Conformally Compact Einstein Metrics with Symmetry in Dimension 5

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Abstract of the Dissertation

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We study the Dirichlet problem for conformally compact Einstein metrics on 5-manifolds with globally static isometric circle actions. As an application of our general results we show that any non-flat analytic warped product metric on $S^3 \times S^1$ with non-negative scalar curvature is the conformal infinity of some Einstein metric on $\mathbb{B}^4 \times S^1$.

Contents

1 Introduction

Conformally compact Einstein metrics have recently been of great interest to physicists and mathematicians alike. Understanding the interesting relationship between the geometry of the interior metric of a conformally compact Einstein metric and its conformal boundary is a major issue in the AdS/CFT correspondence, a field that is under intense investigation by string theorists; see Witten [26, 1998] for instance. The mathematical work in this area started with the works of Fefferman and Graham in [16, 1985] who studied the conformal invariants of Riemannian manifolds. In this section we review the basic definitions, results and topics related to this work.

1.1 Conformally compact Einstein manifolds

Let $\bar{N} = N \cup \partial N$ be a connected compact oriented smooth manifold of dimension $(n+1)$ with boundary ∂N. A metric g on N is called *Einstein*, if there exists a constant Λ such that $Ric_g = \Lambda g$.

Definition 1.1.1. A *defining function* for N is a non-negative C^∞ function defined on $\bar{N}$ such that $\rho^{-1}(0) = \partial N$ and $d\rho \neq 0$ on ∂N. □

Definition 1.1.2. (N, g) is called *conformally compact* if there exists a defining function ρ such that $\bar{g} = \rho^2 \cdot g$ extends, at least continuously, to the boundary. The metric g is called $C^{m,\alpha}$ conformally compact if a $C^{m,\alpha}$ extension to the boundary exists, where m is a positive integer and $\alpha \in (0,1)$. □

A *geodesic* defining function is a defining function ρ such that

$$|\bar{\nabla}\rho|_{\bar{g}} = 1 \ . \tag{1.1.1}$$

Any defining function for N induces a boundary metric $\gamma_{\partial N} = \rho^2 g|_{\partial N}$ on ∂N. Conversely, associated to any boundary metric $\gamma_{\partial N}$, there exists a unique geodesic defining function ρ such that $\bar{g} = \rho^2 g$ induces $\gamma_{\partial N}$, *cf.* [18, Lemma 5.2]. If ρ is a geodesic defining function associated with $\gamma_{\partial N}$, then $\bar{g} = \rho^2 g$ is called the geodesic compactification associated with $\gamma_{\partial N}$.

Definition 1.1.3. A metric g on an open manifold N is called AH, or *Asymptotically Hyperbolic*, if the sectional curvatures of two-planes at $x \in N$ converge to -1 as $dist_g(x, x_0) \to \infty$, where $x_0 \in N$ is a fixed point. $\square$

An AH metric which is Einstein at the same time is called AHE. For an AHE metric we have

$$Ric_g = -ng \ . \tag{1.1.2}$$

It is straightforward to show that a C^2 conformally compact Einstein metric that satisfies (1.1.2) is AH. Using this fact, one could construct examples of Einstein metrics which are not conformally compact. For example take any Einstein manifold (N^{n+1}, g) with $Ric_g = -(2n+1)g$, then $N \times N$ with the product metric is Einstein with $Ric_{N\times N} = -(2n+1)g_{N\times N}$ but there are two-planes with sectional curvature $K = 0$ at every point.

Definition 1.1.4. For a positive integer m and $\alpha \in (0,1)$, define $E^{m,\alpha}_{AH}(N)$ to be the set of conformally compact Einstein metrics g on N satisfying (1.1.2) that induce a $C^{m,\alpha}$ metric on the boundary. $\square$

The group $\mathcal{D}_1$ of $C^{m+1,\alpha}$ diffeomorphisms of N that induce the identity on ∂N acts on $E^{m,\alpha}_{AH}(N)$ by usual pullback, and so one can form the moduli space

$$\mathcal{E}^{m,\alpha}_{AH} = E^{m,\alpha}_{AH}/\mathcal{D}_1 \ . \tag{1.1.3}$$

By changing the defining function ρ, the conformal class of the induced boundary metric $\gamma_N = \rho^2 g|_{\partial N}$ does not change. In another words, there is a well defined *boundary map*

$$\Pi : \ E^{m,\alpha}_{AH} \to \mathcal{C}^{m,\alpha}(\partial N) \ , \Pi(g) = [\rho^2 g|_{\partial N}] \tag{1.1.4}$$

where $\mathcal{C}^{m,\alpha}(\partial N)$ is the set of conformal classes of $C^{m,\alpha}$ metrics on ∂N.

An important problem is whether one can solve the Dirichlet problem for this class of metrics, i.e. to find an Einstein metric with a given conformal infinity. This is a difficult analytic problem, since the equations form a non-linear elliptic system which degenerates on the boundary. The moduli spaces $\mathcal{E}_{AH}^{m,\alpha}$, as in (1.1.3), are defined and studied in [3, 7] (and we review them in §2) to approach the Dirichlet problem.

Using inverse function theory arguments, Graham and Lee [18, 1991] proved that if the conformal structure is close enough to that of the round sphere, then an Einstein filling exists, which is unique amongst Einstein metrics close to the Poincaré metric. Biquard [11, 2000] generalized this result to arbitrary non-degenerate Einstein manifolds (these metrics are the regular points of the boundary map; *cf.* §A.3).

On the other hand, uniqueness for the Dirichlet problem fails in general. This was first observed by Hawking and Page [19, 1983]; see also [1] for some examples and discussions.

A more global existence result holds in cases where the boundary metric is conformal to a non-flat metric with non-negative scalar curvature. For example, in [3] it is proved that any such conformal class on S^3 is the conformal infinity of some AH Einstein metric on $\mathbb{B}^4$; see §2.1 for more discussion.

In this work, we are primarily interested in the Dirichlet problem in the context of strictly globally static S^1 actions.

Definition 1.1.5. An S^1 action on (N, g) is called *strictly globally static* if (N, g) has the form

$$N = M \times S^1 \ , \quad g = h + u^2 d\theta^2 \ , \tag{1.1.5}$$

where S^1 action is given by rotations in the S^1 factor and $u : M \to \mathbb{R}$ is a smooth function with $u > 0$. □

Anderson, Chruściel and Delay [8, 9] have studied static circle actions in connection to Lorentzian vacuum solutions of Einstein equations and have proved several local existence results.

Theorem 1.1.6. (Anderson, Chruściel, Delay [9]) *Let (N, g) be a non-degenerate (cf. A.3.2), strictly globally static conformally compact Einstein metric, with conformal infinity $\gamma = [\bar{g}|_{\partial N}]$ (conformal equivalence class). Then any small static perturbation of γ is the conformal infinity of a strictly globally static Riemannian Einstein metric on N.*

On the other hand, when $\dim N = 4$ and $\partial M = S^2$, by existence results of [8], any C^∞ strictly globally static boundary metric γ_N of non-negative scalar curvature is the conformal infinity of a complete Einstein strictly globally static vacuum metric on $\mathbb{R}^3 \times S^1$. Theorem B below is essentially a generalization of this result to dimension 5.

The Einstein equations (1.1.2) on N^{n+1} descend to the following equations on M^n (*cf.* §A.2 for all calculations)

$$\begin{aligned} Ric_h &= \frac{D^2 u}{u} - nh \,, \\ \frac{\Delta_h u}{u} &= n \,. \end{aligned} \tag{1.1.6}$$

These are the Einstein equations on M coupled with a scalar field. It is easy to see that (M, h) has constant scalar curvature $s_h = -n(n-1)$. If (N, g) is conformally compact, one concludes that (M, h) is AH and

$$\frac{D^2 u}{u} = h + O(\rho^2) \,. \tag{1.1.7}$$

The existence of a circle action on the boundary of an AHE metric implies the existence of a circle action in the interior. In fact we have the following Isometry Extension Theorem:

Theorem 1.1.7. (Anderson [3]) *Let (N^{n+1}, g) be a C^2 conformally compact Einstein manifold with C^∞ boundary metric $\gamma_{\partial N}, n \geqslant 3$. Then any connected group G of conformal isometries of $(\partial N, \gamma_{\partial N})$ extends to a group G of isometries of (N, g).*

This theorem implies, in particular, that any AHE manifold (N, g) with the round metric on S^n as its conformal boundary admits an effective isometric $SO(n+1)$ action which implies that (N, g) is the $(n+1)$-dimensional hyperbolic space, *cf.* [3, Theorm B]. Another consequence of this theorem is that any AHE manifold (N, g) with the standard product metric on $S^n \times S^1$ as its conformal infinity admits an effective isometric $SO(n+1) \times S^1$ action.

1.2 The main statements

Our main objective is to obtain existence results in the case of static circle actions in dimension 5. We reviewed a few such results in dimension 4 in the previous subsection. We still lack a general theory in higher dimensions; however, we have the following partial generalization.

Suppose $\dim \partial N$ is even. It follows from [21] that given analytic boundary data on ∂N, there exists a unique smooth polyhomogeneous AHE metric in a neighborhood of ∂N which has the given data as its conformal boundary. The term *polyhomogeneous* means that the compactified metric (with defining function r) has a converging Taylor series near the boundary of the form of an infinite polynomial in r, $r \ln r$ and boundary coordinates.

The question is that whether one can extend the above local solution near the boundary to the entire manifold and obtain global solutions to the Einstein equations. In case of globally static circle actions with analytic boundary metric in dimension 5, we establish similar results to those provided by the theory developed by Anderson in [3, 7] in dimension 4.

We begin describing our main results. Let $\mathcal{C}^0_\omega$ be the space of conformal classes of real analytic and strictly globally static metrics on ∂N which contain a non-flat metric with non-negative scalar curvature, and $\mathcal{E}^0_{1,AH} = \mathcal{E}^0_{S^1,AH} = \Pi^{-1}\mathcal{C}^0_\omega$. Moreover let Π^0 denote the restriction of Π to $\mathcal{E}^0_{1,AH}$.

THEOREM A. *Let $N^5 = M^4 \times S^1$ and the inclusion $i : \partial M \to \bar{M}$ induce a surjection*

$$H_2(\partial M, \mathbb{R}) \to H_2(\bar{M}, \mathbb{R}) \to 0 . \tag{1.2.1}$$

Then the boundary map $\Pi^0 : \mathcal{E}^0_{1,AH} \to C^0_\omega$ is proper.

Using this Theorem, a $\mathbb{Z}$-valued degree for the boundary map Π^0 can be defined as follows:

Definition 1.2.1. Let γ be a regular value of Π^0. Let ind_g denote the maximal dimension of the subspace of $L^{2,2}(N, g)$ on which L is negative-definite, where L is the linearization of the Einstein equations. Define

$$\deg \Pi^0 = \sum_{g_i \in (\Pi^0)^{-1}([\gamma])} (-1)^{ind_{g_i}} . \tag{1.2.2}$$

This definition is independent of the choice of a regular value γ of Π^0, *cf.* [3, Theorem 5.1]. □

One way to obtain existence results is to show that this degree is non-zero; we do this for the manifold $N = \mathbb{H}^5(-1)/\mathbb{Z}$ with the boundary $\partial N = S^3 \times S^1$. The precise definitions of these metrics are given below.

Definition 1.2.2. Think of the hyperbolic space as the upper half space

$$\mathbb{H}^5(-1) = (\mathbb{R}^5)^+ = \{x = (x_1, x_2, x_3, x_4, x_5) | \; x_5 > 0\} , \tag{1.2.3}$$

with the metric

$$g_H = \frac{1}{x_5^2}\{dx_1^2 + \ldots + dx_5^2\} = \frac{1}{x_5^2}\, g_E , \tag{1.2.4}$$

and consider the one parameter group of isometries of $(\mathbb{H}^5, g_H)$ given by dilations

$$Dil = \{\lambda^* : \mathbb{H}^5 \to \mathbb{H}^5 | \; \lambda^*(x) = \lambda x, \lambda \in \mathbb{R}\} . \tag{1.2.5}$$

For $L > 1$ consider the subgroup $(\langle L \rangle, \cdot) \equiv (\mathbb{Z}, +)$ of Dil generated by L, where the identification is given by $L \leftrightarrow 1$. The action of $\langle L \rangle$ on $\mathbb{H}^5(-1)$ is free and, topologically, $\mathbb{H}^5/\mathbb{Z} \equiv \mathbb{R}^4 \times S^1$. Define g_0 to be the hyperbolic metric on this quotient space induced by g_H.

$\square$

The metric g_0 is conformally compact Einstein with conformal infinity equal to $S^3 \times S^1(\log L)$ with the standard product metric. Here is a brief calculation that proves this point. A partial compactification of $\mathbb{H}^5$ is given by

$$\rho(x) = x_5/r(x) \ , \tag{1.2.6}$$

where $r(x)$ is the Euclidean length of x. This function is invariant under the group action and so it gives a compactification of $\mathbb{H}^5/\mathbb{Z}$ with the following conformal boundary

$$\gamma(x) = \frac{1}{r^2(x)}\{dx_1^2 + \ldots + dx_4^2\} = du^2 + g_{S^3(1)} \ , \tag{1.2.7}$$

where $u = \log r$. This metric is a translation invariant product metric on $S^3(1) \times \mathbb{R}$. By identifying $r = 1$ with $r = L$ we get a product metric on $S^3 \times S^1(\log L)$.

As an application of Theorem A, we prove that for $N = \mathbb{R}^5/\mathbb{Z} \simeq \mathbb{R}^4 \times S^1$ as above, the boundary map is surjective. With a slight abuse of notation, let $\mathcal{C}^0_\omega$ denote the connected component containing the standard product metric on $S^3 \times S^1(\log L)$ and $\mathcal{E}^0_{1,AH} = \mathcal{E}^0_{S^1,AH} = \Pi^{-1}\mathcal{C}^0_\omega$ be the connected component containing g_0 and let Π^0 be the restriction of the boundary map to this connected component.

THEOREM B. *For $N = \mathbb{B}^5/\mathbb{Z}$ and $\partial N = S^3 \times S^1$, one has:*

$$deg_N \Pi^0 = 1 \ . \tag{1.2.8}$$

In particular any C^∞ non-flat and non-negative conformal class $[\gamma]$ is the conformal infinity of some conformally compact globally static Einstein metric on $\mathbb{B}^4 \times S^1$.

The structure of this dissertation is as follows. In §2, we consider complete conformally compact Einstein metrics on a 5-manifold that admit an isometric group action. Suppose a group G acts on $\bar{N}$ and consider the space of conformally compact Einstein metrics on N which are G-invariant and induce a G-invariant $C^{m,\alpha}$ boundary metric on ∂N. We denote this space by $E^{m,\alpha}_{G,AH}$ and prove that, if non-empty, this space is an infinite dimensional Banach manifold. Moreover we define $\mathcal{E}^{m,\alpha}_{G,AH}$, the moduli space of G-invariant conformally compact Einstein metrics under diffeomorphisms that fix the boundary and preserve the G-action. We show that, if non-empty, this is a separable C^∞ Banach manifold and the boundary map is Fredholm (*cf.* §2 for exact definitions and details of these statements).

After this general result is proved, we focus on strictly globally static circle actions on N^5. In §3, we consider the interior behavior of the metric and derive a compactness result by studying the Einstein equations coupled with scalar field. For the results in §3 to be correct we need a 'Boundary Condition' on the behavior of the metric near the boundary which gives us a $C^{2,\alpha}$ control over the compactified metric near its conformal infinity. This condition is proved in §4 through Prop. 10 and 11.

In §4, after a brief introduction to the theory of Fuchsian systems, we study the behavior of conformally compact Einstein metrics near the boundary and, at the end, present the proofs of Theorem A and B.

