Einstein Metrics on Non-Simply-Connected 4-Manifolds

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Abstract of the Dissertation

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The main part of the thesis involves Einstein metrics on non-simply connected 4-manifolds. In dimension four there is a strong interplay between the Riemannian structure and symplectic or complex structures. We construct infinitely many topological four spaces which support a differential structure which admits an Einstein metric and infinitely many other structures without Einstein metrics. This result is known for simply connected manifolds, our contribution is to exhibit a similar behavior on no-spin manifolds with finite cyclic fundamental group. We complement this result with theorems about non-existence of Einstein metrics on spin mani-

folds with finite cyclic fundamental group and on non-spin manifolds with finitely presented fundamental group. The main tools are Seiberg-Witten Theory, cyclic coverings of complex surfaces and symplectic surgeries.

Finally, we conclude that certain connected sums of projective planes and projective planes with orientation reversed do not admit Einstein metrics which are invariant under an exotic involution.

In the second part we are analyzing the following question: if we start with a Kähler manifold and we do a symplectic surgery (the rational blow-down), when does the new manifold admit a complex structure? We give a sufficient condition to have a positive answer. We discuss some significant examples.

To you, grandma

Contents

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To all, my warmest gratitude.

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Chapter 1

Introduction

In 1958 René Thom posed the following question:

"Are there any best (or distinguished) Riemannian structures on a C^∞-differentiable manifold M of dimension n?"

For $n = 2$ or 3 there is a consensus that a metric of constant sectional curvature should be considered the "best metric". In the case $n = 4$ the problem becomes more complex. First, we analyze the geometry of four manifolds. Let (M, g) be an oriented Riemannian 4-manifold. Using g, we can define the Hodge star operator:

$$* : \Lambda^2 \to \Lambda^2.$$

This operator has the property that $*^2 = id$, and so yields a decomposition:

$$\Lambda^2 = \Lambda^+ \oplus \Lambda^- \tag{1.1}$$

where $\Lambda^\pm$ is the (± 1)-eigenspace of $*$. Let's assume that (M, g) is a compact oriented Riemannian 4-manifold. The Hodge Theorem tells us that in each de Rham class there is a unique harmonic representative, which can be written as a sum of a positive, respectively negative, eigen-vector with respect to $*$. Therefore we have a direct sum decomposition of the second cohomology:

$$H^2(M,\mathbb{R}) = \mathcal{H}_g^+ \oplus \mathcal{H}_g^-$$

where $\mathcal{H}_g^\pm = \{\psi \in \Gamma(\Lambda^\pm) |\ d\psi = 0\}$. The intersection form on $H^2(M,\mathbb{R})$ restricted to $\mathcal{H}_g^+$ is positive definite, and negative definite when restricted to $\mathcal{H}_g^-$. Let $b_2^\pm(M) = dim\mathcal{H}_g^\pm$. On an oriented 4-manifold (M,g) we can define an invariant, the *signature*, $\tau(M) = b_2^+ - b_2^-$. From the definition of $b_\pm$ it might seem that the signature is dependent on the metric, but $b_2^\pm$ is determined as the the dimension of the largest linear subspace of $H^2(M,\mathbb{R})$ where the intersection form is positive, respectively negative, definite. Therefore $\tau(M)$ is an oriented homotopy invariant.

A second homotopy invariant that we need to consider is the Euler characteristic $\chi(M)$.

By raising an index, the Riemann curvature tensor may be viewed as a linear map $R : \Lambda^2 \to \Lambda^2$, called the *curvature operator.* Then, with respect to the decomposition (1.1) R splits into:

$$\mathcal{R} = \left(\begin{array}{c|c} W^+ + \frac{s}{12} & \overset{\circ}{r} \\ \hline \overset{\circ}{r} & W^- + \frac{s}{12} \end{array} \right)$$

Here s and $\overset{\circ}{r}$ are the scalar curvature and trace-free Ricci, respectively. $W_\pm$ are called the self-dual and anti-self-dual Weyl curvatures.

One of the canonical metrics in dimension 4 is the Einstein metric. A smooth Riemannian 4-manifold (M, g) is said to be *Einstein* if its Ricci curvature, considered as a function on the unit tangent bundle of M, is constant. This is equivalent [Bes87] to saying that g satisfies

$$r = \frac{s}{4} g, \tag{1.2}$$

The decomposition of the Ricci-curvature and the above equation implies that $\overset{\circ}{r} \equiv 0$.

On 4-manifolds there is a strong interplay between the topology of the manifold and the geometric structures. Indeed both the Euler characteristic $\chi(X)$ and the signature $\tau(X)$ can be related to the L^2-norms of the components of the curvature for any metric g.

For the Euler characteristic we have a generalized Gauss-Bonnet-type formula:

$$\chi(X) = \frac{1}{8\pi^2} \int_M \left[\frac{s^2}{24} + |W^+|^2 + |W^-|^2 - \frac{|\overset{\circ}{r}|^2}{2} \right] d\mu_g, \tag{1.3}$$

while for the signature τ the Hirzebruch Signature Theorem tells us that:

$$\tau(X) = \frac{1}{12\pi^2} \int_M \left[|W^+|^2 - |W^-|^2 \right] d\mu_g. \tag{1.4}$$

We can combine these to obtain:

$$(2\chi \pm 3\tau)(X) = \frac{1}{4\pi^2} \int_X \left[\frac{s^2}{24} + 2|W^\pm|^2 - \frac{|\overset{\circ}{r}|^2}{2} \right] d\mu_g. \qquad (1.5)$$

This gives us the classical obstruction to the existence of Einstein metrics on a 4-manifold:

Theorem 1.0.1 (Hitchin-Thorpe Inequality). *If the smooth compact oriented 4-manifold M admits an Einstein metric g, then*

$$(2\chi + 3\tau)(M) \geq 0,$$

with equality if (M, g) is finitely covered by a flat 4-torus T^4 or by $K3$ with a hyper-Kähler metric. Moreover, (M, g) must also satisfy

$$(2\chi - 3\tau)(M) \geq 0,$$

and this inequality is strict unless (M, g) is finitely covered by a flat 4-torus T^4 or by the orientation-reversed version of $K3$ with a hyper-Kähler metric.

This obstruction to the existence of Einstein metrics is homotopy invariant. Using curvature estimates which are deduced from Seiberg-Witten equations, LeBrun [LeB96, LeB97, LeB01] and later, Ishida and LeBrun [IsLe02] were able to find obstructions of the differential structures. A short description of these results is presented in Sec.2. Using these obstructions, Kotschick and his co-authors, and LeBrun were able to construct distinct differential structures on given topological spaces, such that one structure admits an Einstein metric, while the others don't. Their results are on simply-connected manifolds. Our contribution is to prove them for manifolds with finite cyclic fundamental group.

The statements of the main theorems are presented in Section 7. In Section 2 we introduce the main results on Seiberg-Witten theory, the induced obstructions and an existence theorem of Kähler-Einstein metrics. The background statements about the topology of non-simply connected 4-manifolds are given in Section 3. In the next sections 4, 5 and 6 we present the main constructions.

While working on this problem, I came across some interesting results with slightly different flavor. Joint with Rasdeaconu, we found a sufficient condition when symplectic surgeries (rational blow downs) can be described as complex operations.

Chapter 2

Einstein metrics: existence and obstructions to existence

The work of Freedman, Donaldson and others on the topology of 4-manifolds gives us a complete classification of simply connected topological 4-spaces. The next brake-through in the study of 4-manifolds was done by the use of Gauge theories, Donaldson's polynomials and later Seiberg-Witten invariant. They provided tools to distinguish the differential structures. Later on, they were also used to obtain obstructions on the existence of certain canonical Riemannian metrics or to the computation of metric invariants, for example the Yamabe invariant.

2.1 Seiberg-Witten invariants and properties

In this section we introduce the *Seiberg-Witten invariant* of a smooth, closed, oriented 4-manifold X. We briefly present the main ingredients, and outline the most important results of the theory. We refer the reader to [LaMi89, Tau94, Wit94] and others for more details and the proofs. We follow the exposition given by LeBrun in [LeB02].

Let L be a Hermitian line bundle $L \to X$ such that $c_1(L) \equiv w_2(TX)$ mod 2. For any such L and Riemannian metric g on X we can define a rank-2 Hermitian vector bundles $\mathbb{V}_\pm$ which formally satisfy

$$\mathbb{V}_\pm = \mathbb{S}_\pm \otimes L^{1/2} \to X,$$

where $\mathbb{S}_\pm$ are the locally defined left-/right-handed spinor bundles of (X, g). $\mathbb{V}_\pm$ is completely determined, up to isomorphism, by $c_1(L) = c_1(\mathbb{V}_\pm) \in H^2(X, \mathbb{Z})$ if $H_1(X, \mathbb{Z})$ does not contain any elements of order 2. This is called the *Spin*^{c}-structure given by $c = c_1(L)$ on X.

If our manifold is endowed with an almost-complex structure, then there is a canonical way to define a *Spin*^{c}-structure. Let $J : TM \to TM,\ J^2 = -1$ be an almost complex structure. This induces a natural decomposition of $TM \otimes \mathbb{C} = T^{1,0} \oplus T^{0,1}$ as the $(\pm i)$ eigen-spaces of J. Setting $\Lambda^{0,p} = \Lambda^p T^{1,0}$, the bundles $V_\pm$ are given by:

$$V_+ = \ \Lambda^{0,0} \oplus \Lambda^{0,2}, \quad V_- = \ \Lambda^{0,1}.$$

The group $H^2(X, \mathbb{Z})$ acts on the set of *Spin*^{c}-structures by tensoring the spinor bundles with the complex line bundle whose first Chern class is the given element in the cohomology. This action is free and transitive. Modulo this action, the *Spin*^{c}-structures correspond [LaMi89] one-to-one with elements of $H^1(X, \mathbb{Z}_2)$. In the case when this cohomology is non-zero, we can fix a class and consider the *Spin*^{c}-structures corresponding to this class.

Every unitary connection A on L induces a connection:

$$\nabla_A : \Gamma(\mathbb{V}_+) \longrightarrow \Gamma(\Lambda^1 \otimes \mathbb{V}_+),$$

which in its turn induces a *Spin^c* Dirac operator [LaMi89, Hit74]:

$$D_A : \Gamma(\mathbb{V}_+) \longrightarrow \Gamma(\mathbb{V}_-).$$

We are now ready to introduce the *Seiberg-Witten Equations:*

$$D_A \Phi = 0 \tag{2.1}$$

$$F_A^+ = i\sigma(\Phi). \tag{2.2}$$

where $\Phi \in \Gamma(\mathbb{V}_+)$, F_A^+ is the self-dual part of the curvature of A, and where $\sigma : \mathbb{V}_+ \longrightarrow \Lambda^+$ is a natural real-quadratic map satisfying

$$|\sigma(\Phi)| = \frac{1}{2\sqrt{2}}|\Phi|^2.$$

We are interested in the *Spin^c* classes for which these equations have non-trivial solution. We introduce a new definition to describe this property:

Definition 1. *Let X be a smooth compact oriented 4-manifold with $b_2^+ > 1$. Let L be a Spin^c-structure for which the corresponding Seiberg-Witten equations (2.1) and (2.2) have a solution for every Riemannian metric g on X then the class*

$$a \in H^2(X, \mathbb{Z})/\text{torsion}, \quad a \equiv c_1(L) \bmod \text{ torsion}$$

is called a **monopole class**.

Curvature estimates, for example see [LeB01], imply that the set of monopole classes is finite. Using the monopole classes we can define a new invariant of a smooth 4-manifold:

Definition 2. *Let X be a smooth compact oriented 4-manifold with $b_2^+ > 1$. Let $\mathcal{C} \subset H^2(X, \mathbb{Z})/torsion$ be the set of monopole classes of X. If $\mathcal{C}$ contains a non-zero element we define the* **bandwidth** *of X to be*

$$\mathcal{B}W(X) = max\{ \ n \in \mathbb{Z}_+ \mid \exists \ a, b \in \mathcal{C}, \ \ a \neq b, \ \text{s.}t. \ 2n \mid (a - b) \ \}.$$

If $\mathcal{C} \subset \{0\}$, we define the bandwidth $\mathcal{B}W(X) = 0$.

Note that as both a and b are monopole classes it follows that $a - b \equiv 0 \bmod 2$, so $(a - b)$ is divisible by 2, hence the $2n$ in the definition.

To be able to define the moduli space of solutions we need to consider a generic perturbation of these equations given by:

$$D_A\Phi = 0 \qquad (2.3)$$

$$iF_A^+ + \sigma(\Phi) = \phi. \qquad (2.4)$$

where ϕ is a smooth self-dual 2-form.

For a given metric g on X, a $Spin^c$ class L and a perturbation ϕ we define $\mathcal{M}_L^\phi(g)$ the moduli space of solutions of the perturbed equations. For a generic metric and perturbation this moduli space [Wit94] is a closed, orientable manifold of dimension $dim\mathcal{M}_L^\phi(g) = \frac{1}{4}(L^2 - (2\chi(X) + 3\tau(X)))$.

A manifold X is said to be of *simple type* if each monopole class satisfies the equation $L^2 = 2\chi(X) + 3\tau(X)$. There is no known example of a simply connected manifold with $b_2^+ > 1$ which is *not* of simple type; for example the existence of a symplectic structure implies that the manifold has simple type.

We can now introduce an invariant of the smooth oriented 4-manifold of simple type with $b_2^+ > 1$ and the $Spin^c$-structures L:

Definition 3. *The Seiberg-Witten invariant $SW_X(L) \in \mathbb{Z}$ is given by the number of points of $\mathcal{M}_L^\phi(g)$ counted with orientations.*

An important feature of this invariant is that it is a *diffeomorphism invariant*, i.e. $SW_X(L)$ does not depend on the chosen metric g or perturbation ϕ. Moreover, for an orientation preserving diffeomorphism $f : X \to X'$ we have $SW_X(L) = \pm(SW_{X'}(f^*(L))$.

In the rest of this section we state the main properties of this invariant.

Theorem 2.1.1 (Non-vanishing). [Tau94]

(i) *If S is a simply connected complex surface (hence $b_2^+(S)$ is odd) and $b_2^+(S) > 1$, then $SW_S(\pm c_1(S)) \neq 0$.*

(ii) *More generally, if (X, ω) is a simply connected symplectic manifold and $b_2^+(X) > 1$, then $SW_S(\pm c_1(X, \omega)) = \pm 1$.*

However, for a large class of manifolds which decompose as connected sums this invariant is trivial.

Theorem 2.1.2 (Vanishing). [GoSt99] *Suppose that X is a smooth, closed, oriented, simply connected 4-manifold with $b_2^+ > 1$ and odd.*

(i) *If* $X = X_1 \# X_2$ *and* $b_2^+(X_i) > 0$ $(i = 1, 2)$, *then* $SW_X \equiv 0$.

(ii) *If* X *admits a metric with positive scalar curvature, then* $SW_X \equiv 0$.

It is important to note that in special cases we can have a nontrivial invariant:

Theorem 2.1.3 (Kotschick, Morgan, Taubes). [KMT95] *Let* Y *be a manifold with a non-trivial Seiberg-Witten invariant, e.g. a symplectic manifold with* $b_2^+ > 1$, *and let* N *be a manifold with* $b_1(N) = b_2^+ = 0$ *whose fundamental group has a nontrivial finite quotient. Then* $X = Y \# N$ *has non-trivial Seiberg-Witten invariant but does not admit any symplectic structure.*

Moreover if $b_2(N) = 0$ the $Spin^c$-structure c on X for which the Seiberg-Witten equations have non-trivial solution is given by $c = c_1(Y)$. If $b_2(N) \neq 0$, let $e_1, \ldots, e_n \in H^2(N, \mathbb{Z})$ descending to a basis of $H^2(N, \mathbb{Z})/torsion$ with respect to which the cup product form is diagonal. Let $c_N = \Sigma e_i$ such that $c \equiv w_2 \bmod 2$. For any b monopole class of Y then $c = b + c_N$ is a monopole class of X.

2.2 Obstructions to the existence of Einstein metrics

If we require that our manifold supports a symplectic structure, or that it has non-trivial Seiberg-Witten invariant, then we have new restrictions on the curvature. For the operator D_A, defined in the previous section,we have an associated Weitzenböck formula:

$$\langle \Phi, D_A^* D_A \Phi \rangle = \frac{1}{2}\Delta|\Phi|^2 + |\nabla_A \Phi|^2 + \frac{s}{4}|\Phi|^2 + 2\langle -iF_A^+, \sigma(\Phi)\rangle \tag{2.5}$$

In conjunction with the Seiberg-Witten equations this implies:

$$0 = 2\Delta|\Phi|^2 + 4|\nabla_A \Phi|^2 + s|\Phi|^2 + |\Phi|^4. \tag{2.6}$$

We can immediately conclude that if a manifold has non-trivial Seiberg-Witten invariant, then it admits no metric g with $s > 0$.

