - e^t \nabla^2 r(\nabla t, \nabla t).
\end{aligned}$$

Let us write $\nabla t = g(\nabla r, \nabla t)\nabla r + \sqrt{1 - g^2(\nabla r, \nabla t)}\nabla u$, where ∇u is a unit vector orthogonal to ∇r . Making this substitution above, and using the asymptotic estimate of $|\nabla^2 r|$ from lemma (3.1.2), we get,

$$\begin{aligned}
\partial_t \phi_1(t) &= \phi_1(t) - e^t(1 - g^2(\nabla r, \nabla t))\nabla^2 r(\nabla u, \nabla u) \\
&= \phi_1(t) - e^t(1 - g(\nabla r, \nabla t))(1 + g(\nabla r, \nabla t))(1 + O(e^{-\frac{3}{2}\beta t})) \\
&= \phi_1(t) - \phi_1(t)(1 + g(\nabla r, \nabla t))(1 + O(e^{-\frac{3}{2}\beta t})).
\end{aligned}$$

Since, $1 + g(\nabla r, \nabla t) \geq 1$, we get the following inequality,

$$\partial_t \phi_1(t) \leq \phi_1(t) - \phi_1(t)(1 + O(e^{-\frac{3}{2}\beta t})),$$

or,

$$\partial_t \phi_1(t) \leq \phi_1(t)(O(e^{-\frac{3}{2}\beta t})).$$

Integrating this,

$$1 - g(\nabla r, \nabla t) = O(e^{-t}). \tag{3.3.1}$$

Next, we define another function $\phi_2(t) = e^{2t}(1 - g(\nabla r, \nabla t))$. Following the same procedure, we get,

$$\partial_t \phi_2(t) = 2\phi_2(t) - e^{2t}(1 - g(\nabla r, \nabla t))(1 + g(\nabla r, \nabla t))(1 + O(e^{-\frac{3}{2}\beta t})). \tag{3.3.2}$$

As before, we replace $e^{2t}(1 - g(\nabla r, \nabla t))$ by $\phi_2(t)$ and use the above estimate, equation (3.3.2) to get,

$$\partial_t \phi_2(t) = 2\phi_2(t) - \phi_2(t)(2 + O(e^{-t}))(1 + O(e^{-\frac{3}{2}\beta t})),$$

or,

$$\partial_t \phi_2(t) = \phi_2(t) O(e^{-t}).$$

Therefore, we get

$$\frac{1}{2}|\nabla c|^2 = 1 - g(\nabla r, \nabla t) = O(e^{-2t})$$

That proves both (1) and (2). To prove the last claim we work with local coordinates,

$$|\bar{\nabla}' c(x)|^2 = (\partial_s c(x))^2 + \bar{g}^{ij} \partial_i c \partial_j (c)(x),$$

or,

$$|\bar{\nabla}' c(x)|^2 = (\partial_s c(x))^2 + |\bar{\nabla}''^t c_t(x)|^2.$$

Therefore, for some constant Λ_1

$$|\bar{\nabla}''^t c_t|^2 \leq \Lambda_1. \tag{3.3.3}$$

□

Under the additional assumption on the Ricci curvature in lemma (3.1.2), we can prove the following lemma. This result is not necessary for the proof of the main result.

Lemma 3.3.2. *There exists $\alpha \in (0,1)$ and a constant Λ_4 such that for all i and t,*

$$||c_t||_{C^{1,\alpha}(S_{i;t}(R))} \leq \Lambda_4. \tag{3.3.4}$$

Proof. We claim that, the scalar curvature $\bar{R}'$ is bounded from both above and below. This is true because, $\bar{R}' = e^{2t}(R + 2n\Delta t - n(n-1))$ and we have already shown in lemma (3.1.2) that $\Delta t = n + O(e^{-2t})$, and $|R + n(n+1)| = o(e^{-2t})$. Therefore, we see that $|\bar{R}'|$ is bounded. Now we will show, $|\bar{\Delta}' c| = O(1)$. We consider the two metrics $\bar{g}$ and $\bar{g} = e^{2c}\bar{g}'$. Therefore the scalar curvature of $(M, \bar{g})$ is given as,

