Nevanlinna Theory and Plücker Identities

A Dissertation, Presented

by

Sergei Panafidin

to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

Stony Brook University

May 2004

Stony Brook University

The Graduate School

Sergei Panafidin

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Blaine Lawson
Distinguished Professor, Department of Mathematics, SBU
Dissertation Director

Anthony Phillips
Professor, Department of Mathematics, SBU
Chairman of Dissertation

Denson Hill
Professor, Department of Mathematics, SBU

Martin Roček
Professor, C.N. Yang Institute for Theoretical Physics, SBU
Outside Member

This dissertation is accepted by the Graduate School.

Graduate School

Abstract of the Dissertation
Nevanlinna Theory and Plücker Identities

by

Sergei Panafidin

Doctor of Philosophy

in

Mathematics

Stony Brook University

2004

The main results of Nevanlinna theory are obtained by integration of the equations given by the local First and Second Main Theorems.

The First Main Theorem for sections of a holomorphic hermitian bundles was proved by Bott and Chern. A section of a vector bundle can be viewed as a special case of a vector bundle homomorphism, and the vanishing of a section is a special (extreme) case of degeneracy of a vector bundle map. In the first chapter we prove the analogue of the local First Main Theorem for degeneracy loci of bundle homomorphisms.

The Second Main Theorem for projective curves relates higher cur-

vature forms of a curve and its singularities. In the second chapter we generalize the local Second Main Theorem to the case of holomorphic curves in grassmannians. To do this we define the sequence of the higher curvature forms on the curve in the grassmanian. These forms are matrix-valued and satisfy the recursive relations similar to the ones that hold for projective curves. In the last section we prove the analogue of the Plücker identities for projective surfaces.

Contents

Acknowledgements

First of all, I would like to thank my advisor, Blaine Lawson, for his support, encouragement and guidance in writing this thesis. He has been a wonderful mentor and I really appreciate the time that he devoted to working with me.

During the years that I spent as a graduate student at Stony Brook, I have benefited from discussions with many people. I would like to thank all of them, especially Tony Phillips, Detlef Gromoll and Leon Takhtajan.

I would also like to express my gratitude to Taki, Onur, Natasa, Fernando, CJ, Kunal, Kasra, Amy, Radu, Yasha, Diyar and Edwige for their friendship, support and insightful conversations.

Finally I would like to give special thanks my parents and Nata for the moral support that I received from them all my life.

Chapter 1

Introduction

Value distribution theory in general studies the behavior of holomorphic maps $f : X \to M$ between a noncompact complex manifold X and a compact complex manifold M. The main object of interest is the growth of the intersection of the image $f(X)$ with the elements of some chosen family D_b of analytic subsets in M, where the parameter b takes values in a compact complex manifold B. The growth of each individual intersection is usually compared with the average over all $b \in B$. The upper estimate for the growth of the intersection set is called the First Main Theorem and the lower estimate for the sum of growth functions for several b_i is the Second Main Theorem. The main results of the value distribution theory were obtained only for quite special cases.

The common choice of the family of analytic subsets is the family of divisors of holomorphic sections of some holomorphic vector bundle E over M. Then the space B is the projective space $P(V)$, where $V \subset \Gamma_{hol}(E)$ is a subspace of the space of holomorphic sections. The preimages $f^{-1}(D_s)$ are exactly the divisors of the pullback sections of f^*E.

In this case the first main theorem can be established for any X and M if E

is a line bundle. Precisely, the result is that the order of growth of dvisor of any section $s \in V$ is bounded by the average over $P(V)$ plus a constant, provided that the space V is "sufficiently ample" (see section 2.2 for the definition). The fact that the upper bound is the average implies that order of growth of divisors of almost all (in the measure sense) sections is the same. The second main theorem is a very strong improvement of the above statement. For line bundles it is established in two cases:

1. X is a riemann surface, M is the projective space P^n and E is the dual of the universal line bundle over P^n. This case is known as equidistribution theory of holomorphic curves due originally to Ahlfors (see [1]).

2. X is affine and M - projective manifolds with $dim(X) \geqslant dim(M)$ and E is any holomorphic line bundle with sufficiently ample space of holomorphic sections. (see [5] and [6]).

For the bundles of rank higher then one even the upper estimate for the growth of divisors is not known. The example of a holomrphic map $\mathbb{C}^2 \to \mathbb{C}^2$ that omits an open set (due to Fatou and Bieberbach) shows that the growth of a divisor may not be bounded by the average (for a trivial bundle over $X = \mathbb{C}^2$) and therefore there is no equidistribution (even in the measure sense) in this case.

In all known cases the proof of the results of value distribution theory uses as a starting point some local formulas on X that are integrated over compact subsets. Our goal is to generalize these local formulas.

First we generalize the local first main theorem (see section 2.3, theorem 4) proved by Bott and Chern for sections of holomorphic vector bundles. Section s of vector bundle E can be considered as a homomorphism from the trivial line bundle over X to E. Therefore vanishing of a section is a special case of vanishing of a homomorphism $\alpha : F \to E$, which in turn is a special (extreme) case of degeneracy of α. For a pair of holomorphic hermitian bundles E and F, classified by maps to grassmannians, we define a natural family of homomorphisms $E \to F$ and prove a result (Theorem 9, Section 2.5) that, in analogy with the theorem of Bott and Chern, relates the k-degeneracy current $\mathbb{D}_k(\alpha)$ to the smooth representative of it's cohomology class. The essential part of our statement (as in the case of sections) is the information about the transgression (such as positivity) that in some cases can be used in value distribution theory. Unfortunately the integration of this local formula meets the same difficulties as in the case of sections. So in general we can not find an upper bound for the growth of the degeneracy sets. However, we can show (Proposition 11) that the average of the order of growth of $\mathbb{D}_k(\alpha)$ over all homomorphisms α from the chosen family is equal to the order of growth of the smooth form that represents them.

The second part of this work is devoted to the generalization of some local formulas related to the second main theorem. For a projective curve $f : X \to P^n$ there is a system of relations between the higher curvature forms of the curve (Theorem 12). The version of these relations that accounts for possible (higher order) singularities of the curve (Theorem 13) is the starting point for the proof of the defect relations for projective curves. We obtain the analogues of the two above mentioned theorems for holomorphic curves in grassmannians. First we define the analogue of the higher curvature forms, which in this case take values in endomorphisms of a bundle E, where E is the pullback of the quotient bundle over the grassmannian. Then we prove Theorems 18 and 19 that generalize Theorems 12 and 13 for the case of holomorphic curves in grassmannians.

In Section 3.6 we consider the case of surfaces in projective space. Our approach is very similar to that used for the case of curves. However we are not able to compute the transgression and therefore obtain only the relations between cohomology classes rather then their specific representatives. Therefore the results of this section can be used only for compact surfaces. In this case we obtain the analogue of Plucker formulas. The result is very similar to Plucker formulas for curves with second differences replaced by the third.

In the last section 3.7 we show that for equidimensional maps $f : X \to P^n$, which are a special case of the maps considered in [5] and [6], our construction also reproduces the result equivalent to the local second main theorem. So, the same general approach reproduces local formulas used in both extreme cases of projective curves and equidimensional maps to projective space.

Chapter 2

Local First Main Theorem

2.1 Currents

In this section we briefly recall some facts about currents on complex manifolds that we shall use later.

On a complex manifold M of dimension m we denote by $A^{p,q}(M)$ the space of C^∞ differential forms of type (p,q) and by $A_c^{p,q}(M)$ the space of such forms with compact support with the C^∞-topology. The space of currents of type (p,q), denoted by $C^{p,q}(M)$ is the topological dual space of $A_c^{m-p,m-q}(M)$. The set of forms of degree r is $A^r = \oplus_{p+q=r} A^{p,q}$ and it's dual is $C^r = \oplus_{p+q=r} C^{p,q}$. The graded space $C^*(M)$ of currents of all degrees form a module over the differential forms $A^*(M)$. For $\varphi \in A_c^*(M)$, $\psi \in A^*(M)$ and $\lambda \in C^*(M)$ the multiplication is defined by:

$$(\psi \wedge \lambda)(\varphi) = \lambda(\psi \wedge \varphi) \tag{2.1}$$

and differentiation by:

$$d\lambda(\varphi) = \lambda(d\varphi) \tag{2.2}$$

A current $\lambda \in C^{n,n}(M)$ is called positive if

$$i^{n(n-1)/2}\lambda(\varphi \wedge \overline{\varphi}) \geqslant 0 \tag{2.3}$$

We shall use the notation:

$$d^c = \frac{i}{4\pi}(\overline{\partial} - \partial) \tag{2.4}$$

There are three important examples of currents.

