Karoubi's Construction for Motivic Cohomology Operations

A Dissertation, Presented

by

Zhaohu Nie

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

August 2005

Stony Brook University

The Graduate School

Zhaohu Nie

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

H. Blaine Lawson, Jr.
Distinguished Professor, Department of Mathematics
Dissertation Director

Sorin Popescu
Associate Professor, Department of Mathematics
Chairman of Dissertation

Mark A. de Cataldo
Associate Professor, Department of Mathematics

Christian Haesemeyer
J.L. Doobs Research Assistant Professor,
Department of Mathematics,
University of Illinois, Urbana-Champaign
Outside Member

This dissertation is accepted by the Graduate School.

Graduate School

Abstract of the Dissertation

Karoubi's Construction for Motivic Cohomology Operations

by

Zhaohu Nie

Doctor of Philosophy

in

Mathematics

Stony Brook University

2005

We use an analogue of Karoubi's construction [7] in the motivic situation to give some cohomology operations in motivic cohomology. We prove many properties of these operations, and we show that they coincide, up to some nonzero constants, with the reduced power operations in motivic cohomology originally constructed by Voevodsky [15]. The relation of our construction to Voevodsky's is, roughly speaking, that of a fixed point set to its associated homotopy fixed point set.

To my beloved wife, Yuhua Wang

Contents

List of Figures vii

Acknowledgements viii

1 Introduction 1

1.1 Reduced power operations 1

1.2 Karoubi's construction 6

1.3 Our main results 12

2 Some background 15

2.1 Some basic notions 15

2.2 Triangulated category of motives 21

2.3 $\mathbb{A}^1$-homotopy category 30

3 Construction and properties of our operations 33

3.1 The total reduced power operation 33

3.2 Lens spaces 39

3.3 The total fixed point set 51

3.4 Bockstein homomorphisms 55

3.5 Basic properties 64

4 **Comparison with Voevodsky's operations** **68**

4.1 Voevodsky's operations 68

4.2 Comparison of the two operations 75

Bibliography **90**

List of Figures

3.1 The power operation P 34

3.2 The exact sequence (3.1.4) 36

3.3 The isomorphism (3.1.6) 38

Acknowledgements

First of all, I would like to thank my advisor, Professor Blaine Lawson. Over the years, I have learned most of my mathematics from him through our weekly meetings. This thesis owes its start to his suggestion that Karoubi's construction in the topological case [7] may be adopted in the situation at our hand (originally we were considering morphic cohomology operations). I especially want to express my deep gratitude to his constant encouragements and support. He has supported me to various conferences to widen my horizon. In particular, he made my one year visit at the Institute for Advanced Study possible by providing financial support. I feel most fortunate to be a student of Blaine.

I thank the faculty members in the department, in particular Professors de Cataldo, LeBrun and Popescu, from whose courses I have learned many aspects of mathematics. Thanks also go to the secretaries, Barbara and Donna, for their patient daily helps.

I would also like to thank the friends in the department, with whom I have had a lot of mathematical discussions and fun. I thank them all, and a partial list is Bin, Ming, Hongyuan, Tao, Zhendong, Kwan Kiu, Jyh-Haur, Li, Luis...

My last year of Ph.D. study is spent at the Institute for Advanced Study.

I thank the Institute for their hospitality. I would also like to thank Patrick Brosnan, Max Karoubi and Vladimir Voevodsky, with whom I had useful discussions about this thesis. In particular, I deeply appreciate Christian Haesemeyer's stimulating discussions, which have helped to simplify the presentations at several places, most notably Proposition 3.1.1. I also thank him very much for his thorough reading of the first draft of this thesis, and for his suggestions of how to correct numerous English errors and mathematical inaccuracies. I also thank Hao Fang, whose friendship made my stay at the Institute more enjoyable.

Finally, I thank my parents for their constant love and encouragements. This thesis would not be possible without the love and care of my wife, Yuhua Wang, and I dedicate it to her.

Chapter 1

Introduction

The goal of this thesis is to use an analogue of Karoubi's construction [7], which is for the reduced power operations in singular cohomology, in the motivic situation to give another construction of Voevodsky's reduced power operations in motivic cohomology [15].

We introduce the subject to study in this thesis, the reduced power operations in motivic cohomology, in Section 1.1. Along the way, we briefly review the notion of cohomology operations and the reduced power operations in singular cohomology. We review the method to follow, Karoubi's construction [7] for the reduced power operations in singular cohomology, in Section 1.2. We outline our analogous construction in the motivic situation and state our main results in Main Theorem 1.3.1 in Section 1.3.

1.1 Reduced power operations

First we introduce the general notion of cohomology operations. Then we review briefly the reduced power operations in singular cohomology [10] and in motivic cohomology [15].

We regard a cohomology theory on a category $\mathcal{C}$ as a family of contravariant functors from $\mathcal{C}$ to the category $\mathcal{Ab}$ of abelian groups. The prototype is the singular cohomology theory on the category $\mathcal{T}op$ of topological spaces, and the case of our interest is the motivic cohomology theory on the category Sm/k of smooth schemes over a field k (see Definition 2.2.11). Speaking roughly, cohomology operations are natural transformations between two cohomology functors (in one cohomology theory). Cohomology operations are not generally assumed to be group homomorphisms, i.e. we only regard the cohomology functors as functors to the category $\mathcal{S}ets$ of sets.

For example, we regard the n-th singular cohomology $H^n(-, A)$ with coefficients in an abelian group A as a contravariant functor

$$H^n(-, A) : (\mathcal{T}op)^{op} \to \mathcal{Ab}$$

from the category of topological spaces to the category of abelian groups. A cohomology operation θ of type $(A, n; B, m)$ is a natural transformation

$$\theta : H^n(-, A) \to H^m(-, B),$$

where we regard both cohomology functors as to the category of sets. In more detail, θ consists of a family of maps

$$\theta_X : H^n(X, A) \to H^m(X, B),$$

one for each space X, satisfying the naturality condition $f^*\theta_Y = \theta_X f^*$ for any continuous map $f : X \to Y$. We denote the set of cohomology operations of type $(A, n; B, m)$ by $\mathcal{O}p(A, n; B, m)$.

Cup product powers and Bockstein homomorphisms are natural examples of cohomology operations.

The reduced power operations consist a very important class of cohomology operations in singular cohomology. Let l be an arbitrary prime. For an integer $i \geq 0$, the i-th reduced power operation P^i is a family of cohomology operations of type $(\mathbb{Z}/l, *; \mathbb{Z}/l, * + 2i(l-1))$ where $*$ is an arbitrary dimension, i.e. we have a family of natural transformations

$$P^i : H^*(-, \mathbb{Z}/l) \to H^{*+2i(l-1)}(-, \mathbb{Z}/l). \tag{1.1.1}$$

These operations are constructed by Steenrod [10], and their constructions use equivariant cohomology and the infinite dimensional lens space as the classifying space BC_l (actually $B\mu_l$ where $\mu_l = \{e^{\frac{2\pi i}{l}} | 0 \leq i \leq l-1\}$ is the group of l-th roots of unity in $\mathbb{C}$). The reduced power operations satisfy the following properties:

1. The P^i are group homomorphisms.

2. $P^0 = id$.

3. $P^n(x) = x^l$ for $x \in H^{2n}(X, \mathbb{Z}/l)$. If $2k > \dim(x)$, then $P^k(x) = 0$.

4. (Cartan formula) $P^i(xy) = \sum_k P^k(x)P^{i-k}(y)$.

5. (Adem relations) See [10, pg. 77].

6. The P^i are stable, i.e. they commute with the suspension isomorphisms.

The reduced power operations in singular cohomology form a fundamental tool in algebraic topology, leading to important progress both in general homotopy theory and in specific geometric applications. We now describe one very important application.

With composition as multiplication, the graded associative algebra generated by the reduced power operations P^i of degree $2i(l-1)$ and the Bockstein homomorphism $\beta : H^*(-,\mathbb{Z}/l) \to H^{*+1}(-,\mathbb{Z}/l)$, associated to the exact sequence of coefficients $0 \to \mathbb{Z}/l \xrightarrow{l} \mathbb{Z}/l^2 \to \mathbb{Z}/l \to 0$, of degree 1 is called the mod l Steenrod algebra $\mathcal{A}_l$. The singular cohomology ring $H^*(X,\mathbb{Z}/l)$ of a space X is a module over the Steenrod algebra $\mathcal{A}_l$, with elements in $\mathcal{A}_l$ acting on $H^*(X,\mathbb{Z}/l)$ as cohomology operations. Through the Adams spectral sequence (see J.F. Adams, Comment. Math. Helv. 32 (1958), 180–214), the existence of such a structure on $H^*(X,\mathbb{Z}/l)$ as a module over $\mathcal{A}_l$ can be used to draw information about the group of stable homotopy classes of maps between spaces, in particular about the stable homotopy groups of spheres.

Now let's start to introduce motivic cohomology and the reduced power operations in motivic cohomology.

The motivic cohomology theory is a bigraded cohomology theory on the category Sm/k of smooth schemes over a field k. The (q,p)-th motivic cohomology $H^{q,p}(-,A)$ with coefficients in an abelian group A is a contravariant functor

$$H^{q,p}(-,A) : (Sm/k)^{op} \to Ab.$$

Here q is called the cohomology dimension, and $p \geq 0$ is called the algebraic dimension.

Motivic cohomology for smooth schemes plays the role of singular cohomology for topological spaces. Actually the construction of algebraic singular homology, a special case of motivic homology, by Suslin and Voevodsky [Invent. Math. 123 (1996), no. 1, 61–94] is inspired by the following Dold-Thom theorem in topology. Given a topological space X, consider the free abelian group $\mathbb{Z} \cdot X$ generated by X. $\mathbb{Z} \cdot X$ has a natural topology to make it a topological abelian group. Then the Dold-Thom theorem [3] says that if X is a CW-complex, one has an isomorphism

$$\pi_i(\mathbb{Z} \cdot X) = H_i(X, \mathbb{Z}). \tag{1.1.2}$$

The definition of algebraic singular homology uses algebraic analogues of the functor $X \to \mathbb{Z} \cdot X$ and the homotopy group π_i. For the precise definition of motivic cohomology, we refer the reader to Section 2.2, especially Definition 2.2.11.

In view of our general notion of cohomology operations (see the beginning of this section), motivic cohomology operations are, roughly speaking, natural transformations between motivic cohomology functors (regarded as to the category of sets). One also has the reduced power operations in motivic cohomology, which we describe below, as an important class of motivic cohomology operations.

We fix a base field k and a prime l different from $char(k)$, the characteristic of k. Voevodsky [15] constructed the reduced power operations

$$P^i : H^{*,*}(-, \mathbb{Z}/l) \to H^{*+2i(l-1),*+i(l-1)}(-, \mathbb{Z}/l), \ i \geq 0$$

in motivic cohomology. Note the similarity of these operations to the reduced power operations in singular cohomology (1.1.1) if we concentrate on the cohomology dimensions. Voevodsky's construction of these P^i (loc. cit.) follows the classical construction of Steenrod [10] (see also [6]) for the reduced power operations in singular cohomology.

The reduced power operations in motivic cohomology play an important role in Voevodsky's proofs of the Milnor conjecture [Publ. Math. IHES 98 (2003), 59–104] and the Bloch-Kato conjecture. The bigraded motivic Steenrod algebra $\mathcal{A}^{*,*}(k, \mathbb{Z}/l)$ generated by the reduced power operations P^i and the Bockstein homomorphism β is very important in the motivic homotopy theory.

1.2 Karoubi's construction

Besides Steenrod's classical construction [10] of the reduced power operations in singular cohomology, Karoubi [7] recently gives a different construction of them, which is very geometric. We review Karoubi's construction in this section.

Recall that the Eilenberg-MacLane space $K(A, n)$ in topology is a "nice" space (a space having the homotopy type of a CW-complex) which has only one nontrivial homotopy group, namely, $\pi_n(K(A, n)) = A$ (A is a group when $n \geq 1$ and abelian when $n \geq 2$).

It's a standard fact that for a space X, there is a one-to-one correspondence

$$Hom_{Ho(\mathcal{T}op)}(X, K(A, n)) \leftrightarrow H^n(X, A), \tag{1.2.1}$$

where $Ho(\mathcal{T}op)$ is the homotopy category of topological spaces (see Section 2.1), and the left hand side is a Hom set in this category. For a proof of (1.2.1), see for example Theorem 1.1 of Mosher and Tangora [Cohomology operations and applications in homotopy theory. Harper & Row, Publishers, 1968] when X has the homotopy type of a CW-complex. The case when X is a general space then follows from this and the construction of $Ho(\mathcal{T}op)$, since every space is weakly equivalent to a CW-complex.

One can therefore establish a one-to-one correspondence

$$Hom_{Ho(\mathcal{T}op)}(K(A,n),K(B,m)) \leftrightarrow \mathcal{O}p(A,n;B,m) \qquad (1.2.2)$$

between the set of morphisms in the homotopy category $Ho(\mathcal{T}op)$ from $K(A,n)$ to $K(B,m)$ and the set of singular cohomology operations of type $(A,n;B,m)$ as follows. For $\phi \in Hom_{Ho(\mathcal{T}op)}(K(A,n),K(B,m))$, we define its associated cohomology operation θ_ϕ of type $(A,n;B,m)$ by the following. Given an element $x \in H^n(X,A)$, by (1.2.1) it corresponds to a morphism $\underline{x} : X \to K(A,n)$ in the homotopy category $Ho(\mathcal{T}op)$. The composition $\phi \circ \underline{x}$ as a morphism from X to $K(B,m)$ in the homotopy category $Ho(\mathcal{T}op)$ corresponds to a class $x' \in H^m(X,B)$ by (1.2.1) again. We define $\theta_{\phi,X}(x) = x'$. It is easy to see that θ_ϕ is a cohomology operation, i.e. it commutes with the homomorphism f^* induced by a continuous map $f : X \to Y$, since for $y \in H^n(Y,A)$ which corresponds to $\underline{y} : Y \to K(A,n)$ through (1.2.2), $f^*(y) \in H^n(X,A)$ is the cohomology class which corresponds to the composition $\underline{y} \circ f : X \to K(A,n)$.

Therefore the reduced power operation $P^i : H^n(-,\mathbb{Z}/l) \to H^{n+2i(l-1)}(-,\mathbb{Z}/l)$

in (1.1.1) corresponds to a map

$$P^i : K(\mathbb{Z}/l, n) \to K(\mathbb{Z}/l, n + 2i(l-1)). \tag{1.2.3}$$

There is a nice model of $K(\mathbb{Z}/l, n)$, which we describe now. Denote by $(\mathbb{Z}/l \cdot S^n)_0$ the $\mathbb{Z}/l$ module over the n-th sphere S^n with the base point $\infty \in S^n$ identified to 0. $(\mathbb{Z}/l \cdot S^n)_0$ has a naturally defined topology to make it a topological abelian group (see [3]). By the Dold-Thom theorem (or rather an obvious variation of it, see loc. cit.), one has

$$\pi_i((\mathbb{Z}/l \cdot S^n)_0) = \tilde{H}_i(S^n, \mathbb{Z}/l) = \begin{cases} \mathbb{Z}/l, & \text{for } i = n, \\ 0, & \text{for } i \neq n. \end{cases}$$

$(\mathbb{Z}/l \cdot S^n)_0$ can be seen to have the homotopy type of a CW complex, and therefore $(\mathbb{Z}/l \cdot S^n)_0$ is a model of the Eilenberg-MacLane space $K(\mathbb{Z}/l, n)$, i.e.

$$K(\mathbb{Z}/l, n) = (\mathbb{Z}/l \cdot S^n)_0. \tag{1.2.4}$$

The cup product

$$H^m(X, \mathbb{Z}/l) \otimes H^n(X, \mathbb{Z}/l) \to H^{m+n}(X, \mathbb{Z}/l),$$

in view of (1.2.1), is represented by the following map on Eilenberg-MacLane spaces

$$K(\mathbb{Z}/l, m) \wedge K(\mathbb{Z}/l, n) \to K(\mathbb{Z}/l, m+n).$$

In terms of our model (1.2.4) of $K(\mathbb{Z}/l, n)$, the above map is the following

$$
\begin{gathered}
(\mathbb{Z}/l \cdot S^m)_0 \wedge (\mathbb{Z}/l \cdot S^n)_0 \to (\mathbb{Z}/l \cdot S^{m+n})_0; \\
(\sum n_i x_i, \sum m_j y_j) \to \sum n_i m_j x_i \wedge y_j.
\end{gathered}
$$

In particular, the l-th cup product power $P : K(\mathbb{Z}/l, n) \to K(\mathbb{Z}/l, nl)$ is represented by (abusing the notation P)

$$
\begin{gathered}
P : (\mathbb{Z}/l \cdot S^n)_0 \to (\mathbb{Z}/l \cdot S^{nl})_0; \qquad (1.2.5) \\
\sum n_i x_i \to (\sum_i n_i x_i)^l = \sum_{i_1, \cdots, i_l} n_{i_1} \cdots n_{i_l} x_{i_1} \wedge \cdots x_{i_l}.
\end{gathered}
$$

The symmetric group S_l acts on $S^{nl} = S^n \wedge S^n \wedge \cdots \wedge S^n$ by permuting the l copies of S^n, and this induces an action of S_l on $(\mathbb{Z}/l \cdot S^{nl})_0$. In particular, the cyclic group C_l acts on S^{nl} and $(\mathbb{Z}/l \cdot S^{nl})_0$. By its construction, the above map P in (1.2.5) factorizes through

$$
\mathcal{P} : (\mathbb{Z}/l \cdot S^n)_0 \to (\mathbb{Z}/l \cdot S^{nl})_0^{C_l}, \qquad (1.2.6)
$$

where the right hand side is the the fixed point set of $(\mathbb{Z}/l \cdot S^{nl})_0$ under the C_l action.

For l a prime, the C_l action on S^{nl} is semi-free. It has the diagonal $S^n = \{x \wedge x \wedge \cdots \wedge x | x \in S^n\} \subset S^{nl}$ as the fixed point set, and it is free outside of the diagonal. Let S^{nl}/C_l denote the quotient of S^{nl} by the cyclic group C_l. There is an inclusion $S^n \subset S^{nl}/C_l$ since S^n is fixed. Karoubi calls the quotient $(S^{nl}/C_l)/S^n$ of S^{nl}/C_l with the diagonal S^n identified to the base point the normalized l-th cyclic product of S^n.

If $\phi : S^{nl} \longrightarrow (S^{nl}/C_l)/S^n$ denotes the canonical map, it is clear that $\phi(P(x)) = \phi((\sum_i n_i x_i)^l)$ is the sum of l copies of $\mathcal{P}_1(x)$ (consider $\mathbb{Z}$ coefficients here), where the map

$$\mathcal{P}_1 : (\mathbb{Z}/l \cdot S^n)_0 \to (\mathbb{Z}/l \cdot (S^{nl}/C_l)/S^n)_0 \tag{1.2.7}$$

is defined by the following formula (which can be defined for $\mathbb{Z}/l$ coefficients)

$$\mathcal{P}_1(x) = \mathcal{P}_1(\sum_i n_i x_i) = \sum_{<i_1, i_2, \cdots, i_l>} n_{i_1} n_{i_2} \cdots n_{i_l} \phi(x_{i_1} \wedge x_{i_2} \wedge \cdots \wedge x_{i_l}),$$

$< i_1, i_2, \cdots, i_l >$ denoting the class of the sequence $(i_1, i_2, \cdots, i_l)$, with $i_1, i_2, \cdots, i_l$ not all equal, modulo the action of the cyclic group C_l.

$(S^{nl}/C_l)/S^n$ can be understood very geometrically. It is clear that S^{nl}/C_l is the $(n+1)$-st suspension $\Sigma^{n+1}S^{nl-n-1}/C_l$ of a lens space S^{nl-n-1}/C_l. Therefore the reduced homology with $\mathbb{Z}/l$ coefficients of S^{nl}/C_l is

$$\tilde{H}_i(S^{nl}/C_l, \mathbb{Z}/l) = \mathbb{Z}/l, \text{ for } n+2 \leq i \leq nl.$$

Since S^{nl}/C_l, being an $(n+1)$-st suspension, is $(n+1)$-connected, the inclusion $S^n \subset S^{nl}/C_l$ is homotopically trivial. Therefore the further contraction $(S^{nl}/C_l)/S^n$ is homotopy equivalent to the "wedge" $S^{nl}/C_l \vee \Sigma S^n = S^{nl}/C_l \vee S^{n+1}$. Thus the reduced homology with $\mathbb{Z}/l$ coefficients of $(S^{nl}/C_l)/S^n$ is

$$\tilde{H}_i((S^{nl}/C_l)/S^n, \mathbb{Z}/l) = \mathbb{Z}/l, \text{ for } n+1 \leq i \leq nl.$$

By again the Dold-Thom theorem [3], one sees that

$$\pi_i((\mathbb{Z}/l\cdot(S^{nl}/C_l)/S^n)_0) = \tilde{H}_i((S^{nl}/C_l)/S^n, \mathbb{Z}/l)$$
$$= \quad \mathbb{Z}/l, \text{ for } n+1 \leq i \leq nl.$$

A theorem of J. Moore states that a topological abelian group is homotopy equivalent to a product of Eilenberg-MacLane spaces corresponding to its homotopy groups. Therefore one has a homotopy equivalence

$$(\mathbb{Z}/l\cdot(S^{nl}/C_l)/S^n)_0 \simeq \prod_{i=n+1}^{nl} K(\mathbb{Z}/l, i). \tag{1.2.8}$$

Now define

$$\bar{\mathcal{P}} : (\mathbb{Z}/l\cdot S^n)_0 \to \Delta_l^+(S^n) := (\mathbb{Z}/l\cdot S^n)_0 \oplus (\mathbb{Z}/l\cdot(S^{nl}/C_l)/S^n)_0 \tag{1.2.9}$$

by $\bar{\mathcal{P}}(x) = (x_{(l)}, \mathcal{P}_1(x))$, where $\mathcal{P}_1$ is defined in (1.2.7) and for $x = \sum n_i x_i$, $x_{(l)}$ is defined to be $\sum n_i^l x_i$. There is clearly a homotopy equivalence

$$\Delta_l^+(S^n) \simeq \prod_{i=0}^{n(l-1)} K(\mathbb{Z}/l, n+i) \tag{1.2.10}$$

by (1.2.4) and (1.2.8). Actually this homotopy equivalence can be made canonical by using the canonical generators of $\tilde{H}_i((S^{nl}/C_l)/S^n, \mathbb{Z}/l)$ (see [7, Annexe C]).