It was LeBrun's clever use of Seiberg-Witten equations that gave better estimates for the different components of the curvature operator which led to new obstructions. The novelty of these obstructions is that they impose restrictions on the differential structure of the manifold. The results developed in a sequence of papers. We refer to [LeB01] for the proof of the theorem presented in here and also for additional references.

Theorem 2.2.1. [LeB01] *Let X be a compact oriented 4-manifold with a non-trivial Seiberg-Witten invariant and with $(2\chi + 3\tau)(X) > 0$. Then*

$$M = X \# k\overline{\mathbb{CP}^2} \# l(S^1 \times S^3)$$

does not admit Einstein metrics if $k + 4l \geq \frac{1}{3}(2\chi + 3\tau)(X)$.

By Theorem 2.1.1 the Seiberg-Witten invariant vanishes for manifolds which decompose as connected sums of manifolds with positive b_2^+, so this theorem does not apply. Independently of one another Bauer and Furuta were able to define a refinement of this invariant which behaves nicely under the connected sum. Using similar estimates and this new invariant Ishida and LeBrun were able to prove:

Theorem 2.2.2. [IsLe02] *Let X_j, $j = 1, \ldots, 4$ be smooth, compact almost-complex 4-manifolds for which the mod-2 Seiberg-Witten invariant is non-zero, and suppose that*

$$b_1(X_j) = 0, \tag{2.7}$$

$$b_2^+(X_j) \equiv 3 \bmod 4, \tag{2.8}$$

$$\textstyle\sum_{j=1}^4 b_2^+(X_j) \equiv 4 \bmod 8. \tag{2.9}$$

Let N be any oriented 4-manifold with $b_2^+ = 0$. Then, for any $m = 2, 3$ or 4, the smooth 4-manifold $M = \#_{j=1}^m X_j \# N$ does not admit Einstein metrics if

$$4m - (2\chi + 3\tau)(N) \geq \frac{1}{3}\sum_{j=1}^m c_1^2(X_j).$$

2.3 Examples of Einstein metrics

The simplest examples of Einstein metrics are the metrics of constant sectional curvature. A much larger family of Einstein metrics is given by metrics which are compatible with an extra structure, such as a complex structure. For more details about this topic we refer the interested reader to [Bes87].

Let (M, J) be a complex manifold. A metric g is compatible to the complex structure J if and only if J is an orthogonal transformation with respect to g:

$$g(\cdot,\cdot) = g(J\cdot, J\cdot).$$

We say that a complex n-manifold (M, g, J) is *Kähler* if g is J-compatible, and the associated 2-form $\omega(\cdot,\cdot) = g(\cdot, J\cdot)$ is d-closed.

In a similar way, we can define the Ricci form $\rho(\cdot,\cdot) = r(\cdot, J\cdot)$.

Definition 4. *A Kähler manifold (M, g, J) is called Kähler-Einstein if:*

$$\rho = \lambda\omega.$$

It is immediate from our definitions that a Kähler-Einstein metric is in particular Einstein.

An important fact about the complex manifolds is the relation between their first Chern class and the Ricci form:

Proposition 2.3.1. *Let (M, g, J) a Kähler manifold then:*

$$c_1(M) = [\frac{1}{2\pi}\rho] \in H^2(M, \mathbb{Z}).$$

The case that it is interesting from our point of view is that of a Kähler-Einstein 4-manifolds with $b_2^+ > 1$. In this case we must have by (2.6) $\lambda < 0$. Then the existence of a Kähler-Einstein metric implies that K_M is a positive line bundle, and the Kodaira embedding theorem tells us that K_M is *ample* meaning sections of a sufficiently large power of the canonical bundle give a holomorphic embedding of the manifold.

The amazing fact is that this condition is also sufficient:

Theorem 2.3.2 (Aubin/Yau). [Yau77] *A compact complex manifold* (M, J) *admits a compatible Kähler-Einstein metric with* $s < 0$ *if and only if its canonical line bundle* K_M *is ample. When such metric exists, it is unique, up to homotheties.*

Chapter 3

Background on the topology of 4-manifolds

In this section we introduce some extra topological tools. First, we give a homeomorphism criteria for non-simply connected 4-manifolds, then we introduce some results about the decomposition of manifolds. We finish by exhibiting a certain class of compact non-simply connected manifolds whose rational homology is that of a sphere.

3.1 Homeomorphism criteria

A remarkable result of Freedman [Fre82] in conjunction with results of Donaldson tells us that smooth compact, simply connected, oriented 4-manifolds are classified by their numerical invariants: Euler characteristic χ, signature τ and Stiefel-Whitney class w_2. Maybe not as well-known are the results involving the classification of *non-simply connected* 4-manifolds ([HaKr93], Theorem C):

Theorem 3.1.1 (Hambleton, Kreck). *Let M be a smooth, closed, oriented, 4-manifold with finite cyclic fundamental group. Then M is classified* *up to homeomorphism by the fundamental group, the intersection form on $H_2(M, \mathbb{Z})/Tors$, and the w_2-type. Moreover, any isometry of the intersection form can be realized by a homeomorphism.*

In contrast with simply connected manifolds, there are three w_2-types that can be exhibit: (I) $w_2(\widetilde{M}) \neq 0$, (II) $w_2(M) = 0$, and (III) $w_2(\widetilde{M}) = 0$, but $w_2(M) \neq 0$.

Using Donaldson's and Minkowski-Hasse's classification of the intersection form we can reformulate this theorem on an easier form:

Equivalently: A smooth, closed, oriented 4-manifold with finite cyclic fundamental group and indefinite intersection form is classified up to homeomorphism by the fundamental group, the numbers $b_2^{\pm}$, the parity of the intersection form and the w_2-type.

However, knowing the Euler characteristic χ and signature τ of a 4-manifold with finite fundamental group, is equivalent to knowing the invariants $b_2^{\pm}$.

3.2 Almost complete decomposability

Freedman's results tell us that any non-spin simply connected 4-manifold is homeomorphic to $a\mathbb{CP}^2 \# b\overline{\mathbb{CP}^2}$, for appropriate positive integers a, b. In Section 2.1 we introduced techniques which distinguish between the differential structures. The next question that we have to ask is when two homeomorphic manifolds are diffeomorphic. Spaces which decompose as connected sums of manifolds with positive b_2^+ have trivial Seiberg-Witten invariants, by the Vanishing Theorem 2.1.2. On the other hand the Non-Vanishing Theorem tells us that symplectic 4-manifolds and complex surfaces have non-trivial invariants.

Hence symplectic 4-manifolds and complex surfaces are not diffeomorphic to manifolds of type $a\mathbb{CP}^2\#b\overline{\mathbb{CP}^2}$. So, it is reasonable to introduce the following definitions:

Definition 5. *We say a smooth non-spin 4-manifold M is* completely decomposable *if it is diffeomorphic to a connected sum of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}^2}$'s.*

Definition 6. *We say M is* almost completely decomposable *if $M\#\mathbb{CP}^2$ is completely decomposable.*

To see the relevance of the last definition we want to point out that if M is a symplectic 4-manifold then $M\#\mathbb{CP}^2$ has trivial Seiberg-Witten invariants, so it is not meaningless to ask if it decomposes completely. It has been conjectured by Mandelbaum [Ma80] that simply connected analytic surfaces are almost completely decomposable. In support of this conjecture he gives the following arguments:

Proposition 3.2.1. [Ma80] *Every simply connected complex surface which is diffeomorphic to a complete intersection of hypersurfaces in some $\mathbb{CP}^N$ is almost completely decomposable.*

Proposition 3.2.2. [Ma80] *Let $X \subset \mathbb{CP}^N$ be a compact complex surface and suppose $M \longrightarrow X$ is an r-fold cyclic branched cover manifold of X whose branch locus is homeomorphic to $X \cap H_r$, for some hypersurface H_r of degree r of $\mathbb{CP}^N$. Then if X is almost completely decomposable so is M.*

For a definition of a r-fold cyclic cover see the next section.

3.3 Rational homology spheres

In the following we give a construction of compact oriented 4-manifolds whose rational homologies are those of a sphere and their fundamental group is a finite cyclic group. We learned this construction from [Ue96].

Let $S^3 \subset \mathbb{C}^2$ be the unit sphere , then there is a free $\mathbb{Z}_d$-action on S^3 generated by $(z, w) \to (e^{2\pi i/d}z, e^{2\pi iq/d}w)$. The quotient space is denoted $L(d, q)$ and it's called a lens space.

Let $S = (L(d,q) \setminus D^3) \times S^1 \cup_{id} S^2 \times D^2$, where D^i is a small ball of corresponding dimension. Then we have:

Lemma 3.3.1. [Ue96] *The diffeomorphism type of S does not depend on q it depends only on d. S is a rational homology 4-sphere with fundamental group $\pi_1(S) = \mathbb{Z}_d$, and its universal covering is diffeomorphic to $(d-1)S^2 \times S^2$.*

We denote by S_d the manifold constructed from $L(d, q)$.

Chapter 4

Finite covers of complex manifolds

One technique which yields a large family of examples of complex manifolds is the construction of finite covers. The work of Catanese on complex surfaces [Cat92, Cat84] exhibits the abundance of these varieties in the geography of complex surfaces and also gives a rich class of examples for which the properties of complex surfaces and their moduli spaces can be studied. For a background exposition of the constructions presented in this chapter we refer the reader to [BPV84] for an algebraic geometric point of view, or to [GoSt99] for a more topological description.

Definition 7. *A (non-singular)* d-fold branched cover *consists of a triplet (X, Y, π), denoted by $\pi : X \to Y$, where X, Y are connected compact smooth complex manifold and π a finite, generically $d : 1$, surjective proper holomorphic map.*

The critical set, $R \subset X$, is called the *ramification divisor* of π and its image $D = \pi(R)$ is called the *branch locus.* For any point $y \in Y \setminus D$ there is a connected neighborhood V_y with the property that $\pi^{-1}(V_y)$ consists of d

disjoint subsets of X, each of which is mapped isomorphically onto V_y by π. For each $p \in R$ there are appropriate local coordinate charts such that the map π is $(z,x) \to (z^m, x)$, where $R = (z=0)$ and m is called the *local degree.* One special class of covers are the *cyclic covers.*

4.1 Cyclic covers

A *cyclic* branched cover is a d-fold cover such that $\pi_{|X\setminus R} : X \setminus R \to Y \setminus D$ is a (regular) cyclic covering. So, it is determined by a epimorphism $\pi_1(Y \setminus D) \to \mathbb{Z}_d$, and $Y = X/\mathbb{Z}_d$. Moreover, a cyclic d-cover is a Galois covering, meaning that the function field embedding $\mathbb{C}(Y) \subset \mathbb{C}(X)$ induced by π is a Galois extension.

We have the following construction:

Construction 1. : *Let Y be a connected complex manifold and D an effective divisor on Y, $\mathcal{O}(D)$ the associated line bundle and let $s_D \in \Gamma(Y, \mathcal{O}_Y(D))$ the section vanishing exactly along D. Suppose we have a line bundle $\mathcal{L}$ on Y such that $\mathcal{O}_Y(D) = \mathcal{L}^{\otimes d}$. We denote by L the total space of $\mathcal{L}$ and we let $p : L \to Y$ be the bundle projection. If $z \in \Gamma(L, p^*\mathcal{L})$ is the tautological section, then the zero divisor of $p^*s_D - z^d$ defines an analytical subspace X in L. If D is a smooth divisor then X is smooth connected manifold and $\pi = p|_X$ exhibits X as a d-fold ramified cover of Y with branch locus D. We call (X, Y, π) the d-cyclic cover of Y branch along D, determined by $\mathcal{L}$.*

Given D and Y, X is uniquely determined by a choice of $\mathcal{L}$. Hence X is unique if $Pic(Y)$ has no torsion.

The following lemmas give us the main relations between the two manifolds:

Lemma 4.1.1. [BPV84] *Let $\pi : X \to Y$ be a d-cyclic cover of Y branched along a smooth divisor D and determined by $\mathcal{L}$, where $\mathcal{L}^{\otimes d} = \mathcal{O}_Y(D)$. Let R be the reduced divisor $\pi^{-1}(D)$ on X. Then:*

(i) $\mathcal{O}_X(R) = \pi^*\mathcal{L}$;

(ii) $\pi^*[D] = d[R]$, *in particular d is the branching order along R;*

(iii) $\mathcal{K}_X = \pi^*(\mathcal{K}_Y \otimes \mathcal{L}^{d-1})$.

Lemma 4.1.2. [BPV84] *Let $\pi : X \to Y$ be as in Lemma 4.1.1. Then*

$$\pi_*\mathcal{O}_X \simeq \bigoplus_{j=0}^{d-1} \mathcal{L}^{-j}.$$

As an immediate consequence, we are able to compute the relations between the topological invariants of X and Y in the case of complex surfaces:

Lemma 4.1.3. *Let X, Y complex surfaces and $\pi : X \to Y$ be as in Lemma 4.1.1. Then:*

(i) $c_2(X) = dc_2(Y) - (d-1)\chi(D)$;

(ii) $c_1^2(X) = d(c_1(Y) - (d-1)c_1(\mathcal{L}))^2$

Moreover, since $\sigma(X) = \frac{1}{3}(c_1^2(X) - 2\chi(X))$, we have $\sigma(X) = d\sigma(Y) - \frac{d^2-1}{3d}D^2$.

In a more general set-up, we can define a d-cyclic branch cover $\pi : X \to Y$ branched along a divisor with simple normal crossing singularities and Y

smooth manifold. In this case, X will be a normal complex space ([BPV84] I.17.) with singularities over singular points of D.

Next, we analyze the double cover $\pi : X \to Y$ branched along a divisor D with simple normal crossing singularities. Let $U \subset Y$a neighborhood of a singular point of D and (x, y) local coordinates such that D is defined by the equation $xy = 0$. Then X is a normal surface with isolated singularities over the singularities of D, and an open neighborhood of each of the singularities in local coordinates is modeled by $z^2 = xy \subset \mathbb{C}^3$. There are two techniques to associate a smooth manifold to X. One is given by resolving the singularities, the other is smoothing.

Given a normal surface X there is always a bi-meromorphic map $\pi : X' \to X$, with X' smooth. Even more, if we require that X' is a minimal surface, then X' is uniquely determined by X (see for instance [BPV84] Theorems 6.1, 6.2). $\pi : X' \to X$ is called the *minimal resolution* of singularities of X.

Definition 8. *A smoothing of a normal surface X is a proper flat map $f : \mathcal{X} \to \Delta$ smooth over $\Delta^* = \Delta \setminus 0$ where: $\mathcal{X}$ is a three dimensional complex manifold, Δ is a small open disk in $\mathbb{C}$ centered at 0 and $f^{-1}(0)$ is isomorphic to X.*

If $t, t' \in \Delta^*$ then $f^{-1}(t)$ is diffeomorphic to $f^{-1}(t')$. An even stronger result is true:

Proposition 4.1.4. *If $\pi : X \to Y$ is a double cover branched along a divisor D with simple normal crossing singularities, such that the linear system $\mathbb{P}(H^0(Y, \mathcal{O}(D)))$ is base point free. Then, there is a smoothing $\varpi : \mathcal{X} \to \Delta$ of*

X such that the generic $X_t = \varpi^{-1}(t)$ is diffeomorphic to the minimal resolution X' of X.

Proof. First, we remark that X has a finite number of singular points, corresponding to the singular points of D. Resolving these singularities is a local process. The singularity, modeled by $(z^2 = xy) \subset \mathbb{C}^3$, is a quotient singularity. It is isomorphic to $\mathbb{C}^2/\mathbb{Z}_2$, where the $\mathbb{Z}_2$ action is just multiplication by (-1). The isomorphism is given by the map:

$$\mathbb{C}^2/\mathbb{Z}_2 \to U, \quad \widehat{(u,v)} \mapsto (u^2, v^2, uv) = (x, y, z).$$

Hence, it is enough to resolve the singularities of type $\mathbb{C}^2/\mathbb{Z}_2$. A resolution of this singularity is given by blowing-up the origin of $\mathbb{C}^2$, then extending the $\mathbb{Z}_2$ action trivially on the exceptional divisor. The total space of the blow-up of $\mathbb{C}^2$ is, in fact, the line bundle $\mathcal{O}_{\mathbb{CP}^1}(-1)$, and factoring by the $\mathbb{Z}_2$ action corresponds to squaring (tensor product) the line bundle. The resulting manifold after taking the quotient is $\mathcal{O}_{\mathbb{CP}^1}(-2)$. So, we resolved the singularities of X by introducing exceptional divisors of self-intersection (-2).