1. Smooth forms themselves. The action of $\psi \in A^{p,q}(M)$ on $\varphi \in A_c^{m-p,m-q}(M)$ is defined by:

$$\psi(\varphi) = \int_M \psi \wedge \varphi \tag{2.5}$$

By Stokes' theorem, $d\psi$ in the sense of currents agrees up to sign with $d\psi$ in the sense of differential forms.

2. An analytic subvariety $Z \subset M$ of pure codimension n defines a current $[Z] \in C^{n,n}(M)$ by the formula:

$$[Z](\varphi) = \int_{Z_{reg}} \varphi \tag{2.6}$$

where $\varphi \in A_c^{m-p,m-q}(M)$ and Z_{reg} is the set of nonsingular points of Z.

3. Forms with locally integrable coefficients on M define currents by integration in the same way as smooth forms. The space of such forms we shall

denote by $L^1_{loc}(M)$. We shall be particularly interested in locally integrable forms that are smooth outside of some analytic subset of M.

If we have a smooth map $f : N \to M$ there is the pullback map $f^* : A^*(M) \to A^*(N)$. It's dual $f_* : C^*(N) \to C^*(M)$ is called the pushforward map. For currents in general the pullback map is not defined. However for the first two cases described above (with some restrictions on the map f) there is a natural way to define the pullback map.

1. If the current is represented by a smooth form it's pullback is defined for any smooth f.

2. If the current $[Z]$ corresponds to an analytic subvariety Z (possibly with multiplicities) locally defined as $Z = \{g_1 = ... = g_k = 0\}$, and f is holomorphic and has the property:

$$codim(f^{-1}(Z)) = codim(Z) \tag{2.7}$$

then the pullback $f^*([Z])$ is defined as an analytic subvariety (with multiplicities) defined by the vanishing of the functions $f^*(g_i)$, $i = 1, ..., k$.

Later we shall need the following

Proposition 1 *Whenever the pullback operation f^* is defined on currents it commutes with the differentiation d and if f is holomorphic it also commutes with d^c. Furthemore, if the currents ω, $[Z]$ and λ are as above (ω - smooth, Z - analytic subvariety and λ - L^1_{loc}, smooth outside of Z) and satisfy the current equation*

$$[Z] - \omega = dd^c\lambda \tag{2.8}$$

on M, then their pullbacks by a holomorphic map $f : N \to M$ satisfy the corresponding equation on N, provided $f^*\lambda$ is defined and L^1_{loc}.

2.2 Holomorphic classifying maps

Let X be a complex manifold and $E \to X$ a holomorphic n-dimensional vector bundle over X. Any such bundle can be classified by a map to a grassmannian $G_n(\mathbb{C}^N)$ of n-planes in $\mathbb{C}^N$. Which means that there is a map $f : X \to G_n(\mathbb{C}^N)$ such that E is the pullback of the universal n-plane bundle $U(G_n(\mathbb{C}^N))$ over the grassmannian along the map f.

However in general f can not be chosen holomorphic. It is easy to see that if $E = f^*U(G_n(\mathbb{C}^N))$ for some holomorphic map f, then E^* should have nonzero holomorphic sections (which are pullbacks of holomorphic sections of U^*). Furthermore the space of holomorphic sections of E^* is "sufficiently ample" in the following sense:

Definition 1 *A subspace $V \subset \Gamma_{hol}(F)$ of the space of holomorphic sections of a vector bundle F over X is called* **sufficiently ample** *if the evaluation map $e : X \times V \to F$ is a surjective vector bundle homomorphism.*

The evaluation map e is defined as $e(x, s) = s(x)$ for $x \in X$ and $s \in V$.

The precise condition for existence of a holomorphic classifying map is as follows:

Proposition 2 *Holomorphic vector bundle $E \to X$ can be classified by a holomorphic map to a grassmannian if and only if the space of holomorphic sections of E^* is sufficiently ample (the classifying map may not be unique).*

Proof. We have the standard exact sequence of bundles over the grassmannian:

$$0 \to U(G_n(\mathbb{C}^N)) \to G_n(\mathbb{C}^N) \times \mathbb{C}^N \to Q(G_n(\mathbb{C}^N)) \to 0 \tag{2.9}$$

It's dual is:

$$0 \to Q^*(G_n(\mathbb{C}^N)) \to G_n(\mathbb{C}^N) \times (\mathbb{C}^N)^* \to U^*(G_n(\mathbb{C}^N)) \to 0 \tag{2.10}$$

But $(\mathbb{C}^N)^*$ is canonically isomorphic to the space of holomorphic sections of $U^*(G_n(\mathbb{C}^N))$. The sequence is exact, so the third homomorphism is surjective and therefore the space $(\mathbb{C}^N)^*$ of all holomorphic sections of $U^*(G_n(\mathbb{C}^N))$ is sufficiently ample.

If the holomorphic classifying map $f : X \to G_n(\mathbb{C}^N)$ exists then $f^*((\mathbb{C}^N)^*)$ is a sufficiently ample subspace of $\Gamma_{hol}(E^*)$ and therefore $\Gamma_{hol}(E^*)$ itself is sufficiently ample.

Conversely, if there is a sufficiently ample $V \subset \Gamma_{hol}(E^*)$, $dimV = N$, then the kernel of the evaluation map $e : X \times V \to E^*$ has dimension $(N-n)$ at every point $x \in X$ and we can define the map $f : X \to G_{N-n}(V)$ by $f(x) = ker(e|_x)$. This definition implies that $ker(e) = f^*U(G_{N-n}(V))$. Therefore we have the exact sequence:

$$0 \to f^*U(G_{N-n}(V)) \to X \times V \to E^* \to 0 \tag{2.11}$$

Therefore $E^* = f^*(V/U(G_{N-n}(V))) = f^*(Q(G_{N-n}(V)))$. But $G_{N-n}(V) =$

$G_n(V^*)$ and $Q^*(G_{N-n}(V)) = U(G_n(V^*))$. So if we use these identifications to interpret f as a map to $G_n(V^*)$, it is a holomorphic classifying map for E. $\square$

2.3 Connections and characteristic classes

In this section we shall give a brief overview of the geometric theory of characteristic classes as developed by Chern and Weil. Although the general theory does not require the bundle E to be holomorphic we shall consider only this special case. Let E be an n-dimensional holomorphic vector bundle over a complex manifold M. We shall denote by $\Gamma(E)$ the space of all C^∞ sections of E, by $\Gamma_{hol}(E)$ the space of holomorphic sections and by $A^*(M,E)$ the graded space of differential forms on M with values in E.

Definition 2 *A connection on E is a first order differential operator $D : \Gamma(E) \to \Gamma(T^* \otimes E)$ that satisfies the Leibnitz rule:*

$$D(fs) = df \cdot s + f \cdot Ds \tag{2.12}$$

for any $f \in A^0(M)$, $s \in \Gamma(E)$.

Relative to a local holomorphic frame in E a section s is represented by a vector function $\tilde{s}$. Then Ds is represented by the vector valued 1-form $d\tilde{s} + \theta \cdot \tilde{s}$ where θ is a locally defined (frame dependent) $n \times n$-matrix valued 1-form.

Under the change of local frame θ transforms by the gauge transformation:

$$\theta \to g^{-1}\theta g + g^{-1}dg \tag{2.13}$$

The matrix $\tilde{\Omega} = d\theta - \theta \wedge \theta$ transforms as $\tilde{\Omega} \to g^{-1}\tilde{\Omega}g$ and therefore defines a global (frame independent) $End(E)$-valued 2-form on M. It is called the curvature of E.

Definition 3 *A polynomial on a Lie algebra is called* **invariant** *if it is constant on the orbits of the adjoint action of the corresponding Lie group.*

The construction of representatives of characteristic classes of E is based on the following

Theorem 3 *For any invariant polynomial φ on $gl(n,\mathbb{C})$ of degree k, $\varphi(\tilde{\Omega})$ is well defined closed $2k$-form on M and it's cohomology class does not depend on the choice of connection.*

The representative of the total Chern class given by the connection D is

$$c(D) = det\left(I + \frac{i}{2\pi}\Omega\right) \tag{2.14}$$

and for the Chern character it is

$$ch(D) = Tr(e^{\frac{i}{2\pi}\Omega}) \tag{2.15}$$

If the bundle E is equipped with an hermitian metric H, then there is unique connection of type (1,0) that is compatible with the metric. And if the matrix h represents the metric in a local frame then the connection and curvature matrices are given by:

$$\theta = h^{-1}\partial h, \quad \tilde{\Omega} = \overline{\partial}\theta = \overline{\partial}(h^{-1}\partial h) \tag{2.16}$$

For a holomorphic section s of E, whose zero set has codimension equal to $rank(E)$, we shall denote by $Div(s)$ the divisor of s. It is known that the current $Div(s)$ represents in cohomology the top chern class $c_n(E)$ of E, and therefore

$$c_n(E, h) - Div(s) = dT \tag{2.17}$$

for some current T, where $c_n(E, h) = det(\frac{i}{2\pi}\Omega)$ is the representative of the top Chern class corresponding to the connection compatible with the metric h. For the compact manifold M this would be sufficient to compute the number of zeros (or the volume of $Div(s)$) of any section and conclude that all sections vanish the same number of times.

