

The Selberg Trace Formula and Selberg Zeta-Function for Cofinite Kleinian Groups with Finite Dimensional Unitary Representations

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Abstract

For cofinite Kleinian groups, with finite-dimensional unitary representations, we derive the Selberg trace formula. As an application we define the corresponding Selberg zeta-function and compute its divisor, thus generalizing results of Elstrodt, Grunewald and Mennicke to non-trivial unitary representations. We show that the presence of cuspidal elliptic elements sometimes adds ramification point to the zeta function. In fact, if $\mathcal{O} = \mathbb{Z}[-\frac{1}{2} + \frac{\sqrt{-3}}{2}]$ is the ring of Eisenstein integers, then the Selberg zeta-function of $\mathrm{PSL}(2, \mathcal{O})$ contains ramification points.

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Chapter 1

Introduction

The Selberg theory (Selberg trace formulas, Selberg zeta-functions, and related applications) has been well studied in both the two-dimensional scalar case ([Iwa02]) and the two-dimensional vector case¹ ([Roe66], [Ven82], [Hej83], [Fis87]). By “Two-dimensional vector case” we mean: cofinite Fuchsian groups with finite-dimensional unitary representations, and the “Scalar case” refers to the case with the trivial representation. Elstrodt, Grunewald and Mennicke extended the Selberg theory to the three-dimensional scalar case in [EGM98]. By the “Three-dimensional case” we mean: cofinite Kleinian groups. The main goal of this thesis is to extend the Selberg theory to the three-dimensional vector case.

In this thesis we derive the Selberg trace formula for cofinite Kleinian groups², with finite-dimensional unitary representations. As an application we define the corresponding Selberg zeta-function and compute its divisor, thus generalizing results of Elstrodt, Grunewald and Mennicke [EGM98] to non-trivial unitary representations.

Much of the two-dimensional vector and three-dimensional scalar cases extends in a straight forward manner to the three-dimensional vector case. However, the extension of several parts of the Selberg theory are more subtle in the three-dimensional case, especially in the vector case. One reason for this is because the set of finite-dimensional unitary representations of a fixed cofinite Kleinian groups is not well understood. Another reason is related to the structure of the stabilizer subgroup of a cusp. In the two-dimensional case the stabilizer subgroup of a cusp is a purely parabolic group that is isomorphic to a rank-one lattice, while in three dimensions the stabilizer subgroup of a cusp is a non-abelian group that contains elliptic elements, with a finite-index purely parabolic subgroup that is isomorphic to a rank-two lattice. The presence of elliptic elements in the stabilizer subgroup introduces some subtleties to the three-dimensional vector case, particularly in the computation of the divisor of the Selberg zeta-function. In addition, the fact that the stabilizer subgroup (in the three-dimensional case) contains a rank-two parabolic subgroup forced us to prove some additional estimates involving two-dimensional lattice sums.

A Kleinian group is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm I\}$. Each element of $\mathrm{PSL}(2, \mathbb{C})$

¹The works [Roe66], [Hej83], [Fis87], contain not only the two-dimensional vector case, but also its generalization, the case of unitary multiplier systems of arbitrary real weight.

²A Kleinian groups is referred to in some texts as a discrete group of isometries acting on hyperbolic three-space, or a discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$.

is identified with a Möbius transformation, and has a well-known action on hyperbolic three-space \mathbb{H}^3 and on its boundary at infinity – the Riemann sphere \mathbb{P}^1 (see [EGM98, Section 1.1]). A Kleinian group is *cofinite* iff it has a fundamental domain $\mathcal{F} \subset \mathbb{H}^3$ of finite hyperbolic volume.

We use the following coordinate system for hyperbolic three-space,

$$\mathbb{H}^3 \equiv \{(x, y, r) \in \mathbb{R}^3 \mid r > 0\} \equiv \{(z, r) \mid z \in \mathbb{C}, r > 0\} \equiv \{z + rj \in \mathbb{R}^3 \mid r > 0\},$$

with the hyperbolic metric

$$ds^2 \equiv \frac{dx^2 + dy^2 + dr^2}{r^2},$$

and volume form

$$dv \equiv \frac{dx dy dz}{r^3}.$$

The Laplace-Beltrami operator is defined by

$$\Delta \equiv -r^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) + r \frac{\partial}{\partial r},$$

and it acts on the space of smooth functions $f : \mathbb{H}^3 \mapsto V$, where V is a finite-dimensional complex vector space with inner-product $\langle \cdot, \cdot \rangle_V$.

Suppose that Γ is a cofinite Kleinian group and $\chi \in \text{Rep}(\Gamma, V)$ ($\text{Rep}(\Gamma, V)$ is the space of finite-dimensional unitary representations of Γ in V). Then the Hilbert space of χ -*automorphic* functions is defined by

$$\mathcal{H}(\Gamma, \chi) \equiv \{f : \mathbb{H}^3 \rightarrow V \mid f(\gamma P) = \chi(\gamma)f(P) \forall \gamma \in \Gamma, \\ P \in \mathbb{H}^3, \text{ and } \langle f, f \rangle \equiv \int_{\mathcal{F}} \langle f(P), f(P) \rangle_V dv(P) < \infty\}.$$

Here \mathcal{F} is a fundamental domain for Γ in \mathbb{H}^3 , and $\langle \cdot, \cdot \rangle_V$ is the inner product on V . Finally, let $\Delta = \Delta(\Gamma, \chi)$ be the corresponding positive self-adjoint Laplace-Beltrami operator on $\mathcal{H}(\Gamma, \chi)$.

Our first result is the spectral decomposition of Δ on $\mathcal{H}(\Gamma, \chi)$ (see Theorem 3.8.1). Except for one important point, the proof of the spectral decomposition theorem is analogous to the two-dimensional vector and three-dimensional scalar cases. The one important point being, *singularity at a cusp*. To the best of the author's knowledge, prior to this thesis, the notion of singularity at a cusp was only defined for cofinite Fuchsian groups [Sel56] [Roe66] [Ven82] [Hej83]. In §3.1 we extend the notion of singularity to cofinite Kleinian groups.

In §4.1 we give an explicit form of the Selberg trace formula for cofinite Kleinian groups with finite-dimensional unitary representations (see Theorem 3.7.1). The new feature in the trace formula is a term of the form,

$$\sum_{\alpha=1}^{\kappa} \frac{g(0)}{|\Gamma_{\alpha} : \Gamma'_{\alpha}|} \sum_{k=l_{\alpha}+1}^{\dim_{\mathbb{C}} V} L(\Lambda_{\alpha}, \psi_{k\alpha}).$$

The above term comes about from *regularity* at a cusp, and its value is computed using Kronecker's second limit formula (see §4.2).

As an application of the spectral decomposition theorem we derive an identity involving conjugacy relations of cuspidal elliptic elements. This identity is used in the proof of the Selberg trace formula and to show that under certain conditions, the Selberg zeta function admits a meromorphic continuation (see Lemma 4.4.4 for the identity).

For $\operatorname{Re}(s) > 1$ the Selberg zeta-function $Z(s, \Gamma, \chi)$ is defined by the following product

$$Z(s, \Gamma, \chi) \equiv \prod_{\{T_0\} \in \mathcal{R}} \prod_{j=1}^{\dim_{\mathbb{C}} V} \prod_{\substack{l, k \geq 0 \\ c(T, j, l, k) = 1}} \left(1 - \mathfrak{t}_j a(T_0)^{-2k} \overline{a(T_0)^{-2l}} N(T_0)^{-s-1} \right).$$

In §5.1 we introduce the various definitions and notations that are needed in order to define the zeta function, and meromorphically³ continue $Z(s, \Gamma, \chi)$ to $\operatorname{Re}(s) \leq 1$ while computing its divisor. The main new difficulty is handling the contribution of the cuspidal elliptic elements to the topological (or trivial) zeros and poles of $Z(s, \Gamma, \chi)$. We show the following in §5.3:

Corollary. *Let $\Gamma = \operatorname{PSL}(2, \mathbb{Z}[-\frac{1}{2} + \frac{\sqrt{-3}}{2}])$, and $\chi \equiv 1$ (the trivial representation). Then $Z(s, \Gamma, \chi)$ is not a meromorphic function⁴ (it is the 6-th root of a meromorphic function).*

In addition, the methods of §5.3 imply that the Selberg zeta-function of the Picard group is meromorphic:

Corollary. *Let $\Gamma = \operatorname{PSL}(2, \mathbb{Z}[\sqrt{-1}])$, and let $\chi \in \operatorname{Rep}(\Gamma, V)$. Then $Z(s, \Gamma, \chi)$ is a meromorphic function.*

³We give the meromorphic continuation for certain cases, and for others show that the zeta function is a rational root of a meromorphic function.

⁴This is the first example that the author is aware of where the Selberg zeta-function is not meromorphic.

Chapter 2

Prerequisite Material

In order to make our presentation self-contained, we present some well-known results concerning hyperbolic three-space, the Laplace-Beltrami operator, and cofinite Kleinian groups. For more details see [EGM98].

2.1 Hyperbolic three-space \mathbb{H}^3

Let \mathbb{H}^3 denote the upper half space (of \mathbb{R}^3) model of hyperbolic three-space. The space \mathbb{H}^3 is parametrized by the following coordinates:

$$\mathbb{H}^3 \equiv \{(x, y, r) \in \mathbb{R}^3 \mid r > 0\} \equiv \{(z, r) \mid z \in \mathbb{C}, r > 0\}.$$

A point $P \in \mathbb{H}^3$ will be denoted by (x, y, r) , (z, r) , or $z + rj$. The standard hyperbolic metric¹ hyperbolic metric and volume form are written respectively as

$$ds^2 \equiv \frac{dx^2 + dy^2 + dr^2}{r^2} \text{ and } dv \equiv \frac{dx dy dz}{r^3}.$$

Remark 2.1.1. *An alternate model of hyperbolic three-space is the open three-ball, $\mathbb{B}^3 \equiv \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 < 1\}$. When equipped with the metric*

$$4 \frac{dx^2 + dy^2 + dz^2}{(1 - x^2 - y^2 - z^2)^2},$$

\mathbb{B}^3 is isometric to \mathbb{H}^3 .

The boundary (at infinity) of \mathbb{H}^3 can be realized (by Remark 2.1.1) as the Riemann sphere $\mathbb{P} = \mathbb{C} \cup \infty$.

The Laplace-Beltrami Δ operator lies at the heart of this thesis. In our coordinates it can be written explicitly as the following differential operator²:

$$\Delta \equiv -r^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial r^2} \right) + r \frac{\partial}{\partial r}.$$

¹The metric with -1 sectional curvature.

²In our notation Δ is a positive self-adjoint operator

The orientation preserving isometry group of \mathbb{H}^3 can be identified with the group $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm I\}$. Each element

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{C})$$

acts on \mathbb{H}^3 as follows: $M(z + rj) = w + tj$, where

$$w = \frac{(az + b)(\bar{c}\bar{z} + \bar{d}) + a\bar{c}r^2}{|cz + d|^2 + |c|^2r^2} \quad \text{and} \quad t = \frac{r}{|cz + d|^2 + |c|^2r^2}.$$

The element M also acts on the boundary at infinity of \mathbb{H}^3 via standard Möbius action on \mathbb{P} ,

$$M\zeta = \frac{a\zeta + b}{c\zeta + d}$$

for $\zeta \in \mathbb{P}$.

2.2 Harmonic Analysis on \mathbb{H}^3

For $P = z + rj, P' = z' + r'j \in \mathbb{H}^3$ denote by $d(P, P')$ the (hyperbolic) distance in \mathbb{H}^3 between P and P' , and let $\delta(P, P')$ be defined by

$$\delta(P, P') \equiv \frac{|z - z'|^2 + r^2 + r'^2}{2rr'}.$$

It follows that $\cosh(d(P, P')) = \delta(P, P')$, and that δ is a point-pair invariant³.

We can use the concept of a point-pair invariant to construct the resolvent kernel for Δ . For $s \in \mathbb{C}, t > 1$, set

$$\varphi_s(t) \equiv \frac{(t + \sqrt{t^2 - 1})^{-s}}{\sqrt{t^2 - 1}}.$$

Lemma 2.2.1. [EGM98, Lemma 4.2.2] Let $u \in C_c^2(\mathbb{H}^3)$, $s \in \mathbb{C}$ and $\lambda = 1 - s^2$. Then for all $Q \in \mathbb{H}^3$

$$u(Q) = \frac{1}{4\pi} \int_{\mathbb{H}^3} \varphi_s(\delta(P, Q))(\Delta - \lambda)u(P) dv(P).$$

(2) The point-pair invariant $\varphi_s \circ \delta$ is the resolvent kernel for Δ on the Hilbert space of square integrable functions on \mathbb{H}^3 .

Let $\mathcal{S} \equiv \mathcal{S}[1, \infty)$ denote the Schwartz space of smooth functions $k : [1, \infty) \rightarrow \mathbb{C}$ that satisfy $\lim_{x \rightarrow \infty} x^n k^{(m)}(x) = 0$ for all $n, m \in \mathbb{N}_{\geq 0}$. For each $k \in \mathcal{S}$, $K \equiv k \circ \delta$ is a point-pair invariant and the kernel of an operator $\mathcal{K} : L_{\text{loc}}^2(\mathbb{H}^3) \mapsto L_{\text{loc}}^2(\mathbb{H}^3)$ defined by

$$\mathcal{K}f(P) = \int_{\mathbb{H}^3} K(P, Q)f(Q) dv(Q).$$

We have

³A point-pair invariant is a map $f : \mathbb{H}^3 \times \mathbb{H}^3 \rightarrow \mathbb{C}$ defined almost everywhere satisfying $f(MP, MQ) = f(P, Q)$ for all $P, Q \in \mathbb{H}^3, M \in \text{PSL}(2, \mathbb{C})$.

Lemma 2.2.2. [EGM98, Lemma 3.5.3] Let $k \in \mathcal{S}$, $K(P, Q) = k(\delta(P, Q))$ for $P, Q \in \mathbb{H}^3$, $f : \mathbb{H}^3 \rightarrow \mathbb{C}$ be a solution of $\Delta f = \lambda f$, $\lambda = 1 - s^2$, and

$$h(\lambda) \equiv h(1 - s^2) \equiv \frac{\pi}{s} \int_1^\infty k\left(\frac{1}{2}\left(t + \frac{1}{t}\right)\right) (t^s - t^{-s}) \left(t - \frac{1}{t}\right) \frac{dt}{t}. \quad (2.2.1)$$

Then (2.2.2) converges absolutely and

$$\int_{\mathbb{H}^3} K(P, Q) f(Q) dv(Q) = h(\lambda) f(P).$$

The lemma above says that if f is an eigenfunction⁴ of Δ with eigenvalue λ then f is also an eigenfunction of \mathcal{K} with eigenvalue $h(\lambda)$ depending only on the eigenvalue λ and not on the particular eigenfunction.

The map sending $k \in \mathcal{S}$ to holomorphic function h above is called the Selberg-Harish-Chandra transform of k .

2.3 Kleinian Groups

A subgroup $\Gamma < \text{PSL}(2, \mathbb{C})$ is called a Kleinian group if for each $P \in \mathbb{H}^3$ the orbit ΓP has no accumulation points in \mathbb{H}^3 . An equivalent formulation is that Γ is a discrete subset of $\text{PSL}(2, \mathbb{C})$ in the topology induced from \mathbb{C}^4 .

A closed subset $\mathcal{F} \subset \mathbb{H}^3$ is called a fundamental domain of Γ if

- \mathcal{F} meets each Γ -orbit at least once,
- the interior \mathcal{F}° meets each Γ -orbit at most once,
- the boundary of \mathcal{F} has Lebesgue measure zero.

For each $Q \in \mathbb{H}^3$ the set

$$\mathcal{P}_Q(\Gamma) \equiv \{P \in \mathbb{H}^3 \mid d(P, Q) \leq d(\gamma P, Q) \quad \forall \gamma \in \Gamma\}$$

is a fundamental domain for Γ that is centered at the point Q .

We say that Γ is *cocompact* if it has a fundamental domain \mathcal{F} that is compact, and *cofinite*⁵ if it has a fundamental domain \mathcal{F} with

$$\text{vol}(\Gamma) \equiv \int_{\mathcal{F}} dv < \infty$$

(dv is the volume form of \mathbb{H}^3).

⁴The function f need not be in any Hilbert space.

⁵Note that cocompact groups are also cofinite.

2.4 Möbius Transformations

Each element

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}(2, \mathbb{C})$$

falls into exactly⁶ one of the following categories:

- *parabolic* if $|\mathrm{tr}(\gamma)| = 2$ and $\mathrm{tr} \gamma \in \mathbb{R}$,
- *elliptic* if $0 \leq |\mathrm{tr}(\gamma)| < 2$ and $\mathrm{tr} \gamma \in \mathbb{R}$,
- *loxodromic* if it is neither *elliptic* nor *parabolic*.

There is a useful geometric characterization of the above notions. An element $\gamma \in \mathrm{PSL}(2, \mathbb{C})$ is

- *parabolic*⁷ iff it has exactly one fixed point in \mathbb{P} ,
- *elliptic* iff it has two fixed points in \mathbb{P} and fixes the geodesic line in \mathbb{H}^3 connecting the two points.
- *loxodromic* iff it has two fixed points in \mathbb{P} and has no fixed points in \mathbb{H}^3 .

2.5 Stabilizer Subgroups and Cusps

Let Γ be a Kleinian group. For each $Q \in \mathbb{H}^3 \cup \mathbb{P}$ the stabilizer subgroup of Q is denoted by $\Gamma_Q \equiv \{\gamma \in \Gamma \mid \gamma Q = Q\}$ and for $\zeta \in \mathbb{P}$,

$$\Gamma'_\zeta \equiv \{\gamma \in \Gamma_\zeta \mid \gamma \text{ is parabolic or the identity}\}.$$

Set

$$B(\mathbb{C}) \equiv \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mid 0 \neq a \in \mathbb{C}, b \in \mathbb{C} \right\} / \{\pm I\} < \mathrm{PSL}(2, \mathbb{C}),$$

$$N(\mathbb{C}) \equiv \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid b \in \mathbb{C} \right\} / \{\pm I\} < \mathrm{PSL}(2, \mathbb{C}).$$

The group $B(\mathbb{C})$ is the stabilizer subgroup of $\mathrm{PSL}(2, \mathbb{C})$ that fixes ∞ and $N(\mathbb{C})$ is the maximal parabolic subgroup of $B(\mathbb{C})$. It contains all parabolic elements fixing ∞ , and the identity element.

A point $\zeta \in \mathbb{P}$ is called a *cusps* of Γ if Γ'_ζ is a free abelian group of rank two. The set of cusps is denoted by C_Γ . Two cusps, α , and β of Γ called equivalent or Γ -equivalent if⁸ $\alpha \in \Gamma\beta$. The equivalence class of cusps is denoted by $\Gamma \setminus C_\Gamma$.

The following is well known (see [EGM98, Chapter 2]).

Lemma 2.5.1. *Let Γ be a Kleinian group.*

⁶The identity element is an exception. Its trace is ± 2 but it is not usually thought of as a parabolic element.

⁷See the previous footnote.

⁸ $\Gamma\beta$ denotes the orbit of the point $\beta \in \mathbb{P}$.

1. If Γ contains a parabolic element then Γ is not cocompact.
2. If Γ is cofinite and is not cocompact then Γ contains a parabolic element.
3. If Γ is cofinite, $\zeta \in \mathbb{P}$, and Γ_ζ contains a parabolic element then ζ is a cusp of Γ .
4. If Γ is cofinite then Γ has only finitely many Γ -equivalent classes of cusps.

Next we study the structure⁹ of the stabilizer subgroup of a cusp for cofinite Kleinian groups.

Lemma 2.5.2. [EGM98, Theorem 2.1.8 part (3)] *Let Γ be cofinite with a cusp at ∞ . Then Γ'_∞ is a lattice in $B(\mathbb{C}) \approx \mathbb{C}$ and one of the following three holds.*

1. $\Gamma_\infty = \Gamma'_\infty$
2. Γ_∞ is conjugate in $B(\mathbb{C})$ to a group of the form

$$\left\{ \left(\begin{array}{cc} \epsilon & \epsilon b \\ 0 & \epsilon^{-1} \end{array} \right) \mid 0 \neq b \in \Lambda, \epsilon \in \{1, i\} \right\} / \{\pm I\}$$

where $\Lambda < \mathbb{C}$ is an arbitrary lattice. As an abstract group Γ_∞ is isomorphic to $\mathbb{Z}^2 \rtimes \mathbb{Z}/2\mathbb{Z}$ where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts by multiplication by -1 .

3. Γ'_∞ is conjugate in $B(\mathbb{C})$ to a group of the form

$$\Gamma(n, t) = \left\{ \left(\begin{array}{cc} \epsilon & \epsilon b \\ 0 & \epsilon^{-1} \end{array} \right) \mid \begin{array}{l} b \in \mathcal{O}_n \\ \epsilon = \exp\left(\frac{\pi i v t}{n}\right) \text{ for } 1 \leq v \leq 2n \end{array} \right\} / \{\pm I\}$$

where $n = 4$ or $n = 6$ and $t|n$ and where \mathcal{O}_n is the ring of integers in the quadratic number field $\mathbb{Q}\left(\exp\left(\frac{2\pi i}{n}\right)\right)$. Hence, as an abstract group Γ'_∞ is isomorphic to the group $\mathbb{Z}^2 \rtimes \mathbb{Z}/m\mathbb{Z}$ for some $m \in \{1, 2, 3, 4, 6\}$. An element

$$\epsilon \in \mathbb{Z}/m\mathbb{Z} \cong \left\{ \exp\left(\frac{\pi i v}{n}\right) \mid v \in \mathbb{Z} \right\}$$

acts on $\mathbb{Z}^2 \cong \mathcal{O}_{m'}^+$ by multiplication with ϵ^2 , where $m' = 4$ in case $m \in \{1, 2, 4\}$ and $m' = 6$ otherwise.

2.6 Cofinite Kleinian Groups

Let Γ be a cofinite Kleinian group. By Lemma 2.5.1 Γ has finitely many equivalence classes of cusps.

Notation 2.6.1. Unless otherwise noted Γ is a cofinite Kleinian group with

$$\kappa \equiv |\Gamma \setminus C_\Gamma|,$$

and a maximal set $\{\zeta_\alpha\}_{\alpha=1}^\kappa$ of representatives for the equivalence classes of cusps. We set $\Gamma_\alpha \equiv \Gamma_{\zeta_\alpha}$ and $\Gamma'_\alpha \equiv \Gamma'_{\zeta_\alpha}$.

⁹The following structure theorem is extremely important, and without it we would be unable to extend the Selberg theory to cofinite Kleinian groups.

The following is elementary.

Lemma 2.6.2. *Let ζ_α be a cusp of Γ . Then there exists $B_\alpha \in \text{PSL}(2, \mathbb{C})$ and a lattice $\Lambda_\alpha = \mathbb{Z} \oplus \mathbb{Z}\tau_\alpha$, $\text{Im}(\tau_\alpha) > 0$ satisfying the following.*

1. $\zeta_\alpha = B_\alpha^{-1}\infty$,
2. $B_\alpha\Gamma_\alpha B_\alpha^{-1}$ acts discontinuously on $\mathbb{P} \setminus \{\infty\} = \mathbb{C}$.
- 3.

$$B_\alpha\Gamma'_\alpha B_\alpha^{-1} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \mid 0 \neq b \in \Lambda_\alpha \right\}.$$

Notation 2.6.3. For each cusp class $\alpha = 1 \dots \kappa$ we fix B_α and Λ_α from Lemma 2.6.2 and set $\Gamma_{\alpha_\infty} \equiv B_\alpha\Gamma_\alpha B_\alpha^{-1}$, and $\Gamma'_{\alpha_\infty} \equiv B_\alpha\Gamma'_\alpha B_\alpha^{-1}$.

While the action of Γ'_{α_∞} on \mathbb{C} is exactly the action of the (additive) lattice Λ_α on¹⁰ \mathbb{C} , the action of Γ_{α_∞} is a combination of the lattice action of Λ_α and possibly finite ordered euclidean rotations of \mathbb{C} .

The fundamental domain for a cofinite Kleinian group can be realized as the union of a compact (hyperbolic) polyhedron and κ cusp sectors. Let $\mathcal{P}_\alpha \subset \mathbb{C}$ be a fundamental domain¹¹ for the action of Γ_{α_∞} on \mathbb{C} , and let \mathcal{P}'_α be the fundamental parallelogram with base point at the origin of the lattice Λ_α . For $Y > 0$ set

$$\tilde{\mathcal{F}}_\alpha(Y) \equiv \{z + rj \mid z \in \mathcal{P}_\alpha, r \geq Y\}$$

and define the cusp sector, $\mathcal{F}_\alpha(Y) \equiv B_\alpha^{-1}\tilde{\mathcal{F}}_\alpha(Y)$.

Lemma 2.6.4. [EGM98, Prop. 2.3.9] *Let $\Gamma < \text{PSL}(2, \mathbb{C})$ be cofinite with $\kappa = |\Gamma \setminus C_\Gamma|$. Then there exist $Y > 0$ and a compact set $\mathcal{F}_Y \subset \mathbb{H}^3$ such that*

$$\mathcal{F} \equiv \mathcal{F}_Y \cup \mathcal{F}_1(Y) \cup \dots \cup \mathcal{F}_\kappa(Y)$$

is a fundamental domain for Γ . The compact set \mathcal{F}_Y can be chosen such that $\mathcal{F}_Y \cap \mathcal{F}_\alpha(Y_\alpha)$ all are contained in the boundary of \mathcal{F}_Y and hence have Lebesgue measure 0. Also, $\mathcal{F}_\alpha(Y_\alpha) \cap \mathcal{F}_\beta(Y_\beta) = \emptyset$ if $\alpha \neq \beta$.

