

The Thurston Boundary of Teichmüller Space and Complex of Curves

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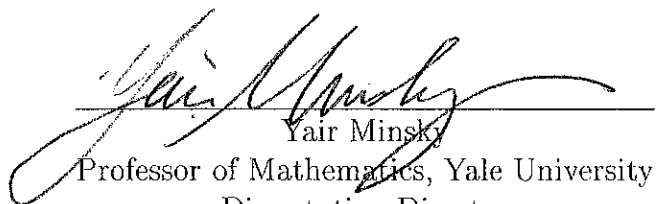
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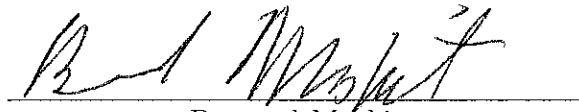
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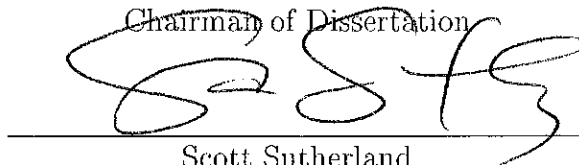
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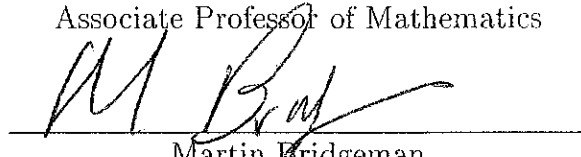
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Abstract of the Dissertation

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Let S be a closed orientable surface with genus $g \geq 2$. Let $\mathcal{T}(S)$ be the Teichmüller space and $\mathcal{C}(S)$ the complex of curves. We write $\mathcal{PML}(S)$ to denote the Thurston boundary of $\mathcal{T}(S)$, and $\partial_\infty \mathcal{C}(S)$ to denote the Gromov boundary of $\mathcal{C}(S)$. Let $\mathcal{UML}(S)$ be the quotient space of $\mathcal{PML}(S)$ by forgetting measure with the quotient map u . Let L be a fixed Bers constant. For $\sigma \in \mathcal{T}(S)$, let $\Phi(\sigma)$ be a pants decomposition whose total length is bounded by L , where all pants curves are geodesics in σ . Suppose that $\sigma_i \in \mathcal{T}(S)$ converges to $[\lambda] \in \mathcal{PML}(S)$. Consider the decomposition

$$u([\lambda]) = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_m$$

as a finite disjoint union of minimal laminations. Suppose that α_i is a pants curve in $\Phi(\sigma_i)$. In this thesis we show:

1. If $\Phi(\sigma_i)$ converges to a geodesic lamination ν in Hausdorff metric topology, then $u([\lambda]) \subset \nu$.
2. Suppose that λ_1 is a simple closed curve. Consider an annular covering Y of S in which a neighborhood of λ_1 lifts homeomorphically. Suppose that α_i meets λ_1 for all i . Then the absolute value of algebraic intersection number $|a_1 \cdot a_i|$ approaches to ∞ , where a_1 is a lift of α_1 which connects the two boundaries of Y , and so is a_i .
3. Suppose that λ_2 is not a simple closed curve. Let F be the essential subsurface of S which is filled by λ_2 . Let β_i be a component of $\alpha_i \cap F$. We can define a simple closed curve $\tilde{\beta}_i$ from β_i in a canonical way. Then the geodesic representative of $\tilde{\beta}_i$ converges to λ_2 in $\mathcal{UML}(F)$.
4. In particular, if λ is a filling lamination then α_i converges to $u([\lambda])$ in $\mathcal{C}(S) \cup \partial_\infty \mathcal{C}(S)$, where $u([\lambda])$ is considered as an element of $\partial_\infty \mathcal{C}(S)$.

To Inae Park, Yoonsuh Kim and my mother

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Chapter 1

Introduction

Let S denote a closed orientable surface with genus $g \geq 2$, and $\mathcal{T}(S)$ the Teichmüller space of S . Although there exist some similarities between $\mathcal{T}(S)$ and a complete negatively curved space (see [6, 31, 54, 66]), Masur ([65]) showed that $\mathcal{T}(S)$ is not negatively curved. In [36], Gromov introduced Gromov hyperbolic spaces which include complete negatively curved spaces and metric trees, but Masur and Wolf ([70]) showed that $\mathcal{T}(S)$ is not Gromov hyperbolic, either (see [72], for another proof by McCarthy and Papadopoulos).

In [57], Luo classified surface theories into geometric theory, i.e. $\mathcal{T}(S)$, algebraic theory, i.e. the mapping class group $\mathcal{MCG}(S)$, and topological theory, i.e. the complex of curves $\mathcal{C}(S)$. The complex of curves, which was introduced by Harvey in [39], is a finite dimensional simplicial complex whose vertices are non-trivial homotopy classes of simple closed curves which are not boundary-parallel, and k -simplices are $k+1$ distinct vertices with disjoint representatives. Let $\mathcal{C}_0(S)$ denote the set of vertices and $\mathcal{C}_1(S)$ its 1-skeleton. In [67], Masur and Minsky defined a metric on $\mathcal{C}(S)$ by making each simplex regular Euclidean with side length 1 and taking shortest-path metric, and showed that

$\mathcal{C}(S)$ is a non-proper Gromov hyperbolic space (see [16], for a shorter proof by Bowditch). Every Gromov hyperbolic space has a natural boundary which is called Gromov boundary (see [23, 36, 98]). We write $\partial_\infty \mathcal{C}(S)$ to denote the Gromov boundary of $\mathcal{C}(S)$.

Since $\mathcal{T}(S)$ is not Gromov hyperbolic, we can not define its Gromov boundary. For example, Kerckhoff([47]) showed that the Teichmüller boundary, which is the set of endpoints of geodesic rays, depends on the choice of base point. In [102], Thurston introduced a compactification of $\mathcal{T}(S)$ with the boundary equal to the space of projective measured laminations $\mathcal{PML}(S)$, on which the action of $\mathcal{MCG}(S)$ extends continuously. Throughout this thesis, we will write $\overline{\mathcal{T}}(S)$ to denote this compactification of Thurston. See [7, 21] for a similar but different compactification by Bers.

Thurston boundary $\mathcal{PML}(S)$ is the space of projective classes of measured laminations. We write $\mathcal{ML}(S)$ to denote the space of measured laminations, and $[\lambda]$ the projective class of $\lambda \in \mathcal{ML}(S)$. A measured lamination consists of a geodesic lamination and a transverse measure with full support on it. The topology on the space of geodesic laminations $\mathcal{GL}(S)$ is the Hausdorff metric topology on closed subsets. On $\mathcal{ML}(S)$, Thurston gave the weak-topology induced by the measures on transverse arcs. Note that $\mathcal{PML}(S)$ has the natural quotient topology. Let $\mathcal{UML}(S)$ be the quotient space of $\mathcal{PML}(S)$ by forgetting measure. Although $\mathcal{UML}(S)$ is a subset of $\mathcal{GL}(S)$, the quotient topology on $\mathcal{UML}(S)$ is not equal to the subspace topology.

In [102] §5, Thurston wrote “... *Intuitively, the interpretation is that a sequence of hyperbolic structures on S can go to infinity by “pinching” a certain geodesic lamination λ ; then it converges to λ . As a lamination is pinched*

toward 0, lengths of paths crossing it are forced toward infinity. The ratios of these lengths determine the transverse invariant measure ...".

More clearly, a sequence $\sigma_i \in \mathcal{T}(S)$ converges to $[\lambda] \in \mathcal{PML}(S)$ in $\overline{\mathcal{T}}(S)$ if and only if for all simple closed curves α, β on S ,

$$\frac{\ell_{\sigma_i}(\alpha)}{\ell_{\sigma_i}(\beta)} \text{ converges to } \frac{i(\alpha, \lambda)}{i(\beta, \lambda)},$$

where $\ell_{\sigma_i}(\alpha)$ is the length of closed σ_i -geodesic which is homotopic to α , and $i(\alpha, \lambda)$ is the intersection number of α and λ which is a generalization of the geometric intersection number of simple closed curves (see [11, 89, 90]).

The geometry of $\mathcal{T}(S)$ and complex of curves are well described by Minsky in [78]. They are related by the collar lemma (see [22, 46]). The collar lemma implies that there is a universal constant $\epsilon > 0$ such that for any distinct $\alpha, \beta \in \mathcal{C}_0(S)$ and $\sigma \in \mathcal{T}(S)$, if $\ell_{\sigma}(\alpha) < \epsilon$ and $\ell_{\sigma}(\beta) < \epsilon$ then the two geodesic representatives of α and β are disjoint, i.e. α and β are on a same simplex in the complex of curves. Therefore $\mathcal{C}_1(S)$ could be considered as the nerve of the family of regions

$$T(\alpha) = \{\sigma \in \mathcal{T}(S) \mid \ell_{\sigma}(\alpha) < \epsilon\}, \quad \alpha \in \mathcal{C}_0(S).$$

Although $\mathcal{T}(S)$ is not Gromov hyperbolic, Masur and Minsky showed that $\mathcal{T}(S)$ is Gromov hyperbolic modulo this family of regions (see [67, 78, 79]).

A lamination $\mu \in \mathcal{ML}(S)$ is called a *filling lamination* if $i(\mu, \mu') = 0$ then

$$\text{support}(\mu) = \text{support}(\mu'),$$

for any $\mu' \in \mathcal{ML}(S)$. Minsky wrote $\mathcal{EL}(S)$ to denote the image of filling laminations in $\mathcal{UML}(S)$. In the celebrated proof of Thurston's ending lamination conjecture, by Brock, Canary and Minsky, the laminations in $\mathcal{EL}(S)$ are appeared as ending laminations of Kleinian surface groups without accidental parabolics (see [86, 88]).

Ending Lamination Conjecture. *A hyperbolic 3-manifold with finitely generated fundamental group is uniquely determined by its topological type and its end invariants.*

The proof of the ending lamination conjecture and the recent proof of Marden's tameness conjecture by Agol([2]) give us rough picture of hyperbolic 3-manifolds. See [60] for the original conjecture, and see [24] for another proof by Calegari-Gabai.

Tameness Conjecture. *A hyperbolic 3-manifold with finitely generated fundamental group is homeomorphic to the interior of a compact manifold with boundary.*

The Gromov boundary of $\mathcal{C}(S)$ is homeomorphic to the Gromov boundary of its 1-skeleton $\mathcal{C}_1(S)$ because $\mathcal{C}(S)$ is quasi-isometric to $\mathcal{C}_1(S)$. In [67], Masur and Minsky showed that the relative hyperbolic space, which is roughly $\mathcal{T}(S)$ modulo the regions $T(\alpha)$, is quasi-isometric to $\mathcal{C}_1(S)$. Therefore we can expect some relation between Thurston boundary of $\mathcal{T}(S)$ and Gromov boundary of $\mathcal{C}(S)$ (see [78] for the first question of Minsky on this relation). In fact, Klarreich showed that $\partial_\infty \mathcal{C}(S)$ is homeomorphic to $\mathcal{EL}(S)$ (see [37] for a new proof by Hamenstädt).

Theorem 1.1 (Klarreich [51]). *There is a homeomorphism*

$$k : \partial_\infty \mathcal{C}(S) \rightarrow \mathcal{EL}(S)$$

such that for any sequence α_i in $\mathcal{C}_0(S)$, α_i converges to $\alpha \in \partial_\infty \mathcal{C}(S)$ if and only if α_i , considered as a subset of $\mathcal{UML}(S)$, converges to $k(\alpha)$.

In the following theorem of Bers, $\mathcal{T}(\Sigma)$ stands for the Teichmüller space of Σ with geodesic boundaries (see [8, 9, 22]).