Next, we construct explicitly a smoothing of X. The idea is smoothing the branch locus in a family of smooth curves, and to construct the corresponding double-covers. Because the linear system $\mathbb{P}\big(H^0(Y, \mathcal{O}(D))\big)$ is base point free, there exists a holomorphic path of sections of $\mathcal{O}(D)$ and a parametrization of this path given by $\varphi : \Delta \to \Gamma(Y, \mathcal{O}(D))$ such that $\varphi(0) = \varphi_D$ and $d\varphi_{|0, Sing D} \neq 0$. The last condition just says that the parametrization is "nice", i.e. the section $\varphi(t), t \neq 0$, doesn't contain any of the singularities of D and we can also assume that $\varphi(t) = \varphi_t, t \neq 0$ corresponds to a smooth divisor, maybe after restricting Δ.

Then $\varpi : \mathcal{X} \to \Delta$, where $\mathcal{X} \subset \mathcal{L} \times \Delta$ given locally by the equation $z^2 - \varphi(t)(x, y) = 0$ is a smoothing of X. First, let's notice that $\mathcal{X}$ is a smooth manifold as:

$$d(z^2 - \varphi(t)(x,y)) = 2zdz - (\frac{d}{dt}\varphi(t)(x,y))dt - \left(\frac{d\varphi_t}{dx}(x,y)dx + \frac{d\varphi_t}{dy}(x,y)dy\right) \neq 0$$

The reason for which this is never zero is that for $t \neq 0$ the section φ_t is smooth, hence the last parenthesis is non-zero and for $t = 0$ we have $\frac{d\varphi}{dt} |_{0,SingD} \neq 0$ and $(\frac{d\varphi}{dx} + \frac{d\varphi}{dy}) |_{0,D \setminus SingD} \neq 0$.

An immediate consequence of Theorem 9.11 in [Har77] implies that the morphism ϖ is flat.

The fact that the two constructions yield diffeomorphic manifolds is a local statement about the differential structures of the manifolds in a neighborhood of the singularities. So, our proof is in local coordinates. Because the morphism ϖ is a submersion away from the central fiber it is enough to show that one of the fibers is isomorphic to X'.

In local coordinates the singularity is given by the equation

$$(z^2 - xy = 0) \subset \mathbb{C}^3.$$

Because the linear system associated to $\mathcal{O}(D)$ is base point free, then the zero locus of a generic section is smooth, and all the divisors are diffeomorphic to each another. We can consider a preferred smoothing given in local coordinates by $(z^2 - xy = 1) \subset \mathbb{C}^3$. If we change the local coordinates $(x, y, z) \to (u, v, z)$, such that $x = iu - v$, $y = iu + v$. Then the smoothing is written in the canonical form $(z^2 + u^2 + v^2 = 1)$. Let $\xi = Re(u, v, z)$, $\eta = Im(u, v, z)$, $\xi, \eta \in \mathbb{R}^3$ the real, respectively imaginary part, of (u, v, z). Then:

$X_1 = \{ (u, v, z) \in \mathbb{C}^3 \mid z^2 + u^2 + v^2 = 1 \}$

$\quad = \{ (\xi, \eta) \in \mathbb{R}^3 \times \mathbb{R}^3 \mid \parallel \xi \parallel^2 - \parallel \eta \parallel^2 = 1, < \xi, \eta >= 0 \}$.

The map $f : X_1 \to T^*S^2 \subset \mathbb{R}^3 \times \mathbb{R}^3$ defined by $f(\xi, \eta) = (\frac{\xi}{\|\xi\|}, \parallel \xi \parallel . \eta)$ is a diffeomorphism. It is a well known fact that T^*S^2 is in fact $\mathcal{O}_{\mathbb{CP}^1}(-2)$, where $\mathbb{CP}^1$is identified to the sphere S^2. Moreover, we consider the standard symplectic structures on X_1, T^*S^2, then the map f is a symplectomorphism.

□

A similar statement is true if the divisor D has *simple singularities.* A singularity of a curve is called *simple* if it is a double or triple point with two, three different tangents or a simple triple point with one tangent. Then the double cover branched along such a divisor has $A - D - E$ singularities, respectively. We call these singularities *rational double points.* It can be proved [HKK86] that for these singularities, and only for these singularities the resolution and deformation manifold coincide.

If we allow other types of singularities then it is easy to construct manifolds for which the two techniques yield manifolds of different Kodaira dimensions.

4.2 Fundamental group

The results presented in this subsection are due to Catanese ([Cat84], Sec.2). We are interested in the case when the branch locus is smooth, but because the results are true for a branch locus with simple normal crossing and manifolds of arbitrary dimension greater than two, we are going to state them in the most general form. We will need to introduce a definition:

Definition 9. *A smooth divisor D is said to be flexible if there exists a divisor $D' \equiv D$ such that $D \bigcap D' \neq \emptyset$ and D' intersects D transversally in codimension* 2.

We remark that if D is a flexible divisor then it must be connected.

From now on all the branch loci that we will consider are smooth and flexible, unless otherwise specified.

First, let's analyze the complement of the branch locus on the base manifold.

Theorem 4.2.1. [Cat84] *Let Y be a simply connected algebraic variety and $D = D_1 \cup \cdots \cup D_k$ a divisor smooth in codimension* 1, *with normal crossing in codimension* 2. *If the D_i's are flexible, then $\pi_1(X \setminus D)$ is abelian.*

Moreover, $\pi_1(X \setminus D)$ is generated by the loops defined by the boundary of a fiber of the tubular neighborhood fibration over D_i.

Theorem 4.2.2. [Cat84] *Let $\pi : X \to Y$ a d-cyclic cover branched along a smooth flexible divisor D, then if Y is simply connected so is X.*

Chapter 5

Main constructions

In this section we introduce a new construction, called the simple bi-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$. Then, we study the analytical and topological properties of this class of manifolds. Finally, we give a construction of complex surfaces of general type with infinite fundamental group.

5.1 Bi-cyclic covers

The manifolds that we want to consider are bi-simple-cyclic covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$. They are inspired by the manifolds studied by Catanese. In his papers [Cat84], [Cat92], etc, he extensively studies the bi-double covers. For our purposes we need to consider cyclic covers of arbitrary degrees. We have the following:

Construction 2. : *Given two smooth curves C, D in $\mathbb{CP}^1 \times \mathbb{CP}^1$, first we consider the d-cyclic cover, X, branched along C, and then we construct our manifold N as the p-cyclic cover of X branched along the proper transform of D. The line bundles $\mathcal{O}(C), \mathcal{O}(D)$ are d, p-tensor powers of some line bundles on $\mathbb{CP}^1 \times \mathbb{CP}^1$, respectively. It can easily be checked that if C, D are smooth and* *intersect transversally, then both X and N are smooth. We call the manifold N a* (simple) bi-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ of type (d, p) branched along (C, D).

Let $\pi_1, \pi_2 : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^1$ be the projections on the first, second factor, respectively. The line bundles on $\mathbb{CP}^1 \times \mathbb{CP}^1$ are of the form $\pi_1^*(\mathcal{O}_{\mathbb{CP}^1}(a)) \otimes \pi_2^*(\mathcal{O}_{\mathbb{CP}^1}(b))$, which we denote by $\mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(a, b)$ or simply $\mathcal{O}(a, b)$. With this notations, $\mathcal{O}(C) = \mathcal{O}(da, db)$ and $\mathcal{O}(D) = \mathcal{O}(pm, pn)$ where a, b, m, n are positive integers, and there are sections $\varphi_C \in \Gamma(\mathcal{O}(C))$ and $\varphi_D \in \Gamma(\mathcal{O}(D))$ such that $C = \{\varphi_C = 0\}$ and $D = \{\varphi_D = 0\}$. Then, our manifold N is a smooth compact submanifold of $\mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(a, b) \oplus \mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(m, n)$. If (x, y) are local coordinates on a chart of $\mathbb{CP}^1 \times \mathbb{CP}^1$ and z, w are coordinates on the fiber in a local trivialization of $\mathcal{O}(C) \oplus \mathcal{O}(D)$ on the (x, y)-chart, then N is defined locally by the equations $\{z^d = \varphi_C(x, y), w^p = \varphi_D(x, y)\}$.

Let π be the projection from N to $\mathbb{CP}^1 \times \mathbb{CP}^1$ induced by the fibration projection. Using Lemma 4.1.3, we can easily compute the topological invariants of N.

Lemma 5.1.1. *Let $\pi : N \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be a bi-simple-cover as above. Then:*

(i) $K_N = \pi^*\Big(\mathcal{O}\Big((d-1)a + (p-1)m - 2\ ,\ (d-1)b + (p-1)n - 2\Big)\Big)$;

(ii) *If $(d-1)a + (p-1)m \geq 3$ and $(d-1)b + (p-1)n \geq 3$ then K_N is an ample line bundle;*

(iii) $c_1^2(N) = 2pd\Big((d-1)a + (p-1)m - 2\Big)\Big((d-1)b + (p-1)n - 2\Big)$;

(iv) $c_2(N) = pd[4 - 2(d-1)(a + b - dab) - 2(p-1)(m + n - pmn) + (p-1)(d-1)(an + bm)]$.

Proof. The proof of (i,iii,iv) is an immediate consequence of Lemmas 4.1.1, 4.1.3. The extra ingredients that need to be computed are the Euler characteristics of the branch loci: $\chi(C) = 2d((a+b) - dab)$, $\chi(D) = 2p((m+n) - pmn)$. The proper transform of D, D', is a $d-$cover of D branched on $\text{card}(C \cap D) = dp(an+bm)$ points. Then: $\chi(D') = d\chi(D) - (d-1)(C \cdot D) = pd(2(m+n-pmn) - (d-1)(an+bm))$.

The fact that that K_N is ample follows [Har77] from the fact that a pull-back of an ample line bundle through a finite map is ample.

□

Lemma 5.1.2. *If $\pi : N \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be a bi-simple-cover as above and a, b, m, n strictly positive integers, then N is simply connected.*

Proof. If a, b, m, n are strictly positive integers, then the divisors C, D are flexible divisors, and then by applying Theorem 4.2.2 twice the manifold N is simply connected. □

If X is a complex surface, we denote by $\mathcal{O}_X$ the structure sheaf of X, and by $\chi_h(X) = \chi(X, \mathcal{O}_X)$ its *holomorphic Euler characteristic.* By Todd-Hirzebruch formula, this is the same as the *Todd genus* of our manifold X. It can be easily computed [BPV84] in terms of the Chern invariants as $\chi_h(X) = \frac{c_1^2(X)+c_2(X)}{12}$.

On manifolds with finite fundamental group any two numerical invariants completely determine the others. Because the holomorphic Euler characteristic is constant under the blow-up process we prefer to use this invariant instead of the Euler characteristic.

Proposition 5.1.3. *Let $\pi : N \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be a bi-simple-cover of type (d,2) branched along the curves (C, D) such that $\mathcal{O}(C) = \mathcal{O}(da, db)$ and $\mathcal{O}(D) =$*

$\mathcal{O}(2m, 2n)$, where a, b, m, n are positive integers then N is almost completely decomposable.

Proof. We prove this proposition in two steps. First we show that the d-cover $\pi_2 : X \to \mathbb{CP}^1 \times \mathbb{CP}^1$ branched along C is almost completely decomposable. Let $\varphi_{(a,b)} : \mathbb{CP}^1 \times \mathbb{CP}^1 \to \mathbb{CP}^N$, where $N = (a+1)(b+1)-1$, be the Segre-type embedding, and $[p_{ij}]$ be homogeneous coordinates on $\mathbb{CP}^N$ corresponding to (a, b)-bi-homogeneous monomials. Then if φ_D is the bi-degree (da, db) polynomial whose zero locus is D, then $\varphi_D = \varphi^*_{(a,b)}(f(p_{ij}))$, where f is a degree d polynomial. Hence, by Proposition, 3.2.2 X is almost completely decomposable.

We need one more ingredient to be able to finish our proof. The following lemma is proved in [MaMo80]:

Lemma 5.1.4. ([MaMo80], 3.4) *Suppose W is a compact complex 3-manifold and V, X_1, X_2 are closed simply-connected complex submanifolds with normal crossing in W. Let $S = X_1 \bigcap X_2$ and $C = V \bigcap S$. Suppose as divisors V is linearly equivalent to $X_1 + X_2$ and that $C \neq \emptyset$. Set $n = \operatorname{card} C$ and g be the genus of S. Then we have the diffeomorphism:*

$$V \# \mathbb{CP}^2 \cong X_1 \# X_2 \# 2g\ \mathbb{CP}^2 \# (2g+n-1)\overline{\mathbb{CP}^2}.$$

Let $\pi_1 : N \to X$ be the double cover branched along $D' = \pi_2^{-1}(D)$. Then $\mathcal{O}(D') = \pi_2^*(\mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(2m, 2n))$ and let $L = \pi_2^*(\mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(m, n))$. Assuming the notations from Section 4.1, there exists a tautological section $z \in \Gamma(L, p^*(\mathcal{L}))$ such that $N \subset L$ is the zero locus of $z^2 - p'^*(\varphi_D) \in \Gamma(L, p^*(\mathcal{L}^{\otimes 2}))$. We can compactify L as $W = \mathbb{P}(L \oplus \mathcal{O}_L)$ by adding a divisor $W_\infty = W \setminus L$. Then $p' : W \to X$ is a $\mathbb{CP}^1$-bundle and we can try to extend the section z to a section in $\Gamma(W, p'^*(\mathcal{L}))$. As $p'^*(L)$ restricts trivially on the fiber and z has a zero at the origin of $L \to X$ then this section will have a pole of multiplicity one along W_∞. To adjust to this problem we consider the section $z' \in \Gamma(W, p'^*(\mathcal{L}) \otimes \mathcal{O}(W_\infty))$, $z' = z \cdot \varphi_{W_\infty}$. Let $\varphi_{W_\infty} = \varphi_\infty$. Then z' has no zero along W_∞ and rescales the values of z outside W_∞. Hence, our manifold N is the zero locus $(\ z'^2 - p'^*(\varphi_D) \cdot \varphi_\infty^2 = 0\ )$.

Let D_1, D_2 be two smooth curves on $\mathbb{CP}^1 \times \mathbb{CP}^1$, such that $\mathcal{O}(D_1) = \mathcal{O}(D_2) = \mathcal{O}(m,n)$ and which are transversal to each other and to D and C. Then $p'^{-1}(D_i) = D_i'$, $i = 1, 2$ are smooth curves on X transversal to each other and $D_1' + D_2'$ is linearly equivalent to D'. Let

$$X_i = (z' - p'^*(\varphi_{D_i}) \cdot \varphi_\infty = 0), i = 1, 2.$$

We remark that X_i is a cover of X of degree one, hence it is diffeomorphic to X. Then N, X_1, X_2 verify the requirements in the lemma so:

$$N \# \mathbb{CP}^2 \cong X \# X \# r\mathbb{CP}^2 \# s\overline{\mathbb{CP}^2}, \text{ for suitable } r, s.$$

If $R = (\varphi_{D_1} - \varphi_{D_2} = 0) \subset \mathbb{CP}^1 \times \mathbb{CP}^1$. Then S is a d-cover of R branched along $R \cdot C = dam + dbn$ points. So: $\chi(S) = d\,\chi(R) - (d-1)d(am+bn) \leq 0$. Hence genus of S is strictly greater than zero, i.e. $r \geq 1$.

But X is almost completely decomposable and so is N. □

Using the same ideas we can prove a more general statement:

Theorem 5.1.5. *Iterated cyclic covers of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along smooth, flexible, transversal curves are almost completely decomposable.*

Proof. In the proof of the previous proposition we have never used the fact that N was obtained by taking two cyclic covers. The same argument is true for iterated covers. So, all we need to prove is that the same reasoning works for covers of arbitrary degree. We show this by induction on the degree k of the cyclic cover, branched along D with $\mathcal{O}(D) = \mathcal{O}(km, kn)$. The first step, $k = 2$, is proved by the proposition. For the induction step we need a small change in our decomposition, namely: $\mathcal{O}(D_1) = \mathcal{O}(m, n)$, $\mathcal{O}(D_2) = \mathcal{O}\big((k-1)m, (k-1)n\big)$ and $X_1 = (z' - p'^*(\varphi_{D_1}) \cdot \varphi_\infty = 0), X_2 = (z'^{k-1} - p'^*(\varphi_{D_2}) \cdot \varphi_\infty^{k-1}) = 0$. The rest follows. □

5.2 Surfaces of general type with finite cyclic fundamental group

In this subsection we give a recipe for a construction of complex surfaces of general type with fundamental group isomorphic to $\mathbb{Z}_d$. They are quotients by a free $\mathbb{Z}_d$ action of bi-cyclic covers $\pi : N \to \mathbb{CP}^1 \times \mathbb{CP}^1$ of type $(2, d)$. We construct explicitly an action of $\mathbb{Z}_d$.