In view of (1.2.10), denote the following components of $\bar{\mathcal{P}}$ (1.2.9) by

$$D^i : K(\mathbb{Z}/l, n) \to K(\mathbb{Z}/l, n+2i).$$

Then Karoubi [7, Theorem 2.6] proves that the D^i are trivial if i is not a multiple of $(l-1)$, and up to some nonzero constants in $\mathbb{Z}/l$, $D^{j(l-1)}$ coincides with the reduced power operations P^j (1.2.3) of Steenrod.

1.3 Our main results

We now outline our construction in the motivic situation following Karoubi's construction in the topological case (see Section 1.2). The reader will find clear analogy almost everywhere.

We work in the $\mathbb{A}^1$-homotopy category $H_\bullet^{\mathbb{A}^1}(k)$ of Morel and Voevodsky (see Section 2.3 and also [8, 12, 2]). $H_\bullet^{\mathbb{A}^1}(k)$ for the category of smooth schemes over a field k plays the role of the homotopy category $Ho(\mathcal{T}op)$ for the category of topological spaces. In $H_\bullet^{\mathbb{A}^1}(k)$, motivic cohomology is represented by motivic Eilenberg-MacLane spaces (see (2.3.1); compare to (1.2.1)), and so motivic cohomology operations are represented by maps between corresponding motivic Eilenberg-MacLane spaces (compare to (1.2.2)).

Let $n \geq 1$ be an integer. Choose our model of the motivic Eilenberg-MacLane space $K(\mathbb{Z}/l, 2n, n)$ as $z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^n, 0)$ (see (2.3.2); compare to (1.2.4)). The l-th cup product power operation

$$P : K(\mathbb{Z}/l, 2n, n) \to K(\mathbb{Z}/l, 2nl, nl) \tag{1.3.1}$$

is represented by (abusing the notation P)

$$P : z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^n, 0) \to z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^{nl}, 0), \tag{1.3.2}$$

which is the l-th fiber product power of cycles (see (3.1.1); compare to (1.2.5)).

The cyclic group C_l acts on $\mathbb{A}^{nl}$ by cyclically permuting the l copies of $\mathbb{A}^n$, and so acts on $z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^{nl}, 0)$ by functoriality. By its very construction, the image of P in (1.3.2) lies in the fixed point set $z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^{nl}, 0)^{C_l}$ of invariant cycles. Call the factorization $\mathcal{P} : z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^n, 0) \to z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^{nl}, 0)^{C_l}$ our total reduced power operation (see (3.1.3); compare to (1.2.6)). We have the following main theorem.

Main Theorem 1.3.1 *Suppose that the field k satisfies some assumptions (detailed at the beginning of Section 3.2).*

1. *We have an isomorphism in $H_\bullet^{\mathbb{A}^1}(k)$*

$$z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^{nl}, 0)^{C_l} \simeq \prod_{i=0}^{n(l-1)} K(\mathbb{Z}/l, 2n+2i, n+i) \times \prod_{i=0}^{n(l-1)-1} K(\mathbb{Z}/l, 2n+2i+1, n+i).$$

2. *Denote the components of $\mathcal{P} : z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^n, 0) \to z_{equi}^{\mathbb{Z}/l}(\mathbb{A}^{nl}, 0)^{C_l}$, in view of the conclusion of part 1, by*

$$D^i : K(\mathbb{Z}/l, 2n, n) \to K(\mathbb{Z}/l, 2n + 2i, n + i),\ 0 \le i \le n(l-1)$$

   and

$$E^i : K(\mathbb{Z}/l, 2n, n) \to K(\mathbb{Z}/l, 2n + 2i + 1, n + i),\ 0 \le i \le n(l-1) - 1.$$

   Then $E^i = \beta \circ D^i$ for $0 \le i \le n(l-1) - 1$, where $\beta : K(\mathbb{Z}/l, *, *) \to K(\mathbb{Z}/l, * + 1, *)$ represents the Bockstein homomorphism associated to the exact sequence of coefficients $0 \to \mathbb{Z}/l \xrightarrow{\cdot l} \mathbb{Z}/l^2 \to \mathbb{Z}/l \to 0$.

3. *$D^0 : K(\mathbb{Z}/l, 2n, n) \to K(\mathbb{Z}/l, 2n, n)$ is the identity map, and $D^{n(l-1)}$: $K(\mathbb{Z}/l, 2n, n) \to K(\mathbb{Z}/l, 2nl, nl)$ is the l-th cup product power operation P (1.3.1).*

4. *For i not a multiple of $(l-1)$, $D^i = 0$ and thus $E^i = 0$ by the conclusion of part 2. Up to nonzero constants in $\mathbb{Z}/l$,*

   $$D^{j(l-1)} \equiv P^j, \ 0 \leq j \leq n,$$

   where the P^j are the reduced power operations of Voevodsky [15].

To be able to carry out the construction and make computations, we give some preliminaries in Chapter 2. In particular, we describe the constructions and basic properties of the triangulated category of motives and the $\mathbb{A}^1$-homotopy category. In order to prove the comparison result, we review in Section 4.1 Voevodsky's construction of the reduced power operations in [15].

The rest of the thesis is concerned with the proof of our main theorem. The proof of part 1 is given in Sections 3.1, 3.2 and 3.3 (see Theorem 3.3.3), part 2 in Section 3.4 (see Theorem 3.4.5), part 3 in Section 3.5 (see Theorems 3.5.1 and 3.5.2), and part 4 in Section 4.2 (see Theorem 4.2.1).

As will be seen in Section 4.2, the relation of our construction to Voevodsky's is, roughly speaking, that of a fixed point set to its associated homotopy fixed point set.

Chapter 2

Some background

Throughout this thesis, we work in both Voevodsky's triangulated category of motives DM over our base field k [13] and the $\mathbb{A}^1$-homotopy category $H_\bullet^{\mathbb{A}^1}(k)$ constructed by Morel and Voevodsky [8, 12]. We review these constructions in some detail.

2.1 Some basic notions

Let's start with the notions of presheaves and sheaves for a Grothendieck topology. See [1] for more detail. Let C be a general category.

Definition 2.1.1 *A presheaf of sets F on the category C is a contravariant functor $F : C^{op} \to Sets$, where the superscript stands for the opposite category and the right hand side is the category of sets. Similarly a presheaf of pointed sets or abelian groups is a contravariant functor from C to the category $Sets_\bullet$ of pointed sets or the category Ab of abelian groups.*

Usually a presheaf of sets is meant when we say "a presheaf" without further qualification. Many notions we will talk about below hold parallelly for presheaves of pointed sets and abelian groups.

Definition 2.1.1 is Grothendieck's notion of a presheaf on a category, and it generalizes the usual concept of a presheaf on a topological space X if we take the category C to have as objects the open sets of X and as morphisms the inclusions. In fact, there is also a general notion of sheaves with respect to a Grothendieck topology as follows.

Definition 2.1.2 *A Grothendieck topology T on a category C is a set $Cov(T)$ of families $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ of maps in C called coverings satisfying*

1. *If f is an isomorphism then $\{f\} \in Cov(T)$.*
2. *If $\{U_i \xrightarrow{f_i} U\}_{i\in I} \in Cov(T)$ and $\{V_{ij} \xrightarrow{g_{ij}} U_i\}_{j\in J_i} \in Cov(T)$ for each $i \in I$ then the family $\{V_{ij} \xrightarrow{f_i \circ g_{ij}} U\}_{j\in J_i, i\in I}$ is in $Cov(T)$.*
3. *If $\{U_i \xrightarrow{f_i} U\}_{i\in I} \in Cov(T)$ and $h : V \to U$ is a map in C then $U_i \times_U V$ exists and $\{U_i \times_U V \to V\}_{i\in I}$ is in $Cov(T)$.*

The pair (C, T) is called a site.

Abusing notation, we sometimes write T for (C, T) when C is understood, and we also call a Grothendieck topology a topology.

Definition 2.1.3 *A presheaf F on C is called a sheaf for the topology T if for any covering $\{U_i \xrightarrow{f_i} U\}_{i\in I}$ in $Cov(T)$, the following sequence with the natural maps*

$$F(U) \to \prod_i F(U_i) \rightrightarrows \prod_{i,j} F(U_i \times_X U_j)$$

is exact, i.e. $F(U)$ is the equalizer.

In the special case that the topology, on the category of the open sets of a topological space X and the inclusions, has open coverings as coverings, the above definition reduces to the usual sheaf axioms.

We denote the category of presheaves on a category $\mathcal{C}$ by $Preshv(\mathcal{C})$, and the category of sheaves on $\mathcal{C}$ for the Grothendieck topology T by $Shv_T(\mathcal{C})$. Clearly, morphisms between presheaves and sheaves are defined to be natural transformations of functors.

We now recall some basic notions of homological algebra. See [16] for more detail. Recall that an *abelian category* is an additive category where each morphism has a kernel and a cokernel which satisfy some basic properties (see [16, Definition 1.2.2]). The category of abelian groups is a prototype of an abelian category. Over an abelian category, we can do homological algebra.

Definition 2.1.4 *Assume that $\mathcal{A}$ is an abelian category. A cochain complex $C^* = (\{C^i\}, d)$ in $\mathcal{A}$ has the following form*

$$\cdots \longrightarrow C^{i-1} \xrightarrow{d} C^i \xrightarrow{d} C^{i+1} \longrightarrow \cdots ,$$

where $C^j \in \mathcal{A}$ and $d^2 = 0$. Cochain maps between cochain complexes are maps between complexes that commute with the differentials. Define $Ch(\mathcal{A})$ to be the category of cochain complexes in $\mathcal{A}$ and cochain maps. We also introduce the following subcategories $Ch^-(\mathcal{A})$, $Ch^+(\mathcal{A})$ and $Ch^b(\mathcal{A})$ of cochain complexes bounded from above, bounded from below and bounded in both directions.

We adopt the following convention to translate freely between chain complexes and cochain complexes. Given a chain complex $(\{C_i\}, d)$ where the

We now want to recall some basics of the simplicial homotopy theory. See [5] for more detail.

Definition 2.1.8 *The ordinal category* Δ *has as objects* $\underline{n} = \{0, 1, \cdots, n\}$ *for* $n \geq 0$ *and as morphisms nondecreasing maps. A simplicial object in a category* $\mathcal{C}$ *is a contravariant functor* $\Delta^{op} \to \mathcal{C}$*, and a cosimplicial object in* $\mathcal{C}$ *is a covariant functor* $\Delta \to \mathcal{C}$.

Among the morphisms in Δ, the following ones are basic. For $0 \leq i \leq n$, the i-th coface map is

$$\begin{aligned} d^i : \underline{n-1} &\rightarrow \underline{n}; \\ j &\rightarrow \begin{cases} j, & \text{for } j < i \\ j+1, & \text{for } j \geq i, \end{cases} \end{aligned}$$

and the i-th codegeneracy map is

$$\begin{aligned} s^i : \underline{n+1} &\rightarrow \underline{n}; \\ j &\rightarrow \begin{cases} j, & \text{for } j \leq i \\ j-1, & \text{for } j > i. \end{cases} \end{aligned}$$

These maps satisfy some obvious relations, called the cosimplicial relations. All other morphisms in Δ are compositions of such coface and codegeneracy maps. Therefore a simplicial object Y in $\mathcal{C}$ is a collection $\{Y_n\}_{n \geq 0}$ of objects in $\mathcal{C}$, with the following face and degeneracy maps

$$d_i : Y_n \to Y_{n-1}, \ s_i : Y_n \to Y_{n+1}, \text{ for } 0 \leq i \leq n,$$

which are the images of the coface and codegeneracy maps under the functor Y, and these face and degeneracy maps satisfy the simplicial relations corresponding to the cosimplicial ones (see [5, pg. 4] for more detail).

We denote the categories of simplicial sets, simplicial pointed sets and simplicial abelian groups respectively by $s\mathcal{S}ets$, $s\mathcal{S}ets_{\bullet}$ and $s\mathcal{A}b$.

Using simplicial sets, we can do abstract homotopy theory. Model category is the language for this.

Definition 2.1.9 *A model category is a category C equipped with three classes of morphisms respectively called weak equivalences, cofibrations and fibrations which satisfy some axioms (see [5, pg. 66]). The localization of the model category C with respect to the class of weak equivalences is the associated homotopy category, denoted by $Ho(C)$.*

The category $\mathcal{T}op$ of compactly generated topological spaces is a model category if we use the standard definitions of weak equivalences, cofibrations and fibrations. We denote the associated homotopy category by $Ho(\mathcal{T}op)$.

A basic result in the simplicial homotopy theory is that the category $s\mathcal{S}ets$ of simplicial sets is a model category if we make suitable definitions of the classes of weak equivalences, cofibrations and fibrations. See [5, Theorem 11.3]. We denote the associated homotopy category by $Ho(s\mathcal{S}ets)$.

Actually there is an equivalence between $Ho(\mathcal{T}op)$ and $Ho(s\mathcal{S}ets)$ by the pair of adjoint functors of total singular simplicial set and geometric realization.

There is the following Dold-Kan correspondence between the category $s\mathcal{A}b$ of simplicial abelian groups and the category $Ch_{\geq 0}$ of chain complexes in positive dimensions. Namely the following pair of adjoint functors $N : s\mathcal{A}b \to Ch_{\geq 0}$ and $K : Ch_{\geq 0} \to s\mathcal{A}b$ form an equivalence of categories. For a simplicial abelian group A, the corresponding chain complex $N(A)$ has $N(A)_n = \cap_{i=1}^{n} ker(d_i : A_n \to A_{n-1})$ with d_0 as the differential. For a chain complex C, the corresponding simplicial abelian group $K(C)$ has $K(C)_n = \oplus_{n \twoheadrightarrow k} C_k$ with naturally defined face and degeneracy maps. For a general chain complex, we have to apply the good truncation functor $\tau_{\geq 0}$ at position 0.

2.2 Triangulated category of motives

In this section, we review the basic constructions of the triangulated category of motives DM by Voevodsky. The basic reference is [13].

Consider the category Sm/k of smooth schemes over k with morphisms of schemes. On this category, there are three useful Grothendieck topologies, namely the Zariski, Nisnevich and étale topologies, defined below.

Definition 2.2.1 *A Zariski covering is a family $\{U_i \to U\}$ where each U_i is a Zariski open set of U and $\cup_i U_i = U$. An étale covering is a family $\{U_i \xrightarrow{f_i} U\}$ where each $f_i : U_i \to U$ is an étale morphism and $\cup_i f_i(U_i) = U$. Recall that an étale morphism is a finite, flat and unramified morphism, which resembles an unramified covering in topology. A Nisnevich covering is an étale one such that for each point $u \in U$, there is an i and a point $u_i \in U_i$ such that $f_i(u_i) = u$ and such that u_i and u have isomorphic residue fields.*

From the above definition, we see the following relation between these topologies

$$Zar \subset Nis \subset ét,$$

where $\subset$ means inclusion of sets of coverings.

The following notion of a presheaf with transfers is crucial to the whole development of the motivic cohomology theory.

Definition 2.2.2 *Given two smooth schemes X and Y, define $c(X,Y)$ to be the abelian group of algebraic cycles on $X \times Y$ which are finite over X and surjective over a component of X. Define the category $SmCor(k)$ of finite correspondences over k to be the category whose objects are smooth schemes over k, and whose morphisms from X to Y are elements of $c(X,Y)$.*

Notice that $SmCor(k)$ is an additive category: the Hom sets are abelian groups $c(X,Y)$ and we can define $X \oplus Y := X \coprod Y$. The graph Γ_f of a morphism $f : X \to Y$ is an element in $c(X,Y)$, and thus we have a functor $\Gamma : Sm/k \to SmCor(k)$. Notice that if $f : X \to Y$ is a finite morphism which surjects to a component of Y, then the transpose Γ_f^t of the graph of f is an element in $c(Y,X)$.

Definition 2.2.3 *A presheaf with transfers F is an additive contravariant functor $F : (SmCor(k))^{op} \to \mathcal{A}b$ from $SmCor(k)$ to the category of abelian groups. We denote the category of presheaves with transfers by $Preshv(SmCor(k))$.*

Through $\Gamma : Sm/k \to SmCor(k)$, we can regard a presheaf with transfers as an abelian presheaf (i.e. presheaf of abelian groups) on Sm/k. For a finite map $f : X \to Y$, the element $\Gamma_f^t \in c(Y, X)$ gives a map $F(X) \to F(Y)$, and this is the transfer structure in the name.

Definition 2.2.4 *A presheaf with transfers which is also a Nisnevich sheaf when considered as a presheaf on* Sm/k *is called a Nisnevich sheaf with transfers. We denote the category of Nisnevich sheaves with transfers by* $Shv_{Nis}(SmCor(k))$.

For a smooth scheme X, we define $\mathbb{Z}_{tr}(X)$ to be the presheaf with transfers representable by X, i.e. for any smooth U

$$\mathbb{Z}_{tr}(X)(U) = c(U, X).$$

Actually this definition can be generalized to any scheme X of finite type over k (see Definition 2.2.9). As one can show, $\mathbb{Z}_{tr}(X)$ is always an étale sheaf with transfers and therefore a Nisnevich and Zariski one.

The appearance of presheaves with transfers is by the analogy to the Dold-Thom theorem (1.1.2) in topology. Indeed $\mathbb{Z}_{tr}(X)$ should be viewed as the analogue of $\mathbb{Z} \cdot X$ in algebraic geometry. Of course, by doing so we have to leave the category of schemes and land in the enlarged category of sheaves.

Definition 2.2.5 *A presheaf* F *on* Sm/k *is homotopy invariant if* $F(X) \to F(X \times \mathbb{A}^1)$ *induced by the projection* $X \times \mathbb{A}^1 \to X$ *is an isomorphism. A presheaf with transfers is homotopy invariant if it is so when viewed as a presheaf.*

Definition 2.2.6 *The cosimplicial scheme* $\Delta^\bullet : \Delta \to Sm/k$ *is defined by*

$$\Delta^n = Spec(k[x_0, x_1, \cdots, x_n]/\sum x_i = 1)$$

with the obviously defined coface and codegeneracy maps.

Note that the above definition mimics its topological analogue.

Definition 2.2.7 *The total singular complex functor* C_* *takes an abelian presheaf* F *on* Sm/k *to a complex* $C_*(F)$ *of presheaves with* $C_n(F)(U) = F(\Delta^n \times U)$ *with the differential to be the alternating sum of the face maps induced from the coface maps of* $\Delta^\bullet$.

It is standard that the cohomology presheaves of C_*F are homotopy invariant.

Theorem 2.2.1 *One has the Nisnevich sheafification functor* a_{Nis} *such that for a presheaf* F *on* Sm/k, $a_{Nis}(F)$ *is a Nisnevich sheaf.*

1. *[13, Lemma 3.1.6] If* F *is a presheaf with transfers then* $a_{Nis}(F)$ *has a canonical structure of Nisnevich sheaf with transfers.*
2. *[4, Theorem 5.1] If* F *is homotopy invariant presheaf with transfers then* $a_{Nis}(F)$ *is also homotopy invariant.*

In the above theorem, part 1 doesn't hold for the Zariski topology and part 2 doesn't hold for the étale topology, and this in part explains why the Nisnevich topology is used in the construction.

Clearly $Preshv(SmCor(k))$ is an abelian category. By Theorem 2.2.1.1, one sees that $Shv_{Nis}(SmCor(k))$ is also an abelian category (the kernel and the cokernel of a morphism are now defined to be the sheafifications of the kernel and the cokernel of the morphism between presheaves). Therefore we can consider its derived category.

Definition 2.2.8 *Denote by $D^-(Shv_{Nis}(SmCor(k)))$ the derived category of bounded from above complexes of Nisnevich sheaves of abelian groups with transfers over Sm/k. Define the triangulated category of motives DM to be the full subcategory of $D^-(Shv_{Nis}(SmCor(k)))$ which consists of all complexes with homotopy invariant cohomology sheaves.*

Recall that the cohomology sheaves of a complex of sheaves are the sheafifications of the cohomology presheaves. By Theorem 2.2.1.2, we see that the cohomology sheaves of C_*F when F is a Nisnevich sheaf with transfers are homotopy invariant.

Definition 2.2.9 *For a scheme X of finite type over k, we define its motive as*

$$M(X) = C_*\mathbb{Z}_{tr}(X),$$

where $\mathbb{Z}_{tr}(X)$ is the Nisnevich sheaf with transfers whose value on a smooth scheme U is the free abelian group generated by closed integral subschemes Z of $X \times U$ such that Z is finite over U and dominant over an irreducible component of U. It is clear that $M(X) \in DM$.

Definition 2.2.10 *For a scheme X of finite type over k, we define its motive* *with compact support as*

$$M^c(X) = C_* z_{equi}(X, 0),$$

where $z_{equi}(X, 0)$ *is the Nisnevich sheaf with transfers whose value on a smooth scheme* U *is the free abelian group generated by closed integral subschemes* Z *of* $X \times U$ *such that* Z *is quasi-finite over* U *and dominant over an irreducible component of* U. *It is clear that* $M^c(X) \in DM$.

As presheaves on Sm/k, both $\mathbb{Z}_{tr}(X)$ and $z_{equi}(X, 0)$ have well defined pullbacks for any map $f : U' \to U$, and we denote the pullbacks by $Cycl(f)$. The presheaves $\mathbb{Z}_{tr}(X)$ are covariantly functorial with respect to X. The presheaves $z_{equi}(X, 0)$ are covariantly functorial with respect to proper morphisms $X \to X'$ and contravariantly functorial with respect to flat morphisms $X \to X'$ of relative dimension 0. For simplicity of notation, we very often write $z(X)$ for $z_{equi}(X, 0)$. Given an abelian group A, $z^A_{equi}(X, 0) = z_{equi}(X, 0) \otimes A$ is the sheaf of equidimensional cycles with coefficients in A, and again we write $z^A(X)$ for short.

There are several equivalent definitions [14] of the motivic complexes $\mathbb{Z}(n)$ for $n \geq 0$:

$$\begin{aligned}
\mathbb{Z}(n) &= C_* \mathbb{Z}_{tr}(\mathbb{G}_m^{\wedge n})[-n] \\
&= C_*(\mathbb{Z}_{tr}(\mathbb{P}^n)/\mathbb{Z}_{tr}(\mathbb{P}^{n-1}))[-2n] \\
&= C_*(\mathbb{Z}_{tr}(\mathbb{A}^n)/\mathbb{Z}_{tr}(\mathbb{A}^n - \{0\}))[-2n] \\
&= C_* z(\mathbb{A}^n)[-2n] = M^c(\mathbb{A}^n)[-2n]. \qquad (2.2.1)
\end{aligned}$$

Obviously the $\mathbb{Z}(n)$ are in DM, and they are called the *Tate motives* when viewed this way.