Theorem 1.2 (Bers [9]). *Let Σ be a compact Riemann surface with genus $g \geq 2$ from which n points and m disks have been removed. For any $\sigma \in \mathcal{T}(\Sigma)$, there exist $3g - 3 + n + m$ disjoint geodesics, which are not boundary parallel, whose lengths are bounded by a constant L which depends only on g, n, m , and the largest length of the geodesics homotopic to the boundaries of Σ .*

The constant L is called a *Bers constant*. Notice that for the closed surface S , for any $\sigma \in \mathcal{T}(S)$, S has a pants decomposition with total length bounded by a constant L which depends only on the genus g . In fact, Buser and Seppälä([23]) showed that we can choose $L = 21g(3g - 3)$. Throughout this thesis, we write L to denote a fixed Bers constant.

Motivated by Theorem 1.2, we define a function Φ on $\mathcal{T}(S)$ as follows. For $\sigma \in \mathcal{T}(S)$, let

$$\Phi(\sigma) = \text{a pants decomposition whose total length is bounded by } L,$$

where all pants curves are geodesics in σ . Let $u : \mathcal{PML}(S) \rightarrow \mathcal{UML}(S)$ be

the quotient map by forgetting measure. Suppose that

$$\sigma_i \in \mathcal{T}(S) \text{ converges to } [\lambda] \in \mathcal{PML}(S) \text{ in } \overline{\mathcal{T}}(S) \quad (1.1)$$

and α_i is a pants curve in $\Phi(\sigma_i)$. From Theorem 1.1, the following question is immediate.

Question 1.3. *Suppose that λ is a filling lamination. Is the geodesic representative of α_i converging to $u([\lambda])$ in $\mathcal{UMC}(S)$?*

This question is the motivation of all the work in this thesis. In Chapter 5, we will solve this question positively. Notice that if we identify $\partial_\infty \mathcal{C}(S)$ with $\mathcal{EL}(S)$ via the homeomorphism in Theorem 1.1, then we have

Theorem 1.4. *If λ is a filling lamination, then α_i converges to $u([\lambda])$ in $\mathcal{C}(S) \cup \partial_\infty \mathcal{C}(S)$.*

Suppose that λ is not necessarily a filling lamination in eq. (1.1). Is there any relation between the limit point of α_i and $u([\lambda])$? The following simple example shows that a limit of α_i and $u([\lambda])$ could be disjoint.

Example 1.5. Consider a fixed $\gamma \in \mathcal{C}_0(S)$. Suppose that $\sigma_i \in \mathcal{T}(S)$ converges to $[\gamma] \in \mathcal{PML}(S)$ and $\ell_{\sigma_i}(\gamma) \rightarrow 0$, where we consider γ as an element of $\mathcal{ML}(S)$ via the counting measure. Since $\ell_{\sigma_i}(\gamma) \rightarrow 0$, there exists a pants decomposition $\Phi(\sigma_i)$ whose total length is bounded by L and

$$\gamma \in \Phi(\sigma_i) \quad \text{for all large enough } i.$$

Then we can find a pants curve α_i in $\Phi(\sigma_i)$ such that a limit point of α_i is

disjoint from γ in $\mathcal{UML}(S)$.

A nonempty geodesic lamination $\mu \in \mathcal{GL}(S)$ is called *minimal* if no proper subset of μ is a geodesic lamination. For example, any closed geodesic is a minimal lamination. The following theorem on the structure of a geodesic lamination on S works for any hyperbolic surface of finite type (see [25] §4.2, [28] §4 or [101] §8).

Theorem 1.6 (Structure of Geodesic Lamination). *A geodesic lamination on S is the union of finitely many minimal sublamination and of finitely many infinite isolated leaves whose ends spiral along the minimal sublamination.*

Furthermore, if $[\lambda] \in \mathcal{PML}(S)$ then we can decompose $u([\lambda])$ as a finite disjoint union of minimal laminations,

$$u([\lambda]) = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_m. \quad (1.2)$$

An essential subsurface F of S is a subsurface of S whose boundaries are all homotopically non-trivial geodesics. Throughout this thesis, we assume that all essential subsurfaces of S are open, i.e. the boundaries are not included (see Figure 1.1). Note that two distinct boundaries of \overline{F} could be a same curve in S , where \overline{F} is the completion of F with the path-metric in F .

Suppose that $\mu \in \mathcal{UML}(S)$. An essential subsurface F is called *filled by μ* , if for any simple closed curve α in F which is not parallel to a boundary of \overline{F} , α intersects μ .

Suppose that $\sigma_i \in \mathcal{T}(S)$ converges to $[\lambda] \in \mathcal{PML}(S)$, and let $u([\lambda]) =$

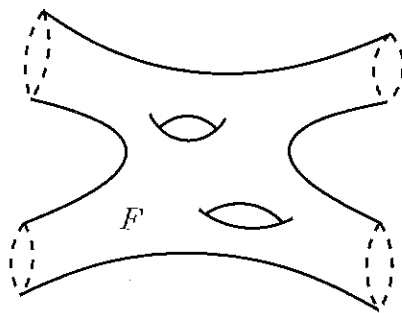


Figure 1.1: An essential subsurface

$\lambda_1 \cup \lambda_2 \cup \dots \cup \lambda_m$ be the decomposition into minimal sublaminations. The main theorem of this thesis is

Theorem 1.7 (Main Theorem). *If $\Phi(\sigma_i)$ converges to a geodesic lamination ν in Hausdorff metric topology, then $u([\lambda]) \subset \nu$.*

There is no minimal lamination which fills a pair of pants (see [91] §2.6), therefore in eq. (1.2), if λ_j is not a simple closed curve then the essential subsurface filled by λ_j is at least a 1-holed torus or 4-holed sphere. For a closed annulus Y , Minsky defined the arc complex $\mathcal{A}(Y)$. In this complex vertices are essential homotopy classes, rel endpoints, of properly embedded arcs, and simplices are sets of vertices with representatives with disjoint interiors. Note that the endpoints are *not* allowed to move in the boundary. Let $\mathcal{A}_0(Y)$ be the set of vertices and $\mathcal{A}_1(Y)$ the 1-skeleton, and give shortest-path metrics to $\mathcal{A}(Y)$ and $\mathcal{A}_1(Y)$ as in $\mathcal{C}(S)$. For $a, b \in \mathcal{A}_0(Y)$, we write $a \cdot b$ to denote the *algebraic* intersection number.

Suppose that in eq. (1.2), λ_1 is a simple closed curve. Consider an annular covering Y of S in which a neighborhood of λ_1 lifts homeomorphically. The following corollary follows from the main theorem.

Corollary 1.8. *Suppose that α_i meets λ_1 for all i . Then $|a_1 \cdot a_i|$ approaches to ∞ , where $a_1 \in \mathcal{A}_0(Y)$ is a lift of α_1 in Y and so is a_i .*

Suppose that in eq. (1.2), λ_2 is not a simple closed curve. Suppose that F is an essential subsurface of S which is filled by λ_2 . Notice that there exists a pants curve α_i in $\Phi(\sigma_i)$ such that $\alpha_i \cap F \neq \emptyset$ for all i . Let β_i be a component of $\alpha_i \cap F$. We can define a simple closed curve $\tilde{\beta}_i$ from β_i canonically (see Chapter 5). We will prove the following theorem which is a generalization of Theorem 1.4.

Theorem 1.9. *The geodesic representative of $\tilde{\beta}_i$ converges to λ_2 in $\mathcal{UML}(F)$.*

•

Plan of the Thesis. In Chapter 2 and Chapter 3, we study some preliminaries. We prove the main theorem in Chapter 4, and we prove Corollary 1.8 and Theorem 1.9 in Chapter 5. In the last chapter, we prove a theorem on discrete, torsion free subgroups of $Isom \mathbf{H}^n$ which is published in *Geometriae Dedicata* (see [50]).

Chapter 2

The Thurston Boundary of Teichmüller Space

Let S be a closed oriented surface of genus $g \geq 2$. In this chapter we study the Teichmüller space $\mathcal{T}(S)$ and its Thurston boundary $\mathcal{PML}(S)$. See [43] and [105] as references.

2.1 Teichmüller space and its boundaries

A conformal structure σ on S is determined by an atlas of coordinate neighborhoods (U_α, z_α) , where $\{U_\alpha\}$ is an open cover of M , and $z_\alpha : U_\alpha \rightarrow \mathbb{C}$ has the property that $z_\alpha \circ z_\beta^{-1}$ is analytic whenever defined. Teichmüller space $\mathcal{T}(S)$ is the space of conformal structures on S , where two structures are considered to be equivalent if there is a conformal map between them isotopic to the identity. Recall that two diffeomorphisms f and g on S are called *isotopic* if there exists a diffeomorphism $H(x, t) = (h_t(x), t) : S \times [0, 1] \rightarrow S \times [0, 1]$ such that $h_0(x) = f(x)$ and $h_1(x) = g(x)$ for all $x \in S$.

By the uniformization theorem, $\mathcal{T}(S)$ can be considered as the space of complete hyperbolic metrics on S with finite area. The area of hyperbolic

surface (S, σ) is equal to $-2\pi\chi(S)$, where $\chi(S) = 2(1 - g)$ is the Euler characteristic of S (see [4] for details).

As usual, two closed curves $\alpha, \beta : [0, 1] \rightarrow S$ are called *free homotopic* if there exists a continuous mapping $F : [0, 1] \times [0, 1] \rightarrow S$ such that

$$F(t, 0) = \alpha(t), \quad F(t, 1) = \beta(t) \quad \text{and} \quad F(0, s) = F(1, s)$$

for all $t, s \in [0, 1]$. The following well-known lemma will be used frequently in this thesis (see [4] §B.4 for a proof).

Lemma 2.1. *Suppose that (S, σ) is a hyperbolic surface. Then each free homotopy class of closed curves contains a unique geodesic representative.*

Two closed curves $\alpha, \beta : [0, 1] \rightarrow S$ are called *isotopic* if there exists a diffeomorphism $G : S \times [0, 1] \rightarrow S \times [0, 1]$ such that for all $x \in S$ and $t, s \in [0, 1]$,

$$G(x, t) = (g_t(x), t), \quad g_0(x) = x \quad \text{and} \quad g_1(\alpha(s)) = \beta(s).$$

Free homotopic simple curves on a connected surface are isotopic, too.

There is well-known classification of $Isom \mathbf{H}^n$ into elliptic, parabolic and hyperbolic isometries. We write $\overline{\mathbf{H}}^2$ to denote the compactification of \mathbf{H}^2 by the circle at infinity. The following two lemmas will be useful in Section 3.2 (see [4] §B.4 for a proof).

Lemma 2.2. *Every non-trivial elements of $\pi_1(S)$ are hyperbolic isometry.*

Lemma 2.3. *Suppose that α is a non-trivial simple closed geodesic in S . Then any two lifts of α into \mathbf{H}^2 can not meet in the whole $\overline{\mathbf{H}}^2$.*

The following lemma will be useful in Chapter 4 (see [4] §B.4 for a proof).

Lemma 2.4. *Suppose that α and β are non-intersecting, non-isotopic and non-trivial simple closed curves in a hyperbolic surface (S, σ) . Then the geodesic representatives of α and β are non-intersecting. In particular, if α is in a subsurface F of S , then the geodesic representative of α is in F , too.*

The standard boundary S_{∞}^{n-1} of \mathbf{H}^n is defined by the equivalence classes of geodesic rays (see [4] §A.5). The Teichmüller boundary of $\mathcal{T}(S)$ is defined in the same way. But there are distinct geodesic rays from a point in $\mathcal{T}(S)$ that always remain within a bounded distance of each other (see [66]).

The mapping class group $\mathcal{MCG}(S)$ is the group of orientation preserving diffeomorphisms of S modulo those which are isotopic to identity, i.e.

$$\mathcal{MCG}(S) = \text{Diff}^+(S)/\text{Diff}^0(S).$$

The mapping class group acts by isometries on $\mathcal{T}(S)$ and its quotient space is called *moduli space* of S . See [63] for a picture of moduli space. Kerckhoff proved in his thesis that the action of $\mathcal{MCG}(S)$ does not in general extend to Teichmüller boundary.