1.3 Conventions and notations

- R denotes the (3,1) tensor (or (4,0) curvature tensor by coupling with the metric) defined as

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \ . \tag{1.3.9}$$

  In addition Ric_g denotes the Ricci curvature, the trace of R, and $s_g = scal_g$ denotes the scalar curvature of the metric g.

- For a (k,0) tensor A, norm of A is denoted by $|A|$ and

$$|A|^2 = \sum |A(e_{i_1}, e_{i_2}, ..., e_{i_k})|^2 , \tag{1.3.10}$$

for *all* ordered possible values of $i_1, \ldots, i_k$ in $\{1, \ldots, \dim\}$, where e_i give an orthonormal basis for the tangent space.

- The divergence of A is denoted by δA and is equal to $tr\nabla A$. The Laplacian $\Delta f = \delta \nabla f$ is defined in such a way that, on $\mathbb{R}$, it gives the usual second derivative.

- The barred (hatted, ...) expressions refer to the barred (hatted, ...) metrics.

- $dist_g(x, y)$ refers to the length of the shortest geodesic connecting x and y. The ball, respectively the sphere, of radius r centered at $x \in M$ is denoted by $B_x^M(r)$, respectively $S_x^M(r)$, and we drop M from this notation when there is no ambiguity.

- For a function ρ on N, let $S(r)$ denote the level set and $B(r)$ denote the upper level set i.e.

$$S(a) = \{x \in N |\ \rho(x) = a\} \ , \quad B(a) = \{x \in N |\ \rho(x) \geqslant a\} \ . \tag{1.3.11}$$

2 The Banach Manifold $\mathcal{E}_{G,AH}$

In this section we give the basic definitions and review the results in dimension 4. In addition we have a brief discussion of boundary regularity issue.

2.1 The Banach manifold $\mathcal{E}_{AH}$ and the boundary map

Throughout this section, we fix a defining function ρ_0 on N. For positive integer k and $0 < \beta < 1$, let $\mathbb{S}_2^{k,\beta}$ be the Banach space of symmetric bilinear forms h on N such that

$$||\rho_0^{-2}h||_{C^{k,\beta}} < C \ , \tag{2.1.1}$$

for some constant $C < \infty$, where the above norm is the usual $C^{k,\beta}$ norm with respect to g. Throughout this section k, m are positive integers and $\alpha, \beta \in (0,1)$ with the conditions $k \geqslant m$ and $k + \beta > m + \alpha$. Let $Met^{m,\alpha}(\partial N)$ be the set of $C^{m,\alpha}$ metrics on ∂N and $\gamma \in Met^{m,\alpha}(\partial N)$.

Definition 2.1.1. For any boundary metric γ, we define a standard corresponding AH metric g_γ on N that induces γ on the boundary as follows. Fix a collar neighborhood U of ∂N and identify U with $[0,1] \times \partial N$ in such a way that the first component is given by ρ_0. For $r = -\log(\rho_0/2)$ let

$$g_U = dr^2 + \sinh^2(r) \cdot \gamma \ . \tag{2.1.2}$$

Next let U' be a thickening of U and let η be a function with the following properties: $\eta = 1$ on U, $\eta = 0$ on $M \setminus U'$ and $d\eta \neq 0$ on $M \setminus U'$. Finally define

$$g_\gamma = (1-\eta) \cdot g_C + \eta \cdot g_U \ , \tag{2.1.3}$$

where g_C is a fixed smooth Riemannian metric on $M \setminus U$. □

Definition 2.1.2. Let $Met^{k,\beta}_{AH}$ denote the set of all metrics on N of the form $g_\gamma + h$ such that $\gamma \in Met^{m,\alpha}(\partial N)$ and $h \in \mathbb{S}^{k,\beta}_2$. Define

$$E^{k,\beta}_{AH} \subset Met^{k,\beta}_{AH} \ , \tag{2.1.4}$$

as the subset of AHE metrics g with $Ric_g = -ng$. In addition let $E^{m,\alpha}_{AH}$ be that subset of $E^{k,\beta}_{AH}$ comprised of metrics which induce $C^{m,\alpha}$ metrics on the boundary. □

In dimension 4, it is proved in [1] that

$$E^{k,\beta}_{AH} = E^{m,\alpha}_{AH} \ . \tag{2.1.5}$$

This statement is not correct in higher dimensions. See §2.3 for a discussion on boundary regularity.

Remark 2.1.3. If there is a compact group action G on $\bar{N} = N \cup \partial N$ and γ is G-invariant, then by requiring the defining function ρ_0, the cut function η and the metric g_C to be G-invariant, the metric g_γ constructed above is G-invariant too; this naturally gives rise to the definition of $Met^{k,\beta}_{G,AH}$ and $E^{k,\beta}_{G,AH}$, *cf.* §2.2.

There are two diffeomorphism groups acting on $E^{k,\beta}_{AH}$ that are of interest to us.

Definition 2.1.4. Let $\mathcal{D}_1$ denote the group of orientation preserving $C^{m+1,\alpha}$ diffeomorphisms that induce identity on the boundary. Let $\mathcal{D}_2$ denote the set of elements $\phi \in \mathcal{D}_1$ that satisfy

$$\lim_{\rho_0 \to 0} \left(\frac{\phi^* \rho_0}{\rho_0}\right) = 1 \ . \tag{2.1.6}$$

□

The group $\mathcal{D}_2$ is a normal subgroup of $\mathcal{D}_1$ and the quotient $\mathcal{D}_1/\mathcal{D}_2$ is naturally isometric to the group of $C^{m,\alpha}$ positive functions on ∂N. The action of the group $\mathcal{D}_i$ on $E^{k,\beta}_{AH}$ is free and proper and the quotient spaces

$$\mathcal{E}^{(i)}_{AH} = E^{k,\beta}_{AH}/\mathcal{D}_i \ , \tag{2.1.7}$$

are C^∞ separable Banach manifolds ($i = 1,2$), with boundary maps:

$$\Pi : \mathcal{E}_{AH}^{(2)} \to Met^{m,\alpha}(\partial N), \quad \Pi[g] = \gamma, \tag{2.1.8}$$

$$\Pi : \mathcal{E}_{AH}^{(1)} \to \mathcal{C}^{m,\alpha}, \quad \Pi[g] = [\gamma], \tag{2.1.9}$$

where $\mathcal{C}^{m,\alpha} = \mathcal{C}$ is the space of conformally equivalent $C^{m,\alpha}$ metrics on the boundary. The boundary maps above are C^∞ and Fredholm of index 0. For the proofs of all the above statements, see [7].

Definition 2.1.5. A linear operator between two Banach spaces is called *Fredholm* if it has finite dimensional kernel and cokernel. The Fredholm index of a Fredholm operator is the difference between the dimension of its kernel and the codimension of its image. A map between two Banach manifolds is a *Fredholm map* if its derivative is a Fredholm operator at every point. □

Let $\mathcal{C}^0 \subset \mathcal{C}$ denote the subset of non-negative conformal classes i.e. classes which contain some non-flat metric with non-negative scalar curvature. Let $\mathcal{E}^0_{AH} = \Pi^{-1}(\mathcal{C}^0)$ and Π^0 denote the restriction of Π to this subset.

Theorem 2.1.6. (Anderson [3]) *Let M be a 4-manifold for which the inclusion $i : \partial M \to \bar{M}$ induces a surjection*

$$H_2(\partial M, R) \to H_2(\bar{M}, R) \to 0 \ . \tag{2.1.10}$$

Then for any $(m,\alpha), m \geqslant 4$, the boundary map $\Pi^0 : \mathcal{E}^0_{AH} \to \mathcal{C}^0$ is proper.

Given this, an integer-valued degree on each component of $\mathcal{E}^0_{AH}$ is defined as in Definition 1.2.2. This degree is computed in the following cases, *cf.* [3].

1. $M = \mathbb{R}^4 \ , \ \partial M = S^3 \ , \ \deg \Pi^0 = 1 \ .$

   This implies that any conformal class $[\gamma] \in \mathcal{C}^0$ on S^3 is the conformal infinity of some AH Einstein metric on $\mathbb{B}^4$.

2. $M = \mathbb{R}^2 \times S^2 \ , \ \partial M = S^1 \times S^2 \ , \ \deg \Pi^0 = 0 \ .$

   In this case Π^0 is not surjective. In fact for large L, the boundary metric

$S^1(L) \times S^2$ is not in the image of Π^0.

3. $M = S^1 \times \mathbb{R}^3$, $\partial M = S^1 \times S^2$, $\deg \Pi^0 = 1$.

   Theorem B is essentially a generalization of this result to dimension 5.

4. $M = \mathbb{CP}^2 \backslash \mathbb{B}^4$, $\partial M = S^3$, $\deg \Pi^0 = 0$.

   Here $\mathbb{CP}^2 \backslash \mathbb{B}^4$ is the disc bundle of degree 1 over S^2.

2.2 Definition and smoothness

In this section we consider the free action of a compact group G on $\bar{N} = N \cup \partial N$.

Definition 2.2.1. Let G be a Lie group. A smooth *group action* on $\bar{N}$ is a smooth map

$$\sigma : \; G \times \bar{N} \to \bar{N}, \quad \sigma(a, x) = x \cdot a \; . \tag{2.2.1}$$

with the following properties: $(x \cdot a_1) \cdot a_2 = x \cdot (a_1 a_2)$ and $x \cdot 1_G = x$, where 1_G is the identity element of group G. The action is called free if $x \cdot a = x$ for all x implies $a = 1_G$. A metric g on $\bar{N}$ is called G-invariant if $\forall a \in G : a^* g = g$, where a^* is the pullback by a. A diffeomorphism $\phi : \bar{N} \to \bar{N}$ is called G-invariant if for every G-invariant metric g, $\phi^* g$ is G-invariant too. A function $f : \bar{N} \to N$ is called G-invariant if $\forall a \in G : a^* f = f \circ a = f$. □

If a group G acts on $\bar{N}$ and the boundary metric $\gamma_{\partial N}$ is invariant under this group action, then the geodesic defining function associated to this G-invariant boundary metric is invariant under the group action too, and so it is a geodesic defining function on the quotient space as well. This can be proved as follows. Since $\gamma_{\partial N}$ is G-invariant, we have $a^* \gamma_{\partial N} = \gamma_{\partial N}$, for any $a \in G$, where a^* denotes the pullback of tensors on N by $a \in Diff(N)$. Hence if ρ is a geodesic defining function associated to $\gamma_{\partial N}$, $a^* \rho$ is a geodesic defining function too. Since geodesic defining function associated to a boundary metric is unique, we conclude $a^* \rho = \rho$ for all $a \in G$ i.e. ρ is G-invariant.

Definition 2.2.2. Let $E^{k,\beta}_{G,AH}$ denote the set of elements of $E^{k,\beta}_{AH}$ that are G-invariant i.e.

$$g \in E^{k,\beta}_{G,AH} \Leftrightarrow g \in E^{k,\beta}_{AH} \wedge (\forall a \in G : a^* g = g) \ . \tag{2.2.2}$$

In a similar manner we define $Met^{k,\beta}_{G,AH}$, the set of G-invariant AH metrics, and $\mathbb{S}^{k,\beta}_{G,2}$, the G-invariant subset of $\mathbb{S}^{k,\beta}_2$. □

Next choose a background G-invariant metric $g_0 \in E^{k,\beta}_{G,AH}$ with a G-invariant boundary metric γ_0. For $\gamma \in Met^{m,\alpha}_{G,AH}(\partial N)$ close to γ_0, following [7], define

$$g(\gamma) = g_0 + \eta(g_\gamma - g_0) \ , \tag{2.2.3}$$

where η is the G-invariant cutoff function in Remark 2.1.3. Now define the Bianchi-gauged Einstein operator at g_0 by

$$\Phi : Met^{m,\alpha}_{AH} \times \mathbb{S}^{m,\alpha}(N) \to \mathbb{S}^{m-2,\alpha}_2(N), \tag{2.2.4}$$

$$\Phi(\gamma, h) = \Phi(g_\gamma + h) = Ric_g + ng + (\delta_g)^* B_{g(\gamma)}(g), \tag{2.2.5}$$

where $B_{g(\gamma)}$ is the Bianchi operator $B_{g(\gamma)}(g) = \delta_{g(\gamma)} g + \frac{1}{2} d(tr_{g(\gamma)} g)$. It is proved in [11] that

$$Z^{k,\beta}_{AH} \equiv \Phi^{-1}(0) \cap \{Ric < 0\} \subset E^{k,\beta}_{AH} \ , \tag{2.2.6}$$

and that $Z^{k,\beta}_{AH}$ provides a local slice to the action of the diffeomorphism group near $g_0 \in \mathcal{E}^{m,\alpha}_{AH}$. The derivative of this map at g_0 with respect to the second factor is the linearized Einstein operator

$$L(h) = D^* D(h) - 2\hat{R}(h) \ , \tag{2.2.7}$$

where $\hat{R}$ is the action of the curvature operator, see (A-39) in the Appendix.

Related to our discussion on group actions, let $Z^{k,\beta}_{G,AH}$ be the G-invariant subset of $Z^{k,\beta}_{AH}$. We first show that the action of G on N induces an action of G on Z_{AH}. To see this, notice that, since g_0 is Einstein, Z_{AH} provides a slice to the action of diffeomorphism group on E_{AH} through g_0, and since g_0 is a fixed point of the action of G, this slice must be invariant under the action of the group G. From [7], we know that $Z^{k,\beta}_{AH}$ is a C^∞ Banach manifold and $Z^{k,\beta}_{G,AH}$ is the fixed point set of the compact group action G. It follows that $Z^{k,\beta}_{G,AH}$ is a C^∞ Banach submanifold of $Z^{k,\beta}_{AH}$.

The set $E^{k,\beta}_{AH}$ differs from $Z^{k,\beta}_{AH}$ by the action of diffeomorphism group, in that, for any $g \in E^{k,\beta}_{AH}$ near g_0, there exists a diffeomorphism ϕ such that $\phi^* g \in Z^{k,\beta}_{AH}$. We prove a similar property for $Z^{k,\beta}_{G,AH}$ in the Lemma below.

Lemma 1. *Suppose $g_0 \in E^{k,\beta}_{G,AH}$, $Isom(g_0) = G$ and $g \in E^{k,\beta}_{G,AH}$ is near g_0, then there exists a G-invariant diffeomorphism ϕ such that $\phi^* g \in Z^{k,\beta}_{G,AH}$.*

Proof. Let $\phi^* g = h \in Z^{k,\beta}_{AH}$. For any $a \in G$ we have

$$(\phi^{-1} a \phi)^*(h) = h \ . \tag{2.2.8}$$

Since $Isom(g_0) = G$ and $\phi^{-1} a \phi$ fixes an element of the slice, we have $\phi^{-1} a \phi \in G$. It follows that the homomorphism $\hat{\phi} : G \to G$ defined by

$$\hat{\phi}(a) = \phi^{-1} a \phi \ , \tag{2.2.9}$$

is well-defined, hence one to one and onto. Then (2.2.8) implies that h is a fixed point of the action as desired. On the other hand (2.2.9) implies that ϕ is in $N(G)$, the normalizer of G in the diffeomorphism group and so, $\forall a \in G$ and for any G-invariant metric g, we have

$$a^*(\phi^* g) = (\phi a)^* g = (b\phi)^* g = \phi^*(b^* g) = \phi^* g \ , \tag{2.2.10}$$

where $b = \phi^{-1} a \phi \in G$. This means that $\phi^* g$ is G-invariant, and so ϕ is G-invariant by definition. ∎

The group $\mathcal{D}_2$ acts on $E^{k,\beta}_{AH}$ and preserves the boundary map in the sense that

$$\forall \phi \in \mathcal{D}_2 : \ \Pi(g) = \Pi(\phi^* g) \ . \tag{2.2.11}$$

With regard to the group action G, we define $\mathcal{D}_2^G$ to be the G-invariant subgroup of $\mathcal{D}_2$. This group acts on $E^{k,\beta}_{G,AH}$ and the discussion above implies that the quotient

$$\mathcal{E}^{(2)}_{G,AH} = E^{k,\beta}_{G,AH}/\mathcal{D}_2^G \ , \tag{2.2.12}$$

is a C^∞ separable Banach manifold, under the condition that there exists a g_0 in this set with $Isom(g_0) = G$. This condition holds if $E^{k,\beta}_{G,AH}$ is non-empty, since the set of such elements are dense and open in $E^{k,\beta}_{G,AH}$. Because of (2.2.11), the boundary map descends to a map

$$\Pi : \mathcal{E}^{(2)}_{G,AH} \to Met^{m,\alpha}_G(\partial N), \Pi([g]) = \gamma \ . \tag{2.2.13}$$

The above discussion leads to the following:

Proposition 2. *The space $\mathcal{E}^{(2)}_{G,AH}$ is a smooth separable Banach manifold, (if non-empty). The boundary map $\Pi : \mathcal{E}^{(2)}_{G,AH} \to Met^{m,\alpha}_G(\partial N)$ is a C^∞ Fredholm map of index 0, and*

$$Ker(D\Pi)_g = K^G_g \ , \tag{2.2.14}$$

where K^G_g is the space of L^2 infinitesimal G-invariant Einstein deformations at g.

