A Symplectic Isotopy of a Dehn Twist on a Product of Projective Spaces

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Abstract of the Dissertation

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The complex manifold $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ with symplectic form $\sigma_\mu = \sigma_{\mathbb{CP}^n} + \mu\sigma_{\mathbb{CP}^{n+1}}$, where $\sigma_{\mathbb{CP}^n}$ and $\sigma_{\mathbb{CP}^{n+1}}$ are normalized Fubini-Study forms, $n \in \mathbb{N}$ and $\mu > 1$ a real number, contains a natural Lagrangian sphere L^μ. We prove that the Dehn twist along L^μ is symplectically isotopic to the identity for all $\mu > 1$. This isotopy can be chosen so that it pointwise fixes a complex hypersurface in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ and lifts to the blow-up of a complex n-dimensional submanifold in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$.

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Chapter 1

Introduction

Given a symplectic manifold (M, ω), the group $\text{Symp}(M, \omega)$ of symplectomorphisms on (M, ω), i.e., the diffeomorphisms that preserve the symplectic structure, is an infinite-dimensional Lie subgroup of the group $\text{Diff}^+(M)$ of orientation-preserving diffeomorphisms of M. We say that a symplectomorphism τ is smoothly isotopic to the identity if there is a path in $\text{Diff}^+(M)$ from the identity to τ, i.e., a smooth family $(\tau_s)_{0\leq s\leq 1}$ such that $\tau_0 = \text{id}$ and $\tau_1 = \tau$. It is symplectically isotopic to the identity if the path can be chosen inside $\text{Symp}(M, \omega)$. This dissertation is motivated by the following natural question:

The symplectic isotopy problem: *Given a symplectic manifold* (M, ω)*, does* M *admit essential symplectomorphisms, i.e., symplectomorphisms that are smoothly isotopic to the identity but not symplectically so? In other words, does the identity component of* $\text{Diff}^+(M)$ *contain more than one component of* $\text{Symp}(M, \omega)$*?*

In this thesis we consider only compact symplectic manifolds. When the dimension of M is 2, no essential symplectomorphisms exist. This follows from Moser's Theorem, since any smooth isotopy $(\tau_s)_{0\leq s\leq 1}$ gives a loop $\tau_s^*\omega$ of cohomologous symplectic forms that contracts because the space of symplectic forms on a 2-dimensional manifold is convex. In dimension 4, Gromov and Abreu-McDuff (see [MS04, p.320-321]) show that the answer to the isotopy problem is once again negative for $\mathbb{CP}^2$, $\mathbb{CP}^1 \times \mathbb{CP}^1$ and the one point blow-up of $\mathbb{CP}^2$.

In 1997, however, Seidel [Sei97] exhibited a large class of 4-dimensional symplectic manifolds (e.g., any complete intersection in some projective space $\mathbb{CP}^N$ except $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$) for which the answer to the symplectic isotopy problem is yes. In fact, under fairly weak conditions on a 4-manifold, if τ_L is the generalized Dehn twist in a Lagrangian sphere L, then τ_L^2 is an essential symplectomorphism. Generalized Dehn twists are higher-dimensional analogues of the well-known Dehn twist along a circle in a 2-dimensional manifold and is a symplectomorphism on M that can be defined whenever M contains a Lagrangian sphere. The generalized Dehn twist (that we call a Dehn twist for short) is compactly supported in a neighborhood of the Lagrangian and restricts to this sphere as the antipodal map. Hence for homological reasons, τ_L cannot be isotopic to the identity when the dimension of L is even. On the other hand, Seidel showed that when M is 4-dimensional, the square of the Dehn twist is always smoothly isotopic to the identity.

In dimensions 6 and above, very little is known. It is not even clear that Dehn twists (or their squares) are smoothly isotopic to the identity. In this dissertation we study a specific example in dimension $4n+2$ for $n \in \mathbb{N}$. Hence the Lagrangian sphere is odd-dimensional and there is no homological obstruction for the Dehn twist to be isotopic to the identity. In fact, we show that the Dehn twist is symplectically isotopic to the identity in this case. Although this does not provide an answer to the symplectic isotopy problem, the construction of the isotopy relies on certain symmetries specific to these examples that suggests they may be higher-dimensional analogues of the special cases $\mathbb{CP}^2$ and $\mathbb{CP}^1 \times \mathbb{CP}^1$ in dimension 4. Furthermore, by understanding how the isotopy could be destroyed (e.g. by blowing up symplectic submanifolds), one may be able to construct examples of essential symplectomorphisms in these dimensions.

We now describe the main results of this dissertation. For $n \in \mathbb{N}$, let $\sigma_{\mathbb{CP}^n}$ and $\sigma_{\mathbb{CP}^{n+1}}$ be the Fubini-Study forms on $\mathbb{CP}^n$ and $\mathbb{CP}^{n+1}$, respectively, normalized to integrate to π on $\mathbb{CP}^1$. Consider $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ with the product symplectic form $\sigma_\mu = \sigma_{\mathbb{CP}^n} + \mu\sigma_{\mathbb{CP}^{n+1}}$ for $\mu > 1$ a real number. As described in Lemma 2.1.6, the graph of the conjugate of the Hopf map $S^{2n+1} \to \mathbb{CP}^n$ embeds as the Lagrangian sphere

$$L^\mu = \{([\overline{z}_0 : \cdots : \overline{z}_n], [z_0 : \cdots : z_n : \sqrt{\mu - 1}]) \mid |z|^2 = 1\}$$

in $(\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma_\mu)$, so the Dehn twist $\tau_{L^\mu} : \mathbb{CP}^n \times \mathbb{CP}^{n+1} \to \mathbb{CP}^n \times \mathbb{CP}^{n+1}$ along L^μ is well-defined. Notice here that when $\mu > 1$, S^{2n+1} embeds naturally into $\mathbb{CP}^{n+1}$ as $\{[z_0 : \cdots : z_n : \sqrt{\mu - 1}] \mid |z|^2 = 1\}$.

Theorem 1.0.1. *For all $\mu > 1$, the Dehn twist $\tau_{L^\mu} \in \mathrm{Symp}(\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma_\mu)$ along the Lagrangian L^μ is symplectically isotopic to the identity by an isotopy* *whose restriction to the complex hypersurface*

$$S = \{([s_0 : \cdots : s_n], [x_0 : \cdots : x_{n+1}]) \in \mathbb{CP}^n \times \mathbb{CP}^{n+1} \mid s_0x_0 + \cdots + s_nx_n = 0\}$$

is the identity.

The isotopy of Theorem 1.0.1 was established in the case $n = 1$ and $\mu \gg 1$ by Corti-Smith [CS05, Section 7]. Corti-Smith construct a certain singular fibration with non-singular fibers isotopic to $(\mathbb{CP}^1 \times \mathbb{CP}^2, \sigma_\mu)$ and such that the monodromy around the only singular fiber of this fibration is symplectically isotopic to the Dehn twist τ_{L^μ}. As the monodromy is also known to be symplectically trivial, this proves the result. The proof of Theorem 1.0.1 uses exactly the same fibration but by examining the construction in more detail, we are able to establish an isotopy for all $\mu > 1$. Furthermore, we observe that the construction generalizes to all $n \in \mathbb{N}$ and that the complex hypersurface S is pointwise-fixed under the isotopy.

The hypersurface S is a $\mathbb{CP}^n$-bundle over $\mathbb{CP}^n$ in which the base coordinates are $[s_0 : \cdots : s_n]$, and each fiber is a linearly embedded copy of $\mathbb{CP}^n$ in $\mathbb{CP}^{n+1}$. Write $\mathbb{CP}^{n+1}$ as $\mathbb{C}^n \sqcup D$ where $\mathbb{C}^n$ is the coordinate chart centered at the point $p_0 = [0 : \cdots : 0 : 1]$ and $D = (x_{n+1} = 0) \cong \mathbb{CP}^n$. Then S contains a section at $0 \in \mathbb{C}^n$ given by $S_0 = \mathbb{CP}^n \times \{p_0\}$ and a section at infinity, namely,

$$S_\infty = \{([s_0 : \cdots : s_n], [x_0 : x_1 : 0 : \cdots : 0]) \mid s_0x_0 + s_1x_1 = 0\} \subset \mathbb{CP}^n \times D.$$

We show that S_0 and S_∞ are not only pointwise-fixed under the isotopy but in fact we have the following result.

Corollary 1.0.2. *Let S_0 and S_∞ be the complex submanifolds in S, each isomorphic to $\mathbb{CP}^n$, defined by*

$$\begin{aligned} S_0 &= \mathbb{CP}^n \times \{p_0\}, \\ S_\infty &= \{([s_0 : \cdots : s_n], [x_0 : x_1 : 0 : \cdots : 0]) \in \mathbb{CP}^n \times \mathbb{CP}^{n+1} \mid s_0 x_0 + s_1 x_1 = 0\} \end{aligned}$$

where $p_0 = [0 : \cdots : 0 : 1]$. The isotopy of Theorem 1.0.1 lifts to the blow-up of $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ along S_0 and S_∞ if the size of the blow-up is sufficiently small.

One should note that the Dehn twist τ_{L^μ} is well-defined on the blow-up since both S_0 and S_∞ are disjoint from L^μ and the Dehn twist is supported near L^μ. In contrast, the proof of [CS05, Proposition 2] shows the following result about S_0 and a different n-dimensional submanifold S'_∞ at infinity.

Proposition 1.0.3. *Consider the complex submanifolds in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ given by*

$$S_0 = \mathbb{CP}^n \times \{p_0\} \quad \textit{and} \quad S'_\infty = \mathbb{CP}^n \times \{p_\infty\},$$

where $p_0 = [0 : \cdots : 0 : 1]$ and $p_\infty = [1 : 0 : \cdots : 0]$. There is no smooth isotopy between the Dehn twist τ_{L^μ} and the identity that simultaneously fixes S_0 and S'_∞ pointwise.

It therefore seems likely that blowing up the submanifolds S_0 and S'_∞ would destroy the symplectic isotopy of Theorem 1.0.1. This does not preclude there being no symplectic isotopies between the Dehn twist along L^μ and the identity in the blow-up. If one could nevertheless exhibit a smooth isotopy between τ_{L^μ} and the identity in the blow-up, then this example would provide a candidate for an essential symplectomorphism in dimension $4n + 2$.

Theorem 1.0.1 is a consequence of Theorem 1.0.4. The proof of the latter relies on the fact that Dehn twists are monodromy maps of certain singular fibrations called Lefschetz fibrations. A Lefschetz fibration is the symplectic analogue of a Morse function; it is a symplectic fibration over $\mathbb{CP}^1$ in which we allow for a finite number of well-behaved critical points. The monodromy maps of non-singular symplectic fibrations are symplectomorphisms that are symplectically isotopic to the identity. In the case of a Lefschetz fibration, however, the monodromy may be non-trivial. Every critical point defines a Lagrangian sphere in each of the non-singular fibers called the vanishing cycle. The monodromy around a positively-oriented loop in $\mathbb{CP}^1$ that circles a critical value exactly once is symplectically isotopic to the Dehn twist in the vanishing cycle. Since the monodromy of a Lefschetz fibration with exactly one critical point is isotopic to the identity, our goal is to construct such a Lefschetz fibration whose monodromy is also given by the Dehn twist τ_{L^μ}.

Theorem 1.0.4. *Let $\mu > 1$. There is a Lefschetz fibration $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ such that:*

1. *π has exactly one critical point $z_{\mathrm{crit}} \in \mathcal{X}_0 = \pi^{-1}(0)$;*

2. *there is a holomorphic trivialization*

$$\Phi : \overline{\mathcal{X}} \setminus \mathcal{X}_0 \to (\mathbb{CP}^n \times \mathbb{CP}^{n+1}) \times \mathbb{C}$$

whose restriction to the fiber $\mathcal{X}_\infty$ over the point at infinity in $\mathbb{CP}^1$ com-

posed with the projection to $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ *is a symplectomorphism*

$$\Phi_\infty : (\mathcal{X}_\infty, \Omega^\mu \mid_{\mathcal{X}_\infty}) \to (\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma_\mu);$$

3. the map Φ_∞ *takes the vanishing cycle* $\mathcal{V}^\mu_\infty$ *in* $\mathcal{X}_\infty$ *to* L^μ *in* $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$.

For $n = 1$, Theorem 1.0.4 is an extended version of [CS05, Proposition 1]. The Lefschetz fibration $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ is an extension of the fibration constructed in [CS05] from a disk $\Delta \subset \mathbb{C}$ to $\mathbb{CP}^1 = \mathbb{C} \cup \{\infty\}$. Note, however, that unless $\mu \gg 1$, the full statement of Theorem 1.0.4 is necessary in order to identify the vanishing cycle with the Lagrangian 3-sphere L^μ in $\mathbb{CP}^1 \times \mathbb{CP}^2$. After deforming Ω^μ to a form $(\Omega^\mu)'$ which is standard in a small neighborhood of the critical point z_{crit}, Corti-Smith compute the vanishing cycle $(\mathcal{V}^\mu_t)'$ in the fibers close to the singular fiber $\mathcal{X}_0$. For each $t \in \Delta$, write $\Phi_t : \mathcal{X}_t \to \mathbb{CP}^1 \times \mathbb{CP}^2$ for the restriction $\Phi \mid_{\mathcal{X}_t}$ of the map Φ described in Theorem 1.0.4 followed by projection to $\mathbb{CP}^1 \times \mathbb{CP}^2$. The map Φ_t is biholomorphic and $\alpha'_t = (\Phi_t^{-1})^*((\Omega^\mu)' \mid_{\mathcal{X}_t})$ is a symplectic form on $\mathbb{CP}^1 \times \mathbb{CP}^2$ in the same cohomology class as σ_μ. Hence by Moser's Theorem, α'_t is isotopic to σ_μ. If $\mu \gg 1$, the fiber $\mathcal{X}_1$ intersects the neighborhood in which $(\Omega^\mu)'$ is standard and Corti-Smith show that the pull-back by Φ_1 of the vanishing cycle $(\mathcal{V}^\mu_1)'$ in $\mathcal{X}_1$ equals the Lagrangian $L^\mu \cong S^3$. Hence the isotopy from α'_1 to σ_μ can be made to fix L^μ.

For general $\mu > 1$, however, it is not clear what happens to the vanishing cycle under the isotopy and, in particular, we cannot identify it with L^μ. On the other hand, by extending the fibration over $\mathbb{CP}^1$ as in Theorem 1.0.4, we see that the form $\alpha_\infty = (\Phi_\infty^{-1})^*(\Omega^\mu)$ is not only isotopic to σ_μ but, in fact, $\alpha_\infty = \sigma_\mu$ and $\Phi_\infty : \mathcal{X}_\infty \to \mathbb{CP}^1 \times \mathbb{CP}^2$ is a symplectomorphism. The key step in our construction is the following. By symplectically embedding a large part of the total space $(\mathcal{X}, \Omega^\mu)$ of the Lefschetz fibration into a toric manifold $(\mathbb{F}, \omega_{\mathbb{F}}^{(1,\mu)})$, we obtain a Darboux chart on $(\mathcal{X}, \Omega^\mu)$ that enables us to compute the horizontal spaces of the symplectic connection coming from Ω^μ in a large neighborhood of the critical point z_{crit}. In particular, we are able to compute the vanishing cycle $\mathcal{V}_\infty^\mu$ explicitly and to see that $\Phi_\infty(\mathcal{V}_\infty^\mu)$ is the Lagrangian L^μ. Our construction generalizes easily from the case $n = 1$ to arbitrary $n \in \mathbb{N}$. Furthermore, with $S \subset \mathbb{CP}^n \times \mathbb{CP}^{n+1}$ as defined in Theorem 1.0.1, we see that the corresponding hypersurface $\Phi_t^{-1}(S)$ in the fiber $\mathcal{X}_t$ is held fixed under symplectic parallel transport in $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$. This implies that the isotopy of Theorem 1.0.1 fixes the complex hypersurface S in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$.