Let $\mathcal{V}(S)$ be the set of representations of $\pi_1(S)$ into $PSL(2, \mathbb{C})$ up to conjugacy with compact-open topology. The product $\mathcal{T}(S) \times \mathcal{T}(S)$ can be identified with an open subset of $\mathcal{V}(S)$, consisting of faithful representations whose images are quasi-Fuchsian groups, by Bers simultaneous uniformization. If we fix the first factor, then we get a holomorphic embedding of $\mathcal{T}(S)$ into $\mathcal{V}(S)$ which is called a *Bers slice*. Although this embedding depends on the fixed first factor, there is a biholomorphic mapping between any two slices. The clo-

sure of this slice is compact, and is called a *Bers compactification* (see [7, 74]). But Kerckhoff and Thurston showed that (see Theorem 1 and Theorem 2 of [49])

1. For each genus $g \geq 2$, there are Bers slices for which the canonical homeomorphisms do not extend to homeomorphisms on their compactifications.
2. For $g = 2$, there is a Bers slice for which the action of the mapping class group does not extend continuously to its compactification.

In the famous 1976 preprint, which is published in [105] later, Thurston introduced the space of projective measured laminations on S , which will be denoted by $\mathcal{PML}(S)$, and a compactification of $\mathcal{T}(S)$ whose boundary is equal to $\mathcal{PML}(S)$. Thurston boundary $\mathcal{PML}(S)$ is a natural boundary of $\mathcal{T}(S)$, in the sense that the action of mapping class group extends continuously to the Thurston compactification $\overline{\mathcal{T}}(S) = \mathcal{T}(S) \cup \mathcal{PML}(S)$. Masur([66]) showed that Teichmüller boundary and Thurston boundary are same almost everywhere, but not everywhere.

With this compactification, Thurston classified surface diffeomorphisms as periodic, reducible or pseudo-Anosov, which is a generalization of the well-known classification of elements of $SL(2, \mathbb{Z})$. Thurston boundary was also used by Kerckhoff to solve the Nielsen realization problem, i.e. *every finite subgroup of $MCG(S)$ can be realized as a group of isometries of some hyperbolic structure on S* (see [48] for the proof).

2.2 Measured laminations

In this section we study measured laminations. See [14, 25, 28, 40, 56] and [101] as references. Consider a fixed hyperbolic structure σ on S . A *geodesic lamination* μ is a closed subset of S , which is a disjoint union of simple geodesics which are called leaves of μ . The leaves of a geodesic lamination are complete, i.e. each leaf is either closed or has infinite length in both of its ends, and a geodesic lamination is determined by its support, i.e. a geodesic lamination is a union of geodesics in just one way. Using S_∞^1 , a geodesic lamination on (S, σ) can be naturally related to a geodesic lamination on (S, σ') for any $\sigma' \in \mathcal{T}(S)$. We write $\mathcal{GL}(S)$ to denote the space of geodesic laminations on S , which is equipped with the Hausdorff metric on closed subsets. Note that $\mathcal{GL}(S)$ is compact and therefore, in particular, every infinite sequence of nontrivial simple closed geodesics has a convergent subsequence.

For an arbitrary topological space X , the Chabauty topology on the set of closed subsets of X has the following sub-bases.

- (i) $O_1(K) = \{A \mid A \cap K = \emptyset\}$ where K is compact.
- (ii) $O_2(U) = \{A \mid A \cap U \neq \emptyset\}$ where U is open.

If X is compact and metrizable, in particular for S , the Chabauty topology agrees with the topology induced by the Hausdorff metric. The following lemma will turn out to be useful (see [25] §3.1).

Lemma 2.5 (Geometric Convergence). *Suppose that X is a locally compact metric space. A sequence A_n of closed subsets of X converges to a closed subset A in Chabauty topology if and only if*

- (i) If $x_{n_k} \in A_{n_k}$ converges to $x \in X$ then $x \in A$.
- (ii) If $x \in A$, then there exists a sequence $x_n \in A_n$ which converges to x .

In \mathbf{H}^2 , a geodesic is determined by an element of the open Möbius band

$$M = (S_\infty^1 \times S_\infty^1 - \Delta) / \mathbf{Z}_2,$$

where $\Delta = \{(x, x)\}$ is the diagonal and \mathbf{Z}_2 acts by interchanging coordinates. A geodesic in \mathbf{H}^2 projects to a simple geodesic on S if and only if the covering translates of its pairs of end points never strictly separate each other. Notice that a geodesic lamination could be considered as a closed subset of M . The Chabauty topology on $\mathcal{GL}(S)$ as closed subsets of M is equivalent to the Chabauty topology on $\mathcal{GL}(S)$ as closed subsets of \mathbf{H}^2 . Therefore

Lemma 2.6. *If μ_i converges to μ in $\mathcal{GL}(S)$ with Hausdorff metric topology, then for any geodesic $\ell \subset \mu$, there exist geodesics $\ell_i \subset \mu_i$ which converge to ℓ .*

A geodesic lamination is called *maximal* if each complementary region is isometric to an ideal triangle. A nonempty geodesic lamination is called *minimal* if no proper subset is a geodesic lamination. For example, any simple closed geodesic is a minimal lamination. The following lemma is about the structure of minimal laminations (see [25] §4.2 for a proof).

Lemma 2.7 (Structure of minimal lamination). *If μ is a minimal lamination then either μ is a single geodesic or consists of uncountable leaves.*

The following theorem is about the structure of geodesic laminations.

Theorem 1.6 (Structure of Geodesic Lamination). *A geodesic lamination on S is the union of finitely many minimal sublaminations and of finitely many infinite isolated leaves whose ends spiral along the minimal sublaminations.*

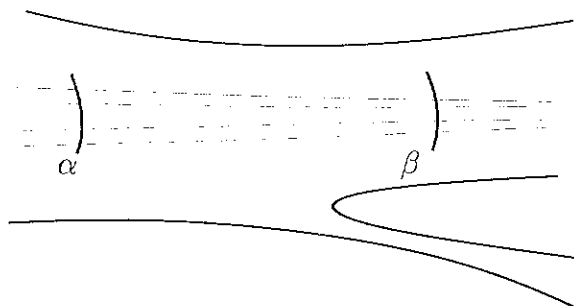


Figure 2.1: α and β have same measure

A transverse measure on a geodesic lamination μ is a rule, which assigns to each transverse arc α a measure that is supported on $\mu \cap \alpha$, which is invariant under a map from α to another arc β if it takes each point of intersection of α with a leaf of μ to a point of intersection β with the same leaf (see Figure 2.1). A measured lamination on S is a geodesic lamination μ with a transverse measure of full support, i.e. if $\alpha \cap \mu \neq \emptyset$ then α has nonzero measure for any transverse arc α . For example, a simple closed geodesic equipped with counting measure is a measured lamination. We write $\mathcal{ML}(S)$ to denote the space of measured laminations on S . There is a natural action of \mathbb{R}^+ on $\mathcal{ML}(S)$. Suppose that $r > 0$. The measured lamination $r\mu$ has the same geodesic lamination as μ with the transverse measure scaled by r . We write $\mathcal{PML}(S)$ to denote the set of equivalence classes of projective measured laminations.

The support of a measured lamination has no infinite isolated leaves, there-

fore by Theorem 1.6, it is a finite disjoint union of minimal sublamination.

For a transverse arc α to a measured lamination μ , let θ be the angle between leaves of μ and α , measured counterclockwise from α to μ . The *total angle* of α is defined by

$$\theta(\alpha, \mu) = \int_{\alpha} \theta d\mu.$$

Thurston gave $\mathcal{ML}(S)$ a topology with following basis,

$$B(\mu, \alpha_1, \dots, \alpha_n, \epsilon) = \{\nu \in \mathcal{ML}(S) :$$

$$|(\mu(\alpha_k), \theta(\alpha_k, \mu)) - (\nu(\alpha_k), \theta(\alpha_k, \nu))| < \epsilon, k = 1, \dots, n\},$$

where $\{\alpha_k\}$ is a finite set of transverse arcs to μ , and $\epsilon > 0$. The following theorem was proved by Thurston.

Theorem 2.8 (Thurston). (1) $\mathcal{ML}(S)$ is homeomorphic to the open ball B^{6g-6} and $\mathcal{PML}(S)$ is homeomorphic to the sphere S^{6g-7} .

(2) $\mathbb{R} \times \mathcal{C}_0(S)$ is dense in $\mathcal{ML}(S)$, and $\mathcal{C}_0(S)$ is dense in $\mathcal{PML}(S)$.

Suppose that $\alpha, \beta \in \mathcal{C}_0(S)$. The *geometric intersection number* $i(\alpha, \beta)$ is the minimal number of intersections of any two their representatives.

For a transverse arc α to $\mu \in \mathcal{ML}(S)$, we write $\int_{\alpha} d\mu$ to denote integration of the transverse measure over α . For a simple closed curve γ , let

$$i(\mu, \gamma) = \inf_{\gamma'} \int_{\gamma'} d\mu,$$

where the infimum is taken over all the simple closed curves γ' which is homo-

topic to γ . For a general transverse arc α , we define

$$i(\mu, \alpha) = \inf_{\alpha'} \int_{\alpha'} d\mu,$$

where the infimum is taken over all the arcs α' which is homotopic to α with endpoints fixed. Note that, in both cases, the infimum is realized by the unique geodesic in the corresponding homotopy class.

Suppose that $\mu \in \mathcal{ML}(S)$, $\gamma \in \mathcal{C}_0(S)$ and $r \in \mathbb{R}^+$, let $i(\mu, r\gamma) = ri(\mu, \gamma)$. The following theorem of Thurston is useful in Chapter 5 (see [104] for a proof).

Theorem 2.9 (Continuity of the intersection number). *The intersection number i extends to a continuous symmetric function on $\mathcal{ML}(S) \times \mathcal{ML}(S)$.*

2.3 Topology of $\overline{\mathcal{T}}(S)$

In this section we study the topology of the Thurston compactification $\overline{\mathcal{T}}(S)$. References are [31, 90] and [104]. The topology on $\overline{\mathcal{T}}(S) = \mathcal{T}(S) \cup \mathcal{PML}(S)$ is determined by the following two properties.

- (P1) $\mathcal{T}(S)$ is open in $\overline{\mathcal{T}}(S)$.
- (P2) $\sigma_i \in \mathcal{T}(S)$ converge to $[\lambda] \in \mathcal{PML}(S)$ if and only if, for all simple closed curves α, β on S with $i(\beta, \lambda) \neq 0$,

$$\frac{\ell_{\sigma_i}(\alpha)}{\ell_{\sigma_i}(\beta)} \text{ converges to } \frac{i(\alpha, \lambda)}{i(\beta, \lambda)},$$

where $\ell_{\sigma_i}(\alpha)$ is the length of closed σ_i -geodesic which is homotopic to α .

The following trivial lemma will turn out to be useful (see [71] for a proof).

Lemma 2.10. *For any infinite sequence of distinct simple closed curves in S , there is a subsequence α_i and $c_i > 0$ such that*

$$c_i \rightarrow 0 \quad \text{and} \quad c_i \alpha_i \rightarrow \mu \quad \text{for some } \mu \in \mathcal{ML}(S) - \{0\}.$$

Suppose that $\mu_i \in \mathcal{ML}(S)$. We write $\mu_i \rightarrow \infty$ to denote that there exists $\alpha \in \mathcal{C}_0(S)$ such that $i(\alpha, \mu_i)$ converges to ∞ . The following theorem is the most useful theorem in this thesis.