Proof. The proof is essentially the same as the proof of [7, Prop. 4.3]; one only needs to replace every set by its G-invariant subset and apply the above discussion.■

Now let $\mathcal{D}_1^G$ be the G-invariant subgroup of $\mathcal{D}_1$. These are $C^{m+1,\alpha}$ G-invariant diffeomorphisms that induce identity on the boundary. It's easy to see that $\mathcal{D}_2^G$ is a normal subgroup of $\mathcal{D}_2^G$ with the quotient group $\mathcal{D}_1^G/\mathcal{D}_2^G$ which is naturally isometric to the positive $C^{m,\alpha}$ G-invariant functions on ∂M. Thus define

$$\mathcal{E}^{(1)}_{G,AH} = E^{k,\beta}_{G,AH}/\mathcal{D}_1^G = \mathcal{E}^{(2)}_{G,AH}/(\mathcal{D}_1^G/\mathcal{D}_2^G) \ . \tag{2.2.15}$$

It's straightforward to show that this is a C^∞ Banach manifold (if nonempty) and the boundary map Π naturally descends to a C^∞ boundary map $\Pi : \mathcal{E}^{(1)}_{G,AH} \to \mathcal{C}_G$ which is Fredholm, of index 0 with kernel as in (2.2.14).

2.3 Boundary regularity

In dimension 4, it is shown in [1] that if g has an $L^{2,p}$ conformal compactification, for some $p > 4$, then it admits a compactification, with the same boundary metric, which is as smooth as the boundary metric. However, in dimension 5, this result is definitely false because, in the asymptotic expansion of the metric, log terms appear from the order 4 on. The Fefferman-Graham expansion of the metric in (total) dimension 5 has the form

$$\bar{g} \sim g_{(0)} + \rho^2 g_{(2)} + \rho^4 (\log \rho) h_{(4)} + \rho^4 g_{(4)} + \dots \tag{2.3.1}$$

While $g_{(2)}$, $h_{(4)}$, the divergence $\delta g_{(4)}$ and $tr(g_{(4)})$ are all determined by $g_{(0)} = \gamma_{\partial N}$, the transverse-traceless part of $g_{(4)}$ is not determined by the boundary metric. At the same time, all other terms in the formal expansion above are determined by $g_{(0)}$ and $g_{(4)}$. The log terms that appear in the above expansion force the metric to be at most $C^{3,\alpha}$ conformally compact. On the positive side, the recent results of Chruściel et al. [13, 2004] show that for smooth boundary metrics, the polyhomogeneous expansion above exists to all orders. Also S. Kichenassamy [21, 2004] proves that, in case of analytic boundary metrics, the expansion (2.3.1) converges to $\bar{g}(x, \rho)$ and $\bar{g}(x, \rho)$ is analytic in ρ and $\rho \log \rho$, where these two are considered as independent parameters.

Although the boundary regularity breaks down in higher dimensions, we have the following Theorem. In the following $\tilde{\mathcal{E}}_{AH}$ is the moduli space of metrics g which are conformally compact with the condition that there exists a compactification $\bar{g}$ which is a smooth function of $(t, t^4 \log t, y)$ where t is the geodesic defining function and y parameterizes the boundary.

Theorem 2.3.1. (Anderson [7]) *Let $\bar{N}$ be a compact oriented $(n+1)$-manifold* *with boundary $\partial N, n > 3$. Then, for a given (m, α), with $2 \leqslant m \leqslant n-1$ the space $\mathcal{E}_{AH}^{m,\alpha}$ (if non-empty) is a smooth infinite dimensional Banach manifold and the boundary map is a smooth Fredholm map of index 0. Moreover the space $\tilde{\mathcal{E}}_{AH}$ is a Fréchet manifold and the boundary map*

$$\Pi : \tilde{\mathcal{E}}_{AH} \to \mathcal{C} = \mathcal{C}^{\infty} \tag{2.3.2}$$

is a smooth Fredholm map of index 0.

The same proof as Proposition 2 gives:

Proposition 2′. *Let $\tilde{\mathcal{E}}_{G,AH}$ be the G-invariant subset of $\tilde{\mathcal{E}}_{AH}$. Then, if nonempty, this is a C^{∞} infinite dimensional Fréchet manifold with the boundary map*

$$\Pi : \tilde{\mathcal{E}}_{G,AH} \to \mathcal{C}_{G}^{\infty}\ , \tag{2.3.3}$$

where $\mathcal{C}_{G}^{\infty}$ is the G-invariant subset of $\mathcal{C}^{\infty}$. The boundary map is smooth and Fredholm of index 0.

3 The Interior Behavior

In this section we study the behavior of AH Einstein metrics in the interior (i.e. away from the boundary) of 5-manifolds with static S^1 actions. The key step, as in dimension 4 (*cf.* [3]), is to control the L^2 norm of the curvature tensor on large balls. Since for W, the Weyl curvature, and z, the trace-free Ricci, we have $|R|^2 = |W|^2 + \frac{1}{2}|z|^2 + 6$, we need a control on the L^2 norm of W and z, and a control on the volume of balls in M. These bounds are obtained in Lemma 4, Proposition 6 and Corollary 7.

3.1 Preliminary Lemmas

In the sequel (N, g) is an AHE metric with a globally static circle action as in (1.1.5) and ρ is the geodesic compactification associated with the boundary metric γ. Throughout this section, until Proposition 9, we assume that $u > 0$ on M. Here and throughout let $B_x^M(R) = B_x(R)$, respectively $B_x^N(R)$, be the ball of radius R centered at x in M, respectively N.

The next Lemma is useful to control u and $|\nabla u|$ on large domains in M. Choose a point $y \in M$ where the minimum of u occurs and set $m = u(y) > 0$. Different choices for y are possible but we fix a y once for all.

Lemma 3. *There exists a constant $\alpha_0 > 0$ such that $\forall x \in M$*

$$|\nabla u(x)| \leqslant \alpha_0 u(x) \cdot (\log \frac{u(x)}{m} + 1) \ . \tag{3.1.1}$$

Moreover, for every $R > 0$, there exists a function $G = G(R)$ such that $\forall x \in B_y(R)$

$$u(x) \leqslant m \cdot G(R) \ , \tag{3.1.2}$$

$$|\nabla u(x)| \leqslant m \cdot G(R) \ . \tag{3.1.3}$$

Proof. Take $\eta(t)$ to be a geodesic in $B_x^N(r)$ of speed one starting at x, and write $\eta(t) = (\phi(t), \theta(t))$, where the curves $\phi(t)$ and $\theta(t)$ are in M and S^1 respectively. Since $|\eta'(t)|^2 = |\phi'(t)|^2 + u^2(\phi(t))|\theta'(t)|^2 = 1$, we have $|\phi'(t)| \leqslant 1$ and $|\theta'(t)| \leqslant 1/u(\phi(t))$. It follows first that the length of the curve $\phi(t)$ is at most r, and secondly since $u(\phi(t)) \geqslant m$,

$$|\theta(r)| \leqslant \int_0^r \frac{dt}{u(\phi(t))} \leqslant \frac{r}{m} \; . \tag{3.1.10}$$

This means that $B_x^N(r) \subset \{(z,\theta) \in M \times S^1 : z \in B_x^M(r), |\theta| \leqslant r/m\}$ and so

$$volB_x^N(r) \leqslant 2 \int_{B_x^M(r)} \frac{r}{m} \cdot u dV_g \leqslant 2rG \cdot volB_x^M(r) \; , \tag{3.1.11}$$

where $G = G(R)$ comes from Lemma 3. This proves one part of (3.1.8). To prove the other inequalities, without loss of generality, by re-scaling u, assume $m = 1$ and notice that $\{(z,\theta) \in B_x^M(r) \times S^1 : |\theta| \leqslant \frac{1}{u(z)}\} \subseteq B_x^N(r+1)$ and so

$$volB_x^M(r) \leqslant volB_x^N(r+1) \; . \tag{3.1.12}$$

Now since (N, g) is Einstein, one has $volB_x^N(r+1) \leqslant V(r)$, where V depends only on r. This proves the second half of (3.1.8). Then (3.1.9) follows from (3.1.11),(3.1.12) and the volume comparison theorem on (N, g). ■

Next we need to prove a non-collapsing result.

Lemma 5. *Let $N = M \times S^1$ be any AH Einstein manifold of dimension 5 as in (1.1.5), with $C^{2,\alpha}$ geodesic compactification $\bar{g} = t^2 \cdot g$ and boundary metric γ_N. In addition assume that*

$$In_{\bar{g}}(\partial N) = dist_{\bar{g}}(\bar{C}, \partial N) \geqslant \bar{\tau} \; , \tag{3.1.13}$$

where $\bar{C}$ is the cutlocus of the boundary, and

$$diam_{\bar{g}} S_N(t_1) \leqslant T \; , \tag{3.1.14}$$

for $t_1 = \bar{\tau}/2$ *and* $S_N(t_1) = \{x \in (N, \bar{g}) : t(x) = t_1\}$. *Then for all* $x \in M$ *with*

$$d \leqslant dist_{\bar{g}}(x, \partial N) \leqslant D \ , \tag{3.1.15}$$

we have

$$volB_x^M(1) \geqslant \nu_0 > 0 \ , \tag{3.1.16}$$

where ν_0 *depends on* $(\partial N, \gamma_N), d, D, T, \bar{\tau}$ *and* $dist_g(x, y)$.

Proof. Define $\bar{E}_N(x,t)$ to be the inward exponential map of $(N, \bar{g})$ at ∂N and $\bar{J}_N(x,t)$ to be its Jacobian. Also let $\tau_0(x)$ be the distance to the cutlocus of $\bar{E}_N$ at $x \in \partial N$. Then using the infinitesimal Bishop-Gromov volume comparison theorem

$$\frac{\bar{J}_N(x,t)}{\tau_0^4(1-(\frac{t}{\tau_0})^2)^4} \uparrow \tag{3.1.17}$$

is monotone non-decreasing in t, for any fixed x. From this it's straightforward to show that

$$\frac{vol_{\bar{g}} S_N(t)}{\bar{\tau}^4(1-(\frac{t}{\bar{\tau}})^2)^4} \uparrow \tag{3.1.18}$$

By taking $t = t_1 = \bar{\tau}/2$, this implies that $vol_{\bar{g}} S_N(t_1) \geqslant (3/4)^4 vol_{\gamma_N} \partial N$, or equivalently

$$vol_N S_N(t_1) \geqslant C \ , \tag{3.1.19}$$

where C depends on γ_N and t_1. Since N is Einstein, there is $c(R)$ such that for $R > 1 : volB_x^N(1) \geqslant c(R) \cdot volB_x^N(R)$. Conditions (3.1.14) and (3.1.15) imply that there exists R such that $S_N(t_1) \subset B_x^N(R-1)$ and so:

$$\begin{aligned} volB_x^N(1) &\geqslant c(R) \cdot volB_x^N(R) \geqslant c(R) \cdot vol(B_x^N(R) \backslash B_x^N(R-2)) \\ &\geqslant \frac{1}{2} c(R) \cdot volS_N(t_1) \geqslant \frac{1}{2} c(R) \cdot C = \nu_0 > 0. \end{aligned} \tag{3.1.20}$$

Now the result follows from (3.1.20) and (3.1.8). ■

3.2 L^2 bounds on Trace-free Ricci and Weyl curvature

We now prove an upper bound on the L^2 norm of the trace-free Ricci. In order to proceed, we impose the following condition on the compactified metric $\bar{g}$ near ∂N; this condition will be removed in the next section (through Propositions 7 and 8).

Boundary Condition. *There exists a geodesic compactification $\bar{g} = \rho^2 g$, and constants $\rho_0, C > 0$ such that for $L = \{x \in N | \; \rho(x) \leqslant \rho_0\}$, the $C^{2,\alpha}$ norm of $\bar{g}$ (with respect to a fixed background metric) is bounded by C for $\alpha \in (0,1)$.*

Notice that the Boundary Condition implies the bounds (3.1.13) and (3.1.14), and so (3.1.16) is valid for $x \in M$ with ν_0 depending on bounds on $dist_{\bar{g}}(x, \partial N)$ and $dist_g(x, y)$.

Definition 3.2.1. The *width* of $(N, \bar{g})$ is

$$Wid_{\bar{g}} = \sup\{dist_{\bar{g}}(x, x_1) | \; x \in N, x_1 \in \partial N\} \ , \tag{3.2.1}$$

i.e. the length of the longest ρ-geodesic from ∂N to points in the interior. □

Proposition 6. *Assume for some $\omega_0 < \infty$,*

$$Wid_{\bar{g}} \leqslant \omega_0 \ . \tag{3.2.2}$$

Then under the Boundary Condition, for any $\rho > 0$ there exists a constant $K = K(\rho, \omega_0)$ such that

$$\int_{B(\rho)} |z|^2 \; d\mathrm{vol}_g \leqslant K \ , \tag{3.2.3}$$

where $B(\rho) = \{x \in M | \; \rho(x) \geqslant \rho\}$.