$$\bar{R} = e^{-2c}(\bar{R}' + 2n\bar{\Delta}' c - n(n-1)|\bar{\nabla}' c|^2)$$

Observe that all except $\bar{\Delta}' c$ are bounded, and hence $|\bar{\Delta}' c| = O(1)$. Next we show $|\bar{\Delta}'^t c_t| = O(1)$. In local coordinates, in each $U_i(2R) \cap M$,

$$\bar{\Delta}' c = \frac{1}{\sqrt{det(\bar{g}')}}(\partial_i \sqrt{det(\bar{g}')}\bar{g}'^{ij}\partial_j c) + \frac{1}{\sqrt{det(\bar{g}')}}(\partial_s(\sqrt{det(\bar{g}')}\partial_s c))$$
$$= \bar{\Delta}'^t c_t + \partial_s(\partial_s c) + \frac{\partial_s \sqrt{det(\bar{g}')}}{\sqrt{det(\bar{g}')}}\partial_s c.$$

Therefore,

$$\partial_s(\partial_s c) = e^{2t}\big((g(\nabla r, \nabla t) - 1) + \partial_t g(\nabla r, \nabla t)\big)$$
$$= e^{2t}\big((g(\nabla r, \nabla t) - 1) + \nabla^2 r(\nabla t, \nabla t)\big) = O(1).$$

We showed in lemma (3.3.1) that, $g(\nabla r, \nabla t) - 1 = O(e^{-2t})$ and by remark (3.1.1), $\nabla^2(\nabla t, \nabla t) = O(e^{-2r}) = O(e^{-2t})$. By equation (2.1.14) and equation (2.1.18) and the estimate, $\Delta t = n + O(e^{-2t})$ we have, $\frac{\partial_t \sqrt{det(\bar{g}')}}{\sqrt{det(\bar{g}')}} = \bar{\Delta}' s = e^{2t}(\Delta t - n) = O(1)$. Furthermore, $|\partial_s c| \leq |\bar{\nabla}' c|$ is uniformly bounded. Therefore, we get,

$$|\bar{\Delta}'^t c_t| = O(1).$$

Finally, we apply Schauder estimates to show that for some constant Λ_4 and for all i, t and $\alpha \in (0,1)$

$$||c_t||_{C^{1,\alpha}(S_{i;t}(R))} \leq \Lambda_3\big(||\bar{\Delta}'^t c_t||_{C^0(S_{i;t}(2R))} + ||c_t||_{C^\alpha(S_{i;t}(2R))}\big) \leq \Lambda_4$$

□

3.4 Regularity of the Boundary Metric

Thus we conclude that $||c_t||_{C^1(S_{k;t}(2R))} \leq \Lambda_2$, for some constant Λ_2 and for all k and t. In other words, we found a Lipschitz bound on c_t. Let $\phi_t : S_k(R) \to S_{k;t}(R)$ be diffeomorphisms and let $\{\partial_i\}_{i=1}^n$ be coordinates on $S_k(R)$. Let us set $(\bar{\gamma}'_t)_{i,j}) = \bar{\gamma}'_t(\partial_i, \partial_j)$ and $\bar{g}_{i,j} = \bar{g}(\partial_i, \partial_j)$. Here we identify $\bar{\gamma}'_t$ with its pull-back $\phi^*_t(\bar{\gamma}'_t)$. Then,

$$|\partial_l(\bar{\gamma}'_t)_{i,j}| \leq 2|(\bar{\gamma}'_t)_{i,j}||\bar{\nabla}'_t c_t| + e^{-2c_t}|\partial_l \bar{g}_{i,j}|$$

The right hand side has a uniform upper bound. Hence, by Arzela-Ascoli's theorem, we conclude that given a sequence $(\bar{\Sigma}'_{t_i}, \bar{\gamma}_{t_i}{}' = \bar{g}'|_{\bar{\Sigma}'_{t_i}})$, there is a subsequence that converges to (Σ, γ), where γ is $C^{0,1} \cap W^{1,p}$ conformal to γ_0, for any $p \geq 1$.