Theorem 2.11 (Theorem 2.2 of [104]). *A sequence $\sigma_i \in \mathcal{T}(S)$ converges to $[\lambda] \in \mathcal{PML}(S)$ if and only if there is a sequence $\mu_i \in \mathcal{ML}(S)$ converging projectively to λ such that $\mu_i \rightarrow \infty$ and $\ell_{\sigma_1}(\mu_i) \rightarrow \infty$ but $\ell_{\sigma_i}(\mu_i)$ remains bounded, and for all $\nu \in \mathcal{ML}(S)$, there exists a constant $C > 0$ such that*

$$i(\nu, \mu_i) \leq \ell_{\sigma_i}(\nu) \leq i(\nu, \mu_i) + C \ell_{\sigma_1}(\nu).$$

In particular, for each $\gamma \in \mathcal{C}_0(S)$, there exists a constant $\Gamma > 0$, which does not depend on i such that

$$i(\gamma, \mu_i) \leq \ell_{\sigma_i}(\gamma) \leq i(\gamma, \mu_i) + \Gamma. \tag{2.1}$$

The measured laminations μ_i were constructed in [31], [90] and [103] §9. In these constructions, the associated measured foliations are constructed first, and the measured laminations μ_i are induced later. In the following paragraphs, we study measured foliations and the construction of μ_i following Papadopoulos (see [90] for details). Another good French reference for this

construction is [31].

A measured foliation F on S is a foliation with finite number of singularities equipped with a invariant transverse measure, i.e. F is determined by a finite number of points $p_k \in S$ and an atlas of coordinate neighborhoods

$$(x_i, y_i) : U_i \rightarrow \mathbb{R}^2$$

on the complement of $\{p_k\}$ such that $x_j = f_{ij}(x_i, y_i)$ and $y_j = \pm y_i + C$ for any overlapping coordinate neighborhoods (x_j, y_j) , where C is a constant and the transverse measure is dy . The singularities have p -pronged saddles with $p \geq 3$. Suppose that F is a measured foliation on S and $\alpha \in \mathcal{C}_0(S)$. Let

$$i(F, \alpha) = \inf_{\alpha'} \int_{\alpha'} |dy|,$$

where the infimum is taken over all the representatives α' in the class α . Two measured foliations F and G are called *equivalent* if $i(F, \alpha) = i(G, \alpha)$ for all $\alpha \in \mathcal{C}_0(S)$. We write $\mathcal{MF}(S)$ to denote the set of equivalence classes of measured foliations. Measured foliations and measured laminations are related by the following theorem (see [56] for the proof).

Theorem 2.12. *There is a homeomorphism $h : \mathcal{MF}(S) \rightarrow \mathcal{ML}(S)$ which is identity on $\mathbb{R} \times \mathcal{C}_0(S)$ and preserves the intersection number.*

Suppose that μ is a maximal geodesic lamination on S and $\sigma \in \mathcal{T}(S)$. Thurston constructed a measured foliation $F_\mu(\sigma)$, which is called *horocyclic foliation* as follows. Since μ is maximal, the complementary components of μ are all isometric to ideal triangles. In each of these components, define a

partial foliation, i.e. a foliation whose support is subsurface, whose leaves are intersections of the triangle and horocycle centered at vertices of the triangle (see Figure 2.2).

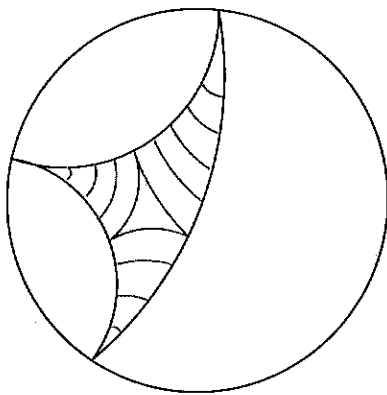


Figure 2.2: The partial foliation

Notice that the horocycle meets the triangle with right angles, and the non-foliated region is equal to a little triangle whose edges are subarcs of horocycles which meet tangentially at their endpoints. These partial foliations in the ideal triangles fit together on the surface and define a partial foliation on S . The transverse measure on this partial foliation is uniquely determined by the fact that on the leaves of μ this transverse measure is equal to the hyperbolic distance.

Suppose that μ is a maximal geodesic lamination on S . Let $\mathcal{MF}(\mu)$ be the subset of $\mathcal{MF}(S)$ consisting of equivalence classes which has representative transverse to μ . In [103] §9, Thurston showed

Theorem 2.13. *The map $\phi_\mu : \mathcal{T}(S) \rightarrow \mathcal{MF}(\mu)$ such that $\phi_\mu(\sigma) = F_\mu(\sigma)$ is a homeomorphism.*

Suppose that $\sigma_i \in \mathcal{T}(S)$ converges to $[\lambda] \in \mathcal{PML}(S)$ in $\overline{\mathcal{T}}(S)$. Then the measured lamination μ_i , obtained from $F_\mu(\sigma_i)$ by Theorem 2.12, is the lamination in Theorem 2.11.

Chapter 3

Complex of Curves

In this chapter we study the complex of curves and its Gromov boundary. All theories in this chapter work for any orientable surface of finite type, i.e. surface with genus g and n punctures. Throughout this chapter, we write $\Sigma = \Sigma_{g,n}$ to denote an orientable surface with genus g and n punctures. As before, we write $\mathcal{MCG}(\Sigma)$ to denote the mapping class group of Σ , and $\mathcal{T}(\Sigma)$ the Teichmüller space of Σ .

3.1 Complex of curves

In [39], Harvey introduced the complex of curve $\mathcal{C}(\Sigma)$ to study the action of $\mathcal{MCG}(\Sigma)$ at the infinity of $\mathcal{T}(\Sigma)$. This complex encodes the asymptotic geometry of Teichmüller space, similarly as the Tits buildings for symmetric spaces. Let $\mathcal{S}(\Sigma)$ be the set of isotopy classes of essential, unoriented, non-boundary parallel simple closed curves in Σ . The vertices of $\mathcal{C}(\Sigma)$ are elements of $\mathcal{S}(\Sigma)$, i.e. $\mathcal{C}_0(\Sigma) = \mathcal{S}(\Sigma)$, and the k -simplices of $\mathcal{C}(\Sigma)$ are subsets $\{\alpha_1, \dots, \alpha_{k+1}\}$ of $\mathcal{S}(\Sigma)$ with mutually disjoint representatives. Notice that $\mathcal{C}(\Sigma)$ is empty if

$g = 0$ and $n \leq 3$. The maximal dimension of simplices is called the *dimension* of $\mathcal{C}(\Sigma)$, and it is equal to $3g + n - 4$. Masur and Minsky defined a metric on $\mathcal{C}(\Sigma)$ by making each simplex regular Euclidean with side length 1 and taking shortest-path metric. Let d_c denote this metric on $\mathcal{C}(\Sigma)$.

The mapping class group acts on $\mathcal{C}(\Sigma)$ and Ivanov proved that, if $g \geq 2$ then all automorphisms of $\mathcal{C}(\Sigma)$ are given by elements of $\mathcal{MCG}(\Sigma)$ (see [44] for the proof, and [52] for the related work of Korkmaz). Luo([59]) generalized this result by showing that, if $3g + n - 4 \geq 1$ and $(g, n) \neq (1, 2)$, then all automorphisms of $\mathcal{C}(\Sigma)$ are given by elements of $\mathcal{MCG}(\Sigma)$. In [38], Harer showed that $\mathcal{C}(\Sigma_{g,n})$ is homotopic to a wedge of spheres of dimension r , where

$$r = \begin{cases} 2g + n - 3 & \text{if } g > 0 \text{ and } n > 0 \\ 2g - 2 & \text{if } n = 0 \\ n - 4 & \text{if } g = 0. \end{cases}$$

If Σ is a torus, once-punctured torus or 4-times punctured sphere, then any two essential simple closed curves intersects, i.e. there is no edge in $\mathcal{C}(\Sigma)$. Masur and Minsky introduced a new definition for these cases so that it has edges. In this definition, $\{\alpha, \beta\}$ is an edge if $\alpha \neq \beta$, and α and β have the lowest possible intersection number. For the tori this is 1, and for 4-holed sphere this is 2 (see [41] for Hatcher-Thurston complex which is related to this definition).

3.2 Relative twist number in annulus complex

In this section we study the relative twist number in annulus complex, which was introduced by Minsky in [30], of two simple closed curves around a fixed simple closed curve. See [30] §2.1, [69] §2.4 and [86] §4 as references. An annular domain in Σ is an annulus with incompressible boundary. A complex, which is called an *annulus complex*, is defined for such annuli to keep track of Dehn twisting around their cores.

Consider an oriented annulus $Y = S^1 \times [0, 1]$. We write $\mathcal{A}_0(Y)$ to denote the set of arcs joining $S^1 \times \{0\}$ to $S^1 \times \{1\}$, up to homotopy with endpoints fixed. As in $\mathcal{C}(\Sigma)$, we put an edge between any two elements of $\mathcal{A}_0(Y)$ which have representatives with disjoint interiors, and define the annular complex $\mathcal{A}(Y)$ as for the complex of curves. We also make $\mathcal{A}(Y)$ a metric space with edge length 1 as in the curve complex. Let d_Y denote the path-metric.

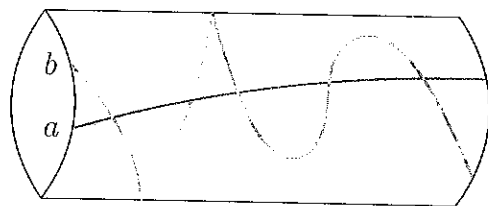


Figure 3.1: $|a \cdot b| = 2$

Suppose that $a, b \in \mathcal{A}_0(Y)$ and they do not share any endpoints. Notice that a and b inherit orientations from the orientation of $[0, 1]$. Therefore we can define the *algebraic intersection number* $a \cdot b$ (for example, see Figure 3.1). Let $a \cdot a = 0$. Consider a lift of $a \in \mathcal{A}_0(Y)$ to the covering space $\tilde{Y} = \mathbb{R} \times [0, 1]$ which has endpoints $(a_0, 0)$ and $(a_1, 1)$. Notice that these endpoints are determined

by a , up to \mathbb{Z} , and

$$a \cdot b = \lfloor b_1 - a_1 \rfloor - \lfloor b_0 - a_0 \rfloor,$$

where $\lfloor x \rfloor$ denotes the largest integer less than or equal to x . It follows that

$$a \cdot c = a \cdot b + b \cdot c + \Delta \quad \text{with } \Delta \in \{0, 1, -1\} \quad (3.1)$$

for all $a, b, c \in \mathcal{A}_0(Y)$ such that the intersection numbers are defined. With an inductive argument, we can also check

$$d_Y(a, b) = 1 + |a \cdot b| \quad \text{for all distinct } a, b \in \mathcal{A}_0(Y). \quad (3.2)$$

Fix $a \in \mathcal{A}_0(Y)$. From eq. (3.1) and eq. (3.2), we can show that the map $f : \mathcal{A}_0(Y) \rightarrow \mathbb{Z}$ with $f(b) = a \cdot b$ is a quasi-isometry. Thus $\mathcal{A}(Y)$ is quasi-isometric to \mathbb{Z} .

For a fixed finite generating set of $\mathcal{MCG}(\Sigma)$, let $\|\cdot\|$ be the minimal word length with respect to these generators. Masur and Minsky introduced the relative twist number, and Farb, Lubotzky and Minsky proved that every Dehn twist has linear growth in $\mathcal{MCG}(\Sigma)$.