2.7 Cuspidal elliptic elements

Throughout this section Γ is a cofinite Kleinian group.

Lemma 2.7.1. *Let Γ be cofinite. Then Γ has only finitely many elliptic conjugacy classes in Γ .*

Proof. Assume not. Then there is an infinite sequence of elliptic Γ -conjugacy classes $\{[e_n]\}$. Next for each n choose representative e_n which fixes a point P_n on the boundary of \mathcal{F} , the fundamental domain of Γ given in Lemma 2.6.4. Since Γ is discrete, the points *must* accumulate at least at one cusp. After conjugating Γ and passing to a subsequence of $\{P_n\}$ we may assume that $P_n \rightarrow \infty$.

¹⁰For $z \in \mathbb{C}$, we have $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} z = z + b$.

¹¹The domain \mathcal{P}_α is a euclidean polygon.

An application of [EGM98, Corollary 2.3.3] implies that $e_n \in \Gamma_\infty$. In short we have constructed infinitely many elliptic Γ -conjugacy classes with a representative fixing the cusp ∞ . An elementary computation using Lemma 2.5.2 shows that there are only finitely many elliptic Γ_∞ -conjugacy classes, a contradiction. \square

Notation 2.7.2. For each $\alpha \in \{1 \dots \kappa\}$ set

$$\Upsilon_\alpha \equiv \left\{ \left(\begin{array}{cc} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{array} \right) \mid \left(\begin{array}{cc} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{array} \right) \in \Gamma_{\alpha_\infty} \right\}.$$

If

$$\left(\begin{array}{cc} \epsilon & 0 \\ 0 & \epsilon^{-1} \end{array} \right) \in \Upsilon_\alpha,$$

then for some non-zero natural number N , $\epsilon^N = 1$. We have the following application of Lemma 2.5.2.

Lemma 2.7.3. *The set Υ_α is isomorphic to a finite subgroup of the unit circle S^1 , cyclic, and isomorphic to $\Gamma_{\infty\alpha}/\Gamma'_{\alpha_\infty}$. Any element $\gamma \in \Gamma_{\alpha_\infty}$ can be written uniquely in the form $\gamma = \alpha\beta$, where $\alpha \in \Upsilon_\alpha$ and $\beta \in \Gamma'_{\alpha_\infty}$.*

Definition 2.7.4. An elliptic element $\gamma \in \Gamma$, is said to be *cuspidal elliptic* if at least one¹² of its fixed points in \mathbb{P} is a cusp of Γ . Otherwise it is called a *non-cuspidal elliptic* element. The set of cuspidal elliptic elements is denoted by Γ^{ce} .

Notation 2.7.5. For $\alpha \in \{1 \dots \kappa\}$ define \mathcal{CE}_α to be the set of elements of Γ which are Γ -conjugate to an element of $\Gamma_\alpha \setminus \Gamma'_\alpha$. We fix representatives of $\mathcal{CE}_\alpha : g_1^\alpha, \dots, g_{d_\alpha}^\alpha$ and define $q_i^\alpha \equiv B_\alpha g_i^\alpha B_\alpha^{-1}$

Since q_i^α fixes ∞ ,

$$q_i^\alpha = \left(\begin{array}{cc} \epsilon_i^\alpha & \epsilon_i^\alpha w_i^\alpha \\ 0 & (\epsilon_i^\alpha)^{-1} \end{array} \right), \quad (2.7.1)$$

where ϵ_i^α is a finite-ordered root of unity and $w_i^\alpha \in \Lambda_\alpha$.

¹²Actually by [EGM98, Cor 2.3.11], if one fixed point is a cusp, then the other fixed point is also a cusp.

Chapter 3

The Spectral Decomposition Theorem

In this chapter we prove the spectral decomposition theorem for cofinite Kleinian groups with finite-dimensional unitary representations. While much of our work is analogous to the two-dimensional vector and three-dimensional scalar cases¹ there are some new complications that arise because of cuspidal elliptic elements².

3.1 Unitary Representations

Throughout this section Γ is a cofinite Kleinian group with $\kappa = |\Gamma \setminus C_\Gamma|$ and cusp representatives $\{\zeta_\alpha\}_{\alpha=1}^\kappa$.

Let V be an n -dimensional complex vector space, with an inner product $\langle \cdot, \cdot \rangle_V$ linear in the first argument and anti-linear in the second argument. For $v \in V$ its norm $|v|_V$ is given by $\sqrt{\langle v, v \rangle_V}$ and the norm of a linear operator $L : V \rightarrow V$ is defined as

$$|L|_V \equiv \sup_{v \in V} \left(\frac{|Lv|_V}{|v|_V} \right).$$

Definition 3.1.1. Define $\text{Rep}(\Gamma, V)$ to be the set of pairs (χ, V) where V a finite dimensional complex inner product space and χ is a unitary representation of Γ in $\text{GL}(V)$.

We will abuse notation and identify χ with (χ, V) .

Definition 3.1.2. Let Γ be cofinite, $\kappa = |\Gamma \setminus C_\Gamma|$ and $\chi \in \text{Rep}(\Gamma, V)$. For each $\alpha \in \{1, \dots, \kappa\}$ define

$$V_\alpha \equiv \{v \in V \mid \chi(\gamma)v = v \quad \forall \gamma \in \Gamma_\alpha\},$$

$$V'_\alpha \equiv \{v \in V \mid \chi(\gamma)v = v \quad \forall \gamma \in \Gamma'_\alpha\},$$

$$k_\alpha \equiv \dim V_\alpha,$$

$$l_\alpha \equiv \dim V'_\alpha, \text{ and}$$

$$k \equiv k(\Gamma, \chi) \equiv \sum_{\alpha=1}^{\kappa} k_\alpha.$$

¹We encourage the reader to first read [EGM98, Sections 4.1,4.2,4.3,6.1,6.2,6.3,6.4] and [Ven82, Chapter 1,2,3]

²Fuchsian groups do not have cuspidal elliptic elements.

The subspace V_α is called a *singular* subspace while V'_α is called an *almost singular*³ subspace, k_α is called the *degree of singularity* of the representation χ at the cusp α , and $k = k(\Gamma, \chi)$ the degree of singularity of Γ relative to χ .

Definition 3.1.3. A representation χ is *singular* at the cusp ζ_α if $k_\alpha > 0$, *regular* at ζ_α if $k_\alpha = 0$. It is *singular* if $k > 0$, and *regular* if $k = 0$.

Unitary representations of the stabilizer subgroup

Lemma 3.1.4. *There exist $E_\alpha, R_\alpha, S_\alpha \in \Gamma_\alpha$ with the following properties:*

- (1) $\Gamma_\alpha = \{ E_\alpha^k R_\alpha^i S_\alpha^j \mid 0 \leq k < m_\alpha, i, j \in \mathbb{Z} \}$. Here R_α, S_α are parabolic elements with $B_\alpha R_\alpha B_\alpha^{-1}(P) = P + 1$ and $B_\alpha S_\alpha B_\alpha^{-1}(P) = P + \tau_\alpha$ (here $\Lambda_\alpha = \mathbb{Z} \oplus \mathbb{Z}\tau_\alpha$) for all $P \in \mathbb{H}^3$, and E_α is elliptic of order m_α .
- (2) $\Gamma'_\alpha = \{ R_\alpha^i S_\alpha^j \mid i, j \in \mathbb{Z} \}$.
- (3) The elements R_α and S_α commute but the group Γ_α is not abelian when $m_\alpha > 1$.
- (4) If in addition, $m_\alpha > 1$, then $\chi(E_\alpha)$ maps V'_α onto itself. Furthermore, there exists a basis of V'_α so that $\chi(E_\alpha)|_{V'_\alpha}$ is diagonal.

Proof. (1), (2), and (3) readily follow from [EGM98, Theorem 2.1.8]. We prove (4): set $E = E_\alpha, R = R_\alpha, S = S_\alpha$. Since ERE^{-1} and R^{-1} are both parabolic and in Γ'_α it follows that the $A \equiv E^{-1}R^{-1}ER \in \Gamma'_\alpha$ is parabolic. Since A is parabolic, $A = R^i S^j$ for some $i, j \in \mathbb{Z}$ (applying (2)). By definition, the restriction $\chi(A)|_{V'_\alpha} = I_{V'_\alpha}$. We have

$$\chi(E)\chi(R) = \chi(R)\chi(E)\chi(A).$$

Next applying an arbitrary $v \in V'_\alpha$ we obtain

$$\chi(E)v = \chi(E)\chi(R)v = \chi(R)\chi(E)\chi(A)v = \chi(R)\chi(E)v \quad (3.1.1)$$

Thus $\chi(R)$ fixes $\chi(E)v$ and similarly $\chi(S)$ fixes $\chi(E)v$, so by definition $\chi(E)v \in V'_\alpha$. Since $\chi(E)$ is unitary, its restriction to V'_α is also unitary, hence V'_α has a diagonalizing basis. \square

We remark that for a cusp α of Γ , the group Γ'_α is abelian. Thus, the unitary representations χ restricted to Γ'_α is diagonalizable and can be thought of as a direct sum of one-dimensional unitary representations of Γ'_α .

3.2 Automorphic Functions

The set of Γ -automorphic functions $A(\Gamma, \chi)$ is the set of all Borel-measurable functions $f : \mathbb{H}^3 \rightarrow V$ that satisfy $f(\gamma P) = \chi(\gamma)f(P)$ for all $\gamma \in \Gamma$. Such functions are uniquely determined by their values on a fundamental domain. With this in mind we fix a fundamental domain $\mathcal{F} \subset \mathbb{H}^3$ for Γ and define

$$\mathcal{H}(\Gamma, \chi) \equiv L_2(\mathcal{F}, V, dv, \chi) \equiv \{ f \in A(\Gamma, \chi) \mid \int_{\mathcal{F}} \langle f(P), f(P) \rangle_V dv(P) < \infty \}.$$

For $f, g \in \mathcal{H}(\Gamma, \chi)$ define the inner product $\langle f, g \rangle \equiv \int_{\mathcal{F}} \langle f(P), g(P) \rangle_V dv(P)$. With this inner product $\mathcal{H}(\Gamma, \chi)$ is a Hilbert Space. For $n \in \mathbb{N}$ let $C^n(\mathbb{H}^3, V, \chi)$ be space of Γ -automorphic functions which are n -times differentiable on \mathbb{H}^3 .

³The notion of *almost singularity* does not occur in the two dimensional case.

The Automorphic Laplacian $\Delta(\Gamma, \chi)$

Recall that the Laplace-Beltrami operator Δ is defined on twice continuously differentiable functions $f : \mathbb{H}^3 \rightarrow \mathbb{C}$. Thus Δ can be defined on the set $A(\Gamma, \chi) \cap C^2(\mathbb{H}^3, V, \chi)$, and for $f \in A(\Gamma, \chi) \cap C^2(\mathbb{H}^3, V, \chi)$ it follows (since Δ commutes with isometries) that $\Delta f \in A(\Gamma, \chi) \cap C^2(\mathbb{H}^3, V, \chi)$.

Consider the dense subspace $D = \{f \in \mathcal{H}(\Gamma, \chi) \cap C^2(\mathbb{H}^3, V, \chi) \mid \Delta f \in \mathcal{H}(\Gamma, \chi)\}$. It follows that $\Delta : D \rightarrow \mathcal{H}(\Gamma, \chi)$ is essentially self-adjoint and has a unique positive self-adjoint extension to a space \tilde{D} (see [EGM98, Section 4.1]).

Definition 3.2.1. Let Γ be a cofinite Kleinian group and $\chi \in \text{Rep}(\Gamma, V)$. We define the automorphic Laplacian

$$\Delta(\Gamma, \chi) : \tilde{D} \rightarrow \mathcal{H}(\Gamma, \chi)$$

to be the self-adjoint extension of $\Delta : D \rightarrow \mathcal{H}(\Gamma, \chi)$.

When there is little possibility for confusion, we will identify Δ with $\Delta(\Gamma, \chi)$. The dense subspace \tilde{D} can be realized as the set of functions in $\mathcal{H}(\Gamma, \chi)$ whose *distributional* Laplacian is in $\mathcal{H}(\Gamma, \chi)$.

3.3 Fourier series expansion at a cusp

The scalar case

Let $f \in A(\Gamma, 1)$ (that is $f : \mathbb{H}^3 \rightarrow \mathbb{C}$, and for all $P \in \mathbb{H}^3$, $\gamma \in \Gamma$, $f(\gamma P) = f(P)$), and let ζ_α be a cusp of Γ with $B_\alpha^{-1}\infty = \zeta_\alpha$. Then it is easily seen that $f \circ B_\alpha^{-1} \in A(B_\alpha \Gamma B_\alpha^{-1}, 1)$, in particular $f \circ B_\alpha^{-1} \in A(B_\alpha \Gamma'_\alpha B_\alpha^{-1}, 1)$. Since $B_\alpha \Gamma'_\alpha B_\alpha^{-1}$ is isomorphic to the lattice Λ_α , $f \circ B_\alpha^{-1}(P + \omega) = f \circ B_\alpha^{-1}(P)$ for all $P \in \mathbb{H}^3$ and $\omega \in \Lambda_\alpha$. That is, $f \circ B_\alpha^{-1}$ is invariant under that lattice Λ_α . If we suppose further that f is smooth then we can expand $f \circ B_\alpha^{-1}$ as

$$f(B_\alpha^{-1}(z + rj)) = \sum_{\mu \in \Lambda_\alpha^0} g_\mu(r) e^{2\pi i \langle \mu, z \rangle}.$$

Here $\langle \cdot, \cdot \rangle$ is the (real) euclidean inner product in $\mathbb{R}^2 = \mathbb{C}$, Λ_α^0 is the lattice dual to Λ_α . If we assume still further that $\Delta f = \lambda f$, and $f(B_\alpha^{-1}(z + rj)) = O(r^N)$ for some $N \in \mathbb{N}$, then a simple separation of variables argument (see [EGM98, Section 3.3]) shows that for $\lambda = 1 - s^2$, $s \neq 0$, g_μ satisfies the Bessel equation

$$\left(r^2 \frac{d^2}{dr^2} - r \frac{d}{dr} + \lambda - 4\pi^2 |\mu|^2 r^2 \right) g_\mu(r) = 0, \quad (3.3.1)$$

whose general solution is $g_\mu(r) = a_\mu r K_s(2\pi|\mu|r) + b_\mu r I_s(2\pi|\mu|r)$. Here

$$I_s(w) = \sum_{m=0}^{\infty} \frac{w^{s+2m}}{m! 2^{s+2m} \Gamma(s+m+1)}, \quad K_s(w) = \frac{\pi}{2 \sin s\pi} (I_{-s}(w) - I_s(w)).$$

The function $K_s(x)$ decreases exponentially, and $I_s(x)$ increases exponentially as $x \rightarrow \infty$. Applying the growth bound of f we obtain

$$f(B_\alpha^{-1}(z + rj)) = a_0 r^{1+s} + b_0 r^{1-s} + \sum_{0 \neq \mu \in \Lambda_\alpha^0} a_\mu r K_s(2\pi|\mu|r) e^{2\pi i \langle \mu, z \rangle}. \quad (3.3.2)$$

We will need the following fact later on:

Lemma 3.3.1. *Let p be a non-negative integer. Then*

$$\frac{d^p}{dr^p} \sum_{0 \neq \mu \in \Lambda_\alpha^0} a_\mu r K_s(2\pi|\mu|r) e^{2\pi i \langle \mu, z \rangle} = O(e^{-|c|r}) \text{ as } r \rightarrow \infty,$$

for some constant $|c| > 0$.

Preliminary Lemmas

Our goal in this section is to come as close as possible to diagonalizing the group of unitary transformations $\{\chi(\Gamma_\alpha)\}$. Recall that

$$V_\alpha \equiv \{v \in V \mid \chi(\gamma)v = v \quad \forall \gamma \in \Gamma_\alpha\},$$

$$V'_\alpha \equiv \{v \in V \mid \chi(\gamma)v = v \quad \forall \gamma \in \Gamma'_\alpha\},$$

$$k_\alpha \equiv \dim V_\alpha, \text{ and}$$

$$l_\alpha \equiv \dim V'_\alpha.$$

For each cusp ζ_α of Γ let $(I - P_\alpha) : V \rightarrow V_\alpha^\perp$ denote the orthogonal projection onto V_α^\perp . By definition of singularity, for each $0 \neq v \in V_\alpha^\perp$ there exists $\gamma \in \Gamma_\alpha$ satisfying $\chi(\gamma)v \neq v$.

Fix a cusp ζ_α of Γ . We can partition $n = \dim V$ as follows:

$$1, \dots, k_\alpha, k_\alpha + 1, \dots, l_\alpha, l_\alpha + 1, \dots, n.$$

We will next choose a particular basis of V that respects the partitions above and is as close as possible to diagonalizing $\{\chi(\Gamma_\alpha)\}$.

Recall from Lemma (3.1.4) that $\Gamma_\alpha = \{E_\alpha^k R_\alpha^i S_\alpha^j \mid 0 \leq k < m_\alpha, i, j \in \mathbb{Z}\}$. Here R_α, S_α are parabolic and E_α is elliptic.

Let $B_{s\alpha} \equiv \{v_{\alpha 1}, \dots, v_{\alpha k_\alpha}\}$ be an orthonormal basis for V_α . By lemma (3.1.4) there exists a set $B_{a\alpha} \equiv \{v_{\alpha(k_\alpha+1)}, \dots, v_{\alpha l_\alpha}\}$ of orthonormal elements in V , pairwise orthogonal to the elements of $B_{s\alpha}$ so that the following conditions are satisfied: For each $v \in B_{a\alpha}$, $\chi(\gamma)v = v$ for all $\gamma \in \Gamma'_\alpha$, and $\chi(E_\alpha)v = \lambda v$, where here E_α is the primitive elliptic element in Γ_α chosen in lemma (3.1.4), and $1 \neq \lambda \in \mathbb{C}$ with $\lambda^{m_\alpha} = 1$. Finally, there exists a set $B_{r\alpha} \equiv \{v_{\alpha(l_\alpha+1)}, \dots, v_{\alpha n}\}$ of orthonormal elements of V , pairwise orthogonal to the elements of $B_{s\alpha} \cup B_{a\alpha}$ so that the following conditions are satisfied: For each $v \in B_{r\alpha}$, $\chi(\gamma)v = \lambda'v$ for each $\gamma \in \Gamma'_\alpha$ and some $1 \neq \lambda' \in \mathbb{C}$ with $|\lambda'| = 1$. We have proven the following.

Lemma 3.3.2. *For each cusp ζ_α let $\Gamma_\alpha = \{E_\alpha^k R_\alpha^i S_\alpha^j \mid 0 \leq k < m_\alpha, i, j \in \mathbb{Z}\}$ (see lemma (3.1.4)). Then there exists an orthonormal basis*

$$\{v_{\alpha 1}, \dots, v_{\alpha k_\alpha}, v_{\alpha(k_\alpha+1)}, \dots, v_{\alpha l_\alpha}, v_{\alpha(l_\alpha+1)}, \dots, v_{\alpha n}\} = B_{s\alpha} \cup B_{a\alpha} \cup B_{r\alpha}$$

of V with the following properties:

(1) For each $v_{\alpha l} \in B_{s\alpha}$ ($1 \leq l \leq k_\alpha$), $\chi(\gamma)v_{\alpha l} = v_{\alpha l}$ for all $\gamma \in \Gamma_\alpha$.

(2) For each $v_{\alpha l} \in B_{a\alpha}$ ($k_\alpha + 1 \leq l \leq l_\alpha$), $\chi(R_\alpha)v_{\alpha l} = \chi(S_\alpha)v_{\alpha l} = v_{\alpha l}$, and $\chi(E_\alpha)v_{\alpha l} = \lambda_{\alpha l}v_{\alpha l}$. Here $1 \neq \lambda_{\alpha l} \in \mathbb{C}$ and $\lambda_{\alpha l}^{m_\alpha} = 1$.

(3) For each $v_{\alpha l} \in B_{r\alpha}$ ($l_\alpha + 1 \leq l \leq n$), $\chi(R_\alpha)v_{\alpha l} = \lambda_{R_\alpha l}v_{\alpha l}$, and $\chi(S_\alpha)v_{\alpha l} = \lambda_{S_\alpha l}v_{\alpha l}$. Here $\lambda_{R_\alpha l}, \lambda_{S_\alpha l} \in \mathbb{C}$, with $|\lambda_{R_\alpha l}| = |\lambda_{S_\alpha l}| = 1$, and $\lambda_{R_\alpha l}, \lambda_{S_\alpha l}$ are not both equal to one.

For each cusp ζ_α and $l_\alpha + 1 \leq l \leq n$ fix $\theta_{R_\alpha l}$ and $\theta_{S_\alpha l}$ so that

$$e^{2\pi i \theta_{R_\alpha l}} = \lambda_{R_\alpha l} \text{ and } e^{2\pi i \theta_{S_\alpha l}} = \lambda_{S_\alpha l}. \quad (3.3.3)$$

We remark that B_{s_α} and B_{r_α} are analogous to the two dimensional *singular* and *regular* cases respectively (see [Ven82]). We call B_{a_α} the *almost singular* case.

Let $P_{\alpha l}$ be the orthogonal projection onto the subspace generated by $v_{\alpha l}$.

Lemma 3.3.3. *Let $f \in A(\Gamma, \chi)$.*

(1) *If $1 \leq l \leq l_\alpha$, then*

$$P_{\alpha l} f(B_\alpha^{-1}(Q + \omega)) = P_{\alpha l} f(B_\alpha^{-1}Q)$$

for all $Q \in \mathbb{H}^3$ and $\omega \in \Lambda_\alpha$.

(2) *If $l_\alpha + 1 \leq l \leq n$, then for $n, m \in \mathbb{Z}$,*

$$P_{\alpha l} f(B_\alpha^{-1}(Q + m + n\tau_\alpha)) = e^{2\pi i(m\theta_{R_\alpha l} + n\theta_{S_\alpha l})} P_{\alpha l} f(B_\alpha^{-1}Q),$$

where $\mathbb{Z} \oplus \mathbb{Z}\tau_\alpha = \Lambda_\alpha$.

Proof. (1) Let $\omega \in \Lambda_\alpha$. Since Λ_α is isomorphic to $B_\alpha \Gamma'_\alpha B_\alpha^{-1}$ there exists $\gamma \in \Gamma'_\alpha$ satisfying $B_\alpha \gamma B_\alpha^{-1}Q = Q + \omega$ for all $Q \in \mathbb{H}^3$ (See lemma (3.1.4)). Since $f \in A(\Gamma, \chi)$, it follows that $P_{\alpha l} f(\gamma Q) = P_{\alpha l} \chi(\gamma) f(Q)$. However,

$$P_{\alpha l} \chi(\gamma) = (\chi(\gamma)^* P_{\alpha l}^*)^* = (\chi(\gamma^{-1}) P_{\alpha l})^* = (P_{\alpha l})^* = P_{\alpha l}.$$

Thus,

$$P_{\alpha l} f(B_\alpha^{-1}(Q + \omega)) = P_{\alpha l} f(B_\alpha^{-1} B_\alpha \gamma B_\alpha^{-1}(Q)) = P_{\alpha l} f(\gamma B_\alpha^{-1}Q) = P_{\alpha l} f(B_\alpha^{-1}Q).$$

(2) A direct calculation similar to (1) shows that $P_{\alpha l} f(B_\alpha^{-1}(Q + m + n\tau_\alpha)) = P_{\alpha l} f(R_\alpha^m S_\alpha^n B_\alpha^{-1}Q) = P_{\alpha l} \chi(R_\alpha)^m \chi(S_\alpha)^n f(B_\alpha^{-1}Q) = e^{2\pi i(m\theta_{R_\alpha l} + n\theta_{S_\alpha l})} P_{\alpha l} f(B_\alpha^{-1}Q)$. \square

Fourier Expansion: The vector case

Proposition 3.3.4. *Let Γ be cofinite, $f \in A(\Gamma, \chi)$ be smooth satisfying $f(B_\alpha^{-1}(z + rj)) = O(r^N)$ for some $N \in \mathbb{N}$ and each cusp ζ_α of Γ , and $\Delta f = \lambda f$ with $\lambda = 1 - s^2$, $s \neq 0$.*

(1) *If $1 \leq l \leq k_\alpha$, then*

$$P_{\alpha l} f(B_\alpha^{-1}(z + rj)) = a_{\alpha l, 0} r^{1+s} + b_{\alpha l, 0} r^{1-s} + \sum_{0 \neq \mu \in \Lambda_\alpha^0} a_{\alpha l, \mu} r K_s(2\pi|\mu|r) e^{2\pi i \langle \mu, z \rangle}.$$

(2) *If $k_\alpha + 1 \leq l \leq l_\alpha$, then*

$$P_{\alpha l} f(B_\alpha^{-1}(z + rj)) = \sum_{0 \neq \mu \in \Lambda_\alpha^0} a_{\alpha l, \mu} r K_s(2\pi|\mu|r) e^{2\pi i \langle \mu, z \rangle}.$$

(3) *If $l_\alpha + 1 \leq l \leq n$, $\theta_{R_\alpha l}, \theta_{S_\alpha l}$ are chosen so that $\exp(2\pi i \theta_{R_\alpha l}) = \lambda_{R_\alpha l}$, and $\exp(2\pi i \theta_{S_\alpha l}) = \lambda_{S_\alpha l}$, then*

$$P_{\alpha l} f(B_\alpha^{-1}(z + rj)) = \sum_{\mu \in \Lambda_\alpha^0} a_{\alpha l, \mu} r K_s(2\pi|\mu'r|) e^{2\pi i \langle \mu', z \rangle}.$$

where $\mu' = \theta_R \mu_1 + \theta_S \mu_2 + \mu$, Λ_α^0 is the lattice dual to Λ_α generated by μ_1, μ_2 , where

$$\langle \mu_1, 1 \rangle = 1, \langle \mu_1, \tau_\alpha \rangle = 0, \langle \mu_2, 1 \rangle = 0, \langle \mu_2, \tau_\alpha \rangle = 1,$$

and the $a_{\alpha l, \mu}$ are scalar multiples of the vector $v_{\alpha l}$.