I would like to say that all the above definitions of the motivic complexes are inspired by the following statement in topology that

$$(\mathbb{Z} \cdot S^n)_0 = K(\mathbb{Z}, n),$$

where $(-)_0$ stands for the degree zero component. This statement follows from the Dold-Thom theorem (1.1.2). (Compare to (1.2.4).) As we said before, $\mathbb{Z}_{tr}$ is an analogue of $\mathbb{Z}\cdot$. Obviously there are analogies between $\mathbb{P}^n/\mathbb{P}^{n-1}$ and S^{2n}, and between $\mathbb{G}_m^{\wedge n}$ and S^n where $\mathbb{G}_m$ is $\mathbb{A}^1 - \{0\}$ pointed at 1.

We have the following definition of motivic cohomology.

Definition 2.2.11 *For a smooth scheme X, its motivic cohomology is defined to be*

$$H^{q,p}(X, \mathbb{Z}) = \mathbb{H}^q_{Nis}(X, \mathbb{Z}(p)),$$

where the motivic complex $\mathbb{Z}(p)$ is considered as a complex of Nisnevich sheaves, and the hypercohomology is computed on the small Nisnevich site Nis/X of X.

It is a theorem[4, Theorem 5.1] that in the above definition, one can use as well the Zariski topology.

By [13, Proposition 3.1.9], the right hand side of the above definition for a smooth X is

$$\begin{aligned} \mathbb{H}^q_{Nis}(X, \mathbb{Z}(p)) &= Hom_{D^-(Shv_{Nis}(SmCor(k)))}(M(X), \mathbb{Z}(p)[q]) \\ &= Hom_{DM}(M(X), \mathbb{Z}(p)[q]). \end{aligned}$$

Thus we can define the motivic cohomology of a general scheme X to be represented by the Tate motives in DM as

$$H^{q,p}(X) = Hom_{DM}(M(X), \mathbb{Z}(p)[q]). \tag{2.2.2}$$

By definition, there is no motivic cohomology of negative weights (the second index). Motivic cohomology of low weights is known. For a smooth connected schemes X, one has

$$\begin{aligned} H^{q,0}(X, \mathbb{Z}) &= \begin{cases} \mathbb{Z} & q = 0 \\ 0 & \text{otherwise;} \end{cases} \\ H^{q,1}(X, \mathbb{Z}) &= \begin{cases} \mathcal{O}^*(X) & q = 1 \\ Pic(X) & q = 2 \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

For a field k, its motivic cohomology $H^{*,*}(k, \mathbb{Z}) := H^{*,*}(Spec(k), \mathbb{Z})$ has the following properties: $H^{q,p}(k, \mathbb{Z}) = 0$ when $q > p$ and $H^{p,p}(k, \mathbb{Z}) = k^M_p(k)$ is the Milnor K-theory of k.

To state further results about DM, we need the following definition of resolution of singularities.

Definition 2.2.12 *Let k be a field. We say that k admits resolution of singularities if the following two conditions hold:*

1. *For any scheme of finite type X over k there is a proper surjective morphism $Y \to X$ such that Y is a smooth scheme over k.*

2. *For any smooth scheme X over k and an abstract blow-up $q : X' \to X$ there exists a sequence of blow-ups $p : X_n \to \cdots \to X_i = X$ with smooth centers such that p factors through q.*

The condition that k admits resolution of singularities at the moment means that $char(k) = 0$ by Hironaka's result.

DM is a tensor triangulated category. If k is a field which admits resolution of singularities, then motives with compact support have the following tensor structure for product of schemes (see [13, Proposition 4.1.7]):

$$M^c(X \times Y) = M^c(X) \otimes M^c(Y). \tag{2.2.3}$$

The following localization distinguished triangle (see [13, Proposition 4.1.5]) is of particular importance to our computations in Section 3.4. Suppose that k is a field which admits resolution of singularities, X is a scheme of finite type over k and Z is a closed subscheme of X. Then we have a canonical distinguished triangle of the form

$$M^c(Z) \to M^c(X) \to M^c(X - Z) \to M^c(Z)[1]. \tag{2.2.4}$$

2.3 $\mathbb{A}^1$-homotopy category

In this section, we review the basic constructions of the $\mathbb{A}^1$-homotopy category $H_\bullet^{\mathbb{A}^1}(k)$. The basic references are [8, 12, 2].

Start with the category $sShv_{Nis}(Sm/k)$ of simplicial Nisnevich sheaves of pointed sets on Sm/k. A Nisnevich sheaf is always regarded in this category as the constant simplicial sheaf, i.e. all the terms of the simplicial sheaf are the given sheaf and all the face and degeneracy maps are the identities.

The $\mathbb{A}^1$-homotopy category $H_\bullet^{\mathbb{A}^1}(k)$ is the successive localization of $sShv_{Nis}(Sm/k)$ with respect to the classes of simplicial weak equivalences and $\mathbb{A}^1$-weak equivalences, which we now describe.

Recall that a *point* of a site $(\mathcal{C}, T)$ is a functor $x^* : Shv_T(\mathcal{C}) \to Sets$ which commutes with finite limits and all colimits. In our case of the site $(Sm/k)_{Nis}$, for any $U \in Sm/k$ and any point $u \in U$, the functor $F \to F(Spec(\mathcal{O}^h_{U,u}))$ from a sheaf to its section on the Henselization of U at u is a point of the site.

Definition 2.3.1 *A morphism of simplicial sheaves $f : \mathcal{X} \to \mathcal{Y}$ is called a simplicial weak equivalence if for any point x of the site $(Sm/k)_{Nis}$ the morphism of simplicial sets $x^*(f) : x^*(\mathcal{X}) \to x^*(\mathcal{Y})$ is a weak equivalence.*

Let $Ho_s((Sm/k)_{Nis})$ denote the category obtained from $sShv_{Nis}(Sm/k)$ by formally inverting the class of simplicial weak equivalences.

Definition 2.3.2 *A simplicial sheaf $\mathcal{X}$ on $(Sm/k)_{Nis}$ is called $\mathbb{A}^1$-local if for any simplicial sheaf $\mathcal{Y}$ the map*

$$Hom_{Ho_s((Sm/k)_{Nis})}(\mathcal{Y}, \mathcal{X}) \to Hom_{Ho_s((Sm/k)_{Nis})}(\mathcal{Y} \times \mathbb{A}^1, \mathcal{X})$$

induced by the projection $\mathcal{Y} \times \mathbb{A}^1 \to \mathcal{Y}$ *is a bijection.*

A morphism of simplicial sheaves $f : \mathcal{X} \to \mathcal{Y}$ *is called an* $\mathbb{A}^1$*-weak equivalence if for any* $\mathbb{A}^1$*-local simplicial sheaf* $\mathcal{Z}$ *the map*

$$Hom_{Ho_s((Sm/k)_{Nis})}(\mathcal{Y}, \mathcal{Z}) \to Hom_{Ho_s((Sm/k)_{Nis})}(\mathcal{X}, \mathcal{Z})$$

is a bijection.

The $\mathbb{A}^1$-homotopy category $H_\bullet^{\mathbb{A}^1}(k)$ is obtained from $Ho_s((Sm/k)_{Nis})$ by formally inverting the class of $\mathbb{A}^1$-weak equivalences.

For an abelian group A, let $K(A, j, i)$ be the simplicial abelian sheaf corresponding to the complex $A(i)[j] = \mathbb{Z}(i)[j] \otimes A$ under the Dold-Kan correspondence. Considered as simplicial sheaf of pointed sets, it defines an object of $H_\bullet^{\mathbb{A}^1}(k)$. It is a motivic Eilenberg-MacLane space, since it represents the corresponding motivic cohomology [2, Theorem 2.3.1] as

$$Hom_{H_\bullet^{\mathbb{A}^1}(k)}(X_+, K(A, j, i)) = H^{j,i}(X, A), \tag{2.3.1}$$

where X is a smooth scheme and X_+ is regarded in $H_\bullet^{\mathbb{A}^1}(k)$ as the constant simplicial representable sheaf associated to X with a disjoint base point.

Actually we could define motivic cohomology for any element in $H_\bullet^{\mathbb{A}^1}(k)$ by the representability (2.3.1). Given an element $F_\bullet \in H_\bullet^{\mathbb{A}^1}(k)$, we define its motivic cohomology by

$$\tilde{H}^{j,i}(F_\bullet, A) = Hom_{H_\bullet^{\mathbb{A}^1}(k)}(F_\bullet, K(A, j, i)).$$

We can choose a model of the motivic Eilenberg-MacLane space $K(A, 2n, n)$ as $K(A(n)[2n]) = K(C_* z^A(\mathbb{A}^n))$ in view of (2.2.1). In addition, by [2, Lemma 2.5.2], we have an $\mathbb{A}^1$-weak equivalence between $K(z^A(\mathbb{A}^n))$ and $K(C_* z^A(\mathbb{A}^n))$, where $z^A(\mathbb{A}^n)$ is considered to be a complex concentrated at dimension 0. It is easy to see by the construction of K that $K(z^A(\mathbb{A}^n))$ is the constant simplicial sheaf at $z^A(\mathbb{A}^n)$. Therefore we have the following nice model for $K(A, 2n, n)$ as

$$K(A, 2n, n) = z^A(\mathbb{A}^n). \tag{2.3.2}$$

The case of our interest is when $A = \mathbb{Z}$ or $\mathbb{Z}/l$. Similarly, another natural choice (2.2.1) of the Eilenberg-MacLane space $K(A, 2n, n)$ is

$$K(A, 2n, n) = \mathbb{Z}_{tr}^A(\mathbb{A}^n)/\mathbb{Z}_{tr}^A(\mathbb{A}^n - \{0\}). \tag{2.3.3}$$

Again we can see clearly the analogy of these models to their topological counterpart $(\mathbb{Z} \cdot S^{2n})_0$.

Because of the representability (2.3.1) of motivic cohomology by motivic Eilenberg-MacLane spaces in the $\mathbb{A}^1$-homotopy category $H_\bullet^{\mathbb{A}^1}(k)$, motivic cohomology operations are the same as morphisms between corresponding motivic Eilenberg-MacLane spaces in this category.

Actually our construction will map the above model $z^{\mathbb{Z}/l}(\mathbb{A}^n)$ of $K(\mathbb{Z}/l, 2n, n)$ (2.3.2) to various Eilenberg-MacLane spaces of other dimensions by performing some geometric constructions on it.

Chapter 3

Construction and properties of our operations

In Section 3.1, we carry out our construction outlined in the introduction (see Section 1.3). In Sections 3.2 and 3.3, we do computations in the triangulated category of motives DM, and the interpretation of these computations in the $\mathbb{A}^1$-homotopy category gives part 1 of Main Theorem 1.3.1. We also prove some properties of our operations. In particular, we prove part 2 and part 3 of Main Theorem 1.3.1 in respectively Section 3.4 and Section 3.5.

3.1 The total reduced power operation

In this section, we work with an arbitrary coefficient ring R, although the ones of interest to us are $\mathbb{Z}/l$ and $\mathbb{Z}$. We omit the coefficient ring from our notation.

Let's start with our model of the motivic Eilenberg-MacLane space $K(2n, n)$ as $z(\mathbb{A}^n)$. Consider the l-th cup product power operation

$$P : K(2n, n) \to K(2nl, nl),$$

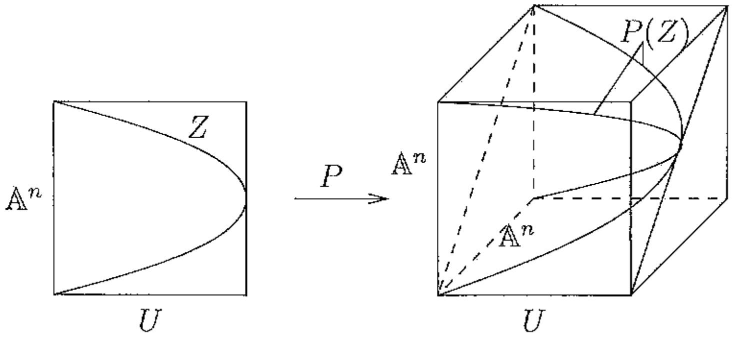

Figure 3.1: The power operation P

which is represented by the l-th fiber product power of cycles

$$P : z(\mathbb{A}^n) \to z(\mathbb{A}^{nl}). \tag{3.1.1}$$

In more detail, for a smooth scheme U and a cycle $Z \in z(\mathbb{A}^n)(U)$, write $Z = \sum n_i Z_i$, where $n_i \in R$ and the Z_i are closed and irreducible subschemes of $U \times \mathbb{A}^n$, which are equidimensional of relative dimension 0 over U. Then $P(Z) = \sum n_{i_1} \cdots n_{i_l} Z_{i_1} \times_U \cdots \times_U Z_{i_l}$, where $Z_{i_1} \times_U \cdots \times_U Z_{i_l}$ is the fiber product of $Z_{i_1}, \cdots, Z_{i_l}$ over U. It is easy to see that $P(Z) \in z(\mathbb{A}^{nl})(U)$. Note that the map P is not linear. Figure 3.1 illustrates the map P (3.1.1) when $l = 2$ and Z is an irreducible subscheme.

The symmetric group S_l acts on $\mathbb{A}^{nl}$ by permuting the l copies of $\mathbb{A}^n$, and in particular the cyclic group C_l acts on $\mathbb{A}^{nl}$ by cyclically permuting the l copies. This naturally induces an action of S_l, and in particular of C_l, on $z(\mathbb{A}^{nl})$ by functoriality. The important thing for us to notice is that the above map (3.1.1) factorizes as

$$P : z(\mathbb{A}^n) \xrightarrow{\mathcal{P}} z(\mathbb{A}^{nl})^{C_l} \xrightarrow{j} z(\mathbb{A}^{nl}), \tag{3.1.2}$$

where $(-)^{C_l}$ stands for the fixed point set under the C_l action on the sheaf level, and j is the natural inclusion map. More precisely $z(\mathbb{A}^{nl})^{C_l}$ is the Nisnevich sheaf whose value on a smooth scheme U is the subgroup of cycles in $z(\mathbb{A}^{nl})(U)$ which are invariant under the C_l action on $U \times \mathbb{A}^{nl}$ as the product of the trivial action on U and the cyclic permutation on $\mathbb{A}^{nl}$. More concretely, an algebraic cycle $Z \in z(\mathbb{A}^{nl})(U)$ is invariant if and only if all its fibers over points of U, as linear combinations of points in $\mathbb{A}^{nl}$, are invariant under the C_l action on $\mathbb{A}^{nl}$. From this we see that $z(\mathbb{A}^{nl})^{C_l}$ is still a presheaf (pullbacks of invariant cycles are invariant). Actually it is a Nisnevich sheaf of abelian groups with transfers as one can easily check. Furthermore the image of $\mathcal{P}$ lies in the similarly defined fixed point set $z(\mathbb{A}^{nl})^{S_l}$ by the whole symmetric group.

The map

$$\mathcal{P} : z(\mathbb{A}^n) \to z(\mathbb{A}^{nl})^{C_l} \tag{3.1.3}$$

is our total reduced power operation. Our task is to analyze the homotopy type of $z(\mathbb{A}^{nl})^{C_l}$ in the $\mathbb{A}^1$-homotopy category $H_{\bullet}^{\mathbb{A}^1}(k)$.

One important feature of the C_l permutation action on $\mathbb{A}^{nl}$ when l is a prime is its semi-freeness. The diagonal of $\mathbb{A}^{nl}$, denoted by $\mathbb{A}^n$, is the fixed point set, and on the complement, $\mathbb{A}^{nl} - \mathbb{A}^n$, the action is free.

Recall that the functor $z(-) = z_{equi}(-, 0)$ is contravariantly functorial for flat morphisms of relative dimension 0, e.g. open embeddings, and covariantly functorial for proper morphisms, e.g. closed embeddings (see [11]). It is easy

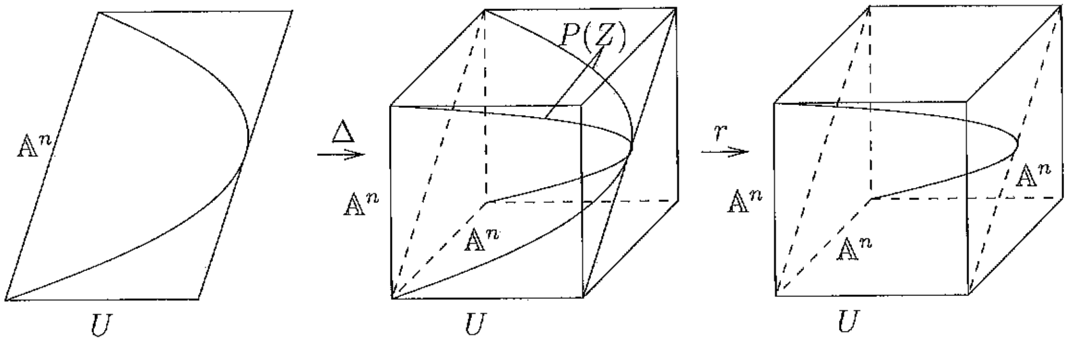

Figure 3.2: The exact sequence (3.1.4)

to see that we have the following exact sequence of sheaves

$$0 \longrightarrow z(\mathbb{A}^n) \xrightarrow{\Delta} z(\mathbb{A}^{nl})^{C_l} \xrightarrow{r} z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l}, \tag{3.1.4}$$

where the inclusion Δ is induced by the closed embedding of the diagonal $\mathbb{A}^n$ in $\mathbb{A}^{nl}$, and the restriction r is induced by the open embedding of the complement $\mathbb{A}^{nl} - \mathbb{A}^n$ in $\mathbb{A}^{nl}$, as illustrated in Figure 3.2.

Proposition 3.1.1 *Let k be a field which admits resolution of singularities. We have a distinguished triangle in DM*

$$C_* z(\mathbb{A}^n) \to C_* z(\mathbb{A}^{nl})^{C_l} \to C_* z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l} \xrightarrow{\delta} C_* z(\mathbb{A}^n)[1]. \tag{3.1.5}$$

Proof. The proof is an analogue of [4, Theorem 5.11], except that we have to take care of the group action here. We write out all the detail for the convenience of the reader. The exact sequence (3.1.4) clearly induces the following diagram with the row as a distinguished triangle in DM

$$\begin{tikzcd}
C_*z(\mathbb{A}^n) \arrow[r, "C_*\Delta"] & C_*z(\mathbb{A}^{nl})^{C_l} \arrow[r] & C_*(z(\mathbb{A}^{nl})^{C_l}/z(\mathbb{A}^n)) \arrow[r] \arrow[d, "C_*r"] & C_*z(\mathbb{A}^n)[1]. \\
& & C_*z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l} &
\end{tikzcd}$$

To prove the proposition, we only need to prove that C_*r is a quasi-isomorphism, or equivalently $C_*(z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l}/z(\mathbb{A}^{nl})^{C_l})$ is acyclic, i.e. quasi-isomorphic to zero. First let's recall the following criterion for acyclicity.

Proposition 3.1.2 *[13, Theorem 4.1.2] Let k be a field which admits resolution of singularities, and F be a presheaf with transfers on Sm/k such that for any smooth scheme U over k and a section $\phi \in F(U)$ there is a proper birational morphism $p : U' \to U$ with U' smooth and $F(p)(\phi) = 0$. Then the complex $C_*(F)$ is acyclic.*

In view of this criterion, it suffices to show that for any smooth scheme U over k and a closed integral subscheme Z in $z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l}(U)$ that there is a blow-up $p : U' \to U$ with U' smooth such that $Cycl(p)(Z)$, the pullback cycle of Z by p, lies in the image of $z(\mathbb{A}^{nl})^{C_l}(U')$.

By the platification theorem (see [11, Theorem 2.2.3] and [9]), there is a blow-up $p : U' \to U$ such that the closure of $Cycl(p)(Z) \in z(\mathbb{A}^{nl} - \mathbb{A}^n)(U')$ lies in the image of $z(\mathbb{A}^{nl})(U')$. It is clear that the closure $\overline{Cycl(p)(Z)} \in z(\mathbb{A}^{nl})^{C_l}(U')$ since $Cycl(p)(Z) \in z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l}(U')$ and the fibers added when taking the closure are all supported in the diagonal $\mathbb{A}^n$, which is the fixed point set under the C_l action.

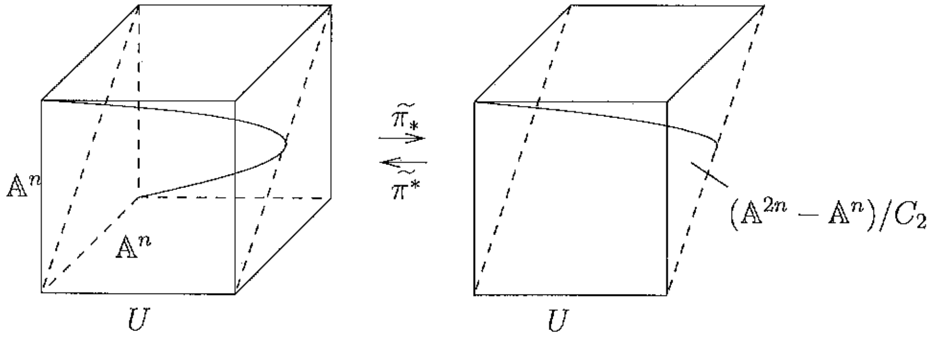

Figure 3.3: The isomorphism (3.1.6)

■

Due to the freeness of the action of C_l on $\mathbb{A}^{nl} - \mathbb{A}^n$, we have canonical mutually inverse isomorphisms

$$z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l} \xrightarrow{\widetilde{\pi}_*} z((\mathbb{A}^{nl} - \mathbb{A}^n)/C_l) \xrightarrow{\widetilde{\pi}^*} z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l}, \qquad (3.1.6)$$

where $\widetilde{\pi}^*$ is the flat pullback whose image is obviously invariant, and $\widetilde{\pi}_*$ is the reduced pushforward, i.e. for $Z \in z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l}(U)$, $\widetilde{\pi}_*(Z)$ is the unique cycle in $z((\mathbb{A}^{nl} - \mathbb{A}^n)/C_l)(U)$ whose flat pullback is Z, as illustrated in Figure 3.3 when $l = 2$.