Theorem 3.1 (Theorem 1.1 of [30]). *For all Dehn twist t , there exists a constant $c > 0$ such that $\|t^m\| \geq c|m|$ for all m .*

The relative twist number is defined as follows. For a fixed essential simple closed curve α in Σ , let g_α be an isometry of \mathbb{H}^2 representing the conjugacy class of α . Let

$$Y = Y_\alpha = \left(\overline{\mathbb{H}^2} \setminus \text{Fix}(g_\alpha) \right) / \langle g_\alpha \rangle,$$

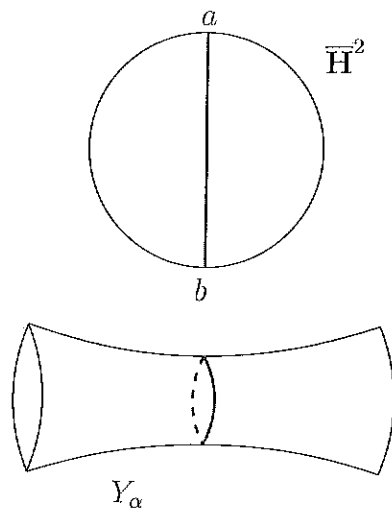


Figure 3.2: Y_α

where $\overline{\mathbf{H}}^2$ is the closed-disk compactification of hyperbolic plane \mathbf{H}^2 . See Figure 3.2, where $\{a, b\} = \text{Fix}(g_\alpha)$. Notice that Y is a closed annulus and a neighborhood of α lifts homomorphically into Y . Suppose that β is an essential simple closed curve in Σ such that $i(\alpha, \beta) \neq 0$. Notice that any lift of α into Y does not share endpoints with a lift of β . Therefore any lift of β extends to a properly embedded arc in Y . We write $\text{lift}_\alpha(\beta)$ to denote the set of lifts of β into Y which connect the two boundaries of Y as elements in $\mathcal{A}_0(Y)$. Let γ be another essential simple closed curve in Σ with $i(\alpha, \gamma) \neq 0$. If β and γ are different, then b and c do not share endpoints for all $b \in \text{lift}_\alpha(\beta)$ and $c \in \text{lift}_\alpha(\gamma)$. The relative twist number is defined by

$$\tau_\alpha(\beta, \gamma) = \{b \cdot c \mid b \in \text{lift}_\alpha(\beta) \text{ and } c \in \text{lift}_\alpha(\gamma)\}.$$

From eq. (3.1), we have $\text{diam}(\tau_\alpha(\beta, \gamma)) \leq 2$. In [30], Farb, Lubotzky and Minsky used the following equations to prove Theorem 3.1.

- (i) If $t = T_\alpha$ is the leftward Dehn twist on α , then $\tau_\alpha(\beta, t^n(\beta)) \subset \{n, n+1\}$.
- (ii) If β and γ intersect α , then their geometric intersection number bounds their relative twisting, i.e.

$$\max |\tau_\alpha(\beta, \gamma)| \leq i(\beta, \gamma) + 1.$$

- (iii) If β, γ and δ intersect α , then

$$\max \tau_\alpha(\beta, \delta) \leq \max \tau_\alpha(\beta, \gamma) + \max \tau_\alpha(\gamma, \delta) + 2$$

$$\min \tau_\alpha(\beta, \delta) \geq \min \tau_\alpha(\beta, \gamma) + \min \tau_\alpha(\gamma, \delta) - 2.$$

3.3 The theorem of Masur and Minsky, and of Klarreich

In this section we study a theorem of Masur and Minsky, and of Klarreich. References are [16, 37, 51, 67] and [78]. If Σ is a torus, once-punctured torus or 4-times punctured sphere, then the complex of curves is the Farey graph (see Figure 3.3). Minsky showed that $(\mathcal{C}(\Sigma), d_{\mathcal{C}})$ is $\frac{3}{2}$ -hyperbolic space for these cases (see [78] §3 for the proof). For $\xi(\Sigma) = 3g + n > 4$, the following lemma gives an upper bound of $d_{\mathcal{C}}$.

Lemma 3.2 (Lemma 1.1 of [16]). *If $\xi(\Sigma) > 4$ then $d_{\mathcal{C}}(\alpha, \beta) \leq i(\alpha, \beta) + 1$.*

Recall the classifications of elements of the mapping class group into periodic, reducible and pseudo Anosov elements. An element $h \in \mathcal{MCG}(\Sigma)$ is called

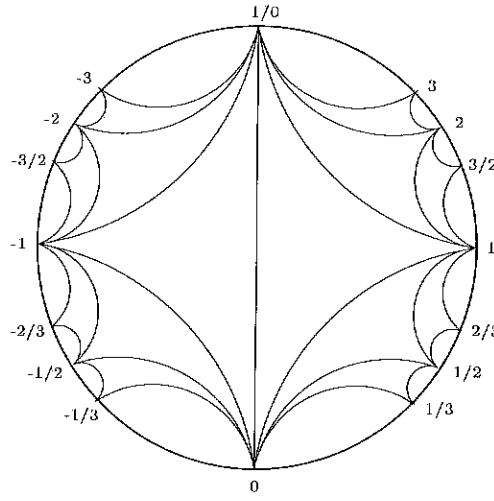


Figure 3.3: Complex of curves of torus

pseudo Anosov if there exist $r > 1$ and a pair of measured foliations \mathcal{F}^s and \mathcal{F}^u such that

$$h(\mathcal{F}^s) = \frac{1}{r} \mathcal{F}^s \quad \text{and} \quad h(\mathcal{F}^u) = r \mathcal{F}^u.$$

If $(\mathcal{C}(\Sigma), d_{\mathcal{C}})$ is a bounded metric space with upper bound K , then it is trivially a K -hyperbolic space. But from the following proposition, it is clear that $\text{diam}(\mathcal{C}(\Sigma)) = \infty$.

Proposition 3.3 (Proposition 4.6 of [68]). *If $\xi(\Sigma) > 4$ then there exists $c > 0$ such that, for any pseudo-Anosov $h \in \mathcal{MCG}(\Sigma)$, $\gamma \in S(\Sigma)$ and $n \in \mathbb{Z}$, we have*

$$d_{\mathcal{C}}(h^n(\gamma), \gamma) \geq c|n|.$$

In [68], Masur and Minsky proved that $(\mathcal{C}(\Sigma), d_{\mathcal{C}})$ is δ -hyperbolic for the case $\xi(\Sigma) = 3g + n > 4$, too.

Theorem 3.4 (Theorem 1.1 of [67]). *$\mathcal{C}(\Sigma_{g,n})$ is a δ -hyperbolic space, where*

δ depends only on g and n .

Masur and Minsky used Teichmüller theory in the proof of Theorem 3.4. In their proof, the constant δ is not constructive because it contains a compactness arguments on the spaces of quadratic differentials. In 2002, Bowditch proved Theorem 3.4 in a more combinatorial way, and showed that the number δ is bounded by a logarithmic function of $3g + n - 4$.

Since $\mathcal{C}(\Sigma)$ is δ -hyperbolic, we can consider its Gromov boundary. For the case of torus, the boundary can be identified with the set of irrational numbers. For general cases, it is clear that a sequence of curves α_n , whose distance from a fixed curve is going to infinity, must converge to a maximal lamination. In fact, Klarreich showed that the Gromov boundary of complex of curves is homeomorphic to the space of topological equivalence classes of filling lamination.

Theorem 1.1 (Klarreich [51]). *There is a homeomorphism*

$$k : \partial_\infty \mathcal{C}(\Sigma) \rightarrow \mathcal{EL}(\Sigma)$$

such that for any sequence α_n in $\mathcal{S}(\Sigma)$, α_n converges to $\alpha \in \partial_\infty \mathcal{C}(\Sigma)$ if and only if α_n , considered as a subset of $\mathcal{UMC}(\Sigma)$, converges to $k(\alpha)$.

Klarreich used Teichmüller theory and the results of Masur and Minsky in [67], to prove Theorem 1.1. It is clear that $\partial_\infty \mathcal{C}(\Sigma)$ is homeomorphic to the Gromov boundary of its 1-skeleton $\mathcal{C}_1(\Sigma)$ because they are quasi-isometric. Suppose that $\epsilon > 0$ satisfies the collar lemma. For each $\alpha \in \mathcal{C}_0(\Sigma)$, let

$$T(\alpha) = \{\sigma \in \mathcal{T}(\Sigma) \mid \ell_\sigma(\alpha) < \epsilon\}.$$

Then a collection of sets $\mathcal{T}(\alpha_1), \dots, \mathcal{T}(\alpha_n)$ has nonempty intersection if and only if $\alpha_1, \dots, \alpha_n$ form a simplex in $\mathcal{C}(S)$. The set $\mathcal{T}_{el}(\Sigma)$ is defined from $\mathcal{T}(\Sigma)$, by adding a new point P_α for each set $T(\alpha)$ and an interval of length $\frac{1}{2}$ from P_α to each point in $T(\alpha)$. $\mathcal{T}_{el}(\Sigma)$ equipped with the minimal path-metric is called the *relative Teichmüller space* following the terminology of Farb([29]). In [67], Masur and Minsky showed that $\mathcal{T}_{el}(\Sigma)$ is quasi-isometric to $\mathcal{C}_1(\Sigma)$, and Klarreich showed that the Gromov boundary of $\mathcal{T}_{el}(\Sigma)$ is homeomorphic to the space of topological equivalence classes of minimal singular foliations on Σ , which is homeomorphic to $\mathcal{EL}(\Sigma)$.

Chapter 4

Proof of Main Theorem (Theorem 1.7)

Suppose that $\sigma_i \in \mathcal{T}(S)$ converges to $[\lambda] \in \mathcal{PML}(S)$ in $\overline{\mathcal{T}}(S)$. Recall the map

$\Phi(\sigma_i) =$ a pants decomposition whose total length is bounded by L ,

where L is a fixed Bers constant and all pants curves are geodesics in σ_i . Recall also the quotient map $u : \mathcal{PML}(S) \rightarrow \mathcal{UML}(S)$ by forgetting measure. Notice that $\Phi(\sigma_i)$ can be considered as a sequence in $\mathcal{GL}(S)$. Since $\mathcal{GL}(S)$ is compact, it has a convergent subsequence. In this chapter we prove our main theorem.

Main Theorem. *If $\Phi(\sigma_i)$ converge to $\nu \in \mathcal{GL}(S)$ in Hausdorff metric topology, then $u([\lambda]) \subset \nu$.*

Consider the decomposition $u([\lambda]) = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_m$ as a finite disjoint union of minimal laminations. To prove $\lambda_j \subset \nu$ for all $1 \leq j \leq m$, it is enough to show that $\lambda_1 \subset \nu$ and $\lambda_2 \subset \nu$, assuming that λ_1 is a simple closed curve and λ_2 is not a simple closed curve.

4.1 Proof of $\lambda_1 \subset \nu$

Recall that λ_1 is a simple closed curve. If $\lambda_1 \subset \Phi(\sigma_i)$ for infinitely many i , then it is clear that $\lambda_1 \subset \nu$. If $\lambda_1 \subset \Phi(\sigma_i)$ for only finitely many i , then there exists $N_1 > 0$ such that $\lambda_1 \not\subset \Phi(\sigma_i)$ for all $i > N_1$. Since $\Phi(\sigma_i)$ is a pants decomposition, there exists a pants curve α_i in $\Phi(\sigma_i)$ such that $\alpha_i \cap \lambda_1 \neq \emptyset$ for all $i > N_1$. Choose $x_i \in \alpha_i \cap \lambda_1$ and a limit point x of x_i . By Lemma 2.5, there exists a leaf ℓ of ν such that $x \in \ell$. If $\ell = \lambda_1$, we are done.

To get a contradiction, suppose that $\ell \neq \lambda_1$. Choose an open neighborhood U of x which is isometric to an open subset of \mathbf{H}^2 (see Figure 4.1).

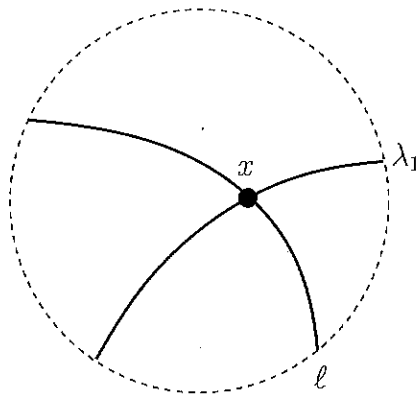


Figure 4.1: The open neighborhood U

Since S is compact, by Lemma 2.6, there exists a pants curve α_i in $\Phi(\sigma_i)$ such that an arc $\beta_i \subset \alpha_i$ approaches to $\ell \cap U$. Therefore there exists $N_2 > 0$ such that

$$i(\alpha_i, \lambda) \geq \int_{\beta_i} d\lambda_1 = r > 0 \quad \text{for all } i > N_2,$$

where $r > 0$ is the transverse measure on λ_1 .

