Heegaard splittings and hyperbolic geometry

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Abstract of the Dissertation
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It is well known that every closed 3-manifold has a Heegaard splitting and the combinatorics of the Heegaard splitting identifies the 3-manifold. Yet it has been hard to use Heegaard splittings to obtain information about topology and geometry of the manifold. We develop a new approach to use hyperbolic geometry and in particular deformation theory of compressible ends of hyperbolic manifolds to study closed 3-manifolds. Using this approach, we have been able to prove that a big class of 3-manifolds which admit a Heegaard splitting with what we call "bounded combinatorics" admit a negatively curved metric with sectional curvatures pinched about -1. This answers some interesting questions about these

manifolds and in fact gives a coarse description of the geometry of these manifolds equipped with the negatively curved metrics.

The description of these geometries is motivated by work of Minsky in constructing models for hyperbolic manifolds with incompressible boundary. In fact, much of our work is aimed at developing a similar theory in the compressible boundary case.

To my Dearest, Roja.

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Chapter 1

Introduction

> "I was unable to find the flaws in my 'proof' for quite a while, even though the error is very obvious. It was a psychological problem, a blindness, an excitement, an inhibition of reasoning by an underlying fear of being wrong. Techniques leading to the abandonment of such inhibitions should be cultivated by every honest mathematician."
>
> — John R. Stallings, How not to prove the Poincaré conjecture

The main motivation for this work is to use combinatorial data from a 3-manifold and arrive at topological and geometrical information about the manifold. In particular, we study this question about closed 3-manifolds by using the *Heegaard splittings*. Our study leads us to the understanding of the deformation theory of hyperbolic structures on three manifolds with compressible boundary and in particular on handlebodies.

Suppose H^+ and H^- are 3-dimensional handlebodies whose boundaries are identified with an oriented closed surface of genus $g \geq 1$ in a way that the orientation of S agrees with the orientation of ∂H^+ and does not agree with the one of ∂H^-. If we glue the handlebodies along S, we obtain a closed oriented 3-manifold $M = H^+ \cup_S H^-$. Such a decomposition is called a *Heegaard splitting* and it is well known that every closed orientable 3-manifold admits such a splitting. We call the surface S a *Heegaard surface* and two Heegaard splittings for M are equivalent if the associated Heegaard surfaces are isotopic in M. The only 3-manifolds with a Heegaard splitting of genus ≤ 1 are S^3, $S^2 \times S^1$ and Lens spaces. These manifolds are not interesting in our discussions and therefore we always assume that a Heegaard splitting has genus at least 2.

An important problem in studying 3-manifolds is using the combinatorics of the Heegaard splitting and obtain topological and geometrical information about the 3-manifold and its geometries. Hempel [He01] introduced an invariant of the Heegaard splitting, which we call *handlebody distance*, and conjectured that the 3-manifold is hyperbolic (admits a hyperbolic metric) if it has a Heegaard splitting with handlebody distance at least 3. The handlebody distance is the distance between sets of meridians of handlebodies H^+ and H^- in the *complex of curves* of S.

Work of Haken [Ha68], Casson-Gordon [CG87], Hempel [He01], Thompson [Tho99] and Moriah-Schultens [MS98] shows that every Heegaard splitting of a 3-manifold that is reducible, toroidal or Seifert fibered has handlebody distance at most 2. Therefore Hempel's conjecture agrees with the description of 3-manifolds given by Thurston's Hyperbolization Conjecture.

For a constant $R > 0$, we restrict ourselves to the Heegaard splittings which satisfy a combinatorial condition called *R-bounded combinatorics.* This is obtained by taking $P^{\pm}$ to be a pants decomposition on S whose components are all compressible in $H^{\pm}$. Then similar to Masur-Minsky [MM00], we associate to the pair P^+, P^- a collection of non-negative integers $\{d_Y(P^+, P^-\}$, where Y runs over all isotopy classes of proper essential subsurfaces in S. We say P^+ and P^- and the Heegaard splitting have R-bounded combinatorics if all these numbers are bounded above by R (see section 2.6). Our main theorem here is the following:

Main Theorem. *Given $\epsilon > 0$ and $R > 0$ there exists $n_\epsilon > 0$ depending only on ϵ, R and $\chi(S)$ that if $M = H^+ \cup_S H^-$ has R-bounded combinatorics and handlebody distance $\geq n_\epsilon$ then M admits a Riemannian metric ν such that the sectional curvature of ν is pinched between $-1-\epsilon$ and $-1+\epsilon$. Moreover ν has a lower bound for the injectivity radius independently of the handlebody distance and ϵ.*

This immediately implies that

Corollary 1.1. *If the Heegaard splitting $M = H^+ \cup_S H^-$ has R-bounded combinatorics and sufficiently large handlebody distance, then $\pi_1(M)$ is infinite and word hyperbolic.* □

On the other hand, Tian [Ti90] has claimed a theorem that in presence of the metric constructed in the Main Theorem for ϵ small implies that M is hyperbolic.

Even when we know that a 3-manifold M is hyperbolic, an important question is to be able to describe the geometry of the hyperbolic metric and use it to get topological information about M. The important feature of our construction of the metric for the Main Theorem is that it gives a concrete description of the metric in terms of known hyperbolic manifolds.

In particular, assume $(M_i = H_i^+ \cup_S H_i^-)$ is a sequence of Heegaard splittings with R-bounded combinatorics and handlebody distances tending to infinity as $i \to \infty$. Using the Main Theorem, we can assume that each M_i is equipped with a Riemannian metric ν_i, whose sectional curvatures are pinched in the interval $[-1 - \epsilon_i, -1 + \epsilon_i]$ and $\epsilon_i \to 0$ as $i \to \infty$. Then we have the following

Theorem 1.2. *Every geometric limit of the sequence $(M_i)_i$ is hyperbolic and either homeomorphic to a genus g handlebody or to the trivial interval bundle $S \times \mathbb{R}$.*

As a matter of fact, we construct a bi-Lipschitz model for the geometry of M_i outside uniform bounded cores of handlebodies H^+ and H^-. The model is described in terms of the *canonical marked hyperbolic surface bundle* over a Teichmüller geodesic, where the Teichmüller geodesic is determined using the combinatorics of the splitting.

Our approach to the proof of the above results is by studying the deformation theory of hyperbolic structures on a handlebody. This approach is highly motivated by works of Minsky and others in proving the Ending Lamination Conjecture and constructing a bi-Lipschitz model for hyperbolic manifolds with incompressible boundary. Once we have a good understanding of the hyperbolic structures on the handlebody, we construct two such structures which are appropriate for our purpose and glue them in a way that we have a manifold homeomorphic to $M = H^+ \cup_S H^-$ and with a Riemannian metric with pinched negative curvature as required by the Main Theorem.

In chapter 3, we prove a version of Thurston's uniform injectivity theorem for hyperbolic structures on handlebodies. This is a starting point for studying these structures. We follow this in chapter 4, by some observations about the pleated surfaces in these structures.

In chapter 5, we prove the following theorem, which is in fact a joint work with Juan Souto. I am thankful that he allowed me to present this here. Suppose N is a hyperbolic structure on handlebody H. The theory of Ahlfors-Bers, Thurston, Bonahon and Canary attaches to N an invariant lying in a combination of Teichmüller space and lamination space of ∂H and its subsurfaces up to actions of $\mathrm{Mod}_0(H)$, where $\mathrm{Mod}_0(H)$ is the subgroup of the mapping class group of ∂H whose elements extend to self-homeomorphisms of H homotopic to identity. When the hyperbolic structure is *geometrically infinite* and without parabolics, it follows from work of Canary [Can89, Can93b] that the associated invariant is a *filling Masur domain lamination*: it is a lamination that intersects every essential simple closed curve on ∂H and is not limit of meridians of H. In particular, the ending lamination is not *realized* in N: there does not exist a map $f : \partial H \to N$ homotopic to the inclusion $\partial H \hookrightarrow H$ that maps every leaf of λ to a geodesic in N. We prove a converse to this statement.

Theorem 1.3. *Suppose λ is a filling Masur domain lamination on ∂H and λ is not realized in N, where N is a hyperbolic structure on H. Then λ is the ending lamination of N (defined up to actions of* $\mathrm{Mod}_0(H)$*).*

This theorem answers a question about these structures which we think has been overlooked. We should remind the reader that Ohshika [Oh] has claimed a proof of the above theorem in a special case where N is a strong limit of convex cocompact structures on H.

Using this theorem, we can prove the following corollary:

Corollary 1.4. *Given a filling Masur domain lamination λ on ∂H, there exists a hyperbolic structure on H, whose ending lamination (defined up to actions of* $\mathrm{Mod}_0(H)$*) is λ.*

A more general version of the above corollary also has been claimed by OhshikaOh. In chapter 6, we introduce the family $\mathcal{B}_0(R)$ of marked hyperbolic structures with R-bounded combinatorics on a handlebody H. The definition of the R-bounded combinatorics for hyperbolic structures is similar to the definition for Heegaard splittings. Suppose N is a hyperbolic structure on H. We assume N has no parabolics and there is no essential short curve on the conformal structure at infinity of N when N is convex cocompact. In either case, the end invariant gives a combinatorial object α which is either a lamination or a *marking* with bounded length on the conformal structure at infinity. There is a *projection* from such an object to a pants decomposition whose components are all meridians of H in the *complex of curves* of ∂H and we take P to be such a projection. We say N is a *hyperbolic structure with R-bounded combinatorics* on H if $d_Y(\alpha, P) \leq R$ for every essential proper subsurface $Y \subset \partial H$. (For a precise definition see 2.6 and definition 6.2.) We prove that this family is compact in the *strong* topology and this is the main tool that helps us make our arguments work.

In chapters 7 and 8, similar to Minsky [Min01], we prove a quasi-convexity result for the set of short curves in a hyperbolic structure in $\mathcal{B}_0(R)$ and then we use it to show that all hyperbolic structures with R-bounded combinatorics on H have uniform *bounded geometry.*

Theorem 1.5. (Bounded geometry) *There exists η depending only on R and $\chi(\partial H)$ such that the injectivity radius of every hyperbolic structure with R-bounded combinatorics on H, $N \in \mathcal{B}_0(R)$, is bounded below by η.*

The above theorem should be compared with the main theorem of Minsky in [Min01] where he proves that such bounded combinatorial condition implies bounded geometry for surface groups.

We use the bounded geometry in chapters 9 and 10 to construct a uniform model for the end of hyperbolic structures in $\mathcal{B}_0(R)$. We use a description of the model which was given by Mosher [Mo03] for the case of hyperbolic structures on $S \times \mathbb{R}$. This gives a description of the structure in terms of the canonical marked hyperbolic surface bundle over a Teichmüller geodesic that is determined by the end invariant of the structure.

We should remark that our results in producing uniform models for this family of hyperbolic structures could not be directly implied from such descriptions for hyperbolic structures on manifolds with incompressible boundary given by Minsky and others. (Cf. Ohshika's [Oh98] description of a bi-Lipschitz model for a single hyperbolic structure with bounded geometry with constants that depend on the structure.) The question of constructing such models in the general case remains an open question.

Finally in chapter 11, we use all these to construct appropriate hyperbolic structures on H^+ and H^-. Then we use the model to show that these two are almost isometric on two subsets homeomorphic to $S\times[0,1]$ and if we glue them along these subsets, we obtain a manifold homeomorphic to $M = H^+ \cup H^-$. All this is provided when the handlebody distance is sufficiently large. This proves the Main Theorem and theorem 1.2 immediately. We briefly describe Tian's result and its consequence in our setting in chapter 12.

We should point out that the first known examples of Heegaard splittings with sufficiently large handlebody distance were constructed by Luo using an idea of Kobayashi (cf. Hempel [He01]). In our construction in the beginning of the introduction we constructed the manifold by gluing H^+ and H^- along S using the identity map; now suppose f is what we call a *generic pseudo-Anosov*: the stable (resp. unstable) lamination is not limit of meridians of H^+ (resp. H^-). Then the handlebody distance for Heegaard splittings $H^+ \cup_{f^n} H^-$ tends to infinity as $n \to \infty$. In fact, in a joint work with Juan Souto [NS], we proved the same results as our Main Theorem and theorem 1.2 for these examples when n is sufficiently large. One can show that all these Heegaard splittings have some bounded combinatorics depending on f. Therefore those results follow from our theorems here; but the proofs there were more elegant and less involved in the analysis of the ends of hyperbolic structures on handlebodies and construction of uniform models for such structures.

On the other hand, work of Farb-Mosher [FM02] produces many more examples of mapping classes S which satisfy our bounded combinatorics condition once used as a gluing map of a Heegaard splitting $H^+ \cup_f H^-$. In their work, they study what they call *Schottky subgroups* of the mapping class group. Using their work and work of Rafi [Ra05], we can see that if G is such a Schottky subgroup of the mapping class group, there exists $R > 0$, such that every Heegaard splitting $H^+ \cup_f H^-$ has R-bounded combinatorics, where $f \in G$. On the other hand Farb-Mosher [FM02, Thm. 1.4] prove that if $\phi_1, \ldots, \phi_n$ are pseudo-Anosov elements of the mapping class group of S whose axes have pairwise disjoint endpoints in Thurston's compactification of Teichmüller space, then for all sufficiently large positive integers $a_1, \ldots, a_n$ the mapping classes $\phi_1^{a_1}, \ldots, \phi_n^{a_n}$ freely generate a Schottky subgroup G. In particular, if these pseudo-Anosovs are generic, then there exists $R > 0$ and we can choose $a_1, \ldots, a_n$ such that $H^+ \cup_f H^-$ satisfies the hypothesis of our theorems for every $f \in G$.

In [NS], we also used our description of the negatively curved metric on $H^+ \cup_S H^-$ to obtain a variety of topological results about the manifolds. Since, all we used was the classification of the geometric limits of these hyperbolic structures and we have a similar classification here, we can prove the same results here.

Theorem 1.6. *If $\Gamma \subset \pi_1(H^+)$ is a finitely generated subgroup of infinite index, then if $M = H^+ \cup_S H^-$ has R-bounded combinatorics and sufficiently large handlebody distance, the map $\Gamma \to \pi_1(M)$ induced by the inclusion $H^+ \hookrightarrow M$ is injective.*

Every minimal generating set for $\pi_1(H^+)$ or $\pi_1(H^-)$ gives a generating set for $\pi_1(M)$ where $M = H^+ \cup_S H^-$ and we call these *standard.*

Theorem 1.7. *The fundamental group of $M = H^+ \cup_S H^-$ has rank g if the Heegaard splitting has R-bounded combinatorics and large handlebody distance. Moreover, every minimal generating set of $\pi_1(M)$ is Nielsen equivalent to a standard generating set.*

For a definition of Nielsen equivalence see [NS].

Theorem 1.8. *For a Heegaard splitting $M = H^+ \cup_S H^-$ with R-bounded combinatorics and large handlebody distance, every proper subgroup $\Gamma \subset \pi_1(M)$ with rank $\leq 2g-2$ is free.*

Theorem 1.9. *If $M = H^+ \cup_S H^-$ has R-bounded combinatorics and large handlebody distance then the Heegaard genus of M is g and every minimal Heegaard surface is isotopic to S.*

Here, we only discus case of Heegaard splittings of closed 3-manifolds and hyperbolic structures on handlebodies. We should point out that most of our arguments easily generalize to a much broader setting. Heegaard splittings can be extended to compact orientable 3-manifolds by decomposing it to a pair of *compression bodies* which are identified along the *exterior boundary.* The definitions of handlebody distance and bounded combinatorics easily extend to this case and the same methods should give similar results in that case. In particular, provided that we have R-bounded combinatorics and large handlebody distance, we can construct Riemannian metrics with sectional curvatures ϵ-pinched about -1. Also similar to theorem 1.2, we are able to describe the geometry of these metrics concretely. In that case, instead of studying the deformation theory of hyperbolic structures on compression bodies. Our results will be still true and in fact some of the results were previously known for the case of compression bodies.

Even more, we could start with a pair of compact atoroidal and orientable 3-manifolds with two boundary components that are homeomorphic. We can develop the same kind of ideas for 3-manifolds which are obtained by gluing these two along the homeomorphic boundary. For sake of bravery and since handlebodies *are* in fact the interesting case, we decided not to allow these generalities. For a similar treatment in the case where the gluing map is obtained by iterations of generic pseudo-Anosovs, we refer the reader to [NS].

Chapter 2

Preliminaries

2.1 Coarse geometry

A metric space is *geodesic* if for any x, y there is a rectifiable path p from x to y whose length is equal to $d(x, y)$.

Let X and Y be metric spaces. A map $f : X \to Y$ is (K, c)-*quasi-isometric embedding* if

$$\frac{1}{K} d_X(x, x') - c \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + c$$

for $x, x' \in X$. We say f is *uniformly proper* with respect to a proper, monotonic function $\rho : [0, \infty) \to [0, \infty)$ and constants K and c if

$$\rho(d_X(x, x')) \leq d_Y(f(x), f(x')) \leq K d_X(x, x') + c \quad \text{for} \quad x, x' \in X.$$

The function ρ is called a *properness gauge* for f. The map f is c-*coarsely surjective* if for all $y \in Y$, there exists $x \in X$ such that $d_Y(f(x), y) \leq c$. The map f is a (K, c)-*quasi-isometry* if it is (K, c)-quasi-isometric embedding and

c-coarsely surjective.

Fact 2.1. *Suppose X and Y are geodesic metric spaces. Any coarsely surjective, uniformly proper map $f : X \to Y$ is a quasi-isometry with constants depending only on the constants in the hypothesis.*

Given a geodesic metric space X, a (λ, c)-*quasigeodesic* in X is a (λ, c)-quasi-isometric embedding $\gamma : I \to X$, where I is a closed connected subset of $\mathbb{R}$. When I is a compact interval we have a *quasigeodesic segment*, when I is a half-line we have a *quasigeodesic ray*, and when $I = \mathbb{R}$ we have a quasigeodesic line.

Recall that the Hausdorff distance between two subsets $A, B \subset X$ is the infimum of $r \in \mathbb{R}_+ \cup \{+\infty\}$ such that A is contained in the r-neighborhood of B, and B is contained in the r-neighborhood of A.

Two paths $\gamma : I \to X, \gamma' : I' \to X$ are *asynchronous fellow travelers* with respect to a (K, c)-quasi-isometry $\phi : I \to I'$ if there is a constant A such that $d(\gamma'(\phi(t)), \gamma(t)) \leq A$ for $t \in I$.

Two paths $\gamma : I \to X, \gamma' : I' \to X$ where γ is a quasigeodesic are asynchronous fellow travelers if and only if γ' is a quasigeodesic and the sets $\gamma(I), \gamma'(I')$ have finite Hausdorff distance in X with constants that are uniformly related.

We say a geodesic metric space is *Gromov hyperbolic* if there exists $\delta > 0$ such that for every triple of points of X and geodesics $[x, y], [y, z]$ and $[z, x]$, which pairwise connect them, every side is in the δ-neighborhood of the union of the other two.

2.2 Laminations and Masur domain

Let S be a closed surface of genus $g > 1$. Let $\mathrm{Diff}(S)$ be the group of diffeomorphisms of S and let $\mathrm{Diff}_0(S)$ be the normal subgroup of homeomorphisms isotopic to the identity. The *mapping class group* of S is $\mathcal{MCG}(S) = \mathrm{Diff}(S)/\mathrm{Diff}_0(S)$.

Suppose S is equipped with a hyperbolic metric τ_0. A *geodesic lamination* on S is a closed subset of S which is a disjoint union of simple geodesics. We denote the space of all these by $\mathcal{GL}(S)$. A *measured lamination* is a geodesic lamination λ together with an invariant (with respect to projection along λ) measure on arcs transversal to λ and supported on λ. $\mathcal{ML}(S)$ is the space of all measured laminations on S and the *projective lamination space* $\mathcal{PML}(S)$ is $(\mathcal{ML}(S)\backslash\{0\})/\mathbb{R}^+$. We identify $\mathcal{PML}(S)$ with the set of measured laminations which have unit length.

We also use the set $\mathcal{UML}(S)$ which is a quotient of $\mathcal{PML}(S)$ obtained by forgetting the measure. We say a geodesic lamination is *filling* or *fills* S if it intersects every essential non-peripheral simple closed curve on S. By a *maximal* lamination we mean an element of $\mathcal{GL}(S)$ whose complementary components are ideal triangles.

It is a standard that the different spaces of laminations defined above do not depend on the hyperbolic metric τ_0. This means that there is a natural homeomorphism from the spaces associated to σ to the ones associated to σ', if σ and σ' are different hyperbolic metrics on S. This homeomorphism is naturally induced from the identification of the circles at infinity of the universal covers $\widetilde{\sigma}$ and $\widetilde{\sigma}'$, via the Gromov boundary of the group $\pi_1(S)$. For more on the spaces of laminations and more see Casson-Bleiler [CB] or Fathi-Laundenbach-Poénorou [FLP].

Also notice that if $\mathcal{C}_0$ represents the set of homotopy classes of essential simple closed curves on S, then there is an embedding $\mathcal{C}_0 \to \mathcal{ML}$, where image of $\alpha \in \mathcal{C}_0(S)$ is a measured lamination whose single leaf is α with total transverse measure 1. This also induces a natural embedding $\mathcal{C}_0 \to \mathcal{PML}$ and an embedding $\mathbb{R}_+ \times \mathcal{C}_0 \to \mathcal{ML}$ whose images are dense. Using the embedding $\mathbb{R}_+ \times \mathcal{C}_0 \to \mathcal{ML}$, we can extend the geometric intersection number for simple closed curves to a continuous intersection number

$$\mathcal{ML} \times \mathcal{ML} \to [0, \infty),$$

denoted $i(\mu_1, \mu_2), \mu_1, \mu_2 \in \mathcal{ML}$. Notice that for elements $\mu, \mu' \in \mathcal{PML}$, it makes sense to say $i(\mu, \mu')$ is zero or nonzero. In particular, we say μ and μ' are *transverse* if $i(\mu, \mu') \neq 0$.

Now assume H is a handlebody of genus g and S its boundary. We denote the group of (isotopy classes of) homeomorphisms of S which extend to homeomorphisms of H, homotopic to identity by $\mathrm{Mod}_0(H)$. In studying, the hyperbolic structures on H a subset of $\mathcal{ML}(S)$ called *Masur Domain* $\mathcal{O}(H)$ appears frequently. Recall that by a *meridian* for H, we mean an essential simple closed curve on ∂H that bounds a disk in H. Let's denote the set of projective measured laminations that are supported on a finite union of meridians by $\mathcal{M} \subset \mathcal{PML}(S)$ and its closure by $\mathcal{M}'$. A measured lamination μ belongs to $\mathcal{O}(H)$ iff it has nonzero intersection with every element of $\mathcal{M}'$. If every measured lamination supported on a (geodesic) lamination is in the

Masur domain, then we say the geodesic lamination is in the Masur domain. The Masur domain has been studied by Masur [Mas86] and Otal [Ota88] and Masur proved the following:

Theorem 2.2. *The Masur domain $\mathcal{O}$ is open and invariant under the action of* $\mathrm{Mod}_0(H)$ *on* $\mathcal{PML}$. *Moreover, the action of* $\mathrm{Mod}_0(H)$ *on* $\mathcal{O}$ *is properly discontinuous.*

We also will make use of the following theorem of Otal [Ota88]. Recall that a multi-curve is a set of finite pairwise disjoint and non-parallel essential simple closed curves on a surface.

Theorem 2.3. *Let α and β be multi-curves in the Masur domain. If α and β are freely homotopic in H then there exists $\phi \in \mathrm{Mod}_0(H)$ such that $\phi(\alpha) = \beta$.*

Kerckhoff [Ker90] proved that $\mathcal{O}(H)$ has full measure in $\mathcal{PML}(S)$. Also, Otal [Ota88] proved that

Lemma 2.4. *The complement of a Masur domain multi-curve is incompressible and acylindrical in H.*

2.3 The complex of curves

For a finite type surface $S = S_{g,b}$, the surface of genus g with b boundary components, the *complex of curves* was originally defined by Harvey [Ha81]. Here we usually use the definitions and description used by Masur-Minsky [MM99, MM00]. The definition is slightly different for an annulus $S = S_{0,2}$ but the complex of curves, which we denote by $\mathcal{C}(S)$ is a locally infinite simplicial complex with a path metric on its 1-skeleton when it is nonempty.

If $3g + b \geq 4$, we consider the vertices of $\mathcal{C}(S)$ to be the set of homotopy classes of essential non-peripheral simple closed curves and essential properly embedded arcs relative to ∂S. Here, non-peripheral curves are those which are not boundary parallel and essential arcs which are not homotopic (rel. ∂S) to subarcs of ∂S. A $(k+1)$-tuple of different vertices makes a k-simplex if they have mutually disjoint representatives on the surface.

Notice that $\mathcal{C}_0 = \mathcal{C}_0(S)$ is the set of vertices of $\mathcal{C}(S)$ and when S is a closed surface it will be the set of homotopy classes of essential simple closed curves on S which is the same as our previous definition for $\mathcal{C}_0$.

When $S = S_{0,2}$ is a compact annulus, we consider the set of vertices of $\mathcal{C}(S)$ to be the homotopy classes of arcs that connect the two boundary components of S relative to their endpoints. This of course will be an uncountable set of vertices; we connect two vertices with an edge when they have representatives with disjoint interiors. For all other surfaces, we define $\mathcal{C}(S)$ to be empty.

To define the metric, we make every edge isometric to the interval $[0, 1]$ and define $d_{\mathcal{C}}(x, y)$ of points x and y in the 1-skeleton of $\mathcal{C}(S)$ to be length of the shortest path in the 1-skeleton that connects them.

Remark 2.1. We should note that what we defined as the curve complex is slightly different from what Masur-Minsky define as the curve complex. (They only allow simple closed curves in the vertices and they call what we define above as the *arc complex.*) Yet it is not hard to see that their complex quasi-isometrically embeds in our complex. Because of this we can translate most of their results about the curve complex to here with possibly different constants.

From now on, by a surface we mean an orientable finite type surface which is an annulus or has negative Euler characteristic. We also assume that every subsurface $Y \subset S$ that we take is *essential*: the map induced on the fundamental groups from the inclusion $Y \hookrightarrow S$ is injective and if Y is an annulus, its core is not peripheral.

Masur-Minsky [MM99] proved that $\mathcal{C}(S)$, when nonempty, has infinite diameter and is hyperbolic in sense of Gromov. In particular, we can define its boundary at infinity in sense of Gromov, which we denote by $\partial\mathcal{C}(S)$.

Notice that since $\mathcal{C}(S)$ is hyperbolic in sense of Gromov, one can consider the Gromov boundary of $\mathcal{C}(S)$ which we denote by $\partial\mathcal{C}(S)$. Recall that this is obtained by fixing a base point $x_0 \in \mathcal{C}(S)$. Then we take sequences $(x_n) \subset \mathcal{C}(S)$ where $d_{\mathcal{C}}(x_0, x_n) \to \infty$ as $n \to \infty$. Two sequences (x_n) and (y_n) are equivalent if the distance from x_0 to a geodesic $[x_n, y_n]$ that connects x_n and y_n tends to infinity as $n \to \infty$. A sequence (x_n) converges to a point of the boundary if it belongs to the equivalence class determined by that point.

E. Klarreich [Kla] gave a description of the Gromov boundary of $\mathcal{C}(S)$. In this description, $\partial\mathcal{C}(S)$ consists of filling laminations in $\mathcal{UML}(S)$. She proved that a sequence $(\alpha_n) \subset \mathcal{C}_0(S)$ converges to $\mu \in \partial\mathcal{C}(S)$, iff the corresponding sequence in $\mathcal{UML}(S)$ converges to μ.

Following Masur-Minsky [MM00], we also define a projection π_Y from $\mathcal{C}_0(S) \cup \mathcal{UML}(S)$ (where $\mathcal{C}_0(S)$ denotes the 0-skeleton of $\mathcal{C}(S)$) to subsets of $\mathcal{C}_0(Y)$ with diameter at most one, where $Y \subset S$ is an essential subsurface.

We assume S is equipped with a finite area hyperbolic metric. If $\alpha \in \mathcal{C}_0(S) \cup \mathcal{UML}(S)$ does not intersect Y essentially or Y is a three-holed sphere, we define $\pi_Y(\alpha) = \emptyset$. If not we have two cases:

- Y *is non-annular.* Consider $\alpha \cap Y$; this is a set of disjoint curves and arcs. At least one of the components is an essential curve or arc in Y since we assumed α intersects Y essentially. Therefore $\alpha \cap Y$ gives a subset of diameter at most one in $\mathcal{C}(Y)$, which we define to be $\pi_Y(\alpha)$.

- Y *is an annulus.* We can identify the universal cover of S with $\mathbb{H}^2$ and we know that the universal cover has a compactification as a closed disk and the action of $\pi_1(S)$ on its universal cover extends to this compactification. Take the annular cover $\widetilde{Y} = \mathbb{H}^2/\pi_1(Y)$ of S to which Y lifts homeomorphically. The group $\pi_1(Y)$ is a cyclic subgroup of isometries of $\mathbb{H}^2$ with two fixed points at infinity. The quotient of the closed disk minus these two points is a closed annulus $\widehat{Y}$ that compactifies $\widetilde{Y}$ naturally. We identify $\mathcal{C}(Y)$ with $\mathcal{C}(\widehat{Y})$ and define π_Y as a map from $\mathcal{C}_0(S)$ to set of subsets of $\mathcal{C}_0(\widehat{Y})$ with diameter at most one. All lifts of the geodesic representative of α to $\widetilde{Y}$ naturally give properly embedded arcs in the closed annulus $\widehat{Y}$. We define $\pi_Y(\alpha)$ to be the set of those which connect the two boundary components. Again this cannot be empty since α intersects Y essentially and has diameter at most one.

We also denote the distance between projections of α and β in $\mathcal{C}(Y)$ by $d_Y(\alpha, \beta)$ and when Y is an annular neighborhood of the simple closed curve γ, we sometimes use the notations $\mathcal{C}(\gamma)$, π_γ and d_γ instead of $\mathcal{C}(Y)$, π_Y and d_Y.

Masur-Minsky [MM00] also proved the following theorem:

Theorem 2.5. (Bounded geodesic image) *Let Y be a proper subsurface of S which is not a three punctured sphere and let g be a geodesic segment, ray or biinfinite line in $\mathcal{C}(S)$ such that $\pi_Y(v) \neq \emptyset$ for every vertex of g.*

There is a constant M only depending on the Euler characteristic of Y, so that

$$\operatorname{diam}_Y(g) \leq M.$$

2.4 Handlebody distance

Suppose H is a handlebody of genus > 1. The set of meridians of H is a subset of the 0-skeleton of $\mathcal{C}(\partial H)$ which we denote by $\Delta(H)$. Masur-Minsky [MM03] proved that:

Theorem 2.6. *There exists a constant $d > 0$ depending only on $\chi(\partial H)$ that $\Delta(H)$ is d-quasiconvex as a subset of $\mathcal{C}(S)$.*

When $M = H^+ \cup_S H^-$ is a Heegaard splitting, since we have identified boundaries of H^+ and H^- with S, we can consider $\Delta(H^+)$ and $\Delta(H^-)$ as subsets of $\mathcal{C}_0(S)$. Following Hempel [He01], we define the *handlebody distance* for the splitting to be $d_{\mathcal{C}}(\Delta(H^+), \Delta(H^-))$.

2.5 Pants decompositions and markings

For a surface S, a *multi-curve* is a subset of $\mathcal{C}_0(S)$ whose elements are simple closed curves with pairwise distance 1. In particular, a *pants decomposition* P is a maximal multi-curve on S. Each component of $S \setminus P$ is called a pair of pants. We sometimes consider a multi-curve or a pants decomposition α as an element of $\mathcal{ML}$ or $\mathcal{PML}$; in this case we assume it is a measured lamination or a projectivized measured lamination supported on α where all the components are equipped with equal transverse measure 1 or is the projection of such element in $\mathcal{PML}$.

Suppose $\alpha_1, \ldots, \alpha_k$ are components of a pants decomposition P. One can see that the component of

$$S \setminus (\alpha_2 \cup \cdots \cup \alpha_k)$$

that contains α_1 is either a 1-holed torus or a 4-holed sphere Y. If we replace α_1 with an essential simple closed curve β in Y that $i(\alpha_1, \beta) = 1$ when Y is a 1-holed torus and $i(\alpha_1, \beta) = 2$ when Y is a 4-holed torus, we obtain another pants decomposition $Q = \beta \cup \alpha_2 \cup \cdots \alpha_k$. We say Q is obtained by an *elementary move* on P and we denote this move by $P \to Q$. By an *elementary move sequence*

$$P_1 \to P_2 \to \cdots \to P_k$$

we mean a sequence of pants decomposition, where P_{i+1} is obtained from P_i by an elementary move for every $i = 1, \ldots, k-1$.

The following lemma is easy and we will be using it in chapter 5.

Lemma 2.7. *Given a path $\alpha_0, \alpha_1, \ldots, \alpha_n$ in $\mathcal{C}(S)$, we can extend it to an elementary-move sequence of pants decompositions $P_0, P_1, \ldots, P_m$ for which:*

every pants decomposition P_i, $0 \leq i \leq m$, contains an element α_j for some $0 \leq j \leq n$ and P_0 and P_m are arbitrary pants decompositions that contain α_0 and α_m respectively.

Following Masur-Minsky [MM00], we define a *marking* on a surface S as follows. A *marking* α on S is a pants decomposition P, denoted by $\mathrm{base}(\alpha)$, together with a *transversal* for every component of P. For every component γ of P, a transversal is a simple closed curve δ such that a regular neighborhood of $\gamma \cup \delta$ is either an essential 1-holed torus or an essential 4-holed sphere and δ does not intersect any other component of P.

Notice that what we have defined above are actually *complete clean markings* in [MM00].

By *support* of a marking α, we mean the union of all the components of base(α) and the transversals. When we consider a marking in $\mathcal{C}_0(S)$ we are in fact considering its support.

We can also extend our definition of π_Y in a way that it includes markings: If Y is an annulus whose core is some $\gamma \in \text{base}(\alpha)$ and δ is the transversal associated to γ, we define $\pi_Y(\alpha) = \pi_Y(\delta)$. In all other cases, $\pi_Y(\alpha) = \pi_Y(\text{base}(\alpha))$. This also defines $d_Y(\alpha, \beta)$ where β is a multi-curve, a geodesic lamination or another marking.

If H is a handlebody, by a *handlebody pants decomposition*, we mean a pants decomposition on ∂H, whose elements are all in $\Delta(H)$. Also by a *handlebody marking*, we mean a marking α such that base(α) is a handlebody pants decomposition.

Proposition 2.8. *For a handlebody H, there exists a finite set of handlebody markings $\mathbf{m}_0(\text{H})$ such that every other handlebody marking is obtained by action of $\text{Mod}_0(H)$ on an element of $\mathbf{m}_0(\text{H})$*

2.6 Bounded combinatorics

Suppose α and β are multi-curves, geodesic laminations on S or markings; we say they have *R-bounded combinatorics* for a constant $R > 0$ if for every proper essential subsurface $Y \subset S$ either $d_Y(\alpha, \beta)$ is undefined or $d_Y(\alpha, \beta) \leq R$.

Suppose H is a handlebody and α is a marking or a multi-curve on $\mathcal{C}_0(\partial H)$, we say α has *R-bounded combinatorics respect with to to H* if there exists a handlebody pants decomposition $P \subset \Delta(H)$ such that α and P have R-bounded combinatorics and

$$d_{\mathcal{C}}(\alpha, \Delta(H)) = d_{\mathcal{C}}(\alpha, P).$$

We say P is the *projection* of α to $\Delta(H)$. (There could be more than one projection for α but they all have bounded distance depending on the quasiconvexity constant of $\Delta(H)$.) When $\mu \in \partial\mathcal{C}(\partial H)$ is given, we say μ has *R-bounded combinatorics with respect to H* if there exists a sequence (α_n) of markings whose base converges to μ in $\mathcal{C}(\partial H) \cup \partial\mathcal{C}(\partial H)$ and every α_n has R-bounded combinatorics with respect to H and the same projection P for every n. It is not hard to see that in this case μ and P have $(R+1)$-bounded combinatorics because $d_Y(\mu, \alpha_n) \leq 1$ for $n \gg 0$ and every proper subsurface $Y \subset \partial H$. Again we call P the *projection* of μ to $\Delta(H)$.

Finally, when $M = H^+ \cup_S H^-$ is a Heegaard splitting, we say two handlebody pants decompositions $P^+ \subset \Delta(H^+)$ and $P^- \subset \Delta(H^-)$ *realize* the handlebody distance if $d_{\mathcal{C}}(P^+, P^-) = d_{\mathcal{C}}(\Delta(H^+), \Delta(H^-))$. We say this Heegaard splitting has *R-bounded combinatorics* if there exists handlebody pants decompositions $P^+ \subset \Delta(H^+)$ and $P^- \subset \Delta(H^-)$ which realize the handlebody distance and have R-bounded combinatorics.

We can easily see that in the above definitions we could replace handlebody pants decompositions with handlebody markings:

Lemma 2.9. *Suppose α is a multi-curve, a marking or an element of $\partial\mathcal{C}$ with R-bounded combinatorics with respect to H and P is the projection of α to $\Delta(H)$. Then we can extend P to a handlebody marking β, $\mathrm{base}(\beta) = P$, such that α and β have R-bounded combinatorics. When $M = H^+ \cup_S H^-$ is a Heegaard splitting with R-bounded combinatorics and P^+ and P^- are the handlebody pants decompositions used in the definition of the R-bounded combinatorics. Then we can extend P^+ and P^- to handlebody markings α^+ and α^- which have R-bounded combinatorics.*

Proof. For every $\gamma \in P$, if $\pi_\gamma(\alpha)$ is empty choose an arbitrary transversal for γ; otherwise choose a transversal that belongs to $\pi_\gamma(P)$. Repeat the same process for every component of P.

The construction for the case of Heegaard splittings is the same except that we have to do the construction for P^+ using projections of P^- and then repeat it for P^- using projections of P^+. □

2.7 Teichmüller space and Thurston's boundary

Like before, assume S is a fixed surface of genus ≥ 2. The *Teichmüller space* of S, denoted $\mathfrak{T}(S)$, is the set of hyperbolic structures on S modulo isotopy, or equivalently the set of conformal structures modulo isotopy. There is a natural actions of $\mathcal{MCG}(S)$ on $\mathfrak{T}(S)$.

The length pairing $\mathfrak{T} \times \mathcal{C}_0 \to \mathbb{R}_+$, assigns to each $\sigma \in \mathfrak{T}$ and $\alpha \in \mathcal{C}_0(S)$ the length of the unique closed geodesic on the hyperbolic surface σ in the isotopy class of α. This induces a $\mathcal{MCG}$-equivariant embedding $\mathfrak{T} \to [0,\infty)^{\mathcal{C}_0}$ and gives $\mathfrak{T}$ the $\mathcal{MCG}$-equivariant structure of a smooth manifold of dimension $6g-6$ diffeomorphic to $\mathbb{R}^{6g-6}$. The action of $\mathcal{MCG}$ on $\mathfrak{T}$ is properly discontinuous and noncocompact, and so the *moduli space* $\mathcal{M} = \mathfrak{T}/\mathcal{MCG}$ is a smooth, noncompact orbifold of dimension $6g-6$.