We now describe the structure of this dissertation. In Chapter 2 we gather some well-known results about Dehn twists and Lefschetz fibrations. The main point here is to show that the monodromy maps of Lefschetz fibrations are isotopic to Dehn twists. Chapter 3 provides the key ingredients for the proofs of the main results. We construct the Lefschetz fibration $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ described in Theorem 1.0.4, and compute explicitly the vanishing cycle $\mathcal{V}_\infty^\mu$ in $\mathcal{X}_\infty$, from which we deduce that $\Phi_\infty(\mathcal{V}_\infty^\mu)$ is the Lagrangian L^μ. Finally, in Chapter 4, we prove the main results, namely Theorems 1.0.1 and 1.0.4, Corollary 1.0.2 and Proposition 1.0.3. We conclude with a discussion of future directions related to the results of this dissertation.

Chapter 2

Dehn twists and Lefschetz fibrations

This chapter describes some well-known results about Dehn twists and Lefschetz fibrations. The main statement of this chapter is Proposition 2.2.3 which shows that the monodromy of a Lefschetz fibration around a loop that circles a critical value once is symplectically isotopic to the Dehn twist along the vanishing cycle coming from the critical value. The discussion in this chapter is based on [Sei97] and [Sei03, Section 1]. Note that although [Sei03] assumes exactness of Lefschetz fibrations, i.e., that each non-singular fiber is a symplectic manifold with boundary and the symplectic form on these fibers is exact, the proofs of the results that we use are easily adapted to ordinary Lefschetz fibrations.

2.1 Generalized Dehn twists

Consider the cotangent bundle of S^N

$$T^*S^N = \{(u,v) \in \mathbb{R}^{N+1} \times \mathbb{R}^{N+1} \mid \langle u,v \rangle = 0,\ \|v\| = 1\}$$

with standard symplectic form $\omega_{T^*S^N} = \sum_j du_j \wedge dv_j$. The zero section L_0 is a Lagrangian submanifold of $(T^*S^N, \omega_{T^*S^N})$. The length function $h : T^*S^N \to \mathbb{R}$ given by $h(u,v) = \|u\|$ generates a Hamiltonian circle action on $T^*S^N \setminus L_0$ whose flow is given by

$$\varphi_\theta^h(u,v) = \left(\cos(\theta)u - \sin(\theta)\|u\|v, \cos(\theta)v + \sin(\theta)\frac{u}{\|u\|}\right). \tag{2.1}$$

The time-π map φ_π^h extends over the zero section by the antipodal map $A(0,v) = (0,-v)$. Now let $R : \mathbb{R} \to \mathbb{R}$ be a smooth function such that $R(s) = 0$ for $s \geq s_R$ for some $s_R > 0$ and $R(-s) = R(s) - s$ for all s. Let $H = R \circ h$. The flow of H is $\varphi_\theta^H(u,v) = \varphi_{\theta R'(\|u\|)}^h(u,v)$. Since $R'(0) = \frac{1}{2}$, $\varphi_{2\pi}^H$ extends continuously to T^*S^N by the antipodal map. By [Sei03, Lemma 1.8] this extension is smooth and hence is a symplectomorphis.

Definition 2.1.1. Let τ be the time-2π flow of $H = R \circ h$ on $T^*S^N \setminus L_0$ as above, extended to L_0 by the antipodal map. The symplectomorphism τ is called a *model Dehn twist.*

Figure 2.1 shows the image of a fiber $F_v = \pi_{T^*S^N}^{-1}(0,v)$ of T^*S^N under a model Dehn twist. Here $\pi_{T^*S^N} : T^*S^N \to L_0 \cong S^N$ is the natural projection. We see that $(u,v) \in T^*S^N$ with $\|u\|$ large are held fixed by τ, but as $\|u\|$ decreases we "turn on" the circle action φ_θ^h of (2.1), with larger and larger $\theta \in (0,\pi)$ until at $\|u\| = 0$, we reach the antipodal map φ_π^h.

Remark 2.1.2. Although the definition of the model Dehn twist depends on the choice of function R, the symplectic isotopy class of τ is independent of R. Indeed, suppose τ_1 and τ_2 are model Dehn twists corresponding to R_1 and R_2

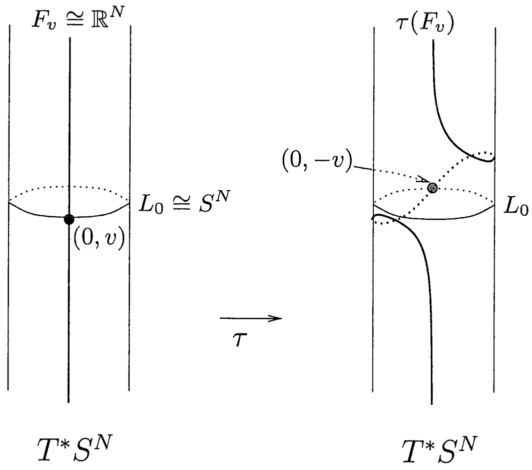

Figure 2.1: The model Dehn twist

respectively. By [Sei03, Proof of Lemma 1.8], $K = (R_2 - R_1) \circ h : T^*S^N \setminus L_0 \to \mathbb{R}$ extends smoothly by 0 to L_0 and the 2π-flow of this extension is $(\tau_1)^{-1}\tau_2$. Hence the flow φ_θ^K of K, $0 \leq \theta \leq 2\pi$, is an isotopy from $(\tau_1)^{-1}\tau_2$ to the identity. Note that τ_i equals the identity on the space $T^*_{\geq s_{R_i}} S^N$ of cotangent vectors with $\|u\| \geq s_{R_i}$ for $i = 1, 2$ and the isotopy between τ_1 and τ_2 is supported in $T^*_{\leq \max\{s_{R1}, s_{R2}\}} S^N$. □

Let $\omega_{\mathbb{C}^N} = \frac{i}{2} \sum_j dz_j \wedge d\overline{z}_j$ denote the standard symplectic structure on $\mathbb{C}^N$. Consider the singular fibration $\pi_{\text{std}} : \mathbb{C}^N \to \mathbb{C}$ given by

$$\pi_{\text{std}}(z_1, \ldots, z_N) = \sum_{j=1}^{N} z_j^2.$$

All fibers of π_{std}, except for $\pi_{\text{std}}^{-1}(0)$, are symplectic submanifolds of $(\mathbb{C}^N, \omega_{\mathbb{C}^N})$. For each $r > 0$,

$$\pi_{\text{std}}^{-1}(r) := \{z \in \mathbb{C}^N \mid \|\Re(z)\|^2 - \|\Im(z)\|^2 = r, \ \sum_j \Re(z_j)\Im(z_j) = 0\}$$

is symplectomorphic to $(T^*S^{N-1}, \omega_{T^*S^{N-1}})$ by the map

$$\Phi_r^{\text{std}}(z) = \left(-\|\Re(z)\|\Im(z), \frac{\Re(z)}{\|\Re(z)\|} \right). \tag{2.2}$$

The symplectic structure $\omega_{\mathbb{C}^N}$ gives a connection on the non-singular fibers of the fibration, where the horizontal space at $z \in \mathbb{C}^N \setminus \pi_{\text{std}}^{-1}(0)$ is

$$\text{Hor}_z = (\text{Ker}\, d(\pi_{\text{std}})_z)^{\omega_{\mathbb{C}^N}} = \text{span}_{\mathbb{C}}\, \overline{z}.$$

Parallel transport maps with respect to this connection are symplectomorphisms, and the monodromy along a contractible loop in $\mathbb{C}^*$ is Hamiltonian isotopic to the identity (see [MS98, Theorem 6.21]). Each fiber $\pi_{\text{std}}^{-1}(t)$ with $t \neq 0$ contains a Lagrangian sphere

$$\Sigma_t = \{\sqrt{r}e^{i\frac{\theta}{2}}z \in \mathbb{C}^N \mid \Im(z) = 0,\ \|z\|^2 = 1\}, \quad \text{where } t = re^{i\theta}. \tag{2.3}$$

Note that for $r > 0$, $\Phi_r^{\text{std}}(\Sigma_r)$ is the zero section L_0. Since for any point $\sqrt{r}e^{i\frac{\theta}{2}} \in \Sigma_t$, $\Im(z) = 0$, parallel transport along any path in $\mathbb{C}^*$ preserves z and hence takes the Σ_t's to each other. The limit of parallel transport along a path in $\mathbb{C}$ ending at 0 takes the Σ_ts to $0 \in \mathbb{C}^N$. In fact, if

$$\Sigma = \{0\} \cup \bigsqcup_{t\neq 0} \Sigma_t, \tag{2.4}$$

then parallel transport makes sense on $\mathbb{C}^n \setminus \Sigma$ and gives symplectomorphisms between $\pi_{\text{std}}^{-1}(t) \setminus \Sigma_t$ and $\pi_{\text{std}}^{-1}(0) \setminus \{0\}$ for all $t \neq 0$. The following result is based on the proof of [Sei03, Lemma 1.10].

Lemma 2.1.3. *Consider the singular symplectic fibration* $\pi_{\text{std}} : \mathbb{C}^N \to \mathbb{C}$ *given by* $\pi_{\text{std}}(z_1, \dots, z_N) = \sum_{j=1}^N z_j^2$*, where* $\mathbb{C}^N$ *has the standard symplectic form* $\omega_{\mathbb{C}^N}$*. For* $r > 0$*, let* $\rho_r : \pi_{\text{std}}^{-1}(r) \to \pi_{\text{std}}^{-1}(r)$ *denote the monodromy around the loop* $\gamma(\theta) = re^{i\theta}$*,* $0 \leq \theta \leq 2\pi$*, and let* $\Phi_r^{\text{std}} : \pi_{\text{std}}^{-1}(r) \to T^*S^{N-1}$ *be defined as in (2.2). Then for all* $r > 0$*,*

$$\widetilde{\tau}_r = \Phi_r^{\text{std}} \circ \rho_r \circ (\Phi_r^{\text{std}})^{-1} : T^*S^{N-1} \to T^*S^{N-1}$$

is symplectically isotopic to a model Dehn twist.

Proof.

Claim 1. For each $r > 0$, the restriction of $\widetilde{\tau}_r$ to $T^*S^{N-1} \setminus L_0$ is the time-2π flow of a Hamiltonian $\widetilde{H}_r = \widetilde{R}_r \circ h$, where $h : T^*S^{N-1} \to \mathbb{R}$ is the length function and $\widetilde{R}_r : \mathbb{R} \to \mathbb{R}$ is given by

$$\widetilde{R}_r(s) = \tfrac{s}{2} - \tfrac{1}{2}\sqrt{\tfrac{r^2}{4} + s^2}. \tag{2.5}$$

Since $\widetilde{R}_r(-s) = \widetilde{R}_r(s) - s$, by Definition 2.1.1, $\widetilde{\tau}_r$ would be a model Dehn twist, if $\widetilde{R}_r$ had compact support. We see, however, that $\widetilde{\tau}_r$ is isotopic to a model Dehn twist as follows. Let $\beta : \mathbb{R} \to [0, 1]$ be an even smooth bump function with compact support such that $\beta = 1$ in a neighborhood of 0 and let

$$R_r(s) = \widetilde{R}_{r \cdot \beta(s)}(s) \quad \text{for } s \in \mathbb{R}.$$

Then $R_r(-s) = R_r(s) - s$ and R_r has compact support and hence the time-2π flow of the Hamiltonian $H_r = R_r \circ h : T^*S^{N-1} \setminus L_0 \to \mathbb{R}$ extends to a model

Dehn twist $\tau_r : T^*S^{N-1} \to T^*S^{N-1}$. Moreover, $R_r = \widetilde{R}_r$ in a neighborhood of 0. Hence if $\varphi_\theta^{H_r}$ and $\varphi_\theta^{\widetilde{H}_r}$ denote the flow of H_r and $\widetilde{H}_r$ respectively,

$$\varphi_\theta^r = \varphi_\theta^{\widetilde{H}_r} \circ (\varphi_\theta^{H_r})^{-1}, \quad 0 \le \theta \le 2\pi,$$

extends to T^*S^{N-1} as the identity map on L_0 and is an isotopy from τ_r to $\widetilde{\tau}_r$.

It remains to prove Claim 1. Recall that parallel transport is well-defined on $\mathbb{C}^N \setminus \Sigma$ with Σ defined as in (2.4). We now trivialize $\mathbb{C}^N \setminus \Sigma$ using parallel transport in the radial directions by a map

$$\Phi^{\text{std}} : \mathbb{C}^N \setminus \Sigma \to \mathbb{C} \times (T^*S^{N-1} \setminus L_0)$$

that is a symplectomorphism on each fiber and whose restriction $\Phi^{\text{std}} \mid_{\pi^{-1}(r)\setminus\Sigma_r}$ extends to the symplectomorphism $\Phi_r^{\text{std}} : \pi_{\text{std}}^{-1}(r) \to T^*S^{N-1}$ for all $r > 0$. In particular, $\widetilde{\tau}_r \mid_{T^*S^{N-1}\setminus L_0}$ is the monodromy of $\mathbb{C} \times (T^*S^{N-1} \setminus L_0)$ along the loop $re^{i\theta}$, $0 \le \theta \le 2\pi$, where the symplectic connection is induced by the form $((\Phi^{\text{std}})^{-1})^*\omega_{\mathbb{C}^N}$.

We use the following explicit expression for Φ^{std}:

$$\begin{aligned} \Phi^{\text{std}}(z) &= (\pi_{\text{std}}(z), (\varphi^h_{\frac{\theta}{2}} \circ \Phi_r^{\text{std}})(e^{-i\frac{\theta}{2}}z)) \\ &= \left(re^{i\theta}, \varphi^h_{\frac{\theta}{2}}\left(-\Im(e^{-i\frac{\theta}{2}}z)\|\Re(e^{-i\frac{\theta}{2}}z)\|, \frac{\Re(e^{-i\frac{\theta}{2}}z)}{\|\Re(e^{-i\frac{\theta}{2}}z)\|}\right)\right) \end{aligned} \tag{2.6}$$

where $\pi_{\text{std}}(z) = re^{i\theta}$ and φ^h is the flow on $T^*S^{N-1} \setminus L_0$ described in (2.1). Because of the use of polar coordinates, it requires an argument to see that Φ^{std} is smooth at $\pi_{\text{std}}^{-1}(0) \setminus \Sigma_0$. Details of this can be found in [Sei03]. By construction, $\Phi^{\text{std}} \mid_{\pi_{\text{std}}^{-1}(re^{i\theta})\setminus\Sigma_{re^{i\theta}}}$ is a symplectomorphism for all r, θ and $\Phi^{\text{std}} \mid_{\pi_{\text{std}}^{-1}(r)\setminus\Sigma_r}$

extends to the symplectomorphism $\Phi_r^{\text{std}} : \pi_{\text{std}}^{-1}(r) \to T^*S^{N-1}$ for all $r > 0$.

Note that

$$\begin{aligned} \omega_{\mathbb{C}^N} &= d\alpha_{\mathbb{C}^N}, &\text{where}\quad \alpha_{\mathbb{C}^N} &= \tfrac{i}{4}\textstyle\sum_j (z_j d\overline{z}_j - \overline{z}_j dz_j), \\ \omega_{T^*S^{N-1}} &= d\alpha_{T^*S^{N-1}}, &\text{where}\quad \alpha_{T^*S^{N-1}} &= \tfrac{1}{2}\textstyle\sum_j (u_j dv_j - v_j du_j). \end{aligned}$$

<u>Claim 2.</u>

$$((\Phi^{\text{std}})^{-1})^*\alpha_{\mathbb{C}^n} = \alpha_{T^*S^{N-1}} - \widetilde{H}_r d\theta$$

where $\widetilde{H}_r = \widetilde{R}_r \circ h$ and $r > 0$. Hence

$$((\Phi^{\text{std}})^{-1})^*\omega_{\mathbb{C}^N} = \omega_{T^*S^{N-1}} - d\widetilde{H}_r \wedge d\theta. \tag{2.7}$$

Assuming we have proved Claim 2, consider $\mathbb{C} \times (T^*S^{N-1} \setminus L_0)$ with symplectic connection induced by the form $((\Phi^{\text{std}})^{-1})^*\omega_{\mathbb{C}^N}$. We see that for each $t \in \mathbb{C}^*$, the horizontal lift of a vector $\xi \in T_t\mathbb{C}$ is of the form

$$(\xi, X) \in T_{(z,u,v)}(\mathbb{C} \times (T^*S^{N-1} \setminus L_0)), \text{with } i(X)\omega_{T^*S^{N-1}} = -d\theta(\xi)d\widetilde{H}_r.$$

Hence for each $(u, v) \in T^*S^{N-1}\setminus L_0$, the path $(re^{i\theta}, \varphi_\theta^{\widetilde{H}_r}(u, v))$ is a horizontal lift of the loop $re^{i\theta}$. It follows that the monodromy along the loop $re^{i\theta}$, $0 \le \theta \le 2\pi$ is the 2π-flow of $\widetilde{H}_r$. This proves Claim 1.