We denote by $x = [x_0 : x_1], y = [y_0 : y_1]$ the homogeneous local coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^1$. We want to consider the action of $\mathbb{Z}_d =< e^{\frac{2\pi i}{d}} >$ generated by:

$$e^{\frac{2\pi i}{d}} * ([x_0 : x_1], [y_0 : y_1]) = ([e^{\frac{2\pi i}{d}} x_0 : x_1], [e^{\frac{2\pi i}{d}} y_0 : y_1]).$$

This action has four fixed points:

$$([0:1],[0:1]),\ ([0:1],[1:0]),([1:0],[0:1]),\ ([1:0],[1:0]).$$

If we require that a smooth flexible divisor $D \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ be invariant under the above $\mathbb{Z}_d$ action, i.e. φ_D is $\mathbb{Z}_d$ invariant, then φ_D is a bi-homogeneous polynomial of bi-degrees divisible by d. So, there are natural numbers (a,b) such that $\mathcal{O}(D) = \mathcal{O}(da, db)$ and

$$\varphi_D = \sum_{\substack{i=\overline{0,a} \\ j=\overline{0,b}}} a_{ij} X_0^{di} X_1^{d(a-i)} Y_0^{dj} Y_1^{d(b-j)} + \sum_{i=\overline{0,da}} \sum_{i \le dj \le db+i} b_{ij} X_0^{i} X_1^{da-i} Y_0^{dj-i} Y_1^{d(b-j)+i}$$

for some complex coefficients a_{ij}, b_{ij}. The linear system of $\mathbb{Z}_d$–invariant sections of $\mathcal{O}(da, db)$ is base point free, hence by Bertini's Theorem the generic section is smooth, and we can choose D such that none of the four points are on D. Hence $\mathbb{Z}_d$ acts freely on D.

Let $U_0 = \{[1:x] | x \in \mathbb{C}\}$ and $U_1 = \{[x:1] | x \in \mathbb{C}\}$ be charts of $\mathbb{CP}^1$. Then the charts $U_0 \times U_0, U_0 \times U_1, U_1 \times U_0, U_1 \times U_1$ form an atlas for $\mathbb{CP}^1 \times \mathbb{CP}^1$. The line bundle $\mathcal{O}(a,b)$ restricted to these charts admits a trivialization. Let $z_{00}, z_{01}, z_{10}, z_{11}$ be the corresponding coordinates on each trivialization.

On the chart $U_1 \times U_1 \times \mathbb{C}$ we let the $\mathbb{Z}_d =< e^{\frac{2\pi i}{d}} >$ action be generated by:

$$e^{\frac{2\pi i}{d}} * ([x_0 : x_1], [y_0 : y_1], z_{11}) \to ([e^{\frac{2\pi i}{d}} x_0 : x_1], [e^{\frac{2\pi i}{d}} y_0 : y_1], e^{\frac{2\pi i}{d}} z_{11})$$

Using the change of coordinates, the above action is generated in the other charts by:

on $U_0 \times U_1$: $(x_1, y_0, z_{01}) \to (e^{\frac{2\pi i(d-1)}{d}} x_1,\ e^{\frac{2\pi i}{d}} y_0,\ e^{\frac{2\pi i(1+a)}{d}} z_{01})$;

on $U_1 \times U_0$: $(x_0, y_1, z_{10}) \to (e^{\frac{2\pi i}{d}} x_0,\ e^{\frac{2\pi i(d-1)}{d}} y_1,\ e^{\frac{2\pi i(1+b)}{d}} z_{10})$;

on $U_0 \times U_0$: $(x_1, y_1, z_{00}) \to (e^{\frac{2\pi i(d-1)}{d}} x_1,\ e^{\frac{2\pi i(d-1)}{d}} y_1,\ e^{\frac{2\pi i(1+a+b)}{d}} z_{00})$.

Hence, we have the following lemma:

Lemma 5.2.1. *The $\mathbb{Z}_d$ action given explicitly above acts freely on $(\mathcal{O}(a,b) \setminus 0$-section) if and only if each of integers $\{a+1,\ b+1,\ a+b+1\}$ is relatively prime to d.*

We also need to consider a weighted action of $\mathbb{Z}_d$ which is defined on $U_1 \times U_1$ as follows: $e^{\frac{2\pi i}{d}} * (x_0, y_0, z_{11}) \longrightarrow (e^{\frac{2\pi i}{d}} x_0, e^{2\frac{2\pi i}{d}} y_0, e^{\frac{2\pi i}{d}} z_{11})$. This action extends on the other coordinate charts as:

on $U_0 \times U_1$: $(x_1, y_0, z_{01}) \to (e^{\frac{2\pi i(d-1)}{d}} x_1,\ e^{2\frac{2\pi i}{d}} y_0,\ e^{\frac{2\pi i(1+a)}{d}} z_{01})$;

on $U_1 \times U_0$: $(x_0, y_1, z_{10}) \to (e^{\frac{2\pi i}{d}} x_0,\ e^{\frac{2\pi i(d-2)}{d}} y_1,\ e^{\frac{2\pi i(1+2b)}{d}} z_{10})$;

on $U_0 \times U_0$: $(x_1, y_1, z_{00}) \to (e^{\frac{2\pi i(d-1)}{d}} x_1,\ e^{\frac{2\pi i(d-2)}{d}} y_1,\ e^{\frac{2\pi i(1+a+2b)}{d}} z_{00})$.

We have a similar lemma:

Lemma 5.2.2. *The weighted $\mathbb{Z}_d$ action defined above acts freely on $(\mathcal{O}(a,b) \setminus 0$-section) if and only if each of integers $\{a+1,\ 2b+1,\ a+2b+1\}$ is relatively prime to d.*

A $d-$cyclic cover $N' \to \mathbb{CP}^1 \times \mathbb{CP}^1$ branched along D with $\mathcal{O}(D) = \mathcal{O}(da, db)$ is a submanifold of $\mathcal{O}(a,b)$. If the branch locus D is invariant under this action, or equivalently φ_D is $\mathbb{Z}_d-$invariant, then the action $\mathbb{Z}_d$ on $\mathcal{O}(a,b)$ restricts to an action on N'. Moreover, if D does not contain any of the fixed four points and the conditions in the Lemma 5.2.1 are satisfied, then $\mathbb{Z}_d$ acts freely on N'.

Now we are ready to construct our examples:

Proposition 5.2.3. *Let C, D smooth, flexible, transversal divisors on $\mathbb{CP}^1 \times \mathbb{CP}^1$ such that both are invariant under the above $\mathbb{Z}_d$ action. Let a, b, m, n be positive integers such that $\mathcal{O}(D) = \mathcal{O}(da, db)$ and $\mathcal{O}(C) = \mathcal{O}(2dm, 2dn)$, and d is relatively prime to any of the numbers $a+1, b+1, a+b+1$. Then, the bi-cyclic cover $\pi : N \to \mathbb{CP}^1 \times \mathbb{CP}^1$ of type $(d, 2)$ branched along (D, C) admits a free $\mathbb{Z}_d$ action. The quotient $M = N/\mathbb{Z}_d$ has the following properties:*

(i) *M is a smooth complex surface, with fundamental group $\pi_1(M) = \mathbb{Z}_d$;*

(ii) *K_M is an ample line bundle if $(d-1)a + dm > 2$ and $(d-1)b + dn > 2$;*

(iii) $c_1^2(M) = 4\Big((d-1)a + dm - 2\Big)\Big((d-1)b + dn - 2\Big)$;

(iv) $c_2(M) = 8 - 4(d-1)(a+b-dab) - 4d(m+n-2dmn) + 2(d-1)d(an+bm)$;

(iv') $\chi_h(M) = \frac{(d-1)(2d-1)}{3}ab + \frac{(d-1)}{2}(dan + dbm - a - b) + d^2mn - dm - dn + 2.$

Proof. First, we need to define the $\mathbb{Z}_d$ action on N. The action that we want to consider is the trivial extension of the action on $\mathcal{O}(a, b)$ to an action on $\mathcal{O}(a, b) \oplus \mathcal{O}(dm, dn)$. The conditions from the theorem imply that the action restricts to N as a free holomorphic action. Its quotient is a smooth complex surface, with fundamental group $\pi_1(M) = \mathbb{Z}_d$. The numerical invariants of N are described by Lemma 5.1.1. The invariants of M are related to those of N by the following relations: $c_2(M) = \frac{1}{d}c_2(N)$, $c_1^2(M) = \frac{1}{d}c_1^2(N)$. The computation from (iii-iv') are immediate.

The conditions that d and $a+1, b+1, a+b+1$ are relatively prime integers imply that $\chi_h(M)$ is an integer number, as expected.

By Lemma 5.1.1, K_N is ample which shows [Har77] that so is K_M.

□

5.3 Surfaces of general type with infinite fundamental group

In this subsection we use the same techniques to construct complex surfaces with ample canonical line bundle and infinite fundamental group, whose universal covers are nontrivial, more precisely have infinite second homology group.

Let Σ_1, Σ_2 be two Riemann surfaces of genus g_1, g_2 greater than one. And let $p_1, \ldots, p_{2n}$ a collection of $2n$ distinct points on Σ_1 and $q_1, \ldots, q_{2m}$ a collection of $2m$ distinct points on Σ_2. Let $pr_i : \Sigma_1 \times \Sigma_2 \to \Sigma_i$, $i = 1, 2$ the projection on the corresponding factors. We denote by D the effective divisor given by $pr_1^{-1}(\{p_1, \ldots, p_{2n}\}) \bigcup pr_2^{-1}(\{q_1, \ldots, q_{2m}\})$. D has $4nm$ normal crossing singularities, and we can compute the important numerical invariants:

$K_{\Sigma_1 \times \Sigma_2} \cdot D = 4n(1-g_1) + 4m(1-g_2)$;

$D^2 = 8nm$;

$g_D = 2mg_1 + 2ng_2 + (2n-1)(2m-1)$, $\chi(D) = -4\Big(g_1(m-1) + g_2(n-1) + 2nm\Big)$;

$\chi(\Sigma_1 \times \Sigma_2, \mathcal{O}(D)) = (2n - g_1 + 1)(2m - g_2 + 1)$.

So, for large enough values of n, m the line bundle $\mathcal{O}(D) \otimes K^{\vee}_{\Sigma_1 \times \Sigma_2}$ is positive, and by Kodaira Vanishing Theorem $H^j(\Sigma_1 \times \Sigma_2, \mathcal{O}(D)) = 0, j = 1, 2$. So its Euler characteristic gives us the number of sections.

Let $f' : N' \to \Sigma_1 \times \Sigma_2$ be a double cover branched along D. To resolve the singularities of N', we have to introduce $4nm$ exceptional spheres of self-intersection (-2). Let $f : N \to N'$ be the resolution of N'.

Proposition 5.3.1. *The surface N has infinite fundamental group and it is diffeomorphic to a complex surface with ample canonical line bundle.*

Proof. To show that it has infinite fundamental group it is enough to find a loop and a 1−form which evaluated on that loop is nonzero. Let $[\gamma] \in \pi_1(\Sigma_1)$ a nontrivial class, and $[\alpha] \in H^1(\Sigma_1, \mathbb{Z})$ such that $\int_\gamma \alpha \neq 0$. Because $\Sigma_1 \times q_1$ is a component of the branch locus then we can embed Σ_1 in N as the proper transform of $f'^{-1}(\Sigma_1 \times q_1)$. We have a corresponding class $[\gamma'] \in \pi_1(N)$ and $\int_{\gamma'} (pr_1 \circ f' \circ f)^*(\alpha) \neq 0$. Moreover, the above reasoning implies that the homomorphism $(f' \circ f)_\# : \pi_1(N) \to \pi_1(\Sigma_1 \times \Sigma_2)$ is onto.

For the second part of the statement, according to Proposition 4.1.4, we have to show that the linear system associated to $\mathcal{O}(D)$ is base point free. This is obvious since our divisor is a union of vertical and horizontal fibers. Hence, our manifold is diffeomorphic to a manifold obtained as a double cover with smooth branch locus. Its canonical bundle is the pull-back of an ample line bundle through a finite morphism, hence is ample. $\square$

A second description of the manifold N can be given as follows: Let Σ_1', Σ_2' be the Riemann surface obtained by taking the double cover of Σ_1,Σ_2 respectively, branched along $\{p_1, \dots, p_{2n}\}, \{q_1, \dots, q_{2m}\}$. Then on $\Sigma_1' \times \Sigma_2'$ we have a natural $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ action interchanging the sheets on each factor. The diagonal subgroup $\mathbb{Z}_2$ acts on $\Sigma_1' \times \Sigma_2'$ with $2n \cdot 2m$ fixed points, and $N' = \Sigma_1' \times \Sigma_2'/\mathbb{Z}_2$. If we first blow-up the fixed points then the $\mathbb{Z}_2$ action extends trivially on the exceptional divisor and the quotient $\left((\Sigma_1' \times \Sigma_2')\#4nm\overline{\mathbb{CP}^2}\right)/\mathbb{Z}_2$ is the smooth manifold N.

Chapter 6

Symplectic 4-manifolds

A *symplectic structure* on a oriented, smooth, real $2n$-manifold M is a non-degenerate closed 2-form $\omega \in \Omega^2(M)$ compatible with orientation. The compatibility condition means that the n-fold wedge product is a positive multiple of the volume form everywhere. Any Kähler manifold is also symplectic, but there are large families of symplectic manifolds which are not Kähler, for example all manifolds with b_1 odd. Most of the examples are obtained via surgeries.

In some cases these surgeries yield Kähler manifolds. We analyze this situation in the last part of the thesis.

6.1 Symplectic surgeries

In this section we briefly recall two important techniques of constructing symplectic 4–manifolds. One is the normal connected sum procedure of Gompf [Gom95] and the other is Fintushel and Stern's [FiSt97] rational blow-down.

6.1.1 The normal connected sum

Let (M_i, ω_i), $i = 1, 2$ be two symplectic 4−manifolds. Suppose there exist $\Sigma_i \subset M_i$, $i = 1, 2$ two closed, smooth 2−dimensional symplectic submanifolds of the same genus, and satisfying the compatibility condition:

$$N_{\Sigma_1|M_1} = N^{\vee}_{\Sigma_2|M_2}.$$

Let $N_1(\Sigma_i)$, $N_2(\Sigma_i)$ be two tubular neighborhoods of Σ_i, such that $\overline{N_1(\Sigma_i)} \subset N_2(\Sigma_i)$. We denote by W_i the tubular shell neighborhood $N_2(\Sigma_i) - \overline{N_1(\Sigma_i)}$ of Σ_i in M_i. Suppose we have an orientation preserving diffeomorphism $\Phi : W_1 \to W_2$ taking the inside boundary of W_1 to the outside boundary of W_2. We define the *normal connected sum* of M_1 and M_2 along Σ_1 and Σ_2 via Φ to be the smooth oriented manifold obtained by gluing $M_1 - \overline{N_1(\Sigma_1)}$ and $M_2 - \overline{N_1(\Sigma_2)}$ along the tubular shell neighborhoods W_1 and W_2 using Φ. Let $M = M_1 \#_\Phi M_2$ be the resulting 4−manifold.

Theorem 6.1.1 (Gompf). *Possibly after rescaling ω_1 or ω_2, there exists a symplectomorphism Φ of the tubular shell neighborhoods of Σ_1 and Σ_2 such that the 4−manifold $M = M_1 \#_\Phi M_2$ admits a symplectic form which agrees with the rescaled symplectic forms on $M_i - \overline{N_1(\Sigma_i)}$.*

Since the disjoint union $M_1 \sqcup M_2$ and M are oriented-cobordant to each other, the signature of the two manifolds are equal:

$$\tau(M) = \tau(M_1) + \tau(M_2).$$

The Euler characteristic can be easily computed, too:

$$\chi(M) = \chi(M_1) + \chi(M_2) - 2\chi(\Sigma_1)$$

So, we immediately have:

$$c_1^2(M) = c_1^2(M_1) + c_1^2(M_2) - 4\chi(\Sigma_1).$$

A particular case is the *fiber sum* operation. This construction gives a symplectic description of the elliptical surfaces without multiple fibers. Let $E(1) = \mathbb{CP}^2 \# 9\overline{\mathbb{CP}^2}$, where the nine points that we blow-up are the intersection of two generic cubics. Then $E(1)$ admits a fibration over $\mathbb{CP}^1$, with the generic fiber $T = T^2$ a real two dimensional torus, induced by the pencil determined by the two cubics. We can form the fiber sum by taking two copies of $E(1)$ and identifying the two corresponding fibers via the identity map. We denote this new manifold by $E(2)$. Notice that $E(2)$ is simply connected, as the complement of a fiber in $E(1)$ is simply connected. Also, $E(2)$ has the same topological invariants as a $K3$ surface. It can be easily proved that in fact is diffeomorphic [GoSt99] to a $K3$ surface. Iterating this construction for n copies of $E(1)$ we obtain the other elliptic surfaces $E(n)$.

Using these surgeries, Gompf was able to show that the symplectic four manifolds are better behaved than complex surfaces with respect to groups that can be obtained as fundamental groups:

Theorem 6.1.2 (Gompf). [Gom95] *Let G any finitely presented group. Then there is a closed , symplectic 4-manifold X with $\pi_1(X) \cong G$. Furthermore, X*

may be chosen to be spin or non-spin.

Proof. We give a sketch of the proof as we need this construction later.