The length pairing can be extended to a continuous function:

$$\mathfrak{T} \times \mathcal{ML} \to (0,\infty)$$
$$(\sigma,\mu) \to l_\sigma(\mu)$$

We also have a $\mathcal{MCG}$-equivariant embedding $i : \mathcal{ML} \to [0,\infty)^{\mathcal{C}_0}$ by considering $i(\mu,\alpha)$ for $\mu \in \mathcal{ML}$ and every $\alpha \in \mathcal{C}_0$. This induces an embedding $\mathcal{PML} \to \mathcal{P}[0,\infty)^{\mathcal{C}_0}$, whose image is homeomorphic to a sphere of dimension $6g-5$. The composed map

$$\mathfrak{T} \to [0,\infty)^{\mathcal{C}_0} \to \mathcal{P}[0,\infty)^{\mathcal{C}_0}$$

is an embedding, the closure of whose image is a closed ball of dimension $6g-6$ with interior $\mathfrak{T}$ and boundary sphere $\mathcal{PML}$ called the *Thurston compactification.*

2.8 Teichmüller geodesics

Suppose S with a conformal structure is given. A *quadratic differential* associates to each conformal coordinate z an expression $q(z)dz^2$ with q holomorphic, such that whenever z, w are two overlapping conformal coordinates we have $q(z) = q(w)(\frac{dw}{dz})^2$. For such a quadratic differential, we have the *area form* expressed in the conformal coordinate $z = x + iy$ as $|q(z)|\,|dx|\,|dy|$, and the integral of this form is a positive number $\|q\|$ called the *area*. We say q is *normalized* if $\|q\| = 1$.

Away from zeros of q, there is a canonical conformal coordinate $\zeta = x + iy$, defined locally up to translation and sign, such that $q = d\zeta^2$ in this coordinate. The lines $\{y = x\}$ and $\{x = c\}$ are thus consistently defined and form what are known as the *horizontal and vertical foliations*, respectively or q_h and q_v. The metric $|q| = |d\zeta|^2 = dx^2 + dy^2$ is also canonically defined, and is Euclidean with isolated singularities at the zeros of q where there is concentrated negative curvature.

Now for every $t \in \mathbb{R}$ consider a new conformal structure obtained by taking the singular Euclidean metric $e^{2t}dx^2 + e^{-2t}dy^2$ and let $g(t)$ be the associated point of $\mathfrak{T}(S)$. This gives path in $\mathfrak{T}$ which we call a *Teichmüller geodesic.* Teichmüller's theorem states that any two points $\sigma \neq \sigma' \in \mathfrak{T}$ lie on a Teichmüller line g, and that line is unique up to an isometry of the parameter in $\mathbb{R}$. Moreover, if $\sigma = g(s)$ and $\sigma' = g(t)$, then $d_{\mathfrak{T}}(\sigma, \sigma') = |s - t|$ defines a proper, geodesic metric on $\mathfrak{T}$, called the *Teichmüller metric.*

For a Teichmüller geodesic constructed as above, the horizontal and vertical foliations of q correspond to two transverse elements of $\mathcal{PML}(S)$, called the *negative and positive ending laminations* or *ideal endpoints* of g. It turns out that the image of g is completely determined by this pair. Also given $\sigma \in \mathfrak{T}$ and $\mu \in \mathcal{PML}$, there is a unique geodesic ray with finite endpoint σ and given by the above description in a way that μ is associated to the vertical foliation and we call μ the *ideal endpoint* of the ray.

The group $\mathcal{MCG}$ acts isometrically on $\mathfrak{T}$, and so the Teichmüller metric descends to a proper geodesic metric on the moduli space $\mathcal{M}$. A subset $A \subset \mathfrak{T}$ is said to be *cobounded* if its projection to $\mathcal{M}$ has bounded image. When $\mathcal{K}$ a bounded subset of $\mathcal{M}$ is given, we say A is $\mathcal{K}$-cobounded if the projection of A to $\mathcal{M}$ is contained in $\mathcal{K}$.

Mumford's theorem provides a criterion for coboundedness. Given $\epsilon > 0$, let $\mathfrak{T}_\epsilon$ be the set of hyperbolic structures σ whose shortest closed geodesic has length $\geq \epsilon$ (we sometimes say σ is ϵ-*thick*) and define $\mathcal{M}_\epsilon$ to be the projected image of $\mathfrak{T}_\epsilon$. Mumford's theorem says that the sets $\mathcal{M}_\epsilon$ are all compact and their union is evidently all of $\mathcal{M}$. It follows that a subset $A \subset \mathfrak{T}$ is cobounded if and only if it is contained in some $\mathfrak{T}_\epsilon$.

2.9 Canonical bundles over Teichmüller space

For the closed surface S of genus ≥ 2, there is a smooth fiber bundle $\mathcal{S} \to \mathfrak{T}(S)$ whose fiber $\mathcal{S}_\sigma$ over $\sigma \in \mathfrak{T}$ is a hyperbolic surface representing the point $\sigma \in \mathfrak{T}$. More precisely, as a smooth fiber bundle we identify $\mathcal{S}$ with $S \times \mathfrak{T}$, and we impose smoothly varying hyperbolic structures on the fibers $\mathcal{S}_\sigma = S \times \sigma$, $\sigma \in \mathfrak{T}$, such that under the canonical homeomorphism $\mathcal{S}_\sigma \to S$ the hyperbolic structure on $\mathcal{S}_\sigma$ represents the point $\sigma \in \mathfrak{T}$. The action of $\mathcal{MCG}$ on $\mathfrak{T}$ lifts to an fiber-wise isometric action of $\mathcal{MCG}$ on $\mathcal{S}$. Each fiber $\mathcal{S}_\sigma$ is a *marked* hyperbolic surface, i.e. it comes equipped with an isotopy class of homeomorphisms to S. The bundle $\mathcal{S} \to \mathfrak{T}$ is called the *canonical marked hyperbolic surface bundle* over $\mathfrak{T}$.

The *canonical hyperbolic plane bundle* $\mathcal{H} \to \mathfrak{T}$ is defined as the composition $\mathcal{H} \to \mathcal{S} \to \mathfrak{T}$ where $\mathcal{H} \to \mathcal{S}$ is the universal covering map. Each fiber $\mathcal{H}_\sigma$, $\sigma \in \mathfrak{T}$, is isometric to the hyperbolic plane, with hyperbolic structures varying smoothly in σ. The group $\pi_1(S)$ acts as deck transformations of the covering map $\mathcal{H} \to \mathcal{S}$ and this action preserves each fiber $\mathcal{H}_\sigma$ with quotient $\mathcal{S}_\sigma$. The action of $\pi_1(S)$ on $\mathcal{H}$ extends to a fiber-wise isometric action of $\mathcal{MCG}(S,p)$ on $\mathcal{H}$, such that the covering map $\mathcal{H} \to \mathcal{S}$ is equivariant with respect to the group homomorphism $\mathcal{MCG}(S,p) \to \mathcal{MCG}(S)$. Bers [Be73] proved that $\mathcal{H}$ can be identified with the Teichmüller space of the once-punctured surface (S,p), and the action of $\mathcal{MCG}(S,p)$ on $\mathcal{H}$ is identified with the natural action of the mapping class group on Teichmüller space.

Suppose $T\mathcal{S}$ denotes the tangent bundle of $\mathcal{S}$ and $T_v\mathcal{S}$ denotes the *vertical subbundle* of $T\mathcal{S}$, i.e. the kernel of the derivative of the fiber bundle projection $\mathcal{S} \to \mathfrak{T}$. It follows from standard methods that there exists an $\mathcal{MCG}$-equivariant connection on $\mathcal{S}$. Choose a locally finite, equivariant open cover of $\mathfrak{T}$, and an equivariant partition of unity dominated by this cover. For each $\mathcal{MCG}$-orbit of this cover, choose a representative $U \subset \mathfrak{T}$ and choose a linear retraction $T\mathcal{S}_U \to T_v\mathcal{S}_U$. Use the action of $\mathcal{MCG}$ to define this retraction on all elements of orbit of U and take a linear combination using the partition of unity to obtain an equivariant linear retraction $T\mathcal{S} \to T_v\mathcal{S}$. The kernel of this retraction is one such connection. Also by lifting to $\mathcal{H}$ we obtain a connection on the bundle $\mathcal{H} \to \mathfrak{T}$, equivariant with respect to the action of the group $\mathcal{MCG}(S, p)$. We fix a choice of such a connection once and forever.

The connection obtains a smooth sub-bundle $T_h\mathcal{S}$ of $T\mathcal{S}$ which is complementary to $T_v\mathcal{S}$: $T\mathcal{S} = T_h\mathcal{S} \oplus T_v\mathcal{S}$. Lifting to $\mathcal{H}$ we also have a sub-bundle $T_h\mathcal{H}$ of the bundle $T\mathcal{H}$.

By a *closed interval*, we mean a closed connected subset of $\mathbb{R}$. Given a closed interval $I \subset \mathbb{R}$, a path $\gamma : I \to \mathfrak{T}$ is *affine* if it satisfies $d_{\mathfrak{T}}(\gamma(s), \gamma(t)) = K|s - t|$ for some constant $K \geq 0$, and γ is *piecewise affine* if γ is affine restricted to pieces of a decomposition of I into subintervals. In particular γ is $\mathbb{Z}$*-piecewise affine* if it is affine restricted to $[n, n+1] \cap I$ for every integer n.

Given an affine path $\gamma : I \to \mathfrak{T}$, by pulling back the canonical marked hyperbolic surface bundle $\mathcal{S} \to \mathfrak{T}$ and its connection $T_h\mathcal{S}$, we obtain a marked hyperbolic surface bundle $\mathcal{S}_\gamma \to I$ and a connection $T_h\mathcal{S}_\gamma$. This connection canonically determines a Riemannian metric on $\mathcal{S}_\gamma$ as follows. Without loss of generality, assume $K = 1$ in the definition of the affine path γ. Since $T_h\mathcal{S}_\gamma$ is 1-dimensional, there is a unique vector field V on $\mathcal{S}_\gamma$ parallel to $T_h\mathcal{S}_\gamma$ such that the derivative of the map $\mathcal{S}_\gamma \to I \subset \mathbb{R}$ takes each vector in V to the positive unit vector in $\mathbb{R}$. The fiber-wise Riemannian metric on $\mathcal{S}_\gamma$ now extends uniquely to a Riemannian metric on $\mathcal{S}_\gamma$ such that V is everywhere orthogonal to the fibration and has unit length.

Even when γ is piecewise affine, the above construction gives a Riemannian metric over each affine subpath, and at any point $t \in I$ where two such subpaths meet, the metrics agree along the fibers, thereby producing a piecewise Riemannian metric on $\mathcal{S}_\gamma$.

We can lift the above construction to $\mathcal{H}_\gamma$ to produce an $\mathcal{MCG}$-equivariant (piecewise affine) Riemannian metric such that the covering map $\mathcal{H}_\gamma \to \mathcal{S}_\gamma$ is local isometry. One can see that these path metrics are proper geodesic metrics.

A *connection line* in either of the bundles $\mathcal{S}_\gamma \to I$, $\mathcal{H}_\gamma \to I$ is a piecewise smooth section of the projection map which is everywhere tangent to the connection. By construction, given $s, t \in I$, a path p from a point in the fiber over s to a point in the fiber over t has length $\geq |s-t|$, with equality only if p is a connection path. It follows that the min distance and the Hausdorff distance between these fibers are both equal to $|s-t|$. By moving points along connection paths, for each $s, t \in I$ we have well-defined maps $\mathcal{S}_s \to \mathcal{S}_t$, $\mathcal{H}_s \to \mathcal{H}_t$, both denoted h_{st}. By a result of Farb-Mosher [FM02, Lem. 4.1], for a bounded set $\mathcal{K} \subset \mathcal{M}$ and $\rho \geq 1$ there exists K such that if $\gamma : I \to \mathfrak{T}$ is a $\mathcal{K}$-cobounded, ρ-Lipschitz, piecewise affine path, then for each $s, t \in I$ the connection map h_{st} is $K^{|s-t|}$-bi-Lipschitz.

2.10 Singular SOLV spaces

When $\gamma : I \to \mathfrak{T}$ is a geodesic there is another pair of natural geometries, the singular SOLV space $\mathcal{S}_\gamma^{\mathrm{SOLV}}$ and its universal cover $\mathcal{H}_\gamma^{\mathrm{SOLV}}$. Recall that a Teichmüller geodesic $\gamma(t)$ (parametrized by length) is given by a quadratic differential q and a family of singular Euclidean metrics

$$ds^2_{\gamma(t)} = e^{2t}|dx|^2 + e^{-2t}|dy|^2$$

where $|dy|$ and $|dx|$ are associated to the horizontal and vertical measured foliations of q and the conformal class of $ds_{\gamma(t)}$ represents $\gamma(t) = \mathcal{S}_t$.

We can use the above to define the *singular* SOLV *metric* on $\mathcal{S}_\gamma$ by

$$ds^2 = e^{2t}|dx|^2 + e^{-2t}|dy|^2 + dt^2$$

and we denote this metric space by $\mathcal{S}_\gamma^{\mathrm{SOLV}}$. The lift of this metric to the universal cover ch_γ produces a metric space denoted by $\mathcal{H}_\gamma^{\mathrm{SOLV}}$.

Farb-Mosher [FM02, Porp. 4.2] proved the following:

Proposition 2.10. *For any $\rho \geq 1$, any bounded subset $\mathcal{K} \subset \mathcal{M}$, and any $A \geq 0$ there exists $K \geq 1, c \geq 0$ such that the following holds. If $\gamma, \gamma' : I \to \mathfrak{T}$ are two ρ-Lipschitz, $\mathcal{K}$-cobounded, piecewise affine paths defined on a closed interval I, and if $d(\gamma(s), \gamma'(s)) \leq A$ for all $s \in I$, then there exists a map $\mathcal{S}_\gamma \to \mathcal{S}_{\gamma'}$ taking each fiber $\mathcal{S}_{\gamma(t)}$ to the fiber $\mathcal{S}_{\gamma'(t)}$ by a homeomorphism in the correct isotopy class, such that any lifted map $\mathcal{H}_\gamma \to \mathcal{H}_{\gamma'}$ is a (K, c)-quasi-isometry.*

If γ' is a geodesic, the same is true with $\mathcal{S}_{\gamma'}$, $\mathcal{H}_{\gamma'}$ replaced by the singular SOLV *spaces $\mathcal{S}_{\gamma'}^{\mathrm{SOLV}}, \mathcal{H}_{\gamma'}^{\mathrm{SOLV}}$.*

2.11 Hyperbolic Manifolds

By a *hyperbolic manifold*, we always mean a Riemannian 3-manifold with finitely generated fundamental group and constant sectional curvature -1. A hyperbolic manifold N is also recognized by the conjugacy class of a discrete and faithful representation

$$\pi_1(N) \to \mathrm{PSL}_2(\mathbb{C}).$$

We recall the definition of the *injectivity radius* of N at a point x, denoted by $\mathrm{inj}(x)$, is half the length of shortest (homotopically nontrivial) loop through x. By Margulis lemma, there exists a universal constant $\epsilon_M > 0$, such that for any $\epsilon \leq \epsilon_M$, every component of the ϵ-*thin part* of N

$$N^{<\epsilon} := \{x \in N | \ \mathrm{inj}(x) < \epsilon\}$$

is either

1. a torus cusp: a horoball in $\mathbb{H}^3$ modulo a parabolic action of $\mathbb{Z} \oplus \mathbb{Z}$,

2. a rank one cusp: a horoball in $\mathbb{H}^3$ modulo a parabolic action of $\mathbb{Z}$, or

3. a solid torus neighborhood of a geodesic

(see Thurston [Thu79] or Benedetti and Petronio [BP].) We also denote the complement of $N^{<\epsilon}$ by $N^{\geq\epsilon}$ that is the ϵ-thick part of N and the complement of all cuspidal parts of the thin part by $\hat{N}^{\epsilon}$. We call the components of type (1) and (2), ϵ-*cusps* or simply *cusps* of N and we call the components of type (3), ϵ-*Margulis-tubes* or simply *Margulis-tubes*. The radius of an ϵ-Margulis tube grows as the length of the core curve shrinks (cf. Brooks-Matelski [BM82] and Meyerhoff [Me87]).

Suppose N is a hyperbolic manifold and

$$\rho : \pi_1(N) \to \mathrm{PSL}_2(\mathbb{C})$$

is the associated representation which gives a discrete subgroup $\Gamma = \rho(\pi_1(N))$ of the group of isometries of $\mathbb{H}^3$. We can consider the limit set $\Lambda(\Gamma)$ and its convex hull $\mathcal{CH}(\Gamma)$; the projection of this set gives a subset $\mathcal{CH}(N) \subset N$, which we call the *convex core* of N. We say N is *convex cocompact* if $\mathcal{CH}(N))$ is compact.

The following lemma is an easy observation using hyperbolic geometry.

Lemma 2.11. *Let N be a hyperbolic manifold and α a homotopically non-trivial closed curve in N and α^* its geodesic representative. Then*

$$\cosh d_N(\alpha, \alpha^*) \leq l_N(\alpha)/l_N(\alpha^*),$$

where $l_N(\alpha)$ is length of α as a curve in N.

2.12 Geometrically finite and infinite

A hyperbolic manifold N is *geometrically finite* if its convex core has finite volume; otherwise it is *geometrically infinite*. Works of Bers, Maskit [Mas71], Kra [Kr72] and Sullivan give a description of the space of geometrically finite structures on a 3-manifold in terms of the Teichmüller space of its boundary.

Even when N is geometrically infinite, there exists a compact submanifold C of $\hat{N}^\epsilon$, called the *relative compact core* such that the inclusion of C into N

is a homotopy equivalence, C intersects each component of $\partial\hat{N}_\epsilon$ in an annulus, if the corresponding component is a rank one cusp, or in a torus, if the corresponding component is a torus cusps (see Feighn-McCullough [FMc87]). The ends of $\hat{N}^\epsilon$ are in one-to-one correspondence with components of $\partial C \setminus P$, where $P := \partial\hat{N}^\epsilon \cap C$ is the *parabolic locus.* In general $\hat{N}^\epsilon$ can have several ends. Each end is either a *geometrically finite end* when it intersects the convex core in a bounded set or a *geometrically infinite end* otherwise.

Suppose $E \subset \hat{N}^\epsilon$ is a neighborhood of an end of $\hat{N}^\epsilon$ associated to a component R of $\partial C \setminus P$. Now suppose we have a sequence of simple closed curves (α_n) on R which converge to an element of $\mathcal{PML}$ that is supported on $\mu \in \mathcal{UML}(R)$. Also assume α_n is homotopic to a closed geodesic α_n^* in N, with a homotopy that stays within E and the sequence (α_n^*) exits the end E. In this case, following Thurston, we say E is simply degenerate and μ is the *ending lamination of* E. It follows that the ending lamination fills R and is unique for a given parametrization of a simply degenerate end. In fact, Thurston proved that E is simply degenerate if and only if there exists a sequence of *simplicial or pleated hyperbolic surfaces* $\{h_n : \overline{R} \to U\}$ leaving every compact set such that for each n, $h_n(\overline{R})$ is homotopic to $\overline{R} \times \{0\}$ within U. (We postpone definitions of these surfaces to 2.14.)

Canary [Can89], [Can93b] proved that if the manifold is *topologically tame,* (it is homeomorphic to the interior of a compact manifold), then each geometrically infinite end is simply degenerate. In this case, we can assume E a neighborhood of an end of $\hat{N}^\epsilon$ is homeomorphic to $\overline{R} \times [0, \infty)$, where $\overline{R}$ is the closure of a component of $\partial C \setminus P$ and $\overline{R} \times \{0\}$ is homotopic to the inclusion $\overline{R} \hookrightarrow \partial C$ with a homotopy that stays away from C. In this situation, Canary

[Can93b, Thm. 10.1] proved E is simply degenerate with an ending lamination $\mu(E)$ that fills R. He also proved that if $\{\gamma_i^*\}$ is a collection of closed geodesics exiting E which are homotopic (within E) to curves γ_i on R, then every limit of the sequence $(\frac{\gamma_i}{l_0(\gamma_i)})$ in the space of *currents* is a measured lamination supported on $\mu(E)$. Here, $l_0(\gamma_i)$ denotes length of γ_i in a fixed finite area hyperbolic metric on R.

Recall that a (geodesic) *current* on a hyperbolic manifold M (in any dimension) is a (positive) transverse invariant measure on the geodesic flow of M whose support is contained within the projective tangent bundle of the *convex core*. (The convex core of a hyperbolic manifold is the smallest convex submanifold such that the inclusion is a homotopy equivalence.) Equivalently, if $M = \mathbb{H}^n/\Gamma$, we may think of a current as a Γ-invariant measure on $L_\Gamma \times L_\Gamma \backslash \Delta$, where L_Γ is the limit set of Γ and Δ is the diagonal. We denote the space of currents on M by $\mathscr{C}(M)$. When the support of a current c is a closed geodesic, we define its length, $l_M(c)$, to be the length of its support times the transverse measure of c. This extends to a continuous map $l_M : \mathscr{C}(M) \to \mathbb{R}_+ \cup \{0\}$, which is continuous when M is *convex cocompact*: it has a compact convex core. If S is a hyperbolic surface and α and β two closed geodesics, we define their geometric intersection number $i(\alpha, \beta)$ to be the number of points in $\alpha \cap \beta$. This extends to a symmetric, bilinear map

$$i : \mathscr{C}(S) \times \mathscr{C}(S) \to \mathbb{R}_+ \cup \{0\}$$

which is again continuous if S is convex cocompact. Note that $\mathcal{ML}(S)$ is naturally a subset of $\mathscr{C}(S)$ consisting of the currents c such that $i(c, c) = 0$.

(See Bonahon [Bo86] for a discussion of currents and for the above definitions and facts.)

We should emphasize here that even when N is topologically tame, the ending lamination depends on the choice of the relative compact core and the provided parametrization of a neighborhood of the end using a homeomorphism from $\overline{R} \times [0, \infty)$. (The parametrization is considered up to homotopy that takes place outside of the relative compact core.) In particular when N has no parabolics the relative compact core is called simply a *compact core* or a *Scott core*. Suppose C_1 is a compact core of N and R is a component of ∂C_1. As we mentioned there is an end of N associated to R and the inclusion $R \hookrightarrow \partial C_1$ determines a parametrization of a neighborhood of this end with a homeomorphism from $R \times [0, \infty)$. In this case, a theorem of McCullough-Miller-Swarup [MMS85] shows that if C_2 are is any other compact core of N, then there exists a homeomorphism $h : C_1 \to C_2$ such that $h_* = (i_2)_*^{-1} \circ (i_1)_*$, where $i_j : C_j \to N$ is the inclusion map. From this one can see that (the homotopy class outside of compact core of) the parametrization of the neighborhood of the end given above is well defined up to actions of $\mathrm{Mod}_0(R)$ which consists of elements of mapping class group of R that extend to a homeomorphism $(C, P) \to (C, P)$ homotopic to identity.

2.13 Hyperbolic structures on handlebodies

A *hyperbolic structure* on the handlebody H (or simply a *structure*) is a complete hyperbolic manifold N with a homeomorphism $\phi : H \to N$. Two structures (N_1, ϕ_1) and (N_2, ϕ_2) are equivalent if there exists an isometry

$f : N_1 \to N_2$ such that $\phi_2^{-1} \circ f \circ \phi_1$ is homotopic to identity. Equivalently, a hyperbolic structure on H is given by the conjugacy class of a representation $\rho : \pi_1(H) \to \mathrm{PSL}_2(\mathbb{C})$. (The equivalence, in fact, follows from the recent proof of the Tameness Conjecture (Agol [Ag] and Calegari-Gabai [CG].) If we choose a *base frame* (a base point together with a basis for the tangent space at the base point), this uniquely determines a representation of $\pi_1(H)$. By a *hyperbolic structure with a base frame*, we mean a hyperbolic structure with a choice of a base frame.

A *marked hyperbolic structure* on the handlebody H (or simply a *marked structure*) is a complete hyperbolic manifold N and an embedding $j : \partial H \to N$, called *marking*, such that j can be extended to an embedding $\bar{j} : H \to N$ and $N \setminus \bar{j}(H)$ is homeomorphic to $\partial H \times \mathbb{R}$. Two marked structures (N_1, j_1) and (N_2, j_2) are equivalent if there exists an isometry $f : N_1 \to N_2$ such that $f \circ j_1$ and j_2 are isotopic. We can think of a marked hyperbolic structure on H as a complete hyperbolic metric on the interior of H defined up to deformations induced by self-homeomorphisms of H isotopic to identity. In this case, the marking j is simply any embedding of ∂H into the interior of H isotopic to the inclusion $\partial H \hookrightarrow H$. When we speak of a marked structure N, always a choice of a marking $j : \partial H \to N$ is implicit. Also when we have a compact core $C \subset N$, we can isotope the marking j and assume that $j(\partial H)$ does not intersect C and the component of $N \setminus j(\partial H)$ that gives a neighborhood of the end of N does not intersect C either. Notice that the class of embeddings that are homotopic to j in $N \setminus C$ is included in the class of maps that are isotopic to j.

Remark 2.2. In the literature, usually a marked hyperbolic structure is a hyperbolic manifold with a marking for the fundamental group (a choice of an isomorphism from a fixed group to the fundamental group of the manifold). Here our markings not only mark the fundamental group but mark the isotopy class of identification of the manifold with a fixed copy of the manifold.

We say a sequence of hyperbolic structures with base frame converge *algebraically* to a hyperbolic manifold N, if the associated representations do. A sequence of hyperbolic structures converge algebraically if by choosing a base frame for each of the elements of the sequence, they converge algebraically (equivalently if the associated representations converge algebraically up to conjugation). Also a sequence of hyperbolic structures (N_i) converge *geometrically* to a hyperbolic manifold N if with an appropriate choice of base points $p_i \in N_i$ and $p \in N$, the pointed manifolds (N_i, p_i) converge to (N, p) in the Gromov-Hausdorff topology. In other terms, there exists a sequence of maps

$$\kappa_i : (\mathcal{N}_i(p), p) \to (N_i, p_i) \quad i \geq 0,$$

called the *approximating maps*, where $\mathcal{N}_i(p)$ is the ball of radius i about p, such that on every compact subset of N the maps κ_i converge to an isometry in the C^∞ topology as $i \to \infty$.

We use the geometric topology and the approximating maps in many places of our arguments; in order to shorten the arguments we are usually careless and consider the approximating maps by

$$\kappa_i : N \to N_i.$$

It should be understood that the approximating maps do not have to be defined on all of N and by the above notation we simply mean that κ_i is defined on larger and larges neighborhoods of the base point as $i \to \infty$.

We say a sequence of hyperbolic structures converge *strongly* to N if for an appropriate choice of base frames, they converge both algebraically and geometrically to N. For a sequence (N_i) of marked structures on H, we say (N_i) converge *strongly*, if there exists a marked hyperbolic structure N and base-points $p_i \in N_i$ and $p \in N$ such that $(N_i, p_i) \to (N, p)$ strongly as a sequence of hyperbolic manifolds and if

$$\kappa_i : (N, p) \to (N_i, p_i)$$

are the approximating maps, then $\kappa_i \circ j$ is isotopic to j_i, where j_i is the marking of N_i and j is the marking of N.

When a (marked) hyperbolic structure N on H is convex cocompact, the associated representation and subgroup of $\mathrm{PSL}_2(\mathbb{C})$ is called a *Schottky group*. Suppose $N = \mathbb{H}^3/\Gamma$ is a convex cocompact hyperbolic structure on H and $j : \partial H \to N$ is a marking. If Ω is the domain of discontinuity for the action of Γ on the boundary at infinity, then we know that $(\mathbb{H}^3 \cup \Omega)/\Gamma$ is homeomorphic to H and gives a compactification for N. Also we know that Ω/Γ has a conformal structure which is induced by the Poincaré metric on Ω. The marking j can be used to obtain a marking for Ω/Γ and this together with the conformal structure uniquely determines a point τ of $\mathfrak{T}(\partial H)$, which we call the *conformal structure at infinity*. We sometimes say τ is *associated* to the marked structure on N.

It follows from works of Bers, Maskit [Mas71], Kra [Kr72] and Sullivan that the above map gives a parametrization of the space of marked convex cocompact hyperbolic structures on H by $\mathfrak{T}(\partial H)$, the Teichmüller space of ∂H. In fact, it follows from their work that the parametrization is a homeomorphism between $\mathfrak{T}(\partial H)$ and the space of marked convex cocompact structures on H with the *quasi-isometric topology*. We should only point out that this topology is finer than the geometric and algebraic topology considered above and therefore this also parametrizes these structures with the strong topology. This also can be used to show that the space of unmarked convex cocompact hyperbolic structures on H is parametrized by $\mathfrak{T}(\partial H)/\mathrm{Mod}_0(H)$. Note that the statements of these results are usually made for unmarked structures. But using standard arguments, we can extend this convergence to the domain of discontinuity and from there it will be clear that the marking has to be preserved.

Suppose $N_i \to N$ strongly and N_i, N are marked convex cocompact structures on H. the convex core of N is compact and we can see that image of $\partial\mathcal{CH}(N)$ by approximating maps is very close to $\partial\mathcal{CH}(N_i)$ for $i \gg 0$. Now we can keep

We can see that the the conformal structure on the boundary of the convex core of N_i marked by using the markings converges to the one of N

once we have strong convergence, then we know that image of the boundary of the convex core the approximating maps take boundary of the convex core to the

In this case, also recall that $\partial\mathcal{CH}(N)$, the boundary of the convex core of N, is equipped with a hyperbolic metric induced by N and there is a natural nearest point retraction from the conformal structure at infinity to $\partial\mathcal{CH}(N)$. The following follows from a theorem of Bridgeman and Canary [BC03]:

Theorem 2.12. *Given ϵ_0 there exists $J > 0$ such that if N is a convex cocompact hyperbolic structure on handlebody H associated to the conformal structure at infinity τ and the injectivity radius of the hyperbolic metric correspondent to τ is bounded below by ϵ_0, then the nearest point retraction*

$$r : (\partial H, \tau) \to \partial\mathcal{CH}(N)$$

is J-Lipschitz and has a J-Lipschitz homotopy inverse.

On the other hand for a marked hyperbolic structure N on H, we can choose a relative compact core $C \subset N$ which is homeomorphic to H and $N \setminus C$ is homeomorphic to $\partial H \times \mathbb{R}$ and is a neighborhood of the end of N. The marking $j : \partial H \to N$ determines a homotopy equivalence from ∂H to $N \setminus C$ and gives an identification of ∂C with ∂H up to isotopy. By Canary's theorem [Can93b] all geometrically infinite ends of N are simply degenerate. In this case P, the parabolic locus, can be represented by a set of disjoint non-parallel essential simple closed curves on ∂H, which we still call the parabolic locus. The ends of $\hat{N}^\epsilon$ correspond to the components of the complement of the parabolic locus in ∂H.

In particular, when N is geometrically infinite without parabolics, $\hat{N}^\epsilon$ has exactly one end with a neighborhood $N \setminus C$. Since we have an identification of ∂C with ∂H (using the marking j), this uniquely determines an ending lamination on ∂H, which we call the ending lamination for the marked structure N. (Notice that for unmarked structures, the ending lamination is defined only up to actions of Mod_0.) Even more, Canary [Can93b, Cor. 10.2] proved that in this case, the ending lamination is in the Masur domain of H and fills ∂H.

2.14 Hyperbolic surfaces in 3-manifolds

Following Thurston [Thu79] (cf. Canary-Epstein-Green [CEG87]), we define a *pleated surface* in a hyperbolic 3-manifold N to be a map $f : S \to N$ together with a hyperbolic metric σ_f on S, called the *induced metric*, and a σ_f-geodesic lamination λ on S, so that the following holds: f is length-preserving on paths, maps leaves of λ to geodesics and is totally geodesic on the complement of λ. We say f realizes λ' if λ' is a sublamination of λ. With an abuse of notation, when we consider a pleated surface $f : S \to N$, we usually assume that S is equipped with the induced metric already.

When N is a hyperbolic structure on H, by $\mathbf{pleat}_N$ we denote the set of pleated surfaces $f : \partial H \to N$ which induce the same map as $\partial H \hookrightarrow H$ on the level of fundamental groups. If μ is a geodesic lamination on ∂H, by $\mathbf{pleat}_N(\mu)$ we denote the subset of $\mathbf{pleat}_N$ whose elements realize μ.

Similar to Bonahon [Bo86] and Canary [Can96], we define simplicial hyperbolic surfaces and recall some facts about them.

First recall a generalized definition of a triangulation for a surface (cf. Harer [Ha86] and Hatcher [Ha91]). Let S be a closed surface and Let $\mathcal{V}$ denote a finite collection of points in S. (We often restrict to the case where $\mathcal{V}$ is a single point.) A *curve system* $\{\alpha_1, \ldots, \alpha_m\}$ is a collection of arcs with disjoint interiors and endpoints in $\mathcal{V}$, no two of which are ambient isotopic (rel $\mathcal{V}$), and none of which is homotopic to a point (rel $\mathcal{V}$). A *triangulation* $\mathcal{T}$ of $(S, \mathcal{V})$ is simply a maximal curve system for $(S, \mathcal{V})$. We say two triangulations are equivalent if they are ambient isotopic (rel $\mathcal{V}$).

Suppose N is a hyperbolic 3-manifold. A continuous map $f : S \to N$ from a closed surface S into N is said to be a *simplicial pre-hyperbolic surface* if there exists a triangulation $\mathcal{T}$ of S such that image of each face of $\mathcal{T}$ is an immersed, totally geodesic, non-degenerate triangle. The map f induces a piecewise Riemannian metric on S, and f is said to be a *simplicial hyperbolic surface* if the angle about each vertex of $\mathcal{T}$ is at least 2π. We say a simplicial hyperbolic surface *realizes* a multi-curve α on S if there exists a subset of the 1-skeleton of $\mathcal{T}$ homotopic to α, and f maps each component of α to a closed geodesic in N..

Here, we only use a special class of simplicial hyperbolic surfaces where all the vertices of $\mathcal{T}$ are contained on a subset of the 1-skeleton that is homotopic to a multi-curve and that multi-curve is realized by the simplicial hyperbolic surface.

We say a complete Riemannian 3-manifold has *pinched negative curvature* if there exist nonzero constants $-a^2$ and $-b^2$ such that the sectional curvatures of N lie between the two constants. When N has pinched negative curvature, instead of simplicial hyperbolic surfaces, we use *simplicial ruled surfaces*. Recall that a *ruled triangle* is constructed by taking 3 totally geodesic arcs e_1, e_2 and e_3 which form a triangle in N and taking the collection of geodesics (in the appropriate homotopy class) with one endpoint v_{12}, the mutual endpoint of e_1 and e_2, and the other endpoint on e_3. A map $f : S \to N$ is called a *simplicial ruled surface* if there exists a triangulation $\mathcal{T}$ of S, such that each face of the triangulation is taken to a non-degenerate ruled triangle and the total angle about each vertex is at least 2π. We say a simplicial ruled surface f *realizes* a simple closed α if there is a closed loop in the 1-skeleton of the triangulation associated to f which is homotopic to α and is mapped to a closed geodesic in N.

Suppose an incompressible map $f : S \to N$ and a multi-curve α on S are given such that $f(\alpha)$ is freely homotopic to a a set of closed geodesics. Then it follows from work of Bonahon [Bo86] that there exists a simplicial ruled surface which realizes α.

For N hyperbolic or with pinched negative curvature by $N^{<\epsilon}$ we mean the set of points of N where the injectivity radius is less than ϵ and $N^{\geq\epsilon}$ denotes its complement. Also, for subsets $X, Y \subset N$, by $d_N^{\geq\epsilon}(X, Y)$ we denote the infimum of length of $P \cap N^{\geq\epsilon}$ among all paths that connect X to Y and by $\mathrm{diam}_N^{\geq\epsilon}(X)$, we denote the supremum of $d_N^{\geq\epsilon}(x, y)$ for points $x, y \in X$. Various versions of the next theorem have been proved by Thurston, Bonahon [Bo86] and Canary [Can93b, Can96].

Lemma 2.13. (Bounded diameter lemma) *Let $f : S \to N$ be a pleated surface or a simplicial hyperbolic surface or a simplicial ruled surface, where in the last case we assume N has pinched negative curvature for constants $-a^2$ and $-b^2$ and otherwise N is hyperbolic. Also assume $f(\gamma)$ has length at least ϵ if γ is a compressible curve on S. Then*

$$\mathrm{diam}_N^{\geq\epsilon}(f(S)) \leq D,$$

where D depends only on ϵ and $\chi(S)$ and the pinching constants in case N

has pinched negative curvature.

2.15 Constants

Suppose H is a fixed handlebody.

We say a collection Γ of simple closed curves and measured laminations *binds* the surface, if it has nonzero intersection with every nonzero element of $\mathcal{ML}$. It is a standard fact that the set of points $\sigma \in \mathfrak{T}(\partial H)$ where the total length of components of Γ is bounded by a constant $K > 0$ is compact and has bounded diameter depending on K and $\chi(\partial H)$. Notice that support of a marking α always binds the surface.

Lemma 2.14. *Given D there exists K depending only on D and $\chi(S)$ such that for every marking α on S,*

$$\{\tau \in \mathfrak{T}(S) | l_\tau(\alpha) \leq D\}$$

has diameter bounded by K in $\mathfrak{T}(S)$, where $l_\tau(\alpha)$ denotes the total length of elements in the support of α.

It also follows from an observation of Kerckhoff [Ker80] that there exists a unique point of $\mathfrak{T}(\partial H)$, where the total length of Γ is minimized. The set of markings on ∂H is finite up to actions of $\mathcal{MCG}$. Using this we can see that the total length of support of α in $\tau(\alpha)$ is bounded by B, where B depends only on $\chi(\partial H)$.

It follows from an observation of Bers [Be74, Be85] (cf. Buser [Buser]) that there exists a universal constant B_0 depending only on $\chi(\partial H)$ where on every

$\sigma \in \mathfrak{T}$ there exists a pants decomposition with total length at most B_0. We also assume B_0 is bigger than the constant B.

We fix ϵ_0 smaller than the Margulis constant for dimensions 2 and 3 and smaller than ϵ above such that two distinct ϵ_0-Margulis tubes are at least distance 1 apart. We also assume ϵ_0 is sufficiently small that on a hyperbolic surface, any simple closed geodesics is disjoint from any ϵ_0-Margulis tube but its own and that if $\tau \in \mathfrak{T}(\partial H)$ has a marking of length at most B_0 then τ is ϵ_0-thick.

Chapter 3

Uniform injectivity for handlebodies

Our purpose in this chapter is to obtain a parallel version of Thurston's uniform injectivity theorem and the efficiency of pleated surfaces for hyperbolic structures on a handlebody H. All we are doing is assuming that our pleated surfaces are incompressible in a nice neighborhood of the end and then once they are far from a compact core, we can argue similar to Thurston [Thu86, Thu98].

Suppose N is a hyperbolic structure on H. Here in this chapter we assume that a compact core $C \subset N$ is already chosen. Following Canary-Minsky [CM96], we say a continuous map $f : \partial H \to N$ is *end-homotopic*, if there exists a neighborhood of the end homeomorphic to $\partial H \times (0, \infty)$ and $f(\partial H)$ is homotopic to $\partial H \times \{0\}$ within $N \setminus C$. We concentrate on end-homotopic pleated surfaces in N. In fact, our proofs work in a more general setting. It is enough to have a closed subset C of any hyperbolic manifold N and consider pleated surfaces which are acylindrical in $N \setminus C$. Then once we are far from C the conclusions of our theorems hold.