Following the language of Karoubi [7], we call $(\mathbb{A}^{nl} - \mathbb{A}^n)/C_l$ the l-th normalized cyclic product of $\mathbb{A}^n$, and sometimes denote it by $CP_l^+(\mathbb{A}^n)$.

In the next section, we will compute the motive type of $M^c((\mathbb{A}^{nl} - \mathbb{A}^n)/C_l) = C_* z((\mathbb{A}^{nl} - \mathbb{A}^n)/C_l)$ for $\mathbb{Z}$ and $\mathbb{Z}/l$ coefficients. Together with (3.1.6) and (3.1.5), we are able to analyze the motive type of $C_* z(\mathbb{A}^{nl})^{C_l}$, which determines the homotopy type of $z(\mathbb{A}^{nl})^{C_l}$. We will see that $z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}$ has the desired homotopy type to be the target of the total reduced power operation

(3.1.3).

3.2 Lens spaces

From now on we assume that k admits resolution of singularities to be able to use some distinguished triangles such as (2.2.4) and (3.1.5). This condition at the moment means that $char(k) = 0$. We furthermore assume that k has a primitive l-th root of unity ζ to simplify our exposition. Note that under the condition that $l \neq char(k)$, this assumption doesn't constitute a real restriction, and can always be achieved by passing to a separable extension of k of degree prime to l. Transfer arguments show that this doesn't affect considerations for motivic cohomology with $\mathbb{Z}/l$ coefficients, which is the main subject in this paper. Under this assumption we have an isomorphism between the cyclic group C_l and μ_l, the group of l-th roots of unity in k, which sends $1 \in \mathbb{Z}/l = C_l$ to $\zeta \in \mu_l$. Note that μ_l has a standard representation on $\mathbb{A}^1$, denoted by ρ, where $\zeta \in \mu_l$ acts as $z \to \zeta \cdot z$ for $z \in \mathbb{A}^1$.

It is clear that if we choose a suitable basis of $\mathbb{A}^{nl}$, the action of C_l on $\mathbb{A}^{nl}$ by cyclically permuting the l copies of $\mathbb{A}^n$ can be written as a direct sum $id \oplus \rho \oplus \rho^2 \oplus \cdots \oplus \rho^{l-1}$. Here id is the trivial action, the action ρ is interpreted through the isomorphism of C_l to μ_l, ρ^i is the i-th tensor product of ρ where $\zeta \in \mu_l$ acts as $z \to \zeta^i \cdot z$, and for notational simplicity we have written ρ^i for $\overbrace{\rho^i \oplus \cdots \oplus \rho^i}^{n}$, which acts on a copy of $\mathbb{A}^n$.

The detail of this process is as follows. Assume that $\mathbb{A}^{nl} = \underline{\mathbb{A}}_0^n \oplus \cdots \oplus \underline{\mathbb{A}}_{l-1}^n$, where $\underline{\mathbb{A}}_i^n$ is the i-th copy of $\mathbb{A}^n$ in $\mathbb{A}^{nl}$ for $i \in \mathbb{Z}/l$. T is a generator of C_l which cyclically permutes the $\underline{\mathbb{A}}_i^n$, and so $T(\underline{\mathbb{A}}_i^n) = \underline{\mathbb{A}}_{i+1}^n$. Let $\{e_0^1, \cdots, e_0^n\}$ be a basis of $\underline{\mathbb{A}}_0^n$. Let $e_i^j = T^i(e_0^j)$. Then $\{e_i^j\}_{j=1}^n$ form a basis of $\underline{\mathbb{A}}_i^n$, and $T(e_i^j) = e_{i+1}^j$. Now we define $b_i^j = \sum_{k=0}^{l-1} \zeta^{-ik} e_k^j$. Then the $\{b_i^j\}_{j=1}^n$ are linearly independent because the transformation matrix is a nondegenerate Vandermonde matrix, and so they form a basis of a new copy of $\mathbb{A}^n$, denoted by $\mathbb{A}_i^n$. Now $T(b_i^j) = T(\sum_{k=0}^{l-1} \zeta^{-ik} e_k^j) = \sum_{k=0}^{l-1} \zeta^{-ik} e_{k+1}^j = \zeta^i \sum_{k=0}^{l-1} \zeta^{-i(k+1)} e_{k+1}^j = \zeta^i b_i^j$, and so the action of C_l on $\mathbb{A}_i^n$ is ρ^i, through its isomorphism to μ_l.

Therefore under this new basis, we can rewrite our normalized cyclic product as

$$CP_l^+(\mathbb{A}^n) = (\mathbb{A}^{nl} - \mathbb{A}^n)/C_l = \mathbb{A}^n \times (\mathbb{A}^{n(l-1)} - \{0\})/C_l,$$

where C_l acts freely on $\mathbb{A}^{n(l-1)} - \{0\}$.

By the tensor structure of motives with compact support (2.2.3), we have

$$\begin{aligned} M^c((\mathbb{A}^{nl} - \mathbb{A}^n)/C_l) &= M^c(\mathbb{A}^n \times (\mathbb{A}^{n(l-1)} - \{0\})/C_l) \\ &= M^c((\mathbb{A}^{n(l-1)} - \{0\})/C_l) \otimes M^c(\mathbb{A}^n) \\ &= M^c((\mathbb{A}^{n(l-1)} - \{0\})/C_l) \otimes \mathbb{Z}(n)[2n]. \quad (3.2.1) \end{aligned}$$

Now we want to decide the motive type of $M^c((\mathbb{A}^{n(l-1)} - \{0\})/C_l)$. This is a special case of the following general consideration. Suppose that μ_l acts on $\mathbb{A}^m$ as a direct sum of nontrivial irreducible representations $\rho^{a_1} \oplus \cdots \oplus \rho^{a_m}$ with $1 \leq a_i \leq l-1$ for $1 \leq i \leq m$, where $m \geq 1$ is a general dimension. Then $(\mathbb{A}^m - \{0\})/\mu_l$ is called a lens space. One has the following result.

Proposition 3.2.1 *The motive with compact support of the lens space* $(\mathbb{A}^m - \{0\})/\mu_l$, *where* μ_l *acts on* $\mathbb{A}^m$ *as a direct sum of nontrivial irreducible repre-*

sentations, is isomorphic to

$$M^c((\mathbb{A}^m - \{0\})/\mu_l, \mathbb{Z}) = \mathbb{Z}(0)[1] \oplus \bigoplus_{i=1}^{m-1} \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(m)[2m].$$

We first give the following lemma, which will be used in the proof of Proposition 3.2.1.

Lemma 3.2.2 *For an integer $m \geq 1$, one has an isomorphism*

$$M^c(\mathbb{A}^m - \{0\}) = \mathbb{Z}(0)[1] \oplus \mathbb{Z}(m)[2m].$$

Proof. Consider the following embeddings

$$\{0\} \xrightarrow{\text{closed}} \mathbb{A}^m \xleftarrow{\text{open}} \mathbb{A}^m - \{0\}.$$

Its localization distinguished triangle (2.2.4) is

$$M^c(\{0\}) \to M^c(\mathbb{A}^m) \to M^c(\mathbb{A}^m - \{0\}) \to M^c(\{0\})[1],$$

which in view of (2.2.1) is

$$\mathbb{Z}(0)[0] \to \mathbb{Z}(m)[2m] \to M^c(\mathbb{A}^m - \{0\}) \to \mathbb{Z}(0)[1].$$

It is easily seen that the first arrow is zero (by (2.2.2) it represents a cohomology class of the field k, which is zero by dimensional considerations), which implies our lemma. ∎

Proof of Proposition 3.2.1. We prove this by induction on m. When $m = 1$, it is clear that there is an isomorphism $(\mathbb{A}^1 - \{0\})/\mu_l \cong \mathbb{A}^1 - \{0\}$. Lemma 3.2.2 shows $M^c((\mathbb{A}^1 - \{0\})/\mu_l) = M^c(\mathbb{A}^1 - \{0\}) = \mathbb{Z}(0)[1] \oplus \mathbb{Z}(1)[2]$. We very often write $\mathbb{A}^*$ for $\mathbb{A}^1 - \{0\}$.

Now assume $m \geq 2$ and the result holds for $m - 1$. Consider the following diagram of embeddings

$$\begin{tikzcd}
\mathbb{A}^{m-1} - \{0\} \arrow[r, "\text{closed}"] \arrow[d] & \mathbb{A}^m - \{0\} \arrow[d] & \mathbb{A}^{m-1} \times \mathbb{A}^* \arrow[l, "\text{open}"'] \arrow[d] \\
(\mathbb{A}^{m-1} - \{0\})/\mu_l \arrow[r, "\text{closed}"] & (\mathbb{A}^m - \{0\})/\mu_l & (\mathbb{A}^{m-1} \times \mathbb{A}^*)/\mu_l . \arrow[l, "\text{open}"']
\end{tikzcd} \tag{3.2.2}$$

Here $\mathbb{A}^{m-1} - \{0\}$ is embedded in $\mathbb{A}^m - \{0\}$ as $(\mathbb{A}^{m-1} - \{0\}) \times \{0\}$ and $\mathbb{A}^{m-1} \times \mathbb{A}^*$ is its open complement. Since both embeddings are equivariant with respect to the μ_l action, which is diagonal, they induce the corresponding embeddings on the quotients. The vertical arrows are the quotient maps.

By (2.2.4), the diagram (3.2.2) gives a diagram of localization distinguished triangles:

$$\begin{tikzcd}
M^c(\mathbb{A}^{m-1} - \{0\}) \arrow[r, "1"] \arrow[d] & M^c(\mathbb{A}^m - \{0\}) \arrow[r, "2"] \arrow[d] & M^c(\mathbb{A}^{m-1} \times \mathbb{A}^*) \arrow[d] \\
M^c((\mathbb{A}^{m-1} - \{0\})/\mu_l) \arrow[r] & M^c((\mathbb{A}^m - \{0\})/\mu_l) \arrow[r] & M^c((\mathbb{A}^{m-1} \times \mathbb{A}^*)/\mu_l),
\end{tikzcd} \tag{3.2.3}$$

whose boundary maps are

$$\begin{tikzcd}
M^c(\mathbb{A}^{m-1} \times \mathbb{A}^*) \arrow[r, "3"] \arrow[d, "4"] & M^c(\mathbb{A}^{m-1} - \{0\})[1] \arrow[d, "5"] \\
M^c((\mathbb{A}^{m-1} \times \mathbb{A}^*)/\mu_l) \arrow[r, "6"] & M^c((\mathbb{A}^{m-1} - \{0\})/\mu_l)[1].
\end{tikzcd} \tag{3.2.4}$$

In light of Lemma 3.2.2, the first row of distinguished triangle in (3.2.3) is

$$\begin{aligned} &\mathbb{Z}(0)[1] \oplus \mathbb{Z}(m-1)[2m-2] \xrightarrow{1} \mathbb{Z}(0)[1] \oplus \mathbb{Z}(m)[2m] \qquad (3.2.5)\\ \xrightarrow{2} \quad &\mathbb{Z}(m-1)[2m-1] \oplus \mathbb{Z}(m)[2m] \xrightarrow{3} \mathbb{Z}(0)[2] \oplus \mathbb{Z}(m-1)[2m-1], \end{aligned}$$

and by obvious considerations we understand that $1 : \mathbb{Z}(0)[1] \to \mathbb{Z}(0)[1]$ is the identity, $1 : \mathbb{Z}(m-1)[2m-2] \to \mathbb{Z}(m)[2m]$ is zero, $2 : \mathbb{Z}(m)[2m] \to \mathbb{Z}(m)[2m]$ is the identity, and $3 : \mathbb{Z}(m-1)[2m-1] \to \mathbb{Z}(m-1)[2m-1]$ is the identity.

Now we want to study the second row of distinguished triangle in (3.2.3). The action of μ_l on $\mathbb{A}^*$ is free, and therefore $(\mathbb{A}^{m-1} \times \mathbb{A}^*)/\mu_l$ is a rank $m-1$ vector bundle over $\mathbb{A}^*/\mu_l = \mathbb{A}^*$. By the projective bundle formula (see [13, Corollary 4.1.11]), we have

$$\begin{aligned} &M^c((\mathbb{A}^{m-1} \times \mathbb{A}^*)/\mu_l) = M^c(\mathbb{A}^*/\mu_l)(m-1)[2(m-1)] \qquad (3.2.6)\\ = \quad &M^c(\mathbb{A}^*)(m-1)[2m-2] = \mathbb{Z}(m-1)[2m-1] \oplus \mathbb{Z}(m)[2m]. \end{aligned}$$

Therefore under the induction hypothesis, the second row in (3.2.3) is

$$\begin{aligned} &\mathbb{Z}(0)[1] \oplus \bigoplus_{i=1}^{m-2} \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(m-1)[2m-2] \qquad (3.2.7)\\ \to \quad &M^c((\mathbb{A}^m - \{0\})/\mu_l) \to \mathbb{Z}(m-1)[2m-1] \oplus \mathbb{Z}(m)[2m]\\ \xrightarrow{6} \quad &\mathbb{Z}(0)[2] \oplus \bigoplus_{i=1}^{m-2} \mathbb{Z}/l(i)[2i+1] \oplus \mathbb{Z}(m-1)[2m-1]. \end{aligned}$$

By the general theory of Tate motives, there are no nontrivial morphisms lowering algebraic dimensions (dimensions in the parentheses). Therefore the map 6 in (3.2.7) is zero on the factor $\mathbb{Z}(m)[2m]$ and only possibly maps the factor $\mathbb{Z}(m-1)[2m-1]$ on the left nontrivially to the $\mathbb{Z}(m-1)[2m-1]$ factor on the right. To study it, let's observe the relevant parts of the boundary maps (3.2.4) involving $\mathbb{Z}(m-1)[2m-1]$, which are

$$\begin{tikzcd}
\mathbb{Z}(m-1)[2m-1] \arrow[r, "3"] \arrow[d, "4"] & \mathbb{Z}(m-1)[2m-1] \arrow[d, "5"] \\
\mathbb{Z}(m-1)[2m-1] \arrow[r, "6"] & \mathbb{Z}(m-1)[2m-1].
\end{tikzcd} \tag{3.2.8}$$

Recall that we understand that the map $3 : \mathbb{Z}(m-1)[2m-1] \to \mathbb{Z}(m-1)[2m-1]$ in the above diagram is the identity (see the paragraph containing (3.2.5)). In Lemma 3.2.6 below, we show that the map $4 : \mathbb{Z}(m-1)[2m-1] \to \mathbb{Z}(m-1)[2m-1]$ is the identity, and the map $5 : \mathbb{Z}(m-1)[2m-1] \to \mathbb{Z}(m-1)[2m-1]$ is the multiplication by l, then by the commutativity of the diagram, we conclude that the map $6 : \mathbb{Z}(m-1)[2m-1] \to \mathbb{Z}(m-1)[2m-1]$ is the multiplication by l.

Now we have a complete knowledge of the map 6, which enables us to understand the whole distinguished triangle (3.2.7). Recall that we have a distinguished triangle

$$\begin{aligned} &\mathbb{Z}(m-1)[2m-2] \to \mathbb{Z}/l(m-1)[2m-2] \\ \xrightarrow{\beta}\ &\mathbb{Z}(m-1)[2m-1] \xrightarrow{\cdot l} \mathbb{Z}(m-1)[2m-1], \end{aligned} \tag{3.2.9}$$

where β is the Bockstein homomorphism associated to the exact sequence of coefficients $0 \to \mathbb{Z} \xrightarrow{\cdot l} \mathbb{Z} \to \mathbb{Z}/l \to 0$.

We conclude that the distinguished triangle (3.2.7) is the direct sum of the

Bockstein distinguished triangle (3.2.9) and the split distinguished triangle

$$\begin{aligned} 0 \ \to \ & \mathbb{Z}(0)[1] \oplus \bigoplus_{i=1}^{m-2} \mathbb{Z}/l(i)[2i] \to \\ \to \ & \mathbb{Z}(0)[1] \oplus \bigoplus_{i=1}^{m-2} \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(m)[2m] \to \mathbb{Z}(m)[2m] \to 0. \end{aligned} \tag{3.2.10}$$

Therefore $M^c((\mathbb{A}^m - \{0\})/\mu_l) = \mathbb{Z}(0)[1] \oplus \bigoplus_{i=1}^{m-1} \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(m)[2m]$. ∎

To prove Lemma 3.2.6 below, let's first prove three preliminary lemmas.

Lemma 3.2.3 *Let $f : X \to Y$ be a morphism of schemes, and $V \to Y$ a vector bundle over Y of rank r. Let $f^*(V)$ be the pullback vector bundle on X, i.e. we have the following Cartesian diagram*

$$\begin{tikzcd} f^*(V) \arrow[r, "g"] \arrow[d] & V \arrow[d] \\ X \arrow[r, "f"] & Y. \end{tikzcd}$$

If f is proper, then so is g and we have the following commutative diagram in DM

$$\begin{tikzcd} M^c(f^*(V)) \arrow[r, "g_*"] \arrow[d, "\cong"] & M^c(V) \arrow[d, "\cong"] \\ M^c(X)(r)[2r] \arrow[r, "f_*\otimes id"] & M^c(Y)(r)[2r], \end{tikzcd}$$

where the vertical isomorphisms are induced by the projective bundle formula, g_* and f_* are the induced maps on motives with compact support by the properness of g and f.

Proof. This is easily seen by the naturality of the construction of the projective bundle formula (using the tautological line bundle on the projective bundle) (see [13, Proposition 3.5.1]). ∎

Lemma 3.2.4 *The quotient map $\pi : \mathbb{A}^* \to \mathbb{A}^*/\mu_l = \mathbb{A}^*$, being proper and flat of relative dimension 0, induces on the motives with compact support*

$$\pi_* : M^c(\mathbb{A}^*) = \mathbb{Z}(0)[1] \oplus \mathbb{Z}(1)[2] \to \mathbb{Z}(0)[1] \oplus \mathbb{Z}(1)[2] = M^c(\mathbb{A}^*/\mu_l),$$

and

$$\pi^* : M^c(\mathbb{A}^*/\mu_l) = \mathbb{Z}(0)[1] \oplus \mathbb{Z}(1)[2] \to \mathbb{Z}(0)[1] \oplus \mathbb{Z}(1)[2] = M^c(\mathbb{A}^*).$$

One has $\pi_* = id \oplus \cdot l$ and $\pi^* = \cdot l \oplus id$, where $\cdot l$ stands for the map of multiplication by l.

Proof. Notice that $\pi : \mathbb{A}^* \to \mathbb{A}^*/\mu_l = \mathbb{A}^*$ is the same as $\mathbb{A}^* \xrightarrow{z \to z^l} \mathbb{A}^*$. The conclusion for π_* holds because of the following diagram (see the proof of Lemma 3.2.2)

$$\begin{tikzcd}
0 \arrow[r] & M^c(\mathbb{A}^1) \arrow[r] \arrow[d, "z \to z^l"] & M^c(\mathbb{A}^*) \arrow[r] \arrow[d, "z \to z^l"] & M^c(\{0\})[1] \arrow[r] \arrow[d, "0 \to 0"] & 0 \\
0 \arrow[r] & M^c(\mathbb{A}^1) \arrow[r] & M^c(\mathbb{A}^*) \arrow[r] & M^c(\{0\})[1] \arrow[r] & 0,
\end{tikzcd}$$

together with the facts that the first vertical arrow is $\mathbb{Z}(1)[2] \xrightarrow{\cdot l} \mathbb{Z}(1)[2]$ and the last vertical arrow is $\mathbb{Z}(0)[1] \xrightarrow{=} \mathbb{Z}(0)[1]$. The conclusion for π^* holds since $\pi_* \circ \pi^* = \cdot l \oplus \cdot l$. ∎

Lemma 3.2.5 *For an integer $j \geq 1$, consider the following pullback diagram*

$$\begin{tikzcd}
\mathbb{A}^{j-1} \times \mathbb{A}^* \arrow[r, "\pi_j"] \arrow[d] & (\mathbb{A}^{j-1} \times \mathbb{A}^*)/\mu_l \arrow[d] \\
\mathbb{A}^* \arrow[r, "\pi"] & \mathbb{A}^*/\mu_l.
\end{tikzcd}$$

Consider $(\pi_j)_* : M^c(\mathbb{A}^{j-1} \times \mathbb{A}^*) = \mathbb{Z}(j-1)[2j-1] \oplus \mathbb{Z}(j)[2j] \to \mathbb{Z}(j-1)[2j-1] \oplus \mathbb{Z}(j)[2j] = M^c((\mathbb{A}^{j-1} \times \mathbb{A}^*)/\mu_l)$ *induced by the properness of* π_j*. Then* $(\pi_j)_* = (\pi_j)'_* \oplus (\pi_j)''_*$*, and*

$$\begin{aligned}
(\pi_j)'_* = id \quad &: \quad \mathbb{Z}(j-1)[2j-1] \to \mathbb{Z}(j-1)[2j-1] \\
(\pi_j)''_* = \cdot l \quad &: \quad \mathbb{Z}(j)[2j] \to \mathbb{Z}(j)[2j]. \qquad (3.2.11)
\end{aligned}$$

Consider $(\pi_j)^* : M^c((\mathbb{A}^{j-1} \times \mathbb{A}^*)/\mu_l) = \mathbb{Z}(j-1)[2j-1] \oplus \mathbb{Z}(j)[2j] \to \mathbb{Z}(j-1)[2j-1] \oplus \mathbb{Z}(j)[2j] = M^c(\mathbb{A}^{j-1} \times \mathbb{A}^*)$ *induced by the flatness of* π_j *of relative dimension 0. Then* $(\pi_j)^* = (\pi_j)^{*\prime} \oplus (\pi_j)^{*\prime\prime}$*, and*

$$\begin{aligned}
(\pi_j)^{*\prime} = \cdot l \quad &: \quad \mathbb{Z}(j-1)[2j-1] \to \mathbb{Z}(j-1)[2j-1] \\
(\pi_j)^{*\prime\prime} = id \quad &: \quad \mathbb{Z}(j)[2j] \to \mathbb{Z}(j)[2j]. \qquad (3.2.12)
\end{aligned}$$

Proof. (3.2.11) follows easily from Lemmas 3.2.3 and 3.2.4. (3.2.12) follows from (3.2.11) and that $(\pi_j)_* \circ (\pi_j)^* = \cdot l \oplus \cdot l$. ∎

Now we are ready for Lemma 3.2.6.