Proof. Let $S(\rho)$ be the set of points $x \in M$ such that $\rho(x) = \rho$. Recall that $uz = D^2 u - uh$ and $tr_h \; z = \langle z, h \rangle = 0$. One has

$$u|z|^2 = \langle z, uz \rangle = \langle z, D^2 u \rangle = \delta\{z(du, .)\} - \langle \delta z, \nabla u \rangle \ , \tag{3.2.4}$$

where $z(du,.)$ is viewed as a 1-form. Since $\delta z = \delta Ric = \frac{1}{2}ds_g = 0$, divergence theorem implies that

$$\int_{B(\rho)} u|z|^2 = \int_{B(\rho)} \delta\{z(du,.)\} = \int_{S(\rho)} z(du,dr) \ . \tag{3.2.5}$$

Without loss of generality we assume $\rho \leqslant \rho_0/2$, where ρ_0 is provided by the Boundary Condition. If ρ is a geodesic defining function, equation (1.1.6) gives rise to the following equation on $(M, \bar{h} = \rho^2 h)$, *cf.* [10].

$$\bar{R}ic = -(n-2)\frac{\bar{D}^2\rho}{\rho} - (\frac{\bar{\Delta}\rho}{\rho})\bar{h} + z \ . \tag{3.2.6}$$

This implies that on $S(\rho)$

$$z(du,dr) \leqslant |\{\bar{R}ic + 2\frac{\bar{D}^2\rho}{\rho} + (\frac{\bar{\Delta}\rho}{\rho})\bar{h}\}(du,dr)| \leqslant K_1 \cdot \rho|du| \ , \tag{3.2.7}$$

where K_1 is independent of ρ and is provided by the Boundary Condition. Also $volS(\rho) = \rho^{-3}vol_{\bar{h}}S(\rho) \leqslant \rho^{-3}v_1$, for a constant v_1 independent of ρ. From (3.2.5) and (3.2.7) one has

$$m\int_{B(\rho)} |z|^2 \leqslant \int_{B(\rho)} u|z|^2 \leqslant \frac{K_1 v_1}{\rho^2} \cdot \max|du| \leqslant \frac{K_1 v_1}{\rho^2} \cdot m \cdot G(R) \ , \tag{3.2.8}$$

where $G(R)$ comes from Lemma 3 and $R = \max dist(y,x)$ for $x \in B(\rho)$. It is left to show that R is bounded from above. Choose y_1 on the intersection of the geodesic connecting y to ∂N in $(N,\bar{g})$ and $S(\rho_0/2)$. Then for any $x \in B(\rho)$ we have

$$dist(y,x) \leqslant dist(y,y_1) + dist(y_1,x) \leqslant \frac{\omega_0 - \rho_0/2}{\rho_0/2} + \frac{2}{\rho_0}diam_{\bar{g}}S(\rho_0/2) \ . \tag{3.2.9}$$

This gives an upper bound for R and the Proposition follows. ∎

Proposition 6 leads to the following corollary which provides a L^2 control on the Weyl curvature.

Corollary 7. *The Boundary Condition and* (3.2.2) *imply*

$$\int_M |W|^2 \leqslant \lambda_0 < \infty \ , \tag{3.2.10}$$

where λ_0 *depends only on* ω_0.

Proof. We refer to the proof of [3, Proposition 4.2] and point out the difference which is the fact that trace-free Ricci is not zero in our case. The idea is to find two bounds, one bound on the domain $\Omega = B(\rho_0/2)$ and another bound on the complement $\Omega^c = M\backslash B(\rho_0/2)$. In the region Ω^c the Boundary Condition provides an upper bound for $\int |\bar{W}|^2 = \int |W|^2$, while on Ω, we use the Chern-Gauss-Bonnet theorem:

$$\int_\Omega |W|^2 = 8\pi^2\chi(\Omega) - 6vol\Omega + \int_\Omega \frac{1}{2}|z|^2 + \int_{\partial\Omega} B(R, A) \ , \tag{3.2.11}$$

where $B(R, A)$ is some boundary term that depends on the curvature R and A, the second fundamental form of $S(\rho_0/2)$. Again by Boundary Condition this boundary term and $vol\partial\Omega$ are uniformly controlled, and by Proposition 6, the term involving z is also controlled. This proves the assertion. ∎

3.3 A counter example

One may attempt to find an intrinsic upper bound for $\int_M u|z|^2$, i.e. an upper bound that depends only on the conformal boundary, and from there conclude a bound for $\int_M |z|^2$. The example below shows that, in general, this integral is not intrinsic to the conformal boundary.

Proof. Define

$$f(x) = \log \frac{u(x)}{m} . \tag{3.1.4}$$

Since $u(x) \geqslant m$, we have $f(x) \geqslant 0$ and $f(y) = 0$. Since (N, g) is Einstein and $\Delta_N f = 4$ (*cf.* §A.2), following [12, Theorem 6], we conclude that $\forall x \in M$

$$|\nabla f(x)| \leqslant \alpha_0 (f(x) + 1) , \tag{3.1.5}$$

where α_0 is a constant. This is the same as (3.1.1). To prove the second inequality, rewrite the above inequality as

$$|\nabla \{\log(f(x) + 1)\}| \leqslant \alpha_0 . \tag{3.1.6}$$

which implies that $\log(f(x) + 1) \leqslant \alpha_0 \cdot dist_g(x, y) \leqslant R\alpha_0$ and so

$$\forall x \in B_y(R) : \quad u(x) \leqslant m \cdot \exp \circ \exp(R\alpha_0) . \tag{3.1.7}$$

Finally this and (3.1.1) give the upper bound on $|\nabla u|$. ∎

For future purposes, we need a volume comparison result which enables us to compare the volume of balls of different radii in M. Such a result holds on (N, g) by the usual Bishop-Gromov volume comparison theorem [24], since (N, g) is Einstein. Hence to obtain a similar result on (M, h) one only needs to compare the volume of balls in M with the volume of balls in N; this is done in the following Lemma.

Lemma 4. *There exists a function $V(r), r > 0$ such that*

$$\forall x \in B^M_y(R) : \ \frac{volB^N_x(r)}{2r \cdot G(R)} \leqslant volB^M_x(r) \leqslant V(r) , \tag{3.1.8}$$

where $G(R)$ comes from Lemma 3. Moreover for $a > b > 0$, there exists a constant c=c(a,b,R) such that

$$\forall x \in B^M_y(R) : \ volB^M_x(a) \leqslant c \cdot volB^M_x(b) . \tag{3.1.9}$$

Example. Consider $N = \mathbb{R}^2 \times S^3$ with the AdS Schwarzschild metric:

$$g = V^{-1}dr^2 + Vd\theta^2 + r^2 g_{S^3(1)}, \tag{3.3.1}$$

$$V = 1 + r^2 - \frac{2m}{r^2}. \tag{3.3.2}$$

If we denote the largest zero of V by r_0, (3.3.2) gives the metric on the domain $(r_0, \infty) \times S^1 \times S^3$. We define $M = (r_0, \infty) \times S^3$, where $\partial M = S^3(1)$ and $\partial N = S^1(\beta) \times S^3(1)$ with the standard product metric, where

$$\beta = \frac{4\pi r_0}{2 + 4r_0^2} \ . \tag{3.3.3}$$

The function $u = V^{1/2}$ gives the length of the orbits and a simple calculation yields

$$D^2(e_i, e_j) = 0, \ D^2u(e_i, e_i) = u(1 + \frac{2m}{r^4}) \ , \tag{3.3.4}$$

$$D^2u(\nabla r, \nabla r) = u(1 - \frac{6m}{r^4})V^{-1}, \ D^2u(\nabla r, X_i) = 0 \ , \tag{3.3.5}$$

where e_i make an orthonormal frame for g. One has

$$\forall X, Y \perp \nabla r : \ z(X,Y) = \frac{2m}{r^4} g(X,Y), \ z(\nabla r, \nabla r) = -\frac{6m}{r^4} g(\nabla r, \nabla r), \tag{3.3.6}$$

which implies that $|z|^2 = 48\frac{m^2}{r^4}$ and so

$$\int_M u|z|^2 = \int_{r_0}^{\infty} \int_{S^3} V^{1/2}(48\frac{m^2}{r^4})V^{-1/2} r^3 dr d\mu = 12m^2 vol(S^3)/r_0^4 \ . \tag{3.3.7}$$

From (3.3.3) we conclude that two values of m may define the same conformal boundary (i.e. the same β) while the above mentioned integrals have different values.

3.4 Compactness

To be able to use Lemma 5 we need to have (3.1.15) which gives a bound on the distance from the boundary in the compactified metric. Here we prove that there exists $\rho_1 > 0$ such that

$$dist_{\bar{g}}(y, \partial N) \geqslant \rho_1 \ , \tag{3.4.1}$$

where y is chosen to be a point where a minimum of u occurs. The Boundary Condition implies that there exist two constants $c, C > 0$ such that, $\forall x$ with $dist_{\bar{g}}(x, \partial N) \leqslant \rho_0/2$, one has

$$c \leqslant \rho u \ . \tag{3.4.2}$$

At y one has $u = \frac{\partial}{\partial \rho}(\rho u)$ which is bounded by some D near the boundary, by the Boundary Condition. This and the above inequality imply that $\rho(y) \geqslant c/D$.

Proposition 6 and Corollary 7 give L^2 control over the trace-free Ricci and Weyl curvature. In the proof of next theorem, which is our main compactness result in this section, we need L^∞ bounds on the Ricci curvature. At this point it's useful to consider the conformally related metric

$$\hat{h} = uh \ , \tag{3.4.3}$$

whose curvature, as we see in the sequel, is easier to control, because of the simplicity of the equations under this conformal change. In fact the Einstein equations on (N, g) induce the following equations on $(M, \hat{h})$

$$\hat{Ric} = (\frac{3}{2u^2})du \circ du - 6h, \quad \hat{\Delta} \log u = \frac{4}{u} \ . \tag{3.4.4}$$

In the following discussion we re-scale u such that $m = \min u = 1$. Notice that Proposition 6 and Corollary 7 are valid for any $m > 0$.

From (3.1.2) we conclude that the two metrics h and $\hat{h}$ are quasi-isometric on

$B_y(R)$, and so

$$diam_{\hat{h}} B_y(R) \leqslant R_1 \ , \tag{3.4.5}$$

where R_1 depends only on R. By Lemma 5 there is a lower bound on the volume of the unit ball $B_y(1)$ in (M, h). It follows

$$vol_{\hat{h}} B_y(R) \geqslant vol_{\hat{h}} B_y(1) \geqslant vol_h B_y(1) > \nu_0 \ , \tag{3.4.6}$$

for $R > 1$ and ν_0 depending on $dist_{\bar{g}}(y, \partial N)$.

As to the Ricci curvature, (3.4.4) and Lemma 3 give

$$|\hat{Ric}| \leqslant \frac{3}{2u^2}|\hat{\nabla} u|^2 + 6|h| \leqslant \lambda \ , \tag{3.4.7}$$

where λ depends only on R. In order to obtain an upper bound on the L^2 norm of the curvature of $\hat{h}$, we notice that $|\hat{R}|^2 = |\hat{W}|^2 + \frac{1}{2}|\hat{z}|^2 + \frac{1}{24}\hat{s}^2$, where $\hat{z}$ and $\hat{s}$ denote the trace-free Ricci and scalar curvature of $\hat{h}$. Now (3.4.4) and Lemma 3 imply

$$\int_{B_y(R)} (\frac{1}{2}|\hat{z}|^2 + \frac{1}{24}\hat{s}^2) dvol_{\hat{h}} \leqslant C \cdot vol_{\hat{h}} B_y(R) \ , \tag{3.4.8}$$

and Lemma 4 implies $vol_{\hat{h}} B_y(R) \leqslant V$, where C and V depend only on R. It follows

$$\int_{B_y(R)} |\hat{R}|^2 dvol_{\hat{h}} \leqslant \int_{B_y(R)} (|\hat{z}|^2 + \frac{1}{24}\hat{s}^2) dvol_{\hat{h}} + \int_M |\hat{W}|^2 dvol_{\hat{h}} \leqslant \hat{C} \ , \tag{3.4.9}$$

where $\hat{C} = C \cdot V + \lambda_0$ depends on R and ω_0.

We are ready to state the main result of this section.

Theorem 8. *Let $g_i = h_i + u_i^2 d\theta^2 \in E^{m,\alpha}_{S^1,AH}$ be a sequence of metrics on $N^5 = M^4 \times S^1$ that satisfy the Boundary Condition, with $u_i > 0$. In addition assume that the inclusion map $i : \partial M \to M$ induces a surjection*

$$H_2(\partial M, R) \to H_2(\bar{M}, R) \to 0 \ , \tag{3.4.10}$$

and that there is a positive constant $\omega_0 < \infty$ such that

$$Wid_{\bar{g}_i} \leqslant \omega_0. \tag{3.4.11}$$

Let $y_i \in M$ be a point where the minimum of u_i occurs. Then there exists a subsequence of (M, h_i, y_i) which converges to a complete metric (L, h, y_∞) with $y_\infty = \lim y_i$. The convergence is in the C^∞ topology, uniformly on compact subsets, and the manifold L weakly embeds in M,

$$L \subset\subset M \ , \tag{3.4.12}$$

i.e. smooth bounded domains in L embed as such in M. Moreover the manifold $(L \times S^1, h + u^2 d\theta^2)$ is Einstein, where u is defined by $u(\lim x_i) = \lim u_i(x_i)/u_i(y_i)$ for any sequence $x_i \in M$.

Proof. Re-scale u_i so that $u(y_i) = 1$. First we show that, after this re-scaling, the Boundary Condition stays valid (for a possibly smaller value of ρ_0). It's enough to show that $u_i(y_i)$ is away from 0. Take any $x \in S(\rho_0/2)$, where ρ_0 is provided by the Boundary Condition. Then Lemma 3 and the upper bound on the width of $(N, \bar{g}_i)$ imply that

$$m_i = u_i(y_i) \geqslant \beta_0 u_i(x) \ , \tag{3.4.13}$$

where β_0 is independent of i. On the other hand, the Boundary Condition gives a lower bound on $\rho_i u_i$ in the ρ_0-distance of the boundary. It follows that there exists a constant $c > 0$ such that $(\rho_0/2)u_i(x) \geqslant c$. This and the above inequality imply that there is a lower bound on m_i. We divide the rest of the proof into four steps.

Step 1. In this step we show that for each $R > 0$, there exists a subsequence of $(B_{y_i}(R), \hat{h}_i, y_i)$ that converges, in the $C^{1,\alpha}$ topology, to a $C^{1,\alpha}$ Riemannian manifold $(V, \hat{h}, y_\infty), y_i \to y_\infty$. Recall y_i is chosen such that $u_i(y_i) = \min u_i$. The bounds (3.4.11) and (3.4.1) give

$$\rho_1 \leqslant dist_{\bar{g}_i}(y_i, \partial N) \leqslant \omega_0 \ . \tag{3.4.14}$$

The Boundary Condition implies that the bounds (3.1.13) and (3.1.14) are valid for some $\bar{\tau}$ and T. Then (3.4.14) and Lemma 5 imply that (3.1.16) is valid on $B^M_{y_i}(R)$ and hence (3.4.6) follows. Now consider the sequence $(B_{y_i}(R), \hat{h}_i, y_i)$ and observe that (3.4.5)–(3.4.9) imply that a subsequence (denoted again by $\hat{h}_i$) is converging, in Gromov-Hausdorff topology, to a 4-dimensional orbifold $(V, \hat{h}, y_\infty)$ which is a C^0 Riemannian metric and $C^{1,\alpha}$ off the singular points (*cf.* [2, Theorem 2.6]). The topological assumption (3.4.10) is made exactly to rule out orbifold degenerations (*cf.* [3]) and so we conclude that a subsequence of the sequence $(B_{y_i}(R), \hat{h}_i, y_i)$ converges in the $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ Riemannian metric $(V, \hat{h}, y_\infty)$, $y_i \to y_\infty$.

Step 2. Now we proceed to control u further in order to control the Ricci curvature of (M, h). By Lemma 3, both u and $|du|$ are bounded in $B_{y_i}(R)$ and so one has a $C^{1,\alpha}$ bound for $v = \log u$. Step 1 implies that there is a uniform lower bound h_0 on the $C^{1,\alpha}$ harmonic radius of $(B_{y_i}(R), \hat{h}_i)$, (*cf.* [2]). For $x \in B_{y_i}(R)$ take the ball of radius h_0 centered at x in $(M, \hat{h})$. From this, Schauder estimates, applied to the elliptic equation $\hat{\Delta} v = 4\exp(-v)$ (*cf.* (3.4.4)), give a $C^{2,\alpha}$ bound on v and hence on u on a smaller ball (*cf.* [17, Theorem 6.2]) i.e.

$$\forall x \in B_x(r_0/2) : |v|_{2,\alpha} \leqslant C \ , \tag{3.4.15}$$

where C is independent of i, although it depends on r_0 and R. By (1.1.6), this gives a uniform C^0 bound on the Ricci curvature of h_i on $B_{y_i}(R)$

$$|Ric_{h_i}| \leqslant \lambda \ , \tag{3.4.16}$$

for some $\lambda > 0$ independent of i.