Fact 3.1. *Suppose N is a homeomorphic to a handlebody and $C \subset N$ is a* *compact core. Then $H \setminus C$ is acylindrical. In particular, if*

$$f : \partial H \to N \setminus C$$

is end-homotopic, we have

(a) *every disk in N whose boundary is the f-image of an essential curve of ∂H intersects C,*

(b) *every homotopy in N between $f(\alpha)$ and $f(\beta)$, where α and β are non-homotopic closed curves on ∂H hits C and*

(c) *every homotopy between $f(\alpha)$ and a non-trivial power in N intersects C, if α is primitive in ∂H.*

By $\mathcal{N}_a(C)$ we denote the set of points of N which have distance at most a from C.

Lemma 3.2. *Suppose N is a hyperbolic structure on H with a compact core C and $f : \partial H \to N$ is an end-homotopic surface such that $f(\partial H) \subset N \setminus \mathcal{N}_a(C)$. Then every compressible curve on $f(\partial H)$ has length at least a.*

Proof. Suppose $f(\alpha)$ is compressible. Choose a point $x \in f(\partial H)$ and by coning from x construct a ruled disk that is bounded by $f(\alpha)$. Length of every geodesic between x and another point of $f(\alpha)$ is bounded by $\frac{1}{2}$ of the circumference of $f(\alpha)$. This gives a disk bounded by $f(\alpha)$ such that every point of the disk has distance at most $\frac{1}{4}$ of the circumference from $f(\alpha)$. If α is nontrivial in ∂H, every such disk has to intersect C and therefore length of $f(\alpha)$ is at least 4 times the distance of $f(\partial H)$ from C. □

The next lemma is a variation of an observation of Thurston about pleated surfaces.

Lemma 3.3. *Suppose N is a hyperbolic structure on the handlebody H and C is a compact core of N. There exists D_0 such that the following holds: for any ϵ there exists $\delta = \delta(\epsilon, \chi(\partial H)) < \epsilon$ with*

$$f((\partial H)^{<\epsilon}) \subset N^{<\epsilon}, \quad f((\partial H)^{\geq\epsilon}) \subset N^{\geq\delta}$$

for every end-homotopic $f \in \mathbf{pleat}_N$ that $f(\partial H) \subset N \setminus \mathcal{N}_{D_0}(C)$.

Proof. Using lemma 3.2, the first statement

$$f((\partial H)^{<\epsilon}) \subset N^{<\epsilon}$$

is immediate once $f(\partial H)$ has distance ϵ or more from C because the f-image of a curve of length $< \epsilon$ will be an essential curve of length $< \epsilon$.

Suppose, x is a point in the ϵ_0-thick part of ∂H then it has two loops through it of length not exceeding some constant $a/4$, depending only on $\chi(\partial H)$, such that the two loops generate a free subgroup of rank 2 in $\pi_1(\partial H)$. The commutator of these two loops also will have length at most a. But by what we said above, if $f(\partial H)$ does not intersect $\mathcal{N}_a(C)$and α is a closed curve of length at most a on ∂H, then $f(\alpha)$ is essential in N by lemma 3.2. Therefore the loops that we considered on ∂H and their commutator map to nontrivial loops in N. This provides two loops of length at most a based at $f(x)$ whose representatives do not commute. Because of Margulis lemma, this immediately implies that the injectivity radius of N at $f(x)$ is greater than some $\delta_0 > 0$ which depends on a.

Now if x is in the ϵ-thick part of ∂H for any ϵ, its distance from the ϵ_0-thick part of ∂H is bounded depending only on ϵ. Hence the distance of $f(x)$ from the δ_0-thick part of N is bounded with the same bound as well. From this, one can easily see that $f(x)$ has to be in the δ-thick part of N for some δ depending only on ϵ and $\chi(\partial H)$. □

If $f : S \to N$ is a pleated surface with pleating locus λ, it naturally lifts to a map $\mathbf{P}\mathrm{f}$ of λ into the tangent line bundle $\mathbf{P}\mathrm{N}$ of the target hyperbolic manifold.

Theorem 3.4. (Uniform injectivity) *Let H be a handlebody and ϵ_0 a given constant. For every hyperbolic structure N on H, a compact core $C \subset N$ and an end-homotopic pleated surface $f : \partial H \to N$, that realizes a geodesic lamination λ, the map*

$$\mathbf{P}\mathrm{f} : \lambda \to \mathbf{P}\mathrm{N}$$

is uniformly injective on the ϵ_0-thick part of ∂H, provided that $d_N(f(\partial H), C)$ is large. That is, for every $\epsilon > 0$, there is D and $\delta > 0$ such that for any N, $C \subset N$, λ and an end-homotopic $f \in \mathbf{pleat}_N$ with $d_N(f(\partial H), C) \geq D$, if x and $y \in \lambda$ are given whose injectivity radii are greater than ϵ_0,

$$d_{\sigma_f}(x, y) \geq \epsilon \Longrightarrow d_{\mathbf{P}\mathrm{N}}(\mathbf{P}\mathrm{f}(\mathrm{x}), \mathbf{P}\mathrm{f}(\mathrm{y})) \geq \delta.$$

Proof. Suppose we are given a sequence of hyperbolic structures N_i on H, and for every i, we have a compact core $C_i \subset N_i$ and an end-homotopic pleated surfaces $f_i : \partial H \to N_i$ realizing a geodesic laminations λ_i. Also assume for every i, there are points x_i and $y_i \in \lambda_i$ with $\text{inj}(x_i), \text{inj}(y_i) \geq \epsilon_0$ and $d_{\mathbf{PN_i}}(\mathbf{Pf_i}(x_i), \mathbf{Pf_i}(y_i)) \to 0$ and $d_{N_i}(f_i(\partial H), C_i) \to \infty$. Theorem 3.4 (Uniform injectivity) will follow when we show that $d_{\sigma_i}(x_i, y_i) \to 0$, where σ_i is the metric induced by f_i.

By lemma 3.3, we know that $\text{inj}(f_i(x_i))$ and $\text{inj}(f_i(y_i))$ are bigger than $\delta_0 = \delta(\epsilon_0, \chi(\partial H))$ for $i \gg 0$. We take x_i and $f_i(x_i)$ to be base points for $(\partial H, \sigma_i)$ and N_i. Therefore these pleated surfaces and the domain and target manifolds converge in the geometric topology (after passing to a subsequence). Suppose Σ, N and $f : \Sigma \to N$ are limits of $(\partial H, \sigma_i)$, N_i and f_i respectively. Notice that Σ and N are not necessarily hyperbolic structures on ∂H and H anymore, but we know that $\chi(\Sigma) \geq \chi(\partial H)$. By taking a further subsequence, we can also assume that the laminations λ_i converge in the Hausdorff topology and λ is the limit lamination on Σ.

Recall Thurston's [Thu86] notion of *weakly doubly incompressible* surfaces: if Σ is a hyperbolic surface of finite area and if $f : \Sigma \to N$ is a map to a hyperbolic 3-manifold which takes cusps to cusps, then f is weakly doubly incompressible if

(a) $f_* : \pi_1(\Sigma) \to \pi_1(N)$ is injective,

(b) homotopy classes of maps $(I, \partial I) \to (\Sigma, \text{cusps}(\Sigma))$ relative to cusps map injectively to homotopy classes of maps $(I, \partial I) \to (N, \text{cusps}(N))$,

(c) for any cylinder $c : S^1 \times I \to N$ with a factorization of its boundary $\partial c = f \circ c_0 : \partial(S^1 \times I) \to \Sigma$ through Σ, if $\pi_1(c)$ is injective then either the image of $\pi_1(c_0)$ consists of parabolic elements of $\pi_1(\Sigma)$, or c_0 extends to a map of $S^1 \times I$ into Σ and

(d) Each maximal cyclic subgroup of $\pi_1(S)$ is mapped to a maximal cyclic subgroup of $\pi_1(N)$.

Lemma 3.5. *The limit pleated surface $f : \Sigma \to N$ is weakly doubly incompressible.*

Proof. The proof is very similar to Thurston's proof [Thu86, Lem. 5.10], where he proves that limits of *doubly incompressible* pleated surfaces are weakly doubly incompressible. The main difference is that here we do not have doubly incompressibility of the maps f_i, but instead using fact 3.1 and the fact that image of f_i is a closed surface, we know that f_i is doubly incompressible in $N_i \setminus C_i$ and its distance from C_i tends to infinity as $i \to \infty$.

Suppose f is not π_1-injective. Then there exists a closed geodesic α on Σ such that $f(\alpha)$ bounds a disk D in N. Since arbitrary large compact subsets of N are approximately isometric to subsets of N_i for large i, we obtain similar disks D_i in N_i. But since D_i has bounded diameter for every i, we conclude that D_i does not intersect C_i for $i \gg 0$. Then fact 3.1 shows that, there is a disk $D'_i \subset \partial H$, whose f_i-image has the the same boundary as D_i. Because ∂D_i has bounded length, we can assume D'_i has bounded diameter in σ_i and therefore they converge to a disk $D' \subset \Sigma$ with boundary α and we have a contradiction.

To check condition (c), suppose we have an incompressible cylinder $A : S^1 \times I \to N$ in the limit manifold, with a factorization of its boundary through f. Again by using the approximating maps, we can push this cylinder to obtain similar cylinders A_i in N_i with bounded diameter. By fact 3.1 (for $i \gg 0$) there is a cylinder A'_i in ∂H, whose f_i-image has the same boundary as A_i. The core curve of these cylinders cannot be inessential in N_i, otherwise we can take a sequence of bounded diameter compressing disks for A_i and in the limit we get a compressing disk for A, which contradicts incompressibility of A. If the lengths of the core curves of these cylinders tend to 0, then the boundary components of A are parabolics and (c) is satisfied. Otherwise since length of ∂A_i is bounded, there is a bounded diameter homotopy in $(\partial H, \sigma_i)$ between boundaries of A_i and in the limit we will have a homotopy in Σ between boundaries of A and (c) is satisfied.

For condition (d), suppose α is a non-trivial element of $\pi_1(\Sigma)$, and $f_*(\alpha) = \beta^k$ for some $\beta \in \pi_1(N)$. If we take representatives $a \subset \Sigma$ and $b \subset N$ for α and β respectively, together with a cylinder C giving the homotopy from b^k to $f(a)$, we can push this configuration to approximates N_i. Again by fact 3.1, it follows that we can push the homotopy to $f_i(\partial H)$. Therefore, if a_i is the approximation to a on $(\partial H, \sigma_i)$, there is a loop c_i on ∂H such that c_i^k is homotopic to a_i. In fact, we can assume that c_i is contained in a small neighborhood of a_i independently of i and the homotopy between c_i^k and a_i does not leave this neighborhood either. Then in the limit a will be a kth power in Σ and this proves (d).

To prove (b), let α and β be two arcs on Σ with ends in cusps(Σ) which represent different homotopy classes of maps

$$(I, \partial I) \to (\Sigma, \mathrm{cusps}(\Sigma))$$

relative to cusps. Suppose that they are mapped to the same element of the homotopy classes of maps $(I, \partial I) \to (N, \mathrm{cusps}(N))$ relative to cusps. This means that there are arcs μ and ν in cusps(N) such that

$$f(\alpha) * \nu * f(\beta^{-1}) * \mu$$

is null-homotopic in N. Note that if we push α, β, $f(\alpha)$ and $f(\beta)$ to the approximates, we get arcs

$$\alpha_i, \beta_i : (I, \partial I) \to (\partial H, \mathrm{thin}(\sigma_i))$$

and

$$f_i(\alpha_i), f_i(\beta_i) : (I, \partial I) \to (N_i, \mathrm{thin}(N_i)).$$

Even more, we can push μ and ν to μ_i and $\nu_i \subset \mathrm{thin}(N_i)$ in the approximates and

$$f_i(\alpha_i) * \nu_i * f_i(\beta_i^{-1}) * \mu_i$$

bounds a disk with bounded diameter in N_i.

Suppose P is the component of cusps(N) which contains $\alpha(0)$ and $\beta(0)$ and P_i is the corresponding component of thin(N_i) in an approximate and note that P is either a rank 1 cusp or a rank 2 cusp and for every i, P_i is either a rank 1 cusp or a Margulis tube. Now we have two different cases:

Case 1: P is a rank 2 cusp. Then ∂P has bounded diameter and in the approximates ∂P_i has bounded diameter as well and therefore P_i is a Margulis tube in N_i whose distance to Γ_i tends to infinity as $i \to \infty$. Because the thin components of $(\partial H, \sigma_i)$ corresponding to $\alpha_i(0)$ and $\beta_i(0)$ map to P_i, some power of their cores are homotopic within P_i. Because of fact 3.1 and lemma 3.3, it is possible only when these cores are homotopic in ∂H and they represent the same component of $\mathrm{thin}(\sigma_i)$. In addition, it easily follows that we can connect $\alpha_i(0)$ and $\beta_i(0)$ with an arc μ_i' such that $f(\mu_i')$ and μ_i are homotopic relative their endpoints with a homotopy that stays in P_i and therefore does not intersect C_i.

Case 2: P is a rank 1 cusp. The components of $\mathrm{cusps}(\Sigma)$ corresponding to $\alpha(0)$ and $\beta(0)$ map to the same component of $\mathrm{cusps}(N)$: P. Hence by condition (d), the images of their cores are homotopic to the core of P and there is a cylinder $A \subset P$ that gives the homotopy. One can easily assume that μ is in A. Now if we push A to the approximates N_i, we get cylinders A_i whose boundaries are on $f_i(\partial H)$ and because they have bounded diameter they stay in $N_i \setminus C_i$ for $i \gg 0$. Also note that $\mu_i \subset A_i$. Again fact 3.1 implies that there is a homotopy within $N_i \setminus C_i$, which fixes ∂A_i and pushes A_i to $f_i(\mathrm{thin}(\sigma_i))$. In particular, μ_i is homotopic (rel. endpoints) to an arc $f_i(\mu_i')$ with a homotopy that does not intersect C_i, where $\mu_i' \subset \mathrm{thin}(\sigma_i)$.

In both these cases, we could find an arc $\mu_i' \subset \mathrm{thin}(\sigma_i)$ such that $f_i(\mu_i')$ is homotopic to μ_i (rel. endpoints) within $N_i \setminus C_i$ for $i \gg 0$. We can do the same for ν_i and find and arc $\nu_i' \subset \mathrm{thin}(\sigma_i)$ such that $f_i(\nu_i')$ is homotopic to ν_i (rel. endpoints) within $N_i \setminus C_i$. Then $f_i(\alpha_i * \nu_i' * \beta_i^{-1} * \mu_i')$ bounds a disk in $N_i \setminus C_i$ and by fact 3.1, α_i and β_i are homotopic as maps from $(I, \partial I)$ to $(\partial H, \mathrm{thin}(\sigma_i))$.

We claim that we can assume μ_i' and ν_i' have bounded length. We certainly can assume that μ_i' and ν_i' are geodesics (since $\mathrm{thin}(\sigma_i)$ is convex). Consider geodesics α_i' and β_i' homotopic to α_i and β_i (rel. endpoints) respectively. Lengths of α_i' and β_i' are bounded uniformly. If μ_i' and ν_i' had very long length then because of hyperbolicity of σ_i they have to get arbitrary close and this means that the two components of thin(σ_i) get arbitrary close using an arc in the homotopy class (rel. cusps) of α and β. Which is impossible since we always assume that components of the thin part have distance at least 1 from each other.

Using the above claim, we can assume that the homotopy between α_i and β_i has bounded diameter in thick(σ_i) and therefore in the limit it gives a homotopy between α and β relative to cusps(Σ) and we have a contradiction.

□

Thurston proved [Thu86, Thm. 5.6] that for a weakly doubly incompressible map $f : \Sigma \to N$ which takes each leaf of λ, a geodesic lamination on Σ, to a geodesic in N, the canonical lifting $\mathbf{P}\mathrm{f} : \lambda \to \mathbf{P}\mathrm{N}$ is an embedding. Once we know this and the above lemma, we can conclude that the limit points $x = \lim_i x_i$ and $y = \lim_i y_i$ must be equal, since their images in $\mathbf{P}\mathrm{N}$ are equal. Therefore, $d_{\sigma_i}(x_i, y_i) \to 0$ and we have proved theorem 3.4. □

If λ is a lamination in S, a *bridge arc* for λ is an arc in S with end points on λ, which is not deformable rel endpoints into λ. A *primitive bridge arc* is a bridge arc whose interior is disjoint from λ. If σ is a hyperbolic metric on S and τ is a bridge arc for λ, let $[\tau]$ denote the homotopy class of τ with endpoints fixed, and let $l_\sigma([\tau])$ denote the length of the minimal representative of $[\tau]$.

Suppose f, f' realize a lamination λ in a hyperbolic manifold N, we say that

f and f' are *homotopic relative to* λ if there is a homotopy between them fixing λ point-wise. One can see that after precomposing f' by a homeomorphism isotopic to identity, we can obtain a map that is homotopic to f relative to λ. (Cf Minsky [Min00, Lem. 3.3].)

Similar to Minsky [Min92, Min00], we can strengthen uniform injectivity as follows:

Corollary 3.6. (Short bridge arcs) *Fix the handlebody H and ϵ_0. Given $\delta_1 > 0$ there exists $\delta_2 > 0$ and D such that the following holds: Let N be a hyperbolic structure on H and $C \subset N$ a compact core. Also let $g \in \mathbf{pleat}_N$ be an end-homotopic map with $d_N(g(\partial H), C) \geq D$ that realizes a lamination λ.*

Suppose that τ a bridge arc for λ is either primitive or contained in the ϵ_0-thick part of σ_g. Then

$$l_{\mathbf{P}\mathrm{N}}(\mathbf{P}\mathrm{g}(\tau)) \leq \delta_2 \Longrightarrow \mathrm{l}_{\sigma_\mathrm{g}}([\tau]) \leq \delta_1.$$

Moreover if f is another map that realizes λ, chosen so it is homotopic to f relative to λ, then

$$l_{\sigma_f}([\tau]) \leq \delta_2 \Longrightarrow l_{\sigma_g}([\tau]) \leq \delta_1.$$

We can use our uniform injectivity theorem and similar to Thurston we can prove a version of efficiency of pleated surfaces. Recall the notion of *alternation number* $a(\lambda, \gamma)$ where λ is a lamination with finitely many leaves and γ is a simple closed curve. This is defined carefully in Thurston [Thu98] and Canary [Can93a]. For our purpose, we need only to know that if γ does not intersect the recurrent part of λ then $a(\lambda, \gamma)$ is bounded by the number of intersection points of λ and γ. We also need a variation in the statement of the theorem to allow short curves in the pleating locus. This will be similar to Minsky's statement and proof in [Min00, Thm. 3.5].

Theorem 3.7. (Efficiency of pleated surfaces) *Given $\epsilon > 0$ smaller than ϵ_0, there exist constants $c > 0$ and $D > 0$ depending only on $\chi(\partial H)$ and ϵ such that the following holds. Let N be a hyperbolic structure on H with a compact core $C \subset N$ and $f \in \mathbf{pleat}_N$ is end-homotopic realizing a maximal finite leaved lamination λ and $d_N(f(\partial H), C) \geq D$. Also suppose γ is a simple closed curve which does not intersect any closed leaf of λ whose length is less than ϵ and $f(\gamma)$ is homotopic to a closed geodesic γ^* with a homotopy that stays in $N \setminus C$. Then*

$$l_N(\gamma^*) \leq l_\sigma(\gamma) \leq l_N(\gamma^*) + ca(\gamma, \lambda), \tag{3.1}$$

where σ is the metric induced by f.

Sketch of proof. The proof follows Thurston's [Thu98] original proof. Similar to Thurston, we can find a polygonal representative γ' of γ on ∂H, equipped with σ, which consists of $a(\gamma, \lambda)$ segments on leaves of λ connected by $a(\gamma, \lambda)$ "jumps" of bounded length. It follows that $l_\sigma(\gamma') \leq l_\sigma(\gamma) + c_0 a(\gamma, \lambda)$. (In what follows constants c_i will depend only on ϵ and $\chi(\partial H)$.)

Thurston observed that by moving the points where the bounded jumps occur, we can assume that those jumps never occur inside any ϵ-Margulis tube whose core is not a leaf of λ. In our situation, when γ does not intersect short closed components of λ, as we will show later, we can assume that that γ' does not enter ϵ_1-Margulis tubes whose cores are leaves of λ, for a uniform ϵ_1.

The image $f(\gamma')$ consists of the images of arcs along λ, which remain geodesic, and of the jumps, which still have bounded length. After straightening the images of the bounded jumps, we obtain a polygonal curve ν in N, and the difference between $l_\sigma(\gamma)$ and length of ν is at most $c_1 a(\gamma, \lambda)$.

There exists a pleated annulus A in N which represents the free homotopy between ν and γ^*. By Gauss-Bonnet theorem the area of A with the induced hyperbolic metric is bounded proportional to the number of corners of ν, which is $2a(\gamma, \lambda)$.

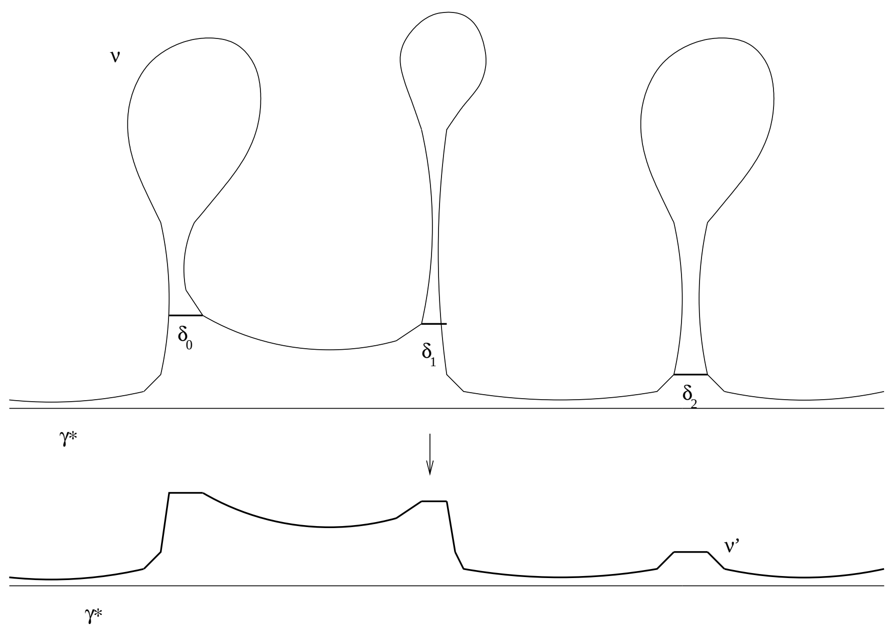

Figure 3.1: Construction of ν'.

Fix some $\epsilon' > 0$ and let ν_0 be the union of segments of γ' which admit collar neighborhoods of width ϵ' in A, together with all the bounded length jumps. Because of the area bound on A, length of ν_0 is bounded by $c_2 a(\gamma, \lambda)/\epsilon'$. Let ν_1 denote those segments that are distance ϵ' from γ^*. The total length of ν_1 is at most $l_N(\gamma^*) + c_3 a(\gamma, \lambda)$. We can see that $\gamma' \setminus (\nu_0 \cup \nu_1)$ consists of pairs of segments $\{\sigma_1, \sigma_1', \ldots, \sigma_m, \sigma_m'\}$ such that σ_i and σ_i' have Hausdorff distance at most ϵ'. We can use a pair σ_i, σ_i' to construct a rectangle R_i on A with two opposite sides that cover σ_i and σ_i' except for a bounded length and the other two sides are ϵ' short. Now we can remove these rectangles from A, we will be left with a number of disks and one annulus. One boundary of the annulus is γ^* and the other boundary consists of subsegments of γ' and jumps $\delta_1, \ldots, \delta_k$ of length ϵ'. We call this other boundary ν'. Notice that except for a set of length at most $c_4 a(\gamma, \lambda)/\epsilon'$ every other point of ν' has distance at most ϵ' from γ^*. In particular, we see that ν' and γ^* are homotopic with a homotopy that is contained in a bounded neighborhood of $f(\partial H)$ depending on ϵ'.

Consider a component δ_i, $i = 0, \ldots, k$ obtained above. We claim that if we choose $\epsilon_1 < \epsilon$ small depending on ϵ' and ϵ then the endpoints of δ_i are images of points in the ϵ_1-thick part of σ_f. Suppose the endpoints of δ_i are images of points in the ϵ_1-thin part of σ_f. First note that the endpoints have to be in the same component of the thin part of σ_f. This is because the distance between their images is small and therefore the homotopy between their cores will stay away from C and since f is end-homotopic, by fact 3.1 they have to be homotopic in ∂H.

Let α be core of this component. Because of our hypothesis, γ does not intersect closed leaves of λ with length $\leq \epsilon$ and therefore α is not realized by f. Leaves of λ which enter the associated ϵ-thin component of σ_f from one side have to exit the other side and the images of their intersections with this component are ϵ-close geodesic segments in N. If l_1 and l_2 are leaves of λ that contain endpoints of δ_i then we can see that l_1 and l_2 (and their images in N) stay ϵ-close to each other along long subsegments centered at endpoints of δ_i. These subsegments are exponentially large depending on ϵ/ϵ_1 which is because of the part where l_1 and l_2 run between ϵ-thin and ϵ_1-thin collars of α. In particular along these subsegments, they cannot be ϵ-parallel to γ^* in opposite directions. But we know that if we follow ν' starting from endpoints of δ_i in a bounded distance (depending on ϵ') and in opposite directions, we get to segments that are parallel to γ^* in opposite directions. We can lift the picture to the universal cover and note that once these two are parallel to γ^* in opposite directions, they cannot be parallel to each other and we have a contradiction with the fact that they were fellow traveling in the ϵ-Margulis tube of the surface. Therefore we should have seen a jump along our way on ν and since the jumps only occur in the ϵ-thick part of σ_f, this gives a lower bound for the injectivity radius of σ_f at endpoints of δ_i depending only on ϵ' and ϵ.

Once we know that endpoints of δ_i are images of points in the ϵ_1-thick part of σ_f, we can use corollary 3.6 (Short bridge arcs) and imply that δ_i is homotopic (rel. endpoints) to a short arc $f(\delta_i')$ in $f(\partial H)$ with a homotopy that stays away from C, if D is large depending on ϵ' and ϵ_1.

Using the arcs δ_i', $i = 0, \ldots, k$, we can follow the recipe given by ν' to construct a closed curve γ'' from γ' such that $f(\gamma'')$ is the same as ν' except for the straightening of the jumps and replacing δ_i with $f(\delta_i')$. From this we also conclude that length of γ'' is at most $l_N(\gamma^*) + c_5 a(\gamma, \lambda)$.

We claim that γ'' is homotopic to γ in ∂H. This is because we showed that each $f(\delta_i')$ is homotopic (rel. endpoints) to δ_i within $N \setminus C$. Therefore $f(\gamma'')$ is homotopic within $N \setminus C$ to ν' and thus to γ^*. But $f(\gamma)$ was homotopic to γ^* within $N \setminus C$ as well and by fact 3.1, γ'' and γ are homotopic in ∂H.

□

Chapter 4

Pleated surfaces in handlebodies

In this chapter, we prove some useful properties of pleated surfaces in hyperbolic structures of handlebodies. The main corollary of these results is corollary 4.7, which allows us sweep through the convex core of a structure once we have a definite distance from the compact core. In these results, we assume that the compact core contains what we call a *diskbusting geodesic*. These were originally used by Canary in [Can89, Can93b]. In particular, he proved:

Proposition 4.1. *Let H be a handlebody. There is a collection Γ of disjoint simple closed curves on ∂H with the following properties:*

1. *Γ intersects at least three times every essential simple closed compressible curve on ∂H,*
2. *Γ intersects the boundary of every essential and properly embedded annulus $(A, \partial A) \subset (H, \partial H)$ and*
3. *$0 = [\Gamma] \in H_1(H; \mathbb{Z})$.*

For a hyperbolic structure N on H, by a *diskbusting geodesic* in N, we mean the geodesic representative of a collection of curves which satisfy the conclusion of the above. In fact, Canary proved that we can always choose Γ such that none of its components represent parabolic elements of N and therefore there always exist a diskbusting geodesic.

Let N be a marked hyperbolic structure on H and Γ a diskbusting geodesic in N which we assume is fixed. Suppose we have chosen a compact core $C \subset N$ which contains a diskbusting geodesic and as always, we assume that image of the marking j does not intersect C and the component of $N \setminus j(\partial H)$ that is a neighborhood of the end of N does not intersect C either. We call such a compact core a *useful* compact core for N. If α is a multi-curve on ∂H, by a geodesic representative of α in $N \setminus C$, we mean a closed geodesic which is freely homotopic to $j(\alpha)$ with a homotopy that stays in $N \setminus C$.

In the rest of this work, except in chapter 5, we assume that our compact cores contain a diskbusting geodesic. It should be pointed out that the next lemma is the only place where we use this property of the compact cores and Canary's idea of branched covers. This lemma is crucial for the rest of the paper, since we use end-homotopic pleated surfaces to control the geometry of the manifold and this lemma gives a sufficient condition for existance of such pleated surfaces.

Lemma 4.2. *Given $\epsilon > 0$ and $d > 0$ there exists a constant $D_1 > 0$ depending only on ϵ, d and $\chi(\partial H)$ such that if N is a marked hyperbolic structure on H with a useful compact core C and if α is a simple closed curve on ∂H which* *has a geodesic representative* α^* *in* $N \setminus C$ *with*

$$d_{\hat{N}}^{\geq \epsilon}(\alpha^*, C) \geq D_1,$$

then there exists $f \in \mathbf{pleat}_N(\mu)$ *that is homotopic to* j *within* $N \setminus C$ *and*

$$d_{\hat{N}}^{\geq \epsilon}(f(\partial H), C) \geq d,$$

for every finite leaved lamination μ *that contains* α*.*

Proof. Suppose μ is a finite leaved maximal lamination that contains α and let γ be the recurrent part of μ. Consider a diskbusting geodesic $\Gamma \subset C$. The idea is to lift to the 2-fold branched cover of N, branched along Γ, which we call $\hat{N}$. Similar to Canary [Can93b], one can see that $\hat{N}$ has a compact core $\hat{C}$ with incompressible boundary which is lift of C. We also know that we can put a Riemannian metric with pinched negative curvature on $\hat{N}$, which is hyperbolic and is lift of the metric of N outside $\hat{C}$. By construction, there is a component of $\hat{N} \setminus \hat{C}$ which is isometric to $N \setminus C$ and we can lift j and α^* to $\hat{j}$ and $\hat{\alpha}^*$. Notice that $\hat{j}$ is incompressible. (Note that there is a minor problem, when Γ has self intersection, but as Canary [Can93b] showed, we can get around this problem by perturbing the metric of N in a small neighborhood of Γ without affecting the metric in $N \setminus C$ and pursue as before.)

As we mentioned in 2.14, it follows from work of Bonahon [Bo86] and Canary [Can93b] that there exists a simplicial ruled surface $\hat{g} : \partial H \to \hat{N}$ homotopic to $\hat{j}$ that realizes γ. Note that $\hat{\alpha}^*$ has to be in the image of this surface. Since the covering map is an isometry when restricted to $\hat{N} \setminus \hat{C}$, we have

$$d^{\geq \epsilon}_{\hat{N}}(\hat{\alpha}^*, \hat{C}) = d^{\geq \epsilon}_{N}(\alpha^*, C) \geq D_1.$$

But $\operatorname{diam}^{\geq \epsilon}_{\hat{N}}(\hat{g}(\partial H))$ is bounded from above depending only on ϵ and $\chi(\partial H)$ because of lemma 2.13 (Bounded diameter lemma). Therefore if we assume D_1 is large enough depending only on ϵ and $\chi(\partial H)$, the image of $\hat{g}$ has to be contained in $\hat{N} \setminus \hat{C}$. Even more, we can see that there is a homotopy between $\hat{g}$ and $\hat{j}$ that stays within $\hat{N} \setminus \hat{C}$. (Note that in our statement of the bounded diameter lemma, D depended on the curvature bounds; but here since $\hat{N}$ is hyperbolic outside $\hat{C}$, it is enough to assume $D_1 \geq D + 1$ where D is the upper-bound for the hyperbolic case.)

Then we can project $\hat{g}$ down to $N \setminus C$ to obtain a simplicial ruled surface $g : \partial H \to N$ which is homotopic to j within $N \setminus C$ and realizes γ. Replace ruled triangles of $g(\partial H)$ with totally geodesic triangles and similar to Thurston's construction of pleated surfaces [Thu79], spiral vertices of the associated triangulation about components of γ in a way that it approximates μ. In the limit, we get a pleated surface that realizes μ and one can see that during this process, we stay in a small neighborhood of $g(\partial H)$ and the obtained pleated surface has all the required properties of our statement. □

For $f \in \mathbf{pleat}_N$, we define $\mathbf{short}(f, B)$ to be the set of essential simple closed curves on ∂H whose length in the induced metric does not exceed B.

Theorem 4.3. *Given $\epsilon > 0$ and d there exists $D_2 > d$ and $A > 0$ depending only on d, R and $\chi(\partial H)$ such that the following holds. Let N be a marked hyperbolic structure on H and $C \subset N$ a useful compact core. If α has a geodesic representative α^* in $N \setminus C$ with $d^{\geq \epsilon}_{N}(\alpha^*, C) \geq D_2$, then for every*

$\beta \in \mathcal{C}_0(S)$ with $d_{\mathcal{C}}(\alpha, \beta) \leq 1$:

(a) *there exists $f \in \mathbf{pleat}_N(\beta)$ homotopic to j within $N \setminus C$*

(b) *there exists $f \in \mathbf{pleat}_N(\alpha) \cap \mathbf{pleat}_N(\beta)$ homotopic to j within $N \setminus C$*

(c) *every $f \in \mathbf{pleat}_N(\beta)$ has $d_N^{\geq\epsilon}(f(\partial H), C) \geq d$*

(d) *for every end-homotopic f and $g \in \mathbf{pleat}_N(\beta)$, the set*

$$\mathbf{short}(f, B) \cup \mathbf{short}(g, B)$$

has diameter bounded by A in $\mathcal{C}(\partial H)$.

Proof. Using lemma 4.2, we know that for every $d > 0$ if we assume D_2 is bigger than the constant obtained there and α, β and N as in the hypothesis, there exists $f \in \mathbf{pleat}_N(\alpha \cup \beta)$ homotopic to j within $N \setminus C$ and with

$$d_N^{\geq\epsilon}(f(\partial H), C) \geq d.$$

This already proves (a) and (b). In particular, this implies that β has a geodesic representative β^* in $N \setminus C$ with

$$d_N^{\geq\epsilon}(\beta^*, C) \geq d.$$

By assuming that d is larger than the constant in lemma 2.13 (Bounded diameter lemma) for ϵ, this implies (c) too. In fact, given $d > 0$, we can choose

D_2 large such that for every $f \in \mathbf{pleat}_N(\beta)$

$$d_N^{\geq \epsilon}(f(\partial H), C) \geq d.$$

For part (d), the idea is to use an argument similar to Minsky's [Min01, Lem. 3.2]. There, the pleated surfaces are doubly incompressible and in particular π_1-injective; but here we use theorem 3.4 and lemma 3.3 instead. Before explaining the proof, note that

$$\operatorname{diam}_{\mathcal{C}}(\mathbf{short}(f, B)) < B' \tag{4.1}$$

for every pleated surface f, where B' depends only on $\chi(\partial H)$ and B.

Suppose f and g are as in the statement and σ_f and σ_g are the hyperbolic metrics induced on ∂H. After possibly precomposing g with a self-homeomorphism of ∂H homotopic to identity, we can assume that f and g are homotopic via a homotopy that fixes image of β point-wise. As we showed in proving part (c), by assuming $d > D_0$ we can make sure that both $f(\partial H)$ and $g(\partial H)$ have distance more than D_0 from Γ_N, where D_0 is the constant in lemma 3.3.

Let ϵ' be a constant smaller than the Margulis constant and the injectivity radii within distance D_1 of Γ_N and let $\delta = \delta(\epsilon', \chi(\partial H))$ be the constant chosen in lemma 3.3. Suppose β meets the δ-thin part of either σ_f or σ_g, say σ_f. Then its f-image meets the δ-thin part of N and so does its g-image (since they agree). But then because of our choice of the constant δ from lemma 3.3, the pleated surface $g(\partial H)$ intersects this component of the thin part of

N in a component of its ϵ'-thin part. The images of the cores of the thin part components of σ_f and σ_g are homotopic (up to taking a power) within $N \setminus C$. By fact 3.1 these have to be homotopic in ∂H and therefore there is a simple closed curve γ that is short in both σ_f and σ_g; hence

$$\gamma \in \mathbf{short}(f, B) \cap \mathbf{short}(g, B).$$

Together with (4.1), this implies a bound on

$$\mathrm{diam}_{\mathcal{C}}(\mathbf{short}(f, B) \cup \mathbf{short}(g, B)).$$

Hence we can assume that β stays in the δ-thick part of σ_f and σ_g. By the second part of corollary 3.6, there exists a $\delta_2 > 0$ and D such that if $d_N(g(\partial H), \Gamma_N) \geq D$ and τ is a bridge arc for β in the δ-thick part of σ_f and whose σ_f-length is at most δ_2, then τ is homotopic rel endpoints to an arc of σ_g-length ϵ'.

Given this δ_2, we may construct a simple closed curve γ_{δ_2} in the ϵ'-thick part of σ_f, whose σ_f length is at most a constant L depending only on δ_2 and $\chi(\partial H)$, and which is composed of at most two arcs of β and at most two bridge arcs of length δ_2 or less (Cf. Minsky [Min01, Lem. 8.5]).

The bridge arcs can be homotoped to have σ_g-length at most ϵ', and hence γ_{δ_2} can be realized in σ_g with length at most $L + 2\epsilon'$. In each surface, this bounds its $\mathcal{C}$-distance from the curves of length B, and together with (4.1) we again obtain a bound on

$$\mathrm{diam}_{\mathcal{C}}(\mathbf{short}(f,B)\cup\mathbf{short}(g,B)).$$

Hence by simply assuming $d > D$ and choosing D_2 accordingly, we will be done. $\square$

We relax our definition of realization of a multi-curve P by considering

$$\mathbf{good}_N(P;A)$$

to denote the set of end-homotopic pleated maps $f \in \mathbf{pleat}_N$ such that

$$l_{\sigma_f}(\alpha) \leq l_N(\alpha) + A$$

for all components α of P, where $l_N(\alpha)$ is length of the geodesic representative of α if there exists one and is zero otherwise.

For a hyperbolic metric σ on ∂H and $\alpha \in \mathcal{C}_0(\partial H)$, we define $\mathbf{collar}(\alpha,\sigma)$ to be the set of points which have distance $\leq \omega(l_\sigma(\alpha))$ from the geodesic representative of α on σ, where

$$\omega := \max(\omega_0/2, \omega_0 - 1)$$

and

$$\omega_0(t) := \sinh^{-1}\left(\frac{1}{\sinh(t/2)}\right).$$

It is well known that this set is always an embedded annulus, and f $\alpha_1,\dots,\alpha_k$

are disjoint and homotopically distinct then $\mathbf{collar}(\alpha_i)$ are pairwise disjoint with a definite distance between every pair. (C.f. Minsky [Min01, Sec. 8] or Buser [Buser, Chap. 4].)