For noncompact M, to define the counting function that measures the growth of $Div(s)$ one has to restrict the integration to compact subsets of M with boundary, and therefore the right side of equation (2.17) would contribute to the estimates. Because of that one needs a more precise statement about the right hand side of the equation. It is usually called the (nonintegrated) First Main Theorem.

Theorem 4 *Let (E, h) be a positive holomorphic hermitian bundle over M. Then for any bounded (in metric h) holomorphic section s there is formula:*

$$c_n(E, h) - Div(s) = dd^c\lambda(s) \tag{2.18}$$

where λ is a positive L^1_{loc} (n-1, n-1)-form on M.

If E is a line bundle then the assumption of positivity of E in the theorem is not necessary and the formula becomes

$$c_1(E,h) - Div(s) = dd^c log \frac{1}{||s||^2} \tag{2.19}$$

where we can assume that the norm of s is bounded by 1 and therefore $log\frac{1}{||s||^2}$ is positive.

Another example is the case when E is the direct sum of n copies of a line bundle L with direct sum metric. Then any section s of E can be written as $(s_1, \cdots, s_n)$ where s_i are the sections of E. If the divisors $Div(s_i)$ intersect in a variety of codimension n then the formula becomes

$$(c_1(L,h))^n - Div(s_1) \cdot Div(s_2) \ldots Div(s_n) = dd^c \lambda \tag{2.20}$$

with

$$\lambda = log \frac{1}{||s||^2} (\sum_{k=1}^{n-1} \omega_0^{n-k-1} \omega^k) \tag{2.21}$$

where the forms $\omega = c_1(L,h)$ and $\omega_0 = \omega + dd^c log||s||^2$ are nonnegative and $||s||^2 = \sum_{i=1}^{n} ||s_i||^2$. In this case the assumption of positivity of E is necessary for positivity of λ. The proof of the theorem can be found in [3] and the last example in [6].

2.4 Integration

Theorem 4 is commonly used in value distribution theory as follows. One considers the holomorphic map $f : X \to M$ from noncompact complex manifold X to compact M. Then if we choose a positive holomorphic vector bundle E over M and a subspace V in the space $\Gamma_{hol}(E)$ of holomorphic sections of E, the problem is to study "the position of the image $f(X)$ relative to the divisors of sections from V". To make it more precise we need to choose an exhaustion function on X.

Definition 4 *An* **exhaustion function** *is a proper* C^∞ *map* $\tau : X \to [0, \infty)$

Then the counting function for the divisor $Div(s)$ of a section $s \in \Gamma(E)$ over M is

$$N(s, r) = \int_{r_0}^{r} \frac{dt}{t} n(s, t) \tag{2.22}$$

where

$$n(s, t) = \int_{B_t} (dd^c\tau)^{m-n} f^* Div(s) \tag{2.23}$$

and

$$B_t = \{x \in X \ : \ \tau(x) \leqslant logt\} \quad and \quad m = dim_C X, \quad n = rank(E) \tag{2.24}$$

The pullback divisor $f^*Div(s) = Div(f^*s)$ is well defined if the zero set of the pullback section f^*s has codimension n in X. Double integration of the current equation yields the formula

$$N(s,r) = T(r) + S(s,r) - m(s,r) + const \tag{2.25}$$

where

$$T(r) = \int_{r_0}^{r} \frac{dt}{t} \int_{B_t} f^* c_n(E,h) \tag{2.26}$$

$$S(s,r) = \int_{B_r} (dd^c\tau)^{m-n+1} f^* \lambda(s) \tag{2.27}$$

$$m(s,r) = \int_{\partial B_r} (dd^c\tau)^{m-n} d^c\tau f^* \lambda(s) \tag{2.28}$$

Positivity of $\lambda(s)$ implies positivity of $m(s,r)$. So in the case when the order of growth of $S(s,r)$ is smaller than that of $T(r)$ this formula gives the upper bound on the growth of divisors of holomorphic sections. This is the case when E is a line bundle and τ is the "special exhaustion function" for which $(dd^c\tau)^{m-n+1} = (dd^c\tau)^m$ (since $n = 1$) has compact support. Unfortunately in general the order of growth of $S(s,r)$ can be larger then that of $T(r)$ so this formula does not give an upper bound on the growth of $N(s,r)$

If we choose a sufficiently ample subspace of holomorphic sections V_E, so that E^* can be clasified by a holomorphic map to the grassmannian of n-planes in V_E and give E the pullback metric, then

Proposition 5 *For any r, $T(r)$ is the average of $N(s,r)$ over the projective space $P(V_E)$ (with the invariant volume form).*

See [3] for the proof.

2.5 Degeneracy loci

Let E and F be holomorphic vector bundles over a complex manifold M with $rank(E) = m$, $rank(F) = n$. And let $\alpha : E \to F$ be a holomorphic bundle homomorphism. We shall recall the definition of the k-th degeneracy current (where $0 \leqslant k \leqslant min(m, n)$) of α following [8]. The k-degeneracy current can be defined for the very general class of k-atomic homomorphisms (see [8] for the definition), however we shall be concerned only with the complex analytic case.

For $r = m - k$ let's consider the bundle $\pi : G_r(E) \to M$ the fibre of which at each point $x \in M$ is the grassmannian of r-planes in the fiber of E over x. The pullback $\pi^* E$ has a tautological subbundle $U(G_r(E))$ whose fibre over $p \in G_r(E)$ consists of all vectors $v \in p$. The homomorphism α lifts to $\pi^*\alpha : \pi^* E \to \pi^* F$ which can be restricted to $U(G_r(E))$:

$$\hat{\alpha} = \pi^*\alpha|_{U(G_r(E))} : U(G_r(E)) \to \pi^* F \tag{2.29}$$

Definition 5 *If the zeros of $\hat{\alpha}$, considered as a section of the bundle $Hom(U(G_r(E)), \pi^* F)$ over $G_r(E)$, are an analytic subset of "correct" codimension rn (so that the divisor of the section is a well defined current $Div(\hat{\alpha})$), the k-degeneracy current of α is defined as:*

$$\mathbb{D}_k(\alpha) = \pi_* Div(\hat{\alpha}) \tag{2.30}$$

Proposition 6 *Whenever the degeneracy current $\mathbb{D}_k(\alpha)$ is well defined*

$$supp\ \mathbb{D}_k(\alpha) \subseteq \{x \in M : rank(\alpha_x) \leqslant k\} \tag{2.31}$$

And if $\hat{\alpha}$ vanishes nondegenerately then

$$\mathbb{D}_k(\alpha) = [\Sigma_k(\alpha)] \tag{2.32}$$

where $[\Sigma_k(\alpha)]$ is the current of integration over

$$\Sigma_k(\alpha) = \{x \in M : rank(\alpha_x) = k\} \tag{2.33}$$

Now suppose that the bundles E^* and F have sufficiently ample subspaces V_{E^*} and V_F in their spaces of holomorphic sections. Then, as we discussed before, there are holomorphic maps $f_E : M \to G_m(V_{E^*})$ and $f_F : M \to G_n(V_F)$ such that

$$E = f_E^* U(G_m(V_{E^*}^*)), \quad F^* = f_F^* U(G_n(V_F^*)) \tag{2.34}$$

The map f_F can be also considered as a map $f_F : M \to G_{N-n}(V_F)$. So

$$F = f_F^* Q(G_{N-n}(V_F)) \tag{2.35}$$

Then there is natural subspace in the space of homomorphisms $\alpha : E \to F$ formed by the pullbacks f^*A of homomorphisms $A : U(G_m(V_{E^*}^*)) \to Q(G_{N-n}(V_F))$ of bundles over the product $G_m(V_{E^*}^*) \times G_{N-n}(V_F)$ where $f = f_E \times f_F : M \to G_m(V_{E^*}^*) \times G_{N-n}(V_F)$.