Step 3. In this step we use standard elliptic regularity results to draw the promised C^∞ convergence. Proposition 6 and Corollary 7 imply

$$\int_{B_{y_i}(R)} |R_{h_i}|^2 dV_{h_i} \leqslant \Lambda \ , \tag{3.4.17}$$

where Λ is independent of i. Now Lemma 5, (3.4.16) and (3.4.17) imply that a subsequence of $(B_{y_i}(R), h_i, y_i)$ converges in the $C^{1,\alpha}$ topology to a $C^{1,\alpha}$ Riemannian metric $(B_{y_\infty}(R), h, y_\infty)$. It follows that $(B_{y_i}(R) \times S^1, g_i, y_i)$ converge to $(B_{y_\infty}(R) \times S^1, g, y_\infty)$, in $C^{1,\alpha}$ topology, where the metric on $B_{y_i}(R) \times S^1$ is given by $g_i = h_i + u_i^2 d\theta^2$. Since these metrics are Einstein, the $C^{1,\alpha}$ convergence implies C^∞ convergence. Now take $R \to \infty$ and the theorem follows.

Step 4. Finally we prove that the function $u(\lim x_i) = \lim u_i(x_i)$ is well defined and equations (1.1.6) are valid on (L, h). For the sequence $\{y_i\}$ we have $u(y_\infty) = \lim u_i(y_i) = 1$. For any sequence $x_i \to x_\infty$ we have $dist_{h_i}(x_i, y_i) \leqslant T$ for some $T > 0$. The discussion in Step 3 then implies that u and its derivatives are bounded on $B_{y_i}(T)$ and so $\{u_i\}$ is an equicontinuous family of functions on compact sets. It follows that $\lim u_i(x_i)$ is well-defined. As soon as u is defined as a weak limit of u_i, the C^∞ convergence obtained in Step 3 and taking limit from both sides of (1.1.6) imply the last assertion in the Theorem. ■

3.5 Static circle actions with fixed points

So far we have assumed $u > 0$. We can remove this assumption as follows. Define

$$T = \{x \in N | \ u(x) = 0\} \ , \tag{3.5.1}$$

and let $M = (N - T)/S^1$ be the quotient manifold. M is a 4-manifold with boundary T. The set T is the fixed point set of the S^1 action and it is a collection of submanifolds of even codimension in N (odd codimension in M), in another words, it consists of a finite collection of smooth 3-dimensional submanifolds $\Sigma = \cup \Sigma_i$, and a finite collection of smooth curves $\gamma = \cup \gamma_i$ in N. The quotient manifold is topologically a 4-dimensional manifold with boundary $\partial M = \Sigma$, while γ corresponds to singular curves in M. Nevertheless, M has a boundary at infinity $\partial_\infty M$ which is the quotient ∂N under the S^1 action. As an example, consider the AdS Schwarzschild metric of

§3.3. The fixed point set of the S^1 action there is a totally geodesic 3-sphere.

The equations (1.1.6) are still valid on M. In the below discussion $B(\rho_0, \rho) = B(\rho)\backslash B(\rho_0)$.

Proposition 9. *Under the same conditions of Theorem 8, this time with the possibility of $u = 0$, the same conclusion holds for a subsequence of $(B_i(\rho_0, \rho), h_i, x_i)$, where x_i satisfy* (3.1.15), *with the difference that L is a manifold with boundary ∂L.*

Proof. In the sequel we drop the subscript i. The Boundary Condition implies that, for all x with $\rho < \rho_0$

$$c < \rho u < C \ , \tag{3.5.2}$$

where $c, C > 0$ are constants. By taking c smaller, we can assume $c < \rho_0$. Since u is subharmonic, the maximum of u on the set $B(\rho_0)$ is realized by some x_0 with $\rho(x_0) = \rho_0$. Hence the above inequality implies that for *all* x

$$u(x) \leqslant C/\rho_0 + C/\rho(x) \ , \tag{3.5.3}$$

and for x with $\rho(x) \leqslant \rho_0$

$$u(x) \geqslant c/\rho(x) \geqslant c/\rho_0 \ . \tag{3.5.4}$$

Now we show that there exists a constant $a > 0$ such that for all $x \in B(c/2)$ one has

$$B_x(a) \subset B(c) \ . \tag{3.5.5}$$

Take $\eta(t)$ to be a minimizing geodesic of speed one from x to $z \in S(c)$. To prove (3.5.5), it is enough to show that the length of this curve in (M, h) has a lower bound. We can assume that $z = \eta(t)$ is the first point of the intersection of η with $S(c)$. One has

$$t = \int_0^t |\dot{\eta}(s)|_g ds = \int_0^t \frac{1}{\rho} |\dot{\eta}(s)|_{\bar{g}} ds \geqslant \frac{1}{c} L(\eta) \ , \tag{3.5.6}$$

where $L(\eta)$ denotes the length of the curve $\eta(t)$ in $(M, \bar{h})$. Now Boundary Condition provides a lower bound for $L(\eta)$ and (3.5.5) follows.

Equations (3.5.4) and (3.5.5) imply $u \geqslant 1$ on $B_x(a)$ for all $x \in B(c/2)$. As in Lemma 3 we define $f(x) = \log u(x) \geqslant 0$ and use [12, Theorem 6] to conclude that for all x with $\rho(x) \leqslant c/2$

$$|du| \leqslant G'(\rho) , \tag{3.5.7}$$

where G' depends on ρ through (3.5.4). From this, the same proof as in Proposition 6 gives

$$\int_{B(\rho)} u|z|^2 dV_g \leqslant K'(\rho) . \tag{3.5.8}$$

This and (3.5.4) imply that, for $\rho < \rho_0$

$$\int_{B(\rho_0,\rho)} |z|^2 dV_g \leqslant K(\rho) . \tag{3.5.9}$$

In this case, the bound on the Weyl curvature is automatic

$$\int_{B(\rho_0,\rho)} |W_h|^2 dV_h = \int_{B(\rho_0,\rho)} |W_{\tilde{h}}|^2 dV_{\tilde{h}} \leqslant \lambda_0'(\rho) . \tag{3.5.10}$$

The above two inequalities give the bound on the L^2 norm of the curvature on $B(\rho_0, \rho)$. From here on the same proof as in Theorem 8 works. ■

4 The Boundary Behavior

Assume g is an AHE metric on N^{n+1} with boundary metric γ and associated geodesic defining function ρ. Let A denote the second fundamental form of the level sets $S(\rho)$ of ρ and $H = trA$. One has the following system of equations, *cf.* [5].

$$\begin{aligned} \rho\ddot{g}_\rho - n\dot{g}_\rho - 2Hg_\rho &= \rho\{2Ric_\rho - H\dot{g}_\rho + (\dot{g}_\rho)^2\} , & (4.0.1) \\ \rho\dot{H} - H &= \rho|A|^2 , & (4.0.2) \\ \delta A &= -dH , & (4.0.3) \end{aligned}$$

where g_ρ is induced by $\bar{g} = \rho^2 g$ on $S(\rho)$, Ric_ρ is its Ricci curvature and dot denotes the differentiation w.r.t. ρ. In case $n = 4$, the formal solution to this system, given by the Fefferman-Graham expansion, is the following

$$(\bar{g})_\rho \sim g_{(0)} + \rho^2 g_{(2)} + \rho^4(\log\rho)h_{(4)} + \rho^4 g_{(4)} + \dots , \qquad (4.0.4)$$

where $g_{(0)} = \gamma$ is the boundary metric and $g_{(2)}, g_{(4)}, h_{(4)}, \ldots$ are 2-tensors on the boundary. This formal series does not need to converge in general. However, if the boundary metric and the $g_{(4)}$ term above are analytic, the following result holds:

Theorem 4.0.1 (Kichenassamy [21]) *Let $n \geqslant 1$.*

1. *Eq.* (4.0.1) *admits formal solutions involving logarithms of ρ.*

2. *The series converge: solutions are holomorphic functions of ρ and $\rho\ln\rho$, when ρ is small.*

3. *The series contains $n(n+1)/2$ arbitrary coefficients, which can be identified with*

   (a) the $n(n+1)/2$ components of the metric tensor γ_N;

   (b) the $(n^2+n-2)/2$ independent components T_{ij} of the trace-free part of the coefficient of ρ^n; and

(c) the trace τ of the coefficient of ρ^{2n}, where the trace is taken with respect to γ_N.

4. The other two equations in the system reduce the arbitrariness in the solutions: they determine τ and $\nabla^i_{\gamma_N} T_{ij}$, where ∇_{γ_N} is the covariant derivative on ∂N determined by γ_N.

This theorem gives solutions to Einstein equations in a collar neighborhood of the boundary, given the analytic boundary metric $g_{(0)}$ and the trace-free part of $g_{(4)}$ ($trg_{(4)}$ is locally calculable by the boundary metric).

4.1 Fuchsian systems and stability of solutions

Let $u = (u_1, ..., u_m)$ be a function of $(x_1, ..., x_n) \in \Omega \subset \mathbb{C}^n$. Let $A(x)$ be a matrix with entries holomorphic in x. Moreover let $t_0, ..., t_l$ be the time variables which we assume are close to zero in $\mathbb{C}$. Then a *Fuchsian PDE* is a partial differential equation of the type

$$(N + A)u = \sum_{p=0}^{l} t_p f_p(x, t_0, ..., t_p, u, u_x) \ , \tag{4.1.1}$$

where $f_1, ..., f_l$ are holomorphic functions in all their arguments and

$$N = \sum_{i,j} n_{ij} t_i \partial/\partial t_j \ , \tag{4.1.2}$$

where n_{ij} are constants, $0 \leqslant i, j \leqslant p$. Regarding such systems the following existence result holds:

Theorem 4.1.1. (Kichenassamy [20]) *Consider the Fuchsian system* (4.1.1) *where $N = \sum_{k=0}^{l}(t_k + kt_{k-1})\partial/\partial t_k$. Suppose f is analytic near* $(0, 0, 0, 0)$ *without constant terms in $t' = (t_0, ..., t_l)$. Moreover assume that A is a constant matrix with no eigenvalues with negative real part. Then* (4.1.1) *has near the origin exactly one analytic solution which vanishes for $t' = 0$.*

The proof is rather technical; we use the proof of this theorem in order to establish Prop. 10 below.

Fuchsian reduction. From now on we focus on the case $n = 4$ although the discussion is correct in all dimensions. Therefore let g be an AHE metric on N^5 with geodesic defining function r and *analytic* boundary metric $g_{(0)} = \gamma_N$. Since by Theorem 4.0.1, the formal solutions to the system above converge, with coefficients being holomorphic functions of r and $r \ln r$, one writes

$$\bar{g} = \rho^2 g = g_{(0)} + r^2 p \ , \tag{4.1.3}$$

where p is given by

$$p = q + \sum_{k=0}^{7} r^7 (\ln r)^k u_k(r, r \ln r) \ , \tag{4.1.4}$$

for some q that contains all terms of lower total degree in r (including terms up to the fourth order). Let $\vec{u} = (u_0, ..., u_7)$; each u_i is a 2-tensor on ∂N. Decomposing $\vec{u} = \vec{u_1} + \vec{u_2}$ to the sum of traceless part and a part proportional to $g_{(0)}$, we have the following first-order Fuchsian system for an unknown v (defined below),

$$(N + B)\vec{v} = r_0 \phi(x, r_0, r_1, \vec{v}, \partial \vec{v}) \ , \tag{4.1.5}$$

where r_0 and r_1 correspond to r and $r \ln r$ respectively, and $N = r_0 \partial / \partial r_0 + r_1 \partial / \partial r_1$ is a differential operator on the algebra B_1 of formal series in r_0 and r_1 (with analytic coefficients), B is a matrix and

$$\vec{v} = (v_K)_{1 \leqslant K \leqslant 12} = (\vec{u}_1, N\vec{u}_1, (r_0 \partial_i \vec{u}_1)_{1 \leqslant i \leqslant 4}, \vec{u}_2, N\vec{u}_2, (r_0 \partial_i \vec{u}_2)_{1 \leqslant i \leqslant 4}) \ , \tag{4.1.6}$$

where ∂_i represent tangential differentiation.

The equation (4.1.5) is the Fuchsian reduction of (4.0.1). Theorem 4.1.1 establishes the existence of holomorphic solutions for small r. Following a suggestion by Kichenassamy [22], in the following Proposition we prove the stability of solutions.

Proposition 10. *Suppose $g^i_{(0)}$ and $g^i_{(4)}$ are analytic on ∂N and*

$$(g^i_{(0)}, g^i_{(4)}) \to (g_{(0)}, g_{(4)}) \ , \tag{4.1.7}$$

with convergence in the C^2 topology on the boundary to analytic tensors $g_{(0)}, g_{(4)}$. Then the associated solutions $\bar{g}_i$, given by Theorem 4.0.1 converge to a solution $\bar{g}$ as holomorphic functions of x, r and $r \ln r$ (polyhomogeneous convergence). Moreover $\bar{g}_i \to \bar{g}$ in the C^∞ topology of metrics in $(0, \epsilon) \times \partial N$, on compact sets, for some $\epsilon > 0$.

Proof. One needs to verify the polyhomogeneous convergence and the second part of the theorem follows. Thus we need to show that if $(g_{(0)}, g_{(4)})$ and $(g'_{(0)}, g'_{(4)})$ are close in the C^2 topology on ∂N, then $\bar{g}$ and $\bar{g}'$ are close in the polyhomogeneous topology in a collar neighborhood of ∂N in N. The Fuchsian equation (4.1.5) for $(g'_{(0)}, g'_{(4)})$ becomes

$$(N + B)\vec{w} = r_0 \phi(x, r_0, r_1, \vec{w}, \partial \vec{w}) \ , \tag{4.1.8}$$

where $\vec{w}$ is defined in terms of $g'_{(0)}, g'_{(4)}$. Define: $\vec{y} = \vec{v} - \vec{w}$. Comparing (4.1.5) and (4.1.8), we have

$$(N + B)\vec{y} = r_0 f_0(x, r_0, r_1, \vec{v}, \vec{w}, \partial \vec{v}, \partial \vec{w}) = r_0 f_0[\vec{y}] \ . \tag{4.1.9}$$

The function f_0 is a function of $r_0, r_1, \vec{y}$ and $\partial \vec{y}$. Since we need the proof of Theorem 4.1.1 to finish the proof, we briefly sketch the ideas of the proof and refer the reader to the reference [20, Theorem 2.2] for more details. The Fuchsian system (4.1.1) with two time variables $(T, Y) = (r_0, r_1)$ becomes:

$$(N + A)z = f(T, Y, X, z, Dz) \ , \ z(0, 0, X) = z_0(X) \in ker(A) \ , \tag{4.1.10}$$

where $D = D_X$ and $f = T f_0 = 0$ for $T = Y = 0$. Notice that $T = Y = 0$ corresponds to the boundary and z corresponds to $\vec{y}$ in our case, while z_0 corresponds to the boundary data which is solely a function of X (a parameter representing the boundary coordinates). Our goal is to control z in terms of z_0 and constants that depend only on the boundary metric.