Let $f, g \in \mathbf{pleat}_N$ and let P be a curve system. We say f and g admit a (K, ϵ)-*good homotopy* with respect to P if there is a homotopy $F : \partial H \times [0, 1] \to N$ such that

(a) F_0 and F_1 are respectively the same as f and g up to precomposing with a homeomorphism of ∂H isotopic to identity.

(b) $\mathbf{collar}(P, \sigma_0) = \mathbf{collar}(P, \sigma_1)$ where σ_i denotes the metric induced by F_i for $i = 0, 1$.

(c) The metrics σ_0 and σ_1 are locally K-bi-Lipschitz outside $\mathbf{collar}(P, \sigma_0)$.

(d) Suppose P_0 denotes the subset of P consisting of curves α with $l_N(\alpha) < \epsilon$. The tracks $F(p \times [0, 1])$ are bounded in length by K when $p \notin \mathbf{collar}(P_0, \sigma)$.

(e) For each $\alpha \in P_0$, the image $F(\mathbf{collar}(\alpha, \sigma_0) \times [0, 1])$ is contained in a K-neighborhood of the Margulis tube $T_\alpha(\epsilon)$.

Similar to Minsky [Min01, Lem. 4.1], we have:

Lemma 4.4. (Homotopy bound) *Given A and $\epsilon > 0$, there exists $K = K(A, \epsilon)$ and $D > 0$ so that for any hyperbolic structure N on the handlebody H, a useful compact core $C \subset N$ and a maximal curve system P on ∂H, if*

$$f, g \in \mathbf{good}_N(P; A)$$

and $d_N^{\geq \epsilon}(f(\partial H), C) \geq D$ *then* f *and* g *admit a* (K, ϵ)*-good homotopy with respect to* P *that stays outside of* C*.*

Sketch of proof. Without loss of generality, we assume $\epsilon \leq \epsilon_0$. Suppose α is a component of P and $f(\alpha)$ is compressible in N. Then $l_N(\alpha) = 0$ and therefore $l_{\sigma_f}(\alpha) \leq A$. Then using lemma 3.2, we can see that $f(\alpha)$ is not compressible by assuming that $d_N(f(\partial H), C) > A$. On the other hand, using lemma 2.11, one can see that the distance from $f(\alpha)$ to the geodesic representative of α or its ϵ-Margulis tube, if it is short, is uniformly bounded depending on A and ϵ. Hence, we can assume that $d_N^{\geq \epsilon}(\alpha^*, C)$ is large by making D large. In particular, then we can use lemma 4.2 and see that there exists some $h \in \mathbf{pleat}_N(P, A)$ which is homotopic to f within $N \setminus C$. (In lemma 4.2, we had a marked structure but here we can simply choose an embedding homotopic to f within $N \setminus C$ and use that as our marking. In fact, by increasing D we can assume that $h(\partial H)$ has large distance from C.

From here we sketch Minsky's proof and point out the modifications in our case. First, we can replace g with $g \circ h$, where $h : \partial H \to \partial H$ is a homeomorphism isotopic to identity, such that $\mathbf{collar}(P, \sigma_f) = \mathbf{collar}(P, \sigma_g)$ (which from now on we call just $\mathbf{collar}(P)$), and σ_f and σ_g are locally K-bi-Lipschitz off $\mathbf{collar}(P)$, and have bounded additive length distortion on $\partial\mathbf{collar}(P)$, with K depending only on A (see Minsky [Min01, Lem. 8.2]).

Then define $F : \partial H \times [0, 1] \to N$ to be the homotopy between f and g whose tracks $F|_{\{x\} \times [0,1]}$ are geodesics parametrized at constant speed and we will bound tracks of F on successively larger parts of the surface.

Let Y be a component of $\partial H \setminus P$ and let $Y_0 = Y \setminus (\mathbf{collar}(\partial Y))^\circ$, where

$(\mathbf{collar}(\partial Y))^\circ$ denotes the interior of $\mathbf{collar}(\partial Y)$. Note that length of any boundary component of Y_0 is at most 2 more than its corresponding geodesic in ∂H and we have

$$l_\sigma(\gamma) \leq l_N(\gamma) + A + 2 \tag{4.2}$$

for $\sigma = \sigma_f$ or σ_g.

Let $\epsilon_1 = \delta(\epsilon, \chi(\partial H))$ and suppose $D \geq D_0$ where $\delta(\epsilon, \chi(\partial H))$ and D_0 are the constants obtained in lemma 3.3. Consider the lift $\widetilde{F} : \widetilde{\partial H} \times [0,1] \to \mathbb{H}^3$ where $\widetilde{\partial H}$ denotes the cover of ∂H associated to $\mathrm{Ker}(f_* : \pi_1(\partial H) \to \pi_1(N))$.

In [Min01], Minsky considers an *essential tripod* on every hyperbolic pair of pants Y. This is obtained by taking a copy of the 1-complex Δ obtained from three disjoint copies of $[0,1]$, called "legs", by identifying the three copies of $\{0\}$. The three copies of $\{1\}$ are called the boundary of Δ. An essential tripod in Y_0 is an embedding of Δ taking $\partial\Delta$ to ∂Y_0, such that each subarc of Δ obtained by deleting on copy of $(0,1]$ is not homotopic rel endpoints into ∂Y_0. Minsky [Min01, Lem. 8.1] proves the following:

Lemma 4.5. *There exists a constant δ such that for any hyperbolic pair of pants Y and each boundary component γ of Y_0, there is an essential tripod $\Delta \subset Y_0$ whose three legs have length at most δ and which meets γ. On the other hand, there exists $\delta' > 0$ such no essential tripod in Y_0 has all three legs of length less than δ'.*

Use the above lemma and consider an essential tripod Δ in Y_0 whose length have σ_f-length at most δ and let X be a component of the preimage of Δ in $\widetilde{\partial H}$.

First we need to show that there is a uniform bound on the lengths of the tracks $\widetilde{F}(\{x\} \times [0,1])$ for $x \in X$.

The image of $\widetilde{F}(X \times \{0\})$ connects three lifts of boundary curves of Y_0, each invariant by a primitive deck translation γ_k, $k = 1,2,3$, in $\pi_1(N)$ considered as a subgroup of isometries of $\mathbb{H}^3$. Let L_k be the axis of γ_k. Notice that L_k and L_l, $k \neq l$, cannot be identified because then the third isometry has to be trivial and this contradicts the fact that components of $f(P)$ are incompressible in N. If γ_k has translation length less than $\epsilon_1/2$, define N_k to be the lift of the corresponding ϵ-Margulis tube in N. Otherwise define $N_k = L_k$.

It follows from [Min01, Lem. 8.4] and (4.2) that the homotopy between image of each boundary curve of Y_0 (under f or g) and its geodesic representative in N can be made in uniformly bounded distance to reach either the geodesic or its ϵ-Margulis tube if it is short. Lifting to the covers, we conclude that the endpoints of $\widetilde{F}(X \times \{i\}), i = 0,1$ are within bounded distance of the corresponding N_k.

Since we have a uniform diameter bound $2K\delta$ on $\widetilde{F}(X \times \{0\})$ and $\widetilde{F}(X \times \{1\})$, we find that

$$\widetilde{F}(X \times \{i\}) \subset \mathcal{N}_{d_1}(N_1) \cap \mathcal{N}_{d_1}(N_2) \cap \mathcal{N}_{d_1}(N_3) \tag{4.3}$$

for $i = 0,1$ with a uniform d_1.

Now similar to Minsky, we claim that the triple intersection above cannot have very large diameter. If two of the L_ks are Margulis tubes this follows from their strict convexity.

Recall that we picked a pleated surface $h \in \mathbf{pleat}_N(P)$. Suppose N_1 corresponds to a curve of length less than $\epsilon_1/2$. Let $N_k(\epsilon_1) \subset N_k$, $k = 1,2,3$, denote a lift of the corresponding ϵ_1-Margulis tube. We claim that L_2 is disjoint from $N_1(\epsilon_1)$. Otherwise in (by our choice of ϵ_1) σ_h the geodesic representative of γ_2 enters the ϵ-Margulis tube associated to γ_1. By our choice of $\epsilon < \epsilon_0$, a simple geodesic cannot meet an ϵ-Margulis tube in a surface unless it is the core of the tube or it intersects the core of the Margulis tube, which is a contradiction. Knowing this we can argue as in the previous case and get a bound on diameter of $\mathcal{N}_{d_1}(N_1) \cap \mathcal{N}_{d_1}(N_2)$.

Finally suppose all three γ_i have translation length at least $\epsilon_1/2$. Then the three axes $\{L_k\}$ come within bounded distance $2d_2$ for some constant d_2. If the intersection of all three $\mathcal{N}_{d_2}(L_k)$ has diameter L then L_1, L_2 and L_3 contain segments of length at least $L - 2d_2$ that remain distance d_2 apart. Therefore, there are two a-prior constants $d_3, b > 0$ such that there exists a point $p \in \mathbb{H}^3$ which is at most $\epsilon(L) = d_3 e^{-bL}$ from all three L_k, and so that the tangent directions to L_k at the points x_k closest to p are at most $2\epsilon(L)$ apart in $\mathbf{P}\mathbb{H}^3$.

We can extend Δ to a tripod Δ' with endpoints in ∂Y (after lifting to $\widetilde{\partial H}$ and applying $\widetilde{h}$) map to x_k. Note that each pair of legs of Δ' is a primitive bridge arc for ∂Y, whose h-image is homotopic rel endpoints to an arc of length at most $2\epsilon(L)$.

Using lemma 3.6 (Short bridge arcs) suppose $\delta_2 = \delta'$ is the constant in lemma 4.5 and consider δ_1 which is obtained there. Also suppose D is large enough that h is further than the constant in lemma 3.6 from C.

If L is sufficiently large that $2\epsilon(L) < \delta_1$, then each pair of legs of Δ' is homotopic to an arc of length at most δ_1 in σ_h. This gives a triangle in Y with the same vertices as endpoints of Δ' whose side lengths are at most δ'. Joining the barycenter to its vertices we obtain a new tripod Δ'' all of whose legs are bounded by δ' which contradicts lemma 4.5. We conclude that the triple intersection of (4.3) has diameter at most L.

The diameter bound and (4.3) now gives a uniform bound on the track lengths of $\widetilde{F}$ restricted to X and hence F restricted to Δ.

The rest of the argument is exactly the same as Minsky's [Min01, Lem. 4.1] where we refer the reader.

□

Lemma 4.6. (Halfway surface) *Given ϵ, there exist constants A_1 and D such that given pants decompositions P and Q on ∂H which differ by an elementary move and a hyperbolic structure N on H with a useful compact core C then*

$$\mathbf{good}_N(P, A_1) \cap \mathbf{good}_N(Q, A_2) \neq \emptyset$$

if there exists an end-homotopic $f \in \mathbf{pleat}_N(P)$ with

$$d_N^{\geq \epsilon}(f(\partial H), C) \geq D.$$

Proof. In [Min01, Lem. 4.2] Minsky considers a finite leaved lamination μ whose recurrent part is $P \cap Q$. To be able to use lemma 4.2, assume we have changed the marking of N to one which is homotopic to f within $N \setminus C$. If $d_N^{\geq \epsilon}(f(\partial H), C)$ is bigger than the constant in lemma 4.2, since $f(P \cap Q)$ is a geodesic representative of $P \cap Q$ it follows that there exists a pleated surface $h \in \mathbf{pleat}_N$ that realizes μ. Because f and h have the same image restricted to $P \cap Q$, and $\mathrm{diam}_N^{\geq \epsilon}(h(\partial H))$ is bounded depending only on ϵ and $\chi(\partial H)$, by assuming that D is large, we can guarantee that $d_N(h(\partial H), C)$ is bigger than the constant in theorem 3.7 (Efficiency of pleated surfaces).

Suppose $\alpha_0 \in P$ and $\alpha_1 \in Q$ are curves that are exchanged by the elementary move. Similar to Minsky, we can see that

$$a(\mu, \alpha_i) \leq 4$$

for $i = 0, 1$. Hence by using theorem 3.7 and noticing that α_1 and α_2 do not intersect any closed leaf of μ, we have

$$l_{\sigma_h}(\alpha_i) \leq l_N(\alpha_i) + A$$

for $i = 0, 1$ and a uniform A. Thus $h \in \mathbf{good}_N(P, A) \cap \mathbf{good}_N(Q, A)$ and the lemma is proved. □

Using the above lemmas, we can prove the following corollary.

Corollary 4.7. *Given ϵ there exists $D > 0$ and $K > 0$ such that for a hyperbolic structure N on H and a useful compact core C the following holds. Let $P_0 \to P_1 \to \cdots \to P_n$ be an elementary-move sequence of pants decompositions on ∂H and let $f_0 \in \mathbf{pleat}_N(P_0)$ be end-homotopic then there exists some $0 \leq k \leq n$ and $F : \partial H \times [0, k] \to N \setminus C$ such that*

- $F_0 = f_0$,
- $F_i = F|_{\partial H \times \{i\}} \in \mathbf{pleat}_N(P_i)$,
- $F_{i-1/2} = F|_{\partial H \times \{i-1/2\}} \in \mathbf{pleat}_N(P_{i-1}) \cap \mathbf{pleat}_N(P_i)$ *and*
- *F is a (K, ϵ)-good homotopy restricted to $\partial H \times [i-1, i-\frac{1}{2}]$ and $\partial H \times [i-\frac{1}{2}, i]$ with respect to P_{i-1} and P_i respectively*

for every $i = 1, \ldots, k$. *Moreover, if* $k \neq n$ *then* $d_N^{\geq \epsilon}(F_k(\partial H), C) < D$.

Chapter 5

Non-realizability and ending laminations

Assume H is a handlebody and let N be a hyperbolic structure on H. Suppose $C \subset N$ is a relative compact core homeomorphic to H, choose a marking $j : \partial H \to N \setminus C$ and as usual we assume that the component of $N \setminus J(\partial H)$ that is a neighborhood of the end does not intersect C.

As we explained in 2.12, Canary [Can93b] proved that if N is geometrically infinite then there exists an ending lamination which is not realizable in N. Recall that a geodesic lamination λ on ∂H is realizable, if there exists a pleated surface $f : \partial H \to N$ homotopic to j such that f is totally geodesic restricted to leaves of λ. He also proved that when N has no parabolics, the ending lamination is filling and in the Masur domain.

One can ask the converse question:

Question. Suppose λ is a filling lamination in the Masur domain of ∂H and λ is non-realizable in N, is λ the ending lamination of N?

Recall that the ending lamination for hyperbolic structures on H is defined only up to actions of $\mathrm{Mod}_0(H)$ and for marked structures where we have a well-defined and unique ending lamination, the translates of the ending lamination are unrealizable too. Hence in the statement of our question above we are considering λ only up to actions of $\mathrm{Mod}_0(H)$ and if N is marked the ending lamination is $\phi(\lambda)$ for a unique element $\phi \in \mathrm{Mod}_0(H)$.

Aside from being an interesting problem, to prove that given a Masur domain filling lamination λ, there exists a hyperbolic structure with ending lamination λ one needs to know a solution to the above question or a variation of that. In fact, this is what we will prove and use in chapter 6. Work of Kleineidam-Souto [KS03] answers a similar question for hyperbolic structures on a compression body that is not a handlebody.

In case of handlebodies, we think this problem was not noticed before. Here we give an affirmative answer to the above question in case of handlebodies and as we explained in the introduction, the proof is a joint work with Juan Souto. We should also point out that Ohshika [Oh] has also recently claimed an answer to the above question when N is the strong limit of a sequence of convex cocompact structures on H.

Theorem 1.3 *Suppose λ is a filling Masur domain lamination on ∂H and λ is not realized in N, where N is a hyperbolic structure on H. Then λ is the ending lamination of N (defined up to actions of* $\mathrm{Mod}_0(H)$*).*

As we mentioned in the introduction, we can use the above theorem and prove the following:

Corollary 1.4 *Given a filling Masur domain lamination λ on ∂H, there exists a hyperbolic structure on H, whose ending lamination (defined up to actions of* $\mathrm{Mod}_0(H)$*) is λ.*

Proof. Choose a sequence of marked convex cocompact structures (N_i) whose conformal structure at infinity (τ_i) converges to a point supported on λ in Thurston's compactification of Teichmüller space, which we denote by λ as well. Let $\rho_i : \pi_1(H) \to \mathrm{PSL}_2(\mathbb{C})$ be the representation associated to N_i: N_i is isometric to $\mathbb{H}^3/\rho_i(\pi_1(H))$.

In [KS02], Kleineidam and Souto proved that a sequence of convex cocompact structures whose conformal structures at infinity converge to a filling lamination in Masur domain has an algebraically convergent subsequence (up to conjugation). This shows that up to passing to a subsequence and conjugation the sequence (ρ_i) converges algebraically to a representation $\rho : \pi_1(H) \to \mathrm{PSL}_2(\mathbb{C})$.

By recent proof of the Tameness Conjecture (Agol [Ag], Calegari-Gabai [CG] and Brock-Souto [BS]), we know that $N = \mathbb{H}^3/\rho(\pi_1(H))$ is topologically tame and is homeomorphic to the interior of a compact 3-manifold. An elementary argument shows that N has to be homeomorphic to the interior of H (cf. Hempel [He86]).

Using the above theorem, it is enough to show that λ is not realized in N. This would follow immediately if we had continuity of the length function for these structures similar to Brock [Br00]. Here in lack of such statement, we assume λ is realized in N and seek a contradiction.

By convergence of (τ_i) to λ, there exists a sequence of simple closed curves (γ_i) in Masur domain which converge to λ in $\mathcal{PML}$ and

$$l_{\tau_i}(\gamma_i)/l_{\tau_0}(\gamma_i) \to 0, \text{ as } i \to \infty. \tag{5.1}$$

From convergence of τ_i to λ, we know that there exists a sequence of measured laminations $(\mu_i) \in \mathcal{ML}(\partial H)$ and $K > 0$, such that for any closed curve m on ∂H

$$i(\mu_i, m) \leq l_{\tau_i}(m) \leq i(\mu_i, m) + K l_{\tau_0}(m) \tag{5.2}$$

such that $l_{\tau_i}(\mu_i)$ stays bounded and the sequence (μ_n) converges to λ in $\mathcal{PML}$. We claim that given $A > 0$ there exists $i_A > 0$ such that $l_{\tau_i}(m) \geq A$ for every $i \geq i_A$ and every meridian m. Otherwise, there exist compressible curves (m_i) with length less than A for arbitrary large i. Suppose the sequence $(m_i/l_{\tau_0}(m_i))$ converges (up to subsequence) to $\mu \in \mathcal{ML}$ that is in the closure of set of meridians. Then $i(\lambda, \mu) = 0$ which contradicts our assumption that λ is in the Masur domain.

In this situation, Canary [Can91] proved that the lengths of curves in the conformal structure at infinity of N_i gives an upper-bound for their length inside N_i, up to a multiplicative error that depends on A. In other terms, there exists $c > 0$ such that for all $i \geq i_A$

$$l_{N_i}(\gamma_i) \leq c l_{\tau_i}(\gamma_i) \tag{5.3}$$

where $l_{N_i}(\gamma_i)$ is the length of the geodesic representing $\rho_i(\gamma_i)$ in N_i.

Let $\bar{\lambda}$ be the Hausdorff-limit of the sequence (γ_i). The geodesic lamination $\bar{\lambda}$ contains λ as a sublamination and $\bar{\lambda} \backslash \lambda$ consists of a finite number of biinfinite geodesics. Seeking a contradiction, assume λ is realized in N. Then $\bar{\lambda}$ is also realized in N by work of Otal [Ota88] (cf. Kleineidam-Souto [KS03, Lem. 4.2]). Suppose $f \in \mathbf{pleat}_N$ realizes $\bar{\lambda}$. This implies that for all $i \gg 0$ the curve

$f(\gamma_i)$ is nearly geodesic and in particular we have $l_N(\gamma_i) \asymp l_{\sigma_f}(\gamma_i)$, i.e. there is a constant $c_1 > 1$ with

$$c_1^{-1} l_N(\gamma_i) \leq l_{\sigma_f}(\gamma_i) \leq c_1 l_N(\gamma_i)$$

for all $i \gg 0$.

Since (ρ_i) converges to ρ algebraically, on the level of manifolds there are smooth homotopy equivalences $h_i : N \to N_i$, compatible with ρ_i and ρ, such that on any compact subset of N, h_i tends C^∞ to a local isometry for all $i \gg 0$. (Cf. McMullen [McM].)

Therefore for large i, the curve $h_i(f(\gamma_i))$ has small geodesic curvature and we have

$$l_{N_i}(\gamma_i) \asymp l_N(\gamma_i) \asymp l_{\sigma_f}(\gamma_i).$$

But $l_{\sigma_f}(\gamma_i) \to \infty$ as $i \to \infty$. This contradicts (5.3) and we have proved that λ is not realized. □

Now we start proof of theorem 1.3 which takes the rest of this chapter. For now on we assume N, marking $j : \partial H \to N$ and the relative compact core $C \subset N$ are fixed and suppose λ is not realized in N.

Recall that for a hyperbolic structure N on a handlebody H a lamination $\lambda \in \mathcal{UML}(\partial H)$ is an ending lamination if there exists a sequence of simple closed curves α_i on ∂H which converge to a lamination supported on λ in $\mathcal{PML}$, such that $j(\alpha_i)$ is homotopic to a closed geodesic α_i^* within $N \setminus C$ and (α_i^*) exits the end of N. In our case, non-realizability of λ gives us a sequence of closed geodesics (and in fact pleated surfaces which realize them) that exit the end and they have representatives on $j(\partial H)$ which converge to a lamination supported on λ; but the homotopy between these and $j(\partial H)$ may pass through the core and the difficulty is to prevent that from happening. Equivalently, we have a sequence of pleated surfaces that exit the end but we cannot use them since they are not necessarily end-homotopic.

Here our idea is to use the fact that λ is a Masur domain filling lamination and seek a contradiction using topological arguments.

Otal [Ota88] extended Thurston's construction of pleated surfaces to the compressible boundary case and proved that if N has no parabolics then every multi-curve α in the Masur domain is realized in N. Kleineidam-Souto [KS03] generalized this and proved that if a multi-curve α in the Masur domain intersects every simple closed curve γ on ∂H where $j(\gamma)$ represents a parabolic in N, then α is realized in N. If α satisfies the above condition and a lamination μ is union of support of α and finitely many noncompact leaves, then is not hard to see and can be deduced from arguments of Otal [Ota94] that μ is realized too.

Using this we can prove the following:

Fact 5.1. *Suppose a sequence of multi-curves γ_n converges to a lamination supported on λ then for n sufficiently large, γ_n and every lamination μ that is a union of γ_n and finitely many noncompact leaves is realized in N.*

Proof. Using what we just said, it is enough to prove that for n sufficiently large γ_n intersects every simple closed curve γ, where $j(\gamma)$ represents a parabolic in N. If this is not the case then, up to taking a subsequence, we can assume for every n there exists a simple closed curve δ_n disjoint form γ_n such that $j(\delta_n)$

represents a parabolic in N.

After passing to a further subsequence, assume the sequence (δ_n) converges to λ' in $\mathcal{PML}$. It should be obvious that $i(\lambda', \lambda) = 0$ and since λ is filling, λ' and λ have the same support and λ' is in the Masur domain $\mathcal{O}(H)$. Since the Masur domain is open, δ_n is in the Masur domain too, for $i \gg 0$.

On the other hand by Sullivan's finiteness theorem [Sul81], there are only a finite number of free homotopy classes of parabolics in N. Hence we may extract a subsequence such that for all $i \geq i_0$, δ_i and δ_{i_0} are freely homotopic in N. Then by theorem 2.3 the curves δ_i are in a single orbit of $\mathrm{Mod}_0(H)$ and they are bounded in $\mathcal{O}(H)$ which contradicts theorem 2.2. □

Choose a sequence of multi-curves (α_n) that converges to a lamination supported on λ. Using the above suppose $f_n : \partial H \to N$ realizes α_n for every $n \gg 0$. The next proposition directly follows from work of Kleineidam-Souto [KS03, Prop. 4.3].

Proposition 5.2. *A filling Masur domain lamination λ is realized in N if there is a sequence (α_n) of multi-curves converging to λ and a compact set $K \subset N$ such that α_i is realized by a pleated surface $f_i : \partial H \to N$ with $f(\partial H) \cap K \neq \emptyset$ for all i.*

They proved this by showing that for the sequence of pleated surfaces (f_i) in the hypothesis, there exists a uniform diameter bound. Then up to passing to a subsequence, the pleated surfaces converge to a pleated surface and they prove that this limit pleated surface realizes λ.

From this, we conclude that the sequence of maps (f_n) *exits the end of* N, i.e. every compact subset of N intersects at most a finite number of pleated surfaces $f_n(\partial H)$.

Claim 5.3. *We can assume, perhaps after considering sufficiently large indexes, that f_n and f_m are homotopic in $N \setminus C$ for every m and n.*

Proof. Using Klarreich's work [Kla], we know that λ represents a point on the Gromov boundary of the curve complex of ∂H and α_n converges to this point in sense of Gromov. Also we can assume that the sequence (α_n) is an infinite path in the curve complex, i.e. $d_{\mathcal{C}}(\alpha_n, \alpha_{n+1}) = 1$ for every n. Then we can extend it to an elementary-move sequence of pants decompositions (P_n) as in lemma 2.7. Notice that still every limit of the sequence (P_n) in $\mathcal{PML}$ is supported on λ.

Now for $n \gg 0$ choose a maximal lamination μ_n that contains P_n as a sublamination and such that all noncompact leaves of μ_n that approach a component γ of P_n (from either side) spiral about γ in the same direction. Fact 5.1 shows that μ_n is realized for $n \gg 0$. Suppose f_n realizes μ_n and f_{n+1} realizes μ_{n+1}. It will be enough to prove that for n sufficiently large, there exists a homotopy between f_n and f_{n+1} that stays away from C.

Suppose $\alpha, \alpha_1, \ldots, \alpha_k$ and $\beta, \alpha_1, \ldots, \alpha_k$ are components of P_n and P_{n+1} respectively and let $Y \subset \partial H$ be the closure of a component of $\partial H \setminus \{\alpha_1, \ldots, \alpha_k\}$ that contains α and β. We know that Y is either a 4-holed sphere or a 1-holed torus.

First we construct a triangulation $\mathcal{T}$ on ∂H as follows. Restricted to Y, we assume that $\mathcal{T}$ is one of the triangulations in figure 5.1. Then extend this to a triangulation for the entire ∂H in a way that all the vertices are on components of $P_n \cap P_{n+1} = \{\alpha_1, \ldots, \alpha_k\}$.

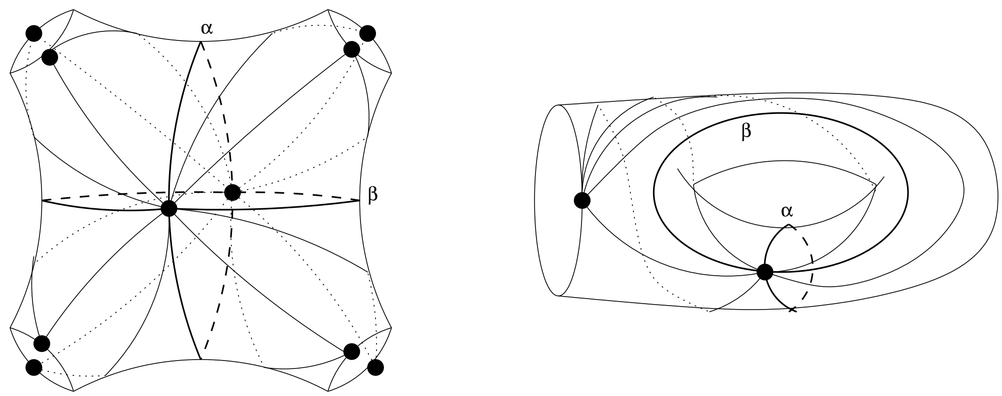

Figure 5.1: The triangulation on Y.

What is important for us about this triangulation is that P_n and P_{n+1} are homotopic to subgraphs of the 1-skeleton of $\mathcal{T}$ and every triangle on Y has at least one vertex on ∂Y.

For any choice for images of vertices of $\mathcal{T}$ on geodesic representative of P_n (resp. P_{n+1}) that preserves their ordering, we can construct a simplicial hyperbolic surface with associated triangulation $\mathcal{T}$ that realizes P_n (resp. P_{n+1}). The construction is standard, simply make the map identical to f_n (resp. f_{n+1}) restricted on P_n (reps. P_{n+1}). Change the triangulation by an ambient isotopy such that the vertices get mapped to the chosen points on the image of P_n (resp. P_{n+1}). Extend this to the 1-skeleton of $\mathcal{T}$ by sending each edge to a geodesic in the homotopy class of its f_n (resp. f_{n+1}) image (rel. endpoints). Finally extend the map to the entire surface by mapping the 2-simplices totally geodesically.

Using an idea of Thurston, we can construct a continuous family g_n^t (resp. g_{n+1}^t of simplicial hyperbolic surfaces as above that converge to f_n (resp. f_{n+1}) in Hausdorff topology. This is possible by starting from one such map g_n^0 (resp. g_{n+1}^0) and for each component γ of P_n (resp. P_{n+1}) continuously twist the images of vertices on γ about the geodesic representative of γ in the direction that noncompact leaves of μ_n (resp. μ_{n+1}) spiral about γ when approaching γ and then construct the simplicial hyperbolic surface g_n^t (resp. g_{n+1}^t) as above. (Here we are assuming that the image of P_n (resp. P_{n+1}) is fixed and we are twisting the vertices of the triangulation about the components.)

Notice that a maximal lamination containing P_n (resp. P_{n+1}) is identified by the direction that its noncompact leaves spiral about components of P_n (resp. P_{n+1}) on each side. Hence the limit of the simplicial hyperbolic surfaces g_n^t (resp. g_{n+1}^t) has to be a pleated surface that realizes μ_n and since μ_n is maximal it has to be identical with f_n (resp. f_{n+1}) up to precomposition with a self-homeomorphism of ∂H isotopic to the identity. From here we can see that when t is large there is a homotopy with bounded length tracks between f_n and g_n^t (resp. f_{n+1} and g_{n+1}^t). Fix t large such that this homotopy stays away from C and denote $h = g_n^t$, $h' = g_{n+1}^t$

It will be enough to show the existence of a homotopy between h and h' whose image is contained in a uniformly bounded neighborhood of $h(\partial H) \cup h'(\partial H)$. First of all, we precompose h or h' with a self-homeomorphism of ∂H isotopic to identity to make h and h' identical restricted to $P_n \cap P_{n+1}$. We know that there is a homotopy between h and h' and we can consider this as a map from $\partial H \times [0,1] \to N$, where restricted to $\partial H \times \{0\}$ and $\partial H \times \{1\}$ the map induces h and h'. The simplicial structure of h and h' makes $\partial H \times \{0,1\}$ triangulated with two triangulations which are isotopic to $\mathcal{T}$ on ∂H. Extend this to a triangulation of $\partial H \times [0,1]$ first connecting every vertex on $\partial H \times \{0\}$ to the corresponding vertex on $\partial H \times \{1\}$. Then add faces homeomorphic to rectangles where two opposite sides of the rectangle are corresponding edges of the triangulations on $\partial H \times \{0\}$ and $\partial H \times \{1\}$. Finally we are left with regions that are homeomorphic to a triangle times an interval, we call them *prisms*, and simply divide each of these to 3 tetrahedral arbitrarily. Now we can assume that the homotopy is totally geodesic restricted to the 1-skeleton and 2-skeleton of the constructed triangulation and extend it to the 3-skeleton (the prisms) by coning of from a vertex of each tetrahedral and map every line segment geodesically.

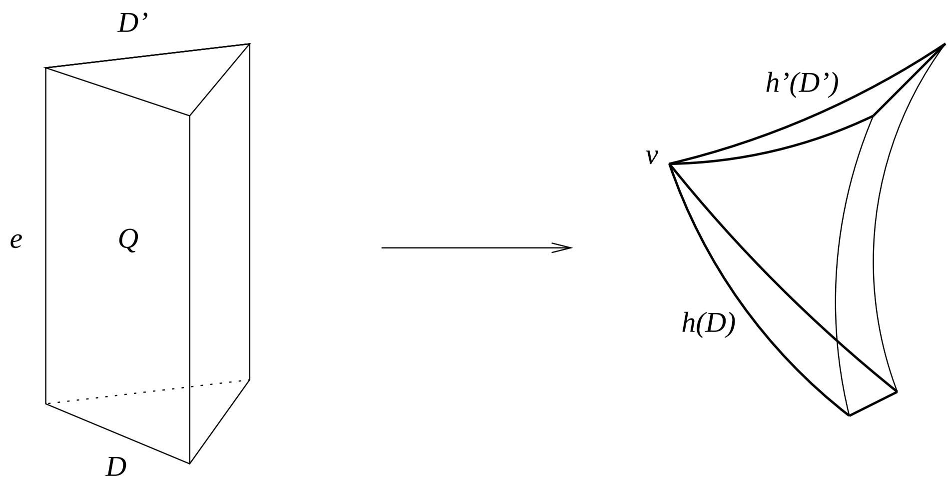

Figure 5.2: The image of a prism.

It will be enough to show that image of every prism stays in a bounded diameter neighborhood of $h(\partial H) \cup h'(\partial H)$. In fact it is enough to do this for faces of the prism. Every prism Q has two triangular faces D and D' which we call them *horizontal* and we call the other faces and edges that connect these horizontal faces *vertical*. The image of horizontal faces are contained in $h(\partial H) \cup h'(\partial H)$. In our construction of the triangulation $\mathcal{T}$, each triangle has at least one vertex on a component of $P_n \cap P_{n+1}$ and the image of the vertical edge e associated to this vertex will be a single point v on the geodesic representative of $P_n \cap P_{n+1}$. The picture of image of a prism is suggested in figure 5.2.

Every point in the image of the prism is contained in a triangle with a vertex v and two sides on $h(D)$ and $h'(D')$ and from this and hyperbolicity of N we can see that it has to have bounded distance from $h(D) \cup h'(D')$ and therefore from $h(\partial H) \cup h'(\partial H)$ and we are done.

□

Note that every $\alpha_n^* := f_n(\alpha_n)$ is homotopic to a closed curve $j(\beta_n)$ in $N \setminus C$, because $j : \partial H \to N \setminus C$ is a homotopy equivalence. Then because the sequence α_n^* exits the end and the homotopy stays outside of the compact core C, the sequence of closed curves (β_n) has to converge in the space of currents to a lamination supported on μ: an ending lamination for some end of N. (See 2.12.) On the other hand, because maps (f_n) are homotopic outside of C and they are homotopic to the inclusion $\partial H \hookrightarrow H$ in H, there exists a single map $g : \partial H \to \partial H$ that extends to a map $\overline{g} : H \to H$ homotopic to identity and $g(\alpha_n)$ is homotopic to β_n in ∂H for every n.

Remark 5.1. Note that if g is homotopic to a homeomorphism, since g extends to the handlebody, it has to be in $\mathrm{Mod}_0(H)$. Then $\mu = g(\lambda)$ is the ending lamination and we have nothing more to prove. So it will be enough to prove that g is homotopic to a homeomorphism on ∂H.

Without loss of generality, we can assume that the sequence (α_n) converges to a lamination $\bar{\lambda}$ in the Hausdorff topology. Note that $\bar{\lambda}$ contains λ as a sublamination.

Fix a marked convex cocompact structure N_0 on H with a marking $j_0 : \partial H \to N_0$. By a theorem of Otal [Ota88], $\bar{\lambda}$ is realized in N_0 by a map $f_0 : \partial H \to N_0$ homotopic to j_0.

Suppose ∂H is equipped with the hyperbolic metric σ_0, induced from f_0. Recall that a *train track* on ∂H is an embedded 1-complex $\tau \subset \partial H$ whose edges (called *branches*) are smooth arcs with well-defined tangent vectors at endpoints. At any vertex (called a *switch*) the incident edges are mutually tangent. The branches which are incident to a switch are divided into two nonempty subsets: "incoming" and "outgoing" branches according to their inward pointing tangent at the switch.

A geodesic lamination ν is *carried* by the train track τ, if there is a C^1-map $F : \partial H \to \partial H$ which is isotopic to the identity and maps ν to τ in such a way that the restriction of its differential dF to every tangent line of ν is non-singular. We say τ *fully carries* ν if the image of ν by F in the above definition surjects to τ.

Here for $\bar{\lambda}$, we consider a family of special train tracks which carry $\bar{\lambda}$ and are obtained in the following way. Fix a point x_0 on $\lambda \subset \partial H$ throughout the construction. Take a transversal κ to $\bar{\lambda}$ of some definite length centered at x_0. For $\epsilon > 0$ small take a subarc I_ϵ of κ of length 2ϵ centered at x_0. Now we squeeze all intersections of I_ϵ with $\bar{\lambda}$ to x_0 using a smooth map $\partial H \to \partial H$ which preserves x_0 and the tangent to λ at x_0 and collapses I_ϵ to x_0. After doing this we get a smooth embedded graph in ∂H with a single vertex x_0. Some of loops of this graph may be parallel, we identify all these. Then we have a set of simple loops and it is not hart to see that the number of loops is bounded depending on the genus of ∂H. This gives a train track τ_ϵ and it should be obvious that τ_ϵ fully carries λ. In fact our process gives a well-defined map from λ to the train track τ_ϵ for every ϵ, which we refer to it as the *projection map*. Notice that if a positive $\epsilon' < \epsilon$ is given the train track $\tau_{\epsilon'}$ is also carried by τ_ϵ (it can be isotoped to a small neighborhood of τ_ϵ).

There is a single leaf of λ which passes through x_0 and we call it *exceptional*; in some part of our argument we exclude this leaf. Now we claim that if ϵ is sufficiently small, all branches of τ_ϵ will have length at least L for a given constant L. If not then we have a sequence of subarcs of λ with length bounded and endpoints ϵ_i-close to each other, where $\epsilon_i \to 0$ as $i \to \infty$. But if we consider the Hausdorff limit of these subarcs as $i \to \infty$, it is either a simple closed geodesic or a monogon. Neither of these two can be a sublamination of λ and we have a contradiction. (λ is not disjoint from any simple closed curve or monogon.)

In fact using the above claim, it is not hard to see that given $\delta > 0$, if $\epsilon > 0$ is small, f_0 *δ-realizes* τ_ϵ in N_0. With a slight modification to Otal's definition [Ota96], we say a continuous map $F : \partial H \to N_0$, δ-realizes τ in N_0 if:

(i) Restricted to every branch, F is injective and its image is a $(1+\delta, \delta)$-quasigeodesic in N.

(ii) If two branches b_1 and b_2 lie on opposite sides of the same switch v, then the angle between the tangent vectors to the geodesics $F(b_1)$ and $F(b_2)$ at their intersection point $F(v)$ is in the interval $[\pi - \delta, \pi]$.

(In our definition is hidden that the angle in (ii) should be well-defined.) Notice that f_0 and $f_0 \circ g$ are homotopic in N_0 where g is the map defined after proof of claim 5.3. Suppose $F : \partial H \times [0, 1] \to N_0$ is this homotopy.

Claim 5.4. *There exist constants $\epsilon_1 > 0$, K and c such that for every $\epsilon \leq \epsilon_1$, the image of every route of τ_ϵ by g is a (K, c)-quasigeodesic in σ_0.*

Proof. Suppose ϵ is small and fix a route of τ_ϵ. Also assume β is a long closed curve carried along this route. Then $f_0(\beta)$ is nearly geodesic and therefore its length is nearly the smallest among the closed curves in its free homotopy class in N_0. In particular, since $f_0(g(\beta))$ is freely homotopic to $f_0(\beta)$, its length has to be bigger than $l_0(\beta)(1-\epsilon')$ where ϵ' is small depending on ϵ and l_0 denotes the length in σ_0. On the other hand g has bounded Lipschitz constants and increases the length in bounded proportion and therefore length of $g(\beta)$ is at most a uniform constant times length of β.