Finally, we prove Claim 2. The inverse of Φ^{std},

$$(\Phi^{\text{std}})^{-1} : \mathbb{C} \times T^*S^{N-1} \setminus L_0 \to \mathbb{C}^N \setminus \Sigma$$

is given by

$$
\begin{aligned}
(\Phi^{\mathrm{std}})^{-1}(re^{i\theta},(u,v)) &= e^{i\frac{\theta}{2}}((\Phi_r^{\mathrm{std}})^{-1}((\varphi^h_{\frac{\theta}{2}})^{-1}(u,v)) \\
&= \left(\cos(\tfrac{\theta}{2})\lambda\tilde{v}+\sin(\tfrac{\theta}{2})\tfrac{\tilde{u}}{\lambda}\right)+i\left(\sin(\tfrac{\theta}{2})\lambda\tilde{v}-\cos(\tfrac{\theta}{2})\tfrac{\tilde{u}}{\lambda}\right)
\end{aligned} \tag{2.8}
$$

where

$$
\begin{aligned}
&(\tilde{u},\tilde{v}) = (\varphi^h_{\frac{\theta}{2}})^{-1}(u,v) = \left(\cos(\tfrac{\theta}{2})u+\sin(\tfrac{\theta}{2})v\|u\|, \cos(\tfrac{\theta}{2})v-\sin(\tfrac{\theta}{2})\tfrac{u}{\|u\|}\right), \\
&\lambda^2 = \tfrac{r}{2}+\sqrt{\tfrac{r^2}{4}+\|u\|^2}, \quad \lambda>0.
\end{aligned} \tag{2.9}
$$

In vector notation we have that

$$
\alpha_{\mathbb{C}^N} = \tfrac{i}{4}(zd\overline{z}-\overline{z}dz) \text{ and } \alpha_{T^*S^{N-1}} = \tfrac{1}{2}(udv-vdu).
$$

It follows from (2.8) that if $z=(\Phi^{\mathrm{std}})^{-1}(re^{i\theta},(u,v))$, then $z=e^{i\frac{\theta}{2}}\tilde{z}$ where $\tilde{z}=\lambda\tilde{v}-i\frac{\tilde{u}}{\lambda}$ with $(\tilde{u},\tilde{v})$ and λ defined by (2.9). Note that λ is chosen precisely such that $\pi_{\mathrm{std}}(\tilde{z})=r$, i.e., such that λ^2 satisfies the equation

$$
\|\lambda\tilde{v}\|^2-\|\tfrac{\tilde{u}}{\lambda}\|^2 = \lambda^2-\tfrac{\|\tilde{u}\|^2}{\lambda^2} = r.
$$

We see that

$$
\begin{aligned}
((\Phi^{\mathrm{std}})^{-1})^*\alpha_{\mathbb{C}^N} &= \tfrac{i}{4}\left(e^{i\frac{\theta}{2}}\tilde{z}d(e^{-i\frac{\theta}{2}}\overline{\tilde{z}})-e^{-i\frac{\theta}{2}}\overline{\tilde{z}}d(e^{i\frac{\theta}{2}}\tilde{z})\right) \\
&= \tfrac{i}{4}\left(e^{i\frac{\theta}{2}}\tilde{z}(e^{-i\frac{\theta}{2}}d\overline{\tilde{z}}+\overline{\tilde{z}}d(e^{-i\frac{\theta}{2}}))-e^{-i\theta}\overline{\tilde{z}}(e^{i\frac{\theta}{2}}d\tilde{z}+\tilde{z}d(e^{i\frac{\theta}{2}}))\right) \\
&= \tfrac{i}{4}(\tilde{z}d\overline{\tilde{z}}-\overline{\tilde{z}}d\tilde{z})+\tfrac{1}{4}\|\tilde{z}\|^2d\theta \\
&= \tfrac{1}{2}\left(\tfrac{\tilde{u}}{\lambda}d(\lambda\tilde{v})-\lambda\tilde{v}d(\tfrac{\tilde{u}}{\lambda})\right)+\tfrac{1}{4}(\|\lambda\tilde{v}\|^2+\|\tfrac{\tilde{u}}{\lambda}\|^2)d\theta.
\end{aligned}
$$

Since $\|\tilde{v}\| = \|v\| = 1$, $\|\tilde{u}\| = \|u\|$ and $\langle u, v\rangle = \langle \tilde{u}, \tilde{v}\rangle = 0$,

$$\begin{aligned}((\Phi^{\mathrm{std}})^{-1})^*\alpha_{\mathbb{C}^N} &= \tfrac{1}{2}\left(\tfrac{\tilde{u}}{\lambda}(\lambda d\tilde{v} + \tilde{v}d\lambda) - \lambda\tilde{v}(\tfrac{1}{\lambda}d\tilde{u} + \tilde{u}d(\tfrac{1}{\lambda}))\right) + \tfrac{1}{4}(\lambda^2 + \tfrac{\|\tilde{u}\|^2}{\lambda^2})d\theta \\ &= \tfrac{1}{2}(\tilde{u}d\tilde{v} - \tilde{v}d\tilde{u}) + \tfrac{1}{4}(\lambda^2 + \tfrac{\|u\|^2}{\lambda^2})d\theta.\end{aligned}$$

A similar computation shows that $\tilde{u}d\tilde{v} - \tilde{v}d\tilde{u} = udv - vdu - \|u\|d\theta$. Using the fact that $\lambda^2 - \frac{\|u\|}{\lambda^2} = r$ and the definition of λ^2, we see that

$$\begin{aligned}\tfrac{1}{2}\|u\| - \tfrac{1}{4}(\lambda^2 + \tfrac{\|u\|^2}{\lambda^2}) &= \tfrac{1}{2}\|u\| - \tfrac{1}{2}\lambda^2 + \tfrac{1}{4}(\lambda^2 - \tfrac{\|u\|^2}{\lambda^2}) \\ &= \tfrac{1}{2}\|u\| - \tfrac{1}{2}\sqrt{\tfrac{r^2}{4} + \|u\|^2}.\end{aligned}$$

Hence Claim 2 holds. □

Definition 2.1.4. Let (M^{2N}, ω) be a symplectic manifold and $L \subset M$ a Lagrangian sphere with a chosen identification $\iota : S^N \to L$. By the Lagrangian Neighborhood Theorem, for some small $\lambda > 0$, we can extend ι to a symplectic embedding of the space $T^*_{\leq\lambda}S^N$ of cotangent vectors with $\|u\| \leq \lambda$ to a neighborhood $\mathcal{N}(L)$ of L. Choose a model Dehn twist τ whose support is in $T^*_{\leq\frac{\lambda}{2}}S^N$. The *generalized Dehn twist* (or simply *Dehn twist*) along L is the map

$$\tau_L(z) = \begin{cases} \iota \circ \tau \circ \iota^{-1}, & \text{if } z \in \mathcal{N}(L); \\ z & \text{otherwise.} \end{cases}$$

Remark 2.1.5. The Dehn twist is a symplectomorphism of M that depends on the choice of identification $\iota : S^N \to L$. We say that two identifications ι_1 and ι_2 are equivalent if $\iota_2^{-1} \circ \iota_1 : S^N \to S^N$ can be deformed inside the group of diffeomorphisms of S^N to an element of $O(N+1)$. An equivalence class $[\iota]$ is called a *framing* of the Lagrangian sphere. If ι_1 and ι_2 give the same framing of L, then the corresponding Dehn twists $\tau_{L,1}$ and $\tau_{L,2}$ are symplectically isotopic and $\tau_L = \tau_{(L,[\iota_1])}$ is well-defined as an isotopy class. If $N = 1$, then there is only one choice of framing since $\text{Diff}(S^2) \cong O(3)$. However, in general it is not known how a change of framing affects the isotopy class of the Dehn twist and we have to specify a framing in order to define the Dehn twist. We omit the choice of framing in our notation as there is often a natural choice in the situations we describe below. □

Now consider $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ as a $4n+2$-dimensional real manifold with the product symplectic form $\sigma_\mu = \sigma_{\mathbb{CP}^n} + \mu\sigma_{\mathbb{CP}^{n+1}}$, where $\mu > 1$ and $\sigma_{\mathbb{CP}^n}, \sigma_{\mathbb{CP}^{n+1}}$ are the normalized Fubini-Study forms on $\mathbb{CP}^n$ and $\mathbb{CP}^{n+1}$, respectively. As shown in [CS05, Lemma 1], this manifold contains a natural Lagrangian sphere L^μ, namely the graph of the complex conjugate of the Hopf map $H_{\mathbb{CP}^n} : S^{2n+1} \to \mathbb{CP}^n$.

Lemma 2.1.6. *If $\mu > 1$, then*

$$L^\mu := \{[\overline{z}_0 : \cdots : \overline{z}_n][z_0 : \cdots : z_n : \sqrt{\mu - 1}] \in \mathbb{CP}^n \times \mathbb{CP}^{n+1} \mid |z|^2 = 1\}$$

is a Lagrangian sphere in $(\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma_\mu)$.

Proof. For $\lambda > 0$, let $B(\lambda) = \{z \in \mathbb{C}^{n+1} \mid |z|^2 < \lambda^2\}$. Then the map $i : B(\sqrt{\mu}) \hookrightarrow \mathbb{CP}^{n+1}$ given by

$$i : (z_0, \ldots, z_n) \mapsto [z_0 : \cdots : z_n : \sqrt{\mu - |z_0|^2 - \cdots - |z_n|^2}]$$

is a symplectic embedding $(B(\sqrt{\mu}), \omega_{\mathbb{C}^{n+1}}) \hookrightarrow (\mathbb{CP}^{n+1}, \mu\sigma_{\mathbb{CP}^{n+1}})$. This follows from the fact that i factors through the Hopf map $H_{\mathbb{CP}^{n+1}} : S^{2n+3} \to \mathbb{CP}^{n+1}$

$$\begin{array}{ccc} & & S^{2n+3} \\ & \overset{\tilde{i}}{\nearrow} & \downarrow H_{\mathbb{CP}^{n+1}} \\ B(\sqrt{\mu}) & \xrightarrow{\ i\ } & \mathbb{CP}^{n+1} \end{array}$$

where

$$\tilde{i}(z_0, \ldots, z_n) = \left(\frac{z_0}{\sqrt{\mu}}, \ldots, \frac{z_n}{\sqrt{\mu}}, \sqrt{1 - |\tfrac{z}{\sqrt{\mu}}|^2}\right).$$

Hence,

$$i^*(\mu\sigma_{\mathbb{CP}^{n+1}}) = \tilde{i}^*((H_{\mathbb{CP}^{n+1}})^*(\mu\sigma_{\mathbb{CP}^{n+1}})) = \tilde{i}^*(\mu\omega_{\mathbb{C}^{n+2}}|_{S^{2n+3}}) = \omega_{\mathbb{C}^{n+1}}|_{B(\sqrt{\mu})}.$$

Thus, $S^{2n+1} := \partial B(1) \subset B(\sqrt{\mu})$ embeds symplectically into $(\mathbb{CP}^{n+1}, \mu\sigma_{\mathbb{CP}^{n+1}})$. Therefore, the graph of the conjugate of the Hopf map $H_{\mathbb{CP}^n} : S^{2n+1} \to \mathbb{CP}^n$ embeds symplectically into $(\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma^n_\mu)$ as L^μ . Since $(\overline{H}_{\mathbb{CP}^n})^*(\sigma_{\mathbb{CP}^n}) = -\omega_{\mathbb{C}^{n+1}}|_{S^{2n+1}}$,

$$\sigma_\mu|_{L^\mu} = \omega_{\mathbb{C}^{n+1}}|_{S^{2n+1}} + (\overline{H}_{\mathbb{CP}^n})^*(\sigma_{\mathbb{CP}^n}) = 0,$$

i.e., L^μ is Lagrangian. □

2.2 Lefschetz fibrations

Definition 2.2.1. A *Lefschetz fibration* is a smooth fibration $\pi : E \to \mathbb{CP}^1$ such that

- the set E_{crit} of critical points of π is finite and no two critical points lie in the same fiber;

- E is compact and has a closed 2-form Ω that restricts to a symplectic form on the non-singular fibers;

- there is a complex structure J defined in a neighborhood of each critical point such that Ω is J-Kähler;

- if j is the complex structure on $\mathbb{CP}^1$, then π is (J, j)-holomorphic and at each critical point the Hessian of π is non-degenerate.

Note that by the Morse Lemma, the fourth condition is equivalent to the condition that for each $z_{\text{crit}} \in E_{\text{crit}}$ we can find a J-holomorphic coordinate chart Ψ in a neighborhood of z_{crit} and a chart ψ on $\mathbb{CP}^1$ centered at $\pi(z_{\text{crit}})$ in $\mathbb{CP}^1$ such that

$$\psi \circ \pi \circ \Psi^{-1}(z) = \sum_{i=1}^{N} z_i^2.$$

Such a pair of charts (Ψ, ψ) will be called a *Morse chart.* In the following we often suppress the choice of ψ in our notation.

Away from the critical points, (E, π, Ω) is a symplectic fiber bundle. Hence it admits a symplectic connection, i.e., the connection with the horizontal space at $z \in E \setminus E_{\text{crit}}$ given by

$$(T_z E)^h = \{X \in T_z E \mid \Omega(X, Y) = 0 \text{ for all } Y \in \ker D\pi_z\}.$$

Therefore, we can define parallel transport maps

$$\rho_\gamma : E_{\gamma(a)} \longrightarrow E_{\gamma(b)}$$

along embedded paths $\gamma : [a, b] \to \mathbb{CP}^1 \setminus \pi(E_{\text{crit}})$.

Each critical point $z_{\text{crit}} \in E_{\text{crit}}$ gives rise to a Lagrangian sphere in the non-singular fibers called the vanishing cycle. Let $\gamma : [a, b] \to \mathbb{CP}^1$ be an embedded path which avoids $\pi(E_{\text{crit}})$ except at the endpoint $\gamma(b) = \pi(z_{\text{crit}})$. Define

$$B_\gamma = \{z_{\text{crit}}\} \cup \bigcup_{a \leq s < b} \{z \in E_{\gamma(s)} \mid \lim_{s' \to b} \rho_{\gamma|_{[s,s']}}(z) = z_{\text{crit}}\}.$$

By [Sei03, Lemma 1.13 and 1.14], B_γ is an embedded closed N-ball in E with $\Omega|_{B_\gamma} = 0$ whose boundary

$$V_\gamma = \partial B_\gamma = B_\gamma \cap E_{\gamma(a)}$$

is a Lagrangian sphere in $(E_{\gamma(a)}, \Omega|_{E_{\gamma(a)}})$ that comes with a natural framing (see Remark 2.1.5). We call V_γ the *vanishing cycle* associated to γ. (See Figure 2.2). If γ' is path-homotopic to γ in $\mathbb{CP}^1 \setminus \pi(E_{\text{crit}})$, then $V_{\gamma'}$ with its natural framing is symplectically isotopic to V_γ. Hence, for each $z_{\text{crit}} \in E_{\text{crit}}$ and $t \in \mathbb{CP}^1 \setminus \pi(E_{\text{crit}})$, the symplectic isotopy class of V_γ with its natural framing depends only on the path-homotopy class.

Definition 2.2.2. Let (E, π, J, Ω) be a Lefschetz fibration. Let $\gamma : [a, b] \to \mathbb{CP}^1$ be an embedded path that avoids $\pi(E_{\text{crit}})$ except at the endpoint $\gamma(b) = \pi(z_{\text{crit}}) \in \pi(E_{\text{crit}})$. We say that a loop $\ell : [c, d] \to \mathbb{CP}^1 \setminus \pi(E_{\text{crit}})$ *doubles* γ if $\ell(c) = \ell(d) = \gamma(a)$, ℓ is positively oriented with respect to the standard orientation of $\mathbb{CP}^1$ and ℓ circles the point $\pi(z_{\text{crit}})$ exactly once and circles no other critical values of π.