So far we have not used the structure of the boundary explicitly, that is all our results hold under the assumption that boundary $(\partial M, [\bar{g}|_{\partial M}])$ is smooth, not necessarily $(S^n, [\gamma_0])$. Now for the first time we will use the assumption that the boundary is $(S^n, [\gamma_0])$ to show that γ is in fact smooth. Now that we have γ, a $C^{0,1} \cap W^{1,p}$ metric, for any $p \geq 1$, we can define scalar curvature of (Σ, γ), in the sense of distributions. We choose a representative element from the conformal class $[\gamma_0]$. Without any loss of generality we pick the round metric γ_0. To simplify our computations below, we define $\gamma = u^{\frac{4}{n-2}}\gamma_0$, where u is $C^{0,1} \cap W^{1,p}$,for all $p \geq 1$.

Lemma 3.4.1. *If $(\partial M, [\gamma]) = (S^n, [\gamma_0])$, where $\gamma = u^{\frac{4}{n-2}}\gamma_0$ for some $u \in W^{1,p}(S^n)$, for any $p \geq 1$, then γ is smooth.*

Proof. The scalar curvature of (Σ, γ), in the weak sense is given by,

$$\bar{R}' = u^{-\frac{n+2}{n-2}}\big(n(n-1)u - \frac{4(n-1)}{n-2}\Delta_{\gamma_0}u\big), \tag{3.4.1}$$

where $\Delta_{\gamma_0}u$ is in $W^{-1,p}$, since $\Delta_{\gamma_0} : W^{1,p} \to W^{-1,p}$. Therefore we can integrate $\bar{R}'$. Let $d\eta$ and $d\eta_0$ be the volume elements of (Σ, γ) and (Σ, γ_0) respectively. Therefore, $d\eta = u^{\frac{2n}{n-2}}d\eta_0$. Integrating the scalar curvature over $\Sigma = S^n$,

$$\int_{S^n} \bar{R}' u^{\frac{2n}{n-2}} d\eta_0 = \int_{S^n} \big(n(n-1)u^2 + \frac{4(n-1)}{n-2}|\nabla_{\gamma_0}u|^2\big)d\eta_0 \tag{3.4.2}$$

Recall, we showed earlier in lemma (3.2.1) that the scalar curvatures of level sets of t,

$$\begin{aligned} \int_{\bar{\Sigma}'_t} \bar{R}'_t d\bar{\eta}'_t &\leq n(n-1)Vol(\bar{\Sigma}'_t) + o(1), \\ Vol(\bar{\Sigma}'_t) &\leq \omega_n. \end{aligned}$$

Therefore, at the boundary we have

$$\begin{aligned} \int_{S^n} \bar{R}' u^{\frac{2n}{n-2}} d\eta_0 &\leq n(n-1)\int_{S^n} u^{\frac{2n}{n-2}} d\eta_0, \\ \int_{S^n} u^{\frac{2n}{n-2}} d\eta_0 &\leq \omega_n. \end{aligned}$$

So,

$$n(n-1)\int_{S^n} u^{\frac{2n}{n-2}} d\eta_0 \;\geq\; \int_{S^n} \big(n(n-1)u^2 + \frac{4(n-1)}{n-2}|\nabla_{\gamma_0} u|^2\big) d\eta_0,$$

$$n(n-1)\big(\int_{S^n} u^{\frac{2n}{n-2}} d\eta_0\big)^{\frac{2}{n}} \;\geq\; \frac{\int_{S^n} \big(n(n-1)u^2 + \frac{4(n-1)}{n-2}|\nabla_{\gamma_0} u|^2\big) d\eta_0}{\big(\int_{S^n} u^{\frac{2n}{n-2}} d\eta_0\big)^{\frac{n-2}{n}}}.$$