Proposition 7 *Over $G_m(V_{E^*}^*) \times G_{N-n}(V_F)$ the globally defined holomorphic bundle maps $A : U(G_m(V_{E^*}^*)) \to Q(G_{N-n}(V_F))$ are in natural one to one correspondence with the matrices $\hat{A} \in Hom(V_{E^*}^*, V_F) = V_{E^*} \otimes V_F$. Furthermore, the* *homomorphisms corresponding to the matrices of maximal rank have the degeneracy loci $\Sigma_k(A)$ of correct dimension and their degeneracy currents $\mathbb{D}_k(A)$ are the currents of integration over the regular subset of $\Sigma_k(A)$.*

Proof. Any matrix $\hat{A} \in Hom(V^*_{E^*}, V_F)$ defines the homomorphism A by restriction of $\hat{A}$ to the subbundle $U(G_m(V^*_{E^*}))$ and then projecting to the quotient $Q(G_{N-n}(V_F))$. The space of all holomorphic bundle maps is $\Gamma(U^*(G_m(V^*_{E^*}))\otimes Q(G_{N-n}(V_F))) = \Gamma(U^*(G_m(V^*_{E^*})))\otimes\Gamma(Q(G_{N-n}(V_F))) = V_{E^*}\otimes V_F$ therefore all the homomorphisms A are generated by the matrices $\hat{A}$ in the above manner.

The k-degeneracy locus of A is:

$$\Sigma_k(A) = \{(p,q) \in G_m(V^*_{E^*}) \times G_{N-n}(V_F) \;:\; dim(\hat{A}p \cap q) \geqslant n-k\} \quad (2.36)$$

To prove that the k-degeneracy current is the current of integration over $\Sigma_k(A)$ with multiplicity one we should consider the universal bundle $U_r \to G_{m,r}(V^*_{E^*})$, over the flag manifold

$$G_{m,r}(V^*_{E^*}) = \{(p,q) \in G_m(V^*_{E^*}) \times G_r(V^*_{E^*}) \;:\; q \subset p\} \quad (2.37)$$

and the homomorphism $U_r \to Q(G_{N-n}(V_F))$ defined by restriction of $\hat{A}$ to U_r and projection to the quotient. If $\hat{A}$ has maximal rank, this homomorphism vanishes to first order on

$$X_k = \{(p,q,x) \in G_m(V^*_{E^*}) \times G_r(V^*_{E^*}) \times G_{N-n}(V_F) \;:\; q \subset p, \;\; Aq \subset x\} \quad (2.38)$$

Therefore, by the previous proposition, the degeneracy current is the current of integration over $\Sigma_k(A)$.

□

It is known (see [8] for the proof) that

Proposition 8 *$\mathbb{D}_k(A)$ is represented in cohomology by the Shur polynomial $\Delta_{n-k}^{(m-k)}(c(Q)c(U)^{-1})$ in Chern classes of $Q = Q(G_{N-n}(V_F))$ and $U = U(G_m(V_{E^*}^*))$.*

We are going to prove the following

Theorem 9 *For a homomorphism A corresponding to the matrix $\hat{A}$ of maximal rank there is the current equation:*

$$\Delta_{n-k}^{(m-k)}(c(Q)c(U)^{-1}) - \mathbb{D}_k(A) = dd^c\lambda_k \tag{2.39}$$

where λ_k is a positive form with L^1_{loc} coefficients.

Proof. We shall first consider the case when $dimV_{E^*} \leqslant dimV_F$ so that the assumption that $\hat{A}$ has maximal rank means that it is injective. Let Y be the product $Y = G_m(V_{E^*}^*) \times G_{N-n}(V_F) \times P(Hom(V_{E^*}^*, V_F)) \times G_m(V_F) \times G_k(V_F)$. And define

$$Y_k = \{(p, q, \hat{A}, x, y) \in Y \ : \ \ \hat{A}p \subset x, \ y \subset x, \ y \subset q\} \tag{2.40}$$

Lemma 10 *Y_k is a smooth manifold.*

Proof.

Let's consider

$$X = \{(p, \hat{A}, x) \in G_m(V_{E^*}^*) \times Hom(V_{E^*}^*, V_F) \times G_m(V_F) \ : \ \ \hat{A}p \subset x\} \quad (2.41)$$

X is a smooth vector bundle over $G_m(V_{E^*}^*) \times G_m(V_F)$. So it's projectivization $P(X)$ is a smooth manifold. Let

$$\sigma : G_{N-n}(V_F) \times P(X) \times G_k(V_F) \to G_{N-n}(V_F) \times G_m(V_F) \times G_k(V_F) \quad (2.42)$$

be the projection. Then $Y_k = \sigma^{-1}(F_k)$ where

$$F_k = \{(q, x, y) \in G_{N-n}(V_F) \times G_m(V_F) \times G_k(V_F) \ : \ \ y \subset x, \ y \subset q\} \quad (2.43)$$

Since $F_k \to G_k(V_F)$ is smooth fiber bundle and $d\sigma$ is surjective, we conclude that Y_k is a smooth manifold.

□

Let's consider two projections:

$$\begin{array}{ccc} & Y_k & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ G_m(V_{E^*}^*) \times G_{N-n}(V_F) & & P(Hom(V_{E^*}^*, V_F)) \end{array} \quad (2.44)$$

The differential $d\pi_1$ is surjective and $d\pi_2$ is surjective for all points in Y_k that correspond to injective $\hat{A} \in Hom(V_{E^*}^*, V_F)$

On $P(Hom(V_{E^*}^*, V_F))$ we have the equation:

$$[a] = v + dd^c\lambda \tag{2.45}$$

where $[a]$ is a current corresponding to the point $a \in P(Hom(V_{E^*}^*, V_F))$, v is the volume form on $P(Hom(C^M, C^N))$ and λ is some positive $(n-1, n-1)$ form with L_{loc}^1 coefficients. The explicit formula for λ is a special case of the formula (2.20):

$$\lambda = \log\left(\frac{\sum_{i=1}^n |z_i|^2}{\sum_{i=0}^n |z_i|^2}\right)\left(\sum_{k=1}^{n-1} \omega_0^{n-k-1}\omega^k\right) \tag{2.46}$$

where $\omega = dd^c log(\sum_{i=0}^n |z_i|^2)$ and $\omega_0 = dd^c log(\sum_{i=1}^n |z_i|^2)$ in projective coordinates z_i such that $z_i(a) = 0$ for $i = 1, \ldots, n$.

So, if we choose a corresponding to an injective homomorphism, then the singular sets of π_2 and λ do not intersect and therefore we can define $\lambda_k = \pi_{1*}(\pi_2^*(\lambda))$. Then λ_k is a well defined positive form with L_{loc}^1 coefficients that satisfies the equation

$$\pi_{1*}(\pi_2^*(v)) - \pi_{1*}(\pi_2^*([a])) = dd^c\lambda_k \tag{2.47}$$

But $\pi_{1*}(\pi_2^*([a]))$ is precisely the k-degeneracy current of the bundle homomorphism A corresponding to a. And $\pi_{1*}(\pi_2^*(v))$ is a smooth $U_M \times U_N$ invariant form on $G_m(V_{E^*}^*) \times G_{N-n}(V_F)$. Invariant forms on the product of the grassmannians are harmonic and unique in their cohomology class. Therefore

$\pi_{1*}(\pi_2^*(v))$ should be equal to $\Delta_{n-k}^{(m-k)}(c(Q)c(U)^{-1})$ and therefore λ_k defined in this way is positive and satisfies the equation (2.39).

□

The pullback of the equation (2.39) to M is:

$$\Delta_{n-k}^{(m-k)}(c(F)c(E)^{-1}) - \mathbb{D}_k(\alpha) = dd^c\Lambda_k \tag{2.48}$$

again with positive L^1_{loc} form Λ_k. It can be integrated twice as in the case of divisors of sections of a vector bundle to obtain the equation

$$N_k(\alpha, r) = T(r) + S(\alpha, r) - m(\alpha, r) + const \tag{2.49}$$

Where $N_k(\alpha, r)$ represents the order of growth of the k-degeneracy set of α, $T(r)$ does not depend on α and $m(\alpha, r)$ is positive. Unfortunately the presence of the term $S(\alpha, r)$ does not allow us to use this equation to obtain the upper bound on $N_k(\alpha, r)$.