The following result is the main result of this chapter and is described in

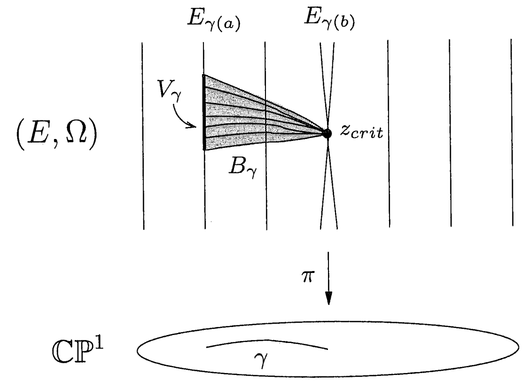

Figure 2.2: The vanishing cycle

[Sei03, Proposition 1.15].

Proposition 2.2.3. *Let (E, π, J, Ω) be a Lefschetz fibration and $V_\gamma \subset E_{t_0}$ the vanishing cycle of a path $\gamma : [a, b] \to \mathbb{CP}^1$ with $\gamma(a) = t_0$ and $\gamma(b) = \pi(z_{\text{crit}}) \in E_{\text{crit}}$. Let ℓ be a loop that doubles γ. The monodromy $\rho_\ell : E_{t_0} \to E_{t_0}$ around ℓ is isotopic to the Dehn twist τ_{V_γ} along the vanishing cycle V_γ.*

Proof. For simplicity assume that the Lefschetz fibration has a single critical point z_{crit} and $\pi(z_{\text{crit}}) = 0$. Let $(z_1, \ldots, z_N)$ denote the coordinates of the Morse chart around z_{crit} with respect to which π has the form $\sum_j z_j^2$. By [Sei03, Lemma 1.6], there is a smooth family Ω^λ, $0 \le \lambda \le 1$, such that

- $\Omega^0 = \Omega$,
- for all λ, $\Omega^\lambda = \Omega$ outside a small neighborhood of z_{crit},
- each $(E, \pi, J, \Omega^\lambda)$ is a Lefschetz fibration,
- Ω^1 equals $c\omega_{\mathbb{C}^N} = \frac{i}{2} \sum_j dz_j \wedge d\overline{z}_j$ near z_{crit} for some constant $c > 0$.

The monodromy $\rho_\ell : E_{t_0} \to E_{t_0}$ in (E, π, J, Ω) along ℓ is isotopic to the monodromy $\rho_\ell^1 : E_{t_0} \to E_{t_0}$ in (E, π, J, Ω^1) under an isotopy taking the vanishing cycle V_γ with its natural framing to the vanishing cycle V_γ^1 for Ω^1. Thus, by Definition 2.1.4, it suffices to show that ρ_ℓ^1 is isotopic to the Dehn twist along V_γ^1.

In fibers close to the singular fiber, the monodromy of (E, π, J, Ω^1) can be computed using the standard local model $(\mathbb{C}^N, \pi_{\text{std}}, J_0, \omega_{\mathbb{C}^N})$ described in Section 2.1. Take a small $\varepsilon > 0$. We claim that the monodromy ρ_ε^1 around the loop $l(\theta) = \varepsilon e^{i\theta}$, $0 \leq \theta \leq 2\pi$, is isotopic to the Dehn twist $\tau_{V_\varepsilon^1} : E_\varepsilon \to E_\varepsilon$ where V_ε^1 is the vanishing cycle in the fiber E_ε. Let $\overline{D}(\varepsilon)$ denote the closed disk of radius ε in $\mathbb{CP}^1$ and $E_{\overline{D}(\varepsilon)} = \pi^{-1}(\overline{D}(\varepsilon))$. Let $\Psi : W \to E$, $W \subset \mathbb{C}^N$, be a Morse chart on E in a neighborhood of z_{crit}. For each $t \in \overline{D}(\varepsilon)$, let Σ_t^E be the Lagrangian sphere in E_t defined by (2.3) in the Morse coordinates. Note that Σ_ε^E is the vanishing cycle V_η^1 for the path $\eta(r) = \varepsilon - r$, $0 \leq r \leq \varepsilon$. Let

$$\Sigma^E = \{z_{\text{crit}}\} \cup \bigcup_{t \in \overline{D}(\varepsilon) \setminus \{0\}} \Sigma_t^E.$$

As in the proof of Lemma 2.1.3, we use parallel transport in the radial direction to construct a trivialization $\Phi^E : E_{\overline{D}(\varepsilon)} \setminus \Sigma^E \to \overline{D}(\varepsilon) \times (E_\varepsilon \setminus \Sigma_\varepsilon)$ that is the identity on the fiber E_ε. Hence for fixed $r > 0$,

$$((\Phi^E)^{-1})^*(\Omega^1) = \Omega^1 \mid_{E_\varepsilon \setminus \Sigma_\varepsilon} - \beta_r \wedge d\theta, \tag{2.10}$$

where β_r is a closed 1-form on $E_\varepsilon \setminus \Sigma_\varepsilon$. We claim that β_r is exact. If not, there exists a 1-cycle δ such that

$$\int_{\partial\overline{D}(r)\times\delta}\beta_r\wedge d\theta=-2\pi r\int_\delta\beta_r\neq 0.$$

On the other hand, by (2.10), $\beta_r\wedge d\theta$ extends to a closed form η on $\overline{D}(r)\times\delta$ and hence by Stokes' Theorem,

$$\int_{\partial\overline{D}(r)\times\delta}\beta_r\wedge d\theta=\int_{\overline{D}(r)\times\delta}d\eta=0.$$

This is a contradiction. Thus, for each $r>0$ there exists a function $\widetilde{H}_r^E$ such that $\beta_r=d\widetilde{H}_r^E$. The restriction $\rho_l\mid_{E_\varepsilon\setminus\Sigma_\varepsilon}$ of the monodromy around l is the time 2π-flow of the function $\widetilde{H}_\varepsilon^E$.

The map $\Phi_\varepsilon^{\mathrm{std}}$ defined in (2.2) gives a natural embedding $\iota: T^*_{\leq\lambda}S^{n-1}\hookrightarrow E_\varepsilon$, where $\lambda>0$ is chosen so that $\mathrm{Im}(\iota)\subset\Psi(W)$. With Φ^{std} as defined in (2.6), the diagram

$$\begin{array}{ccc}
\mathbb{C}^N\setminus\Sigma & \xrightarrow{\Phi^{\mathrm{std}}} & \mathbb{C}\times(T^*S^{N-1}\setminus L_0)\\
\big\uparrow & & \big\uparrow\\
W\setminus\Sigma & & \overline{D}(\varepsilon)\times(T^*_{\leq\lambda}S^{N-1}\setminus L_0)\\
\Big\downarrow{\scriptstyle \Psi|_{W\setminus\Sigma}} & & \Big\downarrow{\scriptstyle \mathrm{id}\times\iota|_{T^*_{\leq\lambda}S^{N-1}\setminus L_0}}\\
E_{\overline{D}(\varepsilon)}\setminus\Sigma^E & \xrightarrow{\Phi^E} & \overline{D}(\varepsilon)\times(E_\varepsilon\setminus\Sigma_\varepsilon)
\end{array}$$

where Φ^{std} is the trivialization (2.6). By Claim 2 in Lemma 2.1.3, it follows that

$$(\mathrm{id}\times\iota\mid_{T^*_{\leq\lambda}S^{N-1}\setminus L_0}\circ(\Phi^E)^{-1})^*\Omega^1=\omega_{T^*S^{N-1}}-d\widetilde{H}\wedge d\theta,$$

where $\widetilde{H} = \widetilde{R} \circ h$ is as in the proof of Lemma 2.1.3. Hence

$$\widetilde{H}^E_\varepsilon(\theta, \iota(u,v)) = \widetilde{H}_\varepsilon(u,v) \quad \text{for } 0 \leq \theta \leq 2\pi,\ (u,v) \in T^*_{\leq\lambda}S^{N-1} \setminus L_0.$$

Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth bump function such that $F = f \circ h$ has support in W. Let $H^E = F \cdot \widetilde{H}^E$. By the proof of Lemma 2.1.3, the flow of H^E induces an isotopy between the monodromy $\rho^1_l : E_\varepsilon \to E_\varepsilon$ along the loop l and the Dehn twist $\tau_{V^1_\varepsilon}$.

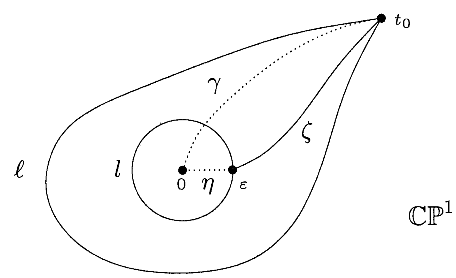

Figure 2.3: Deforming the loop ℓ

The original loop ℓ can be deformed to a loop of the form $\widehat{\ell} = \zeta * l * \zeta^{-1}$, where ζ is a path from the basepoint t_0 of ℓ to ε (see Figure 2.3). The monodromy along this loop is $\rho_{\widehat{\ell}} = \rho_\zeta^{-1} \circ \rho^1_l \circ \rho_\zeta$, where ρ_ζ denotes parallel transport along ζ. Since we have just seen that ρ^1_l is isotopic to the Dehn twist $\tau_{V^1_\varepsilon}$ and ρ_ζ maps $V^1_{\zeta*\eta}$ to V^1_η. Definition 2.1.4 shows that $\rho_{\widehat{\ell}}$ is isotopic to the Dehn twist $\tau_{V^1_\gamma}$. □

Chapter 3

The Lefschetz fibration $\pi : \overline{\mathcal{X}} \to \mathbb{CP}^1$

This chapter contains the essential steps in the proofs of the main results of this dissertation. In the first three sections we construct the Lefschetz fibration $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ of Theorem 1.0.4. Its total space $\overline{\mathcal{X}}$ is a complex hypersurface in $\mathbb{F} \times \mathbb{CP}^1$, where $\mathbb{F}$ is a toric manifold of complex dimension $2n+2$. The manifold $\overline{\mathcal{X}}$ fibers over $\mathbb{CP}^1$ as a subbundle of the bundle $\mathbb{F} \times \mathbb{CP}^1 \to \mathbb{CP}^1$. For each $\mu > 1$, the closed two-form Ω^μ on $\overline{\mathcal{X}}$ comes from a toric symplectic structure on $\mathbb{F}$, and the symplectic isotopy class of the non-singular fibers of $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ depends on the cohomology class of this toric structure. For each $\mu > 1$, it is possible to choose a toric structure on $\mathbb{F}$ such that the non-singular fibers of $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ are isotopic to $(\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma_\mu)$.

Section 3.1 defines the toric manifold $\mathbb{F}$. We then construct the Lefschetz fibration in two steps. In Section 3.2 we define the part of the fibration that fibers over $\mathbb{C}$. Then, in Section 3.3 we extend the fibration over the point at infinity in $\mathbb{CP}^1$. By symplectically embedding a large part of $(\overline{\mathcal{X}}, \Omega^\mu)$ into the toric manifold $(\mathbb{F}, \omega_{\mathbb{F}}^{(1,\mu)})$, and using a Darboux chart on $\mathbb{F}$, we show in Section 3.4 that the vanishing cycle $\mathcal{V}_\infty$ in the fiber at infinity is the Lagrangian L^μ.

3.1 The toric manifold $\mathbb{F}$

In this section we apply the theory of toric manifolds in both the algebro-geometric and symplectic categories. Elementary introductions can be found in [Cox95], [Tha94], [Aud04, Chapter VII], and [MS04, Section 11.3.1].

Let $\mathbb{C}^{2n+4}$ have coordinates $(\underline{s}, q, \underline{x}, x_{n+1}) = (s_0, \ldots, s_n, q, x_0, \cdots, x_{n+1})$ and consider the $(\mathbb{C}^*)^2$ action on $\mathbb{C}^{2n+4}$ with weights

$$\begin{pmatrix} \overbrace{1 \; \cdots \; 1}^{n+1} & 1 & \overbrace{0 \; \cdots \; 0}^{n+2} \\ 0 \; \cdots \; 0 & 1 & 1 \; \cdots \; 1 \end{pmatrix}. \tag{3.1}$$

The effective cone C_{eff} of the action is the cone in $\mathbb{R}^2$ generated by the columns of the matrix (3.1), i.e., C_{eff} is the positive quadrant. This cone decomposes naturally into a union of 2-dimensional cones, each generated by a pair of column vectors. The interior of these cones are the chambers C_1 and C_2 for the action depicted in Figure 3.1.

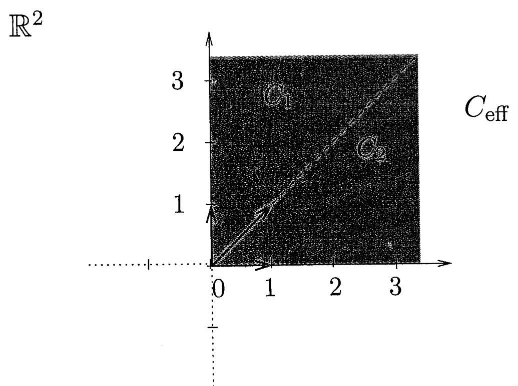

Figure 3.1: The effective cone

We define the complex manifold $\mathbb{F}$ to be the GIT quotient

$$\mathbb{F} = \mathbb{C}^{2n+4} /\!\!/_{\kappa} (\mathbb{C}^*)^2,$$

where κ is an integral point of C_1. As a complex manifold, the GIT quotient is the same for different choices of κ in the same chamber [Tha94, Theorem 3.9]. In this case,

$$\mathbb{F} = (\mathbb{C}^{n+2} \setminus \{0\} \times \mathbb{C}^{n+2} \setminus \{0\})/(\mathbb{C}^*)^2.$$

Since the action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^{n+2} \setminus \{0\} \times \mathbb{C}^{n+2} \setminus \{0\}$ is free and proper, this quotient is a smooth complex manifold. We denote the points of $\mathbb{F}$ by $[[\underline{s}, q, \underline{x}, x_{n+1}]]$, where $(\underline{s}, q, \underline{x}, x_{n+1}) \in \mathbb{C}^{n+2} \setminus \{0\} \times \mathbb{C}^{n+2} \setminus \{0\}$ and

$$[[\underline{s}, q, \underline{x}, x_{n+1}]] = [[\alpha \underline{s}, \alpha\beta q, \beta \underline{x}, \beta x_{n+1}]] \text{ for } (\alpha, \beta) \in (\mathbb{C}^*)^2$$

We now describe a Kähler structure on $\mathbb{F}$ by constructing it as a symplectic quotient. Let $\mathbb{C}^{2n+4}$ have the standard symplectic form $\omega_{\mathbb{C}^{2n+4}}$ and suppose T^2 acts on $\mathbb{C}^{2n+4}$ with weights (3.1). The moment map $\Psi : \mathbb{C}^{2n+4} \to (\mathfrak{t}^2)^* \cong \mathbb{R}^2$ of the action is given by

$$\Psi(\underline{s}, q, \underline{x}, x_{n+1}) = \left(\sum_{j=0}^{n} |s_j|^2 + |q|^2, |q|^2 + \sum_{j=0}^{n+1} |x_j|^2 \right).$$

Its image is the effective cone C_{eff} illustrated in Figure 3.1 and the regular values are the elements of the chambers C_1 and C_2. By the standard symplectic quotient construction [MS98, Section 5.4], for each $\kappa \in C_1$, the form $\omega_{\mathbb{C}^{2n+4}} |_{\Psi^{-1}(\kappa)}$ descends to a symplectic form on the quotient manifold $\Psi^{-1}(\kappa)/T^2$. The elements of $\Psi^{-1}(\kappa)/T^2$ are denoted $[\underline{s}, q, \underline{x}, x_{n+1}]$ where $(\underline{s}, q, \underline{x}, x_{n+1}) \in \Psi^{-1}(\kappa)$ and

$$[\underline{s}, q, \underline{x}, x_{n+1}] = [e^{i\theta_1}\underline{s}, e^{i(\theta_1+\theta_2)}q, e^{i\theta_2}\underline{x}, e^{i\theta_2}x_{n+1}] \text{ for } (e^{i\theta_1}, e^{i\theta_2}) \in T^2.$$

By [Aud04, Theorem VII.2.1], as a complex manifold, the quotient $\Psi^{-1}(\kappa)/T^2$ equals $\mathbb{C}^{2n+4}/\!\!/_{\kappa'}(\mathbb{C}^*)^2$, where κ' is any integral point in the chamber of κ. In fact, the map $\Psi^{-1}(\kappa)/T^2 \to \mathbb{C}^{2n+4}/\!\!/_{\kappa'}(\mathbb{C}^*)^2$ given by

$$[\underline{s}, q, \underline{x}, x_{n+1}] \mapsto [[\underline{s}, q, \underline{x}, x_{n+1}]]$$

is biholomorphic and the quotient symplectic form is a Kähler structure $\omega_{\mathbb{F}}^{\kappa}$ on $\mathbb{F} = \mathbb{C}^{2n+4}/\!\!/_{\kappa'}(\mathbb{C}^*)^2$.