The proof is in five steps. Define

$$F[z] = f(T, Y, X, z, Dz) \ . \tag{4.1.11}$$

Step 1. First one shows that the system

$$(N + A)z(T, Y) = k(T, Y) \ , \ z(0, 0) = 0 \ , \tag{4.1.12}$$

where k is analytic and independent of X and vanishes for $T = Y = 0$, has a unique analytic solution, given by

$$z(T, Y) = H[k] = \int_0^1 \sigma^{A-I} k(\sigma T, \sigma(T \ln \sigma + Y)) d\sigma \ . \tag{4.1.13}$$

Therefore rewrite the equation (4.1.10) as $z = H[F[z]]$. Let $u = F[z]$ and consider

$$u = G[u] = F[H[u]] \ . \tag{4.1.14}$$

One proceeds to solve this equation by a fixed point argument.

Step 2. One defines two norms as follows. Assume f is analytic for $X \in C^n$ and $d(X, \Omega) < 2s_0$ and $u \in C^m$ with $|u| < 2R$, for some $s_0, R > 0$, where Ω is a bounded open neighborhood of 0. For a function $u = u(X)$ define the s-norm

$$||u||_s = \sup\{|u(X)| \ : d(X, \Omega) < s\} \ . \tag{4.1.15}$$

For a function $u = u(T, Y, X)$, and $\alpha > 0$, sufficiently small, define the α-norm

$$|u|_\alpha = \sup_{\delta_0(T,Y) < \alpha(s_0 - s),\ 0 \leqslant s < s_0} \left\{ \delta_0^{-1} ||u||_s(T, Y)(s_0 - s)\sqrt{1 - \frac{\delta_0}{\alpha(s_0 - s)}} \right\} \ , \tag{4.1.16}$$

where $\delta_0 = \delta_0(T, Y) = |T| + \theta|Y|$ and $0 < \theta < 1$ is fixed.

Steps 3. One proves the following estimate for the s-norm of $H[u]$ in terms of the α-norm of u

$$||H[u]||_s(T,Y) \leqslant C_1\alpha|u|_\alpha \ , \tag{4.1.17}$$

where C_1 is a constant.

Step 4. Moreover, for $0 < s' < s < s_0$, if $||u||_s, ||v||_s \leqslant R$ one has

$$||F[u]-F[v]||_{s'}(T,Y) \leqslant \frac{C_2\delta_0(T,Y)}{s-s'}||u-v||_s \tag{4.1.18}$$

Step 5. Now assume that $|u|_\alpha, |v|_\alpha < R/(2C_2\alpha)$ and let $G[u] = F[H[u]]$. One proves the following inequality

$$|G[u]-G[v]|_\alpha \leqslant C_3\alpha|u-v|_\alpha \ , \tag{4.1.19}$$

for some constant C_3. Now define $u_0 = 0$ and $u_1 = G[u_0]$ and choose R_0 so that

$$||u_1||_{s_0} \leqslant R_0\delta_0(T,Y) \ , \tag{4.1.20}$$

if $|T| + \theta|Y| = \delta_0$. Now choose α so small that

$$C_3\alpha < 1/2 \text{ and } R_0 s_0 < R/4C_1\alpha \ . \tag{4.1.21}$$

This implies that G is a contraction in the α-norm on the set $\{|u|_\alpha \leqslant R/(4C_1\alpha)\}$ and so by contraction mapping principle, a unique solution exists in this set.

End of proof of Prop. 10. Since y is a solution inside the set above we have

$$|y|_a \leqslant R_0 s_0 \ . \tag{4.1.22}$$

Since the two initial pairs $(g_{(0)}, g_{(4)})$ and $(g'_{(0)}, g'_{(4)})$ are uniformly close in the C^2 topology on ∂N, $u_1 = G[0] = F[0]$ is small and so correspondingly one chooses R_0 small. It follows from the above inequality and definitions in Step 2 of the proof above that

$$||y||_s < C(s, s_0, \delta_0, R_0, a) , \quad (4.1.23)$$

where $C(s_0, \delta_0, R_0, a) = o(R_0)$.

This implies that the u_k terms (and in fact their first derivatives) are uniformly close to each other. Since q involves only a finite number of terms and, by assumption, the first few terms and their derivatives are uniformly close, we conclude that $\bar{g}$ and $\bar{g}'$ are close in the C^0 topology. In another words:

$$||g_{(0)} - g'_{(0)}||_{C^2} + ||g_{(4)} - g'_{(4)}||_{C^2} < \epsilon \Rightarrow ||\bar{g} - \bar{g}'||_{C^0} < \delta(\epsilon) , \quad (4.1.24)$$

and the first part of the theorem follows. The second part of the theorem follows from the first part, since $\ln r$ is a C^∞ function with bounded derivatives away from the boundary, *cf.* (4.1.3) and (4.1.4). Notice that C^0 closeness of these metrics implies their C^∞ closeness since these are solutions to elliptic PDE away from the boundary. ■

A consequence of the above Proposition is that, under the same assumptions, we have $\bar{g}_i \to \bar{g}$ in the $C^{3,\beta}$ topology on ∂N for $\beta \in (0,1)$; this convergence cannot be strengthened to C^4.

4.2 The logarithmic $C^{4,\alpha}$ harmonic radius

In this section we define the logarithmic $C^{4,\alpha}$ harmonic radius of a conformally compact Einstein metric and prove the Boundary Condition.

Definition 4.2.1. Let ρ be a geodesic defining function for an AHE metric g on N^5 and define $\bar{g} = \rho^2 g$. Let $N_\rho = \{x \in N |\ \rho(x) \geqslant \rho\}$ and $N^\rho = \{x \in N |\ \rho(x) \leqslant \rho\}$. Let $\gamma, g_{(2)}, \ldots$ be as in the Fefferman-Graham expansion in (2.3.1). Fix $\rho_0 > 0$ and define a *good r-covering* of N to be a covering $U_s, s \in J = J_1 \cup J_2$ satisfying the following conditions:

- (r1) $\delta = \frac{1}{10}e^{-C}r$ is a Lebesgue number for the covering $\{U_s\}$.
- (r2) $\forall s \in J_1$
  - (r2-1) There are charts $\phi_s : B^4(r) \times [0, r) \to U_s \subset N$ such that the coordinate along $[0, r)$ is just ρ and other coordinates x^i are constant along integral curves of ρ.
  - (r2-2) $\phi_s(B^4(r)) \subset \partial N$ and $\phi_s(B^4(r/10)), s \in J_1$, cover ∂N.
  - (r2-3) $\phi_s^{-1} : U_s \cap \partial N \to B^4(r)$ are harmonic coordinates on the boundary.
  - (r2-4) For all $j = 0, \ldots, 6$ and $\gamma = (\gamma_1, \gamma_2)$ with $|\gamma| = j, |\gamma_1| \leqslant 4$ and $\partial^\gamma = \partial_\rho^{\gamma_1}\partial_x^{\gamma_2}$:

$$\sup_{U_s} r^{j+\alpha}||\partial^\gamma f||_\alpha \leqslant C \ , \tag{4.2.1}$$

    where $f = \bar{g}_\rho - \gamma - \rho^2 g_{(2)} - (\rho^4 \log \rho) h_{(4)}$.

- (r3) $\forall s \in J_2$
  - (r3-1) $U_s \subset N_{r/10}$.
  - (r3-2) There are charts $\phi_s : B^5(r) \to U_s \subset N$ such that $\phi_s^{-1} : U_s \to B^5(r)$ are harmonic coordinates.
  - (r3-3) For all $j = 0, \ldots, 4$ and γ with $|\gamma| = j$

$$\sup_{U_s} r^{j+\alpha}||\partial^\gamma \bar{g}||_\alpha \leqslant C \ . \tag{4.2.2}$$

- (r4) $\forall s \in J :\ |d\phi_s^{-1}| \leqslant e^C$ and $|d\phi_s| \leqslant e^C$ on their domain of definition.

Finally define the logarithmic $C^{4,\alpha}$ harmonic radius, $r(N; \bar{g})$ to be the supremum of all r such that a good r-covering exists. □

By definition, a lower bound on r implies upper bounds on the C^2 norms of $g_{(0)}$ and $g_{(4)}$.

Now we show that if $g_{(0)}$ is bounded in a fixed background metric, then there exists a lower bound on the logarithmic $C^{4,\alpha}$ harmonic radius r in a collar neighborhood of the boundary. The following Proposition is the analogue of [3, Theorem 4.3].

Proposition 11. *Let (N, g) be a strictly globally static AH Einstein metric satisfying (1.1.5) with a $C^{3,\alpha}$ conformal compactification, and boundary metric γ. Suppose that γ is analytic and that, in a fixed coordinate system on ∂N, $||\gamma||_{C^{6,\alpha}} \leqslant K$. If $\bar{g}$ is the associated geodesic compactification of g determined by γ, then there are constants $\mu_0 > 0$ and $r_0 > 0$, depending on γ, such that*

$$r(N^{\mu_0}; \bar{g}) \geqslant r_0 \ . \tag{4.2.3}$$

Proof. The Proposition follows from the following two lemmas.

Lemma 12. *Let τ denote the distance from the boundary to its cutlocus $\bar{C}$ (defined by the exponential map). For (N, g) as in Proposition 11, there exists a constant c_0 depending only on γ such that*

$$r(N^{\mu_0}; \bar{g}) \geqslant c_0 \tau \ . \tag{4.2.4}$$

Proof. To the contrary, assume for a sequence $\{\bar{g}_i\}$, with the conditions of Proposition 11 satisfied, one has

$$\frac{r(N^{\mu_0}; \bar{g}_i)}{\tau(\bar{g}_i)} = \frac{r_i}{\tau_i} \to 0 \ . \tag{4.2.5}$$

Re-scale the metrics by $\lambda_i = r_i^{-1} \to \infty$ and obtain

$$g_i' = \lambda_i^2 \cdot \bar{g}_i \ . \tag{4.2.6}$$

In the sequel the primed functions, tensors, etc. correspond to the primed metrics. For these re-scaled metrics we have:

$$r_i' = r(g_i') = 1 \ . \tag{4.2.7}$$

Since r/τ is scale-invariant, it follows from (4.2.5) that $\tau_i' \to \infty$. Fix a point x_0 on the boundary of M. The definition 4.2.1 of the logarithmic $C^{4,\alpha}$ harmonic radius, and (r2-4) in particular, implies that the terms $(g_i')_{(4)}$ are bounded in the $C^{2,\alpha}$ topology and so a subsequence will converge in the C^2 topology to a 2-tensor $g_{(4)}'$. On the other hand the boundary metrics $(g_i')_{(0)}$ converge to the flat metric γ' on $\mathbb{R}^4$ which is analytic. It follows that a subsequence of (N^{ρ_0}, g_i', x_0) converges in the C^2 topology to a metric g' on a limit manifold L, while $t_i' = \lambda_i t_i \to t'$, where t' is a smooth positive function on L. Since γ' is analytic (it is flat), Prop. A-2 of Appendix implies that $g_{(4)}'$ is analytic. Now it follows from Proposition 10 that a subsequence of (N^{ρ_0}, g_i', x_0) converges in the polyhomogeneous topology near the boundary and the C^∞ topology away from the boundary. Notice that L is a manifold with boundary ∂L.

To draw a contradiction we first show that this metric is flat. The proof of [3, Prop. 4.4] implies that (L, g') is Ricci flat with $A' = D^2 t' = 0$ and $\nabla t'$ is a parallel vector field. Since $N = M \times S^1$ and $g_i = h_i + u_i^2 d\theta^2$, we have

$$g_i' = (t_i')^2 h_i + (t_i')^2 u_i^2 d\theta^2 = h_i' + (u_i')^2 d\theta^2 \ . \tag{4.2.8}$$

Since (N, g_i', x_0) converges to (L, g', x_0), we have (M, h_i', x_0) converging to (P, h', x_0), where P is a submanifold of L with induced metric h'. Orthogonal to P, there are curves in L which are the limits of the S^1 orbits in (M, h_i). Since $D^2 t' = 0$ on L and t' is constant along these curves, we conclude that $D^2 t' = 0$ on P and so in particular $s_{h'} = 0$. One has

$$g' = h' + v^2 d\theta^2 \ , \tag{4.2.9}$$

for $v = \lim t_i' u_i$. We show that v is constant on L: from (A-33) we have

$$\frac{d}{dt_i'} \log(t_i' u_i) = (s_{h_i'}/6 - s_{g_i'}/8) t_i' \ . \tag{4.2.10}$$

For $x = \lim x_i$, we have $t_i'(x_i) \to t'(x)$ and $(s_{h_i'}/6 - s_{g_i'}/8)(x_i) \to (s_{h'}/6 - s_{g'}/8)(x) = 0$.

Hence the right hand side of the above equation is zero in the limit.

$$\frac{d}{dt'} \log v = 0 \ , \tag{4.2.11}$$

which implies that v is constant.

Since (L, g') is Ricci-flat and v is constant, (P, h') is Ricci-flat too. Now since P splits along t' geodesics, $P = Q \times \mathbb{R}$. It follows that Q is Ricci-flat and hence flat ($\dim Q = 3$). It follows that both P and L are flat. Moreover, since ∂L is equal to $\mathbb{R}^4$ with the flat metric, (L, g') is isometric to $\mathbb{R}^4 \times \mathbb{R}^+$.

It follows from the above discussion that $r' = \infty$ on L, where r' is the logarithmic $C^{4,\alpha}$ harmonic radius. This will contradict (4.2.7) as soon as we show that the logarithmic $C^{4,\alpha}$ harmonic radius is at least semi-lower continuous in the polyhomogeneous topology. To see this, suppose the F-G expansions of $\bar{g}_i$ converge to $\bar{g}'$ as analytic functions of x, ρ and $\log \rho$. We show that $r_i > r - \epsilon$, for i large, where r, r_i are the logarithmic harmonic radii of $\bar{g}$ and $\bar{g}_i$. Recall the definition 4.2.1 for the logarithmic $C^{4,\alpha}$ harmonic radius. Since the polyhomogeneous convergence of g_i to g implies at least the $C^{3,\alpha}$ convergence, we see that the conditions r1, r2-1, r2-2, r3-1 and r4 are all valid on $(N, \bar{g}_i)$ for a slightly smaller choice of r. Also since the boundary metrics $\rho_i^2 g_i|_{\partial N}$ converge to $\rho^2 g|_{\partial N}$ in the C^2 topology on the boundary, both conditions r2-3 and r3-2 are valid (again for a possibly smaller r) along a subsequence. Finally both r2-4 and r3-3 follow from Proposition 10, since the convergence is in the C^∞ topology in a collar neighborhood of the boundary as functions of x, ρ and $\log \rho$. ■

Lemma 13. *For (N, g) as in Proposition 11, there is a constant μ depending only on K and α such that*

$$\tau(N; \bar{g}) \geqslant \mu \ . \tag{4.2.12}$$

Proof. If the conclusion doesn't hold, then there exists a sequence $\{g_i\}$ satisfying the conditions of Proposition 11 such that $\tau_i \to 0$. So re-scale these metrics by $10(\tau_0)^{-1} \to \infty$ so that $\tau_0' \geqslant 10$ for these new metrics. The previous lemma then implies a lower bound on r_i' and so (by Prop. 10 and the proof of Lemma 12) a subsequence of (N, g_i', x_0) converges to a limit (L, g', x_0). Here $\partial L = \mathbb{R}^4$ with the flat metric and x_0 is a fixed point chosen in ∂M. Since $r_0' \geqslant 10c_0$, the definition of the logarithmic $C^{4,\alpha}$ harmonic radius gives

$$(g_i')_{(4)} \to (g')_{(4)}\ , \tag{4.2.13}$$

where the convergence is in the $C^{2,\alpha}$ topology on the boundary. From here on we adopt the same proof as in [3, Prop.4.5] (from (4.24) on) and point out the differences. The main issue is to generalize the claims in [3, (4.31)-(4.36)] to the situation at hand; *cf.* also Prop. A-1 of the Appendix. Thus start by inserting $g_{(4),s} + r_{(g_s)}$ in place of $g_{(3),s}$ in [3, (4.30)], where $r_{(g)}$ is defined in (A-3). Then observe that $r_{(g_s)}$ terms as well as $g_{(4),s}$ terms converge to their limits and so the same argument presented in the reference is applicable to prove that the limit metric (N, g') has a free isometric $\mathbb{R}^4$-action which preserves t' so that (N, g') is foliated by equidistant leaves isometric to $\mathbb{R}^4$. Now a similar argument as in Lemma 12 shows that $N = P \times \mathbb{R}$ where P is a totally geodesic 4-dimensional submanifold of N. $(P, g'|_P)$ is also Ricci flat and admits an isometric $\mathbb{R}^3$ action which preserves the distance t' from the boundary $\partial P = \mathbb{R}^3$. Now the proof of [3, Prop.4.5] implies that then $P \simeq (\mathbb{R}^4)^+$ which, in turn, implies that $N \simeq \mathbb{R}^4 \times \mathbb{R}^+$. This is a contradiction because $\tau = 10$ on the limit and the proof is completed. ∎

Proposition 11 implies the Boundary Condition, since the lower bound on the harmonic radius obtained in the Proposition implies at least the $C^{3,\beta}$ convergence of the geodesic compactifications. Now we are in the position to prove the following Proposition. The proof is immediate from Theorem 8 and Propositions 6 and 7.