This shows that $g(\beta)$ is a quasigeodesic with constants that depend on ϵ. In fact the constants only become better if we choose a smaller ϵ. This argument shows that images of larger and larger subsegments of a route of τ_ϵ are quasigeodesics with the same constants. From here a standard argument shows that the image of the entire route has to be a quasigeodesic with uniform constants. $\square$

Now consider $\widetilde{g} : \mathbb{H}^2 \to \mathbb{H}^2$ the lift of g to the universal covers and assume $\widetilde{\lambda}$ is the lift of λ. The above claim shows that $\widetilde{g}(l)$ is a quasigeodesic for every leaf l of $\widetilde{\lambda}$. Hence $\widetilde{g}(l)$ uniquely determines a geodesic in $\mathbb{H}^2$. We consider $\bar{g}$ to be the map from leaves of $\widetilde{l}$ to geodesics in $\mathbb{H}^2$ induced by g.

Lemma 5.5. *If l_1 and l_2 are leaves of $\widetilde{\lambda}$ then $\bar{g}(l_1)$ and $\bar{g}(l_2)$ do not intersect. Also if l_1 and l_2 are asymptotic leaves of $\widetilde{\lambda}$ then their $\bar{g}$-images are asymptotic geodesics.*

Proof. Suppose $\bar{g}(l_1)$ and $\bar{g}(l_2)$ intersect. Equivalently, this means that the end points of the quasi geodesics $\widetilde{g}(l_1)$ and $\widetilde{g}(l_2)$ at infinity separate each other (they alternate). Without loss of generality we can assume neither of these leaves project to the exceptional leaf of λ. The argument is not hard and basically follows from the fact that a single leaf of $\widetilde{\lambda}$ is an accumulation of lifts of any leaf of λ. Choose lifts of unexceptional leaves which are sufficiently close to l_1 and l_2. Since g is Lipschitz and l_1 and l_2 intersect these new leaves have to intersect too.

Using the fact that l_i, $i = 1, 2$, is not exceptional, we can see that for any subsegment $\kappa_i \subset l_i$, if ϵ is sufficiently small, image of κ_i by the projection map will be contained in a single branch b_i of τ_ϵ. Using this and the fact that $g(l_i)$ is a quasigeodesic, we can assume that images of endpoints of κ_i are close to the endpoints of $g(l_i)$. Suppose a route γ_i, $i = 1, 2$, of τ_ϵ contains b_i; since $\widetilde{g}$ image of γ_i is a (K, c)-quasigeodesic, its endpoints cannot be far from the endpoints of $\widetilde{g}(l_i)$ and therefore $\widetilde{g}(\gamma_1)$ and $\widetilde{g}(\gamma_2)$ intersect each other.

Now assume n is sufficiently large, so that α_n is carried by τ and assume $b_1(\alpha_n)$ and $b_2(\alpha_n)$ denote the number of times that α_n passes through b_1 and b_2 respectively. Note that by what we just said, we can see that

$$i(\beta_n, \beta_n) \geq b_1(\alpha_n) \cdot b_2(\alpha_n)$$

for $n \gg 0$. (Recall that $\beta_n = g(\alpha_n)$ was a closed curve and the sequence (β_n) converges to a lamination supported on μ in the space of currents.)

Because the sequence (α_n) converges in $\mathcal{PML}$ to λ and λ passes through

b_1 and b_2,

$$\frac{b_1(\alpha_n)}{l_0(\alpha_n)} \text{ and } \frac{b_2(\alpha_n)}{l_0(\alpha_n)}$$

converge to positive numbers (the measure deposited on b_1 and b_2 by λ) and are bigger than some $c > 0$ for $n \gg 0$. On the other hand, like in the proof of the previous claim, because g has bounded Lipschitz constants, there exists $c' > 0$ such that

$$l_0(\beta_n) \leq c' l_0(\alpha_n).$$

Hence

$$\frac{i(\beta_n, \beta_n)}{l_0(\beta_n) \cdot l_0(\beta_n)} \geq \frac{1}{(c')^2} \frac{b_1(\alpha_n) \cdot b_2(\alpha_n)}{l_0(\alpha_n) \cdot l_0(\alpha_n)} \geq \frac{c^2}{(c')^2} > 0.$$

But this implies that every limit of the sequence (β_n) has self-intersection and cannot be a lamination, which is a contradiction with the fact that the sequence (β_n) converges to a lamination supported on μ.

The second statement is obvious, because two leaves of $\widetilde{\lambda}$ are asymptotic iff they have bounded Hausdorff distance from each other on one side and again since $\widetilde{g}$ is Lipschitz, their $\widetilde{g}$-images will have bounded Hausdorff distance on one side as well and have to be asymptotic asymptotic geodesics, which immediately proves the claim. ☐

Lemma 5.6. *Even more, $\bar{g}$ is continuous as a map from leaves of $\widetilde{\lambda}$ to the set of geodesics of $\mathbb{H}^2$ with Hausdorff topology.*

Sketch of proof. The idea of the proof is similar to the proof of the first statement of lemma 5.5. Notice that $\widetilde{\lambda}$ is closed in the Hausdorff topology. Also note that every leaf of λ gives a biinfinite route of τ and if l is a leaf of $\widetilde{\lambda}$, the train track routes of other leaves that are close to l (in Hausdorff topology) will share a long segment with the route associated to l. This shows that in the image also images of these leaves are close to image of l on a long subsegment. Then using the fact that images of all these leaves are (K, c)-quasigeodesic implies that their endpoints cannot be far from each other and their geodesic representatives are close in the image with Hausdorff topology. □

Lemma 5.7. *If l_1 and l_2 are leaves of $\widetilde{\lambda}$ and $\bar{g}(l_1) = \bar{g}(l_2)$, then there exists some $h \in \mathrm{Ker}(j_* : \pi_1(\partial H) \to \pi_1(H))$ such that $h(l_1) = l_2$.*

Sketch of the proof. If distinct leaves, l_1 and l_2, of $\widetilde{\lambda}$ project to the same leaf l' of λ in ∂H, then there exists an element $h \in \pi_1(\partial H)$ such that $h(l_1) = l_2$. In this case, we claim that $h \in \mathrm{Ker}(g_*)$ and since $j \circ g$ is homotopic to j, we have to have $h \in \mathrm{Ker}(j_*)$. Suppose g_*h is nontrivial. Since $\bar{g}$ is induced from lifting g we have

$$\bar{g}(l_2) = \bar{g}(h(l_1)) = g_*h(\bar{g}(l_1))$$

and our assumption implies that g_*h preserves $\bar{g}(l_1)$. Since g_*h is non-trivial, it has to be a hyperbolic isometry of $\mathbb{H}^2$ with axis $\bar{g}(l_1)$. This implies that $g(l')$ fellow travels the closed curve that represents g_*h. Since $f_0 \circ g$ and f_0 are homotopic in N_0, this implies that $f_0(l')$ fellow travels a closed curve that represents g_*h in N_0, but $f_0(l')$ is a geodesic and this shows that it is a closed geodesic. We knew that λ has no isolated leaves and because of this l' is noncompact. Otal [Ota88] proved that if a pleated surface f_0 realizes a Masur domain lamination λ, then

$$\mathbf{P}\mathrm{f}_0 : \lambda \to \mathbf{P}\mathrm{N}$$

is a homeomorphism to its image. (Recall that $\mathbf{P}$N is the projective tangent bundle of N.) This contradicts the possibility that f_0 takes a noncompact leaf of λ to a closed geodesic.

On the other hand, suppose l_1 and l_2 project to distinct leaves of λ: l_1' and l_2'. Since $\bar{g}(l_1) = \bar{g}(l_2)$, the images $g(l_1')$ and $g(l_2')$ are asymptotic in ∂H (they have bounded Hausdorff distance) and we have a homotopy with bounded tracks between them. If we concatenate this homotopy with $F|_{l_1' \times [0,1]}$ and $F|_{l_2' \times [0,1]}$, we get a homotopy with bounded tracks in N_0 between $f_0(l_1')$ and $f_0(l_2')$. Since $f_0(l_1')$ and $f_0(l_2')$ are geodesics, this is impossible unless $f_0(l_1') = f_0(l_2')$. This again contradicts Otal's theorem which we mentioned above and we are done.

□

We know that every complementary component of $\widetilde{\lambda}$ is an ideal polygon.

Lemma 5.8. *If P is a complementary component of $\widetilde{\lambda}$ then $\bar{g}$ is injective on sides of P.*

Proof. Suppose $l_1, l_2, \ldots, l_k$ are sides of P. We want to show that $\bar{g}(l_i) \neq \bar{g}(l_j)$ for $i \neq j$. If this is not the case then lemma 5.7 shows that there exists $h \in \ker j_*$ such that $h(l_i) = l_j$. But then if we consider $h(P)$, it is a complementary component of $\widetilde{\lambda}$ too and it cannot be P, since h is not elliptic. The complementary components P and $h(P)$ share a side: l_j. This shows that l_j is an isolated leaf of $\widetilde{\lambda}$ which is impossible, since λ has no isolated leaf. □

The above lemma and lemma 5.5 show that $\bar{g}$ takes complementary domains to complementary domains.

Now we define the *dual tree* for a measured lamination on a surface. (Cf. Skora [Sko90, Sko96] and Otal [Ota96].) The definition is more general for unmeasured laminations and also allows closed leaves but for the sake of bravery, we don't discuss those. Suppose ν is a measured lamination on ∂H. We identify ∂H with $\mathbb{H}^2/\Gamma$, where Γ is a Fuchsian group isomorphic to $\pi_1(\partial H)$. Consider the preimage $\widetilde{\nu}$ in $\mathbb{H}^2$.

Now we consider a partition $\mathcal{T}_\nu$ of $\mathbb{H}^2$ into closed sets: each set is either the closure of a component of $\mathbb{H}^2 \setminus \widetilde{\nu}$, or a leaf of $\widetilde{\nu}$ that is not in the closure of the two components of $\mathbb{H}^2 \setminus \widetilde{\nu}$. We will equip the set $\mathcal{T}_\nu$ with a distance that will turn it into an $\mathbb{R}$-*tree*. Recall that an $\mathbb{R}$-tree is a metric space $(\mathcal{T}, d)$ such that between two arbitrary points x and y there always exists a *unique* arc isometric to the interval $[0, d(x, y)]$.

Consider two points of $\mathcal{T}_\nu$ that are closures of components A and A' of the complement of $\widetilde{\nu}$. Choose two points x and x' in the interiors of components A and A'. The geodesic segment $[x, x']$ between x and x' in $\mathbb{H}^2$ is transverse to $\widetilde{\nu}$ and intersects each leaf in at most one point. Thus the intersection of each closes subset of $\mathcal{T}_\nu$ with $[x, x']$ is a segment. The transverse measure to $\widetilde{\nu}$ assigns to the geodesic $[x, x']$ a positive measure supported on $[x, x'] \cap \widetilde{\nu}$. By integration, this measure induces a distance on the set of closed subsets of $\mathcal{T}_\nu$ that intersect $[x, x']$.

It is easy to check that the distance is independent of the points x, x' and moreover, given two closed sets B and B' in $\mathcal{T}_\nu$, there exist sets A and A' that are the closures of components of the complement of $\widetilde{\nu}$ such that B and B' separate A and A' in $\mathbb{H}^2$. Hence, what we just described defines a number $d(B, B')$ for any two arbitrary closed subsets of $\mathcal{T}_\nu$ and this is independent of the choices of A and A'. Finally, it is not hard to see that this distance turns $\mathcal{T}_\nu$ to an $\mathbb{R}$-tree.

Now consider a measured geodesic laminations supported on λ, which we still call λ and consider the dual tree $\mathcal{T}_\lambda$. Also consider $\mathcal{T}$ to be the dual tree for the $\bar{g}$-image of $\widetilde{\lambda}$. We have, in fact, shown that g induces a $\pi_1(\partial H)$-equivariant morphism $G : \mathcal{T}_\lambda \to \mathcal{T}$.

Claim 5.9. *The map $G : \mathcal{T}_\lambda \to \mathcal{T}$ is locally injective.*

Proof. We will need the following lemma for the proof:

Lemma 5.10. *Suppose l is a leaf of $\widetilde{\lambda}$ then $\bar{g}(l_1) \neq \bar{g}(l_2)$ for distinct leaves l_1, l_2 of $\widetilde{\lambda}$ which are very close to l.*

Proof. Suppose $a_n \neq b_n$ are pairs of leaves of $\widetilde{\lambda}$ whose distance to l tends to zero as $n \to \infty$ and $\bar{g}(a_n) = \bar{g}(b_n)$. Lemma 5.7 shows that there exists $h_n \in \ker j_*$ such that $h_n(a_n) = b_n$. Because of discreteness of action of $\pi_1(\partial H)$, we can see that length of h_n goes to infinity as $n \to \infty$ and from this one can see that the axis of h_n converges to l in the Hausdorff topology.

Hence a_n and b_n project to the same leaf c_n of λ but we have $h_n(a_n) = b_n$. Suppose $\kappa : \mathbb{R} \to \partial H$ parametrizes c_n with arc length. We claim that for n large, there exists a subarc $\kappa([s,t])$ such that $\kappa'(s)$ and $\kappa'(t)$ are ϵ'-close in the tangent bundle of ∂H and there is an arc δ of length at most ϵ' between $\kappa(s)$ and $\kappa(t)$ such that $\kappa([s,t]) \cup \delta$ represents the conjugacy class of either h_n or h_n^2. This perhaps can be seen more easily in the universal cover. Take a point $x \in a_n$ which is ϵ'-close to b_n (in the tangent bundle). By our assumption translate of x by h_n is a point $y \in b_n$. Now make a jump from x to the closest point on b_n and follow b_n to reach y. The path just described projects to a representative of the conjugacy class of h_n in ∂H. The only problem with this representative could be that the tangents at the beginning and the end may not have the same direction but that can be resolved by doing the same process with h_n^2 and we have proved our claim.

Now note that since f_0 realizes λ, image of γ_n will be ϵ'-close to a geodesic. But γ_n represented an element of $\operatorname{Ker} j_*$ and cannot have an almost geodesic representative and we have a contradiction.

$\square$

Now every $x \in \mathcal{T}_\lambda$ either corresponds to a complementary component of $\widetilde{\lambda}$ or to a leaf of $\widetilde{\lambda}$, which is a limit of leaves of $\widetilde{\lambda}$ from both sides. We have to prove local injectivity in small neighborhood of x in both cases.

Suppose x corresponds to a component of $\mathbb{H}^2 \setminus \widetilde{\lambda}$ which is an ideal polygon P with sides $l_1, l_2, \ldots, l_k$. Using lemma 5.8, we know that $\bar{g}(l_i) \neq \bar{g}(l_j)$ when $i \neq j$. Using continuity of $\bar{g}$ from lemma 5.6, we know that leaves which are very close to l_i, will be mapped to leaves which are very close to $\bar{g}(l_i)$. Therefore, images of leaves that are close to l_i cannot be identified with images of those that are close to l_j for $i \neq j$. Finally, using lemma 5.10, we know that $\bar{g}$ is injective in a very small Hausdorff neighborhood of each leaf l_i and this proves the claim in this case.

On the other hand if x corresponds to a leaf l of $\widetilde{\lambda}$ that is a limit from both sides, lemma 5.10 immediately implies that $\bar{g}$ is injective in a small Hausdorff neighborhood of l and we have finished proof of the claim. $\square$

The following lemma shows that $G : \mathcal{T}_\lambda \to \mathcal{T}$ is in fact injective:

Lemma 5.11. *A morphism $\phi : \mathcal{T}_1 \to \mathcal{T}_2$ between $\mathbb{R}$-trees $\mathcal{T}_1$ and $\mathcal{T}_2$ is injective if and only if it is locally injective.*

Proof. Obviously injectivity implies local injectivity. On the other hand, assume ϕ is locally injective but it is not injective. Suppose $\phi(x) = \phi(x')$ for distinct points $x, x' \in \mathcal{T}_1$. Consider the unique geodesic $l : [0,1] \to \mathcal{T}_1$ in $\mathcal{T}_1$ that connects x and x': $l(0) = x$ and $l(1) = x'$. The path $\phi \circ l : [0,1] \to \mathcal{T}_2$ is a closed path in $\mathcal{T}_2$ and it is enough to show that $\phi \circ l$ is not locally injective (because l is injective). If $\phi \circ l$ is locally injective, then there exists a non-degenerate interval $[a,b] \subset [0,1]$ such that $\phi \circ l(a) = \phi \circ l(b)$ but $\phi \circ l$ does not identify any other two points of $[a,b]$. But then we obtain two paths with disjoint interiors between $\phi \circ l(a)$ and $\phi \circ l(\frac{a+b}{2})$ which contradicts the fact that $\mathcal{T}_2$ is an $\mathbb{R}$-tree. $\square$

We use the injectivity of G to show that $g_* : \pi_1(\partial H) \to \pi_1(\partial H)$ is an injective map. Otherwise, assume $h \in \ker g_*$ is nontrivial. If l is a leaf of $\widetilde{\lambda}$, then $h(l) \neq l$; otherwise l would project to a closed curve, which contradicts our assumption about λ. Since h is in $\ker g_*$, we can see that lift of g to the universal covers takes l and $h(l)$ to the same geodesic and we have $\bar{g}(l) = \bar{g}(h(l))$. Let $x, x' \in \mathcal{T}_\lambda$ be the points that correspond to the leaves l and $h(l)$. We know that $x = x'$ if and only if l and $h(l)$ are sides of a complementary component of $\widetilde{\lambda}$ but then we have a contradiction with lemma 5.8. Hence $x \neq x'$ and we have $G(x) = G(x')$ which contradicts injectivity of G. Therefore g_* is injective.

It is a standard fact that every proper subgroup of a surface group is free, therefore $g_*(\pi_1(\partial H))$ cannot be a proper subgroup of $\pi_1(\partial H)$ and therefore g_* is surjective too and in g is a π_1-isomorphism. But again, it is a standard fact about surfaces that every homotopy equivalence from ∂H to itself is homotopic to a homeomorphism. Hence g is homotopic to a homeomorphism and we are done by remark 5.1.

Chapter 6

Hyperbolic structures with bounded combinatorics

In this chapter, we introduce a set of hyperbolic structures on H, whose end-invariants have R-bounded combinatorics respect to H. We prove that this family behaves well in the strong topology. Throughout the rest of this work we will restrict ourselves to this family.

Remark 6.1. In our definition of R-bounded combinatorics with respect to a handlebody in 2.6, we have assumed that α and its projection to $\Delta(H)$ have R-bounded combinatorics. (This a little bit different from what we did in our definition, especially for laminations, but it is essentially the same.) In particular, when α is a marking, we have assumed that there exists a handlebody pants decomposition P, called projection of α to $\Delta(H)$, such that α and P have R-bounded combinatorics and realize the distance between α and $\Delta(H)$. We should point out that we could relax our definition by allowing P to have distance bounded by some constant c from the point of $\Delta(H)$ that realizes the distance between α and $\Delta(H)$. In that case, every thing that we prove here would still be true with constants that will depend on c as well as on R.

In fact, it is interesting that theorem 6.3 is the only place where we use the fact that α and P realize the distance between α and $\Delta(H)$. There we use this condition to show that a lamination with R-bounded combinatorics with respect to H is in the Masur domain.

Lemma 6.1. *Let H be a handlebody and $\Delta(H) \subset \mathcal{C}(\partial H)$ the associated handlebody subcomplex. There exists d_0 only depending on $\chi(\partial H)$ that if $\alpha \subset \mathcal{C}_0(\partial H)$ is a multi-curve with distance bigger than d_0 from $\Delta(H)$ then α is in the Masur domain. In fact, we can consider $d_0 = d + 2$ where d is the quasi-convexity constant of $\Delta(H)$ in theorem 2.6.*

Proof. If α is not in the Masur domain, then it has zero intersection with μ, an element of closure of Δ in $\mathcal{PML}$. Suppose Y is the smallest essential subsurface that contains support of μ. The lamination μ cannot be filling, since it has zero intersection with a simple closed curve and therefore, Y is a proper subsurface of ∂H. Of course, ∂Y has zero intersection with α too and $d_{\mathcal{C}}(\alpha, \partial Y) \leq 1$. We claim that ∂Y has distance at most $d+1$ from Δ and this together with what we said before shows that $d_{\mathcal{C}}(\alpha, \Delta) \leq d+2$ which contradicts our hypothesis. To prove the claim, consider a sequence $(\beta_i) \subset \Delta$ that converges to $\mu \subset Y$ in $\mathcal{PML}$.

Doing surgery, we can replace $\pi_Y(\beta_i)$ with a non-peripheral simple closed curve β_i' supported on Y such that $d_Y(\pi_Y(\beta_i), \beta_i')$ and $i(\beta_i, \beta_i')$ are bounded with universal constants. This implies that every limit of the sequence (β_i') in $\mathcal{PML}$ has zero intersection with μ and has to be supported on μ. Using the fact that μ fills Y, an observation of Luo (cf. Masur-Minsky [MM00]) shows that $d_Y(\beta'_0, \beta'_i) \to \infty$ as $i \to \infty$ and therefore $d_Y(\pi_Y(\beta_i), \pi_Y(\beta_0)) \to \infty$.

Then theorem 2.5 (Bounded geodesic image), implies that for i sufficiently large, any geodesic connecting β_i and β_0 intersects the one-neighborhood of ∂Y. On the other hand because of quasi-convexity of Δ, theorem 2.6, this geodesic is in the d-neighborhood of Δ and this proves the claim. □

Suppose H is a fixed handlebody and Let $R > 0$ be given. Let $\mathcal{A}(R)$ be the set of markings on ∂H and elements of $\partial\mathcal{C}(\partial H)$ which have R-bounded combinatorics with respect to H and base(α) has distance bigger than d_0 from $\Delta(H)$, where d_0 is the constant obtained in lemma 6.1, when α is a marking in $\mathcal{A}(R)$. We also consider the subset $\mathcal{A}_0(R) \subset \mathcal{A}(R)$ to be the subset of $\mathcal{A}(R)$ whose element's projection to $\Delta(H)$ is contained in $\mathbf{m}_0$(H), where $\mathbf{m}_0$(H) is the set defined in proposition 2.8. Notice that every element of $\mathcal{A}(R)$ can be translated to $\mathcal{A}_0(R)$ by an action of $\mathrm{Mod}_0(H)$. We say an element $\alpha \in \mathcal{A}_0(R)$ is *of the first type* if it is a marking; otherwise we say α is *of the second type.*

Proposition 6.2. ($\mathcal{A}_0(R)$ is compact in $\mathcal{C}(\partial H) \cup \partial\mathcal{C}(\partial H)$.) *Let* $(\alpha_n) \subset \mathcal{A}_0(R)$ *be a sequence of elements of* $\mathcal{A}_0(R)$*. There exists a subsequence* $(\alpha_{n_k})_k$ *such that either all its elements are the same or the sequence* $(\mathrm{base}(\alpha_{n_k}))_k$ *converges to a lamination* $\mu \in \mathcal{A}_0(R)$*.*

Proof. Let $(\alpha_n) \subset \mathcal{A}_0(R)$ be given. First suppose all elements of α_n are of the second type. Suppose μ_n is an element of $\mathcal{ML}$ supported on α_n with total measure 1 (we are fixing a hyperbolic metric on ∂H). By definition, since α_n has R-bounded combinatorics with respect to H, there exists a sequence of markings in $\mathcal{A}_0(R)$ that converges to α_n in $\mathcal{C} \cup \partial\mathcal{C}$. Choose an element of this sequence β_n such that $i(\gamma_n, \mu_n) \leq 1/n$ where $\gamma_n \in \mathcal{ML}$ has total measure 1 and is supported on β_n. Now $(\beta_n) \subset \mathcal{A}_0(R)$ consists of markings and it is easy to see that the first possibility in the conclusion of the proposition does not happen for the sequence (β_n). Suppose after passing to a subsequence, which we still denote by (β_n), the sequence (β_n) converges to $\mu \in \partial\mathcal{C}$. Klarreich's description of $\partial\mathcal{C}$ shows that every limit of (γ_n) (in $\mathcal{ML}$) is supported on μ. This proves that every limit of μ_n (in $\mathcal{ML}$) is supported on μ and therefore (α_n) converges to μ in $\partial\mathcal{C}$. Hence it is enough to prove the proposition for a sequence of the first type.

Now suppose $(\alpha_n) \subset \mathcal{A}_0(R)$ is a sequence of markings and suppose $P_n = \text{base}(\alpha_n)$ is the associated pants decomposition for every n. After passing to a subsequence, we can assume that all elements of (α_n) have R-bounded combinatorics with respect to an element $\beta_0 \in \mathbf{m}_0(\mathrm{H})$. (We are using lemma 2.9 to extend the projection to $\Delta(H)$ to a marking.)

Consider the sequence (P_n) as a sequence in $\mathcal{PML}$. If there is a subsequence of (P_n) whose elements are all equal to a single pants decomposition P_0, then we claim that there is a subsequence of (α_n) whose elements are all the same. This is because for every α_n, once we know the pants decomposition $P_n = P_0$ we only have the freedom to choose the transversals. But for each $\gamma \in P_0$, we have only a finite number of choices for the transversal to γ, since $d_\gamma(\alpha_n, \beta_0) \leq R$ and the claim follows.

Now suppose we have extracted a subsequence such that (P_n) converges to a lamination μ in $\mathcal{PML}$. Using Klarreich's description of $\partial\mathcal{C}(\partial H)$ it will be enough to show that μ is filling. Let $Y \subset \partial H$ be the smallest essential subsurface that contains support of μ.

Case 1. Y has a non-annular component $Y' \subsetneq \partial H$. Let μ' be the component of μ that is contained in Y' and notice that μ' fills Y'. We argue similar to the proof of 6.1. For every n, by doing a small number of surgeries on $\pi_{Y'}(P_n)$, construct a non-peripheral simple closed curve a_n on Y' with bounded intersection with P_n and bounded $d_{Y'}(P_n, a_n)$. It follows that every limit of the sequence (a_n) (in $\mathcal{PML}$) has zero intersection with μ' and is supported on Y', therefore it has to have the same support as μ'. Since μ' fills Y', we have $d_{\mathcal{C}(Y')}(a_n, a_0) \to \infty$ as $n \to \infty$ and therefore $d_{Y'}(\alpha_n, \alpha_0) \to \infty$ which contradicts the fact that $d_{Y'}(\alpha_n, \alpha_0) \leq 2R$.

Case 2. All components of Y are annuli. Either infinitely many elements P_n are contained in Y or there exists a subsequence of (P_n) whose elements all intersect a component $Y' \subset Y$. If the former happens, Y is the union of annular neighborhoods of components of a pants decomposition Q and by what we explained earlier there is subsequence of (α_n) whose elements are all equal. If the latter happens, since the sequence (P_n) converges (in $\mathcal{PML}$) to μ which contains core of Y', the only possibility is that P_n spiral about core of Y' more and more as $n \to \infty$. But this implies that $d_{Y'}(P_n, P_0) \to \infty$ as $n \to \infty$ and again we have a contradiction.

The fact that every such limit is in $\mathcal{A}_0(R)$ is by definition of R-bounded combinatorics respect to H for elements of $\partial\mathcal{C}(\partial H)$. $\square$

Let $\mathcal{A}_0'(R)$ be the subset of $\mathcal{PML}(\partial H)$ whose elements are supported on base(α) if $\alpha \in \mathcal{A}_0(R)$ is of the first type or on $\alpha \in \mathcal{A}_0(R)$ when α is of the second type. We call the elements of $\mathcal{A}_0'(R)$ of the first and second type accordingly. Notice that all elements of the second type fill ∂H.

Proposition 6.3. *For every R, the set $\mathcal{A}'_0(R)$ is a compact subset of the Masur domain.*

Proof. Using proposition 6.2 and Klarreich's description of convergence in $\mathcal{C} \cup \partial\mathcal{C}$, we know that every convergent sequence in $\mathcal{A}'_0(R)$ either converges to an element supported on base(α) where $\alpha \in \mathcal{A}_0(R)$ is a marking or converges to a an element supported on $\alpha \in \mathcal{A}_0(R)$ where α is of the second type. Therefore $\mathcal{A}'_0(R)$ is compact and we only need to show that it is a subset of the Masur domain.

First note that if $\mu \in \mathcal{A}'_0(R)$ is of the first type then support of μ has $\mathcal{C}$-distance at least d_0 from $\Delta(H)$ and by lemma 6.1, it is in the Masur domain. Hence we simply need to prove that an element of the second type $\mu \in \mathcal{A}'_0(R)$ is in the Masur domain.

If μ is not in the Masur domain, then it has zero intersection with an element λ in the closure of Δ in $\mathcal{PML}$. Since μ is filling, it has to have the same support as λ, λ is also filling and they represent the same point in $\partial\mathcal{C}(\partial H)$ which we denote by μ.

By definition, there exists a sequence $(\alpha_n) \subset \mathcal{A}_0(R)$ of markings where $(\text{base}(\alpha_n))$ converges to μ in $\mathcal{C} \cup \partial\mathcal{C}$. For each α_i choose a component a_i of base(α_i) and notice that the sequence (a_i) also converges to μ in $\partial\mathcal{C}$. Also let $(b_i) \subset \Delta$ be a sequence that converges to μ in $\mathcal{C} \cup \partial\mathcal{C}$. Since the sequences (a_i) and (b_i) converge to the same point in the Gromov boundary of $\mathcal{C}(S)$, the geodesic segments $[a_i, b_i] \subset \mathcal{C}$ connecting a_i and b_i in $\mathcal{C}(\partial H)$ get further and further from x_0, where x_0 is a fixed point in $\Delta(H)$. Recall Masur and

Minsky's theorem [MM99], where they prove that $\mathcal{C}(\partial H)$ is δ-hyperbolic in sense of Gromov. Consider the geodesic triangle with vertices $\{a_i, b_i, x_0\}$; this triangle is δ-thin. Since the segment $[a_i, b_i]$ is far from x_0, the other two sides $[x_0, a_i]$ and $[x_0, b_i]$ have to be δ-close on subsegment of length D_i of their initial part. The length D_i is comparable to the distance between x_0 and the segment $[a_i, b_i]$ and in particular $D_i \to \infty$ as $i \to \infty$. Because of quasi-convexity of Δ, theorem 2.6, the segment $[x_0, b_i]$ is in the d-neighborhood of Δ. Therefore

$$\begin{aligned} d_{\mathcal{C}}(a_i, \Delta) &\leq d_{\mathcal{C}}(a_i, [b_i, x_0]) + d \\ &\leq d_{\mathcal{C}}(a_i, x_0) - D_i + \delta + d \end{aligned}$$

is much shorter than $d_{\mathcal{C}}(x_0, a_i)$ for $i \gg 0$. Recall from proposition 2.8 that $\mathbf{m}_0(\mathrm{H})$ consisted of a finite set of handlebody markings. Therefore for $i \gg 0$, $d_{\mathcal{C}}(a_i, \Delta)$ is shorter than the distance between a_i and every element of $\mathbf{m}_0(\mathrm{H})$. But this contradicts our choice of $\alpha_i \in A_0(R)$ whose distance from Δ is realized between α_i and an element of $\mathbf{m}_0(\mathrm{H})$. □

Let $\tau(\mathcal{A}_0(R)) \subset \mathfrak{T}(\partial H)$ be the set of points $\tau \in \mathfrak{T}(\partial H)$ where the total length of an element of the first kind $\alpha \in \mathcal{A}_0(R)$ is at most B_0, the Bers constant. Notice that by our assumption about ϵ_0 and B_0 in 2.15, every element in $\tau(\mathcal{A}_0(R))$ is ϵ_0-thick and for every $\alpha \in \mathcal{A}_0(R)$ of the first kind there exists at least one point, namely $\tau(\alpha)$ where the total length of α is at most B_0. Notice that because α binds ∂H, the set of such points has uniformly bounded diameter in $\mathfrak{T}$. As a corollary of the propositions 6.2 and 6.3 we have:

Corollary 6.4. *The set $\tau(\mathcal{A}_0(R))$ is a closed subset of $\mathfrak{T}(\partial H)$ and every limit* *of this set in Thurston's compactification of ∂H is an element of second type in $\mathcal{A}'_0(R)$ and is in a filling Masur domain lamination.*

Definition 6.2. Let $\mathcal{B}_0(R)$ be the set of marked hyperbolic structures on H, which are either

- of the first type: they are convex cocompact with the conformal structure at infinity in $\tau(\mathcal{A}_0(R))$ or
- of the second type: they are geometrically infinite with ending lamination an element of second type in $\mathcal{A}_0(R)$.

We say elements of $\mathcal{B}_0(R)$ are *hyperbolic structures with R-bounded combinatorics on H.*

Notice that, we could start with $\mathcal{A}(R)$ and construct the associated marked structures; but those would be the same as the structures in $B_0(R)$ up to changing the marking and therefore they would provide the same set of structures.

Theorem 6.5. *The set $\mathcal{B}_0(R)$ with an appropriate choice of base points is compact in the set of marked hyperbolic structures on H with strong topology.*

Proof. Suppose $(N_i) \subset \mathcal{B}_0(R)$ is given. For now, we assume that each N_i is an element of the first type and therefore there exists $\tau_i \in \tau(\mathcal{A}_0(R))$ and $\alpha_i \in \mathcal{A}_0(R)$ such that τ_i is the conformal structure at infinity for N_i and $l_{\tau_i}(\alpha_i) \leq B_0$.

By proposition 6.2, up to passing to a subsequence either the sequence (α_i) stabilizes or converges to $\lambda \in \mathcal{A}_0(R)$ of the second type. If the former happens, there exists $\beta \in \mathcal{A}_0(R)$ of the first type such that $l_{\tau_i}(\beta) \leq B_0$ for every i. Since β binds ∂H, up to passing to a subsequence, the sequence (τ_i) converges to a point $\tau \in \mathfrak{T}$. We obviously have $l_\tau(\beta) \leq B_0$ and $\tau \in \tau(\mathcal{A}_0(R))$. In this case, it follows from Ahlfors-Bers theory that the sequence of structures (N_i) converges strongly to a convex cocompact structure N with conformal structure at infinity associated to τ (see section 2.13). Since the convex core of N is compact, it is easy to see that the sequence converges to N as a sequence of marked structures and since $N \in \mathcal{B}_0(R)$ is of first type we are done.

Hence we can assume that the sequence (α_i) converges to $\lambda \in \mathcal{A}_0(R)$ of the second type. We also know that λ is filling and in the Masur domain. We can also assume that (up to passing to a subsequence) the sequence (τ_i) converges to a point in Thurston's compactification supported on λ, which we still call λ. From here, we can argue similar to the proof of corollary 1.4 and imply that the sequence (N_i) converges algebraically to N a hyperbolic structure on H with ending lamination λ (defined up to actions of Mod_0). Note that in particular N has no parabolics and the end is simply degenerate.

Next, we want to prove that the convergence $N_i \to N$ is strong. Suppose $\rho_i, \rho : \pi_1(H) \to \mathrm{PSL}_2(\mathbb{C})$ are the representations associated to N_i and N. Because of the algebraic convergence, we know that we can conjugate the representations ρ_i so that they converge to ρ; we assume from now on that this is the case. It is standard (cf. Canary-Epstein-Green [CEG87]) that if p_i and p are projections of the base point of $\mathbb{H}^3$ to $N_i = \mathbb{H}^3/\rho_i(\pi_1(H))$ and $N = \mathbb{H}^3/\rho(\pi_1(H))$, then there exists a subsequence of the pointed manifolds (N_{i_j}, p_{i_j}) that converges geometrically to a pointed manifold (N_G, p_G), $N_G = \mathbb{H}^3/\Gamma_G$, which is covered by N.

Thurston and Canary's Covering Theorem [Can96] implies that the cover

$$N \to N_G$$

is finitely sheeted. We claim that this covering is trivial, or equivalently that $\Gamma_G = \rho(\pi_1(H))$.

The proof is based on an idea of Thurston. Suppose $\beta \in \Gamma_G$ then $\beta^k \in \rho(\pi_1(H))$ for some $k > 1$ and is equal to $\rho(\gamma)$ for some $\gamma \in \pi_1(H)$. Since $\beta \in \Gamma_G$ there exists a sequence of elements $\alpha_j \in \pi_1(H)$ such that $\beta = \lim_j \rho_{i_j}(\alpha_j)$. So

$$\lim_j \rho_{i_j}(\alpha_j^k) = \beta^k = \rho(\gamma) = \lim_j \rho_{i_j}(\gamma).$$

Because of discreteness and faithfulness of the representations, $\alpha_j^k = \gamma$ for all $j \gg 0$. But in a free group we have at most one k-th root for every element and therefore $\alpha_j = \alpha$ for some fixed $\alpha \in \pi_1(H)$ and $j \gg 0$. Hence $\beta = \lim_j \rho_{i_j}(\alpha) \in \rho(\pi_1(H))$ which proves that the covering the covering is trivial and $\Gamma_G = \rho(\pi_1(H))$.

This proves that every subsequence of (N_i, p_i) has a subsequence that converges to (N, p) geometrically. Therefore the entire sequence (N_i, p_i) converges to (N, p) geometrically and the convergence $N_i \to N$ is strong.

Now it only remains to choose a marking $j : \partial H \to N$ for N such that λ is the ending lamination for the marked structure (N, j) and prove that $N_i \to N$ as a sequence of marked structures.

Let $j : \partial H \to N$ be a marking such that λ is the ending lamination for the marked structure (N, j). (This is always possible by precomposing an arbitrary marking with an element of $\mathrm{Mod}_0(H)$.) Also let C be a useful compact core of N and choose a sequence (Q_n) of pants decompositions that converges to a lamination supported on λ in $\mathcal{PML}$. Since λ is the ending lamination, for n sufficiently large, Q_n has a geodesic representative in $N \setminus C$ and these representatives exit the end of N as $n \to \infty$. (Recall that by a geodesic representative in $N \setminus C$, we mean one that is homotopic to $j(Q_n)$ within $N \setminus C$.) Hence by using lemma 4.2, we know that for $n \gg 0$, there exists a pleated surface $f_n : \partial H \to N$ that realizes Q_n and is homotopic to j within $N \setminus C$. This sequence of pleated surfaces has to exit the end of N because of lemma 2.13 (Bounded diameter lemma). (Notice that in both these, we are using the fact that the function $d_N^{\geq \epsilon}(\cdot, p)$ is proper.) From here on assume, n is sufficiently large such that $d_N^{\geq \epsilon}(f_n(\partial H), C) \geq D + 1$ where D is the constant obtained in corollary 4.7.

Fix n large and let $\kappa_i : (N, p) \to (N_i, p_i)$ be the approximating maps for the geometric convergence $N_i \to N$. It is not hard to see that $C_i = \kappa_i(C)$ is a useful compact core in N_i for $i \gg 0$ (cf. Canary-Minsky [CM96, Prop. 3.3]). Let $j_i : \partial H \to N_i \setminus C_i$ be a representative of the marking of N_i. We know that κ_i approaches an isometry on every compact subset of N. Using this we can see that for i sufficiently large depending on n, there exists $h_{i,n} \in \mathbf{pleat}_{N_i}$ that realizes Q_n in N_i, is ϵ_0-close to $\kappa_i \circ f_n$ and

$$d_{N_i}^{\geq \epsilon_0}(h_{i,n}(\partial H), C_i) \geq D. \tag{6.1}$$

It will be enough to show that $\kappa_i \circ j$ or equivalently $\kappa_i \circ f_n$ or $h_{i,n}$ is homotopic to j_i in $N_i \setminus \kappa_i(C)$ for i sufficiently large.