Let $G = \langle g_1, \ldots, g_k \mid r_1, \ldots, r_l \rangle$ be a finite representation of G with k generators g_i and relators r_i. Let F be a Riemannian surface with genus k and fix a collection of circles $\alpha_i, \beta_i \subset F$ representing a basis of $H_1(F, \mathbb{Z})$ such that $\alpha_i \cdot \beta_j = \delta_i^j$, $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset$. Let γ_j, $j = 1, \ldots, l$ be immersed curves in F representing the relators r_j in the free group $\pi_1(F)/\langle \beta_1, \ldots, \beta_k \rangle$, where g_i are identified with α_i. Let $\gamma_{l+i} = \beta_i$, $i = 1, \ldots, k$.

On the torus $T^2 = S^1 \times S^1$, let α, β be the circles generating it. Now consider the collection of tori $T_i = \gamma_i \times \alpha \subset F \times T^2$, $i = 1, \ldots, k+l$ and $T_0 = pt \times T^2 \subset F \times T^2$. In [Gom95], Gompf shows that for a good choice of these tori the product symplectic form can be perturbed in such a way that the tori forming the family $\{T_i'\}_{i=\overline{0,k+l}}$ are disjoint symplectic submanifolds of $(F \times T^2, \omega')$.

Let X_0 be a $K3$ surface of Kummer type. We can form the fiber sums of $F \times T^2$ and $k+l$ copies of X_0, along the tori T_i' and generic fibers $F \subset X_0$. We denote by X the resulting symplectic 4-manifold. The complement of a generic fiber in X_0 is simply connected, then by Seifert-Van Kampen Theorem the manifold X has fundamental group $\pi_1(X) = G$. Moreover, X is a spin manifold as all the manifolds involved in surgeries are spin.

If we replace the $K3$ surface by a rational elliptic surface $\mathbb{CP}^2\#9\overline{\mathbb{CP}^2}$ then the resulting manifolds are non-spin, and the numerical invariants are smaller.

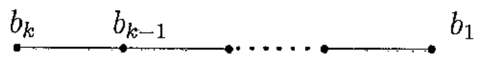
$\square$

We remark that for both spin and non-spin manifolds $c_1^2(X) = 0$. If X

is the spin, respectively non-spin, manifold constructed above, then it has $c_2(X) = 24k$, respectively $c_2(X) = 12k$.

6.1.2 The rational blow-down

Let $C_{p,q}$ be an open smooth 4-manifold obtained by plumbing disk bundles over a chain configuration of 2-sphere prescribed by the following diagram:



Here $p > q > 0$ are two relatively prime integers and $\frac{p^2}{pq-1} = [b_k, b_{k-1}, \ldots, b_1]$ is the unique continued fraction with all $b_i \geq 2$. Each vertex represents a disk bundle over the 2-sphere of self-intersection $-b_i$. Then $C_{p,q}$ is a negative definite simply connected 4−manifold whose boundary is the lens space $L(p^2, 1-pq)$. The lens space $L(p^2, 1-pq)$ bounds a rational ball $B_{p,q}$ with $\pi_1(B_{p,q}) \cong \mathbb{Z}_p$.

Suppose X is a smooth 4-manifold containing a configuration $C_{p,q}$. Then we may construct a new smooth 4-manifold $X_{p,q}$, called the *(generalized) rational blow-down* of X, by replacing $C_{p,q}$ with the rational ball $B_{p,q}$. $X_{p,q}$ is uniquely determined (up to diffeomorphism) by X, as each diffeomorphism of $\partial B_{p,q}$ extends over the rational ball $B_{p,q}$. It is proved in [Sym01] that if $C_{p,q}$ is symplectically embedded in X, then the rational blow-down carries a symplectic structure.

6.2 Geography of symplectic 4-manifolds

Using iterated cyclic-covers of $\mathbb{CP}^2$ and symplectic surgeries introduced by Gompf in [Gom95], Braungardt and Kotschick [BrKo05] were able to show:

Theorem 6.2.1. [BrKo05] *For every $\epsilon > 0$, there is a constant $c(\epsilon) > 0$ such that every lattice point (x, y) in the first quadrant satisfying*

$$y \leq (9 - \epsilon)x - c(\epsilon)$$

is realized by the Chern invariants (χ_h, c_1^2) of infinitely many pairwise non-diffeomorphic simply connected minimal symplectic manifolds, all of which are almost completely decomposable.

Chapter 7

Main results

Kotschick and his collaborators [Kot98], and LeBrun constructed pairs of homeomorphic simply connected manifolds such that one of them admits an Einstein metric, while the other one does not admit any Einstein metric. In this thesis we show that a similar statement holds for manifolds with non-trivial fundamental group.

7.1 Einstein metrics on compact non-simply-connected manifolds

For any finite cyclic fundamental group we construct infinitely many classes of manifolds such that each class supports a differential structure that admits an Einstein metric and infinitely many other differential structures that do not admit any Einstein metrics:

Theorem 7.1.1. *For any finite cyclic group $\mathbb{Z}/d\mathbb{Z}$ there exist infinitely many pairs of compact oriented smooth 4-manifolds $(Z_i, M_{i,j}), i, j \in \mathbb{N}$ satisfying:*

1. *The fundamental group of* Z_i *and* $M_{i,j}$ *is* $\mathbb{Z}/d\mathbb{Z}$ *for any* $i, j \in \mathbb{N}$*;*

2. *For* i *fixed and any* j*,* Z_i *and* $M_{i,j}$ *are homeomorphic, but no two are diffeomorphic to each other;*

3. Z_i *admits an Einstein metric, while no* $M_{i,j}$ *admits any Einstein metrics.*

Moreover their universal covers $\widetilde{Z}_i$ *and* $\widetilde{M_{i,j}}$*, respectively satisfy:*

4. $\widetilde{M_{i,j}}$ *is diffeomorphic to* $n\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$*, where* $n = b_2^+(\widetilde{Z}_i)$ *and* $m = b_2^-(\widetilde{Z}_i)$*;*

5. $\widetilde{Z}_i$ *and* $\widetilde{M_{i,j}}$ *are not diffeomorphic, but become diffeomorphic after connected sum with one copy of* $\mathbb{CP}^2$*.*

Proof. First we construct the manifolds Z_i. They are complex surfaces of general type with ample canonical line bundle. Hence, by Aubin-Yau's Theorem 2.3.2 these manifolds admit Kähler-Einstein metrics.

We need two consider two cases, determined by the parity of d.

For d odd we have the following construction: Let $M(d; a, b, m, n)$ the manifold constructed in Proposition 5.2.3 as a $\mathbb{Z}_d$-quotient of a bi-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ of type $(d, 2)$, branched along (D, C) with $\mathcal{O}(D) = \mathcal{O}(da, db)$, $\mathcal{O}(C) = (2dm, 2dn)$. Let $Z_i = M(d; d, d, i, i)$. $d+1, 2d+1$ are relatively prime to d so the conditions in Proposition 5.2.3 are satisfied. An easy computation of the numerical invariants of Z_i yields:

- $c_1^2(Z_i) = 4(d(d-1) + di - 2)^2$;

- $\chi_h(Z_i) = \frac{1}{3}d^2(d-1)(2d-1) + d(d-1)(di-1) + d^2i^2 - 2di + 2$.

We can compute the signature in terms of these invariants as $\tau(Z_i) = (-8\chi_h + c_1^2)(Z_i)$. If we consider just the intersection forms, then we know that the signature associated to an even intersection satisfies $\tau \equiv 0 \bmod 8$, and Rohlin's Theorem states that on a spin manifold we have the following relation: $\tau \equiv 0 \bmod 16$. For i odd, $c_1^2(Z_i) \neq 0 \bmod 8$, hence $\tau(Z_i) \neq 0 \bmod 8$. Its universal cover $\widetilde{Z}_i$ has signature $\tau(\widetilde{Z}_i) = d\tau(Z_i)$, so for d, i odd numbers $\tau(\widetilde{Z}_i) \neq 0 \bmod 16$. Hence Z_i is of w_2-type (I) and odd intersection form.

As i increases then $\frac{c_1^2}{\chi_h}(Z_i)$ approaches $\frac{4d^2i^2}{d^2i^2} = 4$.

In the case d even we need a different construction. The idea is, though, the same. Let $\pi : N(d; a, b, m, m) \to \mathbb{CP}^1 \times \mathbb{CP}^1$ be a bi-cyclic cover of type $(d, 3)$ branched along (D, C) with $\mathcal{O}(D) = (da, db)$, $\mathcal{O}(C) = (3dm, 3dm)$ and such that D intersects C transversally. To simplify the notation we use $N = N(d; a, b, m, m)$ whenever we want to prove a general statement about the whole class of manifolds. Proposition 5.1.1 tells us that the canonical line bundle is $K_N = \pi^*(\mathcal{O}((d-1)a+2m-2, (d-1)b+2m-2))$, hence [Har77] N is a surface of general type, with ample canonical line bundle if $(d-1)a+2m-2 > 0$, $(d-1)b + 2m - 2 > 0$. Using Proposition 5.1.2, we can also conclude that N is simply connected.

We want N to be non-spin. We show that this is true if d even and b odd. The method which we want to implement is by finding a class $[A] \in H_2(N, \mathbb{Z})$ such that $[A] \cdot w_2(N) \neq 0 \bmod 2$. We construct N in two steps:

$$N \xrightarrow{\pi_2} X \xrightarrow{\pi_1} \mathbb{CP}^1 \times \mathbb{CP}^1$$

where first we consider a 3-cyclic cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along C, and then a d-cyclic cover branched along $\pi_1^{-1}(D)$, the proper transform of D.

We construct a 1-parameter family of deformation of N in the following way. Let $D" = A \cup B \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ such that $A = \{pt\} \times \mathbb{CP}^1$, B a smooth curve of bi-degree $(da-1, db)$, and such that A, B, C intersect transversally at all points. Then $D, D"$ are linearly equivalent, and the pencil associated to D and $D"$ determines a 1-parameter family of deformation of $D"$ with at most a finite number of singular members. Pulling this back to X we have a 1-parameter family associated to $D' = \pi_1^{-1}(D") = \pi_1^{-1}(A) \cup \pi_1^{-1}(B)$ and $\pi_1^{-1}(D)$. Let X_0 be the d-cyclic cover of X branched along D'. As in the proof of Proposition 4.1.4, we have a 1-parameter smoothing of X_0, $\varpi : \mathcal{X} \to \Delta \subset \mathbb{C}$ such that $X_0 = \varpi^{-1}(0)$ and $\varpi^{-1}(1) = N$. The branch locus D' has normal crossing singularities at $\pi_1^{-1}(A \cup B)$ locally given by $x \cdot y = 0$. This implies that X_0 has isolated singularities above these points, and locally they look like $z^d = x \cdot y$, where (z, x, y) are local coordinates on $\pi_1^*(\mathcal{O}_{\mathbb{CP}^1 \times \mathbb{CP}^1}(D))$. This singularity is classified as of A_{d-1} type. We obtain a resolution, $N' \xrightarrow{p_2} X$ by introducing A_{d-1}-strings of exceptional divisors. We denote these divisors by E_i. Let $A"$ be the proper transform of A'. Proposition 4.1.2 tells us that:

$$\begin{aligned}
\mathcal{O}(A") &= \frac{1}{d}(\pi_1 \circ p_2)^*(\mathcal{O}(1,0)) + \Sigma a_i E_i, \ a_i \leq 0 \\
K_{N'} &= (\pi_1 \circ p_2)^*(\mathcal{O}((d-1)a + 2m - 2, (d-1)b + 2m - 2)) \\
K_{N'} \cdot A" &= c_1\Big((\pi_1 \circ p_2)^*(\mathcal{O}((d-1)a + 2m - 2, (d-1)b + 2m - 2))\Big) \cup \\
&\quad c_1\big(\frac{1}{d}(\pi_1 \circ p_2)^*(\mathcal{O}(1,0)) + \Sigma a_i E_i\big) \\
&= \frac{1}{d} 3d c_1\big(\mathcal{O}((d-1)a + 2m - 2, (d-1)b + 2m - 2))\big) \cup c_1(\mathcal{O}(1,0))
\end{aligned}$$

$$= 3((d-1)b + 2m - 2)$$

$$\equiv 1 \bmod 2 \quad \text{if } b \text{ odd}, \ d \text{ even}$$

For these singularities we know [HKK86] that the resolution and the smoothing are diffeomorphic. Hence, N' and N are diffeomorphic non-spin manifolds.

The number d decomposes as $d = 2^k d'$ such that $d' \in \mathbb{N}$ is odd.

We want to consider the manifolds:

$$N_i = N(d; 2d', d', i, i) \subset \mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(2d', d') \oplus \mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(di, di) \text{ if } d' \neq 1,$$

$$\text{and} \quad N_i = N(d; 6, 3, i, i) \text{ if } \ d = 2^k.$$

On these manifolds, we can define a $\mathbb{Z}_d$ action as in Lemma 5.2.2, where we extend the action trivially on the second factor. For this action to be well-defined we need D and C to be invariant under the induced action on $\mathbb{CP}^1 \times \mathbb{CP}^1$. Such curves always exist, moreover we can choose D, C such that $\mathbb{Z}_d$ acts freely on them. The conditions in the lemma 5.2.2 are automatically satisfied by our choice of degrees.

Let $Z_i = N_i/\mathbb{Z}_d$. Z_i is complex surface of general type, with ample canonical line bundle, finite fundamental group $\pi_1(Z_i) = \mathbb{Z}_d$ and of w_2-type (I).

Its numerical invariants can be computed using Lemma 5.1.1 to be:

- $c_1^2(Z_i) = 6\big(2(d-1)d' + 2i - 2\big)\big((d-1)d' + 2i - 2\big)$;
- $c_2(Z_i) = 3[4 - 2(d-1)(3d' - 2d'^2 d) - 4(2i - 3i^2) + 3(d-1)d'i]$;
- $\tau(Z_i) = \frac{1}{3}(c_1^2 - 2c_2)(Z_i) \equiv -6(d-1)d'i \bmod 4$.

From the last relation we see that if i is odd, then $\tau(Z_i) \neq 0 \bmod 8$ hence the intersection form is odd.

For the special case $d = 2^k$ the numerical invariants are computed by the same formulas, for $d' = 3$. We take Z_i to be the subsequence indexed by odd coefficients.

As i increases $\frac{c_1^2}{\chi_h}(Z_i) = \frac{12c_1^2}{c_1^2+c_2}(Z_i)$ approaches $\frac{12 \cdot 6d4i^2}{6d4i^2+3d \cdot 12i^2} = \frac{24}{5} = 4.8$

Hence considering both cases, d odd or even, there is a constant $n_0 > 0$ such that for any $i > n_0$ we have $c_1^2(Z_i) \leq 5\chi_h(Z_i)$.

By Theorem 6.2.1 there exist a constant $n_1 > 0$ such that for any lattice point (x, y) in the first quadrant verifying $x > n_1,\ y \leq 8.5x$ there exists a infinite family of homeomorphic, non-diffeomorphic simply connected minimal symplectic manifolds M_j such that $y = c_1^2(M_j),\ x = \chi_h(M_j) = \frac{(c_1^2+c_2)(M_j)}{12}$. Eventually after truncating and relabeling the sequence Z_i, we can construct $M'_{i,j},\ i, j \in \mathbb{N}$, a family of simply connected symplectic manifolds satisfying:

- for fixed i, $M'_{i,j}$ are homeomorphic, but no two are diffeomorphic;
- $\chi_h(M_{i,j}) = \chi_h(Z_i)$ for any $j \in \mathbb{N}$;
- $c_1^2(M'_{i,j}) \geq 8\chi_h(M'_{i,j})$;
- $c_1^2(Z_i) \leq 5\chi_h(Z_i)$.

Let S_d be the rational homology sphere with fundamental group $\pi_1(S_d) = \mathbb{Z}_d$ described in 3.3. The manifolds $M_{i,j}$ are constructed as:

$$M_{i,j} = M'_{i,j} \# S_d \# k\overline{\mathbb{CP}^2}, \quad \text{where } k = c_1^2(M'_{i,j}) - c_1^2(Z_i).$$

We remark that by Theorem 2.1.3 the manifolds $M_{i,j}$ are not symplectic, but they have non-trivial Seiberg-Witten invariant.

For fixed i the manifolds Z_i and $M_{i,j}$ are all of w_2-type (I), with odd intersection form, fundamental group $\pi_1 = \mathbb{Z}_d$ and have the same numerical invariants: Todd-genus and Euler characteristic. Hence by Theorem 3.1.1, these manifolds are homeomorphic.

In the construction theorem for $M'_{i,j}$ the differential structures were distinguished by different *bandwidths*, after connected sums with $\overline{\mathbb{CP}^2}$'s and S_d, and using Theorem 2.1.3 we see that the bandwidths remain different for most of $M_{i,j}$. We can eventually select a new subsequence to get the desired sequence of manifolds.