Proof. (1) By Lemma 3.3.3 $P_{\alpha l} f(B_{\alpha}^{-1}(z + rj))$ is a scalar function multiplied by the vector $v_{\alpha l}$, thus an application of equation (3.3.2) concludes the proof.

(2) By (1)

$$P_{\alpha l} f(B_{\alpha}^{-1}(z + rj)) = a_{\alpha l, 0} r^{1+s} + b_{\alpha l, 0} r^{1-s} + \sum_{0 \neq \mu \in \Lambda_{\alpha}^0} a_{\alpha l, \mu} r K_s(2\pi|\mu|r) e^{2\pi i \langle \mu, z \rangle}.$$

We will show that $a_{\alpha l, 0}$ and $b_{\alpha l, 0}$ are both zero. To see this first, observe that (by Lemma 3.3.1)

$$\begin{aligned} P_{\alpha l} f(B_{\alpha}^{-1}(z + rj)) &= a_{\alpha l, 0} r^{1+s} + b_{\alpha l, 0} r^{1-s} + \sum_{0 \neq \mu \in \Lambda_{\alpha}^0} a_{\alpha l, \mu} r K_s(2\pi|u|r) e^{2\pi i \langle u, z \rangle} \\ &= a_{\alpha l, 0} r^{1+s} + b_{\alpha l, 0} r^{1-s} + O(e^{-|c|r}) \text{ as } r \rightarrow \infty. \end{aligned}$$

Next,

$$P_{\alpha l} f(B_{\alpha}^{-1}(B_{\alpha} E_{\alpha} B_{\alpha}^{-1})(z + rj)) = a_{\alpha l, 0} r^{1+s} + b_{\alpha l, 0} r^{1-s} + O(e^{-|c|r}) \text{ as } r \rightarrow \infty$$

since $B_{\alpha} E_{\alpha} B_{\alpha}^{-1}(z + rj) = z' + rj$ for some $z' \in \mathbb{Z}$, that is $B_{\alpha} E_{\alpha} B_{\alpha}^{-1}$ fixes the r -coordinate in \mathbb{H}^3 (this follows from elementary computations and that fact that $B_{\alpha} E_{\alpha} B_{\alpha}^{-1} \infty = \infty$). However, we also have

$$\begin{aligned} P_{\alpha l} f(B_{\alpha}^{-1}(B_{\alpha} E_{\alpha} B_{\alpha}^{-1})(z + rj)) &= P_{\alpha l} \chi(E_{\alpha}) f(B_{\alpha}^{-1}(z + rj)) = \\ \lambda_{\alpha l} P_{\alpha l} f(B_{\alpha}^{-1}(z + rj)) &= \lambda_{\alpha l} (a_{\alpha l, 0} r^{1+s} + b_{\alpha l, 0} r^{1-s} + O(e^{-|c|r})), \end{aligned}$$

a contradiction since $\lambda_{\alpha l} \neq 1$.

(3) Let $h(Q) \equiv h(z) \equiv \exp(\langle \mu_1, z \rangle \theta_{R\alpha l} + \langle \mu_2, z \rangle \theta_{S\alpha l})$, where $Q = z + rj$. Then $h(Q)$ satisfies

$$h(Q + m + \tau_{\alpha} n) = \exp(2\pi i(m\theta_{R\alpha l} + n\theta_{S\alpha l})) h(Q),$$

thus by lemma (3.3.3)

$$\frac{P_{\alpha l} f \circ B_{\alpha}^{-1}}{h}(Q + \omega) = \frac{P_{\alpha l} f \circ B_{\alpha}^{-1}}{h}(Q)$$

for all $Q \in \mathbb{H}^3$ and $\omega \in \Lambda_{\alpha}$. Thus we can write

$$\frac{P_{\alpha l} f \circ B_{\alpha}^{-1}}{h}(Q) = \sum_{\mu \in \Lambda_{\alpha}^0} g_{\mu}(r) e^{2\pi i \langle \mu, z \rangle},$$

which is equivalent to

$$P_{\alpha l} f \circ B_{\alpha}^{-1}(Q) = \sum_{\mu \in \Lambda_{\alpha}^0} g_{\mu}(r) h(z) e^{2\pi i \langle \mu, z \rangle}.$$

To conclude the proof recall that $\Delta f = \lambda f$, f grows by at most $O(r^N)$ near each cusp, and apply the separation of variables technique (see [EGM98, page 105]). \square

3.4 The Eisenstein series

As usual for this section let Γ be cofinite with cusp representatives $\{\zeta_{\alpha}\}_{\alpha=1}^{\kappa}$, $B_{\alpha}^{-1} \infty = \zeta_{\alpha}$, and $(\chi, V) \in \text{Rep}(\Gamma, V)$.

Definitions

For $P \in \mathbb{H}^3$, $\operatorname{Re}(s) > 1$, and $v \in V_\alpha$ we define the *Eisenstein series* by

$$E(P, s, \alpha, v) \equiv E(P, s, \alpha, v, \Gamma, \chi) \equiv \sum_{M \in \Gamma_\alpha \backslash \Gamma} (r(B_\alpha M P))^{1+s} \chi(M)^* v.$$

The series $E(P, s, \alpha, v)$ converges uniformly and absolutely on compact subsets of $\{\operatorname{Re}(s) > 1\} \times \mathbb{H}^3$, and is a χ -automorphic function that satisfies

$$\Delta E(\cdot, s, \alpha, v) = \lambda E(\cdot, s, \alpha, v).$$

In addition

$$E(B_\beta^{-1}(z + rj), s, \alpha, v) = O(r^N)$$

for any cusp ζ_β .

Fourier expansion of the Eisenstein series

In this section we give the explicit Fourier expansion of the Eisenstein series.

For ζ_α a cusp of Γ and $1 \leq l \leq k_\alpha$, set $E_{\alpha l} \equiv E_{\alpha l}(s) \equiv E(\cdot, s, \alpha, v_{\alpha l})$

Lemma 3.4.1. *Let $\zeta_\alpha, \zeta_\beta$ be cusps of Γ , $1 \leq l \leq k_\alpha$, and $1 \leq k \leq k_\beta$. Then*

$$\begin{aligned} P_\beta E_{\alpha l}(B_\beta^{-1}(z + rj)) &= \delta_{\alpha\beta} r^{1+s} v_{\beta l} + \\ &\frac{\pi}{|\Gamma_\alpha : \Gamma'_\alpha| |\Lambda_\beta|^s} \left(\sum_{M \in \Gamma'_\alpha \backslash \Gamma / \Gamma'_\beta} P_\beta \chi^*(M) v_{\alpha, l} |c|^{-2-2s} \right) r^{1-s} + \\ g(s) \sum_{0 \neq |\mu| \in \Lambda_\beta^0} |\mu|^s &\left(\sum_{M \in \Gamma'_\alpha \backslash \Gamma / \Gamma'_\beta} P_\beta \chi^*(M) v_{\alpha, l} \frac{e^{2\pi i \langle \mu, \frac{d}{c} \rangle}}{|c|^{2+2s}} \right) r K_s(2\pi |\mu| r) e^{2\pi i \langle \mu, z \rangle} \end{aligned}$$

where

$$g(s) \equiv \frac{2\pi^{1+s}}{|\Gamma_\alpha : \Gamma'_\alpha| |\Lambda_\beta| \Gamma(1+s)},$$

$c = c(B_\alpha M B_\beta^{-1})$ and $d = d(B_\alpha M B_\beta^{-1})$ are the bottom left and bottom right indices respectively of the two-by-two matrix $B_\alpha M B_\beta^{-1}$.

Proof. An application of Proposition 3.3.4 part (1) gives us the general form of the Fourier series expansion. Then apply the argument found in [EGM98, page 111]. \square

A *Maaß form* is a smooth function $f \in A(\Gamma, \chi)$ satisfying $f(B_\alpha^{-1}(z + rj)) = O(r^N)$ for some $N \in \mathbb{N}$ and for each cusp ζ_α of Γ , and $\Delta f = \lambda f$ with $\lambda = 1 - s^2$, $s \neq 0$.

We saw in proposition (3.3.4) that for $1 \leq l \leq k_\alpha$, a Maaß form f can be expanded as

$$P_{\alpha l} f(B_\alpha^{-1}(z + rj)) = a_{\alpha l, 0} r^{1+s} + b_{\alpha l, 0} r^{1-s} + \sum_{0 \neq \mu \in \Lambda_\alpha^0} a_{\alpha l, \mu} r K_s(2\pi |\mu| r) e^{2\pi i \langle \mu, z \rangle}.$$

The r^{1+s} , r^{1-s} terms will play a prominent role in the what follows. Define $u_{f_{\alpha l}}$ by

$$u_{f_{\alpha l}}(B_\alpha^{-1}(z + rj)) = a_{\alpha l, 0} r^{1+s} v_{\alpha l} + b_{\alpha l, 0} r^{1-s} v_{\alpha l}.$$

The scattering matrix

Since the Eisenstein series is a Maaß form, we can apply the function u to read off its constant term in its Fourier expansion. Define the *scattering matrix* $\mathfrak{S}_{\alpha,l,\beta,k}(s)$ by

$$u_{(E_{\alpha l})_{\beta k}}(B_{\beta}^{-1}(z + rj)) = \delta_{\alpha\beta}\delta_{lk}r^{1+s}v_{\beta k} + \mathfrak{S}_{\alpha,l,\beta,k}(s)r^{1-s}v_{\beta k}.$$

We can put the scattering matrix into a $k(\Gamma, \chi) \times k(\Gamma, \chi)$ matrix. Let $\mathfrak{S}(s)$ be the $k(\Gamma, \chi) \times k(\Gamma, \chi)$ matrix given block-wise,

$$(\mathfrak{S}_{\alpha\beta}(s)) \equiv (\mathfrak{S}_{\alpha,l,\beta,k}(s))$$

where $1 \leq l \leq k_{\alpha}, 1 \leq k \leq k_{\beta}$.

3.5 The Maaß-Selberg Relations

Our main goal in this section is to prove the Maaß-Selberg relations in the case of cofinite Kleinian groups with finite dimensional unitary representations. The relations will be needed to prove the Selberg trace formula. For this section assume Γ is a cofinite Kleinian group and $(\chi, V) \in \text{Rep}(\Gamma, V)$.

Greene's Theorem

The basic tool for proving the Maaß-Selberg relations is Greene's Theorem. Let $D \subset \mathbb{H}^3$ be compact with a piece-wise smooth boundary, f, g be complex valued smooth functions, dv the volume form on \mathbb{H}^3 , and \widetilde{dv} the induced volume form on the boundary of D , ∂D . Then Greene's theorem is

$$\int_D (f \Delta \bar{g} - \bar{g} \Delta f) dv = \int_{\partial D} (f \frac{\partial}{\partial n} \bar{g} - \bar{g} \frac{\partial}{\partial n} f) \widetilde{dv}.$$

Our application of Greene's theorem will be to the compact part of the fundamental domain \mathcal{F} of Γ . Writing Greene's theorem in the global coordinates of \mathbb{H}^3 using $dv = \frac{dx \wedge dy \wedge dr}{r^3}$, $\frac{\partial}{\partial n} = r \frac{\partial}{\partial r}$, $\widetilde{dv} = \frac{\partial}{\partial n} \perp dv = \frac{dx \wedge dy}{r^2}$ and Greene's Theorem becomes,

$$\int_D (f \Delta \bar{g} - \bar{g} \Delta f) \frac{dx dy dr}{r^3} = \int_{\partial D} (f \frac{\partial}{\partial r} \bar{g} - \bar{g} \frac{\partial}{\partial r} f) \frac{dx dy}{r}$$

when ∂D is a two-dimensional region parallel to \mathbb{C} on the boundary of \mathbb{H}^3 .

The scalar Maaß-Selberg relations

For this section, assume Γ has only one class of cusps and that $\chi = 1$. Let $f \in A(\Gamma, 1)$ and assume that f has polynomial growth as $z + rj$ approaches the cusp at ∞ , is smooth, and satisfies $\Delta f = \lambda f$ (in other words f is a Maaß form). Let Λ be the lattice corresponding to Γ'_{∞} . Then we can expand

$$f(z + rj) = \sum_{\mu \in \Lambda_0} g_{\mu}(r) e^{2\pi i \langle \mu, z \rangle}$$

where

$$(r^2 \frac{d^2}{dr^2} - r \frac{d}{dr} + \lambda - 4\pi^2 |\mu|^2 r^2) g_{\mu}(r) = 0.$$

Lemma 3.5.1. *Let $\Delta f = \lambda f$, $\Delta g = \nu g$ with $\lambda = 1 - s^2$, $\nu = 1 - t^2$, $s \neq 0$, $t \neq 0$. Assume Γ has only one cusp at ∞ and*

$$f(P) = ar^{1+s} + br^{1-s} + \sum_{\mu \in \Lambda_0} f_\mu(r) e^{2\pi i \langle \mu, z \rangle}$$

$$g(P) = cr^{1+t} + dr^{1-t} + \sum_{\mu \in \Lambda_0} g_\mu(r) e^{2\pi i \langle \mu, z \rangle}$$

Let $u_f(P) = ar^{1+s} + br^{1-s}$, $u_g(P) = cr^{1+t} + dr^{1-t}$

$$h^Y(P) = \begin{cases} h(P) - u_h(P), & P \in \mathcal{F}(Y) \\ h(P), & P \in \mathcal{F}_0 \end{cases}$$

for $h = f, g$. Then

$$\begin{aligned} (s^2 - \bar{t}^2) \int_{\mathcal{F}} f^Y \bar{g}^Y dv &= \frac{|\Lambda|}{|\Gamma_\infty : \Gamma'_\infty|} \left((s - \bar{t}) a \bar{c} Y^{s+\bar{t}} + (s + \bar{t}) a d \bar{Y}^{s-\bar{t}} \right) \\ &\quad - \frac{|\Lambda|}{|\Gamma_\infty : \Gamma'_\infty|} \left((s + \bar{t}) b \bar{c} Y^{-s+\bar{t}} + (s - \bar{t}) b d \bar{Y}^{-s-\bar{t}} \right) \end{aligned}$$

Proof. Let $D = \mathcal{F}_0 = \mathcal{F} \setminus \mathcal{F}(Y)$ (here $\mathcal{F}(Y)$ is the cusp sector that goes to infinity). Then by Greene's theorem

$$\int_D (f \Delta \bar{g} - \bar{g} \Delta f) dv = \int_{\partial D} \left(f \frac{\partial}{\partial n} \bar{g} - \bar{g} \frac{\partial}{\partial n} f \right) \widetilde{dv}.$$

Since the (vertical) sides of D are pair-wise identified by isometries (with opposite orientation) the only surviving boundary term is the horizontal cross section (the boundary where \mathcal{F} was separated into \mathcal{F}_0 and $\mathcal{F}(Y)$). Greene's theorem becomes

$$\int_D (f \Delta \bar{g} - \bar{g} \Delta f) \frac{dx dy dr}{r^3} = \int_P \left(f \frac{\partial}{\partial r} \bar{g} - \bar{g} \frac{\partial}{\partial r} f \right) \frac{dx dy}{r}, \quad (3.5.1)$$

where $dv = \frac{dx \wedge dy \wedge dr}{r^3}$, $\frac{\partial}{\partial n} = r \frac{\partial}{\partial r}$, $\widetilde{dv} = \frac{\partial}{\partial n} \perp dv = \frac{dx \wedge dy}{r^2}$, and $P \equiv \partial D$ is the horizontal boundary region described above.

The set P is a vertical translation of a fundamental domain of the action of Γ_∞ on \mathbb{C} . That is the r -coordinate projection is constant and equal to Y . Set

$$P' \equiv \bigcup_{\gamma \in \Gamma'_\infty \setminus \Gamma_\infty} \gamma P.$$

Then P' is a fundamental domain for $\Lambda = \Lambda_\infty$.

With our notation the right side of equation (3.5.1) becomes

$$\begin{aligned} &\frac{1}{|\Gamma_\infty : \Gamma'_\infty|} \int_{P'} \left(\left(\sum_{\mu \neq 0} f_\mu(Y) e^{2\pi i \langle \mu, z \rangle} \right) \left(\sum_{\mu \neq 0} \bar{g}'_\mu(Y) e^{-2\pi i \langle \mu, z \rangle} \right) \right. \\ &\quad \left. - \left(\sum_{\mu \neq 0} \bar{g}_\mu(Y) e^{-2\pi i \langle \mu, z \rangle} \right) \left(\sum_{\mu \neq 0} f'_\mu(Y) e^{2\pi i \langle \mu, z \rangle} \right) \right) \frac{dx dy}{Y} \end{aligned}$$

$$\begin{aligned}
& + \frac{|\Lambda|}{|\Gamma_\infty : \Gamma'_\infty|} \left((s - \bar{t}) a \bar{c} Y^{s+\bar{t}} + (s + \bar{t}) a \bar{d} Y^{s-\bar{t}} \right) \\
& - \frac{|\Lambda|}{|\Gamma_\infty : \Gamma'_\infty|} \left((s + \bar{t}) b \bar{c} Y^{-s+\bar{t}} + (s - \bar{t}) b \bar{d} Y^{-s-\bar{t}} \right).
\end{aligned}$$

In the terms above involving multiplications of lattice sums, after integration over P' the only surviving terms are (by the orthogonality of the family $\{e^{2\pi i \langle \mu, z \rangle}\}$)

$$\frac{|\Lambda|}{|\Gamma_\infty : \Gamma'_\infty|} \sum_{\mu \neq 0} (f_\mu(Y) \bar{g}'_\mu(Y) - \bar{g}_\mu(Y) f'_\mu(Y)) \frac{1}{Y}.$$

A direct calculation using the fact that g_μ and f_μ satisfy the Bessel equation (3.3.1) shows that

$$\frac{d}{dr} \left(\frac{f_\mu(r) \bar{g}'_\mu(r) - \bar{g}_\mu(r) f'_\mu(r)}{r} \right) = \frac{1}{r^3} (s^2 - \bar{t}^2) f_\mu(r) \bar{g}_\mu(r)$$

thus

$$\begin{aligned}
& \frac{|\Lambda|}{|\Gamma_\infty : \Gamma'_\infty|} \sum_{\mu \neq 0} (f_\mu(Y) \bar{g}'_\mu(Y) - \bar{g}_\mu(Y) f'_\mu(Y)) \frac{1}{Y} \\
& = (s^2 - \bar{t}^2) \frac{|\Lambda|}{|\Gamma_\infty : \Gamma'_\infty|} \sum_{\mu \neq 0} \int_Y^\infty f_\mu(r) \bar{g}_\mu(r) \frac{dr}{r^3} = (s^2 - \bar{t}^2) \int_{\mathcal{F}(Y)} f^Y \bar{g}^Y dv.
\end{aligned}$$

By definition $\int_D f^Y \bar{g}^Y dv = \int_D f \bar{g} dv$ and thus combing the integral of D and $F(Y)$ we obtain the lemma. \square

The vector form of the Maaß-Selberg relations

Let Γ be cofinite with cusps $\zeta_\alpha = B_\alpha^{-1} \infty$ for $\alpha = 1 \dots \kappa$, with fundamental domain $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}_1(Y) \cup \dots \cup \mathcal{F}_\kappa(Y)$. For $(\chi, V) \in \text{Rep}(\Gamma, V)$.

Let $f \in A(\Gamma, \chi)$ be a Maaß form, that is f is smooth, satisfies $f(B_\alpha^{-1}(z + rj)) = O(r^N)$ for some $N \in \mathbb{N}$ and for each cusp ζ_α of Γ , and $\Delta f = \lambda f$ with $\lambda = 1 - s^2$, $s \neq 0$. We define

$$f^Y(P) = \begin{cases} f(P) - \sum_{l=1}^{k_\alpha} u_{f, \alpha l}(P) & \text{if } P \in \mathcal{F}_\alpha(Y), \alpha = 1 \dots \kappa \\ f(P) & \text{otherwise} \end{cases}$$

Theorem 3.5.2. (Maaß-Selberg Relations) Let Γ be cofinite, $(\chi, V) \in \text{Rep}(\Gamma, V)$, f and g be Maaß forms satisfying $\Delta f = \lambda f$, $\Delta g = \nu g$ with $\lambda = 1 - s^2$, $\nu = 1 - t^2$, $s \neq 0$, $t \neq 0$. Suppose that $u_{f, \alpha l}(B_\alpha^{-1}P) = a_{\alpha l} r^{1+s} + b_{\alpha l} r^{1-s}$ and $u_{g, \alpha l}(B_\alpha^{-1}P) = c_{\alpha l} r^{1+s} + d_{\alpha l} r^{1-s}$. Then

$$\begin{aligned}
& (s^2 - \bar{t}^2) \int_{\mathcal{F}} \langle f^Y, g^Y \rangle_V dv \\
& = \sum_{\alpha=1}^{\kappa} \frac{|\Lambda_\alpha|}{|\Gamma_\alpha : \Gamma'_\alpha|} \sum_{l=1}^{k_\alpha} \left[(s - \bar{t}) \langle a_{\alpha l}, c_{\alpha l} \rangle_V Y^{s+\bar{t}} + (s + \bar{t}) \langle a_{\alpha l}, d_{\alpha l} \rangle_V Y^{s-\bar{t}} \right. \\
& \quad \left. - (s + \bar{t}) \langle b_{\alpha l}, c_{\alpha l} \rangle_V Y^{-s+\bar{t}} + (s - \bar{t}) \langle b_{\alpha l}, d_{\alpha l} \rangle_V Y^{-s-\bar{t}} \right]
\end{aligned}$$

Proof. For notational simplicity we assume Γ has only one class of cusps at ∞ . We can write $P_\infty = P_1 + \cdots + P_{k_\infty}$ (see §3.3).

$$\begin{aligned} (s^2 - \bar{t}^2) \int_{\mathcal{F}} \langle f^Y, g^Y \rangle_V dv & \\ &= (s^2 - \bar{t}^2) \int_{\mathcal{F}} \langle (P_\infty + (1 - P_\infty))f^Y, (P_\infty + (1 - P_\infty))g^Y \rangle_V dv \\ &= (s^2 - \bar{t}^2) \int_{\mathcal{F}} (\langle P_\infty f^Y, P_\infty g^Y \rangle_V + \langle (1 - P_\infty)f^Y, (1 - P_\infty)g^Y \rangle_V) dv. \end{aligned} \quad (3.5.2)$$

The equality follows from the fact that P_∞ is an orthogonal projection. We will first show

$$(s^2 - \bar{t}^2) \int_{\mathcal{F}} \langle (1 - P_\infty)f^Y, (1 - P_\infty)g^Y \rangle_V dv = 0 \quad (3.5.3)$$

Since $(1 - P_\infty)f$ has no *constant* Fourier expansion coefficients (see Proposition 3.3.4) $(1 - P_\infty)f^Y = (1 - P_\infty)f$, and by Lemma 3.3.1

$$\frac{d^p}{dr^p} (1 - P_\infty)f(B_\alpha^{-1}(z + rj)) = O(e^{-|c|r}) \quad (3.5.4)$$

for $p \geq 0$ and as $r \rightarrow \infty$.

Next to prove equation (3.5.3) we apply the vector form of Greene's theorem,

$$\int_D (\langle f, \Delta g \rangle_V - \langle g, \Delta f \rangle) dv = \int_{\partial D} \left(\left\langle f, \frac{\partial}{\partial n} g \right\rangle_V - \left\langle g, \frac{\partial}{\partial n} f \right\rangle_V \right) \widetilde{dv}$$

and we obtain (by (3.5.4))

$$\begin{aligned} \lim_{R \rightarrow \infty} (s^2 - \bar{t}^2) \int_{\mathcal{F}_R} \langle (1 - P_\infty)f, (1 - P_\infty)g \rangle_V dv & \\ &= \lim_{R \rightarrow \infty} \int_{P(R)} \left(\left\langle (1 - P_\infty)f, \frac{\partial}{\partial r} (1 - P_\infty)g \right\rangle_V \right. \\ &\quad \left. - \left\langle (1 - P_\infty)g, \frac{\partial}{\partial r} (1 - P_\infty)f \right\rangle_V \right) \frac{dx dy}{R} = 0. \end{aligned}$$

To conclude the proof write

$$(s^2 - \bar{t}^2) \int_{\mathcal{F}} \langle P_\infty f^Y, P_\infty g^Y \rangle_V dv = (s^2 - \bar{t}^2) \sum_{j=1}^k \int_{\mathcal{F}} \langle P_j f^Y, P_j g^Y \rangle_V dv$$

and apply Lemma (3.5.1) to $(s^2 - \bar{t}^2) \int_{\mathcal{F}} \langle P_j f^Y, P_j g^Y \rangle_V dv$. \square

3.6 The resolvent kernel

The definition of the resolvent

One of our main goals in this paper is to understand the spectrum of Δ . The spectrum is best understood in terms of the *resolvent* operator. The resolvent set of Δ , $\rho(\Delta)$, is the set of all $z \in \mathbb{C}$ so that $(\Delta - zI)$ has a bounded inverse, defined on the entire Hilbert space $\mathcal{H}(\Gamma, \chi)$. The spectrum of Δ , $\sigma(\Delta)$, is the complement of the resolvent set $\mathbb{C} \setminus \rho(\Delta)$. For each $z \in \rho(\Delta)$ we have the resolvent operator $R_z = (\Delta - zI)^{-1}$.