Suppose $g_i : \partial H \to N_i$ parametrizes $\partial \mathcal{CH}(N_i)$, the boundary of the convex core of N_i, and g_i is homotopic to j_i in $N_i \setminus C_i$. Let $P_i = \text{base}(\alpha_i)$ be the pants decomposition associated to α_i. By our construction, we know that α_i and therefore P_i has length bounded by B_0 in τ_i. We also know that τ_i is ϵ_0-thick. Hence, we can use Bridgeman-Canary's theorem 2.12 [BC03] and conclude that P_i has length at most JB_0 on σ_{g_i}, where J depends only on ϵ_0.

We obviously know that $d_{N_i}^{\geq \epsilon_0}(\partial \mathcal{CH}(N_i), C_i) \to \infty$ as $i \to \infty$. Suppose $A = JB_0$ and $\epsilon = \epsilon_0$; if i is sufficiently large such that $d_{N_i}^{\geq \epsilon_0}(\partial \mathcal{CH}(N_i), C_i) \geq D'$, where D' is the constant obtained in lemma 4.4 (Homotopy bound) and there exists an end-homotopic $g' \in \textbf{pleat}_{N_i}$ that realizes P_i, then there exists a (K, ϵ_0)-good homotopy with respect to P_i between g_i and g' that stays away from C_i.

Consider the geodesic that connects Q_n and P_i in $\mathcal{C}(\partial H)$. As in lemma 2.7, we can extend this to an elementary-move sequence

$$Q_n = P_0^i \to P_1^i \to \cdots \to P_{m_i}^i = P_i.$$

Using corollary 4.7, since $h_{i,n}$ is end-homotopic, realizes Q_n and satisfies (6.1), we know that either

(1) there exists $g' \in \textbf{pleat}_{N_i}$ that realizes P_i and is homotopic to $h_{i,n}$ within $N_i \setminus C_i$ or

(2) there exists a pleated surface $g'' \in \textbf{pleat}_{N_i}$ homotopic to $h_{i,n}$ within $N_i \setminus C_i$ realizing P_k^i for some $0 \leq k \leq m_i$ and with

$$d_{N_i}^{\geq \epsilon}(g''(\partial H), C_i) \leq D.$$

If case (1) happens, by what we said earlier, g' is homotopic to g_i within $N_i \setminus C_i$ and therefore $h_{i,n}$ is homotopic to g_i within $N \setminus C_i$ and finally $h_{i,n}$ is homotopic to j_i within $N \setminus C_i$. This is what we wanted and if this happens for all sufficiently large i, we are done.

So, we seek a contradiction if this is not the case for every n. Suppose there exists $i(n) \geq n$ such that case (2) happens. Then we get a pleated surface $g''_n \in \mathbf{pleat}_{N_{i(n)}}$ that realizes an element γ_n on the $\mathcal{C}(\partial H)$-geodesic path between Q_n and $P_{i(n)}$, is homotopic to $h_{i(n),n}$ and equivalently to $\kappa_{i(n)} \circ j$ within $N_i \setminus C_i$ and

$$d^{\geq \epsilon}_{N_{i(n)}}(g''_n(\partial H), C_{i(n)}) \leq D.$$

If K is a compact subset of N and i is sufficiently large,

$$d^{\geq \epsilon}_{N}(x, C) - 1 \leq d^{\geq \epsilon}_{N_i}(\kappa_i(x), C_i) \leq d^{\geq \epsilon}_{N}(x, C) + 1.$$

Using this together with lemma 2.13 (Bounded diameter lemma), we can see that $g''_n(\partial H)$ is contained in a bounded neighborhood of $p_{i(n)}$ in $N_{i(n)}$ independently of n. Therefore, $\kappa^{-1}_{i(n)} \circ g''_n$ is ϵ close to a pleated surface f'_n that realizes γ_n in N and intersects a compact subset of N.

When $n \to \infty$ by definition $i(n) \to \infty$. Therefore the sequences (Q_n) and $P_{i(n)}$ both converge to elements supported on λ in $\mathcal{PML}$. Since λ is in $\partial\mathcal{C}(\partial H)$ and γ_n is on the geodesic connecting Q_n and $P_{i(n)}$, it follows from the definition of the Gromov boundary of $\mathcal{C}(\partial H)$ that the sequence (γ_n) converges to λ as well. But since f'_n realizes γ_n and intersects a compact subset of N, it follows from Kleineidam-Souto's [KS03] result, proposition 5.2 that λ is realized in N and we have a contradiction with the fact that λ is the ending lamination of

N.

This finishes the proof of theorem 6.5 for the case where (N_i) is a sequence of convex cocompact structures (elements of the first type) or perhaps has such a subsequence. Now suppose each N_i is an element of the second type with ending lamination λ_i. After passing to a subsequence we can also assume that λ_i converges to an element of the second type $\lambda \in \mathcal{A}_0(R)$. Also assume each λ_i is equipped with a transversal measure with total length 1. What we did above shows that we can approach each N_i using convex cocompact structures in $\mathcal{B}_0(R)$ in the strong topology of marked structures. From there a standard argument by taking a diagonal subsequence proves that (N_i) has to converge strongly to an element of the second type in $\mathcal{B}_0(R)$ as a sequence of marked structures.

$\square$

Remark 6.3. Notice that in the above statement, we have chosen appropriate base point p_N for every $N \in B_0(N)$ and from now on, whenever we speak of N we consider it as a pointed manifold (N, p_n).

Since elements of $\mathcal{B}_0(R)$ are marked structures on H, the conformal structure at infinity or the ending lamination are defined uniquely on $\mathfrak{T}(\partial H)$ or $\partial\mathcal{C}(\partial H)$. Hence we have a map

$$\mathcal{E} : \mathcal{B}_0(R) \to \mathcal{A}_0(R), \tag{6.2}$$

and we call $\mathcal{E}(N)$ the *end invariant* of N.

Chapter 7

Quasiconvexity

Let H be a handlebody as before and $\mathcal{B}_0(R)$ the set of marked hyperbolic structures on H introduced in the last chapter. In our discussions, we usually consider an element $N \in \mathcal{B}_0(R)$ to be the interior of H equipped with a complete hyperbolic metric and we assume the marking $j : \partial H \to N$ is simply an embedding isotopic to the inclusion $\partial H \hookrightarrow H$. Therefore, we use the same marking j for all structures in $\mathcal{B}_0(R)$; yet, because j is defined up to isotopy, we feel free to isotope j whenever needed. In particular, when we make a choice of a compact core, we assume that $j(\partial H)$ and the isotopy between j and $\partial H \hookrightarrow H$ stay away from the compact core.

We also assume that we have fixed a choice of Γ (a diskbusting curve) in H that satisfies proposition 4.1. Then for every $N \in \mathcal{B}_0(R)$, we take Γ_N to be the geodesic representative of $j(\Gamma)$ in N. (We are using the fact that elements of $\mathcal{B}_0(R)$ have no parabolics.) From now on, when we speak of the diskbusting geodesic for $N \in \mathcal{B}_0(R)$, we refer to Γ_N. The next proposition follows easily from compactness of $\mathcal{B}_0(R)$ in the strong topology.

Proposition 7.1. (Uniform compact core) *There exists a constant $d_0 > 1$*

such that for every $N \in \mathcal{B}_0(R)$.

(1) *the diskbusting geodesic Γ_N has total length at most d_0 and is contained in the d_0-neighborhood of the base point p_N and*

(2) *there exists a compact core $C \subset N$ homeomorphic to H that contains a 1-neighborhood of Γ_N and is contained in the $(d_0 - 1)$-neighborhood of Γ_N.*

For now on, we fix a choice of d_0 that satisfies the above proposition and is bigger than the constant D_0 in lemma 3.3 and for $N \in \mathcal{B}_0(R)$ we always assume that a useful compact core is one that satisfies the second part of the above proposition and in particular has uniformly bounded diameter. We also define the set $\overline{\mathbf{pleat}}_N$ to be the set of pleated surfaces $f : \partial H \to N$ homotopic to j within $N \setminus C$ for some useful compact core $C \subset N$ and with $d_N(f(\partial H), \Gamma_N) \geq d_0$. Notice that these pleated surfaces satisfy the conclusion of lemma 3.3 by our assumption about d_0:

$$f((\partial H)^{\geq \epsilon}) \subset N^{\geq \delta} \tag{7.1}$$

for every $f \in \overline{\mathbf{pleat}}_N$ and where $\delta = \delta(\epsilon, \chi(\partial H))$.

If α is a multi-curve on ∂H and $N \in \mathcal{B}_0(R)$, by a *(geodesic) representative for α in $N \setminus \Gamma_N$*, we mean a closed (geodesic) curve freely homotopic within $N \setminus C$ to $j(\alpha)$ for some useful compact core C.

The purpose of this chapter is prove a result in parallel to Minsky's [Min01, Thm. 3.1] that shows for every $N \in \mathcal{B}_0(R)$ and B bigger than or equal to the

Bers' constant B_0, the set

$$\mathcal{C}(B,N) := \bigcup_{f\in\overline{\mathbf{pleat}}_N} \mathbf{short}(f,B)$$

is L-quasiconvex for some constant L that depends only on R and $\chi(\partial H)$. Recall that $\mathbf{short}(f,B)$ is the set of simple closed curves whose σ_f-length is at most B.

When N is geometrically infinite, it follows from the definition and description of the ending lamination of N in Canary's work [Can93b] that there exists a subsequence of $\mathcal{C}(B,N)$ which converges to $\mathcal{E}(N) \in \partial\mathcal{C}(\partial H)$.

By $\overline{\mathbf{pleat}}_N(\mu)$, we denote the set of pleated surfaces in $\overline{\mathbf{pleat}}_N$ whose pleating locus contains μ. If $f \in \overline{\mathbf{pleat}}_N(\mu)$ is given, we say f *realizes* μ in $N \setminus \Gamma_N$. We define $\overline{\mathbf{pleat}}_N^{<D}$ to be the subset of $\overline{\mathbf{pleat}}_N$ whose elements have distance less than D from Γ_N.

Lemma 7.2. *There exists a monotonic function $\rho : [0,\infty) \to [0,\infty)$ such that $d_N(x,p_N) \le \rho(d_N^{\ge\epsilon}(x,p_N))$ for every $N \in \mathcal{B}_0(R)$ and $\epsilon \le \epsilon_0$.*

Proof. It will be enough to show that for every $a > 0$ there exists $b > 0$ such that if $d_N^{\ge\epsilon}(x,p_N) \le a$ then $d_N(x,p_N) \le b$ for every $N \in \mathcal{B}_0(N)$, $x \in N$ and $\epsilon \le \epsilon_0$. Also note that it is enough to prove it for $\epsilon = \epsilon_0$.

Suppose, this is not the case and we have a sequence of counter examples (N_i, x_i) such that $N_i \in \mathcal{B}_0(R)$ and $d_{N_i}^{\ge\epsilon}(x_i, p_{N_i}) \le a$ but $d_{N_i}(x_i, p_{N_i}) \to \infty$. We can assume that the sequence (N_i) converges strongly to $N \in \mathcal{B}_0(R)$ and suppose

$$\kappa_i : (N, p_N) \to (N_i, p_{N_i})$$

are the approximating maps. One can show that we can assume for every ϵ_0-Margulis tube of N, κ_i takes it to an ϵ_0-Margulis tube in N_i for $i \gg 0$. In N, the function $d_N^{\geq\epsilon}(\cdot, p_N)$ is proper. This is because each component of $N^{<\epsilon}$ is compact and they are uniformly separated. Therefore, there exists a compact set $K \subset N$ such that for every $x \notin K$, $d_N^{\geq\epsilon}(x, p_N) \geq a + 1$.

Take $x \in \partial K$ and a path P_i between $\kappa_i(x)$ and p_{N_i} which minimizes $d_N^{\geq\epsilon}(\kappa_i(x), p_{N_i})$. When i is sufficiently large κ_i is very close to an isometry and therefore the injectivity radii of $x \in K$ and $\kappa_i(x)$ are extremely close.

Then except for a subset of very small length which is contained in a small neighborhood of boundary of ϵ-thin components every point $y \in N^{\geq\epsilon} \cap \kappa_i^{-1}(P)$ maps to $N_i^{\geq\epsilon}$ and therefore length of $P \cap N_i^{\geq\epsilon}$ is $\geq a + 1/2$. This shows that

$$d_{N_i}^{\geq\epsilon}(x, p_{N_i}) \geq a + 1/2$$

for every $x \notin \kappa_i(K)$ and we have a contradiction. ☐

Note that $\rho(x) \geq x$ for every $x \in [0, \infty)$. This result helps us to get an actual diameter bound for elements of $\overline{\mathbf{pleat}}_N$ close to Γ_N.

Lemma 7.3. (Uniform bounded diameter lemma) *For every $D > 0$, there exists $L(D) > 0$ such that* $\operatorname{diam}_N(f(\partial H)) \leq L(D)$ *for every $f \in \overline{\mathbf{pleat}}_N^{<D}$ and $N \in \mathcal{B}_0(R)$.*

Proof. The result is immediate by knowing lemma 2.13 (Bounded diameter lemma), lemma 7.2 and noticing that length of compressible curves in σ_f is at least d_0 because of lemma 3.2.

☐

We can also translate our results in lemmas 4.2 and 4.3 by replacing the distance in the thick part of manifolds to the actual distance. Note that we also use the fact that we have a bounded diameter useful compact core for every element of $\mathcal{B}_0(R)$.

Lemma 7.4. *Given $d > 0$ there exists a constant $D_1 > 0$ depending only on d, R and $\chi(\partial H)$ such that if $N \in \mathcal{B}_0(R)$ and if α is a simple closed curve on ∂H with a geodesic representative α^* in $N \setminus \Gamma_N$ that*

$$d_N(\alpha^*, \Gamma_N) \geq D_1,$$

then $\overline{\mathbf{pleat}}_N(\mu)$ is nonempty and

$$d_N(f(\partial H), \Gamma_N) \geq d$$

for every $f \in \overline{\mathbf{pleat}}_N(\mu)$, where μ is any finite leaved lamination that contains α.

Theorem 7.5. *Given $d > 0$ there exists $D_2 > d$ and $A > 0$ depending only on d, R and $\chi(\partial H)$ such that the following holds. If $N \in \mathcal{B}_0(R)$ and α has a geodesic representative α^* in $N \setminus \Gamma_N$ with $d_N(\alpha^*, \Gamma_N) \geq D_2$, then for every $\beta \in \mathcal{C}_0(S)$ with $d_{\mathcal{C}}(\alpha, \beta) \leq 1$:*

(a) $\overline{\mathbf{pleat}}_N(\beta) \neq \emptyset$,

(b) $\overline{\mathbf{pleat}}_N(\alpha) \cap \overline{\mathbf{pleat}}_N(\beta) \neq \emptyset$

(c) *every $f \in \overline{\mathbf{pleat}}_N(\beta)$ has $d_N(f(\partial H), \Gamma_N) \geq d$*

(d) f *and* $g \in \overline{\mathbf{pleat}}_N(\beta)$, *the set*

$$\mathbf{short}(f, B) \cup \mathbf{short}(g, B)$$

has diameter bounded by A *in* $\mathcal{C}(\partial H)$.

Next, we need to have some control over the pleated maps that are nearby the diskbusting geodesic.

Lemma 7.6. *For every* D *there exists* K *such that for every two pleated surfaces*

$$f \in \overline{\mathbf{pleat}}_N^{<D} \text{ and } g \in \overline{\mathbf{pleat}}_M^{<D}$$

the induced metrics σ_f *and* σ_g *on* ∂H *are* K*-bi-Lipschitz up to isotopy for every* $M, N \in \mathcal{B}_0(R)$.

Note that f and g are pleated surfaces in different elements of $\mathcal{B}_0(R)$. Yet we have a K-bi-Lipschitz map between the induced metrics once we know that f and g are near the base points.

Proof. Suppose we have a sequence of counter examples $(N_i, M_i, f_i, g_i)_{i\geq 1}$ that satisfy the hypothesis. We can assume that the sequences (N_i) and (M_i) converge in the geometric topology to N and $M \in \mathcal{B}_0(R)$ respectively. Because of lemma 7.3, the pleated surfaces $f_i(\partial H)$ and $g_i(\partial H)$ are contained in a bounded neighborhood of the base points and therefore they also converge to pleated surfaces f and g in N and M.

Suppose $\kappa_i : N \to N_i$ are the approximating maps for the convergent sequence (N_i). We claim that the sequence (f_i) is convergent as a sequence of *marked pleated surfaces.* (For a description of different types of convergence for pleated surfaces, see Canary-Epstein-Green [CEG87].) In our situation it means that in addition to the fact that $f_i(\partial H)$ converge to $f(\partial H)$ geometrically, $\kappa_i \circ f$ is very close to $f_i \circ \phi_i$ as a map for $i \gg 0$, where ϕ_i is a self-homeomorphism of ∂H whose isotopy class does not depend on i.

Let C be a useful compact core for N. Then it easily follows that f is π_1-injective into $N \setminus C$. Otherwise the image of a compressing disk would give a compressing disk for $f_i(\partial H)$ in $N_i \setminus \Gamma_{N_i}$. Hence there exists a homotopy between $f \circ \psi$ and j within $N \setminus C$ for some ψ a self-homeomorphism of ∂H. By applying κ_i on these, we get a homotopy between $\kappa_i \circ f \circ \psi$ and $\kappa_i \circ j$ within $N_i \setminus \Gamma_{N_i}$. We know that $\kappa_i \circ j$ is isotopic to j and therefore they are homotopic outside a useful compact core and in the complement of Γ_{N_i} (the convergence $N_i \to N$ was in the strong topology for marked structures). On the other hand, $\kappa_i \circ f(\partial H)$ is extremely close to $f_i(\partial H)$ and they have distance at least 1 from Γ_N for $i \gg 0$. Therefore, there is a homotopy between $f_i \circ \phi_i$ and $\kappa_i \circ f$ for a self-homeomorphism ϕ_i of ∂H (within $N_i \setminus \Gamma_{N_i}$). These show that

$$j \sim \kappa_i \circ j, \quad \kappa_i \circ j \sim \kappa_i \circ f \circ \psi, \quad \kappa_i \circ f \circ \psi \sim f_i \circ \phi_i \circ \psi,$$

where $\sim$ denotes homotopy and all the above homotopies take place in $N_i \setminus \Gamma_{N_i}$. But j and f_i were homotopic and π_1-injective therefore $\phi_i \circ \psi$ is isotopic to identity and we have proved the claim. We assume that we have replaced f with $f \circ \psi$ and then the sequence (f_i) converges as a sequence of marked pleated surfaces to f. Once this is the case, one can easily see that the metrics induced by f and f_i are very close up to isotopy for $i \gg 0$. This is because

$\kappa_i \circ f$ and $f_i \circ \phi_i$ are very close as maps, where ϕ_i is isotopic to identity and also that κ_i is very close to an isometry for $i \gg 0$.

The same argument shows that after possibly precomposing g with a homeomorphism of ∂H, $g_i \to g$ as a sequence of marked pleated surfaces. Then, we know that the metrics induced by g_i and g are 1-bi-Lipschitz up to isotopy for $i \gg 0$. But the metrics induced by f and g are bi-Lipschitz and this shows that the metrics induced by f_i and g_i (up to isotopy) are bi-Lipschitz with a bounded bi-Lipschitz constant. $\square$

Corollary 7.7. *Suppose $\mathcal{B}_0(R)$ is as before. For every D and $B > 0$ the set*

$$A_{D,B} := \{\alpha | \alpha \in \mathbf{short}(f, B) \text{ for some } N \in \mathcal{B}_0(R) \text{ and } f \in \overline{\mathbf{pleat}}_N^{<D}\}$$

is finite.

Proof. This is immediate after lemma 7.6. Choose a fixed pleated surface $g \in \overline{\mathbf{pleat}}_M^{<D}$ for some $M \in \mathcal{B}_0(R)$. Now if f and α are as above, since the metrics induced by f and g are K-bi-Lipschitz, we have

$$l_{\sigma_g}(\alpha) \leq KB.$$

But for fixed g, there is only a finite set of closed curves whose length do not exceed KB and we are done. $\square$

In the next lemma, we show that there are pleated surfaces in $\overline{\mathbf{pleat}}_N$ within a uniformly bounded distance from Γ_N. As a matter of fact, if

$$d_N(\partial \mathcal{CH}(N), \Gamma_N)$$

is small, this is false. But in such a case, we can use compactness of $\mathcal{B}_0(R)$ in strong topology and prove that $\mathcal{CH}(N)$ has uniformly bounded diameter. Then most of the things that we need become trivial. In particular, using the above corollary, we can see that $\mathcal{C}(B, N)$ is finite depending on the diameter of $\mathcal{CH}(N)$ and theorem 7.11 (Quasiconvexity) is obvious. Hence, in all our discussions, we assume $d_N(\partial\mathcal{CH}(N), \Gamma_N)$ is uniformly large, when appropriate.

Lemma 7.8. *If D is sufficiently large independently of N, then $\overline{\mathbf{pleat}}_N^{<D}$ is nonempty for every $N \in \mathcal{B}_0(R)$. Even more, suppose $d > 0$ is given then if D is sufficiently large, there exists $f \in \overline{\mathbf{pleat}}_N < D$ whose distance from Γ_N is at least d.*

Proof. Again the idea of the proof is taking a geometric limit. Suppose $(N_i) \subset \mathcal{B}_0(R)$ is a sequence such that for every i, every $f \in \overline{\mathbf{pleat}}_N$ has distance at least i from Γ_{N_i} or $d_N(f(\partial H), \Gamma_N) \leq d$. After extracting a subsequence, which we still call (N_i), we can assume (N_i) converges strongly to $N \in \mathcal{B}_0(R)$. In N, take a pleated surface $f \in \overline{\mathbf{pleat}}_N$ that has distance at least $d + 1$ from Γ_N. If we use the approximating maps to push $f(\partial H)$ to N_i, the image has to be close to a pleated surface for $i \gg 0$ with distance more than d from Γ_{N_i}. The obtained pleated surface has to be in $\overline{\mathbf{pleat}}_N$ and we have a contradiction. $\square$

Fix a constant D_1 that satisfies lemma 7.4 for $d = d_0$ and let D_2 be the constant obtained in theorem 7.5 for $d = D_1$ and let $\eta > 0$ be a lower bound for the injectivity radius in the D_1-neighborhood of Γ_N for every N. Finally fix D_3 to be large enough to satisfy the conclusion of the above lemma and be bigger than

$$\max\{D_2, \cosh^{-1}(\frac{B}{\eta}) + B + D_1\}.$$

Now we define a projection from $\mathcal{C}(\partial H)$ to $\mathcal{C}(B, N)$ as follows:

$$\Pi_{N,B}(\alpha) := \bigcup_{f \in \overline{\mathbf{pleat}}_N(\alpha)} \mathbf{short}(f, B),$$

if α has a geodesic representative α^* in $N \setminus \Gamma_N$ with $d_N(\alpha^*, \Gamma_N) \geq D_1$ and

$$\Pi_{N,B}(\alpha) := A_{D_3,B} \cap \mathcal{C}(B, N),$$

otherwise.

The first part of the above definition always gives nonempty projections, since $\overline{\mathbf{pleat}}_N(\alpha)$ is nonempty by lemma 4.2 and B is bigger than the Bers' constant. Also because of lemma 7.8 and our assumption about D_3, the second part gives nonempty projections as well.

Similar to Minsky [Min01], we can prove that Π is a coarse Lipschitz projection:

Proposition 7.9. (Coarse Projection) *There exists $c > 0$ depending only on $\chi(\partial H)$, R and B such that*

(i) (Coarse idempotence) *If $\alpha \in \mathcal{C}(B, N)$ then $\alpha \in \Pi_{N,B}(\alpha)$.*

(ii) (Coarse Lipschitz) *For α and $\beta \in \mathcal{C}_0(\partial H)$ with $d_{\mathcal{C}}(\alpha, \beta) \leq 1$,*

$$\operatorname{diam}_{\mathcal{C}}(\Pi_{N,B}(\alpha) \cup \Pi_{N,B}(\beta)) \leq c.$$

Proof. Proof of part (i) is easy. Notice that there is always a useful compact core within distance d_0 of Γ_N. If α has a geodesic representative α^* in $N \setminus \Gamma_N$

with $d_N(\Gamma_N, \alpha^*) \geq D_1$, by lemma 7.4 there exists $f \in \overline{\mathbf{pleat}}_N$ that realizes α and then

$$\alpha \in \mathbf{short}(f, B) \subset \Pi_{N,B}(\alpha).$$

On the other hand suppose α does not have a geodesic representative in $N \backslash \Gamma_N$ with distance $\geq D_1$ from Γ_n. By definition of $\mathcal{C}(B, N)$ there exists $f \in \overline{\mathbf{pleat}}_N$ such that $\alpha \in \mathbf{short}(f, B)$. Then either $f(\alpha)$ is compressible and by lemma 3.2, $f(\partial H)$ has distance at most $B < D_3$ from Γ_N or $f(\alpha)$ is incompressible and α^*, the geodesic representative of α, has distance $\leq D_1$ from Γ_N. Then by lemma 2.11,

$$\begin{aligned}
d_N(f(\partial H), \Gamma_N) &\leq d_N(f(\alpha), \alpha^*) + l_N(\alpha^*) + d_N(\alpha^*, \Gamma_N) \\
&\leq \cosh^{-1}\left(\frac{l_{\sigma_f}(\alpha)}{l_N(\alpha^*)}\right) + B + D_1 \\
&\leq \cosh^{-1}(\frac{B}{\eta}) + B + D_1 \\
&\leq D_3.
\end{aligned}$$

In either case, $d_N(f(\partial H), \Gamma_N) \leq D_3$ and therefore

$$\alpha \in \mathbf{short}(f, B) \subset A_{D_3,B} = \Pi_{N,B}(\alpha).$$

For part (ii), first suppose that either α or β, say α, has a geodesic representative with distance $> D_2$ of Γ_N. By theorem 7.5 and our assumption that $d = D_1$, we know that β has a geodesic representative with distance more than D_1 from Γ_N. Therefore we have used the first definition for projection of both α and β. Statements (b) and (d) of theorem 7.5 imply that

$$\mathbf{short}(f, B) \cup \mathbf{short}(g, B)$$

has diameter bounded by $2A$ in $\mathcal{C}(\partial H)$ for every $f \in \overline{\mathbf{pleat}}_N(\alpha)$ and $g \in \overline{\mathbf{pleat}}_N(\beta)$. Hence

$$\operatorname{diam}_{\mathcal{C}}(\Pi_{N,B}(\alpha) \cup \Pi_{N,B}(\beta)) \leq 2A.$$

On the other hand suppose neither α nor β have geodesic representatives with distance $> D_2$ of Γ_N. We claim that $\Pi_N(\alpha)$ and $\Pi_N(\beta)$ are both included in $A_{D_3,B}$ and therefore their union has diameter bounded by diameter of $A_{D_3,B}$.

If α does not have a geodesic representative with distance $\geq D_1$ of Γ_N then the claim for α follows by definition of $\Pi_{N,B}(\alpha)$. If α does have a geodesic representative α^* with $d_N(\alpha^*, \Gamma_N) \geq D_1$, then

$$D_1 \leq d_N(\alpha^*, \Gamma_N) \leq D_2.$$

In particular, every $f \in \overline{\mathbf{pleat}}_N(\alpha)$ has distance $\leq D_2$ from Γ_N. Then

$$\mathbf{short}(f, B) \subset A_{D_2,B} \subset A_{D_3,B}$$

by corollary 7.7 and therefore

$$\Pi_{N,B}(\alpha) \subset A_{D_3,B},$$

and we have proved our claim for α. The same argument proves the claim for β and finishes proof of part (ii) by setting

$$c = \max\{\mathrm{diam}_{\mathcal{C}}(A_{D3,B}), 2A\}.$$

□

Lemma 7.10. (Minsky [Min01, Lem. 3.3]) *Let X be a δ-hyperbolic geodesic metric space and $Y \subset X$ a subset admitting a map $\Pi : X \to Y$ which is coarse-Lipschitz and coarse-idempotent. That is, there exists $C' > 0$ such that*

- *If $d(x, x') \leq 1$ then $d(\Pi(x), \Pi(x')) \leq C'$, and*
- *If $y \in Y$ then $d(y, \Pi(y)) \leq C'$. Then Y is K-quasi-convex, and furthermore if g is a geodesic in X whose endpoints are within distance a of Y then*

$$d(x, \Pi(x)) \leq b$$

for some $b = b(a, \delta, C')$, and every $x \in g$.

Similar to Minsky's [Min01], this proves:

Theorem 7.11. (Quasi-convexity Theorem) *There exists L depending only on R and $\chi(\partial H)$ such that for every B bigger than the Bers' constant B_0 and $N \in \mathcal{B}_0(R)$, the set*

$$\mathcal{C}(B, N) := \bigcup_{f \in \overline{\mathbf{pleat}}_N} \mathbf{short}(f, B),$$

is L-quasi-convex. Moreover, if β is a geodesic in $\mathcal{C}(\partial H)$ with endpoints in $\mathcal{C}(B, N)$ then $d_{\mathcal{C}}(x, \Pi_N(x, B)) \leq L$ for each $x \in \beta$.

Chapter 8

Bounded geometry

Here in this chapter, we prove:

Theorem 1.5 (Bounded geometry) *There exists η depending only on R and $\chi(\partial H)$ such that the injectivity radius of every hyperbolic structure with R-bounded combinatorics on H, $N \in \mathcal{B}_0(R)$, is bounded below by η.*

The proof is the same as Minsky's proof of the main theorem in [Min01]. We will discuss the differences in our setting. We can use lemma 7.2 and translate lemmas 4.4 (Homotopy bound) and 4.6 (Halfway surface) and corollary 4.7 (Interpolation) into our setting and in particular we have:

Corollary 8.1. (Interpolation) *Given $\epsilon > 0$ there exists $D > 0$ and $K > 0$ depending on ϵ, R and $\chi(\partial H)$ such that for a hyperbolic structure $N \in \mathcal{B}_0(R)$ and a useful compact core C the following holds. Let $P_0 \to P_1 \to \cdots \to P_n$ be an elementary-move sequence of pants decompositions on ∂H and let $f_0 \in \overline{\mathbf{pleat}}_N(P_0)$ then there exists $0 \leq k \leq n$ such that there exists $F : \partial H \times [0, k] \to N \setminus C$ with*

- $F_0 = f_0$,

- $F_i = F|_{\partial H \times \{i\}} \in \overline{\mathbf{pleat}}_N(P_i)$,
- $F_{i-1/2} = F|_{\partial H \times \{i-1/2\}} \in \overline{\mathbf{pleat}}_N(P_{i-1}) \cap \overline{\mathbf{pleat}}_N(P_i)$ *and*
- F *is a* (K, ϵ)*-good homotopy restricted to* $\partial H \times [i-1, i-\frac{1}{2}]$ *and* $\partial H \times [i-\frac{1}{2}, i]$ *with respect to* P_{i-1} *and* P_i

for every $i = 1, \ldots, k$. *Moreover,* $d_N(F_k(\partial H), C) < D$ *if* $k \neq n$.

Using the above corollary, for every point in N, we can find a continuous family of surfaces that cover that point. We use such an interpolation to rule out possibility of Margulis tubes with large diameter, which gives a lower bound for the injectivity radius of N.

8.1 The resolution sequence

Let $P_0 \to P_1 \to \cdots \to P_n$ be an elementary-move sequence and $\beta \in \mathcal{C}_0(\partial H)$, we denote

$$J_\beta := \{i \in [0, n] : \beta \in P_i\}.$$

Note that if J_β is an interval $[k, l]$, then the elementary move $P_{k-1} \to P_k$ exchanges some α for β and $P_l \to P_{l+1}$ exchanges β for some α', and we call them *predecessor* and *successor* of β, respectively.

We also use the notation

$$J_{[s,t]} := \bigcup_{i=s}^{t} J_{\beta_i},$$

where $\beta_0, \ldots, \beta_m$ is a sequence of vertices in $\mathcal{C}(\partial H)$. The following theorem is a consequence of work of Masur-Minsky [MM00].

Theorem 8.2. (Controlled Resolution Sequences) *[Min01, Thm. 5.1] Let P and Q be pants decompositions in ∂H. There exists a geodesic in $\mathcal{C}_1(\partial H)$ with vertex sequence $\beta_0, \ldots, \beta_m$, and an elementary move sequence $P_0 \to \cdots \to P_n$, with the following properties:*

1. *$\beta_0 \in P_0 = P$ and $\beta_m \in P_n = Q$.*

2. *Each P_j contains some β_i.*

3. *J_β, if nonempty, is always an interval, and if $[s,t] \subset [0,m]$ then*

$$|J_{[s,t]}| \leq b(t-s) \sup_Y d_Y(P,Q)^a,$$

   where the suprimum is over only those non-annular subsurfaces Y whose boundary curves are components of some P_k with $k \in J_{[s,t]}$.

4. *If β is a curve with non-empty J_β, then its predecessor and successor curves α and α' satisfy*

$$|d_\beta(\alpha, \alpha') - d_\beta(P,Q)| \leq \delta.$$

The constants a, b, δ depend only on $\chi(\partial H)$. The expression $|J|$ for an interval J denotes its diameter.

Let γ be a closed curve on ∂H such that $l_N(\gamma)$ is very small. We will try to bound the diameter of the Margulis tube $\mathbf{T}_\gamma(\epsilon_0)$ depending only on R and $\chi(\partial H)$ and if we are successful, we have proved theorem 1.5. We also fix a useful compact core $C \subset N$.

8.2 Initial pants

We know that $N \in \mathcal{B}_0(R)$ is associated to some $\alpha \in \mathcal{A}_0(R)$, where either α is the ending lamination of N or α is a marking that has length bounded by B_0 on the conformal structure at infinity of N. We also had $\beta \in \mathbf{m}_0(\mathrm{H})$ such that α and β have $(R+1)$-bounded combinatorics. Recall that $\mathbf{m}_0(\mathrm{H})$ was a finite set of handlebody markings on ∂H.

Let $P_- = \mathrm{base}(\beta)$ be the pants decomposition of β. Choose an element $M \in \mathcal{B}_0(R)$ and $f \in \overline{\mathbf{pleat}}_M^{<D}$. (For now we assume D is bigger than the constant in lemma 7.8 and therefore there exists one such f.) Lemma 7.6 shows that the σ_f is bi-Lipschitz to σ_f (there is a bi-Lipschitz map homotopic to identity from σ_f to σ_g) for every $N \in \mathcal{B}_0(R)$ and $g \in \overline{\mathbf{pleat}}_N^{<D}$, and the bi-Lipschitz constant does not depend on g. In particular this proves that

Fact 8.3. *Given D there exists $B_1 > 0$ such that $l_{\sigma_f}(P_-) \leq B_1$ for every $N \in \mathcal{B}_0(R)$ and $f \in \overline{\mathbf{pleat}}_N^{<D}$.*

Suppose N is convex cocompact (equivalently α is a marking). If γ is not a component of $\mathrm{base}(\alpha)$ then choose $P_+ = \mathrm{base}(\alpha)$. If γ is a component of $\mathrm{base}(\alpha)$, replace it with its transversal and let P_+ be the new pants decomposition. In either case, P_+ has length bounded by B_0 on the conformal structure at infinity and γ is not a component of P_+. As we explained in the last chapter, if $d_N(\partial\mathcal{CH}(N), \Gamma_N)$ is bounded by a uniform constant then we can use strong compactness of $\mathcal{B}_0(R)$ and imply that $\mathcal{CH}(N)$ has uniformly bounded diameter and it has uniform bounded geometry. Hence, we assume $d_N(\partial\mathcal{CH}(N), \Gamma_N)$ is uniformly large when required.

Recall that in the convex cocompact case the conformal structure at infinity is ϵ_0-thick and therefore we can use Bridgeman-Canary's theorem 2.12 [BC03] and see that

$$l_{\sigma_g}(P_+) \le JB_0$$

for some J depending only on ϵ_0, where $g \in \overline{\mathbf{pleat}}_N$ parametrizes $\partial\mathcal{CH}(N)$.

By assuming $d_N(\partial\mathcal{CH}(N), \Gamma_N)$ is large, we can make sure that P_+ has a geodesic representative in $N \setminus \Gamma_N$ with large distance from Γ_N and by lemma 7.4 $\overline{\mathbf{pleat}}_N(P_+) \neq \emptyset$. Choose some $f_+ \in \mathbf{pleat}_N(P_+)$; we have

$$f_+, g \in \mathbf{good}_N(P_+, JB_0).$$

Then because of lemma 4.4 (Homotopy bound), there exists a uniform constant K and a (K, ϵ_0)-good homotopy between f_+ and g which stays away from a useful compact core of N. Since γ is not a component of P_+, this good homotopy can only penetrate a distance K_1 into $\mathbf{T}_\gamma(\epsilon_0)$ and hence does not meet $\mathbf{T}_\gamma(\epsilon_1)$, with $\epsilon_1 < \epsilon_0$ depending only on K_1.

If N is geometrically infinite, let $Q_1, Q_2, \ldots$ be a sequence of pants decompositions which converge to a measured lamination supported on α. If i is sufficiently large $\overline{\mathbf{pleat}}_N(Q_i)$ will be nonempty and every element of $\overline{\mathbf{pleat}}_N(Q_i)$ will be far into the end of N. Choose $P_+ = Q_i$ and $f_+ \in \overline{\mathbf{pleat}}_N P_+$ such that $f_+(\partial H)$ encloses a compact subset of N that contains $\mathbf{T}_\gamma(\epsilon_0)$.

8.3 The interpolation

We fix $\epsilon_2 = \delta(\epsilon_1, \chi(\partial H))$ to be the constant obtained in lemma 3.3 and in particular

$$f((\partial H)^{\geq \epsilon_1}) \subset N^{\geq \epsilon_2} \tag{8.1}$$

whenever $f \in \overline{\mathbf{pleat}}_N$ because of our assumption in definition of $\overline{\mathbf{pleat}}_N$ that $d_N(f(\partial H), \Gamma_N) \geq D_0$ for every $f \in \overline{\mathbf{pleat}}_N$.

Now join P_+ and P_- with a resolution sequence $P_+ = P_0 \to \cdots \to P_n = P_-$ as in theorem 8.2. Then we can use corollary 8.1 for ϵ_2 and see that there exist constants K and D depending only on ϵ_2, R and $\chi(\partial H)$ and there exists a continuous family

$$F : \partial H \to [0,k] \to N \setminus C$$

of surfaces such that

- $F_0 = f_+$,
- $F_i = F|_{\partial H \times \{i\}} \in \overline{\mathbf{pleat}}_N(P_i)$,
- $F_{i-1/2} = F|_{\partial H \times \{i-1/2\}} \in \overline{\mathbf{pleat}}_N(P_{i-1}) \cap \overline{\mathbf{pleat}}_N(P_i)$ and
- $F|_{\partial H \times [i-1,i-1/2]}$ and $F|_{\partial H \times [i-1/2,i]}$ are (K, ϵ_2)-good homotopies with respect to P_{i_1} and P_i

for every $i = 1, \ldots, k$. Moreover, we know that $k < n$ and $F_k(\partial H)$ has distance at most D from C because the last pants decomposition P_- consists of a set of meridians and cannot be realized in N.