An estimate for the number k of copies of $\overline{\mathbb{CP}^2}$ is given by:
$k = c_1^2(M_{i,j}) - c_1^2(Z_i) \geq 8\chi_h(Z_i) - 5\chi_h(Z_i) = 3\chi_h(Z_i) = 3\chi_h(M_{i,j})$ We also know that the manifolds are under the Bogomolov-Miyaoka-Yau line, which implies: $\chi_h(M_{i,j}) \geq \frac{1}{9}c_1^2(M_{i,j})$.
Hence $k \geq \frac{1}{3}c_1^2(M_{i,j}) = \frac{1}{3}(2\chi + 3\tau)(M_{i,j})$.

Then Theorem 2.2.1 implies that $M_{i,j}$ does not admit any Einstein metric. As a consequence we also get that Z_i and $M_{i,j}$ are not diffeomorphic.

For the results from the second part of the theorem we have to look at the universal covers $\widetilde{Z_i}$, and $\widetilde{M_{i,j}}$ respectively. From our construction, the universal cover of Z_i is a simply connected minimal complex surface of general type. It can not be diffeomorphic to connected sums of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}^2}$'s as it has non-trivial Seiberg-Witten invariants, but Theorem 5.1.3 tells us that after connected sum with one copy of $\mathbb{CP}^2$ it decomposes completely.

The universal cover of S_d is diffeomorphic to $(d-1)S^2 \times S^2$, hence the manifold $\widetilde{M_{i,j}} \cong dM'_{i,j}\#dk\overline{\mathbb{CP}^2}\#(d-1)S^2 \times S^2$. But $S^2 \times S^2\#\overline{\mathbb{CP}^2}$ is the complex surface $\mathbb{CP}^1 \times \mathbb{CP}^1$ blown-up at one point, which can also be presented as $\mathbb{CP}^2 \# 2\overline{\mathbb{CP}^2}$. So:

$$\widetilde{M_{i,j}} \cong dM'_{i,j} \# dk\overline{\mathbb{CP}^2} \# (d-1) S^2 \times S^2 \cong dM'_{i,j} \# (d-1)\mathbb{CP}^2 \# (dk + d - 1)\overline{\mathbb{CP}^2}.$$

But the manifolds $M'_{i,j}$ are almost completely decomposable, hence $\widetilde{M_{i,j}}$ is diffeomorphic to the connected sums of a number of $\mathbb{CP}^2$'s and $\overline{\mathbb{CP}^2}$'s. □

The manifolds used in Theorem 7.1.1 are necessarily non-spin. However, using obstructions implied by a non-trivial Bauer-Furuta invariant one can also find examples of spin manifolds M_i each of which support infinitely many smooth structures which don't admit any Einstein metric.

Theorem 7.1.2. *For any finite cyclic fundamental group, $\mathbb{Z}_d$, there is an infinite family of spin topological spaces M_i with fundamental group $\pi_1(M_i) = \mathbb{Z}_d$ such that each space supports infinitely many differential structures which do not admit an Einstein metric. All these manifolds satisfy the Hitchin-Thorpe Inequality.*

Proof. The manifolds M_i are constructed as connected sums and fiber sums of different blocks.

The first block is given by the spin elliptic surfaces $E(2n)$. With this notation $E(2)$ is a $K3$ surfaces. Their numerical invariants [GoSt99] are given by $c_2(E(2n)) = 24n$, $\tau(E(2n)) = -16n$ and $b_2^+ = 4n - 1$ ($\equiv 3 \bmod 4$).

The second block is obtained from $E(2)$ after performing a logarithmic transformation of order $2n + 1$ on one non-singular elliptic fiber. We denote the new manifolds by Y_n. All Y_n are simply connected spin manifolds with

$b_2^+ = 3$ and $b_2^- = 19$, hence they are all homeomorphic. Moreover, Y_n are Kähler manifolds and $c_1(Y_n) = 2nf$, where f is the multiple fiber introduced by the logarithmic transformation (see [BPV84]). $SW(\pm c_1(Y_n)) = \pm 1$, by Theorem 2.1.1, and then the bandwidth $BW(Y_n) \geq n$.

The third block is the one used by Gompf [Gom95] in the proof of Theorem 6.1. Let F_1, F_2 be two Riemann surfaces of genera $k+1, k > 1$ and 2, respectively. Let C_i, $i = 1, \ldots, 2k+2$ be homologically nontrivial embedded circles in F_1 generating $H_1(F_1, \mathbb{Z})$ and the circles $C_i' \subset F_2$, $i = 1, \ldots, 4$ be the generators of $H_1(F_2, \mathbb{Z})$, as in the proof of Theorem 6.1.2. Let T_i be the collection of tori given by $C_1 \times C_1', C_2 \times C_2', C_3 \times C_3', C_4 \times C_4', C_i \times C_1'$, $i = 5, \ldots, 2k+2$. We can perturb this collection to a new collection of *disjoint* tori $\{T_i'\}$, where T_i' is homologous to T_i. Because $T_i \subset F_1 \times F_2$ is a Lagrangian torus we may choose $T_i' \subset F_1 \times F_2$ to be Lagrangian, too. These tori are also homologically non-trivial. Then the product symplectic form on $F_1 \times F_2$ can be perturbed [Gom95] such that these tori become symplectic submanifolds. Let X_k be the manifold obtained by performing symplectic connected sum of $F_1 \times F_2$ and $2k+2$ copies of $E(2)$ along the family $\{T_i'\}_{i=\overline{1,2k+2}}$ and generic fibers of $E(2)$. Then the manifold X_k is a spin, symplectic 4-manifold, and by Seifert-Van Kampen Theorem it is also simply connected.

The numerical invariants of X_k are:

$$c_2(X_k) = 52k + 48, \;\; \tau(X_k) = -32(k+1).$$

The first manifold in this family is X_1, with invariants:

$$c_2(X_1) = 152,\ \tau(X_1) = -96,\ c_1^2 = 16,\ \text{and } b_2^+ = 27\ (\equiv 3 \bmod 4).$$

The last block is given by rational homology spheres of w_2-type (II)(i.e.$w_2 = 0$). Their existence is proved in [HaKr93] Proposition 4.1. We denote them by S_d.

We may now define our manifolds:

$$M_{i,j} = X_1 \# E(2i) \# Y_j \# S_d.$$

For fixed i, the manifolds $M_{i,j}$ are all homeomorphic as we take fiber sums of homeomorphic manifolds. We denote this homeomorphism type by M_i.

If we look at the basic classes of the Bauer-Furuta invariant, then both $a = c_1(X_1) + c_1(E(2i)) + c_1(Y_j)$ and $b = c_1(X_1) + c_1(E(2i)) - c_1(Y_j)$ are basic classes. Then $2j \mid (a-b)$, so $BW(M_{i,j}) \geq j$. But $BW(M_{i,j})$ is a finite number, so we can always choose a subsequence $\{j_k\} \to \infty$ such that the manifolds M_{i,j_k} have different bandwidths, hence they are not diffeomorphic.

By Theorem 2.2.2 these manifolds do not support any Einstein metrics. □

In some cases, for a small fundamental group $\mathbb{Z}_d$, some of the manifolds constructed in the proposition support a differential structure which admits an Einstein metric.

If we are not interested to construct pairs of manifolds, but only homotopy types which do not admit Einstein metrics then we can prove a similar statement for manifolds with arbitrary finitely generated fundamental group:

Theorem 7.1.3. *Let G be any finitely presented group. There exists an in-*

finite sequence of manifolds $M_{i,j}$, $i, j \in \mathbb{N}$ *, all non-spin, which satisfies the following:*

1. *The fundamental group* $\pi_1(M_{i,j}) = G$*;*

2. *For fixed* i*,* $\{M_{i,j}\}_j$ *are all homeomorphic, but no two are diffeomorphic.*

Let M_i *be the homeomorphism class of* $M_{i,j}$ *for* i *fixed.*

3. *No two* M_i *are homeomorphic.*

4. *No* $M_{i,j}$ *admits an Einstein metric, but they all satisfy Hitchin-Thorpe Inequality.*

Proof. The proof is similar in concept to the other proofs in this section. We are going to use the second and third block constructed in the proof of Theorem 7.1.2.

We also need to consider $E(4)$. This manifold has an important feature [Gom95, proof of Theorem 6.2]: it contains a torus and a genus 2 surface as disjoint symplectic submanifolds. We denote them by T and F respectively. Both T and F have self-intersection zero and their complement $E(4) \setminus (F \cup T)$ is simply connected.

The extra ingredient is given by Theorem 6.1.2. For an arbitrary finitely presented group G this theorem says there exists a symplectic manifold X_G with fundamental group $\pi_1(X_G) = G$, and the numerical invariants given by: $c_1^2(X_G) = 0$, $c_2(X_G) = 12k$ where k is the number of generators of a given representation of G. X_G also contains an embedded symplectic torus T^2 of self-intersection 0.

We are now ready to construct our manifolds. Let:

$$M_i = X_G \ \#_{T^2} \ E(4) \ \#_F \ X_i \ \#_{T^2} \ E(2)\#p\overline{\mathbb{CP}^2},$$

where the first fiber sum is done along a torus of self-intersection zero, the second surgery is along the surface F in $E(4)$ and one generic $\{pt\} \times F_2 \subset X_i$, the next surgery is done along another symplectic representative of T in $E(4)$ and a generic fiber of $E(2)$, and p is a constant satisfying:

$$c_1^2(X_i) \ > \ p \ > \frac{1}{3}c_1^2(X_i) \ > 0.$$

The fundamental group of M_i can be easily computed by Seifert-Van Kampen Theorem to be G.

To obtain different differential structures on M_i, we take logarithmic transformations of different multiplicities along a generic fiber of $E(2)$. Then by the gluing formula for the Seiberg-Witten invariants the manifolds have different bandwidths hence we have constructed infinitely many non-diffeomorphic manifolds. We denote them by $M_{i,j}$. For fixed i, these manifolds are all homeomorphic. To show this, we can first do the logarithmic transformations on $E(2)$, this yields homeomorphic manifolds. Then, we take the fiber sum along a generic fiber with $X_G \ \#_{T^2} \ E(4) \ \#_F \ X_i\#p\overline{\mathbb{CP}^2}$. We obtain a new homeomorphism by extending the one we had before.

Theorem 2.2.1 implies that no $M_{i,j}$ admits an Einstein metric. □

There are no known obstructions to the existence of Einstein metrics on $n\mathbb{CP}^2\#m\overline{\mathbb{CP}}^2$, except for the topological Hitchin-Thorpe Inequality. But in some cases, corresponding to the constructions in Theorem 7.1.1 there are no Einstein metrics which are invariant under the action of $\mathbb{Z}/d\mathbb{Z}$. To be more precise, one of the examples with the smallest topology would be the following:

Proposition 7.1.4. *On* $5\mathbb{CP}^2\#25\overline{\mathbb{CP}^2}$*, there exists an exotic involution* σ*, such that* $5\mathbb{CP}^2\#25\overline{\mathbb{CP}^2}$ *does not admit any Einstein metric invariant under the involution* σ.

Proof. Let N be the double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$ branched along a smooth divisor D, such that $\mathcal{O}(D) = \mathcal{O}(6,6)$. Then N is a simply connected, almost completely decomposable surface of general type and by 4.1.3 its numerical invariants are:

$$c_2(N) = 14,\ c_1^2(N) = 4,\ \tau(N) = -8.$$

Hence $b_2^+ = 2$ and $b_2^- = 10$. By Theorem 2.2.1 the manifold $M = N\#2\overline{\mathbb{CP}^2}\#S_2$ does not admit any Einstein metric.

Let $\widetilde{M}$ be the universal cover of M. Then we have the following diffeomorphisms:

$$\widetilde{M} \cong 2N\#4\overline{\mathbb{CP}^2}\#(S^2 \times S^2) \cong 2N\#\mathbb{CP}^2\#5\overline{\mathbb{CP}^2} \cong 5\mathbb{CP}^2\#25\overline{\mathbb{CP}^2}.$$

The fact that M does not admit any Einstein metrics implies that $\widetilde{M}$ does not admit any Einstein metrics invariant under the covering involution σ. $\square$

7.2 Complete Einstein metrics on non-compact 4-manifolds

The problem of the existence of Einstein metrics can also be addressed for non compact manifolds. In [AKL89] Anderson, Kronheimer and LeBrun were able to construct an infinite family of complete Ricci-flat simply connected Kähler manifolds, with infinitely generated second homology. Their metrics are explicit and admit a semi-free isometric S^1-action. Recently there has been a lot of interest manifested in this subject both by mathematicians and physicists. For example Calderbank and Singer studied the same problem for Einstein metrics of negative scalar curvature; their construction uses a free T^2-action on a dense open set of the manifold and some intricate analysis.

However, it seems that manifolds of infinite topological type which admit Einstein metrics of negative scalar curvature are quite common. One can use the existence of Kähler-Einstein metrics on complex surfaces with ample canonical line bundle and infinite fundamental group.

Theorem 7.2.1. *There are infinitely many non-compact* 4*-manifolds with infinitely generated second homology which admit complete Einstein metrics of negative scalar curvature.*

Proof. Our proof consists in showing that for any natural number k there is a sequence $\{Z_1, \ldots, Z_k\}$ of non-homeomorphic non-compact complete Einstein manifolds with infinitely generated second homology.

Let $f : N \to \Sigma_1 \times \Sigma_2$ be the minimal resolution of the double cover of $\Sigma_1 \times \Sigma_2$ branched along a divisor D with normal crossing singularities, as in the construction from Proposition 5.3.1. Σ_1, Σ_2 are Riemann surfaces of genus $k, 2$ respectively, and:

$$D = \bigcup_{i=1}^{2n} \{p_i\} \times \Sigma_2 \cup \bigcup_{j=1}^{2m} \Sigma_1 \times \{q_j\}, \ n, m \in \mathbb{N} \text{ sufficiently large}$$

Let $\alpha_i, \beta_i, i = 1, \dots N$ be generators of $\pi_1(\Sigma_1, x_0), (x_0 \notin Sing(D))$ such that $\alpha_i \cdot \beta_j = \delta_i^j$ and $\alpha_i \cap \alpha_j = \beta_i \cap \beta_j = \emptyset, \ i \neq j$. Then α_i, β_i give rise to a basis for $H_1(\Sigma_1, \mathbb{Z})$. Let $\gamma_i \in H^1(\Sigma_1, \mathbb{Z}), \ i = 1, \dots, N$ be the dual of α_i. We denote by $\alpha_i' \in \pi_1(N, f^{-1}(x_0, q_1))$ the preimage of $\alpha_i \subset (\Sigma_1 \times \{q_1\})$, and $\gamma_i' \in H^1(N, \mathbb{Z}), \ \gamma_i' = (pr_1 \circ f)^*(\gamma_i)$. Then α_i' and γ_i' are related by $\int_{\alpha_i'} \gamma_j' = \delta_i^j$.

Let $G_1 =< \alpha_1' >, G_2 =< \alpha_1', \alpha_2' >, \dots, G_k =< \alpha_1', \dots, \alpha_k' >$ subgroups of $\pi_1(N)$. To check that G_i is not equal to any of the previous subgroups $G_1, \dots, G_{i-1}$, it is enough to integrate γ_i' on the generators. The integral is zero on any element of a lower subindex group, and non-trivial on $\alpha_i' \in G_i$.

The classification theorems for covering spaces tell us that there is a $1-1$ correspondence between isomorphism classes of coverings of N and the conjugacy classes of subgroups of $\pi_1(N)$. The correspondence is given by $X \leftrightarrow p_{\#}\pi_1(X)$. Let $Z_1, \dots, Z_k$ the coverings corresponding corresponding to $G_1, \dots, G_k \subset \pi_1(N)$.

To show that Z_i admits an Einstein metric, let's remember from Proposition 5.3.1 that the manifold N is diffeomorphic to complex surface with ample canonical line bundle. Hence, by Theorem 2.3.2, it admits a Kähler-Einstein metric of negative scalar curvature, and so do the covering spaces Z_i.

Notice that it is enough to show that the manifold Z_1 has infinitely generated second homology, the others will follow analogously. N is constructed as the minimal resolution of N', introducing exceptional rational divisors with self-intersection -2 at the singular points.

Without loss of generality we can assume that the loop β_1 passes through p_1 and avoids all the other branch points $p_i, i > 1$. Let $\beta'_1 = \overline{f^{-1}((\beta_1 \setminus \{p_1\}) \times \{q_1\})}$, and $P_1 = \beta'_1 \setminus f^{-1}\big((\beta_1 \setminus \{p_1\}) \times \{q_1\}\big)$. Then P_1 is a point on an exceptional sphere, which we denote by E.

Let $f' : Z_1 \to N$ the covering map satisfying $f'_{\#}\pi_1(Z_1) = G_1$. Then $\beta'_1 \notin G_1$ implies that $f'^{-1}(\beta'_1)$ is a line. Hence, $f'^{-1}(P_1) = \{P_1^j | j \in \mathbb{Z}\}$ and $f'^{-1}(E) = \{E^j | j \in \mathbb{Z}\}$, with $P_1^j \in E^j$.