For a fixed r $N_k(\alpha, r)$ is a function on $P(Hom(V^*_{E^*}, V_F))$. Because we were considering only injective matrices, $N_k(\alpha, r)$ is defined only on the corresponding open dense subset of $P(Hom(V^*_{E^*}, V_F))$. However

Proposition 11 *$N_k(\alpha, r)$ has an integrable continuation to the whole space $P(Hom(V^*_{E^*}, V_F))$ and it's average is equal to $T(r)$.*

Proof.

We can pull back to M the smooth fiber bundle $\pi_1 : Y_k \to G_m(V^*_{E^*}) \times G_{N-n}(V_F)$. Then we have the diagram:

$$\begin{array}{ccc} & f^*Y_k & \\ {}^{\pi}\swarrow & & \searrow^{\pi_2} \\ M & & P(Hom(V^*_{E^*}, V_F)) \end{array} \tag{2.50}$$

The function $N_k(\alpha, r)$ can be rewritten as:

$$N_k(\alpha, r) = \int_M \mathbb{D}_k(\alpha) \cdot \mu_r = (\pi_2)_* \pi^*(\mu_r) \tag{2.51}$$

where

$$\mu_r = \int_{r_0}^{r} \frac{dt}{t} \chi_t \cdot (dd^c \tau)^{d_k} \tag{2.52}$$

and χ_t is the characteristic function of $B_t = \{x \in M \ : \ \tau(x) \leqslant \log x\}$ (τ is the exhaustion function, $d_k = dim(\mathbb{D}_k)$).

The form μ_r is continuous therefore $\pi^* \mu_r$ is continuous. $d\pi_2$ is surjective almost averywhere, so the fiber integral of $\pi^* \mu_r$ has integrable continuation across the singular set of π_2.

□

Chapter 3

Local Second Main Theorem

3.1 Overview of the classic theory of projective curves

Let's consider a holomrphic map $f : X \to P^n$ from a smooth Riemann surface X to n-dimensional projective space. Then the object of interest for value distribution theory is the counting function $N(a, r)$ that measures the order of growth of the intersection of the image $f(X)$ with a hyperplane a in P^n. There are two basic questions. One is to find an upper bound on $N(a, r)$ independent of a. And the second is to find the lower bound for the sum of several $N(a_i, r)$ where a_i form a system of hyperplanes in general position. The first question is simpler and can be answered in terms of the function f itself. To answer the second question (or to prove the second main theorem of the value distribution theory) one has to consider the maps involving the derivatives of higher order of the original map f. They are called associated curves and below we shall briefly recall their construction. Let's choose a local coordinate z on X and use the homogeneous coordinates on P^n. Then the map f is locally represented by the vector valued function $\hat{f}(z) = (f_0(z), f_1(z), \ldots, f_n(z))$. Then we can define

$$\hat{f}_k(z) = \hat{f}(z) \wedge \hat{f}'(z) \wedge \cdots \wedge \hat{f}^{(k)}(z) \tag{3.1}$$

where $\hat{f}^{(k)}(z)$ is the k-th derivative of $\hat{f}(z)$ with respect to local coordinate z. The ambiguity in the choice of local coordinate makes $\hat{f}_k(z)$ well defined only up to nonvanishing scalar holomorphic function. Therefore the subsets

$$\Sigma_k = \{x \in X \ : \ \hat{f}_k(x) = 0\} \tag{3.2}$$

are well defined in X, and outside of Σ_k, $\hat{f}_k(z)$ unambiguously defines a $(k+1)$-plane in $\mathbb{C}^{n+1}$. Thus we obtain the function $\tilde{f}_k : X \setminus \Sigma_k \to G_{k+1}(\mathbb{C}^{n+1})$. The map $\hat{f}_k$ can be also considered as a (locally defined) vector valued function $\hat{f}_k : X \to \wedge^{k+1}\mathbb{C}^{n+1}$. Therefore it has well defined divisor D_k that is a current supported on the set Σ_k. Because the map f is holomorphic, Σ_k is a finite set of points x_k^i in X, and

$$D_k = \sum n_k^i [x_k^i] \tag{3.3}$$

where n_k^i is the order of vanishing of $\hat{f}_k$ at x_k^i. From the construction one can see that $\Sigma_k \subset \Sigma_{k+1}$ and the difference $D_{k+1} - D_k$ is a positive current.

In the neighbourhood of a point x_k^i the function $\hat{f}_k$ can be written as $\hat{f}_k(z) = g(z)h(z)$ where $g(z)$ is a scalar function vanishing at x_k^i to the order n_k^i and $h(z)$ is nonvanishing vector function. Therefore the previously defined maps $\tilde{f}_k$ can be extended to all X. We shall denote the extended maps f_k : $X \to G_{k+1}(\mathbb{C}^{n+1})$ and

Definition 6 *The map $f_k : X \to G_{k+1}(\mathbb{C}^{n+1})$ constructed as above is called k-th associated curve for the curve $f : X \to P^n$.*

Let α_k denote the Käehler form on $G_{k+1}(\mathbb{C}^{n+1})$. Then

Definition 7 *The forms $\omega_k = f_k^* \alpha_k$ on X are called the k-th curvature forms of the curve $f : X \to P^n$.*

Any (1,1) form on X can be interpreted as an hermitian form on TX. As the pullbacks of the positive forms on grassmannians along holomorphic maps, all forms ω_k are nonnegative. And the first curvature form ω_0 is an hermitian metric on X (which is singular on Σ_1) induced from the metric on P^n. If the map f is an immersion, then ω_0 is a nonsingular hermitian metric on TX. Similarly, other curvature forms ω_k are singular metrics on TX induced by the corresponding maps to grassmannians.

It turns out that the curvature forms ω_k are not independent but can all be computed starting from ω_0. The recursive realations are given by the following theorem (see [4]).

Theorem 12 *If the curve $f : X \to P^n$ is not contained in any hyperplane in P^n, then*

$$dd^c log \tilde{\omega}_k = \omega_{k-1} - 2\omega_k + \omega_{k+1} \tag{3.4}$$

where $\omega_{-1} = \omega_n = 0$ and $\tilde{\omega}_k$ is locally defined function such that $\omega_k = \tilde{\omega}_k dz \wedge d\bar{z}$ The equation holds outside of the discrete set Σ_k on which $\tilde{\omega}_k$ vanishes.

If we choose an hermitian metric h on X locally represented as $h = \tilde{h}|dz|^2$, then we can rewrite these relations as

$$dd^c log \frac{\tilde{\omega}_k}{\tilde{h}} + K = \omega_{k-1} - 2\omega_k + \omega_{k+1} \tag{3.5}$$

where K is the curvature of X corresponding to the metric h and again the equation holds outside of the singular set Σ_k. In this equation $\frac{\tilde{\omega}_k}{\tilde{h}}$ is globally well defined nonnegative function on X vanisning at discrete set Σ_k.

We shall be interested in the integration of this equation on X (or compact subsets of X if it is not compact). Therefore we need an equation valid on all X. To obtain such an equation we need to understand the left hand side of (3.5) as a current and add terms that account for the presence of singularities. The result is

Theorem 13

$$dd^c log \frac{\tilde{\omega}_k}{\tilde{h}} + K = \omega_{k-1} - 2\omega_k + \omega_{k+1} - (D_{k-1} - 2D_k + D_{k+1}) \tag{3.6}$$

For the proof see for example [11], [10] or [1]. The classical texts do not use currents to state the result. However the proof of an essentially equivalent statment can be found there. The original proof is based on the direct calculation of the forms ω_k. We shall reprove it from a slightly different point of view in the next section.

If X is compact then the integration of this equation on X yields the Plücker formulas:

$$\chi(X) = \nu_{k-1} - 2\nu_k + \nu_{k+1} - (n_{k-1} - 2n_k + n_{k+1}) \tag{3.7}$$

where $\nu_k = \int_X \omega_k$ is the volume of the k-th associated curve and $n_k = \sum_i n_k^i$ is it's total k-singularity.

If X is not compact but has a parabolic exhaustion function τ where

Definition 8 *An exhaustion function τ on a Riemann surface X is called* **parabolic** *if $dd^c\tau = 0$ outside of a compact set.*

then double integration (first over the ball B_t defined by the exhaustion function τ and then over the parameter t) of equation (3.6) produces result that is the starting point for the proof of the defect relations which is the main result of the value distribution theory. Unlike the case of the first main theorem, where the upper estimates on the counting function (when they can be obtained at all) follow quite easily from the local first main theorem, the proof of defect relations is difficult. However the local formula (3.6) plays an important role in the theory. Our goal will be to obtain some generalizations of it.

3.2 Derivative bundles

In this section we shall review some facts about the derivative bundles following (with some change of notations) the papers [2] and [9] where all the further details can be found.