Replacing $\int_{S^n} u^{\frac{2n}{n-2}} d\eta_0$ by ω_n, we get

$$n(n-1)(\omega_n)^{\frac{2}{n}} \geq \frac{\int_{S^n} \big(n(n-1)u^2 + \frac{4(n-1)}{n-2}|\nabla_{\gamma_0} u|^2\big) d\eta_0}{\big(\int_{S^n} u^{\frac{2n}{n-2}} d\eta_0\big)^{\frac{n-2}{n}}}. \tag{3.4.3}$$

The right-hand side is the Yamabe quotient, $Q(\gamma)$ on S^n. So, $\inf\{Q(\gamma)|\gamma \in [\gamma_0]\} = n(n-1)(\omega_n)^{\frac{2}{n}}$. This can be easily shown by taking the stereographic projection $\pi : S^n \to \mathbf{R}^{n+1}$ and writting out the right-hand side in R^n with respect to the pulled-back metric, $\pi^*\gamma_0$, and comparing the integral with the Sobolev constant.

Therefore, we conclude that the above inequality is an equality, that is $\bar{R}' = n(n-1)$ (a.e.) on S^n. But that would imply that $\bar{R}' \in W^{1,p}(S^n)$ for all $p > 1$. This implies, by Sobolev embedding, that $\bar{R}'$ is continuous on S^n. Clearly, the subset $\mathcal{U}$ of the boundary S^n where $\bar{R}' = n(n-1)$ is a set of full measure. Therefore for any $x \in S^n - \mathcal{U}$, we can find a sequence of points in $\mathcal{U}$ converging to x. Thus, by continuity of $\bar{R}'$, x is in $\mathcal{U}$, and therefore $\mathcal{U} = S^n$. Thus,

$$\bar{R}' = n(n-1) \tag{3.4.4}$$

Finally, a standard boot-strapping argument applied to the semi-linear elliptic P.D.E.,

$$\bar{R}' = n(n-1) = u^{-\frac{n+2}{n-2}}\left(n(n-1)u - \frac{4(n-1)}{n-2}\Delta_{\gamma_0} u\right) \tag{3.4.5}$$

shows that u is in fact smooth. □

Hence we have shown that the two metrics γ and γ_0 on Σ are smoothly conformal to one another and hence γ is smoothly conformal to the round metric. Applying Obata's theorem [13] we conclude that (Σ, γ) is isometric to (S^n, γ_0).

3.5 Proof of the Main Theorem

Proof. We proved the following inequality earlier, equation (3.2.14)

$$Vol(\bar{\Sigma}_t') = \frac{Vol(\Sigma_t)}{\frac{e^{nt}}{2^n}} \leq \frac{Vol(\Sigma_t)}{\sinh^n t} \leq \frac{Vol(S_t)}{\sinh^n t} = \omega_n.$$

Furthermore, we have just shown, that the level sets $(\bar{\Sigma}_t', \bar{\gamma}_t')$ converge to (S^n, γ_0) at the boundary, therefore $Vol(\bar{\Sigma}_t') \to \omega_n$ as $t \to \infty$. So, combining the two and the fact that volume ratio is strictly non increasing, we see that,

$$\frac{Vol(B(p,t)}{Vol(B_0(0,t))} \equiv 1 \text{ for all } t.$$

This implies that $\Delta t = n\coth(t)$. Therefore,

$$-n \geq \partial_t(\Delta t) + |\nabla^2 t|^2 \geq \partial_t(\Delta t) + \frac{(\Delta t)^2}{n} = -n. \tag{3.5.1}$$