By the uniqueness of the lifting property the spheres E^j are all disjoint. They all have self-intersection -2, so they generate different classes in $H_2(Z_1)$. Hence Z_1 has infinitely generated second homology, and implicitly the same result is true for any Z_i. □

We remark that these manifolds can not be obtained as the complement of a divisor in a complex surface, as they have infinite homology.

Chapter 8

The rational blow-down

We saw in the previous sections that the methods to construct symplectic manifolds are via surgeries: normal connected sums or rational blow-downs. It is a interesting problem to decide the existence or the non-existence of complex structures on symplectic 4-manifolds constructed by these two techniques. The question we would like to address here is the following: if the initial manifold admits an integrable complex structure, when does the resulting manifold admit a integrable complex structure? We provide a sufficient condition:

Theorem 8.0.2. *Let G be a finite group acting with only isolated fixed points on a smooth, compact, complex surface S with $H^2(S, \Theta_S) = 0$. If the singularities of S/G are of class T, then the full rational blow down $\widetilde{S}$ of the minimal resolution of S/G admits complex structures. Moreover, as a smooth $4-$manifold, $\widetilde{S}$ is oriented diffeomorphic to the generic fiber of a 1-parameter $\mathbb{Q}-$Gorenstein smoothing of S/G.*

To explain our result, we should recall that the singularities of type T are either rational double points or quotient singularities of a particular type.

The exceptional divisor of the minimal resolution of this last type of quotient singularities is a linear chain of rational curves on which the rational blow down can be performed. What we mean by the full rational blow down is rationally blowing down *all* of the exceptional divisors which appear resolving the singularities which are not ordinary double points.

As an application we study a series of examples obtained by letting $\mathbb{Z}_4$, the multiplicative group of roots of order 4 of unity, act on a product of two appropriated Riemann surfaces. As a particular case, we find a complex structure on an example constructed by Gompf, on which the existence of complex structures was still an open problem.

To briefly recall Gompf's example, we start with a simply connected, relatively minimal, elliptic surface, with no multiple fibers, and with Euler characteristic $c_2 = 48$. Up to diffeomorphisms [FrMo94], there is only one such elliptic surface, which we call $E(4)$. It is known that $E(4)$ admits at most nine rational $(-4)-$curves as disjoint sections. We can form the normal connected sum of it with n copies of $\mathbb{CP}_2$, the complex projective plane, identifying a conic in each $\mathbb{CP}_2$ with one $(-4)-$curve of $E(4)$. This operation is the same as rationally blowing-down n $(-4)-$curves, and we obtain Gompf's examples [Gom95, page 564] denoted by $W_{4,n}$, where $n = 1, \ldots, 9$. The manifold $W_{4,1}$ is does not admit any complex structure, as it violates the Noether Inequality, but the existence of complex structures is topologically unobstructed in the other cases. In Gompf's paper, he shows that the 4$-$manifolds $W_{4,n}$, for $n = 2, 3, 4$, and 9 are diffeomorphic to complex surfaces. We find a complex structure on $W_{4,8}$.

In this thesis we discuss the above examples from the point of view of deformation theory. We prove:

Theorem 8.0.3. *The* $4-$*manifolds* $W_{4,n}, n = 2,3,4,8,9$ *admit a complex structure.*

The emphasis is on the methods we employ. One technique comes from Manetti's interpretation of the rational blow down in algebraic setting as the 1-parameter $\mathbb{Q}-$Gorenstein smoothing of a certain class of normal surface singularities. The second method is a natural approach towards the normal connected sum construction and it consists in viewing it as the smoothing of a simple normal crossing algebraic variety.

8.1 An algebraic description of the rational blow-down

The point of view we adopt in this paper is that the generalized rational blow down procedure and the smoothing of isolated complex surface singularities are essentially the same in the algebraic setting. Here we consider only the case of some quotient singularities, a case well documented in the literature.

Following Manetti's presentation [Man01] of the results of [KoS-B88], we begin by recalling the terminology and some general results.

Definition 10. *A normal variety* X *is* $\mathbb{Q}-$*Gorenstein if it is Cohen-Macaulay and a multiple of the canonical divisor is Cartier.*

In this context we need a generalization of the definition of a smoothing:

Definition 11. *A flat map $\pi : \mathcal{X} \to \mathbb{C}$ is called an one-parameter $\mathbb{Q}-$Gorenstein smoothing of a normal singularity (X, x) if $\pi^{-1}(0) = X$ and there exists $U \subset \mathbb{C}$ an open neighborhood of 0 such that the following conditions are satisfied:*

i) $\mathcal{X}$ is $\mathbb{Q}-$Gorenstein,

ii) The induced map $\mathcal{X} \to U$ is surjective,

iii) $X_t = \pi^{-1}(t)$ is smooth for every $t \in U - \{0\}$.

Definition 12. *A normal surface singularity is* of class T *if it is a quotient singularity and admits a $\mathbb{Q}-$Gorenstein one-parameter smoothing.*

The following result of Kollár and Shepherd-Barron [KoS-B88] gives a complete description of the singularities of class T.

Proposition 8.1.1 (Kollár, Shepherd-Barron). *The singularities of class T are the following:*

1) Rational double points;

2) Cyclic singularities of type $\frac{1}{dn^2}(1, dna - 1)$, for $d > 0$, $n \geq 2$ and $(a, n) = 1$.

Remember in Section 4.1 we introduced the rational double point singularities as the singularities for which the deformation and the resolution manifolds are diffeomorphic.

Next, we introduce the above cyclic singularities. Let $a, d, n > 0$ be integers with $(a, n) = 1$, and $\mathcal{Y} \subset \mathbb{C}^3 \times \mathbb{C}^d$ be the hypersurface of equation

$$uv - y^{dn} = \sum_{k=0}^{d-1} t_k y^{kn},$$

where $u, v, y, t_0, \ldots, t_{d-1}$ are linear coordinates on $\mathbb{C}^3 \times \mathbb{C}^d$. $\mathbb{Z}_n$ acts on $\mathcal{Y}$, the action being generated by:

$$(u, v, y, t_0, \ldots, t_{d-1}) \mapsto (\zeta u, \zeta^{-1} v, \zeta^a y, t_0, \ldots, t_{d-1}), \text{where } \zeta = e^{\frac{2\pi i}{n}}$$

Let $\mathcal{X} = \mathcal{Y}/\mathbb{Z}_n$ and $\phi : \mathcal{X} \to \mathbb{C}^d$ the quotient of the projection $\mathcal{Y} \to \mathbb{C}^d$.

Proposition 8.1.2. *$\phi : \mathcal{X} \to \mathbb{C}^d$ is a $\mathbb{Q}-$Gorenstein deformation of the cyclic singularity (X, x) of type $\frac{1}{dn^2}(1, dna - 1)$.*

Moreover, every $\mathbb{Q}-$Gorenstein deformation $\mathcal{X} \to \mathbb{C}$ of a singularity (X, x) of type $\frac{1}{dn^2}(1, dna - 1)$ is isomorphic to the pullback of ϕ for some germ of holomorphic map $(\mathbb{C}, 0) \to (\mathbb{C}^d, 0)$.

If a global $\mathbb{Q}-$ Gorenstein smoothing exists then the following proposition due to Manetti [Ma80] gives an algebraic interpretation of the algebraic rational blow-down.

Proposition 8.1.3. *Let X be a compact complex surface with singularities of class T, $\widetilde{X}$ be its minimal resolution and $\pi : \mathcal{X} \to U$ a $\mathbb{Q}-$Gorenstein smoothing of X. Then the full rational blow-down of $\widetilde{X}$ is oriented diffeomorphic to $X_t = \pi^{-1}(t)$, for any $t \neq 0$.*

Manetti's algebraic description of the rational blow-down procedure gets more substance by providing a global criterion for smoothing of singularities.

Let X be a compact, reduced analytic space with $\text{Sing}(X) = \{x_1, \ldots, x_n\}$. By restriction, for each $i = 1, \ldots, n$, any deformation of X defines a deformation of the singularity (X, x_i) with *the same base space.* Thus, if $\text{Def}\, X$ and

$\mathrm{Def}(X, x_i)$ denote the base space of the versal deformation of X and (X, x_i), respectively, there exists a natural morphism:

$$\Phi : \mathrm{Def}\, X \to \prod_{i=1}^{n} \mathrm{Def}(X, x_i)$$

If all of the singularities (X, x_i) are of class T, for each i we can choose in $\mathrm{Def}(X, x_i)$ the (smooth) component corresponding the $\mathbb{Q}$–Gorenstein deformations. What we are interested in here is when these deformations of singularities lift to a deformation of the total space. The answer is given by a general criterion of Wahl:

Proposition 8.1.4. *Let Θ_X be the tangent sheaf of X. If $H^2(X, \Theta_X) = 0$, then the morphism Φ is smooth. In particular, every deformation of the singularities (X, x_i), $i = i, \ldots, n$ may be globalized.*

For the proof of this proposition we refer the interested reader to either [Wah81, page 242] or to [Man91, page 93].

We will use this criterion in the following situation.

Suppose we start with a smooth compact complex surface X. Let G be a finite group acting on X with isolated fixed points. Let $Y = X/G$ be the quotient, $\{y_1, \ldots, y_n\} = \mathrm{Sing}(Y)$ the singular locus of Y, and $f : X \to Y$ the quotient map. We denote by Θ_X the holomorphic tangent bundle of X, and by $\Theta_Y = (\Omega^1_Y)^\vee$, the tangent sheaf of Y.

Lemma 8.1.5. *In the above notations, $H^2(X, \Theta_X) = 0 \Longrightarrow H^2(Y, \Theta_Y) = 0$.*

Proof. To prove the vanishing of $H^2(Y, \Theta_Y) = 0$, we need to have a convenient description of the tangent sheaf Θ_Y of Y. In our case, this is provided by

Schlessinger [Sch71]:

$$\Theta_Y = (f_*\Theta_X)^G.$$

Now, by averaging, we get a map $f_*\Theta_X \to \Theta_Y = (f_*\Theta_X)^G$. But this means Θ_Y is a direct summand of $f_*\Theta_X$. To finish the proof, since f is a finite map, the Leray spectral sequence provides an isomorphism:

$$H^2(X, \Theta_X) \simeq H^2(Y, f_*\Theta_X),$$

and the conclusion of the lemma follows. □

Proof of Theorem 8.0.2. Assume now G acts on a smooth complex surface S with fixed points only. If the singularities of S/G are of class T only, we can look at the components of each versal deformation space of any such singular point, and pick the one corresponding to $\mathbb{Q}$−Gorenstein deformations. Theorem 8.0.2 follows immediately from the algebraic description of the rational blow down, the globalization criterion 8.1.4 and Lemma 8.1.5. □

Remark : We should point out that the smoothing of rational double points are diffeomorphic to their minimal resolution. Thus, for the singularities of class T the full rational blow down, essentially performed only on the minimal resolution of the quotient singularities which are not rational double points, coincides with the simultaneous smoothing of all the singular points. □

8.1.1 A family of examples

In this section, by adopting the above viewpoint, we will exhibit a complex structure on Gompf's example $W_{4,8}$.

Let $C \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ be the smooth curve of genus 3 given by the equation

$$F(\mathbf{z}, \mathbf{w}) = z_0^4(w_0^2 + w_1^2) + z_1^4(w_0^2 - w_1^2) = 0,$$

where $(\mathbf{z}, \mathbf{w}) = ([z_0 : z_1], [w_0 : w_1])$ are the standard bi-homogeneous coordinates on $\mathbb{CP}^1 \times \mathbb{CP}^1$.

An easy calculation shows that the action of the cyclic group $\mathbb{Z}_4$ on $\mathbb{CP}^1 \times \mathbb{CP}^1$ generated by

$$([z_0 : z_1], [w_0 : w_1]) \mapsto ([iz_0 : z_1], [w_0 : w_1])$$

fixes two lines $[0 : 1] \times \mathbb{CP}^1, [1 : 0] \times \mathbb{CP}^1$. Its restriction to C is well-defined and has four fixed points: two points $P_j = ([0 : 1], [1 : (-1)^{j+1}]),\ j = 1, 2$ where in local coordinates $\mathbb{Z}_4$ acts by multiplication with i, and two other points $Q_j = ([1 : 0], [1 : (-1)^{j+1}i]),\ j = 1, 2$ where in local coordinates our group acts by multiplication with $-i$.

We will be interested in the manifold obtained by taking the quotient of $C \times C$ under the diagonal action of $\mathbb{Z}_4$. Let $X = (C \times C)/\mathbb{Z}_4$. This action has 16 fixed points. At eight of them, (P_k, P_j), and $(Q_k, Q_j),\ k, j = 1, 2$ the group $\mathbb{Z}_4$ acts (in local coordinates) as

$$(z_1, z_2) \mapsto (iz_1, iz_2),$$

while at the other eight (P_k, Q_j) and (Q_k, P_j), $k, j = 1, 2$ it acts as

$$(z_1, z_2) \mapsto (iz_1, -iz_2).$$

Thus, the singular complex 2-dimensional variety X has 8 singular points of type A_3 and 8 quotient singularities of type $\frac{1}{4}(1,1)$. The minimal resolution of the last type of singularities consists in replacing each such singular point by a smooth rational curve of self-intersection (-4). Let $\widehat{X}$ be the minimal resolution of X.

Proposition 8.1.6. *$\widehat{X}$ is a simply connected, minimal, elliptic complex surface with no multiple fibers and with the Euler characteristic $c_2 = 48$.*

Proof. The quotient $C/\mathbb{Z}_4$ is a rational curve, and we denote by P_1', P_2', Q_1' and Q_2' the image of P_1, P_2, Q_1 and Q_2, respectively, under the projection map. Let:

$$\pi_1 : \widehat{X} \longrightarrow C/\mathbb{Z}_4 \cong \mathbb{CP}_1$$

the projection on the first factor. This fibration has four singular fibers above P_j' and Q_j', $j = 1, 2$. The generic fiber is a Riemann surface of genus 3, while each of the singular fibers consist of a chain of nine spheres, one of which is the quotient $C/\mathbb{Z}_4$ and the other eight are exceptional spheres introduced by the resolution of singularities. It follows that our manifold is simply connected.

To see the elliptic fibration, we consider first the covering:

$$\pi_1 \times \pi_2 : \widehat{X} \longrightarrow (C \times C)/(\mathbb{Z}_4 \oplus \mathbb{Z}_4) = C/\mathbb{Z}_4 \times C/\mathbb{Z}_4 \cong \mathbb{CP}^1 \times \mathbb{CP}^1.$$

As the construction is symmetric in the two factors, we will use freely the identification of $C/\mathbb{Z}_4$ with $\mathbb{CP}_1$ with four marked points $P'_j, Q'_j,\ j = 1, 2$. Let $B,\ D \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ be the following divisors:

$$B = \{P'_1\} \times \mathbb{CP}^1 \cup \{P'_2\} \times \mathbb{CP}^1 \cup \mathbb{CP}^1 \times \{Q'_1\} \cup \mathbb{CP}^1 \times \{Q'_2\}$$
$$D = \{Q'_1\} \times \mathbb{CP}^1 \cup \{Q'_2\} \times \mathbb{CP}^1 \cup \mathbb{CP}^1 \times \{P'_1\} \cup \mathbb{CP}^1 \times \{P'_2\}.$$

$\widehat{X}$ can be presented as a bi-double cover of $\mathbb{CP}^1 \times \mathbb{CP}^1$, first branched over the union of B and D, then branched over the union of the total transforms of B and D.

Remark that $\mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(B) \cong \mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(D) = \mathcal{O}_{\mathbb{CP}^1\times\mathbb{CP}^1}(2,2)$ and the generic element of the associated linear system is a smooth elliptic curve. Let L be the pencil generated by B and D. The base locus of this pencil, $B \cap D = \{(P'_i, P'_j), (Q'_i, Q'_j),\ i, j = 1, 2\}$, is a subset of the set of singular points of the branch locus of the first double cover, hence the linear system $(\pi_1 \times \pi_2)^*(L)$ will be base point free, with smooth elliptic curve as generic section. This gives us an elliptic fibration. The exceptional divisors introduced above $B \cap D$ are the eight sections of the elliptic fibration, all of self-intersection -4.

Next, we need to know two topological invariants, the Euler characteristic $\chi(\widehat{X})$ and signature $\sigma(\widehat{X})$, for example. Both computations are immediate. We obtain $\chi(\widehat{X}) = 48$ and $\sigma(\widehat{X}) = -32$, which imply $c_1^2(\widehat{X}) = 0$. Because of the existence of the elliptic fibration we know that the manifold can not be of general type. Hence the classification of complex surfaces and the topological invariants imply that $\widehat{X}$ is minimal.

□

In particular, by [FrMo94], $\widehat{X}$ is diffeomorphic to $E(4)$. Applying Theorem 8.0.2, we can conclude now the existence of complex structures on $W_{4,8}$.