Lemma 3.2.6 *In digram* (3.2.8)*, the map* $4 : \mathbb{Z}(m-1)[2m-1] \to \mathbb{Z}(m-1)[2m-1]$ *is the identity, and the map* $5 : \mathbb{Z}(m-1)[2m-1] \to \mathbb{Z}(m-1)[2m-1]$ *is the multiplication by* l

Proof. It is observed that the map $4 : \mathbb{Z}(m-1)[2m-1] \to \mathbb{Z}(m-1)[2m-1]$ is $(\pi_m)'_*$ in (3.2.11), and therefore it is the identity by Lemma 3.2.5.

Meanwhile in the case of the map 5, the factors $\mathbb{Z}(m-1)[2m-1]$ are represented by the fundamental classes of the varieties in question shifted by [1]. The quotient map $\mathbb{A}^{m-1} - \{0\} \to (\mathbb{A}^{m-1} - \{0\})/\mu_l$ has degree l and so maps the fundamental class of $\mathbb{A}^{m-1} - \{0\}$ to the l-th multiple of the fundamental class of $(\mathbb{A}^{m-1} - \{0\})/\mu_l$. Therefore the map 5 is the multiplication by l on the factors $\mathbb{Z}(m-1)[2m-1]$.

A more precise reasoning is as follows. Consider the following diagram

$$\begin{tikzcd}
M^c(\mathbb{A}^{m-1} - \{0\}) \arrow[r] \arrow[d] & M^c(\mathbb{A}^{m-2} \times \mathbb{A}^*) \arrow[d] \\
M^c((\mathbb{A}^{m-1} - \{0\})/\mu_l) \arrow[r] & M^c((\mathbb{A}^{m-2} \times \mathbb{A}^*)/\mu_l),
\end{tikzcd} \tag{3.2.13}$$

which is the second square in (3.2.3) for dimension $m-1$. It is observed that two horizontal maps induce the identities on the factors $\mathbb{Z}(m-1)[2m-2]$. For the upper one, this is because of the fact about the map 2 stated after (3.2.5) (applied to dimension $m-1$). For the lower one, this is because of the split distinguished triangle (3.2.10) (applied to dimension $m-1$), which in turn is basically because the map 6 in (3.2.7) is zero on the top dimension. Now the right vertical arrow in (3.2.13) on the factors $\mathbb{Z}(m-1)[2m-2]$ is just $(\pi_{m-1})''_*$ in (3.2.11), and therefore is the multiplication by l by Lemma 3.2.5. By the commutativity, so is the left vertical arrow of (3.2.13). Shifting by 1, we get our conclusion about the map 5 on the factor $\mathbb{Z}(m-1)[2m-1]$. ∎

Remark 3.2.7 In view of Proposition 3.2.1, for $m \geq 2$ the natural map

$$M^c((\mathbb{A}^{m-1} - \{0\})/\mu_l, \mathbb{Z}) \to M^c((\mathbb{A}^m - \{0\})/\mu_l, \mathbb{Z}),$$

induced by the natural inclusion $(\mathbb{A}^{m-1} - \{0\})/\mu_l \to (\mathbb{A}^m - \{0\})/\mu_l$ with the last coordinate zero, has the form

$$\mathbb{Z}(0)[1] \oplus \bigoplus_{i=1}^{m-2} \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(m-1)[2m-2] \to \mathbb{Z}(0)[1] \oplus \bigoplus_{i=1}^{m-1} \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(m)[2m].$$

We can choose the isomorphisms in Proposition 3.2.1 in such a way that the above map is the direct sum of the natural inclusions of the first $m-1$ factors and the natural map $\mathbb{Z}(m-1)[2m-2] \to \mathbb{Z}/l(m-1)[2m-2]$.

Now let's consider $\mathbb{Z}/l$ coefficients.

Proposition 3.2.8 *The motive with compact support with $\mathbb{Z}/l$ coefficients of the lens space $(\mathbb{A}^m - \{0\})/\mu_l$, where μ_l acts on $\mathbb{A}^m$ as a direct sum of nontrivial irreducible representations, is isomorphic to*

$$M^c((\mathbb{A}^m - \{0\})/\mu_l, \mathbb{Z}/l) = \bigoplus_{i=0}^{m-1} \mathbb{Z}/l(i)[2i+1] \oplus \bigoplus_{i=1}^{m} \mathbb{Z}/l(i)[2i].$$

Proof. When $m = 1$, one has

$$M^c((\mathbb{A}^1 - \{0\})/\mu_l, \mathbb{Z}/l) = M^c(\mathbb{A}^*, \mathbb{Z}/l) = \mathbb{Z}/l(0)[1] \oplus \mathbb{Z}/l(1)[2].$$

We now use induction on m. Observe that the projective bundle formula (3.2.6) also holds for $\mathbb{Z}/l$ coefficients, and we have

$$M^c((\mathbb{A}^{m-1} \times \mathbb{A}^*)/\mu_l, \mathbb{Z}/l) = \mathbb{Z}/l(m-1)[2m-1] \oplus \mathbb{Z}/l(m)[2m].$$

Utilize the same localization distinguished triangles (3.2.3) as before but with $\mathbb{Z}/l$ coefficients. The same reasoning as in the proof of Proposition 3.2.1 shows that under the induction hypothesis, the boundary map 6 in the distinguished triangle (3.2.4) is the multiplication by l on $\mathbb{Z}/l(m-1)[2m-1]$, and hence zero for $\mathbb{Z}/l$ coefficients. The conclusion follows easily. ∎

Remark 3.2.9 The isomorphisms in Proposition 3.2.8 can be chosen in such a way that the natural map $M^c((\mathbb{A}^{m-1}-\{0\})/\mu_l, \mathbb{Z}/l) \to M^c((\mathbb{A}^{m}-\{0\})/\mu_l, \mathbb{Z}/l)$, in view of Proposition 3.2.8, is the natural inclusion.

Propositions 3.2.1 and 3.2.8 combined with (3.2.1) give the following two corollaries concerning the motives with compact support of the normalized cyclic product $(\mathbb{A}^{nl} - \mathbb{A}^n)/C_l$ with $\mathbb{Z}$ and $\mathbb{Z}/l$ coefficients.

Corollary 3.2.10 *In the triangulated category of motives DM, we have an isomorphism*

$$M^c((\mathbb{A}^{nl} - \mathbb{A}^n)/C_l, \mathbb{Z}) = \mathbb{Z}(n)[2n+1] \oplus \bigoplus_{i=n+1}^{nl-1} \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(nl)[2nl] \quad (3.2.14)$$

Corollary 3.2.11 *In the triangulated category of motives DM, we have an* *isomorphism*

$$M^c((\mathbb{A}^{nl}-\mathbb{A}^n)/C_l,\mathbb{Z}/l)=\bigoplus_{i=n}^{nl-1}\mathbb{Z}/l(i)[2i+1]\oplus\bigoplus_{i=n+1}^{nl}\mathbb{Z}/l(i)[2i]. \quad (3.2.15)$$

3.3 The total fixed point set

In this section, we use the distinguished triangle (3.1.5) in Proposition 3.1.1

$$C_*z(\mathbb{A}^n)\to C_*z(\mathbb{A}^{nl})^{C_l}\to C_*z(\mathbb{A}^{nl}-\mathbb{A}^n)^{C_l}\xrightarrow{\delta}C_*z(\mathbb{A}^n)[1]$$

together with our computations in the last section to analyze the type of $C_*z(\mathbb{A}^{nl})^{C_l}$. We will study the boundary map δ, which will determine the whole distinguished triangle. Again we will work with $\mathbb{Z}$ coefficients first, and then take up the situation for $\mathbb{Z}/l$ coefficients.

Proposition 3.3.1 *With $\mathbb{Z}$ coefficients, we have an isomorphism in DM*

$$C_*z(\mathbb{A}^{nl})^{C_l}=\bigoplus_{i=n}^{nl-1}\mathbb{Z}/l(i)[2i]\oplus\mathbb{Z}(nl)[2nl]. \quad (3.3.1)$$

Proof. (3.1.6) and Corollary 3.2.10 tells us that

$$\begin{aligned}C_*z(\mathbb{A}^{nl}-\mathbb{A}^n)^{C_l} &= C_*z((\mathbb{A}^{nl}-\mathbb{A}^n)/C_l)=M^c((\mathbb{A}^{nl}-\mathbb{A}^n)/C_l)\\ &= \mathbb{Z}(n)[2n+1]\oplus\bigoplus_{i=n+1}^{nl-1}\mathbb{Z}/l(i)[2i]\oplus\mathbb{Z}(nl)[2nl].\end{aligned}$$

It is well known (2.2.1) that $C_*z(\mathbb{A}^n)[1]=\mathbb{Z}(n)[2n+1]$. Again by weight considerations, the only possible nontrivial part of δ in (3.1.5) is between the factors $\mathbb{Z}(n)[2n+1]$.

We put the distinguished triangle (3.1.5) in the following diagram

$$\begin{tikzcd}
C_*z(\mathbb{A}^n) \arrow[r] \arrow[d] & C_*z(\mathbb{A}^{nl})^{C_l} \arrow[r] \arrow[d] & C_*z(\mathbb{A}^{nl}-\mathbb{A}^n)^{C_l} \arrow[r, "\delta"] \arrow[d, "C_*j"] & C_*z(\mathbb{A}^n)[1] \arrow[d] \\
C_*z(\mathbb{A}^n) \arrow[r] & C_*z(\mathbb{A}^{nl}) \arrow[r] & C_*z(\mathbb{A}^{nl}-\mathbb{A}^n) \arrow[r, "\delta_0"] & C_*z(\mathbb{A}^n)[1].
\end{tikzcd} \tag{3.3.2}$$

Here the second row is a natural localization distinguished triangle (2.2.4), and the vertical arrows are induced by the inclusions of the fixed point set sheaves into the whole sheaves. Now let's try to understand the relevant parts of the third square in (3.3.2) concerning the factors $\mathbb{Z}(n)[2n+1]$, which are

$$\begin{tikzcd}
\mathbb{Z}(n)[2n+1] \arrow[r, "\delta"] \arrow[d, "j_1"] & \mathbb{Z}(n)[2n+1] \arrow[d, "j_2"] \\
\mathbb{Z}(n)[2n+1] \arrow[r, "\delta_0"] & \mathbb{Z}(n)[2n+1].
\end{tikzcd} \tag{3.3.3}$$

It is easy to see that δ_0 (see (3.2.5)) and j_2 are the identities. Let's analyze j_1. Through the isomorphism (3.1.6) between $z(\mathbb{A}^{nl}-\mathbb{A}^n)^{C_l}$ and $z((\mathbb{A}^{nl}-\mathbb{A}^n)/C_l)$, C_*j in (3.3.2) is isomorphic to the following pullback map

$$\pi^* : C_*z((\mathbb{A}^{nl}-\mathbb{A}^n)/C_l) \to C_*z(\mathbb{A}^{nl}-\mathbb{A}^n).$$

Roughly speaking, the factors $\mathbb{Z}(n)[2n+1]$ on both sides are represented by the motives with compact support of $\mathbb{A}^n$ multiplied by the motive of the missing origin. π^* maps a point to l points. Therefore j_1 is the multiplication by l, and so is δ by the commutativity of (3.3.3).

To be more precise about j_1, consider the following diagram

$$\begin{tikzcd}
M^c((\mathbb{A}^1 - \{0\})/\mu_l) \arrow[r] \arrow[d] & M^c((\mathbb{A}^{n(l-1)} - \{0\})/\mu_l) \arrow[d] \\
M^c(\mathbb{A}^1 - \{0\}) \arrow[r] & M^c(\mathbb{A}^{n(l-1)} - \{0\})
\end{tikzcd}$$

where the horizontal arrows are induced by the inclusion of $\mathbb{A}^1 - \{0\}$ in $\mathbb{A}^{n(l-1)} - \{0\}$ with all coordinates but the first one zero, and the vertical arrows are the pullbacks. Then one observes that both the horizontal arrows induce the identities on the factors $\mathbb{Z}(0)[1]$, and the left vertical arrow is $\pi^* : M^c((\mathbb{A}^1 - \{0\})/\mu_l) \to M^c(\mathbb{A}^1 - \{0\})$ in Lemma 3.2.4 and therefore it is the multiplication by l on the factor $\mathbb{Z}(0)[1]$. These imply that the same holds for the right vertical arrow. Tensoring this with $M^c(\mathbb{A}^n) = \mathbb{Z}(n)[2n]$, we get our claim.

Again we have the Bockstein distinguished triangle

$$\mathbb{Z}(n)[2n] \to \mathbb{Z}/l(n)[2n] \xrightarrow{\beta} \mathbb{Z}(n)[2n+1] \xrightarrow{\cdot l} \mathbb{Z}(n)[2n+1] \tag{3.3.4}$$

coming into play. An analysis similar to that at the end of Proposition 3.2.1 finishes the proof. ∎

If we are working with $\mathbb{Z}/l$ coefficients, similar reasonings show that $\delta = 0$ in a similar diagram to (3.3.2) for $\mathbb{Z}/l$ coefficients. Thus one has

Proposition 3.3.2 *With $\mathbb{Z}/l$ coefficients, we have an isomorphism in DM*

$$C_* z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l} = \bigoplus_{i=n}^{nl} \mathbb{Z}/l(i)[2i] \oplus \bigoplus_{i=n}^{nl-1} \mathbb{Z}/l(i)[2i+1] \tag{3.3.5}$$

Now we give an interpretation of the above proposition in the $\mathbb{A}^1$-homotopy category $H_\bullet^{\mathbb{A}^1}(k)$. Recall that in the Dold-Kan correspondence, we have a pair of adjoint equivalence functors N and K between the category of simplicial abelian sheaves and the category of chain complexes of abelian sheaves concentrated in positive dimensions. For a general complex, we need to first apply the good truncation functor at position 0. Let $K(\mathbb{Z}/l, j, i) = K(\mathbb{Z}/l(i)[j])$ be the simplicial abelian sheaf corresponding to the motivic complex $\mathbb{Z}/l(i)[j]$. Then considered as a simplicial sheaf of pointed sets, it serves as the motivic Eilenberg-MacLane space in $H_\bullet^{\mathbb{A}^1}(k)$ (see (2.3.1)).

The functor K respects finite direct sums, i.e. it takes finite direct sums of chain complexes to finite direct sums of simplicial abelian sheaves. The forgetful functor from the category of simplicial abelian sheaves to the category of sheaves of pointed sets takes finite direct sums to products. Therefore using Proposition 3.3.2, we have

$$K(C_* z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}) = K(\bigoplus_{i=n}^{nl} \mathbb{Z}/l(i)[2i] \oplus \bigoplus_{i=n}^{nl-1} \mathbb{Z}/l(i)[2i+1])$$

$$= \prod_{i=n}^{nl} K(\mathbb{Z}/l, 2i, i) \times \prod_{i=n}^{nl-1} K(\mathbb{Z}/l, 2i+1, i).$$

By [2, Lemma 2.5.2], we have an $\mathbb{A}^1$-weak equivalence from $K(z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l})$ to $K(C_* z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l})$, where $z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}$ is considered to be a complex concentrated at dimension 0. It is easy to see by the construction of K that $K(z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l})$ is the constant simplicial sheaf at $z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}$. Thus we arrive at the following theorem.

Theorem 3.3.3 *In $H_{\bullet}^{\mathbb{A}^1}(k)$, we have an isomorphism*

$$z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l} \simeq \prod_{i=n}^{nl} K(\mathbb{Z}/l, 2i, i) \times \prod_{i=n}^{nl-1} K(\mathbb{Z}/l, 2i+1, i). \tag{3.3.6}$$

This implies that $z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}$ has the desired homotopy type to be the target for the total reduced power operation (3.1.3). The above theorem is part 1 of our main theorem 1.3.1.

Similarly, using Proposition 3.3.1 we also have the following proposition.

Proposition 3.3.4 *In $H_{\bullet}^{\mathbb{A}^1}(k)$, we have an isomorphism*

$$z(\mathbb{A}^{nl})^{C_l} \simeq \prod_{i=n}^{nl-1} K(\mathbb{Z}/l, 2i, i) \times K(\mathbb{Z}, 2nl, nl). \tag{3.3.7}$$

3.4 Bockstein homomorphisms

We have constructed the following total reduced power operation (3.1.3)

$$\mathcal{P} : z^{\mathbb{Z}/l}(\mathbb{A}^n) \to z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l},$$

which in the homotopy category by Theorem 3.3.3 is

$$\begin{aligned} \mathcal{P} : K(\mathbb{Z}/l, 2n, n) &\to \prod_{i=n}^{nl} K(\mathbb{Z}/l, 2i, i) \times \prod_{i=n}^{nl-1} K(\mathbb{Z}/l, 2i+1, i) \\ &= \prod_{j=0}^{n(l-1)} K(\mathbb{Z}/l, 2n+2j, n+j) \times \prod_{j=0}^{n(l-1)-1} K(\mathbb{Z}/l, 2n+2j+1, n+j). \end{aligned}$$

We introduce the following notation; let

$$D^i : K(\mathbb{Z}/l, 2n, n) \to K(\mathbb{Z}/l, 2n+2i, n+i),\ 0 \leq i \leq n(l-1) \qquad (3.4.1)$$

and

$$E^i : K(\mathbb{Z}/l, 2n, n) \to K(\mathbb{Z}/l, 2n+2i+1, n+i),\ 0 \leq i \leq n(l-1)-1 (3.4.2)$$

denote the components of our total reduced power operation $\mathcal{P}$.

The goal of this section is to prove Theorem 3.4.5, which is part 2 of our main theorem 1.3.1.

Consider a general dimension $m \geq 1$ and a lens space $(\mathbb{A}^m - \{0\})/\mu_l$, where μ_l acts on $\mathbb{A}^m$ as a direct sum of nontrivial irreducible representations. The natural coefficient reduction map $\tau : M^c((\mathbb{A}^m - \{0\})/\mu_l, \mathbb{Z}) \to M^c((\mathbb{A}^m - \{0\})/\mu_l, \mathbb{Z}/l)$, by Propositions 3.2.1 and 3.2.8, is

$$\begin{aligned} \tau : \ & \mathbb{Z}(0)[1] \oplus \mathbb{Z}/l(1)[2] \oplus \cdots \oplus \mathbb{Z}/l(m-1)[2m-2] \oplus \mathbb{Z}(m)[2m] \\ \to \ & \mathbb{Z}/l(0)[1] \oplus \mathbb{Z}/l(1)[2] \oplus \mathbb{Z}/l(1)[3] \oplus \cdots \oplus \mathbb{Z}/l(m-1)[2m-2] \\ & \oplus \mathbb{Z}/l(m-1)[2m-1] \oplus \mathbb{Z}/l(m)[2m]. \end{aligned}$$

Proposition 3.4.1 *The map τ is the direct sum $\oplus_{i=0}^m \tau_i$ of the following maps. The maps $\tau_0 : \mathbb{Z}(0)[1] \to \mathbb{Z}/l(0)[1]$ and $\tau_m : \mathbb{Z}(m)[2m] \to \mathbb{Z}/l(m)[2m]$ are coefficient reductions by l. The map $\tau_i : \mathbb{Z}/l(i)[2i] \to \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}/l(i)[2i+1]$ for $1 \leq i \leq m-1$ is the direct sum (id, β) of the identity to the first component and the Bockstein homomorphism β to the second one.*

Proof. We proceed by induction on m. When $m = 1$, $\tau : M^c(\mathbb{A}^1 - \{0\}/\mu_l, \mathbb{Z}) \to M^c(\mathbb{A}^1 - \{0\}/\mu_l, \mathbb{Z}/l)$, which is

$$\mathbb{Z}(0)[1] \oplus \mathbb{Z}(1)[2] \to \mathbb{Z}/l(0)[1] \oplus \mathbb{Z}/l(1)[2],$$

is easily seen to consist of coefficient reductions.

Now assume that the conclusion holds for $m-1$. For simplicity of notation, we write L^m for $(\mathbb{A}^m - \{0\})/\mu_l$. Recall from before (see the proofs of Propositions 3.2.1 and 3.2.8) that $M^c(L^m, \mathbb{Z})$ $(M^c(L^m, \mathbb{Z}/l))$ has the same Tate motives of algebraic dimensions $\leq m-2$ (resp. $m-1$) as $M^c(L^{m-1}, \mathbb{Z})$ (resp. $M^c(L^{m-1}, \mathbb{Z}/l)$). This means that the conclusion about τ_i holds for $0 \leq i \leq m-2$ by the induction hypothesis. The conclusion that $\tau_m : \mathbb{Z}(m)[2m] \to \mathbb{Z}/l(m)[2m]$ is the coefficient reduction is easy to see. Therefore we only need to consider τ_{m-1}. Observe the following diagram

$$\begin{tikzcd}
M^c(L^{m-1}) \arrow[r, "i_1"] \arrow[d, "\tau''"] & M^c(L^m) \arrow[r, "r_1"] \arrow[d, "\tau"] & M^c(U^m) \arrow[r, "\delta_1"] \arrow[d, "\tau'"] & M^c(L^{m-1})[1] \arrow[d, "\tau''[1]"] \\
M^c(L^{m-1}, \mathbb{Z}/l) \arrow[r, "i_2"] & M^c(L^m, \mathbb{Z}/l) \arrow[r, "r_2"] & M^c(U^m, \mathbb{Z}/l) \arrow[r, "\delta_2"] & M^c(L^{m-1}, \mathbb{Z}/l)[1]
\end{tikzcd} \quad (3.4.3)$$

where for simplicity we write U^m for $(\mathbb{A}^{m-1} \times \mathbb{A}^*)/\mu_l$, which is the open complement of $L^{m-1} = (\mathbb{A}^{m-1} - \{0\})/\mu_l$ in L^m. Here the horizontal distinguished triangles are the localization ones we had before (see the second row of (3.2.3)), and the vertical arrows are coefficient reduction maps.

The conclusion we want to draw is about the second column of (3.4.3), and in particular we want to see that $\tau_{m-1} = (id, \beta) : \mathbb{Z}/l(m-1)[2m-2] \to \mathbb{Z}/l(m-1)[2m-2] \oplus \mathbb{Z}/l(m-1)[2m-1]$. To see that $\tau_{m-1} : \mathbb{Z}/l(m-1)[2m-2] \to$

$\mathbb{Z}/l(m-1)[2m-2]$ is the identity, we need to observe the relevant parts of the first square in (3.4.3), which are

$$\begin{tikzcd}
\mathbb{Z}(m-1)[2m-2] \arrow[r, "i_1"] \arrow[d, "\tau''"] & \mathbb{Z}/l(m-1)[2m-2] \arrow[d, "\tau"] \\
\mathbb{Z}/l(m-1)[2m-2] \arrow[r, "i_2"] & \mathbb{Z}/l(m-1)[2m-2],
\end{tikzcd}$$

where we understand, from the proofs of Propositions 3.2.1 and 3.2.8, that $i_1 : \mathbb{Z}(m-1)[2m-2] \to \mathbb{Z}/l(m-1)[2m-2]$ is the coefficient reduction, $\tau'' : \mathbb{Z}(m-1)[2m-2] \to \mathbb{Z}/l(m-1)[2m-2]$ is also the coefficient reduction, and $i_2 : \mathbb{Z}/l(m-1)[2m-2] \to \mathbb{Z}/l(m-1)[2m-2]$ is the identity. Then the commutativity of the diagram gives the conclusion.