The resolvent kernel for $\Delta(\Gamma, \chi)$

We will understand the resolvent operator through its kernel. A bounded linear operator $K : \mathcal{H}(\Gamma, \chi) \mapsto \mathcal{H}(\Gamma, \chi)$ is said to have a *kernel* if there exists a function $k : \mathbb{H}^3 \times \mathbb{H}^3 \times V \mapsto V$ so that for all $f \in \mathcal{H}(\Gamma, \chi)$, $P \in \mathbb{H}^3$, $Kf(P) = \int_{\mathcal{F}} k(P, Q)f(Q) dv(Q)$. Recall (from §2.2) that for $t > 1$, $s \in \mathbb{C}$,

$$\varphi_s(t) \equiv \frac{(t + \sqrt{t^2 - 1})^{-s}}{\sqrt{t^2 - 1}},$$

and for $P, Q \in \mathbb{H}^3$, $\delta(P, Q) = \cosh(d(P, Q))$, where d is the hyperbolic distance function.

Definition 3.6.1. Let Γ be a cofinite Kleinian group, $\chi \in \text{Rep}(\Gamma, V)$, and $s \in \mathbb{C}$ with $\text{Re}(s) > 1$. Define the Maaß-Selberg Series

$$F_\chi(P, Q, s) = \frac{1}{4\pi} \sum_{M \in \Gamma} \chi(M) \varphi_s(\delta(P, MQ)). \quad (3.6.1)$$

Since $\sum_{M \in \Gamma} |\varphi_s(\delta(P, MQ))|$ converges uniformly when P, Q , and s are restricted to compact subsets of

$\{\text{Re}(s) > 1\} \times (\mathbb{H}^3 \times \mathbb{H}^3 \setminus \{(P, Q) \in \mathbb{H}^3 \times \mathbb{H}^3 \mid \Gamma P = \Gamma Q\})$ (see [EGM98, page 96]) and χ is a unitary representation ($\chi(\gamma)$ has norm one for all $\gamma \in \Gamma$) the sum in equation (3.6.1) converges absolutely and uniformly when P, Q , and s are restricted to compact subsets of $\{\text{Re}(s) > 1\} \times (\mathbb{H}^3 \times \mathbb{H}^3 \setminus \{(P, Q) \in \mathbb{H}^3 \times \mathbb{H}^3 \mid \Gamma P = \Gamma Q\})$.

The function $F_\chi(P, Q, s)$ is the kernel for the resolvent operator of Δ .

Theorem 3.6.2. Let Γ be cofinite with $\chi \in \text{Rep}(\Gamma, V)$, and $\lambda = 1 - s^2 \in \rho(\Delta)$ with $\text{Re}(s) > 1$. Then $R_\lambda : \mathcal{H}(\Gamma, \chi) \rightarrow \tilde{D}$ has kernel $F_\chi(P, Q, s)$.

To prove the theorem one shows that for $\lambda = 1 - s^2$ with $\text{Re}(s) > 1$, any element $u \in \tilde{D}$ can be represented by a continuous function which satisfies

$$u(Q) = \frac{1}{4\pi} \int_{\mathcal{F}} F_\chi(P, Q, s)(\Delta - \lambda)u(P) dv(P). \quad (3.6.2)$$

Equation (3.6.2) can be seen by unfolding the integral in (3.6.2) and applying Lemma 2.2.1. See [EGM98, theorem 4.2.6, page 150] for more details.

Fourier expansion of the resolvent kernel

In this section we will concern ourselves with a sub-sum of the resolvent kernel that causes the resolvent kernel to not be of Hilbert-Schmidt type. For notational simplicity we assume that Γ is cofinite with only one cusp at $\zeta_\alpha = \infty$. Fix a fundamental domain $\mathcal{F} = \mathcal{F}_Y \cup \mathcal{F}(Y)$, and let $\chi \in \text{Rep}(\Gamma, V)$.

Recall that for $\text{Re}(s) > 1$ the resolvent kernel can be expressed as a sum

$$F_\chi(P, Q, s) = \frac{1}{4\pi} \sum_{M \in \Gamma} \chi(M) \varphi_s(\delta(P, MQ)).$$

Let

$$F_\infty(P, Q) \equiv \frac{1}{4\pi} \sum_{M \in \Gamma_\infty} \chi(M) \varphi_s(\delta(P, MQ)),$$

and

$$F_0(P, Q) = F_\chi(P, Q, s) - F_\infty(P, Q).$$

Let $\{v_{\infty l}\}$ be the basis constructed in §3.3. For convenience we drop the ∞ subscript and consider $\{v_l\}$ and corresponding orthogonal projections $\{P_l\}$.

The following lemmas are all proved using a standard separation of variables technique. See [Ven82, Pages 20-23] and §3.3

Lemma 3.6.3. *Let $P, Q \in \mathbb{H}^3$ satisfy $P \neq Q$. Then*

$$P_\infty F_\infty(P, Q) = P_\infty R_0(r_P, r_Q, s) + P_\infty \sum_{0 \neq \mu \in \Lambda_\alpha^0} a_\mu R_\mu(r_P, r_Q, s) e^{2\pi i \langle \mu, z_P - z_Q \rangle},$$

where

$$R_0 = \frac{c}{s} \begin{cases} r_P^{1+s} r_Q^{1-s} & r_P \leq r_Q \\ r_P^{1-s} r_Q^{1+s} & r_P \geq r_Q \end{cases}$$

for some constant c , and

$$R_\mu(r_P, r_Q, s) = \begin{cases} r_P K_s(2\pi|\mu|r_P) r_Q I_s(2\pi|\mu|r_Q) & r_P \geq r_Q \\ r_P I_s(2\pi|\mu|r_P) r_Q K_s(2\pi|\mu|r_Q) & r_P \leq r_Q \end{cases}.$$

Lemma 3.6.4. *Let $P, Q \in \mathbb{H}^3$ satisfy $P \neq Q$, $k_\infty + 1 \leq l \leq l_\infty$. Then*

$$P_l F_\infty(P, Q) =$$

$$P_l \sum_{0 \neq \mu \in \Lambda_\alpha^0} a_{l,\mu} R_\mu(r_P, r_Q, s) e^{2\pi i \langle \mu, z_P - z_Q \rangle},$$

where

$$R_\mu(r_P, r_Q, s) = \begin{cases} r_P K_s(2\pi|\mu|r_P) r_Q I_s(2\pi|\mu|r_Q) & r_P \geq r_Q \\ r_P I_s(2\pi|\mu|r_P) r_Q K_s(2\pi|\mu|r_Q) & r_P \leq r_Q \end{cases}.$$

Lemma 3.6.5. *Let $P, Q \in \mathbb{H}^3$ satisfy $P \neq Q$, $l_\infty + 1 \leq l \leq n$. Then*

$$P_l F_\infty(P, Q) = \sum_{0 \neq \mu \in \Lambda_\alpha^0} a_{l,\mu} R'_\mu(r_P, r_Q, s) e^{2\pi i \langle \mu', z_P - z_Q \rangle},$$

where

$$R'_\mu(r_P, r_Q, s) = \begin{cases} r_P K_s(2\pi|\mu'|r_P) r_Q I_s(2\pi|\mu'|r_Q) & r_P \geq r_Q \\ r_P I_s(2\pi|\mu'|r_P) r_Q K_s(2\pi|\mu'|r_Q) & r_P \leq r_Q \end{cases},$$

and $\mu' = \theta_R \mu_1 + \theta_S \mu_2 + \mu$. (See Proposition 3.3.4 for an explanation of the notation used).

Estimates of the resolvent kernel

It follows from [EGM98, Theorem 4.5.2] that

$$\int_{\mathcal{F}} |F_0(P, Q)|_V^2 dv(P) dv(Q) < \infty.$$

In other words, $F_0(P, Q)$ is a Hilbert-Schmidt kernel. On the other hand, $F_\infty(P, Q)$ is not. However, from the Fourier series expansion of $F_\infty(P, Q)$, and the decay properties of the Bessel functions $rI_s(2\pi|\mu|r)$, $rK_s(2\pi|\mu|r)$ it follows that the only component of the resolvent kernel that destroys square-integrability is,

$$R_\infty(P, Q) \equiv \frac{c}{s} P_\infty \begin{cases} r_P^{1+s} r_Q^{1-s} & r_P \leq r_Q \\ r_P^{1-s} r_Q^{1+s} & r_P \geq r_Q \end{cases}$$

Definition 3.6.6. Let Γ be cofinite with one cusp at $\zeta_\alpha = \infty$. For $B \geq Y$ decompose $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}(B)$, and let $\chi \in \text{Rep}(\Gamma, V)$, define

$$F^B(P, Q) \equiv \begin{cases} R_\infty(P, Q) & P, Q \in \mathcal{F}(B) \\ 0 & \text{else} \end{cases}, \quad (3.6.3)$$

and $F_B(P, Q) \equiv F_\chi(P, Q, s) - F^B(P, Q)$.

We have shown the following:

Lemma 3.6.7.

$$\int_{\mathcal{F}} |F_B(P, Q)|_V^2 dv(P) dv(Q) < \infty.$$

3.7 Analytic continuation of the Eisenstein Series

The meromorphic continuation of $E_{\alpha l}(P, s)$ is necessary for the proof of the spectral decomposition theorem, and is highly non-trivial. Fortunately, there are several well-known methods available, [Fad69], [Sel89], [Sel91], [CdV81], and [Lan76]. In [EGM98] an adaptation of the methods in [CdV81] is used to prove the three-dimensional scalar case, and a similar adaptation works⁴ for the vector case which we show now.

⁴Alternatively, Faddeev's method can also be used to express the meromorphic continuation of $E_{\alpha l}(P, s)$ by adapting [Ven82, Chapters 2 and 3].

The idea of the proof

As in Faddeev's proof the key idea is the resolvent kernel of Δ . We know from §3.6 that it is not a Hilbert-Schmidt operator. However the key step in the proof is to construct a self adjoint operator Δ_a defined on a closed subspace (the set of functions whose constant Fourier term vanishes for $r > a$) so that $(\Delta_a - \lambda)^{-1}$ is Hilbert-Schmidt and is closely related to Δ .

Statement of the theorem

Theorem 3.7.1. *Let Γ be cofinite, $(\chi, V) \in \text{Rep}(\Gamma, V)$, and $1 \leq l \leq k_\alpha$. Then*

- (1) $E_{\alpha l}(P, s)$ admits a meromorphic continuation in the following sense. For each fixed $P \in \mathbb{H}^3$, $E_{\alpha l}(s, P)$ is a meromorphic function in $s \in \mathbb{C}$. The poles of $E_{\alpha l}(P, s)$ depend only on s and not on P .
- (2) If $U \subset \mathbb{C}$ is open with $E_{\alpha l}(P, s)$ regular on U then $E_{\alpha l}$ is real analytic on $U \times \mathbb{H}^3$.
- (3) For each regular $s \in \mathbb{C}$, $\Delta E_{\alpha l}(P, s) = (1 - s^2)E_{\alpha l}(P, s)$.
- (4) The scattering matrix $\mathfrak{S}(s)$ and $E_{\alpha l}(P, s)$ are both finite ordered meromorphic functions with order ≤ 4 .
- (5) $\mathcal{E}(P, -s) = \mathfrak{S}(-s)\mathcal{E}(P, s)$
- (6) $\mathfrak{S}(s)\mathfrak{S}(-s) = 1$.
- (7) $\mathfrak{S}(s)$ is a unitary matrix on the critical axis $\text{Re}(s) = 0$.

Proof of theorem (3.7.1) part I: Preliminaries

In order to avoid complicated notation (quadruple subscripts) we will assume the following:

Assumption 3.7.2. *The Kleinian group Γ has only one class of cusps at $\zeta = \infty \in \mathbb{P}$, and $\chi \in \text{Rep}(\Gamma, V)$.*

Then we can choose fundamental domain of the form $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}(Y)$ (here $\mathcal{F}_0 = \mathcal{F}_Y$, see §2.6). We can write the orthogonal projection onto V_∞ by

$$P_\infty = \sum_{k=1}^{k_\infty} P_k,$$

where P_k is the orthogonal projection onto $v_k = v_{\infty k}$ (see §3.3).

Since we assume that ∞ is the only cusp we have (by definition) k_∞ distinct Eisenstein series ($E_{\infty l}(P, s)$, $l = 1 \dots k_\infty$) which have an expansion of the form (see §3.3, §3.4).

$$P_k E_{\infty l}(P, s) = \delta_{kl} r^{1+s} v_k + \phi_{kl}(s) r^{1-s} v_k + O(e^{-|c|r}). \quad (3.7.1)$$

We will need various spaces of functions. Let $\pi_\Gamma : \mathbb{H}^3 \rightarrow \Gamma \backslash \mathbb{H}^3$ be the standard (quotient map) projection, and set

$$\mathcal{D}^\infty \equiv \{ \Phi \in A(\Gamma, \chi) \cap C^\infty(\mathbb{H}^3, V) \mid \pi_\Gamma(\text{supp}(\Phi)) \text{ is compact in } \Gamma \backslash \mathbb{H}^3 \}.$$

We will restrict functions in \mathcal{D}^∞ to \mathcal{F} . The space of distributions \mathcal{D}' is the set of all continuous (complex valued) linear functionals on \mathcal{D}^∞ . A function $f \in L^1_{\text{loc}}(\Gamma, \chi)$ can be identified with the distribution U where

$$U(\Phi) \equiv \int_{\mathcal{F}} \langle \Phi(P), f(P) \rangle_V dv(P). \quad (3.7.2)$$

Noting that $\mathcal{H}(\Gamma, \chi) \subset L^1_{\text{loc}}(\Gamma, \chi)$, we say that a distribution $U \in \mathcal{H}(\Gamma, \chi)$ if and only if there exists $f \in \mathcal{H}(\Gamma, \chi)$ so that equation (3.7.2) holds for all $\Phi \in \mathcal{D}^\infty$.

For Y chosen in the decomposition $\mathcal{F} = \mathcal{F}_0 \cup \mathcal{F}(Y)$ define

$$\mathcal{D}(Y, \infty) \equiv C_c^\infty((Y, \infty), \mathbb{C}),$$

and $\mathcal{D}'(Y, \infty)$ the corresponding space of distributions (the space of continuous, complex valued, linear functionals of $\mathcal{D}(Y, \infty)$). For each $h \in L^1_{\text{loc}}(Y, \infty)$ we associate the distribution

$$h(\phi) \equiv \int_Y^\infty \phi(r) \overline{h(r)} \frac{dr}{r^3} \quad (\phi \in \mathcal{D}(Y, \infty)).$$

For $U \in \mathcal{D}'$, $\Phi \in \mathcal{D}^\infty$, $i \in \mathbb{N} \cup \{0\}$, we define the following differential operators at each point $(x, y, r) \in \mathbb{H}^3$:

$$D_x^{[i]}U(\Phi) \equiv (-1)^i U(r^i \frac{\partial^i}{\partial x^i} \Phi), \quad D_y^{[i]}U(\Phi) \equiv (-1)^i U(r^i \frac{\partial^i}{\partial y^i} \Phi),$$

$$D_r^{[i]}U(\Phi) \equiv (-1)^i U(r^3 \frac{\partial^i}{\partial x^i} (r^{i-3} \Phi)),$$

$$\Delta U \equiv \left(D_x^{[2]} + D_y^{[2]} + D_r^{[2]} - D_r^{[1]} \right) U.$$

An unraveling of the definitions shows that $\Delta U(\Phi) = U(\Delta \Phi)$. For $f \in \mathcal{H}(\Gamma, \chi)$ we abuse notation and define

$$\mathbf{grad}(f) \in \mathcal{H}(\Gamma, \chi) \text{ iff } D_x^{[1]}f, D_y^{[1]}f, D_r^{[1]}f \in \mathcal{H}(\Gamma, \chi).$$

Let $f, g \in \mathcal{H}(\Gamma, \chi)$ with $\mathbf{grad}(f), \mathbf{grad}(g) \in \mathcal{H}(\Gamma, \chi)$. Then we define

$$Q(f, g) \equiv \int_{\mathcal{F}} \left(\langle D_x^{[1]}f, D_x^{[1]}g \rangle_V + \langle D_y^{[1]}f, D_y^{[1]}g \rangle_V + \langle D_r^{[1]}f, D_r^{[1]}g \rangle_V \right) dv.$$

An application of Greene's theorem shows that for $\Phi \in \mathcal{D}^\infty$,

$$Q(f, \Phi) = \int_{\mathcal{F}} \langle \Delta \Phi, f \rangle_V dv = (\Delta f)(\Phi),$$

where Δf is taken in the distributional sense.

We next define useful maps between the function spaces $\mathcal{D}^\infty, \mathcal{D}(Y, \infty)$ and the distribution spaces $\mathcal{D}', \mathcal{D}'(Y, \infty)$. In order for the notation to be simple, we will use uppercase Greek letters for functions in \mathcal{D}^∞ , uppercase Roman letters for distributions in \mathcal{D}' , lower case Greek letters for functions in $\mathcal{D}(Y, \infty)$, and lower case Roman letters for distributions in $\mathcal{D}'(Y, \infty)$. It is also useful to think of $\mathcal{D}^\infty, \mathcal{D}'$ as *BIG* spaces and $\mathcal{D}(Y, \infty), \mathcal{D}'(Y, \infty)$ as *small* spaces.

For $F \in \mathcal{H}(\Gamma, \chi)$, $1 \leq l \leq k_\infty$ set

$$L_l[F](r) \equiv \frac{1}{|\Lambda|} \int_{\Lambda \setminus \mathbb{C}} \langle P_l F(x, y, r), v_l \rangle_V dx dy.$$

By Fubini's theorem $L_l[F] \in L^1_{\text{loc}}((Y, \infty), r^{-3} dr)$. For any function

$$\phi : (Y, \infty) \rightarrow \mathbb{C},$$

and $1 \leq l \leq k_\infty$ set

$$B_l[\phi](P) \equiv \begin{cases} \phi(r)v_l & \text{if } P = z + rj \in \mathcal{F}_\infty(Y) \\ 0 & \text{if } P \in \mathcal{F}_0 \end{cases}.$$

We will identify $L_l[\phi](P)$ with its automorphic extension in $A(\Gamma, \chi)$.

For each $l \in 1 \dots k_\infty$ define $B_{l*} : \mathcal{D}' \rightarrow \mathcal{D}'(Y, \infty)$ by

$$B_{l*}[U](\phi) \equiv U(B_l[\phi]),$$

where $U \in \mathcal{D}'$, $\phi \in \mathcal{D}(Y, \infty)$, and define $L_l^* \mathcal{D}'(Y, \infty) \rightarrow \mathcal{D}'$ by

$$L_l^*[u](\Phi) = u(L_l[\Phi]),$$

where $u \in \mathcal{D}'(Y, \infty)$, $\Phi \in \mathcal{D}^\infty$.

Define $\Delta_0 \mathcal{D}'(Y, \infty) \rightarrow \mathcal{D}'(Y, \infty)$ by

$$(\Delta_0 u)(\phi) = u \left(\left(-r^2 \frac{d^2}{dr^2} + r \frac{d}{dr} \right) \phi \right) \quad (u \in \mathcal{D}'(Y, \infty), \phi \in \mathcal{D}(Y, \infty)).$$

Then it follows that for $u \in \mathcal{D}'(Y, \infty)$, $U \in \mathcal{D}'$, $l \in 1 \dots k_\infty$,

$$L_l^*(\Delta_0 u) = \Delta L_l^*(u) \text{ and } B_{l*}(\Delta U) = \Delta_0 B_{l*}(U).$$

For $a > Y$, $s \in \mathbb{C}$ set

$$\eta_{a,s} \equiv \begin{cases} r^{1+s} - a^{2s}r^{1-s} & r > a \\ 0 & r < a \end{cases},$$

and $T_l^a = L_l^*[\delta_a]$, where $\delta_a \in \mathcal{D}'(Y, \infty)$ is the Dirac delta-distribution at the point a .

A simple calculation using Greene's function techniques for second order differential equations (see [Iwa02, Pages 194-195] for details on the technique) shows that

$$(\Delta_0 - (1 - s^2))\eta_{a,s} = -2sa^{s-1}\delta_a,$$

and an application of L_l^* ($l \in 1 \dots k_\infty$) gives us

$$(\Delta - (1 - s^2))B_l[\eta_{a,s}] = -2sa^{s-1}T_l^a. \quad (3.7.3)$$

Proof of theorem (3.7.1) part II: Lemmas

Lemma 3.7.3. *Let*

$$\mathfrak{H}_a \equiv \{F \in \mathcal{H}(\Gamma, \chi) \mid L_l[F](r) = 0, r \geq a, l \in 1 \dots k_\infty\},$$

$$\mathfrak{D}_a \equiv \{F \in \mathfrak{H}_a \mid \mathbf{grad}(F) \in \mathcal{H}(\Gamma, \chi)\},$$

and for $F, G \in \mathfrak{D}_a$

$$Q_a(F, G) \equiv Q(F, G).$$

Then

(1) \mathfrak{H}_a is a closed subspace of $\mathcal{H}(\Gamma, \chi)$.

(2) There exists a self-adjoint operator Δ_a with domain $\mathfrak{D}_{\Delta_a} \subset \mathfrak{D}_a \subset \mathfrak{H}_a$ with

$$\Delta_a : \mathfrak{D}_{\Delta_a} \rightarrow \mathfrak{H}_a$$

that satisfies

$$\int_{\mathcal{F}} \langle \Delta_a F, G \rangle_V dv = Q_a(F, G)$$

for all $F \in \mathfrak{D}_{\Delta_a}, G \in \mathfrak{D}_a$,

(3) \mathfrak{D}_{Δ_a} consists of those $F \in \mathfrak{D}_a$ which satisfy

$$\Delta F = G + \sum_{l=1}^{k_\infty} c_l T_l^a, \quad (3.7.4)$$

for some $G \in \mathfrak{H}_a$. If equation (3.7.4) is satisfied, then $\Delta_a F = G$.

The proof (of the above lemma) is similar to [EGM98, Pages 237-239].

Lemma 3.7.4. (1) For $\operatorname{Re}(s) > 1$, $(\Delta_a - (1 - s^2))^{-1}$ is a Hilbert-Schmidt operator.

(2) The self-adjoint operator Δ_a has a purely discrete spectrum.

(3) The resolvent $(\Delta_a - (1 - s^2))^{-1}$ can be continued to an operator valued meromorphic function on all of \mathbb{C} of order ≤ 4 .

Proof. (1) and (2) follow from a straight forward adaptation [EGM98, Prop. 6.1.8, page 240-241]. (2) follows from standard functional analysis while (3) is proved in [EGM98, Lemma 6.1.9, page 240]. \square

Let $h_s(r) = h(r)r^{1+s}$ where for $Y < Y' < Y''$ $h : (Y, \infty) \rightarrow \mathbb{R}$ smooth and $h(y) = 1$ for $y \geq Y''$ and $h(y) = 0$ for $y < Y'$.

Let $H_l(P, s) = ((\Delta - (1 - s^2))B_l[h_s])(P)$. Then it is in $\mathcal{H}(\Gamma, \chi)$ and smooth. In fact it is also in \mathfrak{H}_a . Let

$$\Omega \equiv \{s \in \mathbb{C} \mid \operatorname{Re}(s) > 0, 1 - s^2 \notin \sigma(\Delta)\}.$$

Lemma 3.7.5. Let $Y < Y' < Y''$ and $h : (Y, \infty) \rightarrow \mathbb{R}$ be a smooth function satisfying $h(r) = 1$ for $r \geq Y''$ and $h(r) = 0$ for $r < Y'$. For each $l \in 1 \dots k_\infty$ set

$$H_l(P, s) \equiv ((\Delta - (1 - s^2))B_l[h_s])(P),$$

and for $s \in \Omega$ set

$$E^l(P, s) \equiv B_l[h_s](P) - ((\Delta - (1 - s^2))^{-1}H_l(P, s)).$$

Then

(1) $B_l[h_s] \in A(\Gamma, \chi)$ is a smooth.

(2) $H_l(z + jr, s) = 0$ for all $r > Y''$ and hence is in $\mathcal{H}(\Gamma, \chi)$. Moreover, if $Y'' < a$ then $H_l(\cdot, s) \in \mathfrak{H}_a$.

(3) The function $H_l(P, s)$ is holomorphic for $s \in \mathbb{C}$.

(4) The function $E^l(P, s)$ is holomorphic for $s \in \Omega$.

(5) The function $E^l(P, s)$ is the unique function in $A(\Gamma, \chi)$ that satisfies

$$(\Delta - (1 - s^2))E^l(P, s) = 0, E^l(P, s) - B_l[h_s](P) \in \mathcal{H}(\Gamma, \chi).$$

(6) For all $P \in \mathbb{H}^3$, $s \in \Omega$, $E^l(P, s) = E_{\infty l}(P, s)$.

See [EGM98, Pages 234-235].

Proof of theorem (3.7.1) part III: The meromorphic continuation

We exhibit the meromorphic continuation of the Eisenstein series to all of \mathbb{C} .

For $s \in \mathbb{C}$ with $1 - s^2 \notin \Omega_a \equiv \sigma(\Delta_a)$ set

$$A_l(P, s) \equiv (\Delta_a - (1 - s^2))^{-1} H_l(P, s).$$

By Lemma 3.7.4 and Lemma 3.7.5, $A_l(P, s)$ varies meromorphically. Next for each $s \in \Omega_a$ $A_l(P, s) \in \mathfrak{D}_{\Delta_a}$ (since it is in the range of the resolvent of Δ_a), and an appeal to Lemma 3.7.4 give us

$$(\Delta - (1 - s^2))A_l(P, s) = H_l(P, s) + \sum_{j=1}^{k_\infty} c_{l,j}(s)T_j^a. \quad (3.7.5)$$

Plugging in (to the distributional equation above) arbitrary test functions $\Phi \in \mathcal{D}^\infty$, which do not vanish at the support of each T_j^a , into equation (3.7.5) shows that the functions $c_{l,j}(s)$ are meromorphic on all of \mathbb{C} (the remaining terms in equation (3.7.5) are meromorphic).