Let E be a holomorphic vector bundle of dimension n over complex m-dimensional manifold X. The cohomology group $H^1(X, End(E) \otimes \Omega)$ contains an obstruction $a(E)$ to trivialization of E (if X is compact Käehler manifold,

$H^1(X, End(E) \otimes \Omega)$ can be identified with $H^{1,1}(X, End(E))$ and the curvature form of E corresponding to any metric connection on E is a representative of the cohomology class $a(E)$).

Definition 9 *An* **extension** *of a holomorphic bundle F'' by F' is an exact sequence of vector bundles:*

$$0 \to F' \to F \to F'' \to 0 \tag{3.8}$$

It is known that

Proposition 14 *The equivalence classes of extensions of F'' by F' are in one-to-one correspondence with the elements of $H^1(X, Hom(F'', F'))$, the trivial extension corresponding to the zero element.*

The element $a(E)$ can be considered as an element of $H^1(X, Hom(E, E \otimes \Omega))$.

Definition 10 *The extension*

$$0 \to E \otimes \Omega \to E_1 \to E \to 0 \tag{3.9}$$

corresponding to $-a(E)$ is called the **derivative** *of the bundle E.*

The bundle E_1 has also a simple description in terms of transition functions corresponding to a local trivialization. Let's choose a covering $\{U_i\}$ of X such that the restriction of E to U_i is trivial for all i. Over each U we can fix the trivialization of E, the local holomorphic frame $\{u_i\}$ and the corresponding local holomorphic connection

$$D_U : \Gamma(E|_U) \to \Gamma((E \otimes \Omega)|_U) \tag{3.10}$$

defined by $D_U u_i = 0$ for all i.

Let V be another trivializing neighbourhood and denote by A_{UV} the $End(E)$-valued (1,0) form on $U \cap V$ that represents the local connection D_U relative to the local frame over V. A (1,0)-type connection on the bundle E can be undertood as a splitting $E_1 \to E \otimes \Omega$ of extension (3.9). Local holomorphic connection D_U defines the local holomorphic splitting $E_1|_U = E \oplus E \otimes \Omega$. Similarly $E_1|_V = E \oplus E \otimes \Omega$ and the transition functions g_{UV} relating two different splittings of E_2 over U and V are:

$$g_{UV} = \left(\begin{array}{c|c} I & 0 \\ \hline A_{UV} & I \end{array}\right) \tag{3.11}$$

The forms A_{UV} can be written in terms of the first partial derivatives of the transition functions of the bundle E.

By its design the bundle E_1 has the property that for any holomorphic section s of E there is a global holomorphic section s_1 of E_1 that can be locally defined as $s_1|_U = (s; D_U s)$.

By analogy with the first derivative E_1 it is possible to define (see [9] for the construction) the sequence of the consecutive extensions:

$$0 \to E \otimes S^{k+1}\Omega \to E_{k+1} \to E_k \to 0 \tag{3.12}$$

whose transition functions can be expressed in terms of the higher derivatives of the transition functions of E. And similarly to E_1 they have the property that s_k locally defined as

$$s_k|_U = (s, D_U s, \dots, D_U^k s) \tag{3.13}$$

is a global holomorphic section of E_k. Here $D_U^k : \Gamma(E|_U) \to \Gamma((E \otimes S^k\Omega)|_U)$ is the k-th derivative in the frame, chosen on U.

3.3 Associated maps

As we discussed before there are two equivalent approaches to studying holomorphic maps $f : X \to G_{N-n}(\mathbb{C}^N)$. One can either start with the map f and obtain the (positive) holomorphic vector bundle E over X, that is the pullback of the quotient bundle over the grassmannian, and the subspace in the space of holomorphic sections of E formed by the pullback sections. Or alternatively we can start with the bundle E and a sufficiently ample subspace V in the space of it's holomorphic sections. Then the map f naturally arises as the holomorphic classifying map for E by mapping $x \in X$ to the subspace $P_x \subset V$ of sections vanishing at x.

Similarly we can define the associated maps for the map f as maps arising out of an attempt to construct the holomorphic classifying maps for the derivative bundles E_k. As in the case of associated curves, these maps will be initially defined outside of some "singular" subset of X. In some cases they can be extended across the singularities.

Let $V_0 = X \times V$ be the trivial bundle and consider the k-th derivative evaluation map $e_k : V_0 \to E_k$ defined by

$$e_k(x,s) = \left(s(x), \frac{\partial s}{\partial z_i}(x)dz_i, \ldots, \frac{\partial^k s}{\partial z_{i_1} \ldots \partial z_{i_k}}(x)dz_{i_1} \ldots dz_{i_k} \right) \tag{3.14}$$

The right hand side of this equation is an element of $E \oplus (E \otimes \Omega) \oplus \ldots (E \otimes S^k\Omega)$, that is locally isomorphic to E_k, where Ω is the holomorphic cotangent bundle on X and $S^k\Omega$ is its symmetric tensor power. This element does depend on the choice of the local coordinates z_i and so does the local splitting of E_k into direct sum of $E \otimes \Omega^i$. And the right hand side, considered as an element of E_k is well defined and independent of the choice of local coordinates. The map $e_0 : V_0 \to E_0 = E$ is the usual evaluation map (priviously denoted by e).

The singular sets are defined by:

$$\Sigma_k = \{x \in X \ : \ e_k|_x \quad \text{is not surjective}\} \tag{3.15}$$

From the definition of the homomorphisms e_k one can see that $\Sigma_k \subset \Sigma_{k+1}$. Over the "nonsingular" subset $X \setminus \Sigma_k$ one can define the smooth vector bundle $V_{k+1} = ker(e_k)$ and the map $f_k : X \to G_{N-d_k}(V)$, where $f_k(x)$ is the fiber of V_k over $x \in X$ and $d_k = dim(E_k)$.

From the exact sequence

$$0 \to V_{k+1} \to V_0 \to E_k \to 0 \tag{3.16}$$

on $X \setminus \Sigma_k$ we see that the restriction $E_k|_{X\setminus\Sigma_k} = V_0/V_{k+1}$. And therefore $E_k|_{X\setminus\Sigma_k} = f_k^*Q(G_{N-d_k}(V))$ and $V_k = f_k^*U(G_{N-d_k}(V))$. If we choose an hermitian metric on V, all subbundles V_k will have natural metrics and the bundles

E_k will have "singular metrics" that are smooth metrics outside of Σ_k.

By their definition, the bundles V_k have the fiber over $x \in X$ formed by all sections in V that vanish at least to order k at x:

$$V_k = \{(x, s) \in (X \setminus \Sigma_k) \times V \ : \ s \text{ vanishes at least to order } k \text{ at } x\} \quad (3.17)$$

The restriction $\tilde{e}_k = e_k|_{V_k}$ is a map to the subbundle $E \otimes S^k\Omega \subset E_k$ that is given by the formula:

$$\tilde{e}_k(x, s) = \frac{\partial^k s}{\partial z_{i_1} \ldots \partial z_{i_k}}(x) dz_{i_1} \ldots dz_{i_k} \quad \text{for} \quad (x, s) \in V_{k-1} \quad (3.18)$$

where the right hand side is independent of local coordinates due to the vanishing of the lower order derivatives of s. So the bundles $V_k = ker(\tilde{e}_k)$ and we have the diagram:

$$\begin{array}{ccccccc} V_0 & \longleftarrow & V_1 & \longleftarrow & V_2 & \longleftarrow & V_3 \cdots \\ \downarrow{\scriptstyle \tilde{e}} & & \downarrow{\scriptstyle \tilde{e}_1} & & \downarrow{\scriptstyle \tilde{e}_2} & & \\ E & & E \otimes \Omega & & E \otimes S^2\Omega & & \end{array} \quad (3.19)$$

From the exact sequence (of bundles over over $X \setminus \Sigma_k$)

$$0 \to V_{k+1} \to V_k \to E \otimes S^k\Omega \to 0 \quad (3.20)$$

and the sequence for the quotient $E_k = V_0/V_{k+1}$

$$0 \to E \otimes S^k\Omega \to E_{k+1} \to E_k \to 0 \quad (3.21)$$

we conclude that the singular projected metrics $h(E \otimes S^k\Omega)$ induced by projections $\tilde{e}_k : V_k \to E \otimes S^k\Omega$ in diagram (3.19) are the same as the metrics that $E \otimes S^k\Omega$ receive as subbundles of E_k (provided that $E_k|_{X\setminus\Sigma_k} = V_0/V_k$ has the quotient metric which is also the pullback of the U_N invariant metric on $Q(G_{N-d_k}(V))$).