Now we want to prove that $\tau_{m-1} = \beta : \mathbb{Z}/l(m-1)[2m-2] \to \mathbb{Z}/l(m-1)[2m-1]$ is the Bockstein homomorphism associated to the exact sequence $0 \to \mathbb{Z}/l \xrightarrow{\cdot l} \mathbb{Z}/l^2 \to \mathbb{Z}/l \to 0$. This follows from the following relevant parts of the second square in (3.4.3),

$$\begin{tikzcd}
\mathbb{Z}/l(m-1)[2m-2] \arrow[r, "r_1"] \arrow[d, "\tau"] & \mathbb{Z}(m-1)[2m-1] \arrow[d, "\tau'"] \\
\mathbb{Z}/l(m-1)[2m-1] \arrow[r, "r_2"] & \mathbb{Z}/l(m-1)[2m-1],
\end{tikzcd}$$

where we understand, from the proofs of Propositions 3.2.1 and 3.2.8, that $r_1 : \mathbb{Z}/l(m-1)[2m-2] \to \mathbb{Z}(m-1)[2m-1]$ is the Bockstein homomorphism associated to $0 \to \mathbb{Z} \xrightarrow{\cdot l} \mathbb{Z} \to \mathbb{Z}/l \to 0$ (see (3.2.9)), $\tau' : \mathbb{Z}(m-1)[2m-1] \to \mathbb{Z}/l(m-1)[2m-1]$ is the coefficient reduction, and $r_2 : \mathbb{Z}/l(m-1)[2m-1] \to \mathbb{Z}/l(m-1)[2m-1]$ is the identity.

∎

Proposition 3.4.1 gives the following proposition.

Proposition 3.4.2 *The coefficient reduction map* $\tau : C_*z(\mathbb{A}^{nl})^{C_l} \to C_*z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}$, *which by Propositions 3.3.1 and 3.3.2 has the following form*

$$\tau : \bigoplus_{i=n}^{nl-1} \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(nl)[2nl] \to \bigoplus_{i=n}^{nl} \mathbb{Z}/l(i)[2i] \oplus \bigoplus_{i=n}^{nl-1} \mathbb{Z}/l(i)[2i+1],$$

is the direct sum $\oplus_{i=n}^{nl} \tau_i$ *of the following maps. The map* $\tau_{nl} : \mathbb{Z}(nl)[2nl] \to \mathbb{Z}/l(nl)[2nl]$ *is the coefficient reduction. The map*

$$\tau_i : \mathbb{Z}/l(i)[2i] \to \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}/l(i)[2i+1]$$

for $n \leq i \leq nl-1$ *is the direct sum* (id, β) *of the identity to the first component and the Bockstein homomorphism* β *to the second one.*

Proof. The statement about τ_i for $n+1 \leq i \leq nl$ follows from the tensor structure (3.2.1), the proof of Proposition 3.3.1 and Proposition 3.4.1. We only need to consider

$$\tau_n : \mathbb{Z}/l(n)[2n] \to \mathbb{Z}/l(n)[2n] \oplus \mathbb{Z}/l(n)[2n+1].$$

Utilize the following diagram

$$\begin{tikzcd}
C_*z(\mathbb{A}^n) \arrow[r, "i_1"] \arrow[d, "\tau''"] & C_*z(\mathbb{A}^{nl})^{C_l} \arrow[r, "r_1"] \arrow[d, "\tau"] & C_*z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l} \arrow[r] \arrow[d, "\tau'"] & C_*z(\mathbb{A}^n)[1] \arrow[d] \\
C_*z^{\mathbb{Z}/l}(\mathbb{A}^n) \arrow[r, "i_2"] & C_*z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l} \arrow[r, "r_2"] & C_*z^{\mathbb{Z}/l}(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l} \arrow[r] & C_*z^{\mathbb{Z}/l}(\mathbb{A}^n)[1],
\end{tikzcd} \quad (3.4.4)$$

where the horizontal distinguished triangles are those from (3.1.5), and the vertical arrows are coefficient reductions. To see that $\tau_n : \mathbb{Z}/l(n)[2n] \to \mathbb{Z}/l(n)[2n+1]$ is the Bockstein β associated to $0 \to \mathbb{Z}/l \xrightarrow{\cdot l} \mathbb{Z}/l^2 \to \mathbb{Z}/l \to 0$, consider the relevant parts of the second square in (3.4.4), which are

$$\begin{tikzcd}
\mathbb{Z}/l(n)[2n] \arrow[r, "r_1"] \arrow[d, "\tau_n"] & \mathbb{Z}(n)[2n+1] \arrow[d, "\tau'"] \\
\mathbb{Z}/l(n)[2n+1] \arrow[r, "r_2"] & \mathbb{Z}/l(n)[2n+1].
\end{tikzcd}$$

Here $r_1 : \mathbb{Z}/l(n)[2n] \to \mathbb{Z}/l(n)[2n+1]$ is the Bockstein associated to $0 \to \mathbb{Z} \xrightarrow{\cdot l} \mathbb{Z} \to \mathbb{Z}/l \to 0$ in light of (3.3.4), $\tau' : \mathbb{Z}(n)[2n+1] \to \mathbb{Z}/l(n)[2n+1]$ is the coefficient reduction by (3.2.1) and Proposition 3.4.1, and $r_2 : \mathbb{Z}/l(n)[2n+1] \to \mathbb{Z}/l(n)[2n+1]$ is the identity by the proof of Proposition 3.2.8. By the commutativity, $\tau_n : \mathbb{Z}/l(n)[2n] \to \mathbb{Z}/l(n)[2n+1]$ is the Bockstein associated to $0 \to \mathbb{Z}/l \xrightarrow{\cdot l} \mathbb{Z}/l^2 \to \mathbb{Z}/l \to 0$.

That $\tau_n : \mathbb{Z}/l(n)[2n] \to \mathbb{Z}/l(n)[2n]$ is the identity can be seen similarly using the first square in (3.4.4). ∎

Consider the following map

$$l\varpi : z(\mathbb{A}^{nl}) \to z(\mathbb{A}^{nl})^{C_l}$$

defined as follows. Given a smooth scheme U and a cycle $Z \in z(\mathbb{A}^{nl})(U)$, define $l\varpi(Z) = l\sum_{\sigma\in C_l} \sigma(Z)$, where $\sigma(Z)$ is the image of Z under the element $\sigma \in C_l$. It is clear that $l\varpi(Z)$ is invariant under the C_l action by its construction and thus is in $z(\mathbb{A}^{nl})^{C_l}(U)$. Denote by $z(\mathbb{A}^{nl})^{C_l}/Im(l\varpi)$ the Nisnevich sheaf whose value on a smooth scheme U is $z(\mathbb{A}^{nl})^{C_l}(U)/Im(l\varpi : z(\mathbb{A}^{nl})(U) \to z(\mathbb{A}^{nl})^{C_l}(U))$, where the quotient is in the category of abelian groups.

Lemma 3.4.3 *In the $\mathbb{A}^1$-homotopy category $H_\bullet^{\mathbb{A}^1}(k)$, one has an isomorphism*

$$z(\mathbb{A}^{nl})^{C_l}/Im(l\varpi) \simeq \prod_{i=n}^{nl-1} K(\mathbb{Z}/l, 2i, i) \times K(\mathbb{Z}/l^2, 2nl, nl). \tag{3.4.5}$$

The natural projection

$$p : z(\mathbb{A}^{nl})^{C_l} \to z(\mathbb{A}^{nl})^{C_l}/Im(l\varpi),$$

in view of (3.3.7) *and* (3.4.5)*, is isomorphic to the product $\prod_{i=n}^{nl} p_i$ of the following maps. $p_i : K(\mathbb{Z}/l, 2i, i) \to K(\mathbb{Z}/l, 2i, i)$ for $n \leq i \leq nl-1$ is the identity map, and $p_{nl} : K(\mathbb{Z}, 2nl, nl) \to K(\mathbb{Z}/l^2, 2nl, nl)$ is the natural coefficient reduction map.*

Proof. Apply the singular complex functor C_* and consider the corresponding statements on the motive level. One has the following distinguished triangle

$$C_*z(\mathbb{A}^{nl}) \overset{C_*(l\varpi)}{\to} C_*z(\mathbb{A}^{nl})^{C_l} \overset{C_*p}{\to} C_*z((\mathbb{A}^{nl})^{C_l}/Im(l\varpi)) \to C_*z(\mathbb{A}^{nl})[1]. \tag{3.4.6}$$

The map $C_*(l\varpi) : C_*z(\mathbb{A}^{nl}) = \mathbb{Z}(nl)[2nl] \to \bigoplus_{i=n}^{nl-1} \mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(nl)[2nl] = C_*z(\mathbb{A}^{nl})^{C_l}$ induces the zero maps to the factors $\mathbb{Z}/l(i)[2i]$ for $n \leq i \leq nl-1$ of lower algebraic dimensions on the right by weight considerations. It induces the multiplication by l^2, $\mathbb{Z}(nl)[2nl] \overset{\cdot l^2}{\to} \mathbb{Z}(nl)[2nl]$, to the last factor. This knowledge about the map $C_*(l\varpi)$ in (3.4.6) enables us to understand the distinguished triangle. In particular,

$$C_*z((\mathbb{A}^{nl})^{C_l}/Im(l\varpi)) = \oplus_{i=n}^{nl-1}\mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}/l^2(nl)[2nl],$$

and $C_*(p) : C_*z(\mathbb{A}^{nl})^{C_l} \to C_*z(\mathbb{A}^{nl})^{C_l}/Im(l\varpi)$ induces the identities on the factors $\mathbb{Z}/l(i)[2i]$ for $n \leq i \leq nl-1$ and the natural coefficient reduction $\mathbb{Z}(nl)[2nl] \to \mathbb{Z}/l^2(nl)[2nl]$ on the last factors. An analysis similar to the proof of Theorem 3.3.3, using the functor K in the Dold-Kan correspondence and [2, Lemma 2.5.2], gives our statements on the homotopy category level. ∎

We now want to prove that our total reduced power operation $\mathcal{P}$ (3.1.3) for $\mathbb{Z}/l$ coefficients factors through $z(\mathbb{A}^{nl})^{C_l}/Im(l\varpi)$ with $\mathbb{Z}$ coefficients. Namely we will show the existence of the following commutative diagram

$$\begin{tikzcd}
z(\mathbb{A}^{nl})^{C_l}/Im(l\varpi) \arrow[dr, "\tau'"] & z(\mathbb{A}^{nl})^{C_l} \arrow[l, "p"'] \arrow[d, "\tau"] \\
z^{\mathbb{Z}/l}(\mathbb{A}^n) \arrow[u, "\wp"] \arrow[r, "\mathcal{P}"] & z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}.
\end{tikzcd} \tag{3.4.7}$$

where τ is the coefficient reduction by l, τ' is defined because $Im(l\varpi)$ is reduced to zero.

The proof of the following proposition is similar to that of [15, Proposition 8.1].

Proposition 3.4.4 *The lift $\wp$ in* (3.4.7) *exists.*

Proof. Recall first that our total reduced power operation $\mathcal{P}$ (3.1.3) is also defined for $\mathbb{Z}$ coefficients, and we have $\mathcal{P} : z(\mathbb{A}^n) \to z(\mathbb{A}^{nl})^{C_l}$. A cycle in

$z^{\mathbb{Z}/l}(\mathbb{A}^n)(U)$ lifts to an integral cycle in $z(\mathbb{A}^n)(U)$ unique up to a multiple of l. To prove the existence of $\wp$, we only need to show that for $Z_1, Z_2 \in z(\mathbb{A}^n)(U)$ such that $Z_1 - Z_2 = lW$ for some $W \in z(\mathbb{A}^n)(U)$, we have $\mathcal{P}(Z_1) - \mathcal{P}(Z_2) \in Im(l\varpi)$, i.e.

$$Z_1^l - Z_2^l = l\varpi(T)$$

for some $T \in z(\mathbb{A}^{nl})(U)$. Observe that any invariant cycle in $z(\mathbb{A}^{nl})^{C_l}(U)$ divisible by l^2, say l^2V with V invariant, is equal to $l\varpi(V)$. Because $Z_1^l - Z_2^l$ is invariant, we only need to take care of the terms in $Z_1^l - Z_2^l = (Z_2 + lW)^l - Z_2^l$ that are not divisible by l^2. Such terms are $l\sum_i Z_2 \times \cdots \times \underset{\text{i-th slot}}{W} \times \cdots \times Z_2$, and it can be written as $l\varpi(W \times Z_2 \times \cdots \times Z_2)$. ▮

Now we can prove our result concerning the Bockstein homomorphism.

Theorem 3.4.5 *$E^i = \beta \circ D^i$ for $0 \le i \le nl - 1$.*

Proof. We have a natural factorization (see (3.4.7))

$$\tau : z(\mathbb{A}^{nl})^{C_l} \xrightarrow{p} z(\mathbb{A}^{nl})^{C_l}/Im(l\varpi) \xrightarrow{\tau'} z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}$$

of the coefficient reduction map. Lemma 3.4.3 says that p induces the identities on the factors $K(\mathbb{Z}/l, 2i, i)$ for $n \le i \le nl - 1$, and thus τ' is the same as τ on these factors. Interpreting Proposition 3.4.2 in the homotopy category, we see that for $0 \le i \le nl - 1$, $\tau_i : K(\mathbb{Z}/l, 2i, i) \to K(\mathbb{Z}/l, 2i, i) \times K(\mathbb{Z}/l, 2i + 1, i)$ is (id, β), and therefore this also holds for τ'. Lemma 3.4.4 gives the following factorization

$$\mathcal{P} : z^{\mathbb{Z}/l}(\mathbb{A}^n) \xrightarrow{\wp} z(\mathbb{A}^{nl})^{C_l}/Im(l\varpi) \xrightarrow{\tau'} z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}.$$

This proves our proposition. ∎

3.5 Basic properties

In this section, we prove some very basic properties of our operations D^i (3.4.1) and E^i (3.4.2). By the representability of motivic cohomology (2.3.1), our operations represent the following cohomology operations:

$$D^i : H^{2n,n}(-, \mathbb{Z}/l) \to H^{2n+2i,n+i}(-, \mathbb{Z}/l),\ 0 \leq i \leq n(l-1);$$

$$E^i : H^{2n,n}(-, \mathbb{Z}/l) \to H^{2n+2i+1,n+i}(-, \mathbb{Z}/l),\ 0 \leq i \leq n(l-1) - 1.$$

Since our construction of the D^i and E^i is through maps between the corresponding motivic Eilenberg-MacLane spaces in the $\mathbb{A}^1$-homotopy category, these operations are natural with respect to morphisms between schemes.

Theorem 3.5.1 *$D^0 : K(\mathbb{Z}/l, 2n, n) \to K(\mathbb{Z}/l, 2n, n)$ is the identity.*

Proof. By our construction (see the proof of Proposition 3.3.2), the $K(\mathbb{Z}/l, 2n, n)$ factor of $z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}$ is represented by $z^{\mathbb{Z}/l}(\mathbb{A}^n)$, the sheaf of equidimensional cycles supported on the diagonal $\mathbb{A}^n$ of $\mathbb{A}^{nl}$. Suppose $Z = \sum n_i Z_i \in z^{\mathbb{Z}/l}(\mathbb{A}^n)(U)$, where U is a smooth scheme and the Z_i are distinct irreducible subschemes.

Then

$$\mathcal{P}(Z) = \sum n_{i_1} \cdots n_{i_l} Z_{i_1} \times_U \cdots \times_U Z_{i_l} \in z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}(U).$$

For $i_1, \cdots, i_l$ not all equal, $Z_{i_1} \times_U \cdots \times_U Z_{i_l}$ must have non-empty intersection with $U \times (\mathbb{A}^{nl} - \mathbb{A}^n)$ (we assume that the Z_i are all distinct). For each i, the fiber product $Z_i \times_U \cdots \times_U Z_i$ may be reducible. The following subvariety, denoted by W_i, of $Z_i \times_U \cdots \times_U Z_i$ defined by $y_1 = \cdots = y_l$, where the y_j are the coordinates to the j-th $\mathbb{A}^n$ component of $\mathbb{A}^{nl}$, is obviously isomorphic to Z_i under the identification of the diagonal of $\mathbb{A}^{nl}$ to $\mathbb{A}^n$, and so irreducible. By definition all other irreducible subschemes of $Z_i \times_U \cdots \times_U Z_i$ have non-empty intersections with $U \times (\mathbb{A}^{nl} - \mathbb{A}^n)$. The coefficient of the irreducible subvariety W_i is the same as the coefficient of $Z_i \times_U \cdots \times_U Z_i$, which is n_i^l. It is well known that $n_i^l = n_i$ in $\mathbb{Z}/l$. Therefore the cycle supported on the diagonal of $\mathbb{A}^{nl}$ is the same as the cycle we started with. This completes the proof. ∎

Theorem 3.5.2 $D^{n(l-1)} = P : K(2n, n, \mathbb{Z}/l) \to K(2nl, nl, \mathbb{Z}/l)$, *where* P *is the* l*-th cup product power operation* (3.1.1).

Proof. By definition (3.1.2), $P = j \circ \mathcal{P} : z^{\mathbb{Z}/l}(\mathbb{A}^n) \to z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l} \to z^{\mathbb{Z}/l}(\mathbb{A}^{nl})$. Here $j : z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l} \to z^{\mathbb{Z}/l}(\mathbb{A}^{nl})$ is the natural inclusion of the fixed point set sheaf into the whole sheaf, which in the $\mathbb{A}^1$-homotopy category by Proposition 3.3.3 is

$$j : \prod_{i=n}^{nl} K(\mathbb{Z}/l, 2i, i) \times \prod_{i=n}^{nl-1} K(\mathbb{Z}/l, 2i+1, i) \to K(\mathbb{Z}/l, 2nl, nl).$$

We will prove that j is isomorphic to the projection to the factor $K(\mathbb{Z}/l, 2nl, nl)$.

Actually we will prove the following integral version. The natural inclusion $j : z(\mathbb{A}^{nl})^{C_l} \to z(\mathbb{A}^{nl})$, which in the $\mathbb{A}^1$-homotopy category by Theorem 3.3.4 is $j : \prod_{i=n}^{nl-1} K(\mathbb{Z}/l, 2i, i) \times K(\mathbb{Z}, 2nl, nl) \to K(\mathbb{Z}, 2nl, nl)$, is isomorphic to the projection to the $K(\mathbb{Z}, 2nl, nl)$ factor. To prove it, let's consider the corresponding map on the motive level $C_*j : C_*z(\mathbb{A}^{nl})^{C_l} \to C_*z(\mathbb{A}^{nl})$. In view of Proposition 3.3.1, the map C_*j has the following form $C_*j : \oplus_{i=n}^{nl-1}\mathbb{Z}/l(i)[2i] \oplus \mathbb{Z}(nl)[2nl] \to \mathbb{Z}(nl)[2nl]$. Clearly C_*j is zero on the factors $\mathbb{Z}/l(i)[2i]$ for $n \leq i \leq nl-1$ for obvious dimensional reasons, and we only have to concentrate on the top dimensional factor $\mathbb{Z}(nl)[2nl]$. Consider the following diagram

$$\begin{tikzcd}
C_*z(\mathbb{A}^{nl})^{C_l} \arrow[r, "r"] \arrow[d] & C_*z(\mathbb{A}^{nl} - \mathbb{A}^n)^{C_l} \arrow[r, "\widetilde{\pi_*}"] \arrow[d] & C_*z((\mathbb{A}^{nl} - \mathbb{A}^n)/C_l) \arrow[r, "\cong"] \arrow[d] & {} \\
C_*z(\mathbb{A}^{nl}) \arrow[r, "r"] & C_*z(\mathbb{A}^{nl} - \mathbb{A}^n) \arrow[r, "="] & C_*z(\mathbb{A}^{nl} - \mathbb{A}^n) \arrow[r, "\cong"] & {}
\end{tikzcd}$$

$$\begin{tikzcd}
C_*z(\mathbb{A}^n) \otimes C_*z((\mathbb{A}^{n(l-1)} - \{0\})/\mu_l) \arrow[r, "id\otimes r"] \arrow[d] & C_*z(\mathbb{A}^n) \otimes C_*z((\mathbb{A}^{n(l-1)-1} \times \mathbb{A}^*)/\mu_l) \arrow[d] \\
C_*z(\mathbb{A}^n) \otimes C_*z(\mathbb{A}^{n(l-1)} - \{0\}) \arrow[r, "id\otimes r"] & C_*z(\mathbb{A}^n) \otimes C_*z(\mathbb{A}^{n(l-1)-1} \times \mathbb{A}^*),
\end{tikzcd}$$

where all the arrows are the obvious ones: the r's are induced by open embeddings. Observe that all the horizontal arrows induce the identities on the factor $\mathbb{Z}(nl)[2nl]$. Now we consider the effect of the last vertical arrow. The $\mathbb{Z}(nl)[2nl]$ factor of $C_*z(\mathbb{A}^n) \otimes C_*z((\mathbb{A}^{n(l-1)-1} \times \mathbb{A}^*)/\mu_l)$ is represented by $\mathbb{Z}(n)[2n]$ tensored with the $\mathbb{Z}(n(l-1))[2n(l-1)]$ factor of $C_*z((\mathbb{A}^{n(l-1)-1} \times \mathbb{A}^*)/\mu_l)$. By Lemma 3.2.5, we see that the last vertical arrow on this factor is just $(\pi_{n(l-1)})^{*''}$ and so it is the identity. Thus the last vertical arrow is the identity on the factor $\mathbb{Z}(nl)[2nl]$. Therefore so is the first vertical one.