Choose $\mu_j \in \mathcal{ML}(S)$ which converges to λ in $\mathcal{PML}(S)$ as in Theorem

2.11. Since $\mu_j \rightarrow \infty$, there exists a sequence $c_j > 0$ such that $c_j \mu_j$ converges to λ in $\mathcal{ML}(S)$ with $\lim_{j \rightarrow \infty} c_j = 0$. Notice that there exists $N_3 > 0$ which does not depend on i such that

$$i(\alpha_i, c_j \mu_j) \geq \frac{r}{2} \quad \text{for all } i, j > N_3.$$

Therefore $i(\alpha_i, \mu_j)$ approaches to ∞ as $i, j \rightarrow \infty$. But from eq. (2.1), we have $i(\alpha_i, \mu_i) \leq \ell_{\sigma_i}(\alpha_i) \leq L$ for all i . This is a contradiction.

4.2 Proof of $\lambda_2 \subset \nu$

Recall that λ_2 is not a simple closed curve. Let F be the essential subsurface of S which is filled by λ_2 . Since $\Phi(\sigma_i)$ is a pants decomposition for all i , we have $\nu \cap F \neq \emptyset$. To get contradictions, suppose that $\lambda_2 \not\subset \nu$ in the next two paragraphs.

Suppose that $\nu \cap \lambda_2 = \emptyset$. Notice that $\nu \cup \lambda_2$ is a geodesic lamination, too. By Theorem 1.6, we can decompose $\nu \cup \lambda_2$ as a finite disjoint union of minimal lamination, including λ_2 , and finite number of infinite isolated leaves. Since λ_2 is a filling lamination in F , any isolated infinite leaf can not intersect F . Thus $\nu \cap F = \emptyset$. This is a contradiction.

Suppose that $\nu \cap \lambda_2 \neq \emptyset$. There exists a leaf ℓ of ν which intersects λ_2 transversely. Choose an open neighborhood V which is isometric to an open subset of \mathbb{H}^2 , and in which ℓ intersects λ_2 transversely (see Figure 4.2). As in Section 4.1, there exists a pants curve α_i in $\Phi(\sigma_i)$ and arc $\beta_i \subset \alpha_i$ such that

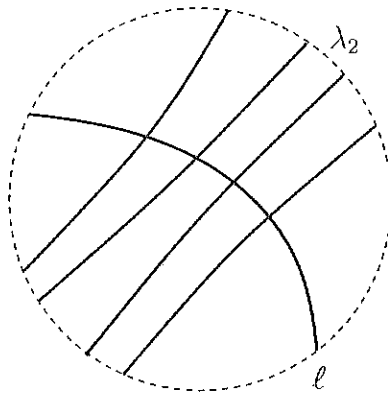


Figure 4.2: The open neighborhood V

β_i converges to $\ell \cap V$. Therefore there exist $r > 0$ and $N > 0$ such that

$$i(\alpha_i, \lambda) \geq \int_{\beta_i} d\lambda_2 = r > 0 \quad \text{for all } i > N.$$

As in Section 4.1, using Theorem 2.11, we can show that this is a contradiction.

Chapter 5

Proof of Corollary 1.8 and Theorem 1.9

In this chapter we prove Corollary 1.8 and Theorem 1.9, and show that Theorem 1.4 comes from Theorem 1.9.

5.1 Proof of Corollary 1.8

Suppose that $\sigma_i \in \mathcal{T}(S)$ converges to $[\lambda] \in \mathcal{PML}(S)$ in $\overline{\mathcal{T}}(S)$, and let $u([\lambda]) = \lambda_1 \cup \lambda_2 \cup \cdots \cup \lambda_m$ be the decomposition as a finite disjoint union of minimal laminations.

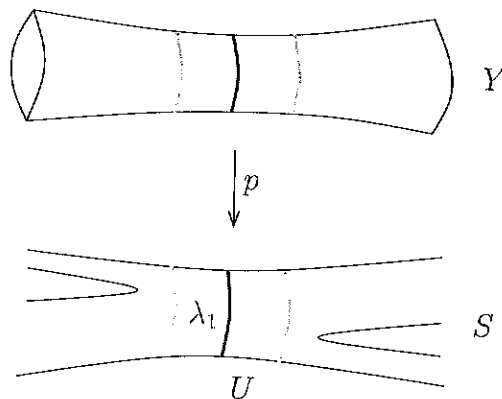


Figure 5.1: The annular cover Y

Suppose that λ_1 is a simple closed curve. Suppose also that α_i is a pants curve in $\Phi(\sigma_i)$ and $\alpha_i \cap \lambda_1 \neq \emptyset$ for all i . Construct an annular covering Y of S in which a neighborhood U of λ_1 lifts homeomorphically (see Figure 5.1). We may assume that U is a closed collar around λ_1 and U does not intersect λ_j for all $j \neq 1$. Let $a_i \in \mathbf{lift}(\alpha_i)$. In this section we prove Corollary 1.8.

Corollary 1.8. $|a_1 \cdot a_i|$ approaches to infinity as i increases.

Proof. To get a contradiction, suppose that $|a_1 \cdot a_i|$ does not approach to ∞ . Then there exists a subsequence of a_i , which we will call a_i again for the sake of simplicity, such that $|a_1 \cdot a_i| = k$ for all i for some $k \in \mathbb{N} \cup \{0\}$. We may assume that $a_1 \cdot a_i = k$ without loss of generality.

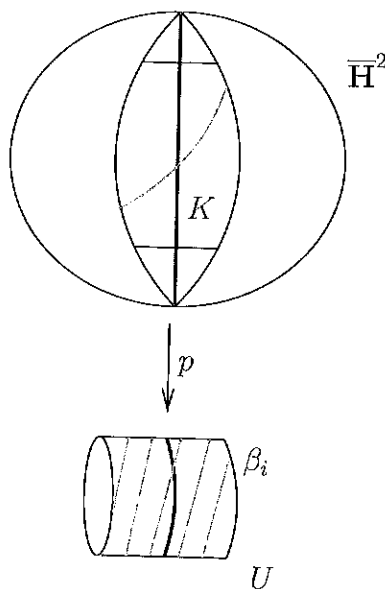


Figure 5.2: A lift of β_i in K

Suppose that $i \geq 3$. From eq. (3.1), we have $a_1 \cdot a_i = a_1 \cdot a_2 + a_2 \cdot a_i + \Delta$,

where $\Delta \in \{-1, 0, 1\}$. Therefore

$$|a_2 \cdot a_i| \leq 1 \quad \text{for all } i \geq 3.$$

This equation implies that $\alpha_i = p(a_i)$ does not change so much in U for $i \geq 2$. Let β_i be a component of $\alpha_i \cap U$ (see Figure 5.2). We can find a compact set $K \subset \mathbf{H}^2$ such that there exists a lift of β_i in K for all i . A geodesic arc in \mathbf{H}^2 is determined by its two endpoints. Therefore there exists a subsequence of β_i , which we will call β_i again, which converges to a geodesic arc β in Hausdorff metric topology with $i(\beta, \lambda_1) \neq 0$. For this subsequence

$$\beta_i \subset \alpha_i \subset \Phi(\sigma_i),$$

$\Phi(\sigma_i)$ still converges to ν in Hausdorff metric topology. Notice that $\lambda_1 \not\subset \nu$. This is a contradiction to Theorem 1.7. \square

5.2 Proof of Theorem 1.9

In this section we prove Theorem 1.9. Suppose that λ_2 is not a simple closed curve and let F be the subsurface of S which is filled by λ_2 (see Figure 5.3). Notice that there exists a pants curve α_i in $\Phi(\sigma_i)$ with $\alpha_i \cap F \neq \emptyset$ for all i . Let β_i be a component of $\alpha_i \cap F$. As in Lemma 2.2 of [69], let

$\tilde{\beta}_i$ be a non-peripheral essential component of boundary
of regular neighborhood of $\beta_i \cup \partial \overline{F}$,

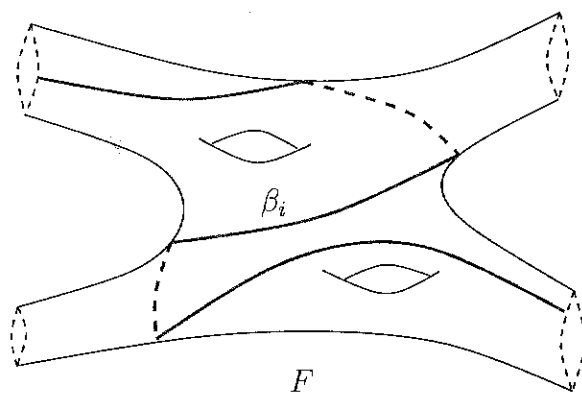


Figure 5.3: β_i

where \overline{F} is the completion of F with path-metric. Note that two different components of $\partial\overline{F}$ could be a same curve in S .

Notice that if β_i is a closed curve, then $\tilde{\beta}_i$ is homotopic to β_i . Notice also that if β_i is an arc, then there are two cases as in Figure 5.4.

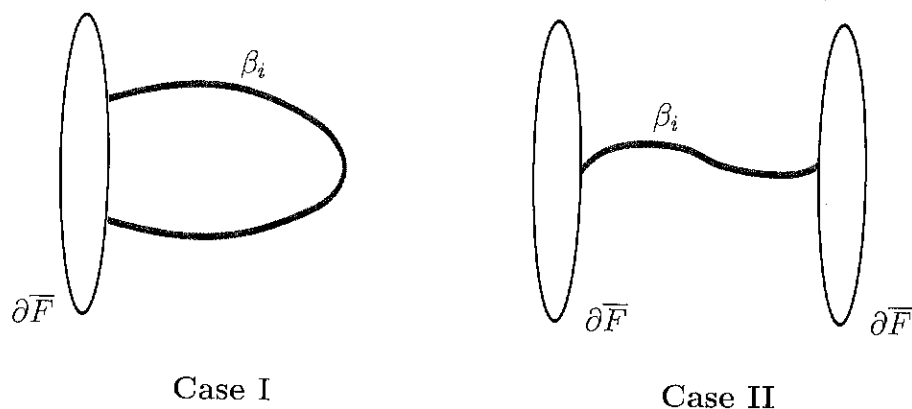


Figure 5.4: Two Cases

Recall that λ_2 is not a closed curve. Since F is filled by λ_2 , it can not be a disk, an annulus or a pants. Therefore for both cases, the regular neighborhood of $\beta_i \cup \partial\overline{F}$ has a boundary component which is non-peripheral and essential in

F . Suppose that $\partial\overline{F} = \{\gamma_1, \dots, \gamma_k\}$ and let

$$\ell_{\sigma_i}(\partial F) = \ell_{\sigma_i}(\gamma_1) + \dots + \ell_{\sigma_i}(\gamma_k).$$

Let $\ell_{\sigma_i}(\tilde{\beta}_i)$ be the length of geodesic representative of $\tilde{\beta}_i$ in σ_i . From the definition of $\tilde{\beta}_i$, we have

$$\ell_{\sigma_i}(\tilde{\beta}_i) < 2\ell_{\sigma_i}(\alpha_i) + \ell_{\sigma_i}(\partial F). \quad (5.1)$$

We now prove Theorem 1.9.

Theorem 1.9. *The geodesic representative of $\tilde{\beta}_i$ converges to λ_2 in $\mathcal{UM}\mathcal{L}(F)$.*

To prove Theorem 1.9, it is enough to show that the geodesic representative of $\tilde{\beta}_i$ converges to λ_2 in $\mathcal{UM}\mathcal{L}(S)$. For any limit point $[\beta]$ of $[\tilde{\beta}_i]$ in $\mathcal{PM}\mathcal{L}(S)$, we will show that $i(\beta, \lambda_2) = 0$. Then Theorem 1.9 follows from the fact that λ_2 fills F .