Remark 3.1.1. Points of $\mathbb{F}$ are denoted $[\underline{s}, q, \underline{x}, x_{n+1}]$, when $\mathbb{F}$ is thought of as a symplectic quotient $\Psi^{-1}(\kappa)/T^2$ and by $[[\underline{s}, q, \underline{x}, x_{n+1}]]$, when we are interested in the complex structure on $\mathbb{F}$. □

The manifold $(\mathbb{F}, \omega_{\mathbb{F}}^{\kappa})$ admits an action by the quotient torus $T^{2n+4}/T^2 \cong T^{2n+2}$. Hence it can be represented by a Delzant polytope Δ^{κ} of dimension $2n+2$. If $p : \mathbb{R}^{2n+4} \to \mathbb{R}^2$ is the linear map given by the matrix (3.1), then

$$\begin{aligned} \Delta^{\kappa} &= \mathbb{R}_+^{2n+4} \cap p^{-1}(\kappa) \\ &= \left\{ (\underline{\xi}, \nu, \underline{\eta}, \eta_{n+1}) \in \mathbb{R}_+^{2n+4} \mid \nu + \textstyle\sum_{j=0}^{n} \xi_j = \kappa_1,\ \nu + \sum_{j=0}^{n+1} \eta_j = \kappa_2 \right\} \end{aligned} \tag{3.2}$$

where $\underline{\xi} = (\xi_0, \ldots, \xi_n)$, $\underline{\eta} = (\eta_0, \ldots, \eta_n)$ and $\kappa = (\kappa_1, \kappa_2)$. Let $A : \mathbb{C}^{2n+4} \to$

$\mathbb{R}^{2n+4}$ be the map

$$A(\underline{s}, q, \underline{x}, x_{n+1}) = (|s_0|^2, \ldots, |s_n|^2, |q|^2, |x_0|^2, \ldots, |x_{n+1}|^2).$$

Then $\Psi^{-1}(\kappa) = A^{-1}(\Delta^\kappa)$. Suppose we have a d-dimensional face of Δ^κ. Then it is given by the vanishing of $2n+4-d$ coordinates in $\mathbb{R}^{2n+4}$. The vanishing of the corresponding coordinates in $\mathbb{C}^{2n+4}$ is a T^2-invariant complex submanifold of $\Psi^{-1}(\kappa)$ and corresponds to a d-dimensional complex submanifold of $\mathbb{F} = \Psi^{-1}(\kappa)/T^2$. Two faces of Δ^κ intersect precisely when the corresponding submanifolds intersect in $\mathbb{F}$.

Lemma 3.1.2. *The manifold $\mathbb{F}$ is a $\mathbb{CP}^{n+1}$ bundle over $\mathbb{CP}^{n+1}$ with fiber coordinates $[s_0 : \cdots : s_n : q]$ and base coordinates $[x_0 : \cdots : x_{n+1}]$. The form $\omega_{\mathbb{F}}^\kappa$ integrates to $\kappa_1\pi$ on a line in the fiber and to $\kappa_2\pi$ on a line in the section $s_0 = \cdots = s_{n-1} = q = 0$. Moreover, the submanifold $\mathbb{F}\cap(q=0)$ is isomorphic to the product $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ and for each $\kappa = (\kappa_1, \kappa_2) \in C_1$, the restriction $\omega_{\mathbb{F}}^\kappa |_{\mathbb{F}\cap(q=0)}$ is the product form $\kappa_1\sigma_{\mathbb{CP}^n} + \kappa_2\sigma_{\mathbb{CP}^{n+1}}$.*

For $n = 1$ the polytope representing $(\mathbb{F}, \omega_{\mathbb{F}}^\kappa)$ is illustrated in Figure 3.2. The polytope that represents $(\mathbb{F}\cap(x_0 = 0), \omega_{\mathbb{F}}^\kappa |_{x_0=0})$ can be illustrated in a 3-dimensional subspace of $\mathbb{R}^6$ as seen on the left hand side of Figure 3.2. This is a $\mathbb{CP}^2$-bundle over $\mathbb{CP}^1$ with base coordinates $[x_1 : x_2]$ and fiber coordinates $[s_0 : s_1 : q]$. We get the polytope associated to $(\mathbb{F}, \omega_{\mathbb{F}}^\kappa)$ by replacing each section $\mathbb{CP}^1$ by a $\mathbb{CP}^2$ as shown in the right hand side of the figure.

Proof of Lemma 3.1.2. First we prove the statement about $\mathbb{F}\cap(q=0)$. We can think of $\mathbb{F}\cap(q=0)$ as $(\Psi |_{(q=0)})^{-1}(\kappa)/T^2$ which is by construction $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$

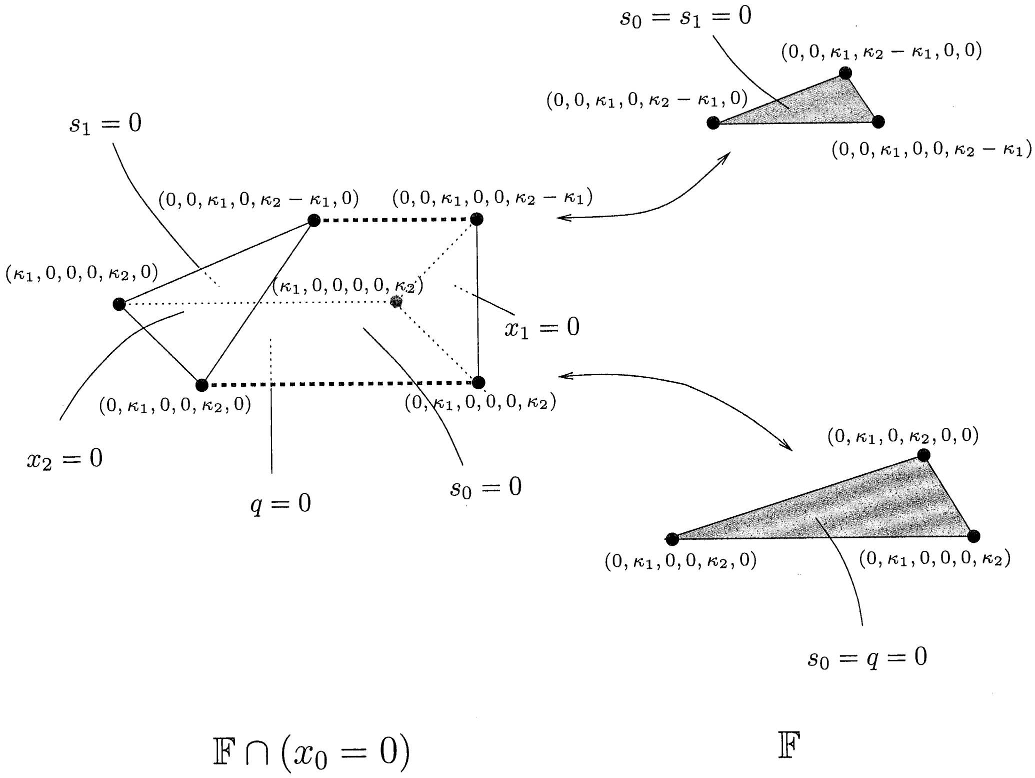

Figure 3.2: Polytope Δ^κ representing $(\mathbb{F}, \omega_{\mathbb{F}}^\kappa)$ for $n = 1$

with symplectic form $\kappa_1 \sigma_{\mathbb{CP}^n} + \kappa_2 \sigma_{\mathbb{CP}^{n+1}}$. More precisely, the map

$$(\mathbb{F} \cap (q = 0), \omega_{\mathbb{F}}^\kappa |_{(q=0)}) \to (\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \kappa_1 \sigma_{\mathbb{CP}^n} + \kappa_2 \sigma_{\mathbb{CP}^{n+1}})$$

given by

$$[\underline{s}, 0, \underline{x}, x_{n+1}] \mapsto ([s_0 : \cdots : s_n], [x_0 : \cdots : x_{n+1}])$$

is a biholomorphic symplectomorphism.

The map $\mathbb{F} \to \mathbb{CP}^{n+1}$ given by $[[\underline{s}, q, \underline{x}, x_{n+1}]] \mapsto [x_0 : \cdots : x_{n+1}]$ is clearly well-defined and makes $\mathbb{F}$ a $\mathbb{CP}^{n+1}$ bundle over $\mathbb{CP}^{n+1}$.

The line $q = s_2 = \cdots = s_n = x_0 = \cdots = x_n = 0$ in the fiber $x_0 = \cdots =$

$x_n = 0$ lies in $(\mathbb{F} \cap (q = 0), \omega_{\mathbb{F}} \mid_{(q=0)})$ and corresponds to

$$\{([s_0 : s_1 : 0 : \cdots : 0], [0 : \cdots : 0 : 1]) \in \mathbb{CP}^n \times \mathbb{CP}^{n+1}\}.$$

Similarly, the line $q = s_0 = \cdots = s_{n-1} = x_0 = \cdots = x_{n-1}$ in the section $q = s_0 = \cdots = s_{n-1} = 0$ corresponds to

$$\{([0 : \cdots : 0 : 1], [0 : \cdots : 0 : x_n : x_{n+1}]) \in \mathbb{CP}^n \times \mathbb{CP}^{n+1}\}.$$

It follows that $\omega_{\mathbb{F}}^{\kappa}$ integrates to $\kappa_1 \pi$ on $q = s_2 = \cdots = s_n = x_0 = \cdots = x_n = 0$ and to $\kappa_2 \pi$ on $q = s_0 = \cdots = s_{n-1} = x_0 = \cdots = x_{n-1} = 0$. □

3.2 The singular fibration $\pi : \mathcal{X} \to \mathbb{C}$

Let

$$\mathcal{X} = \{([[\underline{s}, q, \underline{x}, x_{n+1}]], t) \in \mathbb{F} \times \mathbb{C} \mid s_0 x_0 + \cdots + s_n x_n = tq\}.$$

The holomorphic equation $s_0 x_0 + \cdots + s_n x_n = tq$ is invariant under the $(\mathbb{C}^*)^2$-action on $\mathbb{C}^{2n+4}$ and $\mathcal{X}$ is a complex submanifold of $\mathbb{F} \times \mathbb{C}$. Let

$$X_t = \{[[\underline{s}, q, \underline{x}, x_{n+1}]] \in \mathbb{F} \mid s_0 x_0 + \cdots + s_n x_n = tq\}$$

and $\mathcal{X}_t = X_t \times \{t\}$. Let $\pi : \mathcal{X} \to \mathbb{C}$ be the natural projection map. Figure 3.3 shows how the submanifolds X_t lie inside the polytope representing $(\mathbb{F} \cap (x_0 = 0), \omega_{\mathbb{F}}^{\kappa} \mid_{x_0=0})$ when $n = 1$.

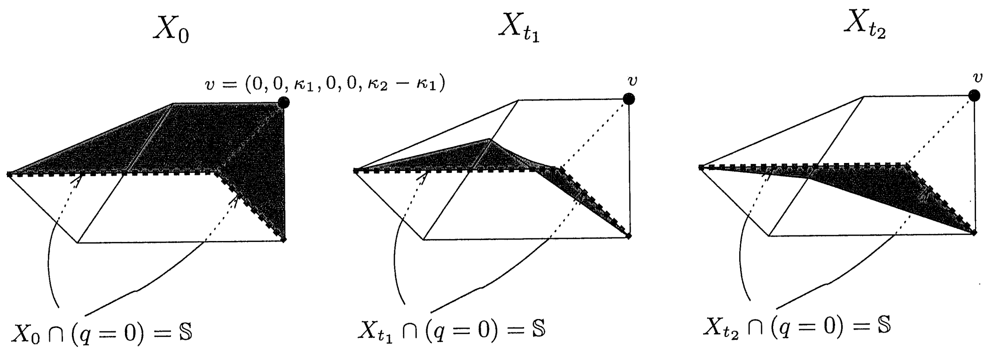

Figure 3.3: X_t in $\mathbb{F} \cap (x_0 = 0)$

Consider the submanifold

$$\mathbb{S} = \{[[\underline{s}, q, \underline{x}, x_{n+1}]] \in \mathbb{F} \mid s_0 x_0 + \cdots + s_n x_n = 0 \text{ and } q = 0\} \tag{3.3}$$

in $\mathbb{F}$. Clearly,

$$X_t \cap (q = 0) = \mathbb{S} \quad \text{for all } t \in \mathbb{C}.$$

When $q \neq 0$, all the X_t's are disjoint in $\mathbb{F}$ and

$$\mathbb{F} \cap (q \neq 0) = \bigsqcup_{t \in \mathbb{C}} X_t \setminus \mathbb{S}.$$

More precisely, if $\pi_{\mathbb{F}} : \mathbb{F} \times \mathbb{C} \to \mathbb{F}$ denotes projection to the first factor, then

$$\pi_{\mathbb{F}} \mid_{\mathcal{X} \setminus (\mathbb{S} \times \mathbb{C})} : \mathcal{X} \setminus (\mathbb{S} \times \mathbb{C}) \to \mathbb{F} \cap (q \neq 0) \tag{3.4}$$

has inverse $(\pi_{\mathbb{F}} \mid_{\mathcal{X} \setminus (\mathbb{S} \times \mathbb{C})})^{-1} : \mathbb{F} \cap (q \neq 0) \to \mathcal{X} \setminus (\mathbb{S} \times \mathbb{C})$ given by

$$[[\underline{s}, q, \underline{x}, x_{n+1}]] \mapsto \left([[\underline{s}, q, \underline{x}, x_{n+1}]], \frac{s_0 x_0 + \cdots + s_n x_n}{q} \right).$$

In particular, $\pi_{\mathbb{F}}\,|_{\mathcal{X}\setminus(\mathbb{S}\times\mathbb{C})}$ is biholomorphic.

Lemma 3.2.1. *The singular fibration $\pi : \mathcal{X} \to \mathbb{C}$ is holomorphic and has exactly one critical point at*

$$z_{\mathrm{crit}} = ([[\underline{0}, 1, \underline{0}, 1]], 0) \in \mathcal{X}_0.$$

In a neighborhood of z_{crit} we obtain holomorphic coordinates $(z_1, \dots, z_{2n+2})$ such that π is the map $(z_1, \dots, z_{2n+2}) \mapsto \sum_{j=1}^{2n+2} z_j^2$.

The non-singular fibers $\mathcal{X}_t$, $t \neq 0$, are biholomorphic to $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$.

Proof. Clearly, $\pi : \mathcal{X} \to \mathbb{C}$ is a holomorphic fibration with exactly one critical point z_{crit}.