Our statement about the $\mathbb{Z}/l$ coefficient case now follows easily from the above by the following commutative diagram

$$\begin{tikzcd}
z(\mathbb{A}^{nl})^{C_l} \arrow[r, "j"] \arrow[d, "\tau"] & z(\mathbb{A}^{nl}) \arrow[d, "\tau"] \\
z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l} \arrow[r, "j"] & z^{\mathbb{Z}/l}(\mathbb{A}^{nl}),
\end{tikzcd}$$

where the j's are the natural inclusions and the τ's are coefficient reductions, together with Proposition 3.4.2. ∎

Chapter 4

Comparison with Voevodsky's operations

In this chapter, we prove that up to nonzero constants, our operations coincide with Voevodsky's reduced power operations, which is part 4 of Main Theorem 1.3.1. The proof is carried out in Section 4.2. In Section 4.1, we review Voevodsky's construction and computations in [15] to facilitate the comparison.

4.1 Voevodsky's operations

In this section, we review Voevodsky's construction of the reduced power operations

$$P^i : H^{*,*}(-, \mathbb{Z}/l) \to H^{*+2i(l-1),*+i(l-1)}(-, \mathbb{Z}/l), \tag{4.1.1}$$

where l is a prime different from $char(k)$. The reference for this section is [15].

Recall that in the $\mathbb{A}^1$-homotopy category there are two circles: the simplicial circle $S^1_s = \mathbb{A}^1/\{0,1\}$ and the geometric circle $S^1_t = \mathbb{A}^1 - \{0\}$ pointed at 1, where each scheme is considered in $H^{\mathbb{A}^1}_\bullet(k)$ as the representable sheaf and the quotient is taken in the category of sheaves. Each circle has a tautological motivic cohomology class: $\sigma_s \in \tilde{H}^{1,0}(S^1_s)$ and $\sigma_t \in \tilde{H}^{1,1}(S^1_t)$. Multiplications with these tautological classes induce two suspension isomorphisms (see [15, Theorem 2.4]).

Voevodsky restricts himself to constructing bistable operations, i.e. cohomology operations that commute with the above mentioned two suspension isomorphisms. He [15, Proposition 2.6] shows that constructing a bistable operation of the form P^i in (4.1.1) is equivalent to constructing a family $\{P^i_n\}_n$ of operations of the form

$$P^i_n : H^{2n,n}(-,\mathbb{Z}/l) \to H^{2n+2i(l-1),n+i(l-1)}(-,\mathbb{Z}/l),$$

such that

$$P^i_{n+1}(x \wedge \sigma_T) = P^i_n(x) \wedge \sigma_T, \tag{4.1.2}$$

where $x \in H^{2n,n}(X,\mathbb{Z}/l)$ is a class, $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\}) \simeq S^1_s \wedge S^1_t$ is the Tate circle, $\sigma_T = \sigma_s \wedge \sigma_t \in \tilde{H}^{2,1}(T,\mathbb{Z}/l)$ is the tautological class of T, and $- \wedge \sigma_T$ is the multiplication by σ_T. Therefore the task now is to construct a family $\{P^i_n\}_n$ which satisfies the stability condition (4.1.2).

Assume that X is a smooth scheme, and that $x \in H^{2n,n}(X,\mathbb{Z}/l)$ is a motivic cohomology class. In view of (2.3.1) and (2.3.3) (and by the Yoneda lemma), such a class is represented by an element in $(\mathbb{Z}_{tr}(\mathbb{A}^n)/\mathbb{Z}_{tr}(\mathbb{A}^n - \{0\}))(X)$, which is the equivalence class of a cycle $Z \in \mathbb{Z}_{tr}(\mathbb{A}^n)(X)$.

Consider the l-th fiber product power Z^l of the cycle Z over X. It is an element in $\mathbb{Z}_{tr}(\mathbb{A}^{nl})(X)$, and its equivalence class in $(\mathbb{Z}_{tr}(\mathbb{A}^{nl})/\mathbb{Z}_{tr}(\mathbb{A}^{nl} -$

$\{0\}))(X)$ represents the l-th cup product power x^l of x.

The symmetric group S_l on l elements acts on $\mathbb{A}^{nl}$ by permuting the l copies of $\mathbb{A}^n$. Let $G < S_l$ be a subgroup of S_l (the ones of interest to us are the cyclic group C_l and the whole group S_l). Assume that U is a smooth scheme with a free G action. Let $pr : U \times X \to X$ be the projection, and $Cycl(pr)(Z^l) \in \mathbb{Z}_{tr}(\mathbb{A}^{nl})(U \times X)$ the pullback.

The action of G on U is free, so is the product action of G on $\mathbb{A}^{nl} \times U \times X$. By its construction, $Cycl(pr)(Z)$ as a cycle on $\mathbb{A}^{nl} \times U \times X$ is invariant under this action. Therefore it comes from (i.e. is the flat pullback of) a cycle Z' on the quotient $(\mathbb{A}^{nl} \times U \times X)/G = \mathbb{A}^{nl} \times_G U \times X$, which is a vector bundle of rank nl under the projection $p : \mathbb{A}^{nl} \times_G U \times X \to U/G \times X$. We denote this vector bundle by E.

Now we have constructed a cycle Z' on the rank nl vector bundle E over $U/G \times X$. Z' is finite over the base by its construction from $Z^l \in \mathbb{Z}_{tr}(\mathbb{A}^{nl})(X)$. We now show that such a cycle Z' gives a cohomology class in $H^{2nl,nl}(U/G \times X, \mathbb{Z}/l)$.

To do this, assume that there is a vector bundle L over $U/G \times X$ such that the sum $E \oplus L = \mathbb{A}^N \times (U/G \times X)$ is trivial. This is always possible when the base is affine, and for a general quasi-projective base we use the Jouanolou trick to introduce an affine torsor.

We first construct a cycle $Z'_L \in \mathbb{Z}_{tr}(\mathbb{A}^N)(L)$ from Z'. With the obvious notation, consider the following pullbacks of vector bundles

$$\begin{tikzcd}
E_L \arrow[r] \arrow[d] & E \arrow[d] & L_L \arrow[r] \arrow[d] & L \arrow[d] \\
L \arrow[r] & U/G \times X, & L \arrow[r] & U/G \times X.
\end{tikzcd}$$

The cycle Z' on E pulls back to a cycle Z'' on E_L which is finite over L. The bundle L_L has a natural cycle, the diagonal cycle Δ, which is obviously finite over L. The fiber product $Z'_L := Z'' \times_L \Delta$ is a cycle on the direct sum $E_L \oplus L_L = \mathbb{A}^N \times L$, the trivial vector bundle of rank N on L. Z'_L is finite over L, i.e. $Z'_L \in \mathbb{Z}_{tr}(\mathbb{A}^N)(L)$. It is easy to see that the restriction of Z'_L to the complement of the zero section, $L - (U/G \times X)$, is in $\mathbb{Z}_{tr}(\mathbb{A}^N - \{0\})(L - (U/G \times X))$ since the diagonal cycle is non-zero there. Therefore Z'_L defines a morphism in the homotopy category

$$Th(L) = L/(L - (U/G \times X)) \to \mathbb{Z}_{tr}(\mathbb{A}^N)/\mathbb{Z}_{tr}(\mathbb{A}^N - \{0\}),$$

and thus by (2.3.1) and (2.3.3) it defines a class $x^l_L \in \tilde{H}^{2N,N}(Th(L), \mathbb{Z}/l)$.

We have Thom isomorphism in motivic cohomology [15, Proposition 4.3]. In our case (note that L has rank $N - nl$, and use the $\mathbb{Z}/l$ coefficient version), it says that there is the Thom class $t_L \in \tilde{H}^{2(N-nl),N-nl}(Th(L), \mathbb{Z}/l)$, such that the following map of multiplication by t_L

$$\cdot t_L : H^{*,*}(U/G \times X, \mathbb{Z}/l) \to \tilde{H}^{*+2(N-nl),*+N-nl}(Th(L), \mathbb{Z}/l)$$

is an isomorphism. Therefore there exist a unique element $x^l_U \in H^{2nl,nl}(U/G \times$

$X, \mathbb{Z}/l)$ whose product with the Thom class t_L is x_L^l. The above construction is carried out in the general situation in [15, Section 5].

Now consider the case that $G = C_l$, and take U to be EC_l and thus U/G is BC_l (see the paragraphs before Definition 4.2.1). Strictly speaking this is not legitimate since EC_l is not a smooth scheme but an inductive scheme. Voevodsky [15, Corollary 6.2] proves that nothing goes wrong in the limit procedure. To be able to do further computations, Voevodsky uses standard transfer arguments and the condition that $l \neq char(k)$ to assume the existence of a primitive l-th root of unity ζ in k to have an isomorphism $C_l \cong \mu_l$ and thus a weak equivalence $BC_l \simeq B\mu_l$. After all these steps, we now have a class $x_{E\mu_l}^l \in H^{2nl,nl}(B\mu_l \times X, \mathbb{Z}/l)$.

Voevodsky [15, Section 6] goes ahead to compute the motivic cohomology ring $H^{*,*}(B\mu_l \times X, \mathbb{Z}/l)$. First of all we have a very geometric picture of $B\mu_l$ using the standard representation ρ of μ_l as

$$B\mu_l = \mathcal{O}(-l)_{\mathbb{P}^\infty} - \mathbb{P}^\infty,$$

where $\mathbb{P}^\infty$ is the zero section of $\mathcal{O}(-l)_{\mathbb{P}^\infty}$. Therefore we have the following cofibration sequence

$$B\mu_l \to \mathcal{O}(-l)_{\mathbb{P}^\infty} \to Th_{\mathbb{P}^\infty}(\mathcal{O}(-l)),$$

which leads to a long exact sequence in motivic cohomology. Using Thom isomorphism and the basic knowledge that $H^{*,*}(\mathbb{P}^\infty) = H^{*,*}(k)[v]$ is the polynomial ring, over the motivic cohomology ring of the base field, on the generator v which corresponds to the hyperplane line bundle on $\mathbb{P}^\infty$, one is able to compute the motivic cohomology of $B\mu_l$. Abusing notation, we also write v for its pullback in $H^{2,1}(B\mu_l, \mathbb{Z}/l)$. It can be seen that v corresponds to the line bundle on $B\mu_l$ associated to the standard representation ρ, i.e. the quotient of $E\mu_l \times \mathbb{A}^1$ with μ_l acting on the component $\mathbb{A}^1$ by ρ. It is clear that $lv = 0$, and it can be shown that there is a unique $u \in H^{1,1}(B\mu_l, \mathbb{Z}/l)$ such that $\beta(u) = v$ and the restriction of u to a rational point of $E\mu_l$ is zero. Then as a ring

$$H^{*,*}(B\mu_l, \mathbb{Z}/l) = H^{*,*}(k, \mathbb{Z}/l)[[u, v]]/(u^2 = \bullet),$$

where $\bullet$ is zero when l is odd and when $l = 2, \bullet$ is a linear combination of u and v with coefficients in $H^{*,*}(k, \mathbb{Z}/l)$. Actually with a little extra care one shows that the Künneth type formula holds for $H^{*,*}(B\mu_l \times X, \mathbb{Z}/l)$ and one has (see [15, Theorem 6.10]) as a ring

$$H^{*,*}(B\mu_l \times X, \mathbb{Z}/l) = H^{*,*}(X, \mathbb{Z}/l)[[u, v]]/(u^2 = \bullet). \tag{4.1.3}$$

In spite of the complication of $\bullet$ when $l = 2$, $H^{*,*}(B\mu_l \times X, \mathbb{Z}/l)$ as an $H^{*,*}(X, \mathbb{Z}/l)$ module always has $\{u^\epsilon v^\delta\}_{\epsilon=0,1,\ \delta\geq 0}$ as a basis.

Thus the element $x^l_{E\mu_l} \in H^{2nl,nl}(B\mu_l \times X, \mathbb{Z}/l)$, which we have obtained, can be written uniquely as a direct sum

$$x^l_{E\mu_l} = \sum_{\epsilon=0,1,\ \delta\geq 0} D_{\epsilon,\delta}(x) u^\epsilon v^\delta, \tag{4.1.4}$$

where the $D_{\epsilon,\delta}(x) \in H^{2nl-2\delta-\epsilon, nl-\delta-\epsilon}(X, \mathbb{Z}/l)$, determined by (4.1.3), are to be defined as the operations of x.

Using furthermore the full invariance of the construction under the whole symmetric group S_l, Voevodsky [15, Theorem 6.16] is able to show that in (4.1.4) only indices of the form $(\epsilon, \delta) = (0, i(l-1))$ and $(\epsilon, \delta) = (1, i(l-1)-1)$ can appear. He fixes the notation to be

$$P_n^i = D_{0,(n-i)(l-1)}, \; B_n^i = D_{1,(n-i)(l-1)-1}.$$

Then the operations P_n^i and B_n^i have the following forms:

$$P_n^i : H^{2n,n}(-, \mathbb{Z}/l) \to H^{2n+2i(l-1),n+i(l-1)}(-, \mathbb{Z}/l);$$

$$B_n^i : H^{2n,n}(-, \mathbb{Z}/l) \to H^{2n+2i(l-1)+1,n+i(l-1)}(-, \mathbb{Z}/l).$$

On the side, Voevodsky [15, Section 3] proves that there are no nontrivial cohomology operations lowering algebraic weights, and so in our situation $P_n^i = 0$ and $B_n^i = 0$ for $i < 0$. He also proves [15, Lemma 9.2] that the stability condition (4.1.2) holds for the families $\{P_n^i\}_n$ and $\{B_n^i\}_n$ of operations and therefore they can be extended to bistable operations. In particular, the family $\{P_n^i\}_n$ gives the bistable operation P^i in (4.1.1). By a formality [15, Corollary 2.10], bistable operations are group homomorphisms.

Voevodsky proves the following further properties of these operations.

1. [15, Lemma 9.6] $B^i = \beta \circ P^i$ where $\beta : H^{*,*}(-, \mathbb{Z}/l) \to H^{*+1,*}(-, \mathbb{Z}/l)$ is the Bockstein homomorphism associated to the exact sequence $0 \to \mathbb{Z}/l \xrightarrow{\cdot l} \mathbb{Z}/l^2 \to \mathbb{Z}/l \to 0$.

2. [15, Lemma 9.5] $P^0 = id$.

3. [15, Lemma 9.8] For $x \in H^{2n,n}(-, \mathbb{Z}/l)$, $P^n(x) = x^l$.

4. (Cartan formula, [15, Proposition 9.7]) $P^i(xy) = \sum_{j+k=i} P^j(x)P^k(y)$ for an odd prime l. When $l = 2$, the formula is more complicated since it involves coefficients from the base field.

5. (Adem relations) These relations deal with the multiplication structure of the algebra of operations. It is complicated, and when $l = 2$ it involves coefficients from the base field. I will spare you the exact formulas, but see [15, Theorem 10.2].

I would also like to mention that one of the purposes that Voevodsky constructed these operations is to use them in his proofs of the Milnor conjecture and the Bloch-Kato conjecture. In those proofs, these operations have beautiful relations with some special varieties through certain characteristic classes.

4.2 Comparison of the two operations

We start by explaining the following commutative diagram

$$\begin{tikzcd} z(\mathbb{A}^n) \arrow[r, "\mathcal{P}"] \arrow[dr, "\Psi"'] & z(\mathbb{A}^{nl})^{C_l} \arrow[d, "\gamma"] \\ & z(\mathbb{A}^{nl})^{hC_l}. \end{tikzcd} \tag{4.2.1}$$

where $\mathcal{P} : z(\mathbb{A}^n) \to z(\mathbb{A}^{nl})^{C_l}$ is our total reduced power operation (3.1.3).

$z(\mathbb{A}^{nl})^{hC_l}$ in (4.2.1) is an analogue of the homotopy fixed point set in topology. To define it, first note that for any linear algebraic group G, Voevodsky has defined its classifying space BG in the homotopy category $H_{\bullet}^{\mathbb{A}^1}(k)$ (see [15, Section 6] and [8]). For the reader's convenience, let's recall its definition.

Suppose that $G \to GL(V)$ is a faithful representation of G. Denote by $\widetilde{V_i}$ the open subset in $\mathbb{A}(V)^i$ where G acts freely. We have a sequence of closed embeddings $f_i : \widetilde{V_i} \to \widetilde{V_{i+1}}$ given by $(v_1, \cdots, v_i) \to (v_1, \cdots, v_i, 0)$. Set $EG = colim_i \widetilde{V_i}$ and $BG = colim_i \widetilde{V_i}/G$, where $\widetilde{V_i}/G$ is the quotient scheme, and the colimits are taken in the category of sheaves. It is known that the homotopy type of BG doesn't depend on the choice of $G \to GL(V)$. In particular, we have the classifying space BC_l for the cyclic group. Under our assumption of the existence of a primitive l-th root of unity ζ, we have an isomorphism between C_l and μ_l, the group of l-th roots of unity in k. Therefore we have a weak equivalence $BC_l \simeq B\mu_l$.

Now let's define $z(\mathbb{A}^{nl})^{hC_l}$. Fix a representation $C_l \to GL(V)$, and use the same notation as above.

Definition 4.2.1 *Given a smooth scheme Y and a scheme X, define $z(Y, X)$ to be the sheaf whose value on a smooth scheme U is $z(Y, X)(U) := z(X)(Y \times U)$.*

Define $z(\widetilde{V_i}, \mathbb{A}^{nl})^{C_l}$ to be the sheaf whose value $z(\widetilde{V_i}, \mathbb{A}^{nl})^{C_l}(U)$ on a smooth scheme U is the group of cycles in $z(\mathbb{A}^{nl})(\widetilde{V_i} \times U)$ which are invariant under the C_l action on $\mathbb{A}^{nl} \times \widetilde{V_i} \times U$ as the product of its natural actions on $\mathbb{A}^{nl}$ and $\widetilde{V_i}$ and the trivial action on U. Concretely, a cycle Z belongs to $z(\widetilde{V_i}, \mathbb{A}^{nl})^{C_l}(U)$ if for any $g \in C_l$, $v \in \widetilde{V_i}$ and $u \in U$, the fiber of Z over the point $(g(v), u) \in \widetilde{V_i} \times U$, as a linear combination of points in $\mathbb{A}^{nl}$, is the image of the fiber of Z over (v, u) under g.

The natural inclusion $f_i : \widetilde{V_i} \to \widetilde{V_{i+1}}$ *induces a natural pullback map* $Cycl(f_i) : z(\widetilde{V_i}, \mathbb{A}^{nl}) \leftarrow z(\widetilde{V_{i+1}}, \mathbb{A}^{nl})$, *and it can be seen to induce a map on the fixed point set* $z(\widetilde{V_i}, \mathbb{A}^{nl})^{C_l} \leftarrow z(\widetilde{V_{i+1}}, \mathbb{A}^{nl})^{C_l}$, *since* $\widetilde{V_i} \times 0 \subset \widetilde{V_{i+1}}$ *is invariant under the* C_l *action.*

We define $z(\mathbb{A}^{nl})^{hC_l} = lim_i z(\widetilde{V_i}, \mathbb{A}^{nl})^{C_l}$, *where the limit is taken in the category of sheaves.*

Furthermore define $z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{hC_l}$ *to be* $z(\mathbb{A}^{nl})^{hC_l} \otimes \mathbb{Z}/l$.

Since the action of C_l on $\widetilde{V_i}$ is free, so is its action on the product $\mathbb{A}^{nl} \times \widetilde{V_i} \times U$ for any scheme U. Then the projection $p : \mathbb{A}^{nl} \times_{C_l} \widetilde{V_i} \times U := (\mathbb{A}^{nl} \times \widetilde{V_i} \times U)/C_l \to (\widetilde{V_i} \times U)/C_l = \widetilde{V_i}/C_l \times U$ is a vector bundle of rank nl. Furthermore any cycle $Z \in z(\widetilde{V_i}, \mathbb{A}^{nl})^{C_l}(U)$, being invariant under the free action of C_l on $\mathbb{A}^{nl} \times \widetilde{V_i} \times U$, comes from a cycle Z' on the quotient (i.e., the flat pullback of Z' is Z, see (3.1.6)). As a cycle on the vector bundle $\mathbb{A}^{nl} \times_{C_l} \widetilde{V_i} \times U \xrightarrow{p} \widetilde{V_i}/C_l \times U$, Z' is equidimensional of relative dimension 0 over the base $\widetilde{V_i}/C_l \times U$ under the projection p, since originally Z is equidimensional over $\widetilde{V_i} \times U$.

This gives the following imprecise interpretation of $z(\mathbb{A}^{nl})^{hC_l}$. Given a smooth scheme U, $z(\mathbb{A}^{nl})^{hC_l}(U)$ can be thought of as the free abelian group on the closed irreducible subschemes on $\mathbb{A}^{nl} \times_{C_l} EC_l \times U$, which are equidimensional of relative dimension 0 over $BC_l \times U$ under the vector bundle projection $p : \mathbb{A}^{nl} \times_{C_l} EC_l \times U \to BC_l \times U$. This is imprecise because EC_l and BC_l are not schemes, but rather sheaves.

$\gamma : z(\mathbb{A}^{nl})^{C_l} \to z(\mathbb{A}^{nl})^{hC_l}$ in (4.2.1) is an analogue of the inclusion of a fixed point set into its associated homotopy fixed point set in topology, and is defined as follows. For a cycle $Z \in z(\mathbb{A}^{nl})^{C_l}(U)$, $Cycl(pr_i)(Z) \in z(\mathbb{A}^{nl})(\widetilde{V_i} \times U)$

is the pullback of Z by the projection $pr_i : \widetilde{V}_i \times U \to U$. It is easily seen that $Cycl(pr_i)(Z) \in z(\widetilde{V}_i, \mathbb{A}^{nl})^{C_l}(U)$, since $Z \in z(\mathbb{A}^{nl})^{C_l}(U)$. Furthermore $Cycl(f_i)(Cycl(pr_{i+1})(Z)) = Cycl(pr_i)(Z)$, where $Cycl(f_i) : z(\widetilde{V_{i+1}}, \mathbb{A}^{nl})^{C_l} \to z(\widetilde{V}_i, \mathbb{A}^{nl})^{C_l}$ is the pullback map by the natural inclusion $f_i : \widetilde{V}_i \to \widetilde{V_{i+1}}$, since clearly $pr_{i+1} \circ (f_i \times id_U) = pr_i : \widetilde{V}_i \times U \to U$. Therefore the system $\{Cycl(pr_i)(Z)\}_i$ defines a cycle in the limit $z(\mathbb{A}^{nl})^{hC_l}(U)$, and this is defined to be $\gamma(Z)$.

$\Psi = \gamma \circ \mathcal{P}$ is the composition. It is understood that Ψ is the same as Voevodsky's construction [15, Section 5].