By fact 8.3, we know that

$$l_{F_k}(P_-) \leq B_1$$

for a constant B_1 that is independent of choice of $N \in \mathcal{B}_0(R)$. Also note that $F_k(\partial H)$ homologically encloses a compact set of bounded diameter independent of our choice of N or the resolution sequence.

Lemma 8.4. *Given D there exists K_1 such that if $N \in \mathcal{B}_0(R)$ and $f \in \overline{\mathbf{pleat}}_N^{<D}$ are given then $f(\partial H)$ homologically encloses a set with diameter bounded by K_1.*

Proof. Suppose (N_i, f_i) is a sequence of counterexamples. Without loss of generality, we can assume that $N_i \to N$ strongly as a sequence of marked hyperbolic structures. We can see that the pleated surfaces $f_i(\partial H)$ converge to a pleated surface $f \in \mathbf{pleat}_N$. The map f is homotopic to j and therefore homologically encloses a set with bounded diameter in N. Pushing this to the approximates we get an upper-bound for the diameter of sets that are homologically enclosed by $f_i(\partial H)$ and we have a contradiction. □

Therefore image of F covers $\mathbf{T}_\gamma(\epsilon_0)$ with degree 1 except for a set of uniformly bounded diameter and it is enough to show that $\mathbf{T}_\gamma(\epsilon_0) \cap F(\partial H \times [0, k])$ has bounded diameter.

Fix $B = \max\{B_1, JB_0\}$ and let ϵ_3 be such that a K-neighborhood of any ϵ_3-Margulis tube is still contained in an ϵ_2-Margulis tube.

Claim 8.5. *There is a subinterval $I_\gamma \subset [0, k]$ of diameter at most $2L$, so that $F(\partial H \times [i, i+1])$ can meet $\mathbf{T}_\gamma(\epsilon_3)$, where L is the quasi-convexity constant of* *theorem 7.11 and depends only on R and $\chi(\partial H)$.*

Proof. Suppose $\{\beta_1, \ldots, \beta_m\}$ is the vertex set of the chosen resolution between P_+ and P_-. We use the notation of theorem 8.2.

Suppose β_i is a component of P_j. If $F_j = F(\partial H \times \{j\})$ meets $\mathbf{T}_\gamma(\epsilon_2)$ then because of (8.1), $l_{\sigma_j}(\gamma) \leq \epsilon_1$ where $\sigma_j = \sigma_{F_j}$ and in particular

$$\gamma \in \mathbf{short}(F_j, B) \subset \Pi_N(\beta_i, B).$$

It follows from theorem 7.11 (Quasiconvexity) that

$$d_{\mathcal{C}}(\beta_i, \gamma) \leq L$$

where L depends only on R and $\chi(\partial H)$. Then because $\{\beta_0, \ldots, \beta_m\}$ are the vertices of a geodesic, the possible values of i lie in an interval of diameter at most $2L$, which we call I_γ and we have $i \in I_\gamma$.

Now notice that because of our choice of ϵ_3 and since $F|_{\partial H \times [i,i+1]}$ is a (K, ϵ_2)-good homotopy, if any track of a block $F(\partial H \times [j, j+1])$ meets $\mathbf{T}_\gamma(\epsilon_3)$ then one of the boundaries must meet $\mathbf{T}_\gamma(\epsilon_2)$ and hence j or $j+1$ is in J_{I_γ}. □

Let us restrict our elementary move sequence to

$$P_{s-1} \to \cdots \to P_{t+1}$$

where $[s,t] = J_{I_\gamma}$ and notice that this subsequence must still encase $\mathbf{T}_\gamma(\epsilon_3)$, since we have thrown away the blocks which avoid $\mathbf{T}_\gamma(\epsilon_3)$. Let $M = t - s = |J_{I_\gamma}|$.

Using part (3) of Theorem 8.2, tells us that

$$M \leq b(2L) \sup_{Y} d_Y(P_+, P_-)^a,$$

where the suprimum is over subsurfaces Y whose boundaries appear among the P_i in our subsequence. Such P_i must lie in a $L+1$ neighborhood of γ, in the $d_{\mathcal{C}}$ metric.

It follows from our assumption about R-bounded combinatorics of α and β respect to each other that

$$d_Y(P_+, P_-) \leq R + 2u \tag{8.2}$$

where u depends only on $\chi(\partial H)$ and $Y \subset \partial H$ is any subsurface. (This is obvious for convex cocompact case and for the geometrically infinite case, we can use Klarreich's theorem in the same way as in Minsky [Min01, Lem. 7.3].) This gives a uniform bound on M.

Suppose γ is not a component of any P_i. Then each block $F|_{\partial H \times [i,i+1]}$ has track lengths of at most $2K$ within $\mathbf{T}_\gamma(\epsilon_3)$. There are only $M+2$ blocks in our restricted sequence and they cover all of $\mathbf{T}_\gamma(\epsilon_3)$, so

$$\operatorname{diam}\mathbf{T}_\gamma(\epsilon_3) \leq 2K(M+2).$$

This bounds $l_N(\gamma)$ from below, and we are done.

When γ does appear among the $\{P_i\}$, the argument is exactly the same as Minsky's [Min01] and we only briefly describe the arguments. In this case J_γ is some nonempty interval of J_{I_γ} and suppose β and β' are the predecessor and successor curves to γ in the sequence. Both of them intersect γ and by part (4) of theorem 8.2, we know that $d_\gamma(\beta, \beta')$ is uniformly approximated by $d_\gamma(P_+, P_-)$, which by (8.2) is uniformly bounded depending on R.

Suppose $J_\gamma = [k, l]$. In $\partial H \times [k - 1/2, l + 1/2]$, the product region above $\mathbf{collar}(\gamma)$ is a solid torus, which we call U. (We are assuming that we have identified $\mathbf{collar}(\gamma)$ for the metrics induced by all integers and half-integers in $[k, l]$.) The map F can take the complement of this solid torus at most $2K(M + 2)$ into $\mathbf{T}_\gamma(\epsilon_3)$. Hence there is a uniform $\epsilon_4 > 0$ so that $F(U)$ must cover $\mathbf{T}_\gamma(\epsilon_4)$ and it will be enough to bound diameter of $F(U)$.

The boundary of U partitions into two horizontal annuli on $\partial H \times \{k - 1/2\}$ and $\partial H \times \{l + 1/2\}$ and a vertical annulus $\partial(\mathbf{collar}(\gamma)) \times [k - 1/2, l + 1/2]$. If we consider ∂U with the induced metric from F, we can show that all these annuli have uniformly bounded geometry. This is because $F_{k-1/2}, F_{l+1/2} \in \mathbf{good}(\gamma, A)$ (for a uniform A obtained from lemma 4.6) and γ has length at least ϵ_4 on these surfaces and final that hight of the vertical annulus is bounded by $K(M + 1)$.

Now we want to control length of a meridian of U. Suppose $\sigma(k - 1/2)$ and $\sigma(l + 1/2)$ are the metrics induced by $F_{k-1/2}$ and $F_{l+1/2}$ and as we said we assume $\mathbf{collar}(\gamma)$ is identified in both of them. Realize β as a geodesic in $\sigma(k - 1/2)$ and let b be an arc of $\beta \cap \mathbf{collar}(\gamma)$ (there may be two). Similarly assume β' is a geodesic in $\sigma(l + 1/2)$ and let b' be an arc $\beta' \cap \mathbf{collar}(\gamma)$. Similar to Minsky[Lem. 4.3]Min01, we have an upper-bound for the length of b in $\sigma(k - 1/2)$ and for the length of b' in $\sigma(l + 1/2)$. The curve

$$m = \partial(b \times [k - 1/2, l + 1/2]),$$

is a meridian of U, and we need to show that its length is uniformly bounded.

The idea here is to observe that length of the arc b in $\sigma(l+1/2)$ is estimated by the number of times it twists around the annulus, which is estimated by $d_\gamma(\beta, \beta')$. This last estimate follows from lemma 6.1 of Minsky [Min01].

It follows that $F(m)$ has uniformly bounded length and therefore spans a disk of bounded diameter. Then by a coning argument we can homotope F on all of U to a new map of bounded diameter. This bounds the radius of $\mathbf{T}_\gamma(\epsilon_4)$ from above and we are done.

Chapter 9

The sweep-out

Here by using the bounded geometry result and similar to our constructions in the last chapter, we want to give an interpolation that covers almost all of the convex core of $N \in \mathcal{B}_0(R)$ and is more efficient. We call such an interpolation a *sweep-out* (see definition 9.1). We prove the following proposition:

Proposition 9.1. *There exists $K > 0$ depending only on R and $\chi(\partial H)$ such that every $N \in \mathcal{B}_0(R)$ with any useful compact core $C \subset N$ admits a K-sweep-out.*

Note that, the bounded geometry theorem tells us that length of all incompressible curves is at least 2η. By definition, the pleated surfaces in $\overline{\mathbf{pleat}}_N$ have distance at least 1 from Γ_N and therefore by lemma 3.2, the length of image of compressible curves by elements of $\overline{\mathbf{pleat}}_N$ is at least 1. We always assume η is smaller than 1 and therefore the metric induced by any element of $\overline{\mathbf{pleat}}_N$ is η-thick.

Lemma 9.2. *There exists a constant $k_0 > 0$ such that if $N \in \mathcal{B}_0(R)$ and $f, g \in \overline{\mathbf{pleat}}_N$ are given with $d_N(f(\partial H), g(\partial H)) \geq k_0$, then the Teichmüller distance between the induced metrics is at least 1.*

Proof. If the distance between $f(\partial H)$ and $g(\partial H)$ is more than $2B_0 + 2d_0$ then we know that at least one of them, say $f(\partial H)$, has distance more than $B_0 + d_0$ from Γ_N. Now let α be the shortest simple closed curve on σ_f. Bers' observation shows that the σ_f-length of α is at most B_0. Therefore $f(\alpha)$ has length at most B_0 and by lemma 3.2, since its distance from a useful compact core of N is more than B_0, α cannot be a meridian and has a geodesic representative α^* in N.

Suppose $d_{\mathfrak{T}}(\sigma_f, \sigma_g) \leq 1$; then similar to Minsky [Min93], we can see that

$$\frac{l_{\sigma_g}(\alpha)}{l_{\sigma_f}(\alpha)} \leq ce^1,$$

where c depends only on $\chi(\partial H)$ and η. Therefore $l_{\sigma_g}(\alpha) \leq ceB_0$. Suppose α^* is the geodesic representative of α in N then by lemma 2.11 and using the fact that length of α^* is at least η, we have

$$d_N(f(\alpha), \alpha^*) \leq \cosh^{-1}(\frac{B_0}{\eta})$$

and

$$d_N(g(\alpha), \alpha^*) \leq \cosh^{-1}(\frac{ceB_0}{\eta}).$$

Since length of α^* is at most B_0, this gives an upper bound for the distance between $f(\partial H)$ and $g(\partial H)$ and we are done. □

From now, we fix the constant $k_0 \geq 1$ which satisfies the above lemma.

Lemma 9.3. *Suppose $\mathcal{B}_0(R)$ is as before. For every $a > 0$ there exists $b > 0$ such that if $N \in \mathcal{B}_0(R)$ with a useful compact core C and $f \in \overline{\mathbf{pleat}}_N$ are* *given and $l : I \to N \setminus C$, I is a compact interval in $\mathbb{R}$, is an arc of length at most a and $l(\partial I) \subset f(\partial H)$ then l is homotopic rel endpoints and within $N \setminus C$ to an arc of length at most b in $f(\partial H)$.*

Proof. The idea of proof is by taking geometric limits. Assume (N_i, f_i, l_i) is a sequence of counter examples. Take a base point $x_i \in N_i$ to be on $f_i(\partial H)$ and assume the sequence of pointed manifolds (N_i, x_i) converges in the geometric topology to (N_∞, x_∞). The sequence of pleated surfaces (f_i) also converge to a pleated image of ∂H in N_∞ (because $f_i(\partial H)$ is η-thick for every i).

If $d_{N_i}(x_i, \Gamma_{N_i})$ stays bounded then the limit N_∞ is in $\mathcal{B}_0(R)$. Similar to lemma 7.6, one can see that the pleated surfaces (f_i) converge to an element f of $\overline{\mathbf{pleat}}_{N_\infty}$ as marked pleated surfaces. We can replace each l_i with another arc l_i' homotopic to l_i relative endpoints and with length bounded depending on a and distance more than d_0 from Γ_{N_i}. Then the arcs l_i' converge to an arc l_∞' with bounded length which has distance more than d_0 from Γ_{N_∞} and has endpoints on $f(\partial H)$.

Let C_∞ be a useful compact core for N_∞. Since f is a homotopy equivalence between ∂H and $N_\infty \setminus C_\infty$, there exists an arc in $f(\partial H)$ which is homotopic to l_∞' (rel endpoints) within $N_\infty \setminus C_\infty$. If we map this arc and the homotopy between these two arcs to the approximates, it will be ϵ-close to an arc on $f_i(\partial H)$ which is homotopic (rel endpoints) to l_i within $N \setminus C$. The length of these arcs are bounded independently of i and we have a contradiction.

On the other hand, assume $d_{N_i}(x_i, \Gamma_{N_i}) \to \infty$ as $i \to \infty$ and $g : \partial H \to N_\infty$ is the limit pleated surface. Also let

$$\kappa_i : (N_i, x_i) \to (N_\infty, x_\infty)$$

be the approximating maps. For every i, consider the representation $\rho_i : \pi_1(\partial H) \to \mathrm{PSL}_2(\mathbb{C})$ induced by $(\kappa_i \circ g)_*$ and also consider $\rho : \pi_1(\partial H) \to \mathrm{PSL}_2(\mathbb{C})$ induced by g_*. It obviously follows that $(\rho_i) \to \rho$ in algebraic topology. Because of bounded geometry, we know that there are no parabolics in N_i and N_∞ and therefore the convergence of $\rho_i \to \rho$ is geometric too. Notice that here ρ_i is not faithful but it is not hard to see that for every $\alpha \in \pi_1(H)$, α is not in the kernel of ρ_i for i sufficiently large. This is because of the fact that the distance of x_i to C tends to infinity as $i \to \infty$. Once we have this we can use standard arguments about geometric and algebraic topology (cf. [NS]). This in particular shows that $g_* : \pi_1(\partial H) \to \pi_1(N_\infty)$ is surjective. Now similar to the last case, by passing to a subsequence we can assume that (l_i) converges to an arc l in N_∞ with endpoints on $g(\partial H)$ and length bounded $a+1$. Since g_* is surjective, we can homotope l (rel endpoints) to an arc in $g(\partial H)$ and the rest of the argument will be the same as the previous case. ☐

Lemma 9.4. *Given $D > 0$, there exists $K > 0$ such that for every f and $g \in \overline{\mathbf{pleat}}_N$, $N \in \mathcal{B}_0(R)$ and a useful compact core C, if*

$$d_N(f(\partial H), g(\partial H)) \leq D$$

then there is a $(K, 0)$-good homotopy between f and g within $N \setminus C$: a homeomorphism $\phi : \partial H \to \partial H$ and a homotopy with tracks length bounded by K between f and $g \circ \phi$ within $N \setminus C$. Also, the identity map on ∂H is K-bi-Lipschitz from the metric induced by f to the metric induced by $g \circ \phi$.

Proof. Using Bers' observation, we know that there exists a pants decomposition P whose σ_f-length is bounded by B_0. Since diameters of $f(\partial H)$ and

$g(\partial H)$ are uniformly bounded, we can represent every component of P with a closed curve with a base point on $g(\partial H)$ and length bounded depending on D, B_0 and diameter of $f(\partial H)$. Now we can use lemma 9.3 and see that this closed curve is homotopic to a closed curve on $g(\partial H)$ whose length is bounded depending on those constants. Hence there exists B_1 depending on D such that the total length of P in σ_g is at most B_1. This in particular implies that $f, g \in \mathbf{good}_N(P; B_1)$. If $f(\partial H)$ has a large distance (and therefore large $N^{\geq \epsilon_0}$-distance) from C depending on B_1 and η then it follows from lemma 4.4 (Homotopy bound) that there exists a (K, η)-good homotopy with respect to P between f and g. Since N has η-bounded geometry this is actually a $(K, 0)$-good homotopy and we are done.

Hence we can assume that the distance between $f(\partial H)$ and C is uniformly bounded depending on D. Suppose we have a sequence of counterexamples (N_i, f_i, g_i) which satisfy the hypothesis but fail the conclusion for bigger and bigger constants K. Since $f_i(\partial H)$ stays within a bounded distance from Γ_{N_i} independently of i, we can assume the sequence (N_i, p_{N_i}) converges strongly to $(N, p_N) \in \mathcal{B}_0(R)$ as marked hyperbolic structures. Suppose C is a useful compact core in N and take $C_i = \kappa_i(C)$, where

$$\kappa_i : N \to N_i$$

are the approximating maps. Since $f_i(\partial H)$ and therefore $g_i(\partial H)$ has bounded distance from Γ_{N_i} by using lemma 7.3 (Uniform bounded diameter lemma), we know that they stay in the D_1-neighborhood of Γ_{N_i} for some constant D_1 independent of i. Since $N_i \to N$ as a sequence of marked structures, we can see that the pleated surfaces (f_i) and (g_i) converge as marked pleated surfaces to $f, g \in \overline{\mathbf{pleat}}_N$. (This is similar to the proof of lemma 7.6.) In other terms, for every i there exists $\phi_i, \psi_i : \partial H \to \partial H$ isotopic to identity such that $\kappa_i \circ f$ and $\kappa_i \circ g$ are ϵ-close to $f_i \circ \phi_i$ and $g_i \circ \psi_i$ (as maps). The track length of the homotopy between f and g and the bi-Lipschitz constant of $\mathrm{id} : \sigma_f \to \sigma_g$ give upper-bounds for the the track length of the homotopy between $f_i \circ \phi_i$ and $g_i \circ \psi_i$ and the bi-Lipschitz constant of $\mathrm{id} : \sigma_{f_i \circ \phi_i} \to \sigma_{g_i \circ \psi_i}$ and we have a contradiction after precomposing with $\phi_i^{-1} \circ \psi_i$.

□

Definition 9.1. Given a useful compact core $C \subset N$, a map $G : \partial H \times I \to N$ for $N \in \mathcal{B}_0(R)$ is a *K-sweep-out*, if G maps into $\mathcal{CH}(N) \setminus C$ and has the following properties:

(a) $I = [0, n]$ for some integer $n \geq 0$ when N is convex cocompact and $I = [0, \infty)$ if N is geometrically infinite,

(b) for each integer $i \in I$

$$G_i = G|_{\partial H \times \{i\}} \in \overline{\mathbf{pleat}}_N,$$

(c) the block $G|_{\partial H \times [i-1,i]}$ has tracks $G(\{x\} \times [i-1, i])$ with length bounded by K for every $x \in \partial H$ and integer $0 < i \in I$,

(d) G covers every point in the convex core of N with degree 1 except a set of diameter bounded by K,

(e) G_0 has distance at most K from p_N and when N is convex cocompact and $I = [0, n]$, G_n gives a parametrization of the boundary of the convex core,

(f) $d_N(G_{i-1}(\partial H), G_i(\partial H)) \geq k_0$ for every positive integer $i \in I$,

(g) $G_i(\partial H)$ separates $G_{i-1}(\partial H)$ from the end of N and

(h) the identity map on ∂H is K-bi-Lipschitz from the metric induced by G_{i-1} to the metric induced by G_i and the Teichmüller distance between these metrics is bounded by K for every positive integer $i \in I$.

Proof of proposition 9.1. Suppose $N \in \mathcal{B}_0(R)$ and a useful compact core $C \subset N$ are given. The theorem is in fact an application of our proof in the last chapter and lemmas 9.2 and 9.4.

Our proof in the last chapter and in fact corollary 8.1 (Interpolation) imply that within distance D of every point in the convex core of N there exists a pleated surface in $\overline{\mathbf{pleat}}_N$, where D depends only on R and $\chi(\partial H)$. Also note that diameter of every element of $\overline{\mathbf{pleat}}_N$ is uniformly bounded by $-2\pi\chi(\partial H)/\eta$. Using this, we can construct a sequence $(G_i)_{i\in J}$ of elements of $\overline{\mathbf{pleat}}_N$, where $J = \{0, 1, \ldots\}$ when N is geometrically infinite and $J = \{0, 1, \ldots, n\}$ is finite when N is convex cocompact, such that

$$k_0 \leq d_N(G_{i-1}(\partial H), G_i(\partial H)) \leq D', \quad 0 < i \in J$$

for a constant D' depending on D, k_0 and diameter bound of pleated surfaces. Furthermore, we can assume that G_0 has distance at most D' from C and when N is convex cocompact, G_n parametrizes the boundary of $\mathcal{CH}(N)$. Note that the sequence (G_i) apparently satisfies (g) as well.

Now use lemma 9.4 to inductively construct $G|_{\partial H\times[i-1,i]}$ for every positive $i \in J$ as a $(K_1, 0)$-good homotopy in $N \setminus C$ for a uniform constant K_1. Note that during this construction, we may change G_i at each integer level by pre-composing with a self-homeomorphism of ∂H isotopic to identity. This gives a map

$$G : \partial H \times I \to N \setminus C$$

and $I = [0, \infty)$ when N is geometrically finite and $I = [0, n]$ when N is convex cocompact. Using lemma 8.4, we can see that $G_0(\partial H)$ homologically encloses a set with diameter uniformly bounded by K_2 and G covers every point outside of this set with degree 1. Finally note that if there is a K_1-bi-Lipschitz map homotopic to identity between two η-thick points of the Teichmüller space, then the Teichmüller distance between these points is bounded by K_3 depending only on K_1, η and the topology of the surface. Setting $K = \max\{K_1, K_2, K_3\}$, we can see that G is a K-sweep-out. □

Chapter 10

The Model Manifold

In this chapter, we want to use the sweep-out constructed in the last chapter to construct a bi-Lipschitz model for the geometry of every $N \in \mathcal{B}_0(R)$, which is determined by $\mathcal{E}(N)$ and the bi-Lipschitz constant depends only on R and $\chi(\partial H)$.

The models are similar to Minsky's models in [Min94]. But we use the description of the model in a way similar to Mosher [Mo03]. We should point out that independently Bowditch [Bo] has given a similar description of these models in terms of marked hyperbolic surface bundles on Teichmüller geodesics as well. Mosher's work is set up to give a model for surface groups with degenerate hyperbolic structure and bounded geometry. But the same construction gives a uniform model for the convex cocompact case as well. Here, we use this to get a uniform model for both convex cocompact and geometrically infinite structures in $\mathcal{B}_0(R)$. Recall the definition of the marked hyperbolic surface bundle on a Teichmüller geodesic and the definition of the SOLV-metric from 2.9 and 2.10.

Theorem 10.1. (The model manifold) *Suppose the handlebody H of genus*

> 1 is given. Given R, there exist constants L and c, for which the following holds. Let $N \in \mathcal{B}_0(R)$ be a hyperbolic structure on H. For a choice of a useful compact core $C \subset N$, there exists a cobounded geodesic ray or segment g in $\mathfrak{T}(\partial H)$, such that:

(1) *There is a map $\Phi : \mathcal{S}_g^{\mathrm{SOLV}} \to N_e$, properly homotopic to a homeomorphism and in the homotopy class determined by j, which lifts to a (L, c)-quasi-isometry of universal covers $\mathcal{H}_g^{\mathrm{SOLV}} \to \widetilde{N}_e$, where $N_e = \mathcal{CH}(N) \setminus C$.*

(2) *The initial point of g is a fixed point $\tau_H \in \mathfrak{T}$ and the "terminal" point is $\tau(\mathcal{E}(N))$. (The "terminal" point is a finite endpoint that corresponds to the conformal structure at infinity, in case N is convex cocompact and an ideal endpoint in Thurston's compactification of Teichmüller space, that corresponds to the ending lamination of N, when N is geometrically infinite.)*

Suppose C is a fixed useful compact core in N and $G : \partial H \times I \to N \setminus C$ is the K-sweep-out constructed in proposition 9.1 and let $\sigma_i = \sigma_{G_i}$ be the metric induced by the pleated surface G_i for every integer $i \in I$. We know that the Teichmüller distance between σ_i and σ_{i+1} is at most K. Hence, there is a $\mathbb{Z}$-piecewise affine, K-Lipschitz path $\gamma : I \to \mathfrak{T}$ with $\gamma(n) = \sigma_n$. Since $\mathrm{inj}(\sigma_n)$ is at most η, it follows that the path γ is $\mathcal{K}$-cobounded, where $\mathcal{K}$ depends only on η and K.

Now consider the canonical hyperbolic surface bundle $\mathcal{S}_\gamma \to I$ and its universal cover, the canonical hyperbolic plane bundle $\mathcal{H}_\gamma \to I$. Note that $\mathcal{S}_n \approx \sigma_n$ for every integer $n \in I$, where $\mathcal{S}_n$ denotes the fiber above $\gamma(n)$. We can identify $\partial H \times I$ with $\mathcal{S}_\gamma$ and from now on we consider the sweep-out as a map

$$G : \mathcal{S}_\gamma \to N.$$

Note that the restriction of G to $\mathcal{S}_n$ is length preserving for every integer $n \in I$ and because of the property (c) of the sweep-out, the restriction of G to any $x \times [n, n+1]$ is a K-Lipschitz map. Also note that property (f) of the sweep-out and lemma 9.2 show that the distance between σ_n and σ_{n+1} is at least 1 and therefore the Hausdorff distance between $\mathcal{S}_n$ and $\mathcal{S}_{n+1}$ is between 1 and K.

We know that image of the sweep-out is contained inside $\mathcal{CH}(N) \setminus C$. Let $N_e = \mathcal{CH}(N) \setminus C$, which is homeomorphic to $\partial H \times [0, \infty)$ or $\partial H \times [0, 1]$ depending on whether N is geometrically infinite or convex cocompact.

Proposition 10.2. *The map $G : \mathcal{S}_\gamma \to N_e$ lifts to a quasi-isometry of universal covers $\widetilde{G} : \mathcal{H}_\gamma \to \widetilde{N}_e$, with constants depending only on R and $\chi(\partial H)$.*

Proof. First we show that $\widetilde{G}$ is coarse surjective. By property (e) of the sweep-out, we actually know that G is surjective to N_e except possibly for a neighborhood of bounded diameter about ∂C. But we know that G_0 is a pleated surface contained in a bounded neighborhood of Γ_N. Therefore by lemma 9.4, there is a homotopy with bounded tracks between ∂C and $G_0(\partial H)$. This homotopy covers every point in the complement of $G(\mathcal{S}_\gamma)$ in N_e. Therefore, by lifting it to the universal cover, one can conclude that every point has bounded distance from image of $\widetilde{G}$.

Using fact 2.1, it is enough to show that $\widetilde{G}$ is uniformly proper with constants and properness gauge independent of N. We know that $\widetilde{G}|_{\mathcal{H}_n}$ is lift of a pleated surface and is distance non-increasing. On the other hand, length of a connection line $x \times [n, n+1]$ is at least 1 in $\mathcal{H}_\gamma$ and its image $\widetilde{G}(x \times [n, n+1])$

has length at most K and therefore $\widetilde{G}$ is Lipschitz along the connection lines as well and these two easily prove that it is Lipschitz everywhere.

The following follows from lemma 9.3 and the fact that lift of a pleated map is distance non-increasing.

Lemma 10.3. (Pleated surfaces are proper) *Given R there exists a properness gauge $\rho : [0, \infty) \to [0, \infty)$, such that if $N \in \mathcal{B}_0(R)$ and $f \in \overline{\mathbf{pleat}}_N$ are given then the lift to the universal covers $\widetilde{f} : \widetilde{\partial H} \to \widetilde{N}_e$ is ρ-uniformly proper.*

This shows that the map $\widetilde{G}$ is uniformly proper along $\mathcal{H}_n$ for every integer n, where $\mathcal{H}_n$ is fiber of $\mathcal{H}_\gamma$ above $\gamma(n)$ and is the universal cover of $\mathcal{S}_n$. Once we know this, we can use property (f) of the sweep-out, to see that if $x \in \mathcal{H}_t$ and $y \in \mathcal{H}_s$ are given then

$$d_{\widetilde{N}_e}(\widetilde{G}(x), \widetilde{G}(y)) \geq k_0 \lfloor s - t \rfloor, \tag{10.1}$$

where $\lfloor x \rfloor$ is the greatest integer $\leq x$. This fact together with the above lemma prove that $\widetilde{G}$ is uniformly proper with constants and properness gauge independent of N and we are done. (cf. Mosher's argument in [Mo03, Claim 4.7].) □

Lemma 10.4. ($\mathcal{H}_\gamma$ is hyperbolic) *The space $\mathcal{H}_\gamma$, or equivalently $\widetilde{N}_e$, is a hyperbolic metric space in sense of Gromov with constants depending only on R and $\chi(\partial H)$.*

Proof. In the proof of the above lemma, we use an idea which was briefly described in Mosher [Mo03, Sec. 4.4] and was partly based on Farb-Mosher [FM02]. The proof is similar to Farb-Mosher's proof of [FM02, Lem. 5.2] with some modifications in our situation and a part which was missing in their proof.

Given $\kappa > 1$, an integer $n \geq 1$ and $A \geq 0$, we say that a sequence of non-negative integers $(r_i)_{i \in J}$ indexed by a subinterval $J \subset \mathbb{Z}$ satisfies the (κ, n, A)-*flaring property* if, whenever the three integers $i-n, i, i+n$ are all in J, we have:

$$r_i > A \quad \Longrightarrow \quad \max\{r_{i-n}, r_{i+n}\} \geq \kappa \cdot r_i.$$

The number A is called the *flaring threshold* and notice that by making n larger, we can make κ as large as we want.

Let $\mathcal{H}_\gamma$ be given for a $\mathbb{Z}$-piecewise affine, K-Lipschitz, $\mathcal{K}$-cobounded path $\gamma : I \to \mathfrak{T}$. A λ-*quasihorizontal path* in $\mathcal{H}_\gamma$, $\lambda \geq 1$, is a λ-Lipschitz path $\alpha : I' \to \mathcal{H}_\gamma$, $I' \subset I$, such that $\alpha(t) \in \mathcal{H}_t$ for every $t \in I'$. We say $\mathcal{H}_\gamma$ satisfies the *horizontal flaring property* if there exists $\kappa > 1$, an integer $n \geq 1$, and a function $A(\lambda) : [1, \infty) \to (0, \infty)$, such that if $\alpha, \beta : I' \to \mathcal{H}_\gamma$ are two λ-quasihorizontal paths with the same domain I', then setting $J = I' \cap \mathbb{Z}$ the sequence

$$d_i(\alpha(i), \beta(i)), \quad i \in J$$

satisfies the $(\kappa, n, A(\lambda))$ flaring property, where d_i is the distance function on $\mathcal{H}_i$, $i \in I$.

Farb-Mosher [FM02, Lem. 5.4] used Bestvina-Feighn Combination Theorem [BF92] to prove that if $\mathcal{H}_\gamma$, for a $\mathcal{K}$-cobounded, K-Lipschitz $\mathbb{Z}$-piecewise affine path γ, satisfies $(\kappa, n, A(\lambda))$-horizontal flaring then $\mathcal{H}_\gamma$ is δ-hyperbolic in sense of Gromov, where δ depends on K, $\mathcal{K}$ and the flaring data $\kappa, n, A(\lambda)$. Therefore it will be enough to show that $\mathcal{H}_\gamma$ satisfies $(\kappa, n, A(\lambda))$ horizontal flaring.

Let's go back and consider the path $\gamma : I \to \mathfrak{T}$ which was determined by taking $\sigma(i) = \sigma_i$ the induced metric from G_i of the sweep-out. We want to prove that $\mathcal{H}_\gamma$ is hyperbolic.

Let $\Sigma_i = G_i(\mathcal{S}_i)$ for every integer $i \in I$ and let $\widetilde{\Sigma}_i$ be its lift in $\widetilde{N}_e$. Recall that it follows from properties of the sweep-out that the Hausdorff distance between Σ_i and Σ_{i+1} (or between $\widetilde{\Sigma}_i$ and $\widetilde{\Sigma}_{i+1}$) is at least k_0 and at most K. Also recall that $G_i : \mathcal{S}_i \to \Sigma_i$ and its lift $\widetilde{G}_i : \mathcal{H}_i \to \widetilde{\Sigma}_i$ are length preserving and we denote the distance in σ_i by d_i and the distance in $\widetilde{N}_e$ by d. We should point out that Σ_i may not be embedded in N_e. Because of this for points $x, y \in \Sigma_i$ we do not use the distance induced from N_e, instead we use the distance on the path-metric σ_i induced by $G_i : \partial H \to N_e$. Note that lemma 10.3, shows properness of this distance when it is mapped by a pleated surface. Also by a d_i-geodesic in $\widetilde{\Sigma}_i$ (or perhaps in Σ_i), we mean a geodesic for the metric d_i, or equivalently an image of a geodesic in $\sigma_i \approx \mathcal{S}_i$.

Notice that $\widetilde{G}$-image of a λ'-quasihorizontal path in $\mathcal{H}_\gamma$ is a λ-Lipschitz path $\alpha : I' \to \widetilde{N}_e$, $I' \subset I$, such that $\alpha(i) \in \widetilde{\Sigma}_i$ for every integer $i \in I'$ and λ depends on λ' and K. By abuse, we call a $\widetilde{G}$-image of a λ-quasihorizontal a λ-quasihorizontal in $\widetilde{N}_e$.

It follows that it is enough to show the existance of flaring data $\kappa, n, A(\lambda)$ such that for every two λ-quasihorizontal paths $\alpha, \beta : I' \to \widetilde{N}_e$, $I' \subset I$, in $\widetilde{N}_e$ the sequence

$$d_i(\alpha(i), \beta(i)) \quad i \in I' \cap \mathbb{Z}$$

satisfies $(\kappa, n, A(\lambda))$ flaring.

Fix a number $\lambda_0 \geq K$; first we obtain constants κ_0, n_0, A_0 such that the above sequence satisfies (κ_0, n_0, A_0)-flaring for λ_0-quasihorizontal paths α, β. Then we use this and prove the existence of uniform flaring data $\kappa, n, A(\lambda)$.

Suppose $\alpha, \beta : I' \to \widetilde{N}_e$, $I' \subset I$, are λ_0-quasihorizontal. Let $J = I' \cap \mathbb{Z} = \{i_-, \ldots, i_+\}$ and assume $i_+ - i_-$ is even and $i_0 = \frac{i_+ + i_-}{2} \in J$. Also define $D_i = d_i(\alpha(i), \beta(i))$.

Recall that the connection map in $\mathcal{H}_\gamma$ gives a map $h_{ij} : \mathcal{H}_i \to \mathcal{H}_j$ which is $L^{|i-j|}$-bi-Lipschitz for every pair of integers $i, j \in I$ and L depends only on $\mathcal{K}$ and K the Lipschitz constant of γ. This gives a similar bi-Lipschitz condition for the distances d_i and d_j. We cannot represent this by a bi-Lipschitz map between $\widetilde{\Sigma}_i$ and $\widetilde{\Sigma}_j$ since they are not embedded; but this shows that if $x, y \in \mathcal{H}_i$ are given

$$d_j(\widetilde{G}(h_{ij}(x)), \widetilde{G}(h_{ij}(y))) \leq L^{|i-j|} d_i(\widetilde{G}(x), \widetilde{G}(y)).$$

For each $i \in J$, let $\rho_i : [0, D_i] \to \widetilde{\Sigma}_i$ be a d_i-geodesic with endpoints $\alpha(i)$ and $\beta(i)$. Similar to [FM02, Claim 5.3], we have:

Claim 10.5. *there is a family of quasihorizontal paths v described as follows:*

- *For each $i \in J$ and each $t \in [0, D_i]$ the family contains a unique quasi-horizontal path $v_{it} : [i_-, i_+] \to \widetilde{N}_e$ that passes through the point $\rho_i(t)$. If we fix $i \in J$, we thus obtain a parametrization of the family v_{it} by points $t \in [0, D_i]$.*

- *The ordering of the family v_{it} induced by the order on $t \in [0, D_i]$ is independent of i. The first path v_{i0} in the family is identified with α, and* *the last path v_{iD_i} is identified with β.*

- *Each v_{it} is λ'_0-quasihorizontal, where λ'_0 depends only on λ_0 and L.*

It is easy to see that a λ'_0-quasihorizontal path in $\widetilde{N}_e$ is a (λ''_0, a) quasi-geodesic for constants λ''_0 and a depending on λ, L and k_0. We already know that it is Lipschitz. On the other hand using property (f) of the sweep-out, we know that $d(\alpha(i), \alpha(j)) \geq k_0|i-j|$ for integers i, j in the domain of α. In fact, the same argument shows that its images in N and its universal cover $\widetilde{N} = \mathbb{H}^3$ are also (λ''_0, a)-quasigeodesic. In $\mathbb{H}^3$, there exists a constant δ_1 depending only on λ''_0 and a such that for any rectangle of the form $v * \sigma * w * \sigma'$ where σ, σ' are geodesics and v, w are (λ''_0, a)-quasigeodesics, any point on v is within distance δ_1 of $\sigma \cup w \cup \sigma'$.

By lemma 10.3 (Pleated surfaces are proper), there exists a constant δ_2 such that:

for all $i \in J$, $x, y \in \widetilde{\Sigma}_i$, if $d(x, y) \leq \lambda''_0(\delta_1 + 1) + \delta_1$ then $d_i(x, y) \leq \delta_2$.
Now consider the flaring parameters κ', n', A' defined below:

$$\kappa' = \frac{3}{2}$$
$$n' = \lfloor \delta_1 + 3\delta_2 \rfloor + 1$$
$$A' = \delta_2$$

First, we show that if all the indices in J are bigger than $6\delta_2$ then the sequence $\{D_i\}_{i \in J}$ has κ', n', A' flaring property. For simplicity and without loss of generality, we assume δ_2 is an integer. Suppose $i\pm = i_0 \pm n'$, we must prove that

- if $D_{i_0} > A'$ then $\max\{D_{i-}, D_{i+}\} \geq \kappa' D_{i_0}$.

Case 1. $\max\{D_{i-}, D_{i+}\} \leq 6\delta_2$. We claim that we can take geodesics $\sigma_\pm$ in the interior of $\widetilde{N}_e$ with the same endpoints as $\rho_{i\pm}$ and length $\leq 6\delta_2$. The reason why $\sigma_\pm$ will be in the interior of $\widetilde{N}_e$ is that all the indices are bigger than $6\delta_2$, therefore the endpoints of $\rho_{i\pm}$ have distance at least $6\delta_2$ from $\widetilde{\Sigma}_0$ and $\partial\widetilde{N}_e$. Therefore the geodesic representative of these arcs (rel endpoints) is inside $\widetilde{N}_e$.

Notice that a geodesic in the interior of $\widetilde{N}_e$ projects to a geodesic in N. This shows that the rectangle $\alpha * \sigma_- * \beta * \sigma_+$ projects to a homotopically trivial rectangle in N_e and its lift to $\widetilde{N} = \mathbb{H}^3$ is a rectangle $\alpha' * \sigma'_- * \beta' * \sigma'_+$ where $\sigma'_\pm$ are geodesics and α' and β' are (λ''_0, a)-quasigeodesics. Hence every point of α' has distance at most δ_1 from $\sigma'_- \cup \beta' \cup \sigma'_+$. This rectangle lifts isometrically to $\widetilde{N}_e$ and therefore, every point of α has distance at most δ_1 from $\sigma_- \cup \beta \cup \sigma_+$. Consider now the point $\alpha(i_0)$ and suppose it has distance at most δ_1 from $z \in \sigma_- \cup \beta \cup \sigma_+$.