Next define the following meromorphic functions,

$$e_{l,j}(s) \equiv \frac{1}{2}s^{-1}a^{1-s}c_{l,j}(s).$$

Finally, let

$$N_l(P, s) \equiv A_l(P, s) + \sum_{j=1}^{k_\infty} e_{l,j}(s)B_l[\eta_{a,s}](P) + B_l[h_s](P).$$

Then (in the distributional sense) $(\Delta - (1 - s^2))N_l(P, s) = 0$, and moreover, by the elliptic regularity theorem $N_l(P, s)$ is smooth as a function of $P \in \mathbb{H}^3$.

Since $e_{l,j}(s)$, $B_l[h_s](P)$ are meromorphic, and of order ≤ 4 , $N_l(P, s)$ is meromorphic, and of order ≤ 4 .

To relate $N_l(P, s)$ to $E_{\infty l}(P, s)$, express $E_{\infty l}(P, s)$ as a linear combination of the $N_l(P, s)$ and apply the uniqueness part of Lemma 3.7.5.

We have proved parts (1)-(4) of Theorem 3.7.1. Part (5) can be seen by plugging in $-s$ into $E_{\infty l}(P, s)$ and applying the uniqueness part of Lemma 3.7.5. Part(6) follows immediately from (5). Part (7): from the Dirichlet series representation of the components of $\mathfrak{S}(s)$ we see that for $\text{Re}(s) > 1$,

$$\mathfrak{S}(s) = \mathfrak{S}^*(\bar{s}).$$

Observing that for $s = it, t \in \mathbb{R}, \bar{it} = -it$ and appealing to (6) proves (7).

3.8 Proof of the spectral decomposition theorem

There are several proofs of the spectral decomposition theorem. All of them use the analytic continuation of the Eisenstein series to the critical line $\text{Re}(s) = 0$ in a crucial manner.

One particularly nice approach is to use the general theory of eigenpackets. This approach makes the deep connection between the Eisenstein series and the spectral decomposition theorem quite clear. We shall see that integral of the Eisenstein series on segments of the critical line ($\text{Re}(s) = 0$) is an eigenpacket.

Statement of the theorem

Let Γ be a cofinite group with $(\chi, V) \in \text{Rep}(\Gamma, V)$.

Let \mathfrak{D} denote a countable indexing set of all $u \in D$ (the domain of Δ) such that $\Delta u = \lambda u$ for some $\lambda \in \mathbb{C}$.

We remark that in general, we do not know if \mathfrak{D} is an infinite set.

Theorem 3.8.1. *Let $f \in D$. Then f has an expansion of the form*

$$f(P) = \sum_{m \in \mathfrak{D}} \langle f, e_m \rangle e_m(P) + \frac{1}{4\pi} \sum_{\alpha=1}^h \sum_{l=1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma'_\alpha]}{|\Lambda_\alpha|} \int_{\mathbb{R}} \langle f, E_{\alpha l}(\cdot, it) \rangle E_{\alpha l}(P, it) dt, \quad (3.8.1)$$

The sum and integrals converge pointwise absolutely and uniformly on compact subsets of \mathbb{H}^3 .

Eigenpackets

Throughout this section, let $A : \mathcal{D}_A \rightarrow \mathbf{H}$ be a self-adjoint operator with domain \mathcal{D}_A in the Hilbert space \mathbf{H} , with inner product $\langle \cdot, \cdot \rangle$.

We will outline the spectral decomposition theorem for eigenpackets. See [EGM98, Section 6.2] for more details.

An eigenpacket of A is a map $v : \mathbb{R} \rightarrow \mathbf{H}$, $\lambda \mapsto v_\lambda$ having the following properties:

(1) $v_0 = 0$ and $v_\lambda \in \mathcal{D}_A$ for all $\lambda \in \mathbb{R}$.

(2) The map $v : \mathbb{R} \rightarrow \mathbf{H}$ is continuous (in the norm sense).

(3) $Av_\lambda = \int_0^\lambda \mu dv(\mu)$, where the integral is the limit in the norm sense of the corresponding Stieltjes sums. By an integration by parts, we can rewrite (3) as $Av_\lambda = \lambda v_\lambda - \int_0^\lambda v_\mu d\mu$, where the integral is the limit of Riemann sums in the norm sense.

Eigenpackets have many similarities to eigenfunctions. If f is an eigenvector of A then $\langle f, v_\lambda \rangle = 0$ for all $\lambda \in \mathbb{R}$. In addition, if w is also an eigenpacket of A and $[\alpha, \beta]$, $[\gamma, \delta]$ are two intervals having at most one point in common, then

$$\langle v_\beta - v_\alpha, w_\delta - w_\gamma \rangle = 0. \quad (3.8.2)$$

Fix an eigenpacket v of A . Then there exists a non-decreasing right continuous function $F : \mathbb{R} \rightarrow \mathbb{R}$ which is unique up to an additive constant satisfying the equation

$$F(\beta) - F(\alpha) = \|v_\beta - v_\alpha\|^2 \quad (3.8.3)$$

for all $\alpha < \beta$.

For $x \in \mathbf{H}$ define a function $\varphi : \mathbb{R} \rightarrow \mathbb{C}$, by

$$\varphi(\lambda) = \langle x, v_\lambda \rangle. \quad (3.8.4)$$

Next let T be a partition of $[\alpha, \beta]$, of the form

$$T : \alpha = \lambda_0 < \lambda_1 < \cdots < \lambda_n = \beta,$$

and define

$$x_T \equiv \sum_{j=1}^n \frac{\varphi(\lambda_j) - \varphi(\lambda_{j-1})}{F(\lambda_j) - F(\lambda_{j-1})} (v_{\lambda_j} - v_{\lambda_{j-1}}).$$

For a sequence of partitions (sub partitions) T_n with maximal width of its subdivision converging to zero set

$$\int_{\alpha}^{\beta} \frac{d\varphi(\lambda) dv(\lambda)}{dF(\lambda)} = \lim_{n \rightarrow \infty} x_{T_n}.$$

Further define

$$P_v x \equiv \int_{-\infty}^{\infty} \frac{d\varphi(\lambda) dv(\lambda)}{dF(\lambda)} \equiv \lim_{\alpha \rightarrow -\infty} \lim_{\beta \rightarrow \infty} \int_{\alpha}^{\beta} \frac{d\varphi(\lambda) dv(\lambda)}{dF(\lambda)}.$$

We can now write down the general spectral decomposition theorem. Let \mathfrak{D} be a countable indexing set for a maximal set of orthonormal eigenvectors A , and \mathfrak{N} a countable indexing set for a maximal set of orthogonal eigenpackets of A .

Lemma 3.8.2. *Let $\{e_m\}_{m \in \mathfrak{D}}$ be a maximal set of orthonormal eigenvectors of A , and $\{v_n\}_{n \in \mathfrak{N}}$ be a maximal set of orthogonal eigenpackets of A . Then every $x \in \mathbf{H}$ can be decomposed as*

$$x = \sum_{m \in \mathfrak{D}} \langle x, e_m \rangle e_m + \sum_{n \in \mathfrak{N}} \int_{-\infty}^{\infty} \frac{d\varphi_n(\lambda) dv_n(\lambda)}{dF_n(\lambda)}.$$

The sum and integrals converge in the norm sense in \mathbf{H} .

Proof of the theorem

Lemma 3.8.3. *Let Γ be cofinite with $\chi \in \text{Rep}(\Gamma, V)$. For $P \in \mathbb{H}^3$, $\alpha \in 1 \dots \kappa$, $l \in 1 \dots k_{\alpha}$, we define*

$$V_{\lambda}^{\alpha l}(P) \equiv \begin{cases} 0 & \text{for } \lambda < 1 \\ \int_0^{\sqrt{\lambda-1}} E_{\alpha l}(P, it) dt & \text{for } \lambda \geq 1 \end{cases}$$

with positive choice of square root. Then $V_{\lambda}^{[\alpha l]}(\cdot) \in \mathcal{H}(\Gamma, \chi)$ and the family of $V^{[\alpha l]}$ are a maximal system of orthogonal eigenpackets for Δ . Further,

$$\|V_{\lambda_1}^{[\alpha l]} - V_{\lambda_2}^{[\alpha l]}\|^2 = 2\pi \frac{|\Lambda_{\alpha}|}{[\Gamma_{\alpha} : \Gamma'_{\alpha}]} (T_1 - T_2) \quad (3.8.5)$$

for $\lambda_1 \geq \lambda_2 \geq 1$ and $T_1 = \sqrt{\lambda_1 - 1}$, $T_2 = \sqrt{\lambda_2 - 1}$.

Proof. For notational simplicity we work under Assumption 3.7.2. From the Fourier expansion of the Eisenstein series we see that $E_{\infty l}(P, s) = A_l(P, s) + O(e^{-|c|r})$ as $r \rightarrow \infty$, where $A_l(P, s) = r^{1+s} v_l + \phi_l(s) r^{1-s}$, and $\phi_l(s)$ is defined by $\langle \phi_l(s), v_k \rangle_V = \mathfrak{S}(s)_{l,k} \equiv \mathfrak{S}(s)_{\infty l, \infty k}$. Since $\mathfrak{S}(t)$ is unitary on the critical axis, $|\phi_l(it)|_V = 1$ for $t \in \mathbb{R}$.

We next show that $V_{\lambda}^{[l]} \equiv V_{\lambda}^{[\alpha l]} \in \mathcal{H}(\Gamma, \chi)$ for each $\lambda \in \mathbb{R}$. Put $T = \sqrt{\lambda - 1}$, and set

$$\beta_l(P, T) \equiv \int_0^T A_l(P, it) dt.$$

An integration by parts shows that⁵

$$\beta_l(P, T) = O\left(\frac{r}{\log r}\right) \quad \text{as } r \rightarrow \infty.$$

Hence $\beta_l(P, T) \in L^2(\mathcal{F})$. The upper bound on $\beta(P, T)$ and the dominated convergence theorem imply that $V_\lambda^{[l]}$ is continuous in the norm sense.

Next, for $\lambda \geq 1$,

$$\Delta V_\lambda^{[l]}(P) = \int_0^{\sqrt{\lambda-1}} (1+t^2) E_{\infty l}(P, it) dt = \int_0^\lambda \mu dV_\mu^{[l]}.$$

The last equality was obtained by substituting $\mu = 1 + t^2$. We have now shown that $V_\mu^{[l]}$ is an eigenpacket.

We next prove the orthogonality relation

$$\langle V_\lambda^{[l]}, V_\lambda^{[k]} \rangle = 0 \quad \text{for } l \neq k,$$

using the Maaß-Selberg relations (Theorem 3.5.2). We have

$$\begin{aligned} \langle V_\lambda^{[l]}, V_\lambda^{[k]} \rangle &= \lim_{Y \rightarrow \infty} \left\langle \int_0^T E_l^Y(P, it) dt, \int_0^T E_k^Y(P, it') dt' \right\rangle \\ &= \lim_{Y \rightarrow \infty} \frac{[\Gamma_\infty : \Gamma'_\infty]}{|\Lambda_\infty|} \int_0^T \int_0^T \langle E_l^Y(P, it), E_k^Y(P, it') \rangle dt dt' \\ &= \int_0^T \int_0^T \lim_{Y \rightarrow \infty} \left[-Y^{it+it'} i \frac{\overline{\mathfrak{S}_{k,l}(it')}}{t+t'} + Y^{-it-it'} i \frac{\overline{\mathfrak{S}_{k,l}(it)}}{t+t'} \right. \\ &\quad \left. - Y^{-it+it'} i \frac{\sum_{j=1}^{k_\infty} \mathfrak{S}_{j,l}(it) \overline{\mathfrak{S}_{j,k}(it')}}{t'-t} \right] dt dt' = 0, \end{aligned}$$

since all three terms above are bounded for $t, t' \in [0, T]$ ($\mathfrak{S}(it)$ is a unitary matrix so its terms are bounded and rows are orthonormal), and by the Riemann-Lebesgue lemma.

Equation (3.8.5) follows in a similar manner. Letting $\lambda_1 = 1 + T_1^2$, $\lambda_2 = 1 + T_2^2$, we have

$$\begin{aligned} \|V_{\lambda_1}^{[l]} - V_{\lambda_2}^{[l]}\|^2 &= \lim_{Y \rightarrow \infty} \int_{T_2}^{T_1} \int_{T_2}^{T_1} \langle E_l^Y(P, it), E_l^Y(P, it') \rangle dt dt' \\ &= \frac{|\Lambda_\alpha|}{[\Gamma_\alpha : \Gamma'_\alpha]} \lim_{Y \rightarrow \infty} \int_{T_2}^{T_1} \int_{T_2}^{T_1} \left[-Y^{it+it'} i \frac{\overline{\mathfrak{S}_{l,l}(it')}}{t+t'} + Y^{-it-it'} i \frac{\overline{\mathfrak{S}_{l,l}(it)}}{t+t'} \right. \\ &\quad \left. + Y^{it-it'} i \frac{1}{t'-t} - Y^{-it+it'} i \frac{\sum_{j=1}^{k_\infty} \mathfrak{S}_{j,l}(it) \overline{\mathfrak{S}_{j,l}(it')}}{t'-t} \right] dt dt' = 2\pi \frac{|\Lambda_\alpha|}{[\Gamma_\alpha : \Gamma'_\alpha]} (T_1 - T_2). \end{aligned}$$

⁵The implicit constant depends on T .

See [EGM98, Pages 272-275] and [Iwa02, Page 97] for the details. The trick is to add and subtract the conjugate to the second to last term (on the left side of the equal sign) and combine one of the extra terms with the last term.

Finally, it remains to prove completeness. Let $A > Y$, and extend the function

$$f_A^{[l]}(P) \equiv \begin{cases} v_l & \text{for } P \in \mathcal{F}(A) \\ 0 & \text{else} \end{cases}$$

to $\mathcal{H}(\Gamma, \chi)$. Then one can verify that $f_A^{[l]}$ is orthogonal to the set of *cuspidal forms*⁶, and to all eigenpackets $V^{[k]}$ for $k \neq l$. The function $f_A^{[l]}$ is not orthogonal to the exceptional eigenfunctions (the residues of the Eisenstein series $E_l(P, s)$ at the poles $\sigma_j \in (0, 1]$) and the eigenpacket $V^{[l]}$. Let T_l denote the orthogonal projection onto the space spanned by the exceptional eigenfunctions and the span of $\{V_\alpha^{[l]} - V_\beta^{[l]}\}_{\alpha, \beta \in \mathbb{R}}$. Then a computation shows that $\|f_A^{[l]}\|^2 = \|T_l f_A^{[l]}\|^2$, that is our system eigenfunctions-eigenpacket system is complete with respect to expanding $f_A^{[l]}$. Now suppose that there exists an eigenpacket W orthogonal to all of the $\{V^{[l]}\}_{l=1 \dots k_\infty}$, and cuspidal forms, and let U be the span of $\{W_\alpha - W_\beta\}_{\alpha, \beta \in \mathbb{R}}$. Then for $u \in U$ by the completeness with respect to $f_A^{[l]}$, $\langle f_A^{[l]}, u \rangle = 0$ for $l = 1 \dots k_\infty$, that is u is a cuspidal form, a contradiction unless $u = 0$. See [EGM98, Pages 274-276] for more details. \square

⁶The space of cuspidal forms is the space spanned by the eigenvalues of Δ that are in $\mathcal{H}(\Gamma, \chi)$, and whose constant term in their Fourier expansion is identically zero. Only finitely many eigenvalues of Δ are not cuspidal forms.

Chapter 4

The Selberg Trace Formula

In this chapter we derive the Selberg Trace Formula for cofinite Kleinian groups with finite dimensional unitary representations.

4.1 The Selberg Trace Formula

Theorem 4.1.1. (*Selberg Trace Formula*) Let Γ be a cofinite Kleinian group, $\chi \in \text{Rep}(\Gamma, V)$, h be a holomorphic function on $\{s \in \mathbb{C} \mid |\text{Im}(s)| < 2 + \delta\}$ for some $\delta > 0$, satisfying $h(1+z^2) = O(1+|z|^2)^{3/2-\epsilon}$ as $|z| \rightarrow \infty$, and let

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(1+t^2) e^{-itx} dt.$$

Then

$$\begin{aligned} \sum_{m \in \mathcal{D}} h(\lambda_m) &= \frac{\text{vol}(\Gamma \backslash \mathbb{H}^3)}{4\pi^2} \dim_{\mathbb{C}} V \int_{\mathbb{R}} h(1+t^2) t^2 dt \\ &+ \sum_{\{R\}_{\text{ncc}}} \frac{\text{tr}_V \chi(R) g(0) \log N(T_0)}{4 |\mathcal{E}(R)| \sin^2(\frac{\pi k}{m(R)})} + \sum_{\{T\}_{\text{lox}}} \frac{\text{tr}_V \chi(T) g(\log N(T))}{|\mathcal{E}(T)| |a(T) - a(T)^{-1}|^2} \log N(T_0) \\ &\quad - \frac{\text{tr}(\mathfrak{S}(0)) h(1)}{4} + \frac{1}{4\pi} \int_{\mathbb{R}} h(1+t^2) \frac{\phi'}{\phi}(it) dt \\ &\quad + \sum_{\alpha=1}^{\kappa} \sum_{k=1}^{e_{\alpha}} \frac{\text{tr}_V \chi(g_{\alpha k})}{|C(g_{\alpha k})|} \left(g(0) c_{\alpha k} + d_{\alpha k} \int_0^{\infty} g(x) \frac{\sinh x}{\cosh x - 1 + \alpha_{\alpha k}} dx \right) \\ &\quad + \sum_{\alpha=1}^{\kappa} \left(\frac{l_{\alpha}}{|\Gamma_{\alpha} : \Gamma'_{\alpha}|} \left(\frac{h(1)}{4} + g(0) \left(\frac{1}{2} \eta_{\Lambda_{\alpha}} - \gamma \right) - \frac{1}{2\pi} \int_{\mathbb{R}} h(1+t^2) \frac{\Gamma'}{\Gamma}(1+it) dt \right) \right) \\ &\quad + \sum_{\alpha=1}^{\kappa} \frac{g(0)}{|\Gamma_{\alpha} : \Gamma'_{\alpha}|} \sum_{k=l_{\alpha}+1}^{\dim_{\mathbb{C}} V} L(\Lambda_{\alpha}, \psi_{k\alpha}). \quad (4.1.1) \end{aligned}$$

Here, $\{\lambda_m\}_{m \in \mathcal{D}}$ are the eigenvalues of Δ counted with multiplicity. Following [EGM98] section 5.2, the summation with respect to $\{R\}_{\text{ncc}}$ extends over the finitely many Γ -conjugacy classes of the non cuspidal elliptic elements (elliptic elements that do not fix a cusp) $R \in \Gamma$, and for such

a class $N(T_0)$ is the minimal norm of a hyperbolic or loxodromic element of the centralizer $\mathcal{C}(R)$. The element R is understood to be a k -th power of a primitive non cuspidal elliptic element $R_0 \in \mathcal{C}(R)$ describing a hyperbolic rotation around the fixed axis of R with minimal rotation angle $\frac{2\pi}{m(R)}$. Further, $\mathcal{E}(R)$ is the maximal finite subgroup contained in $\mathcal{C}(R)$. The summation with respect to $\{T\}_{\text{lox}}$ extends over the Γ -conjugacy classes of hyperbolic or loxodromic elements of Γ , T_0 denotes a primitive hyperbolic or loxodromic element associated with T . The element T is conjugate in $\text{PSL}(2, \mathbb{C})$ to the transformation described by the diagonal matrix with diagonal entries $a(T), a(T)^{-1}$ with $|a(T)| > 1$, and $N(T) = |a(T)|^2$. For $s \in \mathbb{C}$, $\mathfrak{S}(s)$ is a $k(\Gamma, \chi) \times k(\Gamma, \chi)$ matrix-valued meromorphic function, called the *scattering matrix* of Δ , and $\phi(s) = \det \mathfrak{S}(s)$. Furthermore $c_{\alpha k}$, $g_{\alpha k}$, and $d_{\alpha k}$ are constants depending on Γ which will be determined in the case of Γ having only one cusp at ∞ . The remaining notation will be defined in this chapter¹.

For $P = z + rj$, $P' = z' + r'j \in \mathbb{H}^3$ set

$$\delta(P, P') \equiv \frac{|z - z'|^2 + r^2 + r'^2}{2rr'}.$$

It follows that $\delta(P, P') = \cosh(d(P, P'))$, where d denotes the hyperbolic distance in \mathbb{H}^3 . Next, for $k \in \mathcal{S}([1, \infty))$ a Schwartz-class function, define

$$K(P, Q) = k(\delta(P, Q)), \text{ and } K_\Gamma(P, Q) \equiv \sum_{\gamma \in \Gamma} \chi(\gamma) K(P, \gamma Q).$$

The series above converges absolutely and uniformly on compact subsets of $\mathbb{H}^3 \times \mathbb{H}^3$, and is the kernel of a bounded operator $\mathcal{K} : \mathcal{H}(T, \chi) \mapsto \mathcal{H}(T, \chi)$. The Selberg trace formula is essentially² the trace of \mathcal{K} evaluated in two different ways: the first using spectral theory, and the second as an explicit integral.

The function h that appears in the Selberg trace formula is the Selberg–Harish-Chandra transform³ of k , defined as follows:

$$h(\lambda) = h(1 - s^2) \equiv \frac{\pi}{s} \int_1^\infty k\left(\frac{1}{2}\left(t + \frac{1}{t}\right)\right) (t^s - t^{-s}) \left(t - \frac{1}{t}\right) \frac{dt}{t}, \quad \lambda = 1 - s^2. \quad (4.1.2)$$

Expansion of K_Γ

We now *diagonalize* \mathcal{K} . For $v, w \in V$ define a map $v \otimes \bar{w} : V \rightarrow V$ by

$$v \otimes \bar{w}(x) = \langle x, w \rangle v. \quad (4.1.3)$$

An immediate application of the spectral decomposition theorem and the Selberg transform give us,

¹Please note that there is a typographical error in the loxodromic and non cuspidal elliptic terms in [EGM98] Theorem 6.5.1; both terms are missing a factor of $\frac{1}{4\pi}$.

²We say “essentially” because \mathcal{K} is not of trace class. Selberg’s procedure is used to define and compute the *regularized trace*.

³If $f : \mathbb{H}^3 \mapsto V$ is a smooth function satisfying $\Delta f = \lambda f$, then $\mathcal{K}f = h(\lambda)f$. That is f is an eigenfunction of \mathcal{K} with an eigenvalue that depends only on λ .

Lemma 4.1.2. *Let $k \in \mathcal{S}$ and $h : \mathbb{C} \rightarrow \mathbb{C}$ be the Selberg Transform of k . Then*

$$\begin{aligned} K_\Gamma(P, Q) &= \sum_{m \in \mathcal{D}} h(\lambda_m) e_m(P) \otimes \overline{e_m(Q)} \\ &\quad + \frac{1}{4\pi} \sum_{\alpha=1}^h \sum_{l=1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma'_\alpha]}{|\Lambda_\alpha|} \int_{\mathbb{R}} h(1+t^2) E_{\alpha l}(P, it) \otimes \overline{E_{\alpha l}(Q, it)} dt. \end{aligned} \quad (4.1.4)$$

The sum and integrals converge absolutely and uniformly on compact subsets of $\mathbb{H}^3 \times \mathbb{H}^3$.

Set

$$\begin{aligned} H_\Gamma(P, Q) &= \frac{1}{4\pi} \sum_{\alpha=1}^h \sum_{l=1}^{k_\alpha} \frac{[\Gamma_\alpha : \Gamma'_\alpha]}{|\Lambda_\alpha|} \int_{\mathbb{R}} h(1+t^2) E_{\alpha l}(P, it) \otimes \overline{E_{\alpha l}(Q, it)} dt, \\ L_\Gamma(P, Q) &= \sum_{m \in \mathcal{D}} h(\lambda_m) e_m(P) \otimes \overline{e_m(Q)}. \end{aligned}$$

Then clearly $K_\Gamma = L_\Gamma + H_\Gamma$. We have

Lemma 4.1.3.

$$\begin{aligned} \int_{\mathcal{F}} \|L_\Gamma(P, Q)\|_V^2 dv(P) dv(Q) &< \infty, \\ \int_{\mathcal{F}} \text{tr}_V(L_\Gamma(P, P)) dv(P) &= \sum_{m \in \mathcal{D}} h(\lambda_m) \text{ and}, \\ \sum_{m \in \mathcal{D}} |h(\lambda_m)| &< \infty. \end{aligned}$$

Proof. The first equation follows from the decay properties of h , the orthonormality of the e_m and that fact that

$$\int_{\mathcal{F}} \|L_\Gamma(P, Q)\|_V^2 dv(P) dv(Q) \leq C \sum_{m \in \mathcal{D}} |h(\lambda_m)|^2 < \infty.$$

For the second, note that

$$\text{tr}_V(e_m(P) \otimes \overline{e_m(P)}) = \langle e_m(P), e_m(P) \rangle_V$$

thus by the orthonormality of the e_m ,

$$\int_{\mathcal{F}} \text{tr}_V(L_\Gamma(P, P)) dv(P) = \sum_{m \in \mathcal{D}} h(\lambda_m) \int_{\mathcal{F}} \langle e_m(P), e_m(P) \rangle_V dv(P) = \sum_{m \in \mathcal{D}} h(\lambda_m).$$

By the decay properties of h ,

$$\sum_{m \in \mathcal{D}} |h(\lambda_m)| < \infty \quad (4.1.5)$$

□

If we naively try to take the trace of \mathcal{K} by $\int_{\mathcal{F}} \text{tr}_V(K_\Gamma(P, P)) dv(P)$ we would see that the integral does not converge. However we can *regularize* the trace by subtracting off the term $\int_{\mathcal{F}} \text{tr}_V(H_\Gamma(P, P)) dv(P)$ in the following careful manner.