Proposition 14. *Let g_i be a sequence of strictly globally static AH Einstein metrics on $N = M \times S^1$, with strictly globally static and analytic boundary metrics γ_i on $\partial N = \partial M \times S^1$ and suppose $\gamma_i \to \gamma$ in the C^2 topology on ∂N, where γ is* *analytic. Moreover suppose that the inclusion map $i : \partial M \to \bar{M}$ induces a surjection*

$$H_2(\partial M, R) \to H_2(\bar{M}, R) \to 0 , \tag{4.2.14}$$

and that there is a constant $\omega_0 < \infty$ such that

$$Wid_{\bar{g}_i} N \leqslant \omega_0 . \tag{4.2.15}$$

Then a subsequence of g_i converges smoothly and uniformly on compact subsets to an AH Einstein metric g on N with boundary metric γ. In a collar neighborhood of ∂N, the geodesic compactifications $\bar{g}_i$ converge in the polyhomogeneous topology to the geodesic compactification $\bar{g}$ of g. This convergence is in the C^∞ topology away from the boundary. ■

4.3 Proofs of theorems A and B

In this subsection we present the proofs of Theorems A and B.

Proof of Theorem A. To prove that Π^0 is proper, we need to show that for any sequence of analytic boundary metrics $\gamma_i \to \gamma \in \mathcal{C}^0_\omega$ and AH Einstein metrics $g_i \in E_{S^1,AH}$ with $\Pi^0([g_i]) = [\gamma_i]$ there is a subsequence of g_i converging to an AHE metric g. We consider the following two cases:

Case i. There is a constant $s_0 > 0$ such that for the scalar curvature s_{γ_i} of γ_i

$$s_{\gamma_0} \geqslant s_0 . \tag{4.3.1}$$

In this case one has (*cf.* A.2):

$$s'_{\bar{g}_i} = \frac{8|\bar{D}^2\rho|^2}{\rho} \geqslant \frac{2(\bar{\Delta}\rho)^2}{\rho} = \frac{1}{32}\rho s^2_{\bar{g}_i} . \tag{4.3.2}$$

This gives $(1/s_{\bar{g}_i})' \leqslant \rho/32$. Integrating this inequality implies that the length of each minimizing ρ geodesic is at most $4\sqrt{3}/\sqrt{s_0}$ i.e.

$$Wid_{\bar{g}_i} \leqslant \frac{4\sqrt{3}}{\sqrt{s_0}} . \tag{4.3.3}$$

Hence theorem follows from Prop. 14.

Case ii. The only case remaining is when $s_{\gamma_i} \geqslant 0$ and

$$Wid_{\bar{g}_i} \to \infty . \tag{4.3.4}$$

In this case we use the rigidity result of [1, Remark 5.2, Lemma 5.5]. Very briefly, the sequence (N, g_i, y_i) converges in Gromov-Hausdorff topology to a hyperbolic cusp metric with flat boundary metric. This contradicts the condition $\gamma \in \mathcal{C}^0$ and the proof is completed. ■

At this point we consider connected components of $\mathcal{E}^0_{1,AH}$ and restrict Π^0 to these components. Before we present the proof of Theorem B, we need to show that the definition 1.2.1 of $\deg \Pi^0$ is well-defined in the context of analytic metrics. Notice that we have proved Π^0 is proper on $\mathcal{E}^\omega_{1,AH}$ but it's still an open problem whether this space has a manifold structure. On the other hand Theorem 2.3.1 and Proposition 2′ give a manifold structure for the space $\tilde{\mathcal{E}}_{1,AH}$. By Sard-Smale theorem [25], regular values of the Fredholm map Π are open and dense in $\mathcal{C}^\infty$. On the other hand $\mathcal{C}^\omega$ is dense in $\mathcal{C}^\infty$ and so there exist many analytic metrics amongst the regular values of Π. It follows that the definition 1.2.1 of $\deg \Pi^0$ is meaningful for such analytic regular values of Π and [3, Theorem 5.1] shows that this definition is independent of the choice of these analytic regular values. Now we are ready to present the proof of Theorem B.

Proof of Theorem B. As seed metric we take g_0, the metric induced from the hyperbolic metric on $\mathbb{B}^5/\mathbb{Z}$, where the $\mathbb{Z}$ action is given by a fixed translation along a geodesic (*cf.* 1.2.2). The conformal boundary of g_0 is the standard product metric on $S^3 \times S^1(\log L)$. The isometry extension theorem 1.1.7 implies that any AH Einstein metric with this conformal infinity has $SO(4) \times SU(2)$ acting effectively by isometries. The only such metrics are the hyperbolic metric on $\mathbb{B}^5$ and the AdS-Schwarzschild metric on $S^3 \times \mathbb{R}^2$. The latter is defined on a different manifold than $\mathbb{B}^5$ and so $(\Pi^0)^{-1}[\gamma] = [g_H]$. It follows

$$\deg \Pi^0 = 1 \ . \tag{4.3.5}$$

In particular, Π^0 is onto the component containing the standard product metric. ■

A Appendix

A.1 The Renormalized action

Here we study the renormalized action related to our work in §4 and particularly in the proof of Lemma 13. Let t be the geodesic defining function for a conformally compact Einstein metric g on N^5 and let $B(\epsilon)$ denote the subset of N with $t \geqslant \epsilon$. Then one has the following expansion

$$volB(\epsilon) = v_{(0)}\epsilon^{-4} + v_{(2)}\epsilon^{-2} + L\log\epsilon + V + o(1)\ . \tag{A-1}$$

The constant V is called the renormalized volume. The renormalized Einstein-Hilbert action is given by

$$I^{ren}(g) = 8V\ . \tag{A-2}$$

The constants $v_{(i)}$ and L are explicitly computable from $(\partial N, \gamma)$. In dimension 5, I^{ren} depends on the choice of the boundary metric (while L does not).

The variation of I^{ren} at a given $g \in E^{m,\alpha}_{AH}$ gives rise to a 1-form on the manifold $E^{m,\alpha}_{AH}$ and one has the following formula (*cf.* [14])

$$dI^{ren}(\dot{g}) = -2(g_{(4)} + r_{(g)})\ , \tag{A-3}$$

where $g_{(4)}$ comes from the Fefferman-Graham expansion (4.0.4) and $r_{(g)}$ is a two-tensor computable by the local geometry of γ. Thus

$$dI^{ren}(\dot{g}) = -2\int_{\partial N} < g_{(4)} + r_{(g)}\ , \dot{g} > dv_{\gamma}. \tag{A-4}$$

We prove an analogue to [3, Theorem 2.3].

Proposition A-1. *Let $g \in \mathcal{E}^{\omega}_{AH}$. Then any connected Lie group of conformal isometries of the boundary metric $(\partial N, \gamma)$ extends to an action by isometries on*

(N, g).

Proof. We sketch the proof following the proof of [3, Theorem 2.3] and point out the differences as necessary. Suppose ϕ_s is a 1-parameter group of isometries of $(\partial N, \gamma)$ with $\phi_0 = id$ and $\phi_s^*\gamma = \gamma$. One can extend these isometries to diffeomorphisms of $\bar{N}$ preserving the defining function ρ and obtain a curve $g_s = \phi_s^* g$ of Einstein metrics with a fixed conformal infinity γ (w.r.t. ρ). We need to show that the coefficient $g_{(4),s}$ in the F-G expansion of $\bar{g}_s$ is constant for small s. ($r_{(g_s)}$ remains constant since it's determined by γ). As in dimension 4, two analytic metrics with the same conformal infinity and the same $g_{(4)}$ terms are locally isometric; thus one needs to prove the following:

$$g_{(4),s} = g_{(4)} \ . \tag{A-5}$$

Consider a 2-parameter family of metrics $g_{s,u} = g_s + uh_s$ where h_s is an infinitesimal deformation of g_s so that the induced variation on the boundary is fixed i.e. $h_{(0),s} = h_{(0),0} = h_{(0)}$. To prove the above equation, in place of $\mathcal{W}$, the square of the L^2 norm of the Weyl curvature (which is relevant to dimension 4), we use I^{ren}, the renormalized action. Now we show that the analogue of [3, (2.19)] is valid in our situation. For the $r_{(g)}$ term we have

$$\frac{d}{ds} r_{(g_s)} = 0 \ . \tag{A-6}$$

It follows that

$$\begin{aligned}
\int_{\partial N} \langle \frac{d}{ds} g_{(4),s}, h_{(0)} \rangle dvol_\gamma &= \int_{\partial N} \langle \frac{d}{ds} \{ g_{(4),s} + r_{(g_s)} \}, h_{(0)} \rangle dvol_\gamma \\
&= \frac{d}{ds} \int_{\partial N} \langle g_{(4),s} + r_{(g_s)} \}, h_{(0),s} \rangle dvol_\gamma \\
&= -\frac{1}{2} \frac{d}{ds} dI^{ren}_{g_s}(h_s) \ .
\end{aligned} \tag{A-7}$$

To finish the proof, it's enough to show

$$\frac{d}{ds} \frac{d}{du} I^{ren}(g_s + uh_s) = 0 \ , \tag{A-8}$$

at $s = u = 0$. For fixed u and varying s consider

$$\frac{1}{s}[I^{ren}(g_s + uh_s) - I^{ren}(g_0 + uh_0)] \ . \tag{A-9}$$

One has $g_s = g_0 + s\delta^* X + O(s^2)$ for $X = d(\phi_s)/ds$. Let $q_0 = g_0 + uh_s = q_0 + suh' + O(s^2)$ for $h' = dh_s/ds$ and re-write the above expression as

$$\frac{1}{s}[I^{ren}(q_0 + suh' + s\delta^* X + O(s^2)) - I^{ren}(q_0 + suh')] + \frac{1}{s}[I^{ren}(q_0 + suh') - I^{ren}(q_0)] \ .$$

Take the limit of the above expression as $s \to 0$. The first bracketed term gives:

$$dI^{ren}_{q_0}(\delta^* X) = \int_N \langle \nabla_{q_0} I^{ren}, \delta^* X \rangle - 2 \int_{\partial N} \langle (q_0)_{(4)} + r_{q_0}, \delta^* X \rangle \ , \tag{A-10}$$

where r_{q_0} is the term coming from the F-G expansion of q_0. Since $\delta^* X = 0$ on ∂N, the second term above vanishes. Calculations in the proof of [3, Theorem 2.3] and the fact that I^{ren} is diffeomorphism invariant show that at $u = 0$:

$$\frac{d}{du} \int_N \langle \nabla_{q_0} I^{ren}, \delta^* X \rangle = 0 \ . \tag{A-11}$$

In a similar manner, the limit of the second bracketed term in (A-10), as $s \to 0$, is

$$dI^{ren}_{g_0+uh_0}(uh') = \int_N \langle \nabla_{q_0} I^{ren}, uh' \rangle dV - 2 \int_{\partial N} \langle (g+uh)_{(4)} + r_{g+uh}, uh' \rangle + O(u^2) \ . \tag{A-12}$$

The second term above vanishes since $h' = 0$ on the boundary. Taking the u derivative of the first term above at $u = 0$ gives

$$\int \langle \nabla I^{ren}_{g_0+uh}, uh' \rangle dV + \int \langle \nabla I^{ren}_{g_0}, h' \rangle dV \ . \tag{A-13}$$

The first term vanishes since $u = 0$ and the second term vanishes since g_0 is Einstein and hence a critical point of I^{ren} so that $\nabla I^{ren}_{g_0} = 0$. ■

Now we consider the expansion (A-1) more carefully. One can take an analytic curve g_s of AHE metrics with analytic boundary metrics γ_s and study the dependence of V_s, the re-normalized volume of the metric g_s with conformal boundary γ_s, on s. Since the equation (A-3) relates dI to $g_{(4)}$, this may prove useful in studying the $g_{(4)}$ term. We prove the following:

Proposition A-2. *Let g be a conformally compact Einstein metric on N^5 with a geodesic defining function ρ and analytic boundary metric γ. Then the $g_{(4)}$ term in the Fefferman-Graham expansion* (4.0.4) *is analytic.*

Proof. Let $\bar{g} = \rho^2 g = d\rho^2 + g_\rho$ and write

$$dvol_g = f(x,\rho)d\rho\ dvol_\gamma = \rho^{-5}(\frac{det\ g_\rho}{det\ \gamma})^{1/2}d\rho\ dvol_\gamma\ , \tag{A-14}$$

where $f(x,\rho)$ is analytic in x and $\rho > 0$. Let $\epsilon < \omega_0 < Wid_{\bar{g}}$ and consider the following expansion (*cf.* [15])

$$\int_\epsilon^{\omega_0} f(x,\rho)d\rho = c_4(x)\epsilon^{-4} + c_2(x)\epsilon^{-2} + L(x)\log\frac{1}{\epsilon} + V(x) + o(x,\epsilon)\ . \tag{A-15}$$

Choose $x_0 \in \partial N$. We first show that $V(x)$ is analytic in x at x_0 and for simplicity take $x_0 = 0$ (via a chart). Since $f(x,\rho)$ is analytic in x, for a fixed rho, the following Taylor series of f converges for all $|x| < r$ and $\rho > \epsilon$, where $r > 0$ is a constant independent of ρ:

$$f(x,\rho) = \sum f_i(\rho)x^i\ . \tag{A-16}$$

Hence

$$\int_\epsilon^{\omega_0} f(x,\rho)d\rho = \sum(\int_\epsilon^{\omega_0} f_i(\rho)d\rho)x^i = \sum A_i(\epsilon)x^i\ , \tag{A-17}$$

is the expansion of the integral above and converges on the same region. On the other, (A-15) implies there exist constants a_i, b_i, c_i, V_i such that

$$A_i(\epsilon) = a_i\epsilon^{-4} + b_i\epsilon^{-2} + c_i\log\frac{1}{\epsilon} + V_i + o(\epsilon)\ . \tag{A-18}$$

We would like to show that the following Taylor series of $V(x)$ is convergent:

$$V(x) = \sum V_i x^i \, . \tag{A-19}$$

Since the series in (A-17) converges, there exists a constant C such that

$$\forall m: \; |A_m(\epsilon)| < Cr^{-m} \tag{A-20}$$

From (A-18) and the fact that a_i, b_i, c_i are all determined by γ (which is analytic) we have

$$|V_m| < |A_m(\epsilon)| + |a_m|\epsilon^{-4} + |b_m|\epsilon^{-2} + |c_m| \log \frac{1}{\epsilon} + |o(\epsilon)| < C_1 r_1^{-m}, \tag{A-21}$$

for some $C_1, r_1 > 0$. This proves that (A-19) is convergent on a slightly smaller neighborhood of $x_0 = 0$.