The first inequality comes from the Ricatti equation and $Ric \geq -ng$. This implies that:

$$(\Delta t)^2 = n|\nabla^2 t|^2. \tag{3.5.2}$$

So, if $\{\lambda_1, ...\lambda_n\}$ are the eigenvalues of $\Delta^2 t$ then the above equality implies:

$$(\lambda_1 + ... + \lambda_n)^2 = n(\lambda_1^2 + ... + \lambda_n^2). \tag{3.5.3}$$

This can happen if and only if $\lambda_1 = \lambda_2 = ... = \lambda_n = \lambda$. Which means that

$$\Delta t = n\lambda = n\coth(t),$$

or,

$$\lambda = \coth(t).$$

This, in turn, implies that the metric g on M is a space form and is precisely the hyperbolic metric, $g = dt^2 + \sinh^2(t)\gamma_0$. That completes the proof of the main theorem.

□

Bibliography

[1] M.T.Anderson: *Einstein metrics with Prescribed Conformal Infinity on 4-Manifolds*, Preprint.

[2] M.T.Anderson: *Boundary Regularity, Uniqueness and Non-Uniqueness for AH Einstein Metrics on 4-Manifolds*, Adv. in Math., vol. 179, (2003), 205-249.

[3] M.T.Anderson: *Geometric Aspects of the ADS/CFT Correspondence*, Ed. O. Biquard, Euro. Math. Soc., Zurich, (2005), 1-31.

[4] M.T.Anderson: *Unique Continuation Results for Ricci Curvature and Applications*, Preprint.

[5] L.Andersson and M.Dahl: *Scalar Curvature Rigidity for Asymptotically Locally Hyperbolic Manifolds*, Ann.Global Anal.Geom. 16(1998) 1-27.

[6] T.Aubin: *Some Nonlinear Problems in Riemannian Geometry*, Berlin, New York : Springer, c1998.

[7] A.L. Besse: *Einstein Manifolds*, Ergebnisse Series, vol.3:10, Springer Verlag, New York, 1987.

[8] D.Gilbarg and N.S.Trudinger: *Elliptic Partial Differential Equations of Second Order*, 2^{nd} Edition, Springer Verlag, New York, 1983.

[9] R.E.Greene and H.Wu: *Gap theorems for Noncompact Riemannian Manifolds*, Duke Math. J. 49 (1982), no. 3, 731–756.

[10] A.Kasue and K.Sugahara: *Gap theorems for Certain Submanifolds of Euclidean Spaces and Hyperbolic Space Forms*, Osaka J. Math. 24 (1987), no. 4, 679–704.

[11] M.C.Leung: *Pinching theorem on Asymptotically Hyperbolic Spaces*, Internat.J.Math. 4(1993) No.5. 841-857.

[12] M.Min-OO, *Scalar Curvature Rigidity of Asymptotically Hyperbolic Spin Manifolds*, Math.Ann. 285(1989)527-539.

[13] M.Obata: *The Conjectures on Conformal Transformations of Riemannian Manifolds*, J. Differential Geometry 6 (1971/72), 247–258.

[14] P.Petersen: *Riemannian Geometry*, Grad Texts in Math, vol. 171, Springer Verlag, New York, 1997.

[15] J.Qing: *On the Rigidity for Conformally Compact Einstein Manifolds*, International Mathematics Research Notices, (2003), No. 21.

[16] R.Schoen and S.T.Yau: *On the proof of the positive mass conjecture in general relativity*, Comm. Math. Phys. 65, 45-76 (1979)

[17] R.Schoen and S.T.Yau: *Proof of the positive mass theorem II*, Comm. Math. Phys. 79, 231-260 (1981)

[18] Y.Shi and G.Tian: *Rigidity of Asymptotically Hyperbolic Manifolds*, Comm. Math. Phys. 259 (2005), No. 3, 545-559.

[19] K.Sugahara: *Gap Theorems for Riemannian Manifolds of Constant Curvature Outside a Compact Set*, Hokkaido Math. J. 18 (1989), no. 3, 459–468.

[20] E.Witten: *A new proof of the positive energy theorem*, Comm. Math. Phys. 80, 381-402 (1981)