Lemma 4.1.4. *Let $f(P) = \text{tr}_V(K_\Gamma(P, P)) - \text{tr}_V(H_\Gamma(P, P))$. Then f is in $L^1(\mathcal{F})$. Further,*

$$\begin{aligned} \lim_{A \rightarrow \infty} \int_{\mathcal{F}_A} \text{tr}_V(K_\Gamma(P, P)) dv(P) - \lim_{A \rightarrow \infty} \int_{\mathcal{F}_A} \text{tr}_V(H_\Gamma(P, P)) dv(P) \\ = \int_{\mathcal{F}} f(P) dv(P). \end{aligned} \quad (4.1.6)$$

(The set \mathcal{F}_A is defined in Lemma 2.6.4.) In particular, the singularities arising from each integral must cancel out.

The fact that the singularities arising from each integral must cancel out will be crucial in our derivation of the Selberg trace formula, and imply some interesting group relations of Γ .

Proof of the Spectral Truncated Trace

For this section Assumption 3.7.2 is in effect. So far we know little about the t -dependence of the Eisenstein series on the critical line. The following consequence of Lemma 4.1.2 will remedy the situation.

Lemma 4.1.5. *Let $l \in 1 \dots k_\infty$. Then*

$$\int_{-T}^T |E_l(z + jr, it)|_V^2 dt = O(r^2 T + T^3).$$

Proof. For $0 < \epsilon < 1/16$ let $k_\epsilon : [1, \infty) \mapsto \mathbb{R}$ be a smooth function satisfying

$$\text{supp}(k_\epsilon) \subset [1, 1 + \epsilon], \quad \max(|k_\epsilon|) \leq c\epsilon^{-\frac{3}{2}}, \quad \int_{\mathbb{H}^3} k_\epsilon(\delta(P, Q)) dv(Q) = 1,$$

for some $c > 0$, and for all $P \in \mathbb{H}^3$ (See [EGM98, page 292] for the construction). Then it follows ([EGM98, page 292]) that

$$\int_{\mathcal{F}} \|K_\Gamma(P, Q)\|_V^2 dv(Q) \leq c_1 \left(r_P^2 \epsilon^{-\frac{1}{2}} + \epsilon^{-\frac{3}{2}} \right)$$

for all $P = (z_P, r_P) \in \mathcal{F}$. On the other hand we have (Lemma 4.1.2)

$$K_\Gamma(P, Q) =$$

$$\sum_{m \in \mathcal{D}} h(\lambda_m) e_m(P) \otimes \overline{e_m(Q)} + \frac{1}{4\pi} \sum_{l=1}^{k_\infty} C_\infty \int_{\mathbb{R}} h(1+t^2) E_l(P, it) \otimes \overline{E_l(Q, it)} dt,$$

and by Parseval's equality (for the Hilbert space $\mathcal{H}(\Gamma, \chi)$), setting $\epsilon = (16T^2)^{-1}$, we obtain

$$c_1(r_P^2 T + T^3) \geq \|K_\Gamma(P, \cdot)\|_{\mathcal{H}(\Gamma, \chi)}^2 \geq \frac{C_\infty}{4\pi} \int_{-T}^T |h(1+t^2)|^2 |E_l(P, it)|^2 dt.$$

By [EGM98, page 122] we get

$$|h(1+t^2) - 1| < \frac{7}{2}(1+|t|)\frac{1}{4T}, \quad \text{for } |t| \leq T,$$

which gives a positive lower bound on $|h(1+t^2)|^2$ for $t \in [-T, T]$. The lemma now follows. \square

The following lemma will allow us to compute the spectral trace. More specifically the Maaß-Selberg relations will give the leading terms of the truncated trace while the error term (below) will vanish as $A \rightarrow \infty$. We adapt a nice argument found in [Iwa02, Page 142] which shortens the exposition found in [EGM98, Page 295].

Lemma 4.1.6. *Let $h(1+z^2) = O((1+|z|^2)^{-3/2-\epsilon})$. Then*

$$\int_{\mathbb{R}} \int_{\mathcal{F}(A)} h(1+t^2) |E_l^A(P, it)|_V^2 dv(P) dt = O(A^{-1}) \text{ as } A \rightarrow \infty.$$

Proof. Let $C_\infty = \frac{[\Gamma_\infty: \Gamma'_\infty]}{|\Lambda_\infty|}$ and let \mathcal{P} be a fundamental domain for the lattice Λ_∞ . By Parseval's equality⁴ (for eigenfunctions of the euclidean Laplacian on $\mathbb{C} = \mathbb{R}^2$),

$$\frac{1}{|\Lambda_\infty|} \int_{\mathcal{P}} |E_l^A(z + rj, it)|_V^2 dz = \sum_{\mu \neq 0} |\phi_{l,\mu}(it)|_V^2 |rK_{it}(2\pi|\mu|r)|^2. \quad (4.1.7)$$

We have the well known formula

$$K_{it}(r) = \pi^{1/2} \Gamma(it + \frac{1}{2})^{-1} \left(\frac{r}{2}\right)^{it} \int_1^\infty (x^2 - 1)^{it-1/2} e^{-xr} dx,$$

which implies (by an elementary argument) that for any $0 < n < m$, constant A , sufficiently large, and implied constant *independent* of t ,

$$\int_A^\infty |K_{it}(r)|^2 r^{-n} dr \ll \int_{A/2}^\infty |K_{it}(r)|^2 r^{-m} dr. \quad (4.1.8)$$

An application of Parseval's equality, and (4.1.7) gives

⁴ $\phi_{l,\mu}(s)$ is the Fourier coefficient of $E_l(P, s)$.

$$\begin{aligned}
& \int_{\mathcal{F}(A)} |E_l^A(P, it)|_V^2 dv(P) \\
&= \frac{1}{[\Gamma_\infty : \Gamma'_\infty]} \int_A \int_{\mathcal{P}} |E_l^A(z + rj, it)|_V^2 dz \frac{dr}{r^3} \\
&= \frac{1}{C_\infty} \int_A \sum_{\mu \neq 0} |\phi_{l,\mu}(it)|_V^2 |rK_{it}(2\pi|\mu|r)|^2 \frac{dr}{r^3} \\
&= C_\infty \int_A \sum_{\mu \neq 0} |\phi_{l,\mu}(it)|_V^2 |K_{it}(2\pi|\mu|r)|^2 \frac{dr}{r} \\
&\ll \int_{A/2}^\infty \sum_{\mu \neq 0} |\phi_{l,\mu}(it)|_V^2 |K_{it}(2\pi|\mu|r)|^2 \frac{dr}{r^5} \\
&= \int_{A/2}^\infty \sum_{\mu \neq 0} |\phi_{l,\mu}(it)|_V^2 |rK_{it}(2\pi|\mu|r)|^2 \frac{dr}{r^7} \\
&= \int_{A/2}^\infty \frac{1}{|\Lambda_\infty|} \int_{\mathcal{P}} |E_l^{A/2}(z + rj, it)|_V^2 dz \frac{dr}{r^7} \\
&\leq \int_{A/2}^\infty \frac{1}{|\Lambda_\infty|} \int_{\mathcal{P}} |E_l(z + rj, it)|_V^2 dz \frac{dr}{r^7}.
\end{aligned}$$

Hence, we infer from Lemma 4.1.5, and another application of Parsavel's equality (for eigenfunctions of the Euclidean Laplacian) that

$$\begin{aligned}
& \int_{-T}^T \int_{\mathcal{F}(A)} |E_l^A(P, it)|_V^2 dv(P) dt \\
&\ll \int_{-T}^T \int_{A/2}^\infty \int_{\mathcal{P}} |E_l^{A/2}(z + rj, it)|_V^2 dz \frac{dr}{r^7} \\
&\leq C \int_{A/2}^\infty \int_{-T}^T |E_l(P, it)|_V^2 dt \frac{dr}{r^7} \\
&\ll \int_{A/2}^\infty c_1(r^2T + T^3) \frac{dr}{r^7} \ll \frac{T^3}{A^4}.
\end{aligned}$$

The lemma follows from the bound on $h(1 + z^2)$ using an elementary integration by parts argument. \square

Define $\phi(s) = \det \mathfrak{S}(s)$. By Theorem 3.7.1 $\phi(s)$ is a meromorphic, finite ordered function on \mathbb{C} , $|\phi(s)| = 1$ on $\{\operatorname{Re}(s) = 0\}$, and $\frac{\phi'}{\phi}(s)$ is regular on $\{\operatorname{Re}(s) = 0\}$. Let $h(1 + z^2) = O((1 + |z|^2)^{-3/2-\epsilon})$, and

$$g(x) = \frac{1}{2\pi} \int_{\mathbb{R}} h(1 + t^2) e^{-itx} dt.$$

Then we have

Lemma 4.1.7.

$$\int_{\mathcal{F}_A} \mathrm{tr}_V(H_\Gamma(P, P)) dv(P) = g(0)k(\Gamma, \chi) \ln(A) - \frac{1}{4\pi} \int_{\mathbb{R}} \frac{\phi'}{\phi}(it) h(1+t^2) dt + \frac{h(1) \mathrm{tr} \mathfrak{S}(0)}{4} + o(1)_{A \rightarrow \infty}. \quad (4.1.9)$$

The integral converges absolutely.

Proof.

$$\begin{aligned} 4\pi \int_{\mathcal{F}_A} \mathrm{tr}_V(H_\Gamma(P, P)) dv(P) &= \sum_{l=1}^{k_\infty} \int_{\mathcal{F}_A} \mathrm{tr} C_\infty \int_{\mathbb{R}} h(1+t^2) E_l(P, it) \otimes \overline{E_l(P, it)} dt dv(P) \\ &= \int_{\mathbb{R}} h(1+t^2) \sum_{l=1}^{k_\infty} \int_{\mathcal{F}_A} \mathrm{tr} C_\infty |E_l(P, it)|_V^2 dv(P) dt \\ &= \int_{\mathbb{R}} h(1+t^2) \sum_{l=1}^{k_\infty} \int_{\mathcal{F}_A} \mathrm{tr} C_\infty |E_l^A(P, it)|_V^2 dv(P) dt \\ &= \int_{\mathbb{R}} h(1+t^2) \sum_{l=1}^{k_\infty} \int_{\mathcal{F}} \mathrm{tr} C_\infty |E_l^A(P, it)|_V^2 dv(P) dt \\ &\quad - \int_{\mathbb{R}} h(1+t^2) \sum_{l=1}^{k_\infty} \int_{\mathcal{F}(A)} \mathrm{tr} C_\infty |E_l^A(P, it)|_V^2 dv(P) dt. \end{aligned}$$

By Lemma 4.1.6 the term on the last line tends to zero as $A \rightarrow \infty$. We evaluate the first term above with the Maaß-Selberg relations on the critical line $\mathrm{Re}(s) = 0$. For $t \in \mathbb{R}, t' = t + r, r > 0, C_\infty = \frac{[\Gamma_\infty : \Gamma'_\infty]}{|\Lambda_\infty|}$ and $\mathcal{E}^Y(P, it)$ the column vector of Eisenstein series ($l = 1 \dots k_\infty$). Consider the $k_\infty \times k_\infty$ matrix⁵, $\langle C_\infty \mathcal{E}^Y(P, it), \mathcal{E}^Y(P, it')^t \rangle$. Then

$$\mathrm{tr} \langle C_\infty \mathcal{E}^Y(P, it), \mathcal{E}^Y(P, it')^t \rangle = \sum_{l=1}^{k_\infty} \int_{\mathcal{F}} \mathrm{tr} C_\infty |E_l^A(P, it)|_V^2 dv(P).$$

Applying the Maaß-Selberg relations $k_\infty \times k_\infty$ many times and placing the result in matrix form gives us⁶

$$\langle C_\infty \mathcal{E}^Y(P, it), \mathcal{E}^Y(P, it')^t \rangle =$$

⁵Each entry is the inner product of two truncated Eisenstein series. The inner product takes place in the Hilbert space $\mathcal{H}(\Gamma, \chi)$.

⁶We are only interested in the trace of the matrix $\langle C_\infty \mathcal{E}^Y(P, it), \mathcal{E}^Y(P, it')^t \rangle$. However by working with the entire matrix we can use the identity (4.1.10) and derive a nice expression for the trace.

$$-Y^{it+it'} i \frac{\mathfrak{S}(it')}{t+t'} + Y^{-it-it'} i \frac{\mathfrak{S}(it)^*}{t+t'} + Y^{it-it'} i \frac{I}{t'-t} - Y^{-it+it'} i \frac{\mathfrak{S}(it)\mathfrak{S}^*(it')}{t'-t}.$$

Next applying the approximations,

$$Y^{ir} = 1 + ir \ln Y + \dots,$$

$$Y^{-ir} = 1 - ir \ln Y + \dots,$$

$$\mathfrak{S}(it + ir) = \mathfrak{S}(it) + ir\mathfrak{S}(it)' + \dots,$$

identities,

$$\mathfrak{S}(-it) = \mathfrak{S}^*(it) = \mathfrak{S}^{-1}(it),$$

$$\frac{d}{dt} \mathfrak{S}(it)\mathfrak{S}(-it) = -i\mathfrak{S}(it)\mathfrak{S}'(-it) + i\mathfrak{S}'(it)\mathfrak{S}(-it) = \frac{d}{dt} I = 0,$$

$$\text{tr } \mathfrak{S}'(it)\mathfrak{S}^{-1}(it) = \frac{d}{dt} \log \det \mathfrak{S}(it) = \frac{\phi'}{\phi}(it), \quad (4.1.10)$$

and letting $r \rightarrow 0$ we obtain the lemma. See [EGM98, Page 305] and [Iwa02, Page 139-142] for more details. \square

4.2 Lattice Characters and Sums

As mentioned earlier the parabolic elements of a cofinite Kleinian group have an associated lattice.

$$Z(x, \Lambda, \psi)$$

Let $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$ be a lattice with $\text{Im}(\tau) > 0$. A (lattice) character ψ of Λ is a one-dimensional unitary representation of Λ .

Definition 4.2.1. For $x > 0$ set

$$Z(x, \Lambda, \psi) \equiv \sum_{\substack{\mu \in \Lambda \\ |\mu|^2 \leq x \\ \mu \neq 0}} \frac{\psi(\mu)}{|\mu|^2},$$

and when the limit exists

$$L(\Lambda, \psi) \equiv \lim_{x \rightarrow \infty} Z(x, \Lambda, \psi).$$

Proposition 4.2.2. Let ψ be a character of Λ .

(1) If $\psi = \text{id}$, the trivial character, then

$$Z(x, \Lambda, \psi) = \frac{\pi}{|\Lambda|} (\ln x + \eta_\Lambda) + O\left(x^{-\frac{1}{2}}\right) \quad \text{as } x \rightarrow \infty. \quad (4.2.1)$$

(2) If $\psi \neq \text{id}$ then $\lim_{x \rightarrow \infty} Z(x, \Lambda, \psi)$ exists and

$$Z(x, \Lambda, \psi) = L(\Lambda, \psi) + O\left(x^{-\frac{1}{2}}\right) \quad \text{as } x \rightarrow \infty.$$

Here η_Λ can be thought of as an analogue of the Euler constant γ for the lattice $\Lambda \subset \mathbb{R}^2$.

Proof. (1) This is proven in [EGM98, page 298].

(2) By Lemma 4.2.5

$$\sum_{\substack{\mu \in \Lambda \\ x < |\mu|^2 < p}} \frac{\psi(\mu)}{|\mu|^2} = O(x^{-\frac{1}{2}}).$$

Here the implied constant does not depend on p . Hence, convergence follows from the Cauchy criterion. Letting $p \rightarrow \infty$ shows that

$$Z(x, \Lambda, \psi) = L(\Lambda, \psi) + O\left(x^{-\frac{1}{2}}\right).$$

□

Kronecker's Second Limit Formula

Let u, v be real numbers which are both not integers and write $\tau = x + iy$ ($y > 0$). For $\text{Re}(s) > 1$ set

$$E_{u,v}(\tau, s) \equiv \lim_{x \rightarrow \infty} \sum'_{|m\tau+n|^2 < x} e^{2\pi i(mu+nv)} \frac{y^s}{|m\tau+n|^{2s}}.$$

Here the prime in the sum means to leave out zero. The series converges uniformly and absolutely on compact subsets of $\text{Re}(s) > 1$. Thus we can also define

$$E_{u,v}(\tau, s) = \sum_{(n,m) \neq (0,0)} e^{2\pi i(mu+nv)} \frac{y^s}{|m\tau+n|^{2s}}$$

since the order of summation is not important when a sum converges absolutely. In the sums, n and m are understood to vary over the integers.

We have ([Sie80] or [Lan87, page 276]⁷)

Lemma 4.2.3. *The function $E_{u,v}(\tau, s)$ can be continued to an entire function of $s \in \mathbb{C}$, and one has*

$$E_{u,v}(\tau, 1) = -2\pi \log |g_{-v,u}(\tau)|, \quad (4.2.2)$$

where g_{a_1, a_2} is the Siegel function,

$$g_{a_1, a_2}(\tau) = -q_\tau^{(1/2)B_2(a_1)} e^{2\pi i a_2(a_1-1)/2} (1 - q_z) \prod_{n=1}^{\infty} (1 - q_\tau^n q_z) (1 - q_\tau^n / q_z), \quad (4.2.3)$$

$B_2(X) = X^2 - X + 1/6$, $q_\tau = e^{2\pi i \tau}$, $q_z = e^{2\pi i z}$, and $z = a_1 \tau + a_2$.

We now explain the relationship between $L(\Lambda, \psi)$ and $E_{u,v}(\tau, 1)$. The character ψ is determined by u and v . That is u and v can be chosen so that $\psi(1) = e^{2\pi i u}$ and $\psi(\tau) = e^{2\pi i v}$. We now can rewrite

$$Z(x, \Lambda, \psi) = \sum'_{|m\tau+n|^2 < x} \frac{e^{2\pi i(mu+nv)}}{|m\tau+n|^2}. \quad (4.2.4)$$

⁷There appears to be a typographical error in the definition of the Siegel function on page 276 of the second edition of [Lan87]. The correct definition appears on page 262.

Here the prime means we leave off zero from the sum.

Formally, ignoring convergence and order of summation,

$$\lim_{x \rightarrow \infty} Z(x, \Lambda, \psi) = \sum_{(m,n) \neq (0,0)} \frac{e^{2\pi i(mu+nv)}}{|m\tau + n|^2} = \frac{1}{y} E_{u,v}(\tau, 1)$$

In the next section we will make the above argument rigorous.

Evaluation of $L(\Lambda, \psi)$

Proposition 4.2.4. *Let $\Lambda = \mathbb{Z} \oplus \mathbb{Z}\tau \subset \mathbb{C}$ be a lattice with $\text{Im}(\tau) > 0$, ψ a character of Λ , and $u, v \in \mathbb{R}$ are both not integers satisfying $\psi(1) = e^{2\pi iu}$ and $\psi(\tau) = e^{2\pi iv}$. Then*

$$L(\Lambda, \psi) = \frac{-2\pi}{y} \log |g_{-v,u}(\tau)|. \quad (4.2.5)$$

Proof. For $\text{Re}(s) > 1$ recall that

$$y^{-s} E_{u,v}(\tau, s) = \lim_{x \rightarrow \infty} \sum'_{|m\tau+n|^2 < x} \frac{e^{2\pi i(mu+nv)}}{|m\tau + n|^{2s}}.$$

For $t \in [1, 2]$ set

$$f(t) \equiv \lim_{x \rightarrow \infty} \sum'_{|m\tau+n|^2 < x} \frac{e^{2\pi i(mu+nv)}}{|m\tau + n|^{2t}}.$$

By proposition (4.2.2) $\lim_{x \rightarrow \infty} Z(x, \Lambda, \psi) = L(\Lambda, \psi)$ converges, and by equation (4.2.4) $L(\Lambda, \psi) = f(1)$. By Lemma 4.2.3 $y^{-s} E_{u,v}(\tau, s)$ can be continued to an entire function (also denoted by $y^{-s} E_{u,v}(\tau, s)$). In particular $y^{-s} E_{u,v}(\tau, s)$ is continuous and

$$\lim_{t \rightarrow 1^+} y^{-t} E_{u,v}(\tau, t) = y^{-1} E_{u,v}(\tau, 1) = \frac{-2\pi}{y} \log |g_{-v,u}(\tau)|.$$

By definition, $f(t) = y^{-t} E_{u,v}(\tau, t)$ for $t \in (1, 2]$. If we can show that f is continuous on $[1, 2]$ then

$$\begin{aligned} L(\Lambda, \psi) = f(1) &= \lim_{t \rightarrow 1^+} f(t) = \lim_{t \rightarrow 1^+} y^{-t} E_{u,v}(\tau, t) = y^{-1} E_{u,v}(\tau, 1) \\ &= \frac{-2\pi}{y} \log |g_{-v,u}(\tau)|. \end{aligned}$$

The function f is a continuous function on $[1, 2]$ by Lemma 4.2.5. To see this observe that by the Cauchy criterion the sum f converges uniformly for $t \in [1, 2]$. Since a uniformly convergent sum of continuous functions is continuous, the proposition is proved. \square

Proof of lemma (4.2.5)

Lemma 4.2.5. *For $s \in [1, 2]$, $p > w > 0$, and v, u both not integers, $\tau = x + iy$, $\text{Im}(\tau) > 0$ let*

$$h(w, s) \equiv \sum_{w < |m\tau+n|^2 < p} \frac{e^{2\pi i(mu+nv)}}{|m\tau + n|^{2s}}.$$

Then

$$h(w, s) = O\left(w^{\frac{1}{2}-s}\right)$$

uniformly for $s \in [1, 2]$.

The implied constant does not depend on w or s . Our proof will work for $s \in (\frac{1}{2}, \infty)$. However the interval $[1, 2]$ is sufficient for our application.

Proof. Without loss of generality we assume $v \notin \mathbb{Z}$ and $\operatorname{Re}(\tau) > 0$. We estimate the sum

$$\sum_{w < |m\tau + n|^2 < p} \frac{e^{2\pi i(mu + nv)}}{|m\tau + n|^{2s}}. \quad (4.2.6)$$

Sum (4.2.6) is a sum over lattice points of Λ that are outside of the circle of radius \sqrt{w} but inside the circle of radius \sqrt{p} . It suffices to restrict our sum to the first quadrant of the plane since the sum over the other three quadrants can be estimated similarly. With this assumption $n \geq 0$ and $m \geq 0$ and our sum can be written as an explicit double sum with n and m separated as,

$$\sum_{m=0}^{[\sqrt{w}/|\tau|]} e^{2\pi i(mu)} \sum_{n=q(w)}^{q(p)} \frac{e^{2\pi inv}}{|m\tau + n|^{2s}} + \sum_{m=[\sqrt{w}/|\tau|]+1}^{[\sqrt{p}/|\tau|]} e^{2\pi i(mu)} \sum_{n=0}^{q(p)} \frac{e^{2\pi inv}}{|m\tau + n|^{2s}}, \quad (4.2.7)$$

where $q(\alpha) = [\sqrt{\alpha - m^2y^2 - mx}]$. The brackets, $[\]$ represent the greatest integer function. The sum above is split over m . For in the first case horizontal translations of lattice points will intersect the inner circle $|z| = \sqrt{w}$ and hence n must be restricted while the second sum n is free to start at zero. See figure (4.2). We encourage the reader to write down the explicit double sum in the simple case of $\mathbb{Z} \oplus \mathbb{Z}i$.

We estimate the inner sum using a Stieltjes integral. For $t \geq 0$ let $\pi(t) = [t]$. Then for $0 \leq a < b$

$$\sum_{n=a}^b \frac{e^{2\pi inv}}{|m\tau + n|^{2s}} = \int_{a'}^{b'} \frac{e^{2\pi itv}}{|m\tau + t|^{2s}} d\pi(t) = \int_{a'}^{b'} \frac{e^{2\pi itv}}{((t + mx)^2 + (my)^2)^s} d\pi(t),$$

where $a' = a - \delta$, $b' = b + \delta$ for any $\delta > 0$ sufficiently small (the Stieltjes integral must have a slightly larger integration domain to include the endpoints of the sum). Next using integration by parts with

$$\begin{aligned} U(t) &= \frac{1}{((t + mx)^2 + (my)^2)^s}, \\ dV(t) &= e^{2\pi itv} d\pi(t), \\ V(t) &= \int_0^t e^{2\pi ixv} d\pi(x), \\ dU(t) &= \frac{d}{dt} U(t) dt, \end{aligned}$$

and noting that

$$V(t) = \int_{0'}^{t'} e^{2\pi ixv} d\pi(x) = \sum_{k=0}^{[t]} e^{2\pi ikv}$$

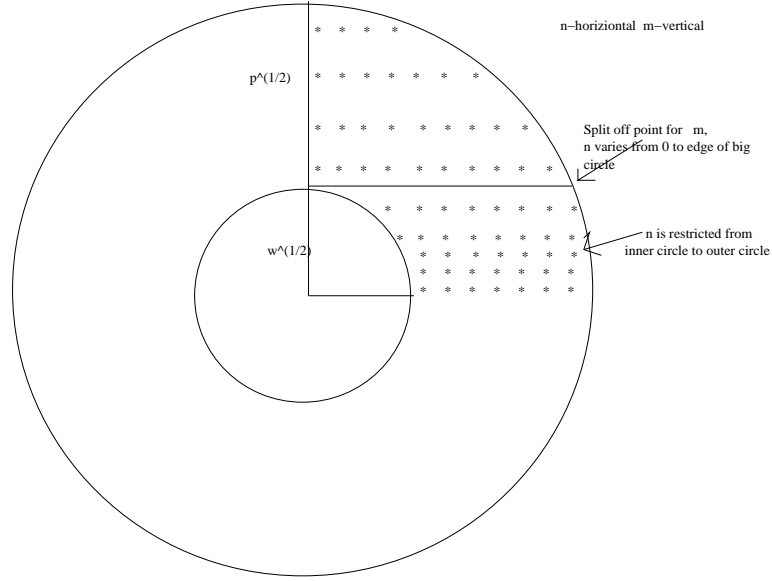


Figure 4.2.1: Lattice points for the sum.

satisfies $-C \leq V(t) \leq C$ since v is not an integer, we can estimate the integral

$$\left| \int_{a'}^{b'} \frac{e^{2\pi itv}}{((t+mx)^2 + (my)^2)^s} d\pi(t) \right| \leq \left| U(t)V(t) \right]_{a'}^{b'} + \left| \int_{a'}^{b'} |V(t)| |dU(t)| \right|$$

$$\leq C \left| U(t) \right]_{a'}^{b'} + C \left| \int_{a'}^{b'} |dU(t)| \right| \leq 2C \left| U(t) \right]_{a'}^{b'} \leq 2C|U(a)| + 2C|U(b)|.$$

Note that $|dU(t)|$ is easily estimated since $U(t)$ is monotone decreasing and that the primes can be dropped since δ can be made arbitrarily small.