Now consider the following map

$$\Phi : Met^{\omega}_{AH} \times \mathbb{S}^{\omega}(N) \to \mathbb{S}^{\omega}_2(N), \tag{A-22}$$

$$\Phi(\gamma, h) = \Phi(g_\gamma + h) = Ric_g + ng + (\delta_g)^* B_{g(\gamma)}(g), \tag{A-23}$$

defined as in (2.2.4), where ω is used to denote the analytic subset of the corresponding sets. Let $\mathcal{V}$ denote the analytic variety $\Phi^{-1}(0) \cap \{Ric_g < 0\}$ and for any fixed $x \in \partial N$, consider $V(x)$ as a functional on $\mathcal{V}$. We show that this functional is analytic on $\mathcal{V}$. Thus let $g_s = g + sh$ be an analytic variation of g which is Einstein to the first order with the induced boundary metric $\gamma_s = \gamma + sh^{(0)}$, where $h^{(0)}$ is an analytic 2-tensor on the boundary. The geodesic defining function ρ_s associated to γ_s is now analytic in s. Let's write $V(x, s)$ for $V(x)$ to show its dependence on g_s. We would like to show that $V(x, s)$ depends analytically on s as well. Going back to the definition of $V(x, s)$ in equation (A-15), by taking m derivatives with respect to s we have

$$\int_\epsilon^{\omega_0} f^{(m)}(x, \rho, s) d\rho = c_4^{(m)}(x, s)\epsilon^{-4} + c_2^{(m)}(x, s)\epsilon^{-2} + L^{(m)}(x, s) \log \frac{1}{\epsilon} + V^{(m)}(x, s) + o(\epsilon) \, .$$

The terms c_4, c_2 are analytic in s since the variation of the boundary metric, $h^{(0)}$, is analytic and these terms are determined by the boundary metric. Now a similar argument as before shows that there are constants C_2, r_2 such that $|V^{(m)}(x,s)| < C_2 r_2^{-m}$ which implies that $V(x,s)$ is analytic in s.

Now one calculates

$$\begin{aligned} dV(h) &= \lim_{s\to 0}\frac{1}{s}(V(g_s) - V(g)) = \lim_{s\to 0}\frac{1}{s}(\int_{\partial N} V(x,s)dvol_{\gamma_s} - \int_{\partial N} V(x,0)dvol_{\gamma}) \\ &= \int_{\partial N}\lim_{s\to 0}\frac{1}{s}(V(x,s)\sqrt{det(\gamma_s)_{ij}} - V(x,0)\sqrt{det(\gamma)_{ij}})dx \\ &= \int_{\partial N}(\frac{d}{ds}V(x,s) + \frac{1}{2}\langle\gamma, h^{(0)}\rangle)dvol_{\gamma} \end{aligned} \tag{A-24}$$

On the other hand from (A-3) we have $dV(h) = -\frac{1}{4}\int_{\partial N}\langle g_{(4)} + r_{(g)}, h^{(0)}\rangle dvol_{\gamma}$. Comparing this with the above, we conclude that

$$\nabla V + \frac{1}{2}\gamma = -\frac{1}{4}(g_{(4)} + r_{(g)}) ,$$

where ∇V denotes the gradient of V on the space of analytic tensors on the boundary defined by $\langle \nabla V, h^{(0)}\rangle = \frac{d}{ds}V(x,s)$. Since V is analytic, ∇V is an analytic 2-tensor and then the above equation implies that $g_{(4)}$ is analytic as well, since γ, and consequently $r_{(g)}$, is analytic. ■

Remark. Propositions A-1 and A-2 are valid in all dimensions. The proofs are the same with $g_{(4)}$ replaced by $g_{(n)}$.

A.2 Calculations

If $N = M \times S^1$ and $g = h + u^2 d\theta^2$, then the curvature quantities of (N, g) and (M, h) are related as discussed below. Take an orthonormal basis $\{e_i\}$ for $T_{(x,\theta)}M$. Let L be the unit tangent along the S^1 orbits at (x,θ). Then $\{e_i\}\cup\{L\}$ is an orthonormal basis for $T_{(x,\theta)}N$. For any $\theta \in S^1$, $M \times \{\theta\}$ is a totally geodesic hypersurface in N

and so we have the following (the tilde expressions refer to (N, g))

$$\tilde{\nabla}_{e_i} L = 0 \ , \tag{A-25}$$

$$\tilde{\nabla}_L e_i = \frac{u_i}{u} N \ , \tag{A-26}$$

$$\tilde{\nabla}_L L = -\nabla u / u \ . \tag{A-27}$$

These imply the following

$$\tilde{R}(e_i, L, L, e_j) = -D^2 u(e_i, e_j)/u \ , \tag{A-28}$$

which is essentially (1.1.6), if $\tilde{Ric} = -4g$, which is the case from here on. Taking trace of the above equality gives $\Delta u/u = -Ric_N(L, L) = 4$.

If f is a C^2 function defined on M, we can extend f to N by defining $f(x, \theta) = f(x)$. A simple calculation yields

$$\tilde{\Delta} f = \Delta f + \langle \nabla f, \nabla u/u \rangle \ . \tag{A-29}$$

Also if $w > 0$, one verifies that $\tilde{\Delta} \log w = (\tilde{\Delta} w)/w - |(\tilde{\nabla} w)/w|^2$ and so the above equation for $f = \log u$ becomes

$$\tilde{\Delta} \log u = 4 \ . \tag{A-30}$$

If ρ is a geodesic defining function for the Einstein manifold (N, g) and $\bar{g} = \rho^2 g$, one has

$$\bar{R}ic_N = -3(\frac{\bar{D}_N^2 \rho}{\rho}) - (\frac{\bar{\Delta}_N \rho}{\rho})\bar{g} \ . \tag{A-31}$$

Taking trace of the above equation gives $\bar{s}_N = -8(\bar{\Delta}_N \rho)/\rho$, where $\bar{s}_N$ is the scalar curvature of $(N, \bar{g})$. A similar calculation gives $\bar{s}_M = -6(\bar{\Delta}_M \rho)/\rho$ for the scalar curvature of $(M, \bar{h})$. These two equations imply

$$-\langle \bar{\nabla} \rho \ , \bar{\nabla}_{\bar{L}} \bar{L} \rangle = \bar{D}^2 \rho(\bar{L}, \bar{L}) = tr_N \bar{D}^2 \rho - tr_M \bar{D}^2 \rho = (\bar{s}_M/6 - \bar{s}_N/8)\rho \ , \tag{A-32}$$

where $\bar{L} = L/\rho$ is the unit tangent vector with respect to the bared metric. On the other hand $\nabla u = -u(\nabla_L L) = -u\rho^2(\bar{\nabla}_{\bar{L}}\bar{L}) - u\rho\bar{\nabla}\rho$. This and the above equation imply

$$\begin{aligned}\frac{d}{d\rho}(\rho u) =&\langle\bar{\nabla}\rho\,,\bar{\nabla}(\rho u)\rangle\\ =&\langle\bar{\nabla}\rho\,,u\bar{\nabla}\rho\rangle + \langle\bar{\nabla}\rho\,,\rho\bar{\nabla}u\rangle\\ =&u + \langle\bar{\nabla}\rho\,,-\rho u\bar{\nabla}_{\bar{L}}\bar{L} - u\bar{\nabla}\rho\rangle\\ =&\rho^2 u(\bar{s}_M/6 - \bar{s}_N/8)\ .\end{aligned} \tag{A-33}$$

Next, we prove the following identity for the scalar curvature of the compactified metric $\bar{s}'_g$

$$\bar{s}'_N = \frac{8|\bar{D}^2\rho|^2}{\rho}\ . \tag{A-34}$$

This is the 5-dimensional version of [1, Prop. 1.4]. The Ricatti equation for the flow lines of $\bar{\nabla}\rho$ is

$$H' + |A|^2 + Ric(\bar{\nabla}\rho, \bar{\nabla}\rho) = 0\ , \tag{A-35}$$

where $H = tr\ A = \bar{\Delta}\rho$ and $H' = \langle H, \bar{\nabla}\rho\rangle$. Replacing the Ricci curvature term by using (A-31) and then dividing by ρ leads to

$$\frac{(\bar{\Delta}\rho)'}{\rho} + \frac{|\bar{D}^2\rho|^2}{\rho} - \frac{\bar{\Delta}\rho}{\rho^2} = 0\ . \tag{A-36}$$

On the other hand $(\bar{\Delta}\rho)'/\rho = (\bar{\Delta}\rho/\rho)' + \bar{\Delta}\rho/\rho^2$. This and the above equation imply

$$(\bar{\Delta}\rho/\rho)' + \frac{|\bar{D}^2\rho|^2}{\rho} = 0\ , \tag{A-37}$$

which is equivalent to (A-34), since $\bar{\Delta}\rho/\rho = -\bar{s}_N/8$.

A.3 Non-degenerate Einstein metrics

Definition A.3.1. Lichnerowicz Laplacian of a symmetric 2-tensor h is given by

$$\Delta_L h = D^*Dh - r_g \circ h - h \circ r_g - 2\hat{R}_g h \ , \tag{A-38}$$

where D denotes the covariant derivative, D^* is its adjoint, r_g is the Ricci curvature and $\hat{R}_g$ is the action of curvature tensor on symmetric 2 tensors given by

$$(\hat{R}_g h)(X,Y) = \sum h(R(X,e_i)e_i, Y) \ , \tag{A-39}$$

where e_i form an orthonormal basis. □

In case of Einstein g with $r_g = -ng$, the formula (A-38) becomes

$$\Delta_L = D^*D - 2n - 2\hat{R}_g \ . \tag{A-40}$$

Lichnerowicz Laplacian is related to the variation of the Ricci curvature in direction of h through the formula

$$r_g' h = \frac{1}{2}\Delta_L h - \delta_g^*(\delta_g h) - \frac{1}{2}D_g d(tr_g h) \ . \tag{A-41}$$

Definition A.3.2. An Einstein metric g with $Ric_g = -ng$ is called *non-degenerate* if the operator

$$P = \Delta_L + 2n \ , \tag{A-42}$$

has trivial L^2 kernel on the space of trace-free symmetric 2-tensors. □

Graham and Lee [18, 1991] proved that if the conformal structure is close enough to that of the round sphere, then an Einstein filling exists, which is unique amongst Einstein metrics close to the Poincaré metric. Biquard [11, 2000] generalized this result to arbitrary non-degenerate Einstein manifolds. Lee [23, 2001] proves the following stability result:

Theorem A.3.3. (Lee [23]) *Let N be the interior of a smooth compact $(n+1)$-dimensional manifold-with-boundary $\bar{N}$, $n \geqslant 3$, and let g_0 be a non-degenerate Einstein metric on N that is conformally compact of class $C^{l,\beta}$ with $2 \leqslant l \leqslant n-1$ and $0 < \beta < 1$. Let ρ be a smooth defining function for ∂N, and let $\gamma_0 = \rho^2 g_0|_{\partial N}$. Then there is a constant $\epsilon > 0$ such that for any $C^{l,\beta}$ Riemannian metric γ on ∂N with $||\gamma - \gamma_0|| < \epsilon$, there exists an Einstein metric g on N that has γ as conformal infinity and is conformally compact of class $C^{l,\beta}$.*

The non-degeneracy condition in the above theorem holds if the sectional curvature of g_0 is non-positive.

References

[1] M. Anderson, *Boundary regularity, uniqueness and non-uniqueness for AH Einstein metrics on 4-manifolds,* Adv. in Math., vol. **179**, (2003), 205-249.

[2] M. Anderson, *Convergence and ridgidity of manifolds under Ricci curvature bounds,* Invent. math. **102**, 429-445 (1990).

[3] M. Anderson, *Einstein metrics with prescribed conformal infinity on 4-manifolds,* (Preprint, May 01/Feb. 04), math.DG/0105243.

[4] M. Anderson, *L^2 curvature and volume renormalization of AHE metrics on 4-manifolds,* Math. Research Letters, vol. **8**, (2001), 171-188.

[5] M. Anderson, *On the structure of asymptotically de Sitter and anti-de Sitter spaces,* Preprint July 04, hep-th/0407087.

[6] M. Anderson, *The L^2 structure of moduli spaces of Einstein metrics on 4-manifolds,* Geom. & Funct. Analysis, **2**, (1992), 29-89.

[7] M. Anderson, *Some results on the structure of conformally compact Einstein metrics,* (Preprint, Jan. 04), math.DG/0402198.

[8] M. Anderson, P. Chruściel and E. Delay, *Nontrivial, static, geodesically complete space-times with a negative cosmological constant,* Jour. High Energy Physics, **10** (2002) 063, 1-27.

[9] M. Anderson, P. Chruściel and E. Delay, *Nontrivial, static, geodesically complete space-times with a negative cosmological constant, II: $n \geqslant 5$,* Preprint, Jan. 04, gr-qc/0401081.

[10] L. Besse, *Einstein Manifolds,* Springer-Verlag, New York.

[11] O. Biquard, *Metriques d'Einstein asymptotiquement symmetriques,* Asterisque, **265**, (2000).

[12] S. Y. Cheng and S. T. Yau, *Differential Equations on Riemannian Manifolds and their Geometric Applications,* Comm. Pure Appl. Math., vol. **28**, (1975), 333-354.

[13] P. Chruściel, E. Delay, J.M. Lee and D. Skinner, *Boundary regularity of conformally compact Einstein metrics,* Preprint, Jan. 04, math.DG/0401386.

[14] S. de Haro, K. Skenderis and S.N. Solodukhin, *Holographic reconstruction of spacetime and renormalization in the AdS/CFT correspondence,* Comm. Math. Phys., **217**, (2001), 595-622, hep-th/0002230.

[15] C.R. Graham, *Volume and area renormalization for Conformally compact Einstein metrics,* Preprint, Sep 1999, math.DG/9909042.

[16] C. Fefferman, and C.R. Graham, *Conformal invariants,* Elie Cartan et les Mathematiques d'aujourd'hui, Asterisque, hors serie, Soc. Math. France, Paris, 1985, pp. 95–116.

[17] D. Gilbarg and N. S. Trudinger, *Elliptic Partial Differential Equations of Second Order,* second edition, Springer-Verlag (1983).

[18] C.R. Graham and J.M. Lee, *Einsten metrics with prescribed conformal infinity on the ball,* Adv. in Math., vol. **87**, (1991), 186-225.

[19] S.W. Hawking and D.N. Page, *Thermodynamics of black holes in Anti-de Sitter space,* Comm. Math. Phys., **87**, (1983), pp. 577–588.

[20] S. Kichenassamy, *Nonlinear Wave Equations,* Marcel Dekker, Inc., New York, (1996).

[21] S. Kichenassamy, *On a conjecture of Fefferman and Graham,* Advances in Math. **184**, (2004), 268-288.

[22] S. Kichenassamy, *Private Communication,* Fall 2004.

[23] J.M. Lee, *Fredholm Operators and Einstein metrics on conformally compact manifolds,* Preprint (2001), math.DG/0105046.

[24] Peter Petersen, *Riemannian Geometry,* Springer-Verlag, New York Inc., 1998.

[25] S. Smale, *An infinite dimensional version of Sards Theorem,* Amer. J. Math. **87**, 861-866 (1965).

[26] E. Witten, *Anti De Sitter spaces and holography,* Adv. Theor. Math. Phys., **2**, (1998), pp. 253–291, hep-th/9802150.

[27] S.T. Yau, *Harmonic functions on complete Riemannian manifolds,* Comm. Pure & Appl. Math., vol. **28**, (1975), 201-228.