For $q \neq 0$ and $x_{n+1} \neq 0$,

$$\tilde{s}_i = \frac{s_i x_{n+1}}{q}, \quad \tilde{x}_j = \frac{x_j}{x_{n+1}} \quad \text{for } 0 \leq i, j \leq n$$

are holomorphic and invariant under the $(\mathbb{C}^*)^2$-action. The map

$$\mathbb{F} \cap (q \neq 0, x_{n+1} \neq 0) \to \mathbb{C}^{2n+2}$$

given by

$$[[\underline{s}, q, \underline{x}, x_{n+1}]] \mapsto (\tilde{s}_0, \dots, \tilde{s}_n, \tilde{x}_0, \dots, \tilde{x}_n)$$

defines a holomorphic coordinate chart on $\mathbb{F}$. Thus, $(\tilde{s}_0, \dots, \tilde{s}_n, \tilde{x}_0, \dots, \tilde{x}_n, t)$ define holomorphic coordinates on $\mathbb{F} \times \mathbb{C}$ in which $\mathcal{X}$ is given by $\tilde{s}_0\tilde{x}_0 + \dots + \tilde{s}_n\tilde{x}_n = t$. Therefore $(\tilde{s}_0, \dots, \tilde{s}_n, \tilde{x}_0, \dots, \tilde{x}_n)$ define holomorphic coordinates on

$\mathcal{X}$. Let

$$\begin{array}{c} z_1 = \frac{1}{2}(\tilde{s}_0 + \tilde{x}_0), \ \dots, \ z_{n+1} = \frac{1}{2}(\tilde{s}_n + \tilde{x}_n), \\ z_{n+2} = \frac{1}{2i}(\tilde{s}_0 - \tilde{x}_0), \ \dots, \ z_{2n+2} = \frac{1}{2i}(\tilde{s}_n - \tilde{x}_n) \end{array} . \tag{3.5}$$

Then $(z_1, \dots, z_{2n+2})$ define holomorphic coordinates on $\mathcal{X}$ in which π has the form $(z_1, \dots, z_{2n+2}) \mapsto \sum_{j=1}^{2n+2} z_j^2$.

Let $\mathcal{X}_t$ be a non-singular fiber of $\pi : \mathcal{X} \to \mathbb{C}$, i.e., fix $t \neq 0$. Then X_t is a complex submanifold of $\mathbb{F}$ in which q is determined by the other coordinates of $\mathbb{F}$ as $q = \frac{s_0 x_0 + \cdots + s_n x_n}{t}$. Since $\mathbb{F} = (\mathbb{C}^{n+2} \setminus \{0\} \times \mathbb{C}^{n+2} \setminus \{0\})/(\mathbb{C}^*)^2$, an element $[[\underline{s}, q, \underline{x}, x_{n+1}]] \in X_t$ cannot have $\underline{s} = 0$ or $\underline{x} = x_{n+1} = 0$. It follows that the map $\Phi_t : \mathcal{X}_t \to \mathbb{CP}^n \times \mathbb{CP}^{n+1}$ given by

$$\Phi_t : ([[\underline{s}, q, \underline{x}, x_{n+1}]], t) \mapsto ([s_0 : \cdots : s_n], [x_0 : \cdots : x_{n+1}]) \tag{3.6}$$

is well-defined. In fact, it is holomorphic. □

Remark 3.2.2. In the notation of Remark 3.1.1,

$$z_{\text{crit}} = (z, 0) = ([\underline{0}, \kappa_1, \underline{0}, \kappa_2 - \kappa_1], 0).$$

In the polytope $\Delta^\kappa \subset \mathbb{R}^{2n+4}$ representing $(\mathbb{F}, \omega_{\mathbb{F}}^\kappa)$, $z \in \mathbb{F}$ corresponds to the vertex $v = (\underline{0}, \kappa_1, \underline{0}, \kappa_2 - \kappa_1)$. (See Figure 3.3 for the case $n = 1$). □

3.3 The extension $\pi : \overline{\mathcal{X}} \to \mathbb{CP}^1$

Regard $\mathbb{CP}^1$ as $\mathbb{C} \sqcup \{\infty\}$.

Lemma 3.3.1. *$\pi : \mathcal{X} \to \mathbb{C}$ extends to a fibration $\pi : \overline{\mathcal{X}} \to \mathbb{CP}^1$ by adding the* *fiber* $\mathcal{X}_\infty = X_\infty \times \{\infty\}$, *where*

$$X_\infty = \mathbb{F} \cap (q = 0).$$

Moreover, when the singular fiber $\mathcal{X}_0$ *is removed,* $\pi : \overline{\mathcal{X}} \to \mathbb{CP}^1$ *is trivialized by a biholomorphic map* $\Phi : \overline{\mathcal{X}} \setminus \mathcal{X}_0 \to (\mathbb{CP}^n \times \mathbb{CP}^{n+1}) \times \mathbb{C}$.

Proof. Let

$$\mathcal{Y} = \{([[\underline{s}, q, \underline{x}, x_{n+1}]], t') \in \mathbb{F} \times \mathbb{C} \mid t'(s_0 x_0 + \cdots + s_n x_n) = q\}.$$

This is a complex manifold that fibers over $\mathbb{C}$ as a subbundle of the trivial bundle $\mathbb{F} \times \mathbb{C} \to \mathbb{C}$. The fibers are $\mathcal{Y}_{t'} = Y_{t'} \times \{t'\}$ where

$$Y_{t'} = X_{\frac{1}{t'}} \quad \text{for all } t' \in \mathbb{C}.$$

Let Φ_t be defined as in (3.6) and let $\Phi_\infty : \mathcal{Y}_0 \to \mathbb{CP}^n \times \mathbb{CP}^{n+1}$ be the map

$$\Phi_\infty([[\underline{s}, 0, \underline{x}, x_{n+1}]], 0) = ([s_0 : \cdots : s_n], [x_0 : \cdots : x_{n+1}]).$$

The bundle $\mathcal{Y} \to \mathbb{C}$ is then trivialized by the biholomorphic map

$$\Phi : \mathcal{Y} \to (\mathbb{CP}^n \times \mathbb{CP}^{n+1}) \times \mathbb{C}$$

given by

$$\Phi([[\underline{s}, q, \underline{x}, x_{n+1}]], t') = (\Phi_{\frac{1}{t'}}([[\underline{s}, q, \underline{x}, x_{n+1}]], t'), t') \quad \text{for } t' \in \mathbb{C} \tag{3.7}$$

By identifying the fiber $\mathcal{Y}_{t'}$ with $\mathcal{X}_{\frac{1}{t'}}$ for all $t' \neq 0$, we get the desired fibration $\pi : \overline{\mathcal{X}} \longrightarrow \mathbb{CP}^1$ which is a subbundle of the fibration $\mathbb{F} \times \mathbb{CP}^1 \longrightarrow \mathbb{CP}^1$. Note that the trivialization $\Phi : \mathcal{Y} \longrightarrow (\mathbb{CP}^n \times \mathbb{CP}^{n+1}) \times \mathbb{C}$ is a trivialization of $\overline{\mathcal{X}}$ with the singular fiber $\mathcal{X}_0$ removed. □

Remark 3.3.2. The submanifold $\mathbb{S}$ defined in (3.3) lies in X_∞ and

$$\mathbb{F} \setminus \mathbb{S} = (X_\infty \setminus \mathbb{S}) \sqcup \bigsqcup_{t \in \mathbb{C}} X_t \setminus \mathbb{S};$$

see Figures 3.3 and 3.4. Furthermore, just as in (3.4), we see that

$$\pi_{\mathbb{F}} \mid_{\overline{\mathcal{X}} \setminus (\mathbb{S} \times \mathbb{CP}^1)} : \overline{\mathcal{X}} \setminus (\mathbb{S} \times \mathbb{CP}^1) \longrightarrow \mathbb{F} \setminus \mathbb{S}$$

is biholomorphic. Here we have extended $\pi_{\mathbb{F}}$ to mean projection to the first factor $\mathbb{F} \times \mathbb{CP}^1 \longrightarrow \mathbb{F}$. □

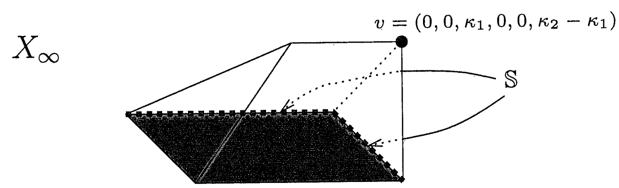

Figure 3.4: X_∞ in $\mathbb{F} \cap (x_0 = 0)$

For each $\mu > 1$ we now introduce a 2-form Ω^μ on $\overline{\mathcal{X}}$ that makes $(\overline{\mathcal{X}}, \pi, J)$ into a Lefschetz fibration. Here J is the complex structure on $\overline{\mathcal{X}}$ coming from $\mathbb{F} \times \mathbb{CP}^1$. Recall that if $\kappa = (\kappa_1, \kappa_2)$ lies in the cone C_1, then we have a toric Kähler structure $\omega_{\mathbb{F}}^{\kappa}$ on $\mathbb{F}$. If $\mu > 1$, then $(1, \mu) \in C_1$ and we define

$$\Omega^\mu = (\pi_{\mathbb{F}} \mid_{\overline{\mathcal{X}}})^* (\omega_{\mathbb{F}}^{(1,\mu)}). \tag{3.8}$$

Proposition 3.3.3. *For any $\mu > 1$, $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ is a Lefschetz fibration whose non-singular fibers are symplectically isotopic to $(\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma_\mu)$. Moreover, the map*

$$\Phi_\infty : (\mathcal{X}_\infty, \Omega^\mu \mid_{\mathcal{X}_\infty}) \to (\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma_\mu)$$

described by (3.7) is a symplectomorphism.

Proof. Let $\mu > 1$. First we prove that $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ is a Lefschetz fibration. Clearly, Ω^μ is closed. Its restriction to each non-singular fiber $\mathcal{Y}_{t'}$ is

$$\Omega^\mu \mid_{\mathcal{Y}_{t'}} = \omega_{\mathbb{F}}^{(1,\mu)} \mid_{Y_{t'}},$$

which is symplectic since all the $Y_{t'}$'s are complex submanifolds of $\mathbb{F}$. We have already shown in Lemma 3.2.1 that π has the form $(z_1, \ldots, z_{2n+2}) \mapsto \sum_{j=1}^{2n+2} z_j^2$ in a neighborhood of the critical point z_{crit}. By Remark 3.3.2 and (3.8), $\pi_{\mathbb{F}} \mid_{\overline{\mathcal{X}} \setminus (\mathbb{S} \times \mathbb{CP}^1)}$ is a biholomorphic map that identifies $(\overline{\mathcal{X}} \setminus (\mathbb{S} \times \mathbb{CP}^1), \Omega^\mu)$ and $(\mathbb{F} \setminus \mathbb{S}, \omega_{\mathbb{F}}^{(1,\mu)})$. Hence Ω^μ is J-Kähler in a neighborhood of the critical point. We conclude that $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ is a Lefschetz fibration.

It remains to show that the non-singular fibers of $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ are symplectically isotopic to $(\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma_\mu)$, i.e., that the forms

$$\alpha_{t'} = (\Phi_{\frac{1}{t'}}^{-1})^* (\omega_{\mathbb{F}}^{(1,\mu)} \mid_{Y_{t'}}) \quad \text{for } t' \in \mathbb{C}$$

are symplectically isotopic to σ_μ for all t'. Since all the $\alpha_{t'}$ are symplectically isotopic, it suffices to prove it for $t' = 0$. But we have already seen in Lemma 3.1.2 that $\alpha_0 = \sigma_\mu$ by construction. □

3.4 The vanishing cycle of $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$

In order to compute the vanishing cycle of $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$, we need to understand the connection coming from the two-form Ω^μ in a neighborhood of the node z_{crit}. By Remark 3.3.2 and (3.8), $\pi_{\mathbb{F}} \mid_{\overline{\mathcal{X}} \setminus (\mathbb{S} \times \mathbb{CP}^1)}$ identifies the neighborhood $(\overline{\mathcal{X}} \setminus (\mathbb{S} \times \mathbb{CP}^1), \Omega^\mu)$ with $(\mathbb{F} \setminus \mathbb{S}, \omega_{\mathbb{F}}^{(1,\mu)})$. Hence Ω^μ is symplectic on $\overline{\mathcal{X}} \setminus (\mathbb{S} \times \mathbb{CP}^1)$ and we can use the toric structure of $\mathbb{F}$ to describe the following Darboux chart:

Lemma 3.4.1. *Let*

$$W^\mu = \{(\underline{s}, \underline{x}) \in \mathbb{C}^{2n+2} \mid |\underline{s}|^2 < 1, \ -|\underline{s}|^2 + |\underline{x}|^2 < \mu - 1\},$$

where $(\underline{s}, \underline{x}) = (s_0, \dots, s_n, x_0, \dots, x_n)$, $|\underline{s}|^2 = \sum_j |s_j|^2$ *and* $|\underline{x}|^2 = \sum_j |x_j|^2$.

The map

$$\psi^\mu_{\overline{\mathcal{X}}} : W^\mu \to \overline{\mathcal{X}} \cap (q \neq 0) \cap (x_{n+1} \neq 0)$$

given by

$$(\underline{s}, \underline{x}) \mapsto \left([\underline{s}, \sqrt{1 - |\underline{s}|^2}, \underline{x}, \sqrt{\mu - 1 + |\underline{s}|^2 - |\underline{x}|^2}], \frac{s_0 x_0 + \dots + s_n x_n}{\sqrt{1 - |\underline{s}|^2}} \right)$$

defines a Darboux chart on $(\overline{\mathcal{X}} \setminus (\mathbb{S} \times \mathbb{CP}^1), \Omega^\mu)$ *such that* $\psi^\mu_{\overline{\mathcal{X}}}(\underline{0}, \underline{0}) = z_{\text{crit}}$.

Proof. As noted in the Remark 3.2.2, $z_{\text{crit}} = (z, 0) \in \mathbb{F} \times \mathbb{C}$ where

$$z = [[\underline{0}, 1, \underline{0}, 1]] = [\underline{0}, 1, \underline{0}, \sqrt{\mu - 1}].$$

By the comments preceding this lemma, it suffices to see that the map $\psi^\mu_{\mathbb{F}}$:

$W^\mu \to \mathbb{F}$, where

$$\psi^\mu_{\mathbb{F}}(\underline{s}, \underline{x}) = [\underline{s}, \sqrt{1-|\underline{s}|^2}, \underline{x}, \sqrt{\mu - 1 + |\underline{s}|^2 - |x|^2}],$$

is a Darboux chart on $(\mathbb{F} \cap (q \neq 0) \cap (x_{n+1} \neq 0), \omega_{\mathbb{F}}^{(1,\mu)})$ centered at z.

We have the following commutative diagram

$$\begin{array}{ccc} W^\mu & \xrightarrow{\widetilde{\psi}^\mu} & \Psi^{-1}((1,\mu)) \subset \mathbb{C}^{2n+4} \\ & {\scriptstyle \psi^\mu_{\mathbb{F}}} \searrow & \downarrow \\ & & \mathbb{F} = \Psi^{-1}((1,\mu))/T^2, \end{array}$$

where $\widetilde{\psi}^\mu$ is given by $(\underline{s}, \underline{x}) \mapsto (\underline{s}, \sqrt{1-|\underline{s}|^2}, \underline{x}, \sqrt{\mu - 1 + |\underline{s}|^2 - |x|^2})$. Hence it suffices to see that $\widetilde{\psi}^\mu$ pulls back $\omega_{\mathbb{C}^{2n+4}}$ to $\omega_{\mathbb{C}^{2n+2}}$. But this is clear since $q, x_{n+1} \in \mathbb{R}$ on $\mathrm{Im}(\widetilde{\psi}^\mu)$, and therefore $dq \wedge d\overline{q}$ and $dx_{n+1} \wedge d\overline{x}_{n+1}$ pull back to 0 under $\widetilde{\psi}^\mu$.