If $z \in \sigma_+$ or σ_-, say σ_+ then it follows that

$$d(\alpha(i_0), \widetilde{\Sigma}_{i+}) \leq \delta_1 + \frac{6\delta_2}{2} < n'$$

which implies that $d(\widetilde{\Sigma}_{i_0}, \widetilde{\Sigma}_{i+}) < n'$ but we know that the distance between these two is at least $n' \cdot k_0 \geq n'$ and we have a contradiction. Therefore $z = \beta(s) \in \beta$.

By (10.1), we know that

$$d(\beta(s), \widetilde{\Sigma}_{i_0}) \geq k_0 \lfloor |s - i_0| \rfloor > |s - i_0| - 1$$

and therefore

$$|s - i_0| - 1 < \delta_1 \;\Rightarrow\; |s - i_0| < \delta_1 + 1 \;\Rightarrow\; d(\beta(s), \beta(i_0)) < \lambda_0''(\delta_1 + 1).$$

Hence

$$d(\alpha(i_0), \beta(i_0)) \leq d(\alpha(i_0), \beta(s)) + d(\beta(s), \beta(i_0)) < \delta_1 + \lambda_0''(\delta_1 + 1),$$

and this implies that $D_{i_0} = d_{i_0}(\alpha(i_0), \beta(i_0)) < A'$.

Case 2. $\max\{D_{i_-}, D_{i_+}\} \geq 3\delta_2$ Suppose v is the family of quasihorizontals constructed in claim 10.5 and assume we consider the parametrization at $i = i_0$. It is not hard to see that there is a discrete subfamily $\alpha = v_{t_0}, v_{t_1}, \ldots, v_{t_k} = \beta$, with $t_0 < t_1 < \cdots < t_k$, such that the following is satisfied: for each $l = 1, \ldots, k$, letting

$$\Delta_{l\pm} = d_{i\pm}(v_{t_{l-1}}(i\pm), v_{t_l}(i\pm))$$

then we have

$$\max\{\Delta_{l-}, \Delta_{l+}\} \in [3\delta_2, 6\delta_2].$$

Since

$$\max\{\Delta_{l-}, \Delta_{l+}\} \leq 6\delta_1,$$

the argument in **Case 1** shows that

$$\Delta_{l0} = d_{i_0}(v_{t_{l-1}}(i_0), v_{t_l}(i_0)) \leq \delta_2$$

for all $l = 1, \ldots, k$. Hence

$$D_{i_0} = \sum_{l=1}^{k} \Delta_{l0} \le k\delta_2$$

$$\begin{aligned} D_{i-} + D_{i+} &= \sum_{l=1}^{k} \Delta_{l-} + \Delta_{l+} \ge \sum_{l=1}^{k} \max\{\Delta_{l-}, \Delta_{l+}\} \\ &\ge k \cdot 3\delta_2. \end{aligned}$$

Then

$$\max\{D_{i-}, D_{i+}\} \ge \frac{3}{2}k\delta_2 \ge \frac{3}{2}D_{i_0}.$$

This proves that the sequence (D_i) has κ', n', A' flaring when we restrict it to indices bigger than $6\delta_2$.

Lemma 10.6. *There exists a constant $b > 0$ depending on K, λ_0 and the properness gauge ρ in lemma 10.3 such that the sequence (D_i) satisfies a (K, b)-*coarse Lipschitz growth condition*: $D_i \le KD_j + b$ given $i, j \in J$ with $|i-j| = 1$.*

Proof. First recall that the map $h_{ij} : \widetilde{\Sigma}_i \to \widetilde{\Sigma}_j$ is $K^{|i-j|}$-bi-Lipschitz. Given $i, j \in J$ with $|i - j| = 1$, let $a = h_{ij}(\alpha(i))$ and $b = h_{ij}(\beta(i))$. The points a and $\alpha(j)$ are connected by a path of length at most $K\lambda_0 + K$, consisting of a segment of α from $\alpha(j)$ to $\alpha(i)$ and a path of length at most K from $\alpha(i)$ to a, and similarly the distance between b and $\beta(j)$ is at most $K\lambda_0 + K$. Suppose

$b = 2\rho(K\lambda_0 + K)$ where ρ is the properness gauge in lemma 10.3. Then

$$\begin{aligned} d_j(a, \alpha(j)) &\le b/2 \\ d_j(b, \beta(j)) &\le b/2 \\ d_j(\alpha(j), \beta(j)) &\le d_j(\alpha(j), a) + d_j(a, b) + d_j(b, \beta(j)) \\ &\le b + K d_i(\alpha(i), \beta(i)) \end{aligned}$$

This finishes the proof of the lemma. □

From the above, we can easily see that there exists b' depending only on K, b and δ_2 such that if $i, j \in J$ and $i - j \le 6\delta_2$ then

$$D_i \ge K^{-m} D_j - b' \tag{10.2}$$

where $m = \lfloor 6\delta_2 \rfloor + 1$.

We knew that the sequence (D_i) has (κ', n', A')-flaring property when restricted to the indices $\ge 6\delta_2$. By choosing n'' to be a multiple of n', we can assume that the sequence $(D_i)_{i \ge m}$ also has (κ'', n'', A') flaring, where $\kappa'' > 4K^m$. Now consider

$$\begin{aligned} \kappa_0 &= 3 \\ n_0 &= n'' + m \\ A_0 &= \max\{A', b'\} \end{aligned}$$

and we claim that the entire sequence $(D_i)_{i \in J}$ has (κ_0, n_0, A_0)-flaring. Suppose $i \pm n_0$ and i are in J. If $i - n_0 \ge m$ then it is obvious. Suppose $D_i > A_0 \ge A'$;

we already know that

$$\max\{D_{i-n''}, D_{i+n''}\} \geq \kappa'' D_i,$$

since indices $i \pm n''$ and i are bigger than $6\delta_2$. Suppose $D_{i+n''} \geq \kappa'' D_i$, then by (10.2)

$$\begin{aligned}
D_{i+n_0} &\geq K^{-m} D_{i+n''} - b' \\
&\geq K^{-m} \kappa'' D_i - b' \\
&> 4D_i - b' \\
&> 3D_i + (D_i - b') \\
&> 3D_i.
\end{aligned}$$

The same argument works in the other case and we have proved that the sequence $(D_i = d_i(\alpha(i), \beta(i)))_{i \in J}$ has (κ_0, n_0, A_0)-flaring property for every pair of λ_0-quasihorizontals $\alpha, \beta : I' \to \widetilde{N}_e$, $I' \subset I$.

Using the above, we want to prove that for arbitrary $\lambda > 0$ and λ-quasihorizontals $\alpha, \beta : I' \to \widetilde{N}_e$, $I' \subset I$, the sequence

$$d_i(\alpha(i), \beta(i)) \quad i \in J = I' \cap \mathbb{Z}$$

has $(\kappa, n, A(\lambda))$ property, where

$$\kappa = 2$$

$$n = n_0$$

$$A(\lambda) = \max\{A_0, 2\rho(K + K \cdot \lambda)\}$$

and ρ is the properness gauge in lemma 10.3 (Pleated surfaces are proper).

Suppose i_0 and $i_0 \pm n_0$ are in J and we define $i\pm = i_0 \pm n_0$,

$$D_0 = d_{i_0}(\alpha(i_0), \beta(i_0)) \text{ and } D_\pm = d_{i\pm}(\alpha(i\pm), \beta(i\pm)).$$

We want to prove that

$$\max\{D_+, D_-\} \geq 3D_0 \quad \text{if } D_0 > A(\lambda).$$

If $\lambda \leq \lambda_0$ then the statement easily follows since $A(\lambda) \geq A_0$; therefore we can assume $\lambda > \lambda_0 \geq K$.

Let

$$\alpha' = \widetilde{G}|_{\{x\}\times[i-,i+]} : [i-, i+] \to \widetilde{N}_e$$

such that $\alpha'(i_0) = \alpha(i_0)$. In the same way define $\beta' : [i-, i+] \to \widetilde{N}_e$. We know that α' and β' are K-quasihorizontals and since $K \leq \lambda_0$

$$\max\{d_{i-}(\alpha'(i-), \beta'(i-)), d_{i+}(\alpha'(i+), \beta'(i+))\} \geq 3d_{i_0}(\alpha'(i_0), \beta'(i_0)) = 3D_0$$

whenever $D_0 \geq A_0$. Similar to proof of lemma 10.6 there is a path of length at most $K + K \cdot \lambda$ connecting $\alpha(i\pm)$ and $\alpha'(i\pm)$ obtained by moving along α from $\alpha(i\pm)$ to $\alpha(i_0)$ and then by moving along α' from $\alpha'(i_0)$ to $\alpha'(i\pm)$. By lemma 10.3 (Pleated surfaces are proper), we have

$$d_{i\pm}(\alpha(i\pm), \alpha'(i\pm)) \leq \rho(K + K \cdot \lambda) \leq \frac{A(\lambda)}{2}.$$

The same argument shows that

$$d_{i\pm}(\beta(i\pm), \beta'(i\pm)) \leq \frac{A(\lambda)}{2},$$

and we have

$$\begin{aligned} D_{\pm} &= d_{i\pm}(\alpha(i\pm), \beta(i\pm)) \\ &\geq d_{i\pm}(\alpha'(i\pm), \beta'(i\pm)) - d_{i\pm}(\alpha(i\pm), \alpha'(i\pm)) - d_{i\pm}(\beta(i\pm), \beta'(i\pm)) \\ &\geq d_{i\pm}(\alpha'(i\pm), \beta'(i\pm)) - A(\lambda) \end{aligned}$$

Hence

$$\begin{aligned} \max\{D_+, D_-\} &\geq \max\{d_{i-}(\alpha'(i-), \beta'(i-)), d_{i+}(\alpha'(i+), \beta'(i+))\} - A(\lambda) \\ &\geq 3D_0 - A(\lambda) \\ &= 2D_0 + (D_0 - A(\lambda)) \\ &\geq 2D_0 \end{aligned}$$

whenever $D_0 \geq A(\lambda)$ and this finishes our proof.

$\square$

Once we know that $\mathcal{H}_\gamma$ is hyperbolic, it follows from work of Mosher [Mo03, Thm 1.1, Prop. 2.3] that γ fellow travels a cobounded geodesic segment or ray g' in $\mathfrak{T}$ with $\gamma(0) = g'(0)$ and we obtain $\pi_1(\partial H)$-equivariant quasi-isometries

$$\mathcal{H}_{g'}^{\mathrm{SOLV}} \to \mathcal{H}_\gamma \xrightarrow{\widetilde{G}} \widetilde{N}_e,$$

with constants independent of N.

Going back to the steps of our construction, we notice that the initial point $\gamma(0)$ was the metric induced by a pleated surface in $\mathbf{pleat}_{N \backslash \Gamma_N}$ with uniformly bounded distance from the useful compact core C independently of N. An immediate, application of lemma 7.6 will be that such initial points are all contained in a compact subset of $\mathfrak{T}(\partial H)$ and therefore they have bounded distance from the base point $\tau_H \in \mathfrak{T}(\partial H)$.

On the other hand, when N is convex cocompact, the terminal point of γ was the metric induced by the boundary of the convex core, in the homotopy class represented by j. Then since τ, the conformal structure at infinity, is ϵ_0-thick, we can use Bridgeman-Canary's result [BC03], theorem 2.12, and conclude that this terminal point has bounded distance from τ independently of N.

Finally, when N is geometrically infinite, we know that the sequence $(\gamma(i))$ represents the metrics induced by a sequence of elements of $\overline{\mathbf{pleat}}_N$ which exit the end of N. Then it follows from Canary's [Can93b] description of the ending laminations for these structures that every limit of the sequence $(\gamma(i))$ in Thurston's compactification of the Teichmüller space, is an element of $\mathcal{PML}$ supported on $\mathcal{E}(N)$, the ending lamination of N. It follows from work of Masur [Mas92] that it is a unique point on the boundary of Teichmüller space.

Suppose g is the geodesic ray or segment in the statement of theorem 10.1. Then what we said above shows that the endpoints of g and g' have either uniformly bounded distance, if they are in the interior of $\mathfrak{T}$, or are the same if they are on the boundary. It follows that g and g' have uniformly bounded Hausdorff distance. (Cf. Minsky [Min96].)

Once, we know that g and g' have uniformly bounded Hausdorff distance, we can see that there exists a fiber preserving map $\mathcal{S}_g \to \mathcal{S}_{g'}$ that lifts to a quasi-isometry of universal covers with uniform constants. (Cf. [FM02, Prop. 4.2].) This finishes the proof of theorem 10.1 (The model manifold).

Chapter 11

Gluing

In this chapter, we prove the main theorem:

Main Theorem 1 *Given $\epsilon > 0$ and $R > 0$ there exists $n_\epsilon > 0$ depending only on ϵ, R and $\chi(S)$ that if $M = H^+ \cup_S H^-$ has R-bounded combinatorics and handlebody distance $\geq n_\epsilon$ then M admits a Riemannian metric ν such that the sectional curvature of ν is pinched between $-1-\epsilon$ and $-1+\epsilon$. Moreover ν has a lower bound for the injectivity radius independently of the handlebody distance and ϵ.*

Suppose a Heegaard splitting $H^+ \cup_S H^-$ with R-bounded combinatorics is given. Recall from 2.6 there exist $P^\pm \subset \Delta(H^\pm)$ pants decompositions which realize the curve complex distance between $\Delta(H^+)$ and $\Delta(H^-)$ and have R-bounded combinatorics. In fact by using lemma 2.9, we can extend $P^\pm$ to a marking $\alpha^\pm$ such that α^+ and α^- still have R-bounded combinatorics. It also follows from our assumptions in 2.15 that there exist points $\tau^\pm \in \mathfrak{T}(S)$ where $\alpha^\pm$ have total length at most B_0.

Now use a homeomorphism $\phi^+ : (H^+, S) \to (H, \partial H)$ to identify H^+ with H and S with ∂H. With an abuse of notation, we denote the induced maps on the corresponding complex of curves, marking spaces and Teichmüller spaces by ϕ^+ as well. Recall from lemma 2.8 that we can translate the handlebody marking $\phi^+(\alpha^+)$ to an element of the finite set $\mathbf{m}_0(\mathrm{H})$ using an action of $\mathrm{Mod}_0(H)$. Then $\phi^+(\alpha^-)$ will have R-bounded combinatorics with respect to H. In fact, by assuming that the handlebody distance is large, $\phi^+(\alpha^-)$ belongs to $\mathcal{A}_0(R)$. (Recall from chapter 6 that $\mathcal{A}_0(R)$ was the set of markings and ending laminations that have R-bounded combinatorics with respect to H and are far enough to be in the Masur domain of H.)

The marking $\phi^+(\alpha^-)$ has length at most B_0 in $\phi^+(\tau^-) \in \mathfrak{T}(\partial H)$ and therefore N, the marked convex cocompact structure on H associated to $\phi^+(\tau^-)$, is in $\mathcal{B}_0(R)$. (Note that we are assuming that the marking of N is determined by the inclusion $\partial H \hookrightarrow H$.) Let g be a Teichmüller geodesic segment that connects τ_H to $\phi^+(\tau^+)$ in $\mathfrak{T}(\partial H)$. Then theorem 10.1 (The model manifold), proves that there is a map in the homotopy class determined by the inclusion $\partial H \hookrightarrow H$,

$$\Phi : \mathcal{S}_g^{\mathrm{SOLV}} \to N_e$$

that lifts to an (L, c)-quasi-isometry $\mathcal{H}_g^{\mathrm{SOLV}} \to \widetilde{N}_e$, where $C \subset N$ is a useful compact core and $N_e = \mathcal{CH}(N) \setminus C$ and also that g is $\mathcal{K}$-cobounded, where $\mathcal{K}$ depends only on R and $\chi(\partial H)$. In other terms, Φ and $\mathcal{S}_g^{\mathrm{SOLV}}$ give a model description of the convex core of N, outside of a small compact core.

Now use ϕ^+ and pull back all these structures to H^+. We get a hyperbolic structure N^+ on H^+, a geodesic segment $g^+ \subset \mathfrak{T}(S)$ and a map

$$\Phi^+ : \mathcal{S}_{g^+}^{\mathrm{SOLV}} \to N_e^+$$

which lifts to an (L, c)-quasi-isometry and is in the homotopy class determined by $S \hookrightarrow H^+$. The set N_e^+ is $\mathcal{CH}(N^+) \setminus (\phi^+)^{-1}(C)$ and is the complement of a small compact core of N^+ which we still call useful.

Obviously the terminal end point of g^+ is τ^-; we claim that the initial point is uniformly close to τ^+. We know that the initial point is $(\phi^+)^{-1}(\tau_H)$. Now it is enough to notice that the upper-bound for the total length of elements of $\mathbf{m}_0(\mathrm{H})$ in τ_H gives an upper bound for the total length of α^+ in $(\phi^+)^{-1}(\tau_H)$. But lemma 2.14 shows that the set of such points has uniformly bounded diameter in $\mathfrak{T}(S)$ as required. This immediately implies that $(\phi^+)^{-1}(\tau_H)$ and τ^+ are uniformly close.

From this, it follows that if we let h to be a geodesic segment in $\mathfrak{T}(S)$ whose initial and terminal endpoints are τ^+ and τ^-, then g^+ and h have bounded Hausdorff distance and they both are cobounded uniformly (cf. Minsky [Min96]). Thus we have a map $\mathcal{S}_h \to \mathcal{S}_{g^+}$ that lifts to a quasi-isometry with appropriate constants. Hence, we can replace g^+ with h and the map Φ^+ with

$$\Psi^+ : \mathcal{S}_h^{\mathrm{SOLV}} \to N_e^+$$

which satisfies all the properties of Φ^+ except possibly with bigger quasi-isometry constants (L_1, c_1).

We can do the same construction for H^- by starting from a homeomorphism $\phi^- : (H^-, S) \to (H, \partial H)$. Notice that in this case ϕ^- itself is orientation preserving but it is orientation reversing when restricted to S. (This is because of our assumption that orientation of S matches the one induced from H^+ and is the opposite of the one induced from H^-.) This gives a convex cocompact structure N^- on H^- and $N_e^- = \mathcal{CH}(N^-) \setminus C^-$, where C^- is a small compact core. We also have a map

$$\Psi^- : \mathcal{S}_h^{\mathrm{SOLV}} \to N_e^-$$

in the homotopy class determined by $S \hookrightarrow H^-$ that lifts to an (L_1, c_1)-quasi-isometry. Notice that in this case the map Ψ^- is orientation preserving if we assume $\mathcal{S}_h$ is oriented by taking the orientation of S times the orientation of h directed from τ^+ to τ^+. (The identification of S with ∂H^- is orientation reversing, but on the other hand we have changed the direction of the geodesic too.)

Also recall that the Teichmüller distance between τ^+ and τ^- (or equivalently length of h) tends to infinity as the curve complex distance between α^+ and α^-, or equivalently the handlebody distance for the Heegaard splitting, goes to infinity. This is essentially the same analysis as what we did in lemma 9.2. We know that $\alpha^\pm$ has length at most B_0 on $\tau^\pm$ and $\tau^\pm$ is ϵ_0-thick. Suppose $d_{\mathfrak{T}}(\tau^+, \tau^-) \leq D$. Then

$$\frac{l_{\tau^+}(\beta)}{l_{\tau^-}(\beta)} \leq ce^D$$

for every simple closed curve β on S, where c depends only on ϵ_0 and $\chi(S)$. Hence $l_{\tau^+}(\alpha^-) \leq D' = ce^D B_0$. But the set of simple closed curves with length bounded by D' on τ^+ has $\mathcal{C}(S)$ diameter bounded depending only on D' and $\chi(S)$. This gives an upper-bound for the $\mathcal{C}$-distance of α^+ and α^-.

Therefore, by assuming that the handlebody distance is large, we can make sure that the Teichmüller distance between $\tau(\alpha^+)$ and $\tau(\alpha^-)$ is large and equivalently the diameter of the convex cores of N^+ and N^- is large.

Proposition 11.1. *Given $\epsilon' > 0, D > 0$ and $R > 0$ there exists $d > 0$ such that the following holds. Suppose $H^+ \cup_S H^-$ is a Heegaard splitting with handlebody distance at least d, $\alpha^\pm$ are handlebody markings for $H^\pm$ which have R-bounded combinatorics and realize the handlebody distance. Also suppose $\tau^\pm \in \mathfrak{T}(S)$ is given such that total length of $\alpha^\pm$ on $\tau^\pm$ is bounded by B_0 and $N^\pm$ is the convex cocompact structure associated to $\tau^\mp$ on $H^\pm$.*
Then there exists a doubly degenerate surface group $\rho : \pi_1(S) \to \mathrm{PSL}_2(\mathbb{C})$ and maps

$$T^\pm : N_\rho \to N_e^\pm$$

which are ϵ'-close to an isometry on the ball of radius D about p_0 in the C^∞-topology and $T^\pm$ is in the homotopy class determined by $S \hookrightarrow H^\pm$, where $N_\rho = \mathbb{H}^3/\rho(\pi_1(S))$ has $\mathrm{inj}(N_\rho) \geq \eta$, $p_0 \in N_\rho$ is the image of $0 \in \mathbb{H}^3$, $N_e^\pm$ is the complement of a useful compact core in the convex core of $N^\pm$ as usual and η is the constant in theorem 1.5 (Bounded geometry) and depends only on R and $\chi(S)$.

Proof. We can prove this by means of taking a geometric limit. Suppose we have a sequence of counter examples: a sequence of R-bounded Heegaard splittings whose handlebody distance goes to infinity and do not satisfy the conclusion of the proposition for any surface group ρ.

Similar to our constructions above, the sequence of splittings gives a sequence of Teichmüller geodesic segments $h_n : [-n, n] \to \mathfrak{T}(S)$ whose length goes to infinity. We also have the corresponding hyperbolic structures $N_n^\pm$ on the handlebodies $H^\pm$ and uniform approximations of neighborhoods of their ends by

$$\Psi_n^{\pm} : \mathcal{S}_{h_n}^{\mathrm{SOLV}} \to N_n^{\pm} \setminus C_n^{\pm},$$

which lift to (L_1, c_1)-quasi-isometries of the universal covers and $C_n^{\pm}$ are useful compact cores of $N_n^{\pm}$. In fact, by proposition 2.10 of Farb-Mosher [FM02], we can replace these maps with

$$\Phi_n^{\pm} : \mathcal{S}_{h_n} \to N_n^{\pm} \setminus C_n^{\pm},$$

which lift to (L_2, c_2)-quasi-isometries of the universal covers, where L_2 and c_2 are uniform constants and $\mathcal{S}_{h_n}$ is equipped with the metric constructed in 2.9 using a fixed $\mathcal{MCG}$-equivariant connection on $\mathcal{S}$.

Notice that actions of mapping class group of S on the Teichmüller space and the geodesic h_n corresponds to precompositions of the embedding $S \hookrightarrow H^+ \cup_S H^-$ with self-homeomorphisms of S. Therefore, these give the same Heegaard splittings and we consider them equivalent. The geodesic segments (h_n) are all uniformly cobounded and therefore up to actions of $\mathcal{MCG}(S)$, we can assume that they converge in the Hausdorff topology to a cobounded biinfinite geodesic $h_\infty : (-\infty, \infty) \to \mathfrak{T}(S)$ in a way that the points $h_n(0)$ converge to $h_\infty(0)$ and the convergence preserves the orientation of the geodesics h_n. Take a point $w_\infty \in \mathcal{S}_{h_\infty(0)}$ and let $w_n \in \mathcal{S}_{h_n(0)}$ be the point obtained by moving along the connection lines between $\mathcal{S}_{h_\infty(0)}$ and $\mathcal{S}_{h_n(0)}$. Then let $x_n^{\pm} = \Psi_n^{\pm}(w_n)$ be the base point of $N_n^{\pm}$. The sequence of pointed manifolds $(N_n^{\pm}, x_n^{\pm})$ converges in the geometric topology to a hyperbolic manifold $(N_\infty^{\pm}, x_\infty^{\pm})$ with $\mathrm{inj}(N_\infty^{\pm}) \geq \eta$. In fact, we will see in a moment that this limit is a doubly degenerate hyperbolic structure on $S \times \mathbb{R}$. Let

$$\kappa_n^{\pm} : (N_\infty^{\pm}, x_\infty^{\pm}) \to (N_n^{\pm}, x_n^{\pm})$$

be the approximating maps. One can see that

$$(\mathcal{S}_{h_n}, w_n) \to (\mathcal{S}_{h_\infty}, w_\infty)$$

in Gromov-Hausdorff topology. This is because we fixed a connection on $\mathcal{S}$ and the Riemannian metric on $\mathcal{S}_{h_n}$ or $\mathcal{S}_{h_\infty}$ is obtained by taking the hyperbolic metric in the vertical directions and in the horizontal direction, we used connection lines which are parametrized by length using the parametrization of h_n or h_∞. It is not hard to see that these metrics on $\mathcal{S}_{h_n}$ on compact subsets converge in C^∞ to the metric on $\mathcal{S}_{h_\infty}$.

Now suppose

$$\rho_n : (\mathcal{S}_{h_\infty}, w_\infty) \to (\mathcal{S}_{h_n}, w_n)$$

are the approximating maps. It follows that the maps

$$(\kappa_n^{\pm})^{-1} \circ \Phi_n^{\pm} \circ \rho_n : (\mathcal{S}_{h_\infty}, w_\infty) \to (N_\infty^{\pm}, x_\infty^{\pm})$$

lift to (L_3, c_3)-quasi-isometries of the universal covers for constants L_3 and c_3 independent of n. In particular images of every compact subset of $\mathcal{S}_{h_\infty}$ are contained in a bounded diameter subset of the image. Then by Arzela-Ascoli theorem on every such compact set, after passing to a subsequence, these maps converge. Then we can consider $\mathcal{S}_{h_\infty}$ as a union of a countable family of compact subsets and repeat the same process for each of the compact subsets. Finally using a standard argument and taking a diagonal sequence, we can assume that a subsequence of these maps converge to

$$\Phi^{\pm}_{\infty} : (\mathcal{S}_{h_\infty}, w_\infty) \to (N^{\pm}_{\infty}, x^{\pm}_{\infty}).$$

It should be clear that these maps also lift to (L_3, c_3)-quasi-isometries of the universal covers. Also note that this immediately implies that $N^{\pm}_{\infty}$ is homeomorphic to $S \times \mathbb{R}$. We can also see that it has to be doubly degenerate, because it can be considered as the geometric limit of convex cores $(\mathcal{CH}(N^{\pm}_n), x^{\pm}_n)$ and therefore both its ends are geometrically infinite.

Finally, we can see that the map

$$\Phi^{-}_{\infty} \circ (\Phi^{+}_{\infty})^{-1} : N^{+}_{\infty} \to N^{-}_{\infty}$$

between these surface groups, lifts to a quasi-isometry of universal covers. Since these two are doubly degenerate, it follows from Sullivan's rigidity that it has to be homotopic to an isometry (cf. Minsky [Min94]). Hence N^{+}_{∞} and N^{-}_{∞} both represent a single doubly degenerate surface group $\rho : \pi_1(S) \to \mathrm{PSL}_2(CC)$.

Now we can consider the approximating maps to be defined on $N_\rho = \mathbb{H}^3/\rho(\pi_1(S))$:

$$\kappa^{\pm}_n : N_\rho \to N^{\pm}_n \setminus C^{\pm}_n.$$

These maps are ϵ' close to an isometry on a ball of radius D for $n \gg 0$ and satisfy the conclusion of the theorem. This contradicts the fact that we started with a sequence of counterexamples and finishes the proof. ☐

It is not hard to prove the next lemma, using a geometric limit argument.

Lemma 11.2. (The gluing region) *There exists a constant D depending only on η and $\chi(S)$ such that the ball of radius D about any point in the convex core of a doubly degenerate hyperbolic structure N_ρ on $S \times \mathbb{R}$ with $\mathrm{inj}(N_\rho) \geq \eta$ contains a subset $V \subset N_\rho$ homeomorphic to $S \times [0,1]$ with $V \hookrightarrow N_\rho$ a homotopy equivalence and the distance at least 1 between the boundary components of V. In addition, there exists a smooth bump function $\theta : V \to [0,1]$ where $\theta|_{\partial_- V} \equiv 0$ and $\theta|_{\partial_+ V} \equiv 1$, where $\partial_- V$ and $\partial_+ V$ are the boundary components of V, and all the first and second derivatives of θ are bounded depending only on η and $\chi(S)$.*

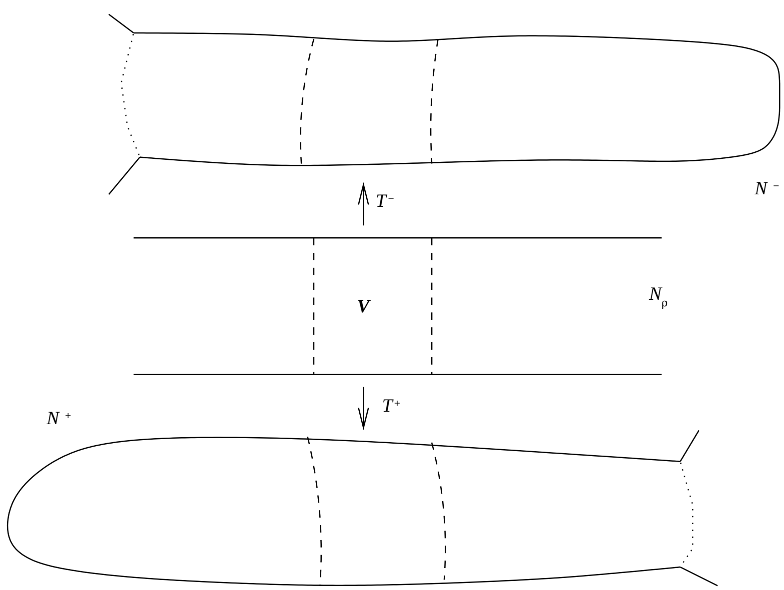

Figure 11.1: The gluing region.

Suppose the handlebody distance for $M = H^+ \cup_S H^-$ is larger than d in the statement of proposition 11.1 when D is the constant obtained in the above lemma and $\epsilon' > 0$ small which will be determined soon. Suppose N_ρ is the associated doubly degenerate hyperbolic structure on $S \times \mathbb{R}$ and

$$T^{\pm} : N_\rho \to N_e^{\pm}$$

are the maps described in proposition 11.1. Choose a subset $V \subset N_\rho$ contained in the D-neighborhood of the ball about p_0 and a bump function

$$\theta : V \to [0, 1]$$

that satisfy lemma 11.2.

The idea is to use restriction of $T^{\pm}$ to V to glue N^+ and N^- and construct a nearly hyperbolic metric on M. We know that $T^{\pm} : V \to N_e^{\pm}$ gives a homotopy equivalence in the homotopy class determined by $S \hookrightarrow N_e^{\pm}$. Without loss of generality, we can assume that $T^+(\partial_2 V)$ separates $T^+(\partial_1 V)$ from the end of N^+. Note that in this case, $T^-(\partial_1 V)$ separates $T^-(\partial_2 V)$ from the end in N^-.

The complement of $T^{\pm}(V)$ in $N^{\pm}$ has two components; one is a bounded diameter set homeomorphic to the interior $H^{\pm}$ and the other one which we call $Y^{\pm}$ is homeomorphic to $S \times \mathbb{R}$ and gives a neighborhood of the end of $N^{\pm}$. Observe that

$$(N^+ \setminus Y^+) \cup_{T^+ \circ (T^-)^{-1}} (N^- \setminus Y^-)$$

is homeomorphic to $M = H^+ \cup_S H^-$. We denote the image of the collar V by V and the two components of $M \setminus V$ by X^+ and X^- which are respectively contained in $N^+ \setminus Y^+$ and $N^- \setminus Y^-$. The hyperbolic metric of $N^{\pm}$ induces a hyperbolic metric $\nu^{\pm}$ on $M \setminus X^{\mp}$. These metrics do not coincide but they are

$2\epsilon'$-close in the C^∞ topology.

Now we can define the metric ν on M to be

$$\nu(x) = \theta(x) \cdot \nu^+(x) + (1 - \theta(x)) \cdot \nu^-(x),$$

for any $x \in M$. This metric is smooth and of course hyperbolic on $M \setminus V$. Moreover on V, the metrics $\nu^\pm$ are ϵ'-close to the metric induced by N_ρ which we call ν_ρ. In particular we have

$$\nu^\pm = \nu_\rho + \xi^\pm,$$

where $\xi^\pm$ is a 2-tensors which is C^2-close to zero. This implies that on V, we have

$$\nu = \nu_\rho + \theta\xi^+ + (1 - \theta)\xi^-.$$

Since the first and second derivatives of θ are bounded from above independently of the Heegaard splitting, by assuming that ϵ' is small, we can make sure that ν and ν_ρ are as C^2-close as we want. The sectional curvatures of ν depend only on the first and second derivatives of the metric and therefore all sectional curvatures of ν stay in the interval $[-1-\epsilon, -1+\epsilon]$ if the handlebody distance is large and ϵ' is small.

In addition to this, it is obvious that the injectivity radius of ν at every point is at least $\eta/2$ and we have proved our main theorem.

In fact, our construction immediately shows the following:

Theorem 11.3. *There are constants K, L_1, c_1 and η depending only on $\chi(S)$ and R such that the following holds.*

Let $M = H^+ \cup_s H^-$ be an R-bounded Heegaard splitting and $\alpha^\pm$ is a handlebody marking for $H^\pm$ such that α^+ and α^- realize the handlebody distance of the splitting and have R-bounded combinatorics. Also assume $\tau^\pm$ are points of $\mathfrak{T}(S)$ where $\alpha^\pm$ has total length at most B_0. Then there exists a Riemannian metric ν on M and an η-cobounded geodesic segment g connecting τ^+ and τ^- such that there is a map

$$\Psi : \mathcal{S}_g \to M \setminus (C^+ \cup C^-)$$

which lifts to an (L_1, c_1)-quasi-isometry of the universal covers, where $C^\pm \subset H^\pm$ is a compact core of $H^\pm$ with ν-diameter bounded by K.

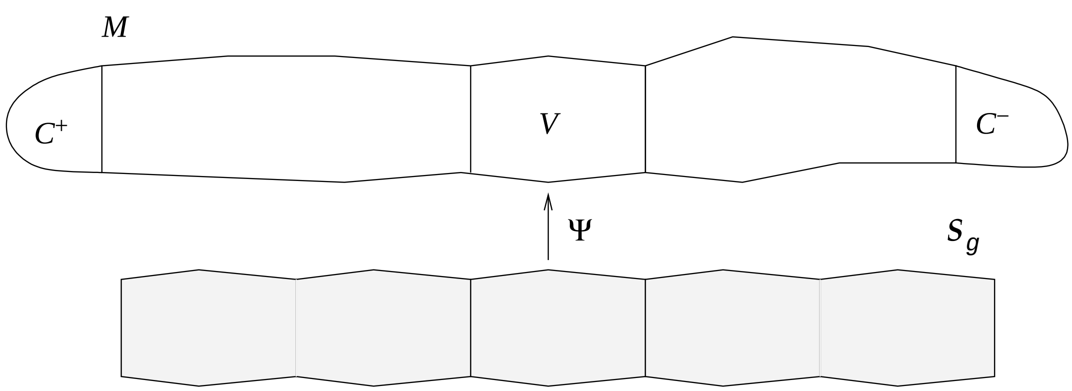

Figure 11.2: The glued manifold with the modeling map.

Notice that theorem 1.2 is simply a corollary of the above theorem.

Chapter 12

Tian's theorem and hyperbolicity

In [Ti90], Tian claims the following theorem:

Theorem 12.1. *Let (M, ν) be a negatively curved Riemannian three manifold and η a Margulis number for negatively curved three manifolds. Denote by M_η the η-thin piece of M. Then, there is a universal constant ϵ' such that if M satisfies*

1. *M_η is a disjoint union of convex neighborhoods $\{C_\alpha\}$ of closed geodesics γ_α with length $\leq 2\eta$ such that the normal injectivity radius of γ_α in C_α is greater than 1.*

2. *let P_α be a smooth function such that P_α is equal to η near the boundary of C_α and $P_\alpha(y)$ is equal to the injectivity radius at y whenever this is less than $\eta/2$ (such P_α always exists). We require that for some choice of P_α,*
$$\int_{C_\alpha} \frac{1}{P_\alpha} |\mathrm{Ric}(\nu) + 2\nu|_\nu^2 \, dV_\nu \leq \epsilon' \qquad \text{for each } \alpha.$$

3. *all sectional curvatures of M lie between $-1-\epsilon'$ and $-1+\epsilon'$.*

4. $\int_M |\mathrm{Ric}(\nu) + 2\nu|_\nu^2 dV_\nu \leq (\epsilon')^2$

then M admits an Einstein metric which is close to ν up to third derivatives.

Here $\mathrm{Ric}(\nu)+2\nu$ is the trace-free Ricci curvature of M. in fact Tian's result is stronger than this and allows dimensions other than 3 and norms other than L^2 norm. However this is more than enough for our application. Note that in dimension three, Einstein manifolds have constant sectional curvature.

Therefore, Tian's theorem implies the following:

Corollary 12.2. *Suppose (M, ν) is a Riemannian three manifold with η-bounded geometry. Also assume (M, ν) is hyperbolic outside a set of volume bounded by some d' and everywhere else the sectional curvatures are between $-1-\epsilon$ and $-1+\epsilon$ for ϵ sufficiently small. Then M admits a hyperbolic metric ν' which is close to ν up to third derivatives.*

Proof. To apply Tian's theorem, we need to verify that (M, ν) satisfies the assumptions. We know that (M, ν) has η-bounded geometry, therefore the η-thin part of the manifold is empty and the first and second assumptions are vacuous. The third assumption is satisfied by the hypothesis too when $\epsilon \leq \epsilon'$. For the last one, note that relative to an orthonormal frame, the entries in the 3×3 matrix for $\mathrm{Ric}(\nu)+2\nu$ are all between -4ϵ and 4ϵ if all sectional curvatures are pinched between $-1-\epsilon$ and $-1+\epsilon$. This follows from the fact that the Ricci tensor may be recovered by polarization from its associated quadratic form $Q(u) = \mathrm{Ric}(u, u)$ and that $\mathrm{Ric}(u, u)$ is simply $< u, u >$ multiplied by the sum of the sectional curvatures of any 2 orthogonal planes containing u.

Therefore, the function in the integral is zero outside a set of volume bound by d' and is small when ϵ is small inside that set. So by making sure that ϵ is small enough, we also have the last assumption and the Tian's theorem proves the claim. $\square$

In particular, putting our main theorem and the last corollary together we have:

Theorem 12.3. *If $M = H^+ \cup_S H^-$ is a Heegaard splitting with R-bounded combinatorics and sufficiently large handlebody distance then M admits a hyperbolic metric ν'. Also similar to theorem 11.3, there is geodesic segment g in $\mathfrak{T}(S)$ determined by combinatorics of the splitting and a map*

$$\Psi : \mathcal{S}_g \to (M \setminus (C_1 \cup C_2), \nu')$$

that lifts to an (L_1', c_1')-quasi-isometry of the universal covers, where $C^\pm \subset H^\pm$ is a compact core with ν'-diameter bounded by K' and constants L_1', c_1' and K' depend only on R and $\chi(S)$.

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