What remains is to apply the above estimate to each of the inner sums and crudely put absolute values around all the other terms.

The first sum can be estimated as follows,

$$\begin{aligned}
& \left| \sum_{m=0}^{[\sqrt{w}/|\tau|]} e^{2\pi i(mu)} \sum_{n=q(w)}^{q(p)} \frac{e^{2\pi inv}}{|m\tau + n|^{2s}} \right| \\
& \leq \sum_{m=0}^{[\sqrt{w}/|\tau|]} |e^{2\pi imu}| \left| \sum_{n=q(w)}^{q(p)} \frac{e^{2\pi inv}}{|m\tau + n|^{2s}} \right| \\
& = \sum_{m=0}^{[\sqrt{w}/|\tau|]} \left| \sum_{n=q(w)}^{q(p)} \frac{e^{2\pi inv}}{|m\tau + n|^{2s}} \right| \\
& = \sum_{m=0}^{[\sqrt{w}/|\tau|]} \left| \int_{q(w)'}^{q(p)'} \frac{e^{2\pi itv}}{|m\tau + n|^{2s}} d\pi(t) \right| \\
& \leq 2C \sum_{m=0}^{[\sqrt{w}/|\tau|]} (|U(q(w))| + |U(q(p))|).
\end{aligned}$$

Simplifying $U(q(w))$ and $U(q(p))$ using

$$U(t) = \frac{1}{((t + mx)^2 + (my)^2)^s}$$

and

$$q(\alpha) = [\sqrt{\alpha - m^2 y^2} - mx]$$

yields

$$\begin{aligned}
& 2C \sum_{m=0}^{[\sqrt{w}/|\tau|]} (|U(q(w))| + |U(q(p))|) \\
& \leq 2C \sum_{m=0}^{[\sqrt{w}/|\tau|]} \left(\frac{1}{[w]^s} + \frac{1}{[p]^s} \right) \\
& \leq 2C[\sqrt{w}/|\tau| + 1] \left(\frac{1}{[w]^s} + \frac{1}{[p]^s} \right) \\
& \leq 2C[\sqrt{w}/|\tau| + 1] \left(\frac{1}{[w]^s} + \frac{1}{[w]^s} \right) = O\left(w^{\frac{1}{2}-s}\right).
\end{aligned}$$

The last inequality follows from $w \leq p$.

We apply the same method to the second sum and obtain,

$$\begin{aligned}
& \left| \sum_{m=[\sqrt{w}/|\tau|]+1}^{[\sqrt{p}/|\tau|]} e^{2\pi i(mu)} \sum_{n=0}^{q(p)} \frac{e^{2\pi inv}}{|m\tau + n|^{2s}} \right| \\
& \leq 2C \sum_{m=[\sqrt{w}/|\tau|]+1}^{[\sqrt{p}/|\tau|]} (|U(0)| + |U(q(p))|) \\
& \leq 2C \sum_{m=[\sqrt{w}/|\tau|]+1}^{[\sqrt{p}/|\tau|]} \left(\frac{1}{((mx)^2 + (my)^2)^s} + \frac{1}{[p]^s} \right) \\
& = 2C \sum_{m=[\sqrt{w}/|\tau|]+1}^{[\sqrt{p}/|\tau|]} \frac{1}{((mx)^2 + (my)^2)^s} + 2C \sum_{m=[\sqrt{w}/|\tau|]+1}^{[\sqrt{p}/|\tau|]} \frac{1}{[p]^s}.
\end{aligned}$$

It remains to show that both sums above are bounded by a term of growth $O(w^{\frac{1}{2}-s})$.

For the right hand sum,

$$2C \sum_{m=[\sqrt{w}/|\tau|]+1}^{[\sqrt{p}/|\tau|]} \frac{1}{[p]^s} \leq 2C[\sqrt{p}/|\tau|] \frac{1}{[p]^s} = O(w^{\frac{1}{2}-s})$$

since $p > w$.

For the left hand sum

$$2C \sum_{m=[\sqrt{w}/|\tau|]+1}^{[\sqrt{p}/|\tau|]} \frac{1}{((mx)^2 + (my)^2)^s} \leq C_1 \sum_{m=[\sqrt{w}]}^{\infty} \frac{1}{m^{2s}} = O(w^{\frac{1}{2}-s}),$$

for some $C_1 > 0$. The last equality holds uniformly for $s \in [1, 2]$ by the integral test. \square

4.3 The Explicit Trace

In this section our main goal is to give an explicit formula for

$$\int_{\mathcal{F}_A} \text{tr}_V(K_\Gamma(P, P)) dv(P).$$

Following Selberg's original method we decompose

$$K_\Gamma(P, Q) = \sum_{\gamma \in \Gamma} \chi(\gamma) K(P, \gamma Q)$$

into various sub-sums. Depending on their *type*. The types are as follows, "id" is the identity, "par" are the parabolic elements, "ce" are the cuspidal elliptic elements⁸, "nce" are the non-cuspidal

⁸These elliptical elements share a common fixed point in \mathbb{P} with some parabolic element in Γ .

elliptical elements, “lox” are the hyperbolic and loxodromic elements, and “cusp” = “par” \cup “ce”. For each $S \in \{\text{id}, \text{par}, \text{ce}, \text{nce}, \text{lox}, \text{cusp}\}$ set

$$K_{\Gamma}^S(P, Q) = \sum_{\gamma \in \Gamma^S} \chi(\gamma) K(P, \gamma Q).$$

Here Γ^S denotes the subset of Γ consisting of elements of type S .

Following [EGM98, Section 5.2, Theorem 6.5.1] we have,

Lemma 4.3.1.

$$\begin{aligned} & \int_{\mathcal{F}_A} \text{tr}_V (K_{\Gamma}^{\text{id}} + K_{\Gamma}^{\text{nce}} + K_{\Gamma}^{\text{lox}}) (P, P) dv(P) \\ &= \frac{\text{vol}(\Gamma \backslash \mathbb{H}^3) \dim_{\mathbb{C}}(V)}{4\pi^2} \int_{\mathbb{R}} h(1+t^2)t^2 dt + \sum_{\{R\}_{\text{nce}}} \frac{\text{tr}(\chi(R))g(0) \log N(T_0)}{4|\mathcal{E}(R)| \sin^2\left(\frac{\pi k}{m(R)}\right)} \\ &+ \sum_{\{T\}_{\text{lox}}} \frac{\text{tr}(\chi(T))g(\log N(T)) \log N(T_0)}{|\mathcal{E}(T)| |a(T) - a(T)^{-1}|^2} + o(1) \text{ as } A \rightarrow \infty. \end{aligned} \quad (4.3.1)$$

The notations above were defined in §4.1.

The Cuspidal Elliptic Elements

Our next immediate goal is to evaluate

$$\int_{\mathcal{F}_A} \text{tr}_V K_{\Gamma}^{\text{ce}}(P, P) dv(P).$$

For notational simplicity, we will adopt Assumption 3.7.2 from this point on until the end of this thesis.

Denote by \mathcal{CE} set of elements of Γ which are Γ -conjugate to an element of $\Gamma_{\infty} \backslash \Gamma'_{\infty} = \{\gamma \in \Gamma_{\infty} \mid \gamma \text{ is not parabolic nor the identity element}\}$. We fix representatives of conjugacy classes of \mathcal{CE} , g_1, \dots, g_d ⁹ that have the form

$$g_i = \begin{pmatrix} \epsilon_i & \epsilon_i \omega_i \\ 0 & (\epsilon_i)^{-1} \end{pmatrix}. \quad (4.3.2)$$

For $g \in \mathcal{CE}$, let $\mathcal{C}(g)$ denote the centralizer in Γ of g . In addition, let $\{p_i, \infty\}$ be the set of fixed points in \mathbb{P} of the element g_i . Since g_i is a cuspidal elliptic element it follows that p_i is a cusp of Γ (see [EGM98] page 52). Hence by Assumption 3.7.2 there is an element $\gamma_i \in \Gamma$ with $\gamma_i \infty = p_i$. Suppose that c_i is the lower left hand (matrix) entry of γ_i . Then we have (see [EGM98, Pages 302-304]),

⁹There are only finitely many distinct conjugacy classes of elliptic elements in a cofinite Kleinian group.

Lemma 4.3.2.

$$\begin{aligned}
\int_{\mathcal{F}_A} \operatorname{tr} K_{\Gamma}^{\text{ce}}(P, P) dv(P) &= \sum_{i=1}^d \frac{\operatorname{tr} \chi(g_i)}{|\mathcal{C}(g_i)|} \left[\frac{2g(0)(\log |c_i| + \log A)}{|1 - \epsilon_i^2|^2} \right. \\
&\quad \left. + \frac{1}{|1 - \epsilon_i^2|^2} \int_0^\infty g(x) \frac{\sinh x}{\cosh x - 1 + \frac{|1 - \epsilon_i^2|^2}{2}} dx \right] \\
&\quad + o(1) \text{ as } A \rightarrow \infty. \quad (4.3.3)
\end{aligned}$$

4.4 The Parabolic Elements

This section contains the new features of the Selberg trace formula that are not present in the two-dimensional vector and three-dimensional scalar cases. We remind the reader that Assumption 3.7.2 is in effect.

Our main goal for this section is to evaluate

$$\int_{\mathcal{F}_A} \operatorname{tr}_V K_{\Gamma}^{\text{par}}(P, P) dv(P). \quad (4.4.1)$$

Evaluation of Integral 4.4.1

Let \mathcal{P} be a fundamental domain for the action¹⁰ of Γ_∞ on \mathbb{C} ,

$$\tilde{\mathcal{P}} \equiv \{(z, r) \in \mathbb{H}^3 \mid z \in \mathcal{P}\},$$

and

$$\mathcal{P}_A \equiv \{(z, r) \in \mathbb{H}^3 \mid z \in \mathcal{P}, r \leq A\}.$$

It follows that $\tilde{\mathcal{P}}$ is a fundamental domain for the action of Γ_∞ on \mathbb{H}^3 .

Recall that Γ'_∞ is canonically isomorphic to a lattice Λ_∞ . For $\mu \in \Lambda_\infty$ let $\hat{\mu}$ denote the corresponding parabolic element in Γ'_∞ . We will need the following (see [EGM98, Pages 300-301])

Lemma 4.4.1.

$$\int_{\mathcal{F}_A} \operatorname{tr} K_{\Gamma}^{\text{par}}(P, P) dv(P) = \sum_{\mu \in \Lambda_\infty} \operatorname{tr}(\chi(\hat{\mu})) \int_{\mathcal{P}_A} K(P, \hat{\mu}P) dv(P) + o(1) \text{ as } A \rightarrow \infty. \quad (4.4.2)$$

Since Γ'_∞ is an abelian group, χ restricted to Γ'_∞ can be diagonalized. In other words, there exist lattice characters $\{\psi_l\}_{l=1 \dots n}$ so that

$$\operatorname{tr} \chi|_{\Gamma'_\infty} = \sum_{l=1}^n \psi_l. \quad (4.4.3)$$

Thus it suffices to consider lattice characters instead of unitary representations.

¹⁰See §2.5 for more details on the action.

Lemma 4.4.2. *Let ψ be a lattice character of Λ_∞ . Then*

(1) *For $\psi = id$,*

$$\begin{aligned} \sum'_{\mu \in \Lambda_\infty} \psi(\mu) \int_{\mathcal{P}_A} K(P, \widehat{\mu}P) = \\ \frac{1}{[\Gamma_\infty : \Gamma'_\infty]} \left(g(0) \log A + \frac{h(1)}{4} + g(0) \left(\frac{\eta_\infty}{2} - \gamma \right) - \frac{1}{2\pi} \int_{\mathbb{R}} h(1+t^2) \frac{\Gamma'}{\Gamma}(1+it) dt \right) \\ + o(1) \text{ as } A \rightarrow \infty. \end{aligned}$$

(2) *For $\psi \neq id$,*

$$\sum'_{\mu \in \Lambda_\infty} \psi(\mu) \int_{\mathcal{P}_A} K(P, \widehat{\mu}P) = \frac{g(0)}{[\Gamma_\infty : \Gamma'_\infty]} L(\Lambda, \psi) + o(1) \text{ as } A \rightarrow \infty.$$

Proof. (1) is proved in [EGM98, pages 300-302].

The proof of (2) is a modification of (1). Let $C_\infty = \frac{|\Lambda_\infty|}{[\Gamma_\infty : \Gamma'_\infty]}$, then by the definition of the action of $\widehat{\mu}$ on \mathbb{H}^3 ,

$$\begin{aligned} \sum'_{\mu \in \Lambda_\infty} \psi(\mu) \int_{\mathcal{P}_A} K(P, \widehat{\mu}P) = C_\infty \sum'_{\mu \in \Lambda_\infty} \psi(\mu) \int_0^A k \left(\frac{|\mu|^2}{2r^2} + 1 \right) \frac{dr}{r^3}. \\ = C_\infty \sum'_{\mu \in \Lambda_\infty} \frac{\psi(\mu)}{|\mu|^2} \int_{\frac{|\mu|^2}{2A^2}}^\infty k(u+1) du. \quad (4.4.4) \end{aligned}$$

Since

$$Z(x, \Lambda, \psi) = \sum_{\substack{\mu \in \Lambda_\infty \\ |\mu|^2 \leq x \\ \mu \neq 0}} \frac{\psi(\mu)}{|\mu|^2},$$

using summation by parts, we can rewrite (4.4.4) as

$$C_\infty \int_0^\infty k(u+1) Z(2A^2u, \Lambda, \psi) du. \quad (4.4.5)$$

Next we apply Proposition 4.2.2 to obtain

$$\begin{aligned} C_\infty \int_0^\infty k(u+1) Z(2A^2u, \Lambda, \psi) du \\ = C_\infty \int_0^\infty k(u+1) \left(L(\Lambda, \psi) + O \left((2A^2u)^{-1/2} \right) \right) du. \quad (4.4.6) \end{aligned}$$

Now we show that the resulting error term is $o(1)$. Since $k(u+1) = O((1+u)^{-4})$ (k is a rapid decay function)

$$\left| \int_0^\infty k(u+1) O \left((2A^2u)^{-1/2} \right) du \right| \leq \frac{D}{A} \int_0^\infty (1+u)^{-4} u^{-1/2} du = o(1) \text{ as } A \rightarrow \infty$$

for some $D > 0$. To complete the proof note that $\int_0^\infty k(u+1) du = g(0)$. \square

Finally we can evaluate (4.4.1):

Lemma 4.4.3.

$$\int_{\mathcal{F}_A} \operatorname{tr} K_{\Gamma}^{\text{par}}(P, P) dv(P) = \frac{l_{\infty}}{|\Gamma_{\infty} : \Gamma'_{\infty}|} \left(g(0) \log A + \frac{h(1)}{4} + g(0) \left(\frac{\eta_{\infty}}{2} - \gamma \right) - \frac{1}{2\pi} \int_{\mathbb{R}} h(1+t^2) \frac{\Gamma'}{\Gamma}(1+it) dt \right) + \frac{g(0)}{|\Gamma_{\infty} : \Gamma'_{\infty}|} \sum_{l=l_{\infty}+1}^n L(\Lambda_{\infty}, \psi_l). \quad (4.4.7)$$

Here $n = \dim_{\mathbb{C}} V$, ψ_l are the lattice characters associated to the lattice Λ_{∞} , $l_{\infty} = \dim_{\mathbb{C}} V'_{\infty}$, and η_{∞} is the analogue of the Euler constant for the lattice Λ_{∞} .

Proof. The proof follows immediately from Lemma 4.4.1, Equation 4.4.3, and Lemma 4.4.2. \square

We have evaluated the truncated trace of \mathcal{K} explicitly as an integral, and by using spectral theory. Notice that as $A \rightarrow \infty$ the integral over the parabolic sum (Lemma 4.4.3) has a divergent term. So does the corresponding cuspidal elliptic integral (Lemma 4.3.2). By Lemma 4.1.4 the divergent terms *must* equal the divergent term of the spectral (truncated) trace (Lemma 4.1.7). It follows that

$$2g(0) \log A \sum_{i=1}^d \frac{\operatorname{tr} \chi(g_i)}{|\mathcal{C}(g_i)||1 - \epsilon_i^2|^2} + g(0) \log A \frac{l_{\infty}}{|\Gamma_{\infty} : \Gamma'_{\infty}|} - g(0)k_{\infty} \log A = 0.$$

By choosing a suitable k so that $g(0) \neq 0$ we obtain

Lemma 4.4.4.

$$2 \sum_{i=1}^d \frac{\operatorname{tr} \chi(g_i)}{|\mathcal{C}(g_i)||1 - \epsilon_i^2|^2} + \frac{l_{\infty}}{|\Gamma_{\infty} : \Gamma'_{\infty}|} = k_{\infty}.$$

The formula¹¹ above is an application of spectral theory to the group relations of a cofinite hyperbolic three-orbifold¹². We will use the above lemma to give a meromorphic continuation of the Selberg zeta-function.

Completion of the proof of the Selberg Trace Formula

The Selberg trace formula now follows: combine Lemma 4.4.3, Lemma 4.3.2, Lemma 4.4.4, Lemma 4.3.1, Lemma 4.1.7, and Lemma 4.1.4. Note that the divergent terms all cancel by Lemma 4.1.4 (or we can use Lemma 4.4.4). Finally take the limit as $A \rightarrow \infty$. See [EGM98, Section 6.5] for more details on combining the lemmas above.

¹¹A similar formula is valid for the general case of κ -many cusps.

¹²Notice that all of the terms above are defined simply in terms of group relations.

Chapter 5

The Selberg Zeta Function

In this section we define the Selberg zeta-function $Z(s, \Gamma, \chi)$ for cofinite Kleinian groups with finite-dimensional unitary representations, in the right half-plane $\operatorname{Re}(s) > 1$. We then evaluate the logarithmic derivative of $Z(s, \Gamma, \chi)$ and show that $Z(s, \Gamma, \chi)$ admits a meromorphic continuation, subject to some technical assumptions concerning the stabilizer subgroup Γ_∞ .

5.1 The Definition and Motivation

In the celebrated paper [Sel56] Selberg first defined what is now called “The Selberg zeta-function¹” as an infinite product over lengths of *primitive closed geodesics*², bearing a strong resemblance to the Riemann zeta-function. Surprisingly, the Selberg zeta-function satisfies a Riemann hypothesis, and encodes both geometric and spectral data of the quotient orbifold³ $\Gamma \backslash \mathbb{H}^2$. The spectral and geometric connection is made clear when one understands the Selberg zeta-function as a by-product of the Selberg trace formula applied to the resolvent kernel of Δ .

In Definition 5.1.1 we will define the Selberg zeta-function for our case of interest. A natural question arises: *what does our zeta-function have in common with the original Selberg zeta-function?* The answer⁴: the logarithmic derivatives of both zeta-functions are directly related to the loxodromic (or hyperbolic) contribution of the Selberg trace formula applied to the resolvent kernel of Δ . The term (from the trace formula) in question for our case has the form

$$\sum_{\{T\}_{\text{lox}}} \frac{\operatorname{tr}(\chi(T)) \log N(T)}{m(T) |a(T) - a(T)^{-1}|^2} N(T)^{-s}.$$

We show in Lemma 5.1.2 that the term above is the logarithmic derivatives of a meromorphic function $Z(s, \Gamma, \chi)$, and that it has a product expansion in the right half-plane $\operatorname{Re}(s) > 1$.

¹More precisely, the Selberg zeta-function of a cocompact Fuchsian group.

²Geodesics that do not trace over themselves multiple times.

³A Riemann surface if Γ is torsion-free.

⁴An alternative answer is that in the cocompact case, both zeta functions are factors of the regularized (functional) determinant $\det(\Delta - (1 - s^2))$. See [Sar87] for more details.

In order to define $Z(s, \Gamma, \chi)$ we will need some notions concerning centralizer subgroups of loxodromic elements. For more details see [EGM98, Sections 5.2, 5.4].

Let Γ be a cofinite Kleinian group and let $\chi \in \text{Rep}(\Gamma, V)$. Suppose $T \in \Gamma$ is loxodromic (we consider hyperbolic elements as loxodromic elements). Then T is conjugate in $\text{PSL}(2, \mathbb{C})$ to a unique element of the form

$$D(T) = \begin{pmatrix} a(T) & 0 \\ 0 & a(T)^{-1} \end{pmatrix}$$

such that $a(T) \in \mathbb{C}$ has $|a(T)| > 1$. Let $N(T)$ denote the *norm* of T , defined by

$$N(T) \equiv |a(T)|^2,$$

and let by $\mathcal{C}(T)$ denote the centralizer of T in Γ . There exists a (primitive) loxodromic element T_0 , and a finite cyclic elliptic subgroup $\mathcal{E}(T)$ of order $m(T)$, generated by an element E_{T_0} such that

$$\mathcal{C}(T) = \langle T_0 \rangle \times \mathcal{E}(T).$$

Here $\langle T_0 \rangle = \{ T_0^n \mid n \in \mathbb{Z} \}$.⁵ Next, let $\mathfrak{t}_1, \dots, \mathfrak{t}_n$, and $\mathfrak{t}'_1, \dots, \mathfrak{t}'_n$ denote the eigenvalues of $\chi(T_0)$ and $\chi(E_{T_0})$ respectively. The elliptic element E_{T_0} is conjugate in $\text{PSL}(2, \mathbb{C})$ to an element of the form

$$\begin{pmatrix} \zeta(T_0) & 0 \\ 0 & \zeta(T_0)^{-1} \end{pmatrix},$$

where here $\zeta(T_0)$ is a primitive $2m(T)$ -th root of unity.

Definition 5.1.1. For $\text{Re}(s) > 1$ the Selberg zeta-function $Z(s, \Gamma, \chi)$ is defined by

$$Z(s, \Gamma, \chi) \equiv \prod_{\{T_0\} \in \mathcal{R}} \prod_{j=1}^{\dim V} \prod_{\substack{l, k \geq 0 \\ c(T, j, l, k)=1}} \left(1 - \mathfrak{t}_j a(T_0)^{-2k} \overline{a(T_0)^{-2l}} N(T_0)^{-s-1} \right).$$

Here the product with respect to T_0 extends over a maximal reduced system \mathcal{R} of Γ -conjugacy classes of primitive loxodromic elements of Γ . The system \mathcal{R} is called reduced if no two of its elements have representatives with the same centralizer⁶. The function $c(T, j, l, k)$ is defined by

$$c(T, j, l, k) = \mathfrak{t}'_j \zeta(T_0)^{2l} \zeta(T_0)^{-2k}.$$

Lemma 5.1.2. For $\text{Re}(s) > 1$,

$$\frac{d}{ds} \log Z(s, \Gamma, \chi) = \sum_{\{T\} \text{ lox}} \frac{\text{tr}(\chi(T)) \log N(T_0)}{m(T) |a(T) - a(T)^{-1}|^2} N(T)^{-s}.$$

⁵Note that by definition $\mathcal{E}(T) = \mathcal{E}(T_0)$, and that T_0 is unique up to multiplication by an element of $\mathcal{E}(T)$.