Lemma 5.1. *If $[\beta]$ is a limit point of $[\tilde{\beta}_i]$ in $\mathcal{PM}\mathcal{L}(S)$, then $i(\beta, \lambda_2) = 0$.*

Proof. Suppose that $[\beta]$ is a limit point of $[\tilde{\beta}_i]$. After possibly restricting to a subsequence, we may assume that $[\tilde{\beta}_i]$ converges to $[\beta]$ in $\mathcal{PM}\mathcal{L}(S)$. There exist a constant $K > 0$ and a sequence $b_i > 0$ such that $b_i\tilde{\beta}_i$ converges to β in $\mathcal{ML}(S)$ with $b_i \leq K$ for all i .

Choose $\mu_i \in \mathcal{ML}(S)$ as in Theorem 2.11. Since μ_i converges to $[\lambda] \in \mathcal{PM}\mathcal{L}(S)$ with $\mu_i \rightarrow \infty$, there exists a sequence $c_i > 0$ such that $c_i\mu_i$ converges to λ in $\mathcal{ML}(S)$ with $c_i \rightarrow 0$. By eq. (2.1), there exists $\Gamma > 0$ which does not

depend on i such that $i(\partial\overline{F}, \mu_i) \leq \ell_{\sigma_i}(\partial\overline{F}) \leq i(\partial\overline{F}, \mu_i) + \Gamma$. Therefore

$$i(\partial\overline{F}, c_i\mu_i) \leq c_i\ell_{\sigma_i}(\partial\overline{F}) \leq i(\partial\overline{F}, c_i\mu_i) + c_i\Gamma.$$

Since $i(\partial\overline{F}, \lambda) = 0$, from the continuity of the intersection number, we have

$$\lim_{i \rightarrow \infty} c_i\ell_{\sigma_i}(\partial\overline{F}) = 0. \quad (5.2)$$

From eq. (2.1) and eq. (5.1), we have

$$i(\tilde{\beta}_i, \mu_i) \leq \ell_{\sigma_i}(\tilde{\beta}_i) \leq 2\ell_{\sigma_i}(\alpha_i) + \ell_{\sigma_i}(\partial\overline{F})$$

Therefore

$$i(b_i\tilde{\beta}_i, c_i\mu_i) \leq K(2c_iL + c_i\ell_{\sigma_i}(\partial\overline{F})).$$

Hence from eq. (5.2) and the continuity of the intersection number, we have $i(\beta, \lambda) = 0$. Thus $i(\beta, \lambda_2) = 0$. \square

5.3 Theorem 1.4 follows from Theorem 1.9

Suppose that $\sigma_i \in \mathcal{T}(S)$ converges to $[\lambda] \in \mathcal{PML}(S)$ in $\overline{\mathcal{T}}(S)$, and α_i is a pants curve in $\Phi(\sigma_i)$. If λ is a filling lamination, from Theorem 1.9, then α_i converges to $u([\lambda])$ in $\mathcal{UML}(S)$. Therefore if we identify $\partial_\infty\mathcal{C}(S)$ with $\mathcal{EL}(S)$ via the homeomorphism k in Theorem 1.1, we have

Theorem 1.4. *If λ is a filling lamination, then α_i converges to $u([\lambda])$ in $\mathcal{C}(S) \cup \partial_\infty\mathcal{C}(S)$.*

Chapter 6

A theorem on discrete, torsion free subgroups of $Isom \mathbf{H}^n$

In this chapter we prove a theorem on discrete, torsion free subgroups of $Isom \mathbf{H}^n$ which is published in *Geometriae Dedicata* 109(2004), 51-57, with kind permission of Springer Science and Business Media. Let \mathbf{H}^n be the hyperbolic n -space with $n \geq 2$. Suppose that $\Gamma < Isom \mathbf{H}^n$ is a discrete, torsion free subgroup and a is a point in the domain of discontinuity $\Omega(\Gamma)$. Let p be the projection map from \mathbf{H}^n to the quotient manifold $M = \mathbf{H}^n/\Gamma$. In this paper we prove that there exists an open neighborhood U of a in $\mathbf{H}^n \cup \Omega(\Gamma)$ such that p is an isometry on $U \cap \mathbf{H}^n$.

6.1 Introduction for Chapter 6

Let $n \geq 2$. Let \mathbf{H}^n be the hyperbolic n -space and $Isom \mathbf{H}^n$ its group of isometries. Suppose that $\Gamma < Isom \mathbf{H}^n$ is a discrete, torsion free subgroup. The action of Γ on \mathbf{H}^n extends to a continuous action on the compactification of \mathbf{H}^n by the sphere at infinity S_∞^{n-1} . The limit set $\Lambda(\Gamma)$ is the set of all

accumulation points of the orbit of a point in \mathbf{H}^n . $\Lambda(\Gamma)$ does not depend on the choice of the point and is a subset of S_∞^{n-1} (see [94], or [64] for the case $n = 3$). The domain of discontinuity $\Omega(\Gamma)$ is the complement of $\Lambda(\Gamma)$ in S_∞^{n-1} .

The quotient space $M = \mathbf{H}^n/\Gamma$ is a manifold because Γ is torsion free and hence acts without fixed points on \mathbf{H}^n (see [5, 94]). Let $p : \mathbf{H}^n \rightarrow M$ be the projection map. Let $d_{\mathbf{H}}$ be the topological metric on \mathbf{H}^n induced by its Riemannian metric and d be the topological metric on M induced by its Riemannian metric. In this paper we prove the following theorem.

Theorem 6.1. *Suppose that $\Gamma < \text{Isom } \mathbf{H}^n$ is a discrete, torsion free subgroup and $a \in \Omega(\Gamma)$. Then there exists an open neighborhood U of a in $\mathbf{H}^n \cup \Omega(\Gamma)$ such that*

$$d(p(x), p(y)) = d_{\mathbf{H}}(x, y) \quad \text{for all } x, y \in U \cap \mathbf{H}^n.$$

The author would like to thank the referee for many valuable suggestions, in particular, those which simplified the proof of Lemma 6.7.

6.2 Proof of Theorem 6.1

Suppose that $n \geq 2$ and $\Gamma < \text{Isom } \mathbf{H}^n$ is a discrete, torsion free subgroup. Let

$$B^n = \{(x_1, \dots, x_n) \in \mathbf{R}^n \mid x_1^2 + \dots + x_n^2 \leq 1\}$$

be the compactification of Poincaré ball model of \mathbf{H}^n by the sphere at infinity S_∞^{n-1} . Let $O = (0, \dots, 0) \in \mathbf{H}^n \subset B^n$ and $\Gamma(O) = \{\gamma O \mid \gamma \in \Gamma\}$. We write d_E to denote the topological Euclidean metric on B^n . We will make use of the following equation (see [94]).

$$d(p(x), p(y)) = \inf_{\gamma \in \Gamma} d_{\mathbf{H}}(x, \gamma y) \quad (6.1)$$

Since Γ is torsion free, we have the following useful theorem (see [94], §12.1).

Theorem 6.2. *Suppose that $a \in \Omega(\Gamma)$. Then there exists open neighborhood V of a in $\mathbf{H}^n \cup \Omega(\Gamma)$ such that $V \cap \gamma V = \emptyset$ for all $\gamma \neq 1$.*

For $a \in \Omega(\Gamma)$, let $B(a, r)$ be the open ball in (B^n, d_E) with center a and radius r . We will need the following trivial lemma.

Lemma 6.3. *Suppose that $a \in \Omega(\Gamma)$ and $C \geq 1$ is any constant. Then there exists open neighborhood U of a in $\mathbf{H}^n \cup \Omega(\Gamma)$ such that*

$$d_E(x, \gamma y) \geq C d_E(x, y) \quad \text{for all } x, y \in U \text{ and } \gamma \neq 1.$$

Proof. Suppose that $a \in \Omega(\Gamma)$. Due to Theorem 6.2, we can choose an open neighborhood V of a in $\mathbf{H}^n \cup \Omega(\Gamma)$ such that $V \cap \gamma V = \emptyset$ for all $\gamma \neq 1$. Choose $r > 0$ such that $B(a, r) \subset V$. Let $U = B(a, \frac{r}{4C})$. Suppose that $x, y \in U$ and $\gamma \neq 1$. Notice that $d_E(a, x) < \frac{r}{4C}$ and $d_E(a, y) < \frac{r}{4C}$. Since $\gamma y \notin V$, we have $d_E(a, \gamma y) > r$. Therefore

$$\begin{aligned} d_E(x, \gamma y) &\geq d_E(a, \gamma y) - d_E(a, x) > r - \frac{r}{4C} = \frac{4Cr - r}{4C} \\ &\geq \frac{4Cr - Cr}{4C} > \frac{2r}{4} > C d_E(x, a) + C d_E(a, y) \geq C d_E(x, y). \end{aligned}$$

Hence

$$d_E(x, \gamma y) \geq C d_E(x, y).$$

□

A Möbius transformation of $\mathbf{R}^n \cup \{\infty\}$ is a finite composition of reflections of $\mathbf{R}^n \cup \{\infty\}$ in spheres or planes. A Möbius transformation of B^n is a Möbius transformation of $\mathbf{R}^n \cup \{\infty\}$ that leaves B^n invariant. We will also make use of the following theorem (see [94], §4.4, §4.5).

Theorem 6.4. *Suppose that $\gamma \in \Gamma$ and $\gamma \neq 1$. Then γ extends to $A_\gamma \circ i_\gamma$, a unique Möbius transformation of B^n (which we also refer to as γ), where A_γ is an orthogonal transformation of \mathbf{R}^n and i_γ is a reflection in a sphere which is orthogonal to S_∞^{n-1} .*

Let $S(c, r)$ be the sphere in \mathbf{R}^n with center c and radius r . We have the following explicit formulae for reflections in spheres (see [94] for a proof).

Theorem 6.5. *Suppose that i is the reflection in the sphere $S(c, r)$. Then for all $x, y \neq c$, we have*

$$i(x) = c + \left(\frac{r}{d_E(x, c)} \right)^2 (x - c) \quad \text{and} \quad d_E(i(x), i(y)) = \frac{r^2 d_E(x, y)}{d_E(x, c) d_E(y, c)}.$$

Since Γ is torsion free and $\Lambda(\Gamma) \subset S_\infty^{n-1}$, we can prove the following proposition.

Proposition 6.6. *Suppose that $\gamma \in \Gamma$ and $\gamma \neq 1$. Let $\gamma = A_\gamma \circ i_\gamma$ be as in Theorem 6.4, where i_γ is the reflection in the sphere $S(c_\gamma, r_\gamma)$ which is orthogonal to S_∞^{n-1} . Then*

$$\sup_{\gamma \neq 1} r_\gamma < \infty.$$

Proof. To get a contradiction, suppose that $\Delta = \{\gamma \in \Gamma \mid \gamma \neq 1, r_\gamma \geq \frac{1}{2}\}$ is an infinite set. Recall that Γ does not contain any element which has a fixed point in \mathbf{H}^n except the identity. Therefore if $\gamma \neq \gamma'$, then $\gamma(O) \neq \gamma'(O)$. Hence

$$\{\gamma O \mid \gamma \in \Delta\} \text{ is an infinite set.} \quad (6.2)$$

Let $\gamma \in \Delta$. Since $S(c_\gamma, r_\gamma)$ is orthogonal to S_∞^{n-1} , we have (see [94], §4.4)

$$d_E(O, c_\gamma)^2 = 1 + r_\gamma^2.$$

Therefore from Theorem 6.5, we have

$$\begin{aligned} i_\gamma(O) &= c_\gamma + \left(\frac{r_\gamma}{d_E(O, c_\gamma)} \right)^2 (-c_\gamma) \\ &= \left(1 - \frac{r_\gamma^2}{d_E(O, c_\gamma)^2} \right) c_\gamma \\ &= \left(1 - \frac{d_E(O, c_\gamma)^2 - 1}{d_E(O, c_\gamma)^2} \right) c_\gamma \\ &= \frac{1}{d_E(O, c_\gamma)^2} c_\gamma. \end{aligned}$$

Hence

$$d_E(O, i_\gamma(O)) = \frac{1}{d_E(O, c_\gamma)}.$$

Any orthogonal transformation is an Euclidean isometry which fixes O . Therefore, for all $\gamma \in \Delta$, from Theorem 6.4 we have

$$\begin{aligned}
d_E(O, \gamma(O)) &= d_E(O, (A_\gamma \circ i_\gamma)(O)) \\
&= d_E(A_\gamma^{-1}(O), i_\gamma(O)) \\
&= d_E(O, i_\gamma(O)) \\
&= \frac{1}{d_E(O, c_\gamma)} \\
&= \frac{1}{\sqrt{r_\gamma^2 + 1}} \\
&\leq \frac{1}{\sqrt{(\frac{1}{2})^2 + 1}} \\
&= \frac{2}{\sqrt{5}}.
\end{aligned}$$

Therefore if $\gamma \in \Delta$, then

$$\gamma(O) \in A = \left\{ x \in B^n \mid d_E(O, x) \leq \frac{2}{\sqrt{5}} \right\}.$$

Thus, by eq. (6.2), $\Gamma(O)$ has an accumulation point in A . This is a contradiction to the fact $\Lambda(\Gamma) \subset S_\infty^{n-1}$. Therefore Δ is a finite set. Hence $\sup_{\gamma \neq 1} r_\gamma < \infty$. \square

We will need the following lemma.