Now let's concentrate on $\mathbb{Z}/l$ coefficients. It is a computation of Voevodsky [15, Sections 5 and 6] that

$$z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{hC_l} \simeq \prod_{i=0}^{nl} K(\mathbb{Z}/l, 2i, i) \times \prod_{i=0}^{nl-1} K(\mathbb{Z}/l, 2i+1, i). \qquad (4.2.2)$$

To be more precise, he (loc. cit.) first proves that for a smooth scheme X

$$Hom_{H_\bullet^{\mathbb{A}^1}}(X_+, z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{hC_l}) \cong H^{2nl,nl}(X \times BC_l, \mathbb{Z}/l). \qquad (4.2.3)$$

This can be informally understood as follows. As we said $z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{hC_l}(X)$ is roughly the group of cycles on $\mathbb{A}^{nl} \times_{C_l} EC_l \times X$, which are equidimensional of relative dimension 0 over $BC_l \times X$ under the projection $p : \mathbb{A}^{nl} \times_{C_l} EC_l \times X \to BC_l \times X$ of a vector bundle of rank nl. Therefore a cycle $Z \in z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{hC_l}(X)$ "should" define a motivic cohomology class of dimension $(2nl, nl)$ of the base $BC_l \times X$, since if the vector bundle were trivial, i.e. it were $\mathbb{A}^{nl} \times BC_l \times X$, such a cycle does define such a class by definition. Actually much of

Voevodsky's proof of (4.2.3) is to add another bundle to make the direct sum trivial, and then use the Thom isomorphism to "pull" the dimension of the motivic cohomology class back.

Under our assumption of the existence of a primitive l-th root of unity ζ, we have a weak equivalence $BC_l \simeq B\mu_l$ (as we explained at the beginning of Section 3.4, this can be assured by just the assumption $l \neq char(k)$ using transfer arguments). Then Voevodsky computed the motivic cohomology of $X \times B\mu_l$ with $\mathbb{Z}/l$ coefficients. His result (loc. cit.) is

$$H^{2nl,nl}(X \times B\mu_l, \mathbb{Z}/l) \cong \bigoplus_{\epsilon=0,1,\ 0\leq\delta\leq nl} H^{*,*}(X, \mathbb{Z}/l)u^{\epsilon}v^{\delta}, \tag{4.2.4}$$

as an $H^{*,*}(X, \mathbb{Z}/l)$ module. Here $v \in H^{2,1}(B\mu_l, \mathbb{Z}/l)$ corresponds to the line bundle on $B\mu_l$ associated to the standard representation ρ, and $u \in H^{1,1}(B\mu_l, \mathbb{Z}/l)$ is the unique class such that its Bockstein $\beta(u) = v$ and its restriction to a rational point $*$ of $B\mu_l$ which lifts to a rational point of some $\widetilde{V}_i$ is zero. In the following table, we count the possible dimensions of $u^{\epsilon}v^{\delta}$ for $\epsilon = 0, 1,\ 0 \leq \delta \leq nl$ and calculate the corresponding dimensions of

$H^{*,*}(X, \mathbb{Z}/l)$ such that the sum of the dimensions is $(2nl, nl)$:

δ	0	0	1	1	$\cdots$	nl
ϵ	0	1	0	1	$\cdots$	0
$\dim u^\epsilon v^\delta$	(0,0)	(1,1)	(2,1)	(3,2)	$\cdots$	$(2nl, nl)$
$\dim H^{*,*}(X)$	$(2nl, nl)$	$(2nl-1, nl-1)$	$(2nl-2, nl-1)$	$(2nl-3, nl-2)$	$\cdots$	(0,0)

Combining (4.2.3), (4.2.4) and the above table of possible dimensions of $H^{*,*}(X, \mathbb{Z}/l)$, we arrive at the conclusion (4.2.2).

Recall our Theorem 3.3.3 about the homotopy type of $z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l}$. To compare the two operations, we need to study the map $\gamma : z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{C_l} \to z^{\mathbb{Z}/l}(\mathbb{A}^{nl})^{hC_l}$ in (4.2.1). Our result is

Theorem 4.2.1 *The map γ in* (4.2.1), *which in view of Theorem 3.3.3 and* (4.2.2) *has the form of*

$$\begin{aligned} \gamma : \; & \prod_{i=n}^{nl} K(\mathbb{Z}/l, 2i, i) \times \prod_{i=n}^{nl-1} K(\mathbb{Z}/l, 2i+1, i) \\ \to \; & \prod_{i=0}^{nl} K(\mathbb{Z}/l, 2i, i) \times \prod_{i=0}^{nl-1} K(\mathbb{Z}/l, 2i+1, i), \end{aligned}$$

induces isomorphisms on the factors $K(\mathbb{Z}/l, 2i, i)$ for $n \leq i \leq nl$. In light of the diagram (4.2.1), we see that our operations D^i (3.4.1) *coincide with the* *corresponding ones of Voevodsky for cohomology classes of dimension $(2n, n)$, up to nonzero constants in $\mathbb{Z}/l$.*

Remark 4.2.2 In view of Theorem 3.4.5 and [15, Lemma 9.6], we see that our E^i (3.4.2) are also the same, up to nonzero constants, as the corresponding ones of Voevodsky for cohomology classes of dimension $(2n, n)$.

Actually the above theorem will be a special case of the following general situation. We will adopt the following notational convention in the rest of this chapter. We assume that $\mathbb{A}^n$ always has a trivial μ_l action, $\mathbb{A}^m$ has a μ_l action which is a direct sum of nontrivial irreducible representations, and $\mathbb{A}^{n+m}$ is the direct sum of them.

Our previous computations in Section 3.4, which correspond to the situation that $m = n(l-1)$, generalize to this general situation. An analogue of Theorem 3.3.3 gives us a weak equivalence

$$z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{\mu_l} \simeq \prod_{i=n}^{n+m} K(\mathbb{Z}/l, 2i, i) \times \prod_{i=n}^{n+m-1} K(\mathbb{Z}/l, 2i+1, i). \tag{4.2.5}$$

Denote by $z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l}$ the similarly defined homotopy fixed point set as in Definition 4.2.1. The following general versions of (4.2.3), (4.2.4) and (4.2.2):

$$Hom_{H_{\bullet}^{\mathbb{A}^1}}(X_+, z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l}) \cong H^{2(n+m),n+m}(X \times B\mu_l, \mathbb{Z}/l), \tag{4.2.6}$$

$$H^{2(n+m),n+m}(X \times B\mu_l, \mathbb{Z}/l) \cong \bigoplus_{\epsilon=0,1,\ 0\leq\delta\leq n+m} H^{*,*}(X, \mathbb{Z}/l)u^{\epsilon}v^{\delta}, \tag{4.2.7}$$

and

$$z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l} \simeq \prod_{i=0}^{n+m} K(\mathbb{Z}/l, 2i, i) \times \prod_{i=0}^{n+m-1} K(\mathbb{Z}/l, 2i+1, i) \qquad (4.2.8)$$

still hold.

Similar to diagram (4.2.1), we can define a natural map $\gamma : z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{\mu_l} \longrightarrow z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l}$. We now prove the following theorem, which in particular gives Theorem 4.2.1.

Theorem 4.2.3 *In view of* (4.2.5) *and* (4.2.8), $\gamma : z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{\mu_l} \longrightarrow z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l}$ *induces isomorphisms on the factors* $K(\mathbb{Z}/l, 2i, i)$ *for* $n \leq i \leq m+n$.

First a lemma.

Lemma 4.2.4 *For arbitrary dimensions* n *and* m, *the map* γ *induces the identity on the top dimensional factor* $K(\mathbb{Z}/l, 2(n+m), n+m)$.

Proof. Consider the following diagram

$$\begin{tikzcd} z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{\mu_l} \arrow[r, "j"] \arrow[d, "\gamma"] & z^{\mathbb{Z}/l}(\mathbb{A}^{n+m}) \\ z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l}, \arrow[ur, "j'"] & \end{tikzcd} \qquad (4.2.9)$$

where j is the inclusion of the fixed point set sheaf into the whole sheaf. j' is defined as follows. Regard $z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l}$ as $z^{\mathbb{Z}/l}(E\mu_l, \mathbb{A}^{n+m})^{\mu_l}$ (see Definition 4.2.1). Let $incl : * \longrightarrow E\mu_l$ be a rational point of $E\mu_l$ which lifts to a rational point of some $\widetilde{V}_i$. Then define $j' = Cycl(incl) : z^{\mathbb{Z}/l}(E\mu_l, \mathbb{A}^{n+m}) \longrightarrow z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})$

to be the pullback map by *incl*. Note that j' doesn't depend on the choice of the rational point $*$ since $E\mu_l$ is contractible.

The diagram (4.2.9) can be seen to commute as follows. Observe that $\gamma = Cycl(pr)$ is the pullback map by $pr : E\mu_l \to *$. Therefore $j' \circ \gamma = Cycl(incl) \circ Cycl(pr) = Cycl(pr \circ incl) = Cycl(id_*)$ is the identity map, except that now we forget the fixed point set structure, which is exactly j.

Exactly the same arguments as in the proof of Proposition 3.5.2 show that j induces the identity on the factor $K(\mathbb{Z}/l, 2(n+m), n+m)$. We now want to prove that j' also induces the identity on this top dimensional factor. For a scheme X, we have the following commutative diagram:

$$\begin{tikzcd}
Hom_{H_\bullet^{\mathbb{A}^1}}(X_+, z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l}) \arrow[r, "\cong"] \arrow[d, "j'\circ"] & H^{2(n+m),n+m}(X \times B\mu_l, \mathbb{Z}/l) \arrow[d, "j'_H"] \\
Hom_{H_\bullet^{\mathbb{A}^1}}(X_+, z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})) \arrow[r, "\cong"] & H^{2(n+m),n+m}(X, \mathbb{Z}/l),
\end{tikzcd}$$

where the first isomorphism is (4.2.6), and the second isomorphism is the obvious one (see (2.3.1) and (2.3.2)). The left vertical map $j'\circ$ is composition by j', and the right vertical map j'_H is induced by the inclusion $* \to E\mu_l \to B\mu_l$. The diagram commutes by the naturality of (4.2.6) (see [15, Lemma 5.12]). (4.2.7) says that j'_H has the form

$$\begin{aligned} j'_H : H^{2(n+m),n+m}(X \times B\mu_l, \mathbb{Z}/l) &\cong \bigoplus_{\epsilon=0,1,\ 0\le\delta\le n+m} H^{*,*}(X, \mathbb{Z}/l)u^\epsilon v^\delta \\ &\to H^{2(n+m),n+m}(X, \mathbb{Z}/l). \end{aligned}$$

Actually j'_H only records the summand $H^{2(n+m),n+m}(X, \mathbb{Z}/l)$, which corresponds to $\epsilon = \delta = 0$, in $\bigoplus_{\epsilon=0,1,\ 0\leq\delta\leq nl} H^{*,*}(X, \mathbb{Z}/l)u^{\epsilon}v^{\delta}$, since both v (by dimension reasons) and u (by definition) are restricted to zero in the motivic cohomology of the rational point $*$. This means that $j' : z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l} \to z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})$ is actually the projection to the $K(\mathbb{Z}/l, 2(n+m), n+m)$ factor.

By the commutativity of the diagram (4.2.9), we see that γ is the identity on the factor $K(\mathbb{Z}/l, 2(n+m), n+m)$. ■

Now let's prove our theorem.

Proof of Theorem 4.2.3. We run induction on m. When $m = 0$ we are done by Lemma 4.2.4. Now assume that the assertion holds for $m - 1$, i.e. (with obvious notation)

$$\gamma_{n+m-1} : z^{\mathbb{Z}/l}(\mathbb{A}^{n+m-1})^{\mu_l} \to z^{\mathbb{Z}/l}(\mathbb{A}^{n+m-1})^{h\mu_l}$$

induces isomorphisms on the factors $K(\mathbb{Z}/l, 2i, i)$ for $n \leq i \leq n+m-1$.

Consider the following diagram

$$\begin{tikzcd}
z^{\mathbb{Z}/l}(\mathbb{A}^{n+m-1})^{\mu_l} \arrow[r, "i"] \arrow[d, "\gamma_{n+m-1}"] & z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{\mu_l} \arrow[d, "\gamma_{n+m}"] \\
z^{\mathbb{Z}/l}(\mathbb{A}^{n+m-1})^{h\mu_l} \arrow[r, "i_h"] & z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l},
\end{tikzcd}$$

where i and i_h are induced by the closed embedding $\mathbb{A}^{n+m-1} \to \mathbb{A}^{n+m}$ with the last coordinate zero. The diagram is clearly commutative by our construction of γ.

We have proved that γ_{n+m} induces the identity on the factor $K(\mathbb{Z}/l, 2(n+m), n+m)$ by Lemma 4.2.4, so by the induction hypothesis we only have to show that i and i_h induce isomorphisms on the factors $K(\mathbb{Z}/l, 2i, i)$ for $n \leq i \leq n+m-1$.

This is clearly true for i by how we compute the type of $z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{\mu_l}$.

Now consider $i_h : z^{\mathbb{Z}/l}(\mathbb{A}^{n+m-1})^{h\mu_l} \to z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l}$. For a scheme X, we have the following commutative diagram

$$\begin{tikzcd}
Hom_{H^{\mathbb{A}^1}_{\bullet}}(X_+, z^{\mathbb{Z}/l}(\mathbb{A}^{n+m-1})^{h\mu_l}) \arrow[r, "\cong"] \arrow[d, "i_h\circ"] & H^{2(n+m-1),n+m-1}(X \times B\mu_l, \mathbb{Z}/l) \arrow[d, "i_H"] \\
Hom_{H^{\mathbb{A}^1}_{\bullet}}(X_+, z^{\mathbb{Z}/l}(\mathbb{A}^{n+m})^{h\mu_l}) \arrow[r, "\cong"] & H^{2(n+m),n+m}(X \times B\mu_l, \mathbb{Z}/l),
\end{tikzcd} \tag{4.2.10}$$

where the two isomorphisms are by (4.2.6), the left vertical map $i_h\circ$ is composition by i_h. The right vertical map i_H is defined as the follow composition:

$$\begin{aligned}
i_H : \quad & H^{2(n+m-1),n+m-1}(X \times B\mu_l, \mathbb{Z}/l) \\
\overset{\cdot t_{L_m}}{\cong} \quad & \widetilde{H}^{2(n+m),n+m}(X_+ \wedge Th(L_m \to B\mu_l), \mathbb{Z}/l) \\
\xrightarrow{i^*} \quad & H^{2(n+m),n+m}(X \times B\mu_l, \mathbb{Z}/l),
\end{aligned}$$

where $Th(L_m \to B\mu_l)$ is the Thom space of the line bundle L_m on $B\mu_l$ associated to the last $\mathbb{A}^1$ component of $\mathbb{A}^m$, the isomorphism is the Thom isomorphism via the multiplication by the Thom class t_{L_m}, and the last arrow is induced by inclusion $i : B\mu_l \to Th(L_m \to B\mu_l)$ as the zero section. The commutativity of the diagram (4.2.10) is seen from the proof of (4.2.6) (see [15, Section 5]), where one adds a vector bundle to make the direct sum trivial and apply the Thom isomorphism theorem to pull back the dimension of the cohomology class.

Therefore

$$i_H = \cdot e(L_m) : H^{2(n+m-1),n+m-1}(X \times B\mu_l) \to H^{2(n+m),n+m}(X \times B\mu_l)$$

is the multiplication by $e(L_m)$, the Euler class of the line bundle L_m. Since the μ_l action on $\mathbb{A}^m$ is a direct sum of nontrivial irreducible representations, we have $e(L_m) = a_m v$, where $1 \leq a_m \leq l-1$ is an integer invertible in $\mathbb{Z}/l$ and v is the Euler class of the line bundle on $B\mu_l$ associated to the standard representation ρ, which by definition is the v in (4.2.7).

Therefore in view of (4.2.7), i_H has the form

$$\begin{aligned} i_H : \quad & H^{2(n+m-1),n+m-1}(X \times B\mu_l) \cong \bigoplus_{\epsilon=0,1,\ 0\leq\delta\leq n+m-1} H^{*,*}(X)u^{\epsilon}v^{\delta} \\ \xrightarrow{\cdot a_m v} & \bigoplus_{\epsilon=0,1,\ 0\leq\delta\leq n+m-1} a_m H^{*,*}(X)u^{\epsilon}v^{\delta+1} \to H^{2(n+m),n+m}(X \times B\mu_l), \end{aligned}$$

which on the components of $H^{p,*}(X)$ for $p \leq 2(n+m-1)$ are multiplications by a_m and thus isomorphisms. This proves our theorem. ∎

Finally let's add some remarks about our operations.

Remark 4.2.5 Our construction doesn't produce any cohomology operations lowering algebraic weights. Meanwhile Voevodsky's approach *a priori* produces such operations (see (4.2.2)). It takes some effort for Voevodsky (see [15, Section 3]) to prove that all such operations lowering algebraic weights are zero.

Remark 4.2.6 Voevodsky's operations $H^{2n,n}(X,\mathbb{Z}/l) \to H^{2n+2i,n+i}(X,\mathbb{Z}/l)$

are nonzero only if $i = j(l-1)$ is a multiple of $l-1$. The proof uses the full symmetry of the construction under the whole symmetric group S_l. By our comparison theorem 4.2.1, we see that this also holds for our operations, i.e. $D^i = 0$ and thus $E^i = 0$ by Theorem 3.4.5 unless $i = j(l-1)$. It would be satisfying to give a direct proof of this fact for our operations. But this eludes my effort so far mainly because in our construction we are dealing with finite dimensional lens spaces, and the homotopy classes of maps between such lens spaces are more complicated than those for the infinite dimensional lens spaces, which were employed in Voevodsky's approach.

Remark 4.2.7 Using a trick of considering only bistable operations, Voevodsky can extend his operations to motivic cohomology $H^{*,*}(X, \mathbb{Z}/l)$ of all dimensions instead of just $H^{2n,n}(X, \mathbb{Z}/l)$, and furthermore by a formality (see [15, Corollary 2.10]) all bistable operations are group homomorphisms. To follow his approach, we need to verify that for us

$$\mathcal{P}_{n+1}(x \wedge \sigma_T) = \mathcal{P}_n(x) \wedge \sigma_T, \tag{4.2.11}$$

where $x \in H^{2n,n}(X, \mathbb{Z}/l)$ is a cohomology class of dimension $(2n, n)$, $\sigma_T \in \widetilde{H}^{2,1}(T, \mathbb{Z}/l)$ is the tautological motivic cohomology class of $T = \mathbb{A}^1/(\mathbb{A}^1 - \{0\})$, i.e. σ_T is represented by the diagonal cycle, and $x \wedge \sigma_T \in \widetilde{H}^{2(n+1),n+1}(X_+ \wedge T, \mathbb{Z}/l)$ is the product. Here we write $\mathcal{P}_n$ to note $\mathcal{P}$ on dimension $(2n, n)$. This "should" be obtained by the Cartan formula, which says that $\mathcal{P}_{n+1}(x \wedge \sigma_T) = \mathcal{P}_n(x) \wedge \mathcal{P}_1(\sigma_T)$, together with the fact that

$$\mathcal{P}_1(\sigma_T) = \sigma_T. \tag{4.2.12}$$

(4.2.12) holds since all the fiber product of the diagonal cycle (being a one-to-one cycle) is in the diagonal $\mathbb{A}^1$ of $\mathbb{A}^l$ and is restricted to zero off the diagonal to $\mathbb{A}^l - \mathbb{A}^1$, and so $\mathcal{P}_1(\sigma_T) = D^0(\sigma_T) = \sigma_T$ by Theorem 3.5.1. But the full powered Cartan formula surely is out of our reach, since it involves some coefficients coming from the base field when $l = 2$ (see [15, Proposition 9.7]). I have a primitive version of the Cartan formula, but it is not clear to me if it is enough for the stability condition (4.2.11).

Remark 4.2.8 I want to say, as my last words, that although our construction can be said to be conceptionally simpler, we lose much control of the actual classes. In particular, our computational ability is very limited using this approach, and for instance, we don't dare to touch the Adem relations at all (see [15, Section 10]).

Bibliography

[1] M. Artin, Grothendieck topologies, notes on a seminar, Harvard University, 1962.

[2] P. Deligne, Voevodsky's lectures on motivic cohomology, *www.math.ias.edu/˜vladimir/seminar.html*, 2000/2001.

[3] A. Dold and R. Thom, Quasifaserungen und unendliche symmetrische Produkte, *Ann. of Math. (2)* **67** (1958), 239–281.

[4] E. Friedlander and V. Voevodsky, Bivariant cycle cohomology, in *Cycles, Transfers, and Motivic Homology Theories, Princeton Univ. Press, Princeton, 2000*, 138-187.

[5] P.G. Goerss and J.F. Jardine, Simplicial homotopy theory, *Progress in Mathematics 174*, Birkäuser Verlag, 1999.

[6] M. Karoubi, Formes differentielles non commutatives et operations de steenrod, *Topology* **34** (1995), 699-715.

[7] M. Karoubi, Produit cyclique d'espaces et opérations de Steenrod, *in The Arnold-Gelfand mathematical seminars, Birkhäuser Boston, Boston, 1997*, 281-323.

[8] F. Morel and V. Voevodsky, $\mathbb{A}^1$-homotopy theory of schemes, *Publ. Math. IHES*, **90** (1999), 45–143.

[9] M. Raynaud and L. Gruson, Criteres de platitude et de projectivite, *Inv. Math.* **13** (1971), 1-89.

[10] N.E. Steenrod and D.B. Epstein, Cohomology operations, *Ann. of Math. Stud. 50*, Princeton Univ. Press, 1962.

[11] A. Suslin and V. Voevodsky, Relative cycles and Chow sheaves, in *Cycles, Transfers, and Motivic Homology Theories, Princeton Univ. Press, Princeton, 2000*, 10-86.

[12] V. Voevodsky, $\mathbb{A}^1$-homotopy theory, *Documenta Mathematica*, **Extra Volume ICM 1998, I**, 579-604.

[13] V. Voevodsky, Triangulated categories of motives over a field, in *Cycles, Transfers, and Motivic Homology Theories, Princeton Univ. Press, Princeton, 2000*, 188-238.

[14] V. Voevodsky, Motivic cohomology groups are isomorphic to higher Chow groups in any characteristic, *Int. Math. Res. Not.* **7** (2002), 351–355.

[15] V. Voevodsky, Reduced power operations in motivic cohomology, *Publ. Math. IHES* **98** (2003), 1-57.

[16] C. Weibel, An introduction to homological algebra, Cambridge Univ. Press, 1994.