⁶See [EGM98] section 5.4 for more details

Proof. It follows from the proof of [EGM98, Lemma 5.4.2] that

$$\begin{aligned} & \sum_{\{T\}_{\text{lox}}} \frac{\text{tr}(\chi(T)) \log N(T_0)}{m(T) |a(T) - a(T)^{-1}|^2} N(T)^{-s} \\ &= \sum_{\substack{\{T_0\} \in \mathcal{R} \\ n \geq 0 \\ 1 \leq v \leq m(T_0)}} \frac{\text{tr} \chi(T_0^{n+1} E_0^v) \log N(T_0)}{m(T_0) |\zeta(T_0)^v a(T_0)^{n+1} - \zeta(T_0)^{-v} a(T_0)^{-n-1}|^2} N(T_0)^{-s(n+1)}. \end{aligned} \quad (5.1.1)$$

Next since T_0 commutes with E_{T_0} we can diagonalize the restriction of χ to $\mathcal{C}(T)$ and continue the equality to

$$\begin{aligned} &= \sum_{\substack{\{T_0\} \in \mathcal{R} \\ n \geq 0 \\ 1 \leq v \leq m(T_0) \\ 1 \leq j \leq \dim V}} \frac{\mathfrak{t}_j^{n+1} \mathfrak{t}'_j{}^v \log N(T_0)}{m(T_0) |\zeta(T_0)^v a(T_0)^{n+1} - \zeta(T_0)^{-v} a(T_0)^{-n-1}|^2} N(T_0)^{-s(n+1)} \\ &= \sum_{\substack{\{T_0\} \in \mathcal{R} \\ n \geq 0 \\ 1 \leq v \leq m(T_0) \\ 1 \leq j \leq \dim V}} \frac{\mathfrak{t}_j^{n+1} \mathfrak{t}'_j{}^v \log N(T_0)}{m(T_0) (1 - d(T_0)) \left(1 - \overline{d(T_0)}\right)} N(T_0)^{-s(n+1)} \\ &= \sum_{\substack{\{T_0\} \in \mathcal{R} \\ n \geq 0 \\ 1 \leq v \leq m(T_0) \\ l, k \geq 0 \\ 1 \leq j \leq \dim V}} \frac{\mathfrak{t}_j^{n+1} \mathfrak{t}'_j{}^v N(T_0)^{-s(n+1)} \log N(T_0) (d(T_0))^k \left(\overline{d(T_0)}\right)^l}{m(T_0)}. \end{aligned}$$

where

$$d(T_0) = \zeta(T_0)^{-2v} a(T_0)^{-2(n+1)}$$

Next we sum over the v -index (note that it is a geometric sum of an $m(T_0)$ -th root of unity) observe that the sum is non-zero only when

$$\mathfrak{t}'_j \zeta(T_0)^{2l} \zeta(T_0)^{-2k} = 1$$

or using our notation $c(T, j, l, k) = 1$. The equality continues as

$$\begin{aligned} &= \sum_{\substack{\{T_0\} \in \mathcal{R} \\ n \geq 0 \\ l, k \geq 0 \\ 1 \leq j \leq \dim V \\ c(T, j, l, k) = 1}} \mathfrak{t}_j^{n+1} N(T_0)^{-s(n+1)} \log N(T_0) \left(a(T_0)^{-2(n+1)}\right)^k \left(\overline{a(T_0)^{-2(n+1)}}\right)^l \\ &= \sum_{\substack{\{T_0\} \in \mathcal{R} \\ l, k \geq 0 \\ 1 \leq j \leq \dim V \\ c(T, j, l, k) = 1}} \frac{\mathfrak{t}_j a(T_0)^{-2k} \overline{a(T_0)^{-2l}} N(T_0)^{-(s+1)} \log N(T_0)}{1 - \mathfrak{t}_j a(T_0)^{-2k} \overline{a(T_0)^{-2l}} N(T_0)^{-(s+1)}} = \frac{Z'(s, \Gamma, \chi)}{Z(s, \Gamma, \chi)}. \end{aligned}$$

□

5.2 The Logarithmic Derivative of the Selberg Zeta-Function

The first step in obtaining the meromorphic continuation of the zeta-function is to relate its logarithmic derivative to the trace formula. From this point on Assumption 3.7.2 is in effect.

We apply the Selberg trace formula to the pair of functions,

$$h(w) = \frac{1}{s^2 + w - 1} - \frac{1}{B^2 + w - 1} \quad \text{and}$$

$$g(x) = \frac{1}{2s} e^{-s|x|} - \frac{1}{2B} e^{-B|x|},$$

where $1 < \operatorname{Re}(s) < \operatorname{Re}(B)$ and obtain

Lemma 5.2.1.

$$\begin{aligned} \frac{1}{2s} \frac{Z'}{Z}(s) - \frac{1}{2B} \frac{Z'}{Z}(B) &= \frac{1}{2s} \sum_{\{T\} \text{ lox}} \frac{\operatorname{tr}(\chi(T)) \log N(T_0)}{m(T) |a(T) - a(T)^{-1}|^2} N(T)^{-s} \\ &\quad - \frac{1}{2B} \sum_{\{T\} \text{ lox}} \frac{\operatorname{tr}(\chi(T)) \log N(T_0)}{m(T) |a(T) - a(T)^{-1}|^2} N(T)^{-B} \\ &= \sum_{n \in D} \left(\frac{1}{s^2 - s_n^2} - \frac{1}{B^2 - s_n^2} \right) - \frac{1}{4\pi} \int_{\mathbb{R}} \left(\frac{1}{s^2 + w^2} - \frac{1}{B^2 + w^2} \right) \frac{\phi'}{\phi}(iw) dw \\ &\quad + \frac{l_\infty}{2\pi[\Gamma_\infty : \Gamma'_\infty]} \int_{\mathbb{R}} \left(\frac{1}{s^2 + w^2} - \frac{1}{B^2 + w^2} \right) \frac{\Gamma'}{\Gamma}(1 + iw) dw + \frac{\operatorname{tr} \mathfrak{S}(0)}{4s^2} - \frac{\operatorname{tr} \mathfrak{S}(0)}{4B^2} \\ &\quad - \frac{l_\infty}{4[\Gamma_\infty : \Gamma'_\infty]s^2} + \frac{l_\infty}{4[\Gamma_\infty : \Gamma'_\infty]B^2} \\ &\quad - \sum_{i=1}^l \frac{\operatorname{tr} \chi(g_i)}{|C(g_i)| |1 - \epsilon_i^2|^2} \int_0^\infty \left(\frac{e^{-sx}}{2s} - \frac{e^{-Bx}}{2B} \right) \frac{\sinh x}{\cosh x - 1 + \frac{|1 - \epsilon_i^2|^2}{2}} dx \\ &\quad - \left(\frac{1}{2s} - \frac{1}{2B} \right) \sum_{\{R\} \text{ nce}} \frac{\operatorname{tr}_V \chi(R) \log N(T_0)}{4|\mathcal{E}(R)| \sin^2\left(\frac{\pi k}{m(R)}\right)} + \frac{\operatorname{vol}(\Gamma \setminus \mathbb{H}^3) \dim V}{4\pi} (s - B) \\ &\quad - \left(\frac{1}{2s} - \frac{1}{2B} \right) \sum_{i=1}^l \frac{2 \operatorname{tr} \chi(g_i) \log |c_i|}{|C(g_i)| |1 - \epsilon_i^2|^2} \\ &\quad - \left(\frac{1}{2s} - \frac{1}{2B} \right) \frac{1}{[\Gamma_\infty : \Gamma'_\infty]} \left(l_\infty \left(\frac{\eta_\infty}{2} - \gamma \right) + \sum_{l=l_\infty+1}^n L(\Lambda_\infty, \psi_l) \right). \quad (5.2.1) \end{aligned}$$

Proof. The first equality follows from Lemma 5.1.2. The second equality follows directly from the Selberg trace formula. □

Equation (5.2.1) is used to exhibit the meromorphic continuation of $Z(s, \Gamma, \chi)$. If we fix B and multiply through by $2s$, it is not hard to see that each term on the right of (5.2.1) is meromorphic.

In order to see that $Z(s, \Gamma, \chi)$ is meromorphic, we must compute the residues of each term on the right of (5.2.1). We will show that the residues are fractional and that for some $N \in \mathbb{N}$, $Z(s, \Gamma, \chi)^N$ is a meromorphic function.

Theorem 5.2.2. *Let Γ be cofinite with one class of cusps at $\zeta = \infty$, and let $\chi \in \text{Rep}(\Gamma, V)$.*

1. *If $[\Gamma_\infty : \Gamma'_\infty] = 1$ or $[\Gamma_\infty : \Gamma'_\infty] = 2$, then $Z(s, \Gamma, \chi)$ is a meromorphic function.*
2. *If $[\Gamma_\infty : \Gamma'_\infty] = 3$, then there exists a natural number N , $1 \leq N \leq 6$, so that $(Z(s, \Gamma, \chi))^N$ is a meromorphic function.*

Proof. The proof follows from a careful study of (5.2.1). We must show that after multiplying by $2s$, each term on the right (of the second equal sign) has at most simple poles with integral or rational residues⁷. This is demonstrated in Lemma 5.2.4, Lemma 5.3.1, Lemma 5.3.2, and Lemma 5.3.4. \square

We remark that the divisor of the Selberg zeta-function is readily read off from Lemma 5.2.4, Lemma 5.3.1, Lemma 5.3.2, and Lemma 5.3.4.

Our zeta function satisfies a functional equation. A standard argument ([Ven82, Theorem 5.1.5, page 85]) using (5.2.1), Lemma 5.2.4, Lemma 5.3.1, Lemma 5.3.2, and Lemma 5.3.4 yields:

Theorem 5.2.3. *Suppose that $[\Gamma_\infty : \Gamma'_\infty] = 1$ or $[\Gamma_\infty : \Gamma'_\infty] = 2$. Then $Z(s, \Gamma, \chi)$ satisfies:*

$$Z(-s, \Gamma, \chi) = Z(s, \Gamma, \chi)\phi(s)\Psi(s, \Gamma, \chi).$$

For $[\Gamma_\infty : \Gamma'_\infty] = 1$,

$$\Psi(s) \equiv \left(\frac{\Gamma(1-s)}{\Gamma(1+s)} \right)^{k_\infty} \exp \left(-\frac{\text{vol}(\Gamma \setminus \mathbb{H}^3) \dim V}{3\pi} s^3 + Es + C \right)$$

and

$$E \equiv \sum_{\{R\}_{\text{ncc}}} \frac{\text{tr}_V \chi(R) \log N(T_0)}{4|\mathcal{E}(R)| \sin^2(\frac{\pi k}{m(R)})} + \left(k_\infty \left(\frac{\eta_\infty}{2} - \gamma \right) + \sum_{l=k_\infty+1}^n L(\Lambda_\infty, \psi_l) \right).$$

For $[\Gamma_\infty : \Gamma'_\infty] = 2$,

$\Psi(s) \equiv$

$$\left(\frac{\Gamma(1-s)}{\Gamma(1+s)} \right)^{l_\infty} \left(\prod_{k=1}^{\infty} \exp(-k(-1)^k \frac{(k-1)^2 - s^2}{(k+1)^2 - s^2}) \right)^{k_\infty/2 - l_\infty/2} \cdot \exp \left(-\frac{\text{vol}(\Gamma \setminus \mathbb{H}^3) \dim V}{3\pi} s^3 + Es + C \right)$$

⁷For case (1) the residues must be integer while for case (2) it suffices to show that the residues are rational with bounded denominator.

and

$$E \equiv \sum_{\{R\}_{\text{nce}}} \frac{\text{tr}_V \chi(R) \log N(T_0)}{4|\mathcal{E}(R)| \sin^2\left(\frac{\pi k}{m(R)}\right)} + \sum_{i=1}^l \frac{2 \text{tr} \chi(g_i) \log |c_i|}{|C(g_i)| |1 - \epsilon_i^2|^2} + \frac{1}{[\Gamma_\infty : \Gamma'_\infty]} \left(l_\infty \left(\frac{\eta_\infty}{2} - \gamma \right) + \sum_{l=l_\infty+1}^n L(\Lambda_\infty, \psi_l) \right).$$

The constant⁸ C satisfies the equation: $\exp(C) = \pm 1$.

Lemma 5.2.4. *The expression*

$$\sum_{n \in D} 2s \left(\frac{1}{s^2 - s_n^2} - \frac{1}{B^2 - s_n^2} \right) - \frac{1}{4\pi} \int_{\mathbb{R}} 2s \left(\frac{1}{s^2 + w^2} - \frac{1}{B^2 + w^2} \right) \frac{\phi'}{\phi}(iw) dw$$

has only simple poles and integral residues:

(a) at the points $\pm s_j$ on the line $\text{Re}(s) = 0$ and on the interval $[-1, 1]$. Each point s_j is related to an eigenvalue λ_j of the discrete spectrum of Δ by $1 - s_j^2 = \lambda_j$. The residue of each s_j is equal to the multiplicity of the corresponding eigenvalue. If $\lambda = 1$, is an eigenvalue of Δ , then the residue of the point $s_j = 0$, is twice the multiplicity of λ ;

(b) at the points ρ_j that are poles of $\mathfrak{S}(s)$, which lie in the half-plane $\text{Re}(s) < 0$. The residue of each ρ_j is non-negative⁹ and equal to its multiplicity as a pole of $\mathfrak{S}(s)$.

Proof. The computation involves elementary complex analysis. See [Ven82, Section 5.1]. \square

The residues above come from terms that are related to the spectral and scattering theory of Δ . The remaining residues are computed using group theoretic data involving Γ and χ . The poles and zeros of $Z(s, \Gamma, \chi)$ that correspond to these residues are commonly called *topological* or *trivial*¹⁰.

5.3 The Topological Zeros and Poles

The computation of the topological residues is considerably more complicated than the corresponding spectral computation. Poles can only arise from the following terms (excluding the spectral terms previously dealt with) of (5.2.1) (note that we multiplied all terms through by $2s$):

$$\begin{aligned} & \frac{l_\infty}{2\pi[\Gamma_\infty : \Gamma'_\infty]} \int_{\mathbb{R}} \frac{2s}{s^2 + w^2} \frac{\Gamma'}{\Gamma}(1 + iw) dw - \frac{l_\infty}{2[\Gamma_\infty : \Gamma'_\infty]s} + \frac{\text{tr} \mathfrak{S}(0)}{2s} \\ & - \sum_{i=1}^l \frac{\text{tr} \chi(g_i)}{|C(g_i)| |1 - \epsilon_i^2|^2} \int_0^\infty e^{-sx} \frac{\sinh x}{\cosh x - 1 + \frac{|1 - \epsilon_i^2|^2}{2}} dx. \quad (5.3.1) \end{aligned}$$

⁸The value of C can be read off by letting $s \rightarrow 0$ in the functional equations. Its value depends on whether $\phi(0)$ is 1 or -1 and the multiplicity of $Z(s, \Gamma, \chi)$ at $s = 0$.

⁹The point ρ_j is a zero of $Z(s, \Gamma, \chi)$ and a pole of $\mathfrak{S}(s)$. We understand the multiplicity of a pole as non-negative number (not as a negative number).

¹⁰We refer to them as topological.

The first two terms come from the parabolic elements of Γ , the third from the spectral trace, and the last from the cuspidal elliptic elements of Γ . It is remarkable that the last three terms need to be taken together in order to compute the residue at $s = 0$, while the first and last are needed to compute the residues on the negative real axis.

It is well known that

$$\begin{aligned} \frac{l_\infty}{2\pi[\Gamma_\infty : \Gamma'_\infty]} \int_{\mathbb{R}} \frac{2s}{s^2 + w^2} \frac{\Gamma'}{\Gamma}(1 + iw) dw \\ = \frac{l_\infty}{[\Gamma_\infty : \Gamma'_\infty]} \left(\frac{\Gamma'}{\Gamma}(1 - s) + \sum_{k=1}^{\infty} \left(\frac{1}{s+k} + \frac{1}{s-k} \right) \right). \end{aligned} \quad (5.3.2)$$

In order to obtain a similarly explicit formula for

$$\sum_{i=1}^l \frac{\text{tr } \chi(g_i)}{|C(g_i)||1 - \epsilon_i^2|^2} \int_0^\infty e^{-sx} \frac{\sinh x}{\cosh x - 1 + \frac{1 - \epsilon_i^2}{2}} dx \quad (5.3.3)$$

we must make some technical assumptions.

Case One: $[\Gamma_\infty : \Gamma'_\infty] = 1$

In this case, (5.3.3) is not applicable and $l_\infty = k_\infty = k(\Gamma, \chi)$ (the last equality follows from our assumption that ∞ is the only cusp). Since $\mathfrak{S}(0)$ is a unitary self-adjoint matrix of dimension $k \times k$, its trace consists of a sum of k terms of the form ± 1 . It follows that $\frac{1}{2}(\text{tr } \mathfrak{S}(0) - k)$ is an integer. After applying equations (5.3.1) and (5.3.2) we have:

Lemma 5.3.1. *Suppose $[\Gamma_\infty : \Gamma'_\infty] = 1$. Then the poles of (5.3.1) are simple and are located at the points $s = -1, -2, \dots$, with residue k_∞ and at the point $s = 0$, with residue $\frac{1}{2}(\text{tr } \mathfrak{S}(0) - k_\infty)$.*

Case Two: $[\Gamma_\infty : \Gamma'_\infty] = 2$

In this case for all i , $\epsilon_i = \epsilon \equiv \sqrt{-1}$ and (5.3.3) becomes

$$\sum_{i=1}^l \frac{\text{tr } \chi(g_i)}{|C(g_i)||1 - \epsilon|^2} \int_0^\infty e^{-sx} \frac{\sinh x}{\cosh x + 1} dx. \quad (5.3.4)$$

An application of Lemma 4.4.4 gives us the coefficient of the integral above

$$\sum_{i=1}^l \frac{\text{tr } \chi(g_i)}{|C(g_i)||1 - \epsilon|^2} = \frac{1}{2} \left(k_\infty - \frac{l_\infty}{2} \right).$$

Next, to evaluate the integral in (5.3.4) we appeal to the following formula:

$$\int_0^\infty e^{-\mu x} (\cosh x - \cos t)^{-1} dx = \frac{2}{\sin t} \sum_{k=1}^{\infty} \frac{\sin kt}{\mu + k}, \quad (5.3.5)$$

valid for $\operatorname{Re}(\mu) > -1$ and $t \neq 2n\pi$, (see [GR65] formula 3.543.2). Averaging for the two values $\mu - 1, \mu + 1$, we obtain

$$\int_0^\infty e^{-sx} \frac{\sinh x}{\cosh x - \cos t} dx = \frac{1}{\sin t} \sum_{k=1}^\infty \sin kt \left(\frac{1}{s-1+k} - \frac{1}{s+1+k} \right). \quad (5.3.6)$$

Finally, we can evaluate the integral in (5.3.4) by taking the limit as $t \rightarrow \pi$

$$\int_0^\infty e^{-sx} \frac{\sinh x}{\cosh x + 1} dx = \sum_{k=1}^\infty k(-1)^{k+1} \left(\frac{1}{s-1+k} - \frac{1}{s+1+k} \right). \quad (5.3.7)$$

Combining the residues above with those from (5.3.2) give us:

Lemma 5.3.2. *Suppose $[\Gamma_\infty : \Gamma'_\infty] = 2$. Then the poles of (5.3.1) are simple and are located at the points $s = n$, with $n = -1, -2, \dots$, and residue¹¹:*

$$m_n = \begin{cases} k_\infty, & \text{if } n \text{ is odd,} \\ l_\infty - k_\infty, & \text{else} \end{cases}$$

and at the point $s = 0$ with residue $\frac{1}{2}(\operatorname{tr} \mathfrak{S}(0) - k_\infty)$.

Since the Picard group $\Gamma = \operatorname{PSL}(2, \mathbb{Z}[\sqrt{-1}])$ satisfies $[\Gamma_\infty : \Gamma'_\infty] = 2$ we have:

Corollary 5.3.3. *Let $\Gamma = \operatorname{PSL}(2, \mathbb{Z}[\sqrt{-1}])$, and let $\chi \in \operatorname{Rep}(\Gamma, V)$. Then $Z(s, \Gamma, \chi)$ is a meromorphic function.*

Case Three: $[\Gamma_\infty : \Gamma'_\infty] = 3$

In this case, the cuspidal elliptic elements g_i that are in (5.3.3) must all be of order three. Hence $\epsilon_i \in \{\sqrt[3]{1}, \sqrt[3]{-1}\}$, but when such ϵ_i is plugged into $|1 - \epsilon_i^2|^2$ one obtains 3. Thus we can rewrite (5.3.3) as

$$\sum_{i=1}^l \frac{\operatorname{tr} \chi(g_i)}{|C(g_i)||1 - \epsilon_i^2|^2} \int_0^\infty e^{-sx} \frac{\sinh x}{\cosh x + \frac{1}{2}} dx. \quad (5.3.8)$$

Next, applying (5.3.6) with $t = \frac{2}{3}\pi$, and Lemma 4.4.4, we can rewrite (5.3.8) as

$$\begin{aligned} \frac{1}{2} \left(k_\infty - \frac{l_\infty}{3} \right) & \left(\left(\frac{1}{s-1+1} - \frac{1}{s+1+1} \right) - \left(\frac{1}{s-1+2} - \frac{1}{s+1+2} \right) \right. \\ & \left. + \left(\frac{1}{s-1+4} - \frac{1}{s+1+4} \right) - \left(\frac{1}{s-1+5} - \frac{1}{s+1+5} \right) + \dots \right). \end{aligned} \quad (5.3.9)$$

Finally combining the residues above with the residues from (5.3.2) we obtain:

¹¹Note that by defintion $l_\infty \geq k_\infty$.

Lemma 5.3.4. *Suppose $[\Gamma_\infty : \Gamma'_\infty] = 3$. Then the poles of (5.3.1) are simple and are located at the points $s = n$, with $n = -1, -2, \dots$, and residue:*

$$m_n = \begin{cases} \frac{2}{3}l_\infty - k_\infty, & \text{if } n \text{ is a multiple of } 3, \\ \frac{1}{6}l_\infty + \frac{1}{2}k_\infty, & \text{else} \end{cases}$$

and at the point $s = 0$ with residue $\frac{1}{2}(\text{tr } \mathfrak{S}(0) - k_\infty)$.

As an application of the lemma above, we have:

Corollary 5.3.5. *Let $\Gamma = \text{PSL}(2, \mathbb{Z}[-\frac{1}{2} + \frac{\sqrt{-3}}{2}])$, and $\chi \equiv 1$ (the trivial representation). Then $Z(s, \Gamma, \chi)$ is not a meromorphic function¹² (it is the 6-th root of a meromorphic function).*

Proof. Since $\mathbb{Q}(\sqrt{-3})$ has class number one, $\text{PSL}(2, \mathbb{Z}[-\frac{1}{2} + \frac{\sqrt{-3}}{2}])$ has one class of cusps ([EGM98, Chapter 7]). In addition it has ∞ for a cusp. Elementary calculations show that $[\Gamma_\infty : \Gamma'_\infty] = 3$. By definition, $\chi \equiv 1$ implies that $k_\infty = l_\infty = 1$. The result follows from Lemma 5.3.4. \square

The Remaining Cases

The cases that remain are: $[\Gamma_\infty : \Gamma'_\infty] = 4$ and $[\Gamma_\infty : \Gamma'_\infty] = 6$. Using the same ideas as in the other three cases one can show the following:

Theorem 5.3.6. *Suppose that $[\Gamma_\infty : \Gamma'_\infty] = 4$. Then for some integer N , $Z(s, \Gamma, \chi)^N$ is a meromorphic function.*

The author does not know a good bound for the integer N (N depends on Γ and χ). On the other hand we conjecture the following:

Conjecture. *Suppose that $[\Gamma_\infty : \Gamma'_\infty] = 6$. Then for some integer N , $Z(s, \Gamma, \chi)^N$ is a meromorphic function.*

5.4 The Entire Function Associated to the Selberg Zeta-Function

As Fischer [Fis87, Chapter 3] observed, it is useful to group the elliptic, parabolic, and identity terms, together with the loxodromic terms to define an entire function associated to the Selberg zeta-function called the *Selberg xi-function* $\Xi(s, \Gamma, \chi)$.

¹²This is the first example that the author is aware of where the Selberg zeta-function is not meromorphic.

Definition 5.4.1. For $\operatorname{Re}(s) > 1$ set¹³

$$\begin{aligned} \frac{\Xi'}{\Xi}(s, \Gamma, \chi) &= \sum_{\{T\}^{\text{lox}}} \frac{\operatorname{tr}(\chi(T)) \log N(T_0)}{m(T)|a(T) - a(T)^{-1}|^2} N(T)^{-s} \\ &\quad + \frac{l_\infty}{2\pi[\Gamma_\infty : \Gamma'_\infty]} \int_{\mathbb{R}} \left(\frac{2s}{s^2 + w^2} \right) \frac{\Gamma'}{\Gamma}(1 + iw) dw \\ &\quad + \frac{1}{2s} \left(\operatorname{tr} \mathfrak{S}(0) - \frac{l_\infty}{[\Gamma_\infty : \Gamma'_\infty]} \right) \\ &\quad - \sum_{i=1}^l \frac{\operatorname{tr} \chi(g_i)}{|C(g_i)||1 - \epsilon_i^2|^2} \int_0^\infty \left(\frac{e^{-sx}}{2s} \right) \frac{\sinh x}{\cosh x - 1 + \frac{|1 - \epsilon_i^2|^2}{2}} dx - E, \end{aligned}$$

where E is defined in Theorem 5.2.3.

Theorem 5.4.2. *The Selberg xi-function can be continued to an entire function $\Xi(s, \Gamma, \chi)$ with*

$$\begin{aligned} \frac{1}{2s} \frac{\Xi'}{\Xi}(s, \Gamma, \chi) - \frac{1}{2B} \frac{\Xi'}{\Xi}(B, \Gamma, \chi) \\ = \sum_{n \in D} \left(\frac{1}{s^2 - s_n^2} - \frac{1}{B^2 - s_n^2} \right) - \frac{1}{4\pi} \int_{\mathbb{R}} \left(\frac{1}{s^2 + w^2} - \frac{1}{B^2 + w^2} \right) \frac{\phi'}{\phi}(iw) dw. \end{aligned}$$

Here $B > 1$, $B > \operatorname{Re}(s)$.

Proof. The second assertion follows from (5.2.1). For the first assertion: by Lemma 5.2.4, after multiplying through by $2s$, the residues of the expression on the right side of the equal sign are all integers and positive. Hence $\Xi(s, \Gamma, \chi)$ is entire. \square

An explicit product formula (for $\operatorname{Re}(s) > 1$) can be obtained for $\Xi(s, \Gamma, \chi)$ in the case of $[\Gamma_\infty : \Gamma'_\infty] = 1, 2, 3$ by integrating, and then exponentiating, the explicit formula for the contribution of the parabolic and cuspidal elliptic elements to the Selberg zeta-function.

Remark 5.4.3. *The Selberg zeta-function for symmetric spaces of rank-one was studied by Gangolli and Warner ([Gan77], [GW80]). More specifically, they studied torsion-free cocompact quotients with finite-dimensional unitary representations, and non-cocompact torsion-free quotients with trivial representations (the scalar case).*

¹³Theorem 5.4.2 shows that we are justified in defining $\Xi(s, \Gamma, \chi)$ by its logarithmic derivative.

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