Lemma 6.7. *Suppose that $a \in \Omega(\Gamma)$. Then there exist an open neighborhood of U of a in $\mathbf{H}^n \cup \Omega(\Gamma)$ and a constant $C \geq 1$ such that*

$$\frac{d_E(\gamma x, \gamma b)}{d_E(x, b)} \leq C \quad \text{for all } x \in U \cap \mathbf{H}^n, b \in U \cap \Omega(\Gamma) \text{ and } \gamma \in \Gamma.$$

Proof. Suppose that $a \in \Omega(\Gamma)$. By Theorem 6.2, we can choose an open neighborhood V of a in $\mathbf{H}^n \cup \Omega(\Gamma)$ such that $V \cap \gamma V = \emptyset$ for all $\gamma \neq 1$. Choose $0 < r < 1$ such that $B(a, r) \subset V$. Let $U = B(a, \frac{r}{2})$.

To get a contradiction, suppose that for all $k \in \mathbf{N}$, there exist $x_k \in U \cap \mathbf{H}^n$, $b_k \in U \cap \Omega(\Gamma)$ and $\gamma_k \in \Gamma$ such that

$$\lim_{k \rightarrow \infty} \frac{d_E(\gamma_k x_k, \gamma_k b_k)}{d_E(x_k, b_k)} = \infty. \quad (6.3)$$

We may assume that $\gamma_k \neq 1$ for all $k \in \mathbf{N}$. For each $k \in \mathbf{N}$, let $\gamma_k = A_k \circ i_k$ be as in Theorem 6.4, where i_k is the reflection in the sphere $S(c_k, r_k)$.

By Proposition 6.6, we can choose a constant $C' \geq 1$ such that $r_k \leq C'$ for all $k \in \mathbf{N}$. Therefore from Theorem 6.5, we have

$$\begin{aligned} \frac{d_E(\gamma_k x_k, \gamma_k b_k)}{d_E(x_k, b_k)} &= \frac{d_E((A_k \circ i_k)(x_k), (A_k \circ i_k)(b_k))}{d_E(x_k, b_k)} \\ &= \frac{d_E(i_k x_k, i_k b_k)}{d_E(x_k, b_k)} \\ &= \frac{r_k^2}{d_E(x_k, c_k) d_E(b_k, c_k)} \\ &\leq \frac{C'^2}{d_E(x_k, c_k) d_E(b_k, c_k)}. \end{aligned}$$

Therefore from eq. (6.3), we have

$$\lim_{k \rightarrow \infty} \frac{C'^2}{d_E(x_k, c_k) d_E(b_k, c_k)} = \infty.$$

Thus $\lim_{k \rightarrow \infty} d_E(x_k, c_k) d_E(b_k, c_k) = 0$. Hence there exists a limit point c of $\{c_k\}$ such that

$$c \in \overline{U} \cap \Omega(\Gamma) \subset \Omega(\Gamma). \quad (6.4)$$

Choose a subsequence $c_{k(j)}$ of c_k such that $\lim_{j \rightarrow \infty} d_E(c, c_{k(j)}) = 0$. Since

$$1 \leq \lim_{j \rightarrow \infty} d_E(O, c_{k(j)}) \leq \lim_{j \rightarrow \infty} (d_E(O, c) + d_E(c, c_{k(j)})) = 1$$

and $S(c_{k(j)}, r_{k(j)})$ is orthogonal to S_∞^{n-1} , we have

$$\lim_{j \rightarrow \infty} r_{k(j)}^2 = \lim_{j \rightarrow \infty} (d_E(O, c_{k(j)})^2 - 1) = 0.$$

Therefore by Theorem 6.5, we have

$$\begin{aligned} \lim_{j \rightarrow \infty} \gamma_{k(j)}^{-1}(O) &= \lim_{j \rightarrow \infty} i_{k(j)}^{-1} \circ A_{k(j)}^{-1}(O) = \lim_{j \rightarrow \infty} i_{k(j)}(O) \\ &= \lim_{j \rightarrow \infty} c_{k(j)} + \lim_{j \rightarrow \infty} \left(\frac{r_{k(j)}}{d_E(O, c_{k(j)})} \right)^2 (O - c_{k(j)}) = c \end{aligned}$$

Thus $c \in \Lambda(\Gamma)$ contradicting eq. (6.4). Hence there exists a constant $C \geq 1$ such that

$$\frac{d_E(\gamma x, \gamma b)}{d_E(x, b)} \leq C \quad \text{for all } x \in U \cap \mathbf{H}^n, b \in U \cap \Omega(\Gamma) \text{ and } \gamma \in \Gamma.$$

□

We can prove the following proposition due to Lemma 6.7.

Proposition 6.8. *Suppose that $a \in \Omega(\Gamma)$. Then there exist an open neighborhood U of a in $\mathbf{H}^n \cup \Omega(\Gamma)$ and a constant $C \geq 1$ such that*

$$C(1 - d_E(O, x)^2) \geq 1 - d_E(O, \gamma x)^2 \quad \text{for all } x \in U \cap \mathbf{H}^n \text{ and } \gamma \in \Gamma.$$

Proof. Suppose that $a \in \Omega(\Gamma)$. By Lemma 6.7, there exist open neighborhood U of a and $C' \geq 1$ such that

$$\frac{d_E(\gamma x, \gamma b)}{d_E(x, b)} \leq C' \quad \text{for all } x \in U \cap \mathbf{H}^n, b \in U \cap \Omega(\Gamma) \text{ and } \gamma \in \Gamma. \quad (6.5)$$

Note that we may assume that $\Gamma(O) \cap U = \emptyset$ and the boundary of U is a sphere orthogonal to S_∞^{n-1} , intersected with B^n .

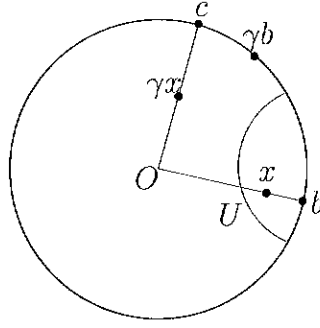


Figure 6.1: Since ∂U is orthogonal to S_∞^{n-1} , we have $b \in U \cap \Omega(\Gamma)$.

Suppose that $x \in U \cap \mathbf{H}^n$ and $\gamma \in \Gamma$. Since $\Gamma(O) \cap U = \emptyset$, we have $x, \gamma x \neq O$. Consider x and γx as vectors in $B^n \subset \mathbf{R}^n$ and let

$$b = \frac{x}{d_E(O, x)} \quad \text{and} \quad c = \frac{\gamma x}{d_E(O, \gamma x)}.$$

so that $b \in U \cap \Omega(\Gamma)$ (see Figure 6.1). Since $\gamma b \in S_\infty^{n-1}$, we have

$$1 - d_E(O, \gamma x) = d_E(\gamma x, c) \quad \text{and} \quad d_E(\gamma x, c) \leq d_E(\gamma x, \gamma b).$$

Therefore by eq. (6.5), we have

$$\begin{aligned}
\frac{1 - d_E(O, \gamma x)^2}{1 - d_E(O, x)^2} &= \frac{(1 + d_E(O, \gamma x))(1 - d_E(O, \gamma x))}{(1 + d_E(O, x))(1 - d_E(O, x))} \\
&\leq 2 \frac{1 - d_E(O, \gamma x)}{1 - d_E(O, x)} \\
&= 2 \frac{d_E(\gamma x, c)}{d_E(x, b)} \\
&\leq 2 \frac{d_E(\gamma x, \gamma b)}{d_E(x, b)} \\
&\leq 2C'.
\end{aligned}$$

Let $C = 2C'$. Then $C(1 - d_E(O, x)^2) \geq 1 - d_E(O, \gamma x)^2$. \square

We will make use of the following theorem (see [94] for a proof).

Theorem 6.9. *Suppose that $x, y \in \mathbf{H}^n \subset B^n$. Then*

$$\cosh d_{\mathbf{H}}(x, y) = 1 + \frac{2d_E(x, y)^2}{(1 - d_E(O, x)^2)(1 - d_E(O, y)^2)}.$$

Now we can prove Theorem 6.1.

Proof of Theorem 6.1. By applying Proposition 6.8 followed by Lemma 6.3, we can choose an open neighborhood U of a in $\mathbf{H}^n \cup \Omega(\Gamma)$ and a constant $C \geq 1$ such that

$$\begin{aligned}
C(1 - d_E(O, y)^2) &\geq 1 - d_E(O, \gamma y)^2 \\
d_E(x, \gamma y)^2 &\geq C d_E(x, y)^2
\end{aligned} \tag{6.6}$$

for all $x, y \in U \cap \mathbf{H}^n$ and $\gamma \neq 1$.

Suppose that $x, y \in U \cap \mathbf{H}^n$ and $\gamma \neq 1$. From eq. (6.6), we have

$$Cd_E(x, \gamma y)^2 (1 - d_E(O, y)^2) \geq Cd_E(x, y)^2 (1 - d_E(O, \gamma y)^2).$$

Hence

$$\frac{d_E(x, \gamma y)^2}{1 - d_E(O, \gamma y)^2} \geq \frac{d_E(x, y)^2}{1 - d_E(O, y)^2}.$$

Therefore

$$\frac{2d_E(x, \gamma y)^2}{(1 - d_E(O, x)^2)(1 - d_E(O, \gamma y)^2)} \geq \frac{2d_E(x, y)^2}{(1 - d_E(O, x)^2)(1 - d_E(O, y)^2)}.$$

Hence from Theorem 6.9, we have $\cosh d_{\mathbf{H}}(x, \gamma y) \geq \cosh d_{\mathbf{H}}(x, y)$. Therefore

$$d_{\mathbf{H}}(x, \gamma y) \geq d_{\mathbf{H}}(x, y). \quad (6.7)$$

Notice that eq. (6.7) is true for $\gamma = 1$, too. Hence $d_{\mathbf{H}}(x, \gamma y) \geq d_{\mathbf{H}}(x, y)$ for all $x, y \in U \cap \mathbf{H}^n$ and $\gamma \in \Gamma$. Therefore from eq. (6.1), we have

$$d(p(x), p(y)) = \inf_{\gamma \in \Gamma} d_{\mathbf{H}}(x, \gamma y) \geq d_{\mathbf{H}}(x, y) \quad \text{for all } x, y \in U \cap \mathbf{H}^n.$$

It is clear from eq. (6.1), that $d(p(x), p(y)) \leq d_{\mathbf{H}}(x, y)$ for any $x, y \in U \cap \mathbf{H}^n$.

Therefore $d(p(x), p(y)) = d_{\mathbf{H}}(x, y)$ for all $x, y \in U \cap \mathbf{H}^n$. ■

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