3.4 Geometric construction of associated maps

Now we shall describe a different construction of the same maps f_k (again defined outside of some singular sets Σ_k).

Let f^* be the pullback of holomorphic 1-forms $f^* : T^*G_{N-n}(V) \to T^*X$. Because $T^*G_{N-b}(V) = UG_{N-n}(V) \otimes Q^*G_{N-n}(V)$ we have the map

$$f^* : UG_{N-n}(V) \otimes Q^*G_{N-n}(V) \to \Omega \tag{3.22}$$

which can also be considered as a map:

$$\hat{f}^* : UG_{N-n}(V) \to \Omega \otimes QG_{N-n}(V) \tag{3.23}$$

Then we can define the first associated map $f_1 : X \to G_{N-d_1}(V)$ on the subset of X where $\hat{f}^*$ is surjective as

$$f_1(x) = \{\text{kernel of } \hat{f}^* \text{ restricted to the fiber over } f(x)\} \tag{3.24}$$

Now let's observe that this definition of the first associated map agrees with the one that we gave before. The pullback f^* defines the homomorphism $\alpha_1 : f^*T^*G_{N-d_1}(V) \to T^*X$ of bundles over X. But $f^*T^*G_{N-d_1}(V) =$

$f^*UG_{N-d_1}(V) \otimes f^*Q^*G_{N-d_1}(V) = V_1 \otimes E^*$. So we have a homomorphism:

$$\alpha_1 : V_1 \otimes E^* \to \Omega \tag{3.25}$$

Or equivalently

$$\tilde{\alpha}_1 : V_1 \to E \otimes \Omega \tag{3.26}$$

The above definition defines the value of f_1 as the kernel of $\tilde{\alpha}_1$. But the homomorphism $\tilde{\alpha}_1$ is precisely the same as $\tilde{e}_1$ described in the previous section.

To proceed with the definition of the higher order associated maps, let's consider the map $g_1 : X \to G_{N-n,N-d_1}(V)$ to the flag manifold of pairs of planes in V defined by $g_1(x) = (f(x), f_1(x))$. Let π_i be the projections:

$$\begin{array}{ccc} & G_{N-n,N-d_1} & \\ {}^{\pi_0}\swarrow & & \searrow^{\pi_1} \\ G_{N-n} & & G_{N-d_1} \end{array} \tag{3.27}$$

Then $f = \pi_0 \circ g_1$ and $f_1 = \pi_1 \circ g_1$. The pullback of the universal bundle $\pi_1^*U(G_{N-d_1})$ is a subbundle of $\pi_0^*U(G_{N-n})$. Therefore $\pi_1^*U(G_{N-d_1}) \otimes \pi_0^*Q^*(G_{N-n})$ is a subbundle of $\pi_0^*U(G_{N-n}) \otimes \pi_0^*Q^*(G_{N-n})$ which is isomorphic to $\pi_0^*(T^*G_{N-n})$.

By the construction of the map g_1 we have

$$\pi_1^*U(G_{N-d_1}) \otimes \pi_0^*Q^*(G_{N-n}) \subset ker(g_1^*) \tag{3.28}$$

The manifold $G_{N-n,N-d_1}$ has two fiber bundle structures:

1. $G_{N-n,N-d_1} = G_{N-d_1}(U(G_{N-n}))$

2. $G_{N-n,N-d_1} = G_{d_1-n}(Q(G_{N-d_1}))$

From the first we have exact sequence:

$$0 \to \pi_1^* U(G_{N-d_1}) \to \pi_0^* U(G_{N-n}) \to Q \to 0 \tag{3.29}$$

where Q is the quotient. The injective map $\pi_1^* U(G_{N-d_1}) \to \pi_0^* U(G_{N-n})$ defines the projection $\pi_1^* Q(G_{N-d_1}) \to \pi_0^* Q(G_{N-n})$. Therefore we have the following exact sequence:

$$0 \to U \to \pi_1^* Q(G_{N-d_1}) \to \pi_0^* Q(G_{N-n}) \to 0 \tag{3.30}$$

corresponding to the second fiber bundle structure. Here U is the kernel of the projection.

Taking tensor product of (21) with $\pi_0^* Q^*(G_{N-n})$ we have:

$$0 \to \pi_1^* U(G_{N-d_1}) \otimes \pi_0^* Q^*(G_{N-n}) \to \pi_0^* T^*(G_{N-n}) \to Q \otimes \pi_0^* Q^*(G_{N-n}) \to 0 \tag{3.31}$$

Similarly the tensor prouct of the sequence dual to (6) with $\pi_1^* U(G_{N-d_1})$ is:

$$0 \to \pi_0^* Q^*(G_{N-n}) \otimes \pi_1^* U(G_{N-d_1}) \to \pi_1^* T^*(G_{N-d_1}) \to U^* \otimes \pi_1^* U(G_{N-d_1}) \to 0 \tag{3.32}$$

Now consider the 1-form pullback map g_1^* restricted to $\pi_1^* T^*(G_{N-d_1})$. According to (20) $\pi_0^* Q^*(G_{N-n}) \otimes \pi_1^* U(G_{N-d_1})$ is contained in its kernel. So we can define the map:

$$\alpha_2 : U^* \otimes \pi_1^* U(G_{N-d_1}) \to T^* X \tag{3.33}$$

Let's consider the corresponding map

$$\tilde{\alpha}_2 : \pi_1^* U(G_{N-d_1}) \to T^* X \otimes U \tag{3.34}$$

We shall define the $(N - d_2)$-dimensional subspace of V necessary to define the map f_2 as $ker(\tilde{\alpha}_2)$ on the subset of X where $\tilde{\alpha}_2$ is surjective.

Now one can consider the flag manifold $G_{N-d_1,N-d_2}$ with two projections to the grassmannians and define the next associated map in the same manner as above. Repeating the procedure until eventually the map $\tilde{\alpha}_k$ fails to be generically surjective one can obtain all the associated maps. As in the case of f_1 one can check that this construction of f_k agrees with the definition from the previous section.

3.5 Holomorphic curves in projective space and grassmannians

In the case of projective curves the associated maps can be extended to the whole curve X. This implies that the bundles V_i defined above over nonsingular sets $X \setminus \Sigma_i$ can be extended to smooth bundles over X. The higher curvature forms ω_k are the pullbacks of the invariant Käehler forms on the grassmannians and therefore equal to the representatives of the first Chern classes of the pullbacks of the quotient bundles:

$$\omega_k = c_1(f_k^* Q(G_{N-k-1})) = c_1(V_0/V_k) = -c_1(V_k) \tag{3.35}$$

From the exact sequence

$$0 \longrightarrow V_{k+1} \longrightarrow V_k \xrightarrow{\tilde{e}_k} E \otimes S^k\Omega \longrightarrow 0 \tag{3.36}$$

we conclude that (see [7] for the proof)

$$c_1(\vec{D}(E \otimes S^k\Omega)) = \sigma_k + (\omega_k - \omega_{k+1}) \tag{3.37}$$

$$c_1(\vec{D}(E \otimes S^k\Omega)) - c_1(E \otimes S^k\Omega) = dd^c\lambda_k \tag{3.38}$$

where σ_k is the 1-degeneracy current of $\tilde{e}_k$, $c_1(\vec{D}(E\otimes S^k\Omega))$ is the c_1 characteristic current of the singular pushforward connection on $E\otimes S^k\Omega$ corresponding to the singular projected metric on it and $c_1(E \otimes S^k\Omega)$ is the smooth representative of the first Chern class corresponding to some metrics on E and Ω. Two last equations combined imply that

$$\omega_{k-1} - 2\omega_k + \omega_{k+1} - (\sigma_k - \sigma_{k-1}) = c_1(E\otimes S^k\Omega) - c_1(E\otimes S^{k-1}\Omega) + dd^c\lambda_k \tag{3.39}$$

However $D_k = \sum_{i=0}^k \sigma_k$, so that

$$\sigma_k - \sigma_{k-1} = D_{k+1} - 2D_k + D_{k-1} \tag{3.40}$$

and $c_1(E \otimes S^k\Omega) - c_1(E \otimes S^{k-1}\Omega) = c_1(\Omega)$. Therefore to reprove equation (3.6) we should compute the transgression λ_k. But because it is a locally integrable function it is sufficient to compute λ_k on the nonsingular subset $X \setminus \Sigma_k$. This is equivalent to computing the projected metric on $E \otimes S^k\Omega$. Therefore the result will follow from the following

Proposition 15 *The projected metrics on the bundles $E \otimes S^k\Omega$ are nonsingular outside of Σ_k and satisfy the recursive relations:*

$$h(E \otimes S^k\Omega) = h(E \otimes S^{k-1}\Omega) \cdot \omega_{k-1} \tag{3.41}$$

where ω_k in the right hand side is understood as a metric on $\Omega = T^*X$ (nonsingular outside of Σ_k).