Our example can be easily generalized in the following way. Consider the smooth curves $C_k,\ C_l \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ given by the equations

$$z_0^4 f_k(w_0, w_1) + z_1^4 g_k(w_0, w_1) = 0$$

and

$$z_0^4 f_l(w_0, w_1) + z_1^4 g_l(w_0, w_1) = 0,$$

respectively. Here $(f_k,\ g_k)$ and $(f_l,\ g_l)$ are generic pairs of homogeneous polynomials of degree k, and l, respectively. The above discussion can be now easily repeated for $C_k \times C_l$ and the induced $\mathbb{Z}_4$ action.

8.2 The algebraic normal connected sum

In this section we approach the normal connected sum procedure from the algebraic point of view. We will test this technique on Gompf's examples $W_{4,n}$, for $n = 2, 3, 4$ and 9.

Mimicking the symplectic normal sum, we start with two pairs of complex varieties $(X_1, Y_1),\ (X_2, Y_2)$, where $X_i,\ i = 1, 2$ are smooth and $Y_i \subset X_i,\ i = 1, 2$ are smooth subvarieties satisfying the following conditions:

- $Y_1 \simeq Y_2$;
- $N_{Y_1|X_1} = N^{\vee}_{Y_2|X_2}$.

Using, the condition on the normal bundles, we can glue X_1 and X_2 to form a normal crossing complex variety Z. Then the symplectic normal sum, in particular the rational blow-down of a (-4) sphere, can be interpreted as a smoothing of Z, in the sense of Friedman [Fri83], as long as the smoothing is of Kähler type. Before we proceed, we recall some basic facts on the deformation theory of singular spaces.

8.2.1 Deformation theory of normal crossing varieties

If Z is a compact, singular, reduced complex space, the deformation theory of normal crossing varieties [Fri83] is given in terms of the global Ext groups $T^i_Z := \text{Ext}^i(\Omega_Z, \mathcal{O}_Z)$. T^1_Z will describe the infinitesimal deformations of the complex structure of Z, and the obstructions lie in T^2_Z. These groups are usually computed from their "local" versions, the sheaves $\mathcal{E}xt^i(\Omega_Z, \mathcal{O}_Z) =: \tau^i_Z$, using the "local to global" spectral sequence $E_2^{p,q} = H^p(\tau^q_Z) \Rightarrow T^{p+q}_Z$.

Suppose Z is a simple normal crossing variety, i.e. Z locally looks like a union of hyperplanes, whose irreducible components are smooth. In this case, the local to global spectral sequence gives:

$$0 \to H^1(\tau^0_Z) \to T^1_Z \to H^0(\tau^1_Z) \to H^2(\tau^0_Z) \to T^2_Z \to 0. \tag{8.1}$$

The space $H^1(\tau^0_Z)$ classifies all "locally trivial" deformations of Z, i.e. for which the singularities remain locally a product. Their obstructions lie in $H^2(\tau^0_Z)$.

In the cases treated below, Z is chosen to be d–semistable, that is $T^1_Z = \mathcal{O}_D$, where $D = \text{Sing}(Z)$ is the singular locus of Z.

Definition 13. *We say that a proper flat map $\pi : \mathcal{Z} \to \Delta$ from a smooth $(n+1)-$fold $\mathcal{Z}$ to $\Delta = \{|z| < 1\} \subset \mathbb{C}$ is a smoothing of a reduced, not necessarily irreducible, complex analytic variety Z, if $\pi^{-1}(0) = Z$ and $\pi^{-1}(t)$ is smooth for $t \in \Delta$ sufficiently small.*

Remark *Before we proceed with our examples, we should point out that if $H^2(\tau_Z^0) = H^1(\mathcal{O}_D) = 0$, then $T_Z^2 = 0$, and so the deformation problem is unobstructed. If Z is $d-$ semistable, it will admit [Fri83] a versal (one-parameter) deformation with smooth total space, and smooth generic fiber. In this case, we say we have a one-parameter smoothing of Z.* □

8.2.2 Gompf's examples

In what follows we are going to treat a very particular situation, and give a simple cohomological criterion suitable to the study of Gompf's examples.

Let S be a complex surface, containing n smooth, disjoint, rational curves of self-intersection -4, which we are going to denote by $D_1, \ldots, D_n$. As discussed, for each of these curves, we can glue in a copy of $\mathbb{CP}^2$ to form a simple normal crossing complex variety denoted by Z with $n+1$ irreducible components, and $\mathrm{Sing}(Z) = D_1 + \cdots + D_n$. Since each copy of $\mathbb{CP}^2$ is glued along a conic, it follows that Z satisfies the $d-$semistability condition. An easy, but useful criterion is:

Proposition 8.2.1. *Z admits a one-parameter smoothing if*

$$H^2(S, \Theta_S \otimes \mathcal{O}_S(-D_1 - \cdots - D_n)) = 0.$$

Proof. From the ending remark of previous section, it suffices to check whether $H^2(\tau_Z^0) = H^1(D, \mathcal{O}_D) = 0$, where $D = D_1 + \cdots + D_n$.

Since the D_i's are smooth, disjoint, rational curves, it follows that $H^1(D, \mathcal{O}_D) = 0$. The sheaf τ_Z^0, naturally sits in the exact sequence:

$$0 \to \Theta_S \otimes \mathcal{O}_S(-D) \oplus \bigoplus_{i=1}^{n} \Theta_{\mathbb{CP}^2} \otimes \mathcal{O}_{\mathbb{CP}^2}(-C_i) \to \tau_Z^0 \to \Theta_D \to 0, \tag{8.2}$$

where by C_i we denoted the smooth conic in $\mathbb{CP}^2$ corresponding to D_i. But, for any smooth conic $C \subset \mathbb{CP}^2$ we have $H^2(\mathbb{CP}^2, \Theta_{\mathbb{CP}^2} \otimes \mathcal{O}_{\mathbb{CP}^2}(-C)) = 0$. A simple inspection of the cohomology sequence associated to (8.2) concludes the proof. □

We take now a look from our perspective at Gompf's examples $W_{4,n}$, for $n = 2, 3, 4$ or 9. In these cases, he found complex structures as appropriate multiple covers of $\mathbb{CP}^2$. Guided by his complex structures, what we do here is merely to reprove this in an algebraic, more conceptual way, using the method described above. We will discuss only the $W_{4,2}$ case, the rest of the cases follow analogously.

We start with the Hirzebruch surface Σ_4. We denote by C_0 the negative section and by f the class of a fiber. Let $f : X \to \Sigma_4$ be the double cover of Σ_4, branched along a smooth member of the linear system $D \in |4(C_0 + 4f)|$. Such a smooth member exists, as a consequence of the standard results on linear systems on Hirzebruch surfaces [Har77]. X is a smooth, simply connected elliptic surface, diffeomorphic to $E(4)$. Moreover, since D and C_0 are disjoint, X contains exactly two smooth rational curves of self-intersection -4, the two irreducible components of the preimage of C_0. We denoted these two curves by D_1 and D_2.

We perform now the algebraic normal connected sum along these two curves gluing in two copies of $\mathbb{CP}^2$, each along a smooth conic, and denote the newly formed singular variety by Z. From Proposition 8.2.1 we know that if $H^2(X, \Theta_X \otimes \mathcal{O}_X(-D_1 - D_2)) = 0$, there is no obstruction to the smoothing of Z . Using the structure of X as a double covering of Σ_4, and the Leray spectral sequence, we get:

$$\begin{aligned} & H^2(X, \Theta_X \otimes \mathcal{O}_X(-D_1 - D_2)) \\ = & H^2(X, \Theta_X \otimes f^*\mathcal{O}_{\Sigma_4}(-C_0)) \\ = & H^2(\Sigma_4, f_*\Theta_X \otimes \mathcal{O}_{\Sigma_4}(-C_0)) \\ = & H^2(\Sigma_4, \Theta_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(-C_0)) \oplus H^2(\Sigma_4, \Theta_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(-3C_0 - 8f)). \end{aligned}$$

To prove the vanishing of the last two cohomology groups, we use the Serre duality:

$$\begin{aligned} & H^2(\Sigma_4, \Theta_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(-C_0)) \\ = & H^0(\Sigma_4, \Omega^1_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(C_0) \otimes \mathcal{O}_{\Sigma_4}(K_{\Sigma_4})) \\ = & H^0(\Sigma_4, \Omega^1_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(-C_0 - 6f)). \end{aligned}$$

Here $K_{\Sigma_4} = -2C_0 - 6f$ denotes the canonical divisor of Σ_4. Since the divisor $C_0 + 6f$ is effective and is linearly equivalent to a smooth curve [Har77], we have the exact sequence of sheaves:

$$0 \to \Omega^1_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(-C_0 - 6f)) \to \Omega^1_{\Sigma_4} \to \mathcal{O}_{C_0+6f} \to 0.$$

Passing to the cohomology sequence and since $H^0(\Sigma_4, \Omega^1_{\Sigma_4}) = 0$, we get the vanishing of $H^2(\Sigma_4, \Theta_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(-C_0))$.

For the vanishing of the second term we proceed in the same fashion:

$$\begin{aligned} &H^2(\Sigma_4, \Theta_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(-3C_0 - 8f)) \\ &= H^0(\Sigma_4, \Omega^1_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(3C_0 + 8f) \otimes \mathcal{O}_{\Sigma_4}(K_{\Sigma_4})) \\ &= H^0(\Sigma_4, \Omega^1_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(C_0 + 2f)). \end{aligned}$$

Arguing by contradiction, if there exists a global non-zero section of $\Omega^1_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(C_0 + 2f)$, then it must exist global non-zero section of

$$\bigwedge^2 (\Omega^1_{\Sigma_4} \otimes \mathcal{O}_{\Sigma_4}(C_0 + 2f)) \cong \mathcal{O}_{\Sigma_4}(K_{\Sigma_4}) \otimes \mathcal{O}_{\Sigma_4}(2C_0 + 4f) \cong \mathcal{O}_{\Sigma_4}(-2f).$$

But this is impossible.

The computations for the other examples go along the same lines, with minor modifications. For $W_{4,3}$ we end up with the complex structure of the $3:1$ covering of $\mathbb{CP}^2$ branched along a smooth sextic curve, $W_{4,4}$ is the simple bi-double cover of $\mathbb{CP}^2$ branched along a transverse pair of conics, and $W_{4,9}$ is a $\mathbb{Z}_3 \oplus \mathbb{Z}_3$ cover of $\mathbb{CP}^2$ branched along 3 transverse conics. With the results obtained in the previous section, the proof of Theorem 8.0.3 is now complete.

Bibliography

[AKL89] M. ANDERSON, P. KRONHEIMER, C. LEBRUN, *Complete Ricci-flat Kähler manifolds of infinite topological type*, Comm. Math. Phy. 125 (1989), 637–642.

[BPV84] W. BARTH, C. PETERS, A. V. DE VEN, *Compact Complex Surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 4, Springer Verlag, Berlin, 1984.

[BaFu04] S. BAUER , M. FURUTA, *A stable cohomotopy refinement of Seiberg-Witten invariants: I*, Invent. Math. 155 (2004), no.1, 1–19.

[Bau] S. BAUER, *A stable cohomotopy refinement of Seiberg-Witten invariants: II*, Invent. Math. 155 (2004), no.1, 21–40.

[Bes87] A. BESSE, *Einstein Manifolds*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 10, Springer Verlag, Berlin, 1987.

[BrKo05] V. BRAUNGARDT, D. KOTSCHICK, *Einstein metrics and the number of smooth structures on a four-manifold*, Topology 44 (2005), no. 3, 641–659.

[Cat84] F. CATANESE, *On the moduli spaces of surfaces of general type*, J. Differential Geom. 19 (1984), no. 2, 483–515.

[Cat92] F. CATANESE, *(Some) old and new results on algebraic surfaces*, First European Congress of Mathematics, Vol. I (Paris, 1992), Progr. Math., 119, Birkhuser, Basel, 1994, 445–490.

[FiSt97] R. FINTUSHEL, R. J. STERN, *Rational blowdowns of smooth 4-manifolds*, J. Differential Geom. 46 (1997), no. 2, 181–235.

[Fre82] M. FREEDMAN, *On the topology of 4-manifolds*, J. Differential Geom.17 (1982), no.3, 357–453.

[Fri83] R. FRIEDMAN, *Global smoothings of varieties with normal crossings*, Ann. of Math. (2) 118 (1983), no. 1, 75–114.

[FrMo94] R. FRIEDMAN, J. MORGAN, *Smooth four-manifolds and complex surfaces*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 27, Springer-Verlag, Berlin, 1994.

[Gom95] R. E. GOMPF, *A new construction of symplectic manifolds*, Ann. of Math. (2) 142 (1995), no. 3, 527–595.

[GoSt99] R. E. GOMPF, A. I. STIPSICZ, *4-manifolds and Kirby calculus.* American Mathematical Society, Providence, RI, 1999.

[HaKr93] I. HAMBLETON, M. KRECK, *Cancellation,elliptic surfaces and the topology of certain four-manifolds*, J. Reine Angew. Math., 444 (1993), 79–100.

[HKK86] J. HARER, A. KAS, R. KIRBY, *Handlebody decompositions of complex surfaces*, Mem. Amer. Math. Soc. 62 (1986), no. 350.

[Har77] R. HARTSHORNE , *Algebraic geometry*, Graduate Texts in Mathematics, No. 52. Springer-Verlag, New York-Heidelberg, 1977.

[Hit74] N. J. HITCHIN, *Harmonic spinors*, Advances in Math., 14 (1974). 1–55.

[IsLe02] M. ISHIDA AND C. LEBRUN, *Spin manifolds, Einstein metrics, and differential topology*, Math. Res. Lett., 9 (2002), no.2-3, 229-240.

[KoS-B88] J. KOLLÁR, N.I. SHEPHERD-BARRON, *Threefolds and deformations of surface singularities*, Invent. Math. 91 (1988), no. 2, 299–338

[Kot98] D. KOTSCHICK, *Einstein metrics and smooth structures.* Geom. Topol., 2 (1998), 1–10.

[KMT95] D. KOTSCHICK, J. W. MORGAN, C. H. TAUBES, *Four-manifolds without symplectic structures but with nontrivial Seiberg-Witten invariants*, Math. Res. Lett. 2 (1995), no. 2, 119–124.

[LaMi89] H. B. LAWSON, M. L. MICHELSOHN, *Spin Geometry*, Princeton Mathematical Series, 38. Princeton University Press, Princeton, NJ, 1989.

[LeB96] C. LEBRUN, *Four-manifolds without Einstein metrics*, Math. Res.Lett. 3 (1996), no. 2, 133–147.

[LeB97] ——, *Yamabe constants and the perturbed Seiberg-Witten equations*, Comm. Anal. Geom., 5 (1997), no. 3, 535–553.

[LeB01] ——, *Ricci curvature, minimal volumes, and Seiberg-Witten theory*, Invent. Math. 145 (2001), no. 2, 279–316.

[LeB02] ——, *Einstein metrics, four-manifolds, and differential topology*, Surveys in differential geometry, Vol. VIII , International Press, Boston, MA, 2002, 235–255.

[MaMo80] R. MANDELBAUM, B. MOISHEZON, *On the topology of simply connected algebraic surfaces*, Trans. Amer. Math. Soc. 260 (1980), no. 1, 195–222.

[Ma80] R. MANDELBAUM, *Four-dimensional topology: an introduction*, Bull.Amer.Math.Society, 2 (1980), 1–159.

[Man91] M. MANETTI, *Normal degenerations of the complex projective plane*, J. Reine Angew. Math. 419 (1991), 89–118.

[Man01] M. MANETTI, *On the moduli space of diffeomorphic algebraic surfaces*, Invent. Math. 143 (2001), no. 1, 29–76.

[Sch71] M. SCHLESSINGER, *Rigidity of quotient singularities*, Invent. Math. 14 (1971), 17–26.

[Sym01] M. SYMINGTON, *Generalized symplectic rational blowdowns*, Algebr. Geom. Topol. 1 (2001), 503–518.

[Tau94] C. H. TAUBES, *The Seiberg-Witten invariants and symplectic forms*, Math. Res. Lett. 1 (1994), no. 6, 809–822.

[Tia90] G. TIAN, *On Calabi's conjecture for complex surfaces with positive first Chern class*, Invent. Math. 101 (1990), no. 1, 101–172.

[Ue96] M. UE, *A remark on the exotic free actions in dimension* 4, J. Math. Soc. Japan 48 (1996), no. 2, 333–350.

[Wah81] J. WAHL, *Smoothings of normal surface singularities*, Topology 20 (1981), no. 3, 219–246.

[Wit94] E. WITTEN *Monopoles and four-manifolds*, Math. Res. Lett. 1 (1994), no. 6, 809–822.

[Yau77] S.-T. YAU, *Calabi's conjecture and some new results in algebraic geometry*. Proc. Nat. Acad. USA 74 (1997), 1789–1799.