□

Remark 3.4.2. Each vertex of the polytope $\Delta^{(1,\mu)}$ gives a natural Darboux chart on $(\mathbb{F}, \omega_{\mathbb{F}}^{(1,\mu)})$, and the chart described in Lemma 3.4.1 is the one corresponding to $v = (\underline{0}, 1, \underline{0}, \mu - 1)$. By (3.2), an element $(\underline{\xi}, \nu, \underline{\eta}, \eta_{n+1})$ of $\Delta^{(1,\mu)}$ satisfies the equations

$$\nu = 1 - \sum_{j=0}^{n} \xi_j, \quad \eta_{n+1} = \mu - 1 + \sum_{j=0}^{n} (\xi_j - \eta_j).$$

Hence the projection $(\underline{\xi}, \nu, \underline{\eta}, \eta_{n+1}) \mapsto (\underline{\xi}, \underline{\eta})$ maps $\Delta^{(1,\mu)}$ isomorphically to a polytope $\widetilde{\Delta}^{(1,\mu)}$ in $\mathbb{R}^{2n+2}$. Here the vertex v is mapped to 0. Let $F^{q,x_{n+1}}$ denote the union of the facets $\nu = 0$ and $\eta_{n+1} = 0$ in $\Delta^{(1,\mu)}$ and let $\widetilde{F}^{q,x_{n+1}}$

be the corresponding facets in $\widetilde{\Delta}^{(1,\mu)}$. If $\widetilde{A} : \mathbb{C}^{2n+2} \to \mathbb{R}^{2n+2}$ is the map $(\underline{s}, \underline{x}) \mapsto (|s_0|^2, \ldots, |s_n|^2, |x_0|^2, \ldots, |x_n|^2)$, then $W^\mu = \widetilde{A}^{-1}(\widetilde{\Delta}^{(1,\mu)} \setminus \widetilde{F}^{q,x_{n+1}})$ and we have the following commutative diagram:

$$\begin{array}{ccccc}
\mathbb{C}^{2n+2} \supset & W^\mu & \xrightarrow{\widetilde{\psi}^\mu} & A^{-1}(\Delta^{(1,\mu)} \setminus F^{q,x_{n+1}}) & \subset \mathbb{C}^{2n+4} \\
 & \Big\downarrow \widetilde{A} & & \Big\downarrow A & \\
\mathbb{R}^{2n+2} \supset & \widetilde{\Delta}^{(1,\mu)} \setminus \widetilde{F}^{q,x_{n+1}} & \xrightarrow{\cong} & \Delta^{(1,\mu)} \setminus F^{q,x_{n+1}} & \subset \mathbb{R}^{2n+4},
\end{array}$$

where $\widetilde{\psi}^\mu$ is the map described in the proof of Lemma 3.4.1. □

By Lemma 3.4.1, we have a commutative diagram of singular fibrations:

$$\begin{array}{ccc}
(W^\mu, \omega_{\mathbb{C}^{2n+2}}) & \xrightarrow[\cong]{\psi^\mu_{\mathcal{X}}} & (\mathcal{X} \cap (q \neq 0) \cap (x_{n+1} \neq 0), \Omega^\mu) \\
 & \pi_{\text{loc}} \searrow \quad \swarrow \pi & \\
 & \mathbb{C} &
\end{array}$$

where

$$\pi_{\text{loc}}(\underline{s}, \underline{x}) = \frac{s_0 x_0 + \cdots + s_n x_n}{\sqrt{1 - |\underline{s}|^2}}.$$

The fiber $W^\mu_t = \pi^{-1}_{\text{loc}}(t)$ in W^μ is given by

$$W^\mu_t = \{(\underline{s}, \underline{x}) \in W^\mu \mid f_1(\underline{s}, \underline{x}) = f_2(\underline{s}, \underline{x}) = 0\},$$

where

$$\begin{aligned}
f_1(\underline{s}, \underline{x}) &= \sum_{j=0}^{n} (\Re(s_j)\Re(x_j) - \Im(s_j)\Im(x_j)) - \Re(t)\sqrt{1 - |\underline{s}|^2}, \\
f_2(\underline{s}, \underline{x}) &= \sum_{j=0}^{n} (\Im(s_j)\Re(x_j) + \Re(s_j)\Im(x_j)) - \Im(t)\sqrt{1 - |\underline{s}|^2}.
\end{aligned}$$

Lemma 3.4.3. *Consider the fibration* $\pi_{\text{loc}} = \pi \circ \psi^{\mu}_{\underline{x}} : W^{\mu} \to \mathbb{C}$ *with connection coming from the standard symplectic form* $\omega_{\mathbb{C}^{2n+2}}$ *on* W^{μ}*. Let* $t > 0$*. If* $(\underline{s}, \underline{x}) \in \pi^{-1}(t)$*, then* $(\overline{\underline{x}}, \overline{\underline{s}})$ *lies in horizontal space* $(T_{(\underline{s},\underline{x})} W^{\mu}_t)^{\omega_{\mathbb{R}^{4n+4}}}$*.*

Proof. The horizontal space at $(\underline{s}, \underline{x}) \in W^{\mu}_t$ is the symplectic complement of $T_{(\underline{s},\underline{x})} W^{\mu}_t$ in $(\mathbb{C}^{2n+2}, \omega_{\mathbb{C}^{2n+2}}) \cong (\mathbb{R}^{4n+4}, \omega_{\mathbb{R}^{4n+4}})$. We have that

$$T_{(\underline{s},\underline{x})} W^{\mu}_t = (\text{span}\, \nabla f_1(\underline{s}, \underline{x}))^{\perp} \cap (\text{span}\, \nabla f_2(\underline{s}, \underline{x}))^{\perp}$$

which has symplectic complement

$$(T_{(\underline{s},\underline{x})} W^{\mu}_t)^{\omega_{\mathbb{R}^{4n+4}}} = \text{span}\{J_0 \nabla f_1(\underline{s}, \underline{x}), J_0 \nabla f_2(\underline{s}, \underline{x})\}.$$

Hence the horizontal space is spanned by the vectors

$$\begin{pmatrix} \Im(x_0) - \Re(t)\frac{\Im(s_0)}{\sqrt{1-|\underline{s}|^2}} \\ \Re(x_0) + \Re(t)\frac{\Re(s_0)}{\sqrt{1-|\underline{s}|^2}} \\ \vdots \\ \Im(x_n) - \Re(t)\frac{\Im(s_n)}{\sqrt{1-|\underline{s}|^2}} \\ \Re(x_n) + \Re(t)\frac{\Re(s_n)}{\sqrt{1-|\underline{s}|^2}} \\ \Im(s_0) \\ \Re(s_0) \\ \vdots \\ \Im(s_n) \\ \Re(s_n) \end{pmatrix}, \begin{pmatrix} -\Re(x_0) - \Im(t)\frac{\Im(s_0)}{\sqrt{1-|\underline{s}|^2}} \\ \Im(x_0) + \Im(t)\frac{\Re(s_0)}{\sqrt{1-|\underline{s}|^2}} \\ \vdots \\ -\Re(x_n) - \Im(t)\frac{\Im(s_n)}{\sqrt{1-|\underline{s}|^2}} \\ \Im(x_n) + \Im(t)\frac{\Re(s_n)}{\sqrt{1-|\underline{s}|^2}} \\ -\Re(s_0) \\ \Im(s_0) \\ \vdots \\ -\Re(s_n) \\ \Im(s_n) \end{pmatrix}.$$

$\square$

Using this, we are able to compute the vanishing cycle of $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$.

Proposition 3.4.4. *Let $t > 0$. The vanishing cycle of $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ in the fiber $\mathcal{X}_t$ is isotopic to*

$$\mathcal{V}_t^\mu = \left\{([\overline{\underline{x}}, \sqrt{1-g(t)}, \underline{x}, \sqrt{\mu - 1}], t) \in \mathcal{X} \mid |\underline{x}|^2 = g(t)\right\}$$

where $\overline{\underline{x}} = (\overline{x}_0, \ldots, \overline{x}_n)$ and

$$g(t) = \frac{-t^2 + \sqrt{t^4 + 4t^2}}{2}. \tag{3.9}$$

The vanishing cycle in $\mathcal{X}_\infty$ is isotopic to

$$\mathcal{V}_\infty^\mu = \left\{([\overline{\underline{x}}, 0, \underline{x}, \sqrt{\mu - 1}], 0) \in \mathcal{Y} \mid |\underline{x}|^2 = 1\right\}.$$

Proof. For each $t > 0$, define

$$L_t^\mu = \left\{(\overline{\underline{x}}, \underline{x}) \in W^\mu \mid |\underline{x}|^2 = g(t)\right\}.$$

The condition on $|\underline{x}|^2$ ensures that $L_t^\mu \subset W_t^\mu$. We claim that parallel transport along the path $\gamma(r) = r$ for $0 < r \leq t$ takes all of L_t^μ to the node $(\underline{0}, \underline{0}) \in W_0^\mu$. Since L_t^μ is diffeomorphic to S^{2n+1} and

$$\psi_{\overline{\mathcal{X}}}^\mu(L_t^\mu) = \mathcal{V}_t^\mu,$$

this shows that $\mathcal{V}_t^\mu$ is the vanishing cycle in $\mathcal{X}_t$.

Fix $t > 0$ and let $\gamma_t(r) = r$ for $0 < r \leq t$. Given $(\overline{x}, \underline{x}) \in L_t^\mu$, define the path

$$\widetilde{\gamma}_t(r) = h_t(r)(\overline{x}, \underline{x}), \quad 0 < r \leq t$$

where

$$h_t(r) = \sqrt{\frac{g(r)}{g(t)}}.$$

This is a lift of γ_t with endpoints $(\overline{x}, \underline{x})$ and $(\underline{0}, \underline{0})$ and tangents

$$\frac{d}{dr}\widetilde{\gamma}_t(r) = h_t'(r)(\overline{x}, \underline{x}).$$

For all r, $\widetilde{\gamma}_t(r) \in L_r^\mu$, and by Lemma 3.4.3, the horizonal space at $\widetilde{\gamma}_t(r)$ contains $\frac{d}{dr}\widetilde{\gamma}_t(r)$. Hence parallel transport along γ_t takes L_t^μ to the node. Since $\lim_{t\to\infty} g(t) = 1$,

$$\mathcal{V}_\infty^\mu = \mathcal{V}_{\gamma_\infty}^\mu = \lim_{t\to\infty} \mathcal{V}_{\gamma_t}^\mu = \left\{ ([\overline{x}, 0, \underline{x}, \sqrt{\mu - 1}], 0) \in \mathcal{Y} \mid |\underline{x}|^2 = 1 \right\}.$$

□

Chapter 4

Main results and concluding remarks

4.1 Proofs of the main results

We are now in a position to prove the main results described in Chapter 1.

Proof of Theorem 1.0.4. Given $\mu > 1$, Proposition 3.3.3 shows that $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ is a Lefschetz fibration with one critical point

$$z_{\text{crit}} = ([\underline{0}, 1, \underline{0}, \sqrt{\mu - 1}], 0) \in \mathcal{X}_0.$$

The biholomorphic trivialization $\Phi : \overline{\mathcal{X}} \setminus \mathcal{X}_0 :\longrightarrow (\mathbb{CP}^n \times \mathbb{CP}^{n+1}) \times \mathbb{C}$ defined by (3.7) induces a symplectomorphism

$$\Phi_\infty : (\mathcal{X}_\infty, \Omega^\mu \mid_{\mathcal{X}_\infty}) \cong (\mathbb{CP}^n \times \mathbb{CP}^{n+1}, \sigma_\mu)$$

and by Proposition 3.4.4, the vanishing cycle in $\mathcal{X}_\infty$ is isotopic to

$$\mathcal{V}^\mu_\infty = \left\{ ([\overline{\underline{x}}, 0, \underline{x}, \sqrt{\mu - 1}], 0) \in \mathcal{Y} \mid |\underline{x}|^2 = 1 \right\}.$$

Under the symplectomorphism Φ_∞ this corresponds to

$$\Phi_\infty(\mathcal{V}^\mu_\infty) = \{([\overline{x}_0 : \cdots : \overline{x}_n], [x_0 : \cdots : x_n : \sqrt{\mu - 1}]) \in \mathbb{CP}^n \times \mathbb{CP}^{n+1} \mid |\underline{x}|^2 = 1\},$$

which is exactly the Lagrangian L^μ in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$. $\square$

Proof of Theorem 1.0.1. Given $\mu > 1$, we consider the Lefschetz fibration $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ from Theorem 1.0.4. By Proposition 2.2.3, the monodromy $\rho_\ell : \mathcal{X}_\infty \to \mathcal{X}_\infty$ along a positively oriented loop ℓ in $\mathbb{CP}^1$ based at $\infty \in \mathbb{CP}^1$ and circling 0 exactly once is symplectically isotopic to the Dehn twist $\tau_{\mathcal{V}^\mu_\infty}$ along the vanishing cycle $\mathcal{V}^\mu_\infty$. Since the fibration has only one singular fiber, by Proposition 2.2.3 ρ_ℓ is symplectically isotopic to the identity through an isotopy that arises from deforming the loop ℓ to the constant loop based at infinity in $\mathbb{CP}^1$. Hence we have an isotopy $(\varphi_\lambda)_{0 \leq \lambda \leq 1}$ of $\mathcal{X}_\infty$ such that φ_1 equals $\tau_{\mathcal{V}_\infty}$ and φ_0 is the identity. Recall that each fiber of $\overline{\mathcal{X}}$ contains the hypersurface $\mathbb{S} \times \{t\}$, with $\mathbb{S}$ defined in (3.3). Parallel transport in $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ along a path in $\mathbb{CP}^1$ from t_1 to t_2 takes any point $(z, t_1) \in \mathbb{S} \times \{t_1\} \subset \overline{\mathcal{X}}_{t_1}$ to $(z, t_2) \in \mathbb{S} \times \{t_2\} \subset \overline{\mathcal{X}}_{t_2}$. Hence φ_λ restricts to the identity on $\mathbb{S} \times \{\infty\}$ for all λ. Observe that $S = \Phi_\infty(\mathbb{S} \times \{\infty\})$ and that by property 3 of Theorem 1.0.4

$$\tau_{L^\mu} = \Phi_\infty \circ \tau_{\mathcal{V}^\mu_\infty} \circ \Phi_\infty^{-1}.$$

Hence $\Phi_\infty \circ \varphi_\lambda \circ \Phi_\infty^{-1}$, $0 \leq \lambda \leq 1$, is the desired isotopy of τ_{L^μ}. $\square$

Under the projection to the first factor, the complex hypersurface S in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ is a $\mathbb{CP}^n$-bundle over $\mathbb{CP}^n$. The fibers are linearly embedded copies of $\mathbb{CP}^n$ in $\mathbb{CP}^{n+1}$. As before, let

$$\begin{aligned} S_0 &= \mathbb{CP}^n \times \{[0 : \cdots : 0 : 1]\} \\ S_\infty &= \{([s_0 : \cdots : s_n], [x_0 : x_1 : 0 : \cdots : 0]) \mid s_0 x_0 + s_1 x_1 = 0\}; \end{aligned}$$

these are sections of S. If $n = 1$, S is a Hirzebruch surface, S_0 is the small section (with negative self-intersection), and S_∞ is the large section (with positive self-intersection). As stated in Corollary 1.0.2, if we symplectically blow up $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ along S_0 and S_∞, then as long as the blow-ups are small, the isotopy of the Dehn twist τ_{L^μ} remains. On the other hand, Proposition 1.0.3 (which is a consequence of the proof of [CS05, Proposition 2]) shows that an isotopy cannot fix the submanifolds S_0 and S'_∞. Figure 4.1 illustrates how the submanifolds S_0, S_∞ and S'_∞ lie inside the polytope representing $(\mathbb{CP}^1 \times \mathbb{CP}^2, \sigma_\mu)$.

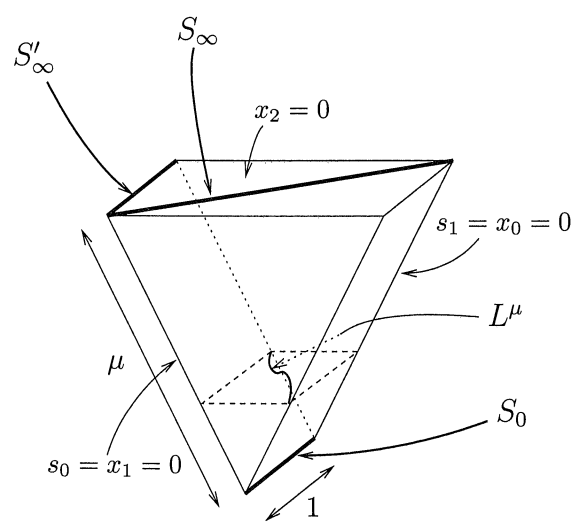

Figure 4.1: Polytope representing $(\mathbb{CP}^1 \times \mathbb{CP}^2, \sigma_\mu)$

Before proving Corollary 1.0.2 and Proposition 1.0.3, we review the symplectic blow-up construction as described in [MS98, Chapter 7]. First we consider the blow-up at the origin in $\mathbb{C}^N$. Roughly speaking, a symplectic blow-up $\widetilde{\mathbb{C}}^N$ is obtained by removing the interior of a (real) $2N$-dimensional ball B and then collapsing the boundary $\partial B \cong S^{2N-1}$ by the Hopf map $S^{2N-1} \to \mathbb{CP}^{N-1}$. The resulting copy of $\mathbb{CP}^{N-1}$ in $\widetilde{\mathbb{C}}^N$ is called the exceptional divisor and the size of the blow-up is the radius of the ball B. Let $\mathcal{L}$ be the tautological line bundle over $\mathbb{CP}^{N-1}$, i.e., the total space of $\mathcal{L}$ is given by

$$\mathcal{L} = \{((z_1, \ldots, z_N), [w_1, \ldots, w_N]) \in \mathbb{C}^N \times \mathbb{CP}^{N-1} \mid w_j z_k = w_k z_j \text{ for all } j, k\}.$$

Let $\mathrm{pr}_{\mathbb{C}^N} : \mathcal{L} \to \mathbb{C}^N$ and $\mathrm{pr}_{\mathbb{CP}^{N-1}} : \mathcal{L} \to \mathbb{CP}^{N-1}$ be the natural projections. For each $\lambda > 0$ the 2-form

$$\omega_{\mathcal{L}}^{\lambda} = \mathrm{pr}_{\mathbb{C}^N}^*(\omega_{\mathbb{C}^N}) + \lambda^2 \mathrm{pr}_{\mathbb{CP}^{N-1}}^*(\sigma_{\mathbb{CP}^{N-1}})$$

is symplectic. For all $\delta \geq 0$, let $\mathcal{L}(\delta) = \mathrm{pr}_{\mathbb{C}^N}^{-1}(B(\delta))$, where $B(\delta)$ is the ball of radius δ in $\mathbb{C}^N$. Note that $\mathcal{L}(0)$ is the zero section in $\mathcal{L}$. By [MS98, Lemma 7.11], for all $\lambda, \delta > 0$ $(\mathcal{L}(\delta) - \mathcal{L}(0), \omega_{\mathcal{L}}^{\lambda})$ is symplectomorphic to the spherical shell $(B(\sqrt{\lambda^2 + \delta^2}) - B(\lambda), \omega_{\mathbb{C}^N})$. The blow-up $\widetilde{\mathbb{C}}^N$ of 0 in $\mathbb{C}^N$ of size λ is obtained by replacing the interior of the ball $B(\sqrt{\lambda^2 + \delta^2})$ in $\mathbb{C}^N$ with $\mathcal{L}(\delta)$. Thus, we have replaced the ball $B(\lambda)$ in $\mathbb{C}^N$ with the zero section $\mathcal{L}(0) \cong \mathbb{CP}^{N-1}$, the exceptional divisor in $\widetilde{\mathbb{C}}^N$. The manifold $\widetilde{\mathbb{C}}^N$ has a natural symplectic structure that is independent of the choice of $\delta > 0$.

If (M^{2N}, ω) is a symplectic manifold and we have a symplectic embedding of the ball $B(\lambda) \subset \mathbb{C}^N$ into M, then we can blow up M by extending the embedding to a ball $B(\lambda+\varepsilon)$ for some $\varepsilon > 0$ and then doing the above construction for some $\delta > 0$ with $\sqrt{\lambda^2+\delta^2}-\lambda < \varepsilon$. One can show that the symplectomorphism class of the resulting manifold only depends on the choice of λ.

If Q is a compact symplectic submanifold in a symplectic manifold (M,ω), then ω can be written in a standard form in a neighborhood of Q. Details can be found in [MS98, p. 249-251]. Suppose Q has codimension $2N$ and let $\nu_{Q|M}$ be the normal bundle of Q in M. Then $\nu_{Q|M}$ has the structure of a $2N$-dimensional symplectic vector bundle with a compatible complex structure J. It is associated to a principal $U(N)$-bundle P, i.e., $\nu_{Q|M} = P \times_{U(N)} \mathbb{C}^N$. This bundle has a symplectic form

$$\omega' = \alpha + \pi_Q^*(\omega\,|_Q), \tag{4.1}$$

where $\pi_Q : \nu_{Q|M} \to Q$ is the natural projection and α is a closed 2-form on $\nu_{Q|M}$ that restricts to the standard symplectic form on the fibers $\mathbb{C}^N$. A small neighborhood of Q in (M,ω) is symplectomorphic to the disk bundle $\nu_{Q|M}^{<\varepsilon} = P \times_{U(N)} B(\varepsilon)$ with symplectic form ω' for some $\varepsilon > 0$. Using the standard form of the symplectic structure near Q, we can define small blow-ups of Q in M. If $\lambda < \varepsilon$, then the size λ blow-up of Q in M is simply the result of blowing up the origin in the fiber $B(\varepsilon)$, i.e. if $\widetilde{B}(\varepsilon)$ is the size λ blow-up of the origin in $B(\varepsilon)$, then the size λ blow-up of Q in M is obtained by removing a neighborhood of Q and glueing in the manifold $P \times_{U(N)} \widetilde{B}(\varepsilon)$ in the obvious way.

Proof of Corollary 1.0.2. Under the trivialization Φ of $(\overline{\mathcal{X}}, \pi, J, \Omega^\mu)$ defined in

(3.7), the section S_0 in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ corresponds to

$$\mathbb{S}_0 = \{[\underline{s}, q, \underline{x}, x_{n+1}] \in \mathbb{F} \mid q = 0,\ \underline{x} = \underline{0}\}.$$

Let $Q = \mathbb{S}_0 \times \mathbb{CP}^1$.

For all $t \in \mathbb{CP}^1$, the fiber $\mathcal{X}_t$ of $\pi : \overline{\mathcal{X}} \to \mathbb{CP}^1$ contains $\mathbb{S}_0 \times \{t\}$ as a symplectic submanifold and the diagram

$$\begin{array}{ccccc} \mathcal{X}_t & \longrightarrow & \overline{\mathcal{X}} & \longrightarrow & \mathbb{CP}^1 \\ \uparrow & & \uparrow & & \| \\ \mathbb{S}_0 \times \{t\} & \longrightarrow & Q & \longrightarrow & \mathbb{CP}^1 \end{array}$$

commutes. It follows that the normal bundle of Q in $\overline{\mathcal{X}}$ can be obtained by glueing the normal bundles of $\mathbb{S}_0 \cong \mathbb{S}_0 \times \{t\}$ in $X_t \cong \mathcal{X}_t$ for all $t \in \mathbb{CP}^1$. Now let $\widetilde{\overline{\mathcal{X}}}$ denote a small symplectic blow-up of Q in $\overline{\mathcal{X}}$ in the neighborhood of Q on which $(\Omega^\mu)'$ has the standard form (4.1). Then each fiber in $\widetilde{\overline{\mathcal{X}}}$ is of the form $\widetilde{X}_t \times \{t\}$, where $\widetilde{X}_t$ is the blow-up of $\mathbb{S}_0$ in X_t. Hence the isotopy of τ_{L^μ} lifts to the blow-up of S_0 in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$.

To see that we can blow up S_∞ in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$, we repeat this construction with $\mathbb{S}_0$ replaced by

$$\mathbb{S}_\infty = \{[\underline{s}, 0, \underline{x}, 0] \in \mathbb{F} \mid x_j = 0 \text{ for } j \in \{2, \ldots, n\},\ s_0 x_0 + s_1 x_1 = 0\}$$

and Q by the complex submanifold $\mathbb{S}_\infty \times \mathbb{CP}^1$ in $\overline{\mathcal{X}}$. □

Proof of Proposition 1.0.3. Assume we have an isotopy $(\varphi_\lambda)_{0 \le \lambda \le 1}$ such that $\varphi_1 = \tau_{L^\mu}$ and φ_λ restricts to the identity on $S_0 \sqcup S'_\infty$ for all λ. We can then deform $(\varphi_\lambda)_{0\le\lambda\le 1}$ to a homotopy $(\widetilde{\varphi}_\lambda)_{0\le\lambda\le 1}$ from τ_{L^μ} to the identity such that each $\widetilde{\varphi}_\lambda$ fixes the neighborhoods

$$\begin{aligned}\mathcal{N}(S_0) &= \mathbb{CP}^n \times B(\varepsilon, p_0) \supset S_0,\\ \mathcal{N}(S'_\infty) &= \mathbb{CP}^n \times B(\varepsilon, p_\infty) \supset S'_\infty,\end{aligned}$$

where $B(\varepsilon, p)$ denotes an embedded size ε (real) $(2n+2)$-ball in $\mathbb{CP}^{n+1}$ centered at $p = p_0, p_\infty$. We show that this leads a to a contradiction as follows. Following [Gom95, p. 534-535], we create a new smooth manifold M by performing a smooth surgery on $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ that takes place in the neighborhoods $\mathcal{N}(S_0)$ and $\mathcal{N}(S_\infty)$. In the manifold M, the Dehn twist corresponds to a diffeomorphism that acts non-trivially on homology but is also homotopic to the identity. This is a contradiction.

Choose a diffeomorphism $\phi : B(\frac{\varepsilon}{2}) \setminus \{0\} \to B(\frac{\varepsilon}{2}) \setminus \{0\}$ of the punctured $(2n+2)$-ball that turns it smoothly inside out, e.g.,

$$\phi : z \mapsto z\sqrt{\frac{\varepsilon^2}{4|z|^2} - 1}.$$

Let $\widehat{\mathbb{CP}}^{n+1}$ be the manifold obtained by removing the points p_0 and p_∞ from $\mathbb{CP}^{n+1}$ and identifying the open sets $B(\frac{\varepsilon}{2}, p_0) \setminus \{p_0\}$ and $B(\frac{\varepsilon}{2}, p_\infty) \setminus \{p_\infty\}$ via the map ϕ. Let

$$M = \mathbb{CP}^n \times \widehat{\mathbb{CP}}^{n+1}.$$

Since the surgery took place on neighborhoods on which the Dehn twist τ_L and the homotopy $(\widetilde{\varphi}_\lambda)_{0\le\lambda\le 1}$ restrict to the identity, τ_L induces a diffeomorphism $\widehat{\tau}_L$ of M homotopic to the identity. The manifold M contains a $(2n+1)$-cycle

C with non-trivial intersection with L. Indeed, let

$$\Gamma = \{[x_0 : 0 : \cdots : 0 : \sqrt{\mu - x_0}] \in \mathbb{CP}^{n+1} \mid x_0 \in [0, \mu]\}$$

be the image of a path from p_0 to p_∞ in $\mathbb{CP}^{n+1}$. Then $\mathbb{CP}^n \times \Gamma$ gives rise to a $(2n+1)$-cycle C in M that intersects the Lagrangian $(2n+1)$-sphere L^μ exactly once, namely at the point

$$([1 : 0 \cdots : 0], [1 : 0 : \cdots : 0 : \sqrt{\mu - 1}]) \in M,$$

and the intersection is transverse. In particular, L represents a non-trivial element in the homology of M. By [ST01, (1.5)], the Dehn twist $\widehat{\tau}_L$ acts on the homology class of C by adding the homology class of L to it. See also Figure 2.1. □

4.2 Concluding remarks

We end this dissertation, by returning once again to the Symplectic Isotopy Problem described in Chapter 1, with a view to finding symplectic manifolds admitting essential symplectomorphisms in dimension 6 and above.

Based on the results of this dissertation, one might hope to construct examples in which an isotopy similar to that of Theorem 1.0.1 does not exist. Of course, even if this is the case, one cannot deduce that no symplectic isotopy exists; for this, one would need to use other techniques such as computing Floer homology. Nevertheless, such examples do provide good candidates for constructions of symplectic manifolds admitting essential symplectomorphisms.

Note first that the construction of Theorem 1.0.4 does not immediately generalize to other examples. One natural attempt is to construct a fibration where the non-singular fiber is a non-trivial $\mathbb{CP}^n$-bundle over $\mathbb{CP}^{n+1}$. Here the natural toric manifold to consider is $\mathbb{F}' = \mathbb{C}^{2n+4}/\!/_{\kappa}(\mathbb{C}^*)^2$, where the action of $(\mathbb{C}^*)^2$ on $\mathbb{C}^{2n+4}$ has weights

$$\begin{pmatrix} \overbrace{1 \ \cdots \ 1}^{m} & \overbrace{1 \ \cdots \ 1}^{n+1-m} & 1 & \overbrace{0 \ \cdots \ 0}^{n+2} \\ 1 \ \cdots \ 1 & 0 \ \cdots \ 0 & 1 & 1 \ \cdots \ 1 \end{pmatrix},$$

and the linearization κ is in the same chamber (see Figure 3.1) as that used to construct $\mathbb{F}$ (this makes sense since the effective cone and its chambers are unchanged). The complex submanifold $q = 0$ in $\mathbb{F}'$ (where once again $(\underline{s}, q, \underline{x}, x_{n+1})$ are the coordinates on $\mathbb{C}^{2n+4}$) is the projectivization of the complex $(n+1)$-bundle

$$\overbrace{\mathcal{O}(1) \oplus \cdots \oplus \mathcal{O}(1)}^{m} \oplus \overbrace{\mathcal{O} \oplus \cdots \oplus \mathcal{O}}^{n+1-m}$$

over $\mathbb{CP}^{n+1}$, i.e., it is a $\mathbb{CP}^n$-bundle over $\mathbb{CP}^{n+1}$. But now there is no longer an obvious way to get a Lefschetz fibration $\overline{\mathcal{X}}' \to \mathbb{CP}^1$ with exactly one singular fiber. For example, consider the complex submanifold $\mathcal{X}'$ in $\mathbb{F}' \times \mathbb{C}$ given by the $(\mathbb{C}^*)^2$-invariant equation

$$\sum_{j=0}^{m-1} s_j + \sum_{l=m}^{n} s_l x_l = tq,$$

where t denotes the coordinate on $\mathbb{C}$ as before. Then the holomorphic fibration $\mathcal{X}' \to \mathbb{C}$ does not have an ordinary double point since the equation cutting out $\mathcal{X}'$ has linear terms.

For non-trivial $\mathbb{CP}^{n+1}$-bundles over $\mathbb{CP}^n$, the construction of Theorem 1.0.4 also seems unlikely to work. If we consider the $(\mathbb{C}^*)^2$-action on $\mathbb{C}^{2n+4}$ with weights

$$\begin{pmatrix} \overbrace{1 \;\cdots\; 1}^{n+1} & 1 & \overbrace{1 \;\cdots\; 1}^{m} & \overbrace{0 \;\cdots\; 0}^{n+2-m} \\ 0 \;\cdots\; 0 & 1 & 1 \;\cdots\; 1 & 1 \;\cdots\; 1 \end{pmatrix}.$$

then in order to ensure that the submanifold $q = 0$ of the resulting toric manifold $\mathbb{F}''$ is a $\mathbb{CP}^{n+1}$-bundle over $\mathbb{CP}^n$, we have to pick the linearization in the chamber C_2 of Figure 3.1. Once again there is no natural way to cut out a hypersurface of $\mathbb{F}'' \times \mathbb{C}$ that fibers over $\mathbb{C}$ in the desired way.

Based on Proposition 1.0.3 it seems likely that the isotopy of τ_{L^μ} in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$ does not lift to the blow-up of S_0 and S'_∞ in $\mathbb{CP}^n \times \mathbb{CP}^{n+1}$. This does not imply that the Dehn twist τ_{L^μ} is essential, as it may not even be smoothly isotopic to the identity, and as mentioned earlier one would also have to prove that no symplectic isotopies exist. An alternative approach to destroying the isotopy of Theorem 1.0.1 would be to perform a large blow-up of the submanifolds S_0 and S_∞ of Corollary 1.0.2. It would be interesting to estimate the bounds of the size of the blow-ups allowed for the isotopy to persist and to see what happens when we exceed the bounds.

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