Proof. For each associated map $f_k : X \to G_{N-k-1}(V)$ let's consider the holomorphic 1-form pullback map

$$f_k^* : f_k^*(T^*G_{N-k-1}(V)) = V_k \otimes (V/V_k)^* \to \Omega \tag{3.42}$$

The inclusion map $V_k \to V_{k-1}$ defines the projection $V/V_k \to V/V_{k-1}$ and therefore $(V/V_{k-1})^*$ is a subbundle of $(V/V_k)^*$. From the geometric construction of the associated maps f_k it follows that:

$$V_k \otimes (V/V_{k-1})^* \subset ker(f_k^*) \tag{3.43}$$

The quotient $(V_k \otimes (V/V_k)^*)/(V_k \otimes (V/V_{k-1})^*)$ is just $V_k \otimes (V_{k-1}/V_k) = V_k \otimes (E \otimes S^{k-1}\Omega)^*$. Therefore f_k^* defines a map:

$$\tilde{\alpha}_k : V_k \otimes (E \otimes S^{k-1}\Omega)^* \to \Omega \tag{3.44}$$

Because $V_k \otimes (E \otimes S^{k-1}\Omega)^*$ is a quotient bundle of $T^*G_{N-k-1}(V)$ with projection metric, the metric on Ω defined by the projection $\tilde{\alpha}_k$ is the same as the metric defined by the map f_k^*, which is the composition of the two projections . Therefore its Käehler form is the pullback along f_k^* of the Käehler form on the grassmannian $G_{n+1-k}(V)$ which is equal to $\omega_k = -c_1(V_k)$. Because $E \otimes S^{k-1}\Omega$ is just a line bundle, we can take tensor product of (3.44) with $E \otimes S^{k-1}\Omega$ to compute the projected metric on $E \otimes S^k\Omega$ stated in the proposition.

□

Next we consider the case of curves in grassmannians. As we did for projective curves we shall start with a riemann surface X, a vector bundle E over it and a sufficiently ample subspace V in the space of holomorphic sections of E. Let n be the rank of E. Then the general construction produces a map $f : X \to G_{N-n}(V)$ such that $E = f^*Q(G_{N-n}(V))$ and it's associated maps f_k. The proof of the fact that the associated maps and corresponding bundles V_k can be extended across the singularities to the entire curve X can be repeated with almost no changes for the bundle E of rank $n > 1$ (but it is important for the argument that X is still 1-dimensional).

Again the derivatives of associated maps $df_k : TX \to TG_{N-n(k+1)}$ define metrics on TX that are singular on Σ_k. But the pullback $f_k^*(TG_{N-n(k+1)})|_{X\setminus\Sigma_k} =$

$E_k^* \otimes V_k|_{X\setminus\Sigma_k}$ has the additional structure of the tensor product which allows us define the $End(E_k)$-valued hermitian form h_k on the tangent bundle of X.

Lemma 16 *On $X \setminus \Sigma_k$ the ($End(E_k)$-valued) curvature form $\hat{\omega}_k$ of E_k corresponding to the metric induced on it by the projection $e_k|_{X\setminus\Sigma_k} : V_0 \to E_k$ is equal to h_k interpreted as a (1,1) form on X.*

The equation (3.43) remains true for $n > 1$ and therefore the image of df_k belongs to the subbundle $(E \otimes S^k\Omega) \otimes V_k^*$ of $TG_{N-n(k+1)}(V)$. Therefore, relative to the orthogonal splitting $E_k = E \otimes S^k\Omega + E_{k-1}$, $\hat{\omega}_k$ is represented by the matrix

$$\hat{\omega}_k = \left(\begin{array}{c|c} \bar{\omega}_k & 0 \\ \hline 0 & 0 \end{array}\right) \tag{3.45}$$

with only one nonzero block, corresponding to the (1,1) form $\bar{\omega}_k$ on X with values in $End(E \otimes S^k\Omega) = End(E)$. Now the last proposition can be repeated almost without changes:

Proposition 17 *The projected metrics on the bundles $E \otimes S^k\Omega$ are nonsingular outside of Σ_k and satisfy the recursive relations:*

$$h(E \otimes S^k\Omega) = h(E \otimes S^{k-1}\Omega) \cdot \bar{\omega}_{k-1} \tag{3.46}$$

where the right hand side is the matrix product of $h(E \otimes S^{k-1}\Omega)$ understood as an hermitian form on $S^{k-1}\Omega$ with values in $Hom(E, E^*)$ and the $End(E)$-valued hermitian form $\bar{\omega}_k$.

Proof. Repeating the argument used for the projective curves, we conclude that the homomorphism

$$\tilde{\alpha}_k : V_k \otimes (E \otimes S^{k-1}\Omega)^* \to \Omega \tag{3.47}$$

corresponding to

$$\tilde{e}_k : V_k \to \Omega \otimes (E \otimes S^{k-1}\Omega) \tag{3.48}$$

induces the projected $End(E \otimes S^k\Omega)$-valued form $\bar{h}_k$ on Ω that is precisely $\bar{\omega}_k$. This implies the statement of the proposition. □

We can use this result in the same way as it was used in the case of projective curves. First corollary is the recursive relation between the curvature forms $\bar{\omega}_k$.

Theorem 18 *If a bundle E of rank n over a complex curve X is classified as a holomorphic hermitian bundle by a map to the grassmannian $G_{n(m-1)}(V)$ of planes of codimension n in an hermitian space V of dimension mn, then the higher curvature forms $\bar{\omega}_k$ satisfy the recursive relations:*

$$\bar{\omega}_{k+1} = \bar{\omega}_k + \frac{i}{4\pi}\bar{\partial}(h_{k+1}^{-1}\partial h_{k+1}), \qquad \bar{\omega}_{-1} = \bar{\omega}_m = 0 \tag{3.49}$$

$$h_k = h(E \otimes S^k\Omega), \ so \quad h_{k+1} = h_k \cdot \bar{\omega}_k, \quad h_0 = h(E) \tag{3.50}$$

The equations hold outside of the singular sets Σ_k. The form $\bar{\omega}_k$ defined by relations (3.49) on $X \setminus \Sigma_k$ has a smooth continuation to all X. All forms $\bar{\omega}_k$

are positive- definite outside of the finite sets Σ_k.

Another application is the analog of the formula (3.6). If we define (as before) D_k as 1-degeneracy current of e_k and σ_k as 1-degeneracy current of $\tilde{e}_k$, then from the general theory of singular connections (see [7]) applied to the diagram (3.19) we obtain

$$\omega^1_{k-1} - 2\omega^1_k + \omega^1_{k+1} - (\sigma_k - \sigma_{k-1}) = c_1(E \otimes S^k\Omega) - c_1(E \otimes S^{k-1}\Omega) + dd^c\lambda_k \quad (3.51)$$

where $\omega^1_i = tr(\bar{\omega}_i)$ and all other notations are as in the formula (3.39) of which this formula is an exact analog. The last proposition allows to compute the transgression λ_k in (3.51) to prove the following

Theorem 19 *Let $f : X \to G_{n(m-1)}(V)$ be a holomorphic curve in the grassmannian. Then for any choice of hermitian metric h on X the higher curvature forms of the curve and it's singularities satisfy the following current equation*

$$dd^c \log \frac{\det(\tilde{\omega}_k)}{\tilde{h}^n} + nK = \omega^1_{k-1} - 2\omega^1_k + \omega^1_{k+1} - (D_{k-1} - 2D_k + D_{k+1}) \quad (3.52)$$

where $\tilde{\omega}_k$ is defined locally by $\bar{\omega}_k = \tilde{\omega}_k dz \wedge d\bar{z}$ and $\frac{\det(\tilde{\omega}_k)}{\tilde{h}^n}$ is a globally well defined function on X.

If X is compact this equation can be integrated to obtain the Plücker identities:

$$n\chi(X) = \nu_{k-1} - 2\nu_k + \nu_{k+1} - (n_{k-1} - 2n_k + n_{k+1}) \quad (3.53)$$

where $\nu_k = \int_X \omega_k^1$ is the volume of the k-th associated curve and $n_k = \sum_i n_k^i$ is it's total k-singularity.

3.6 Surfaces in projective space

In this section X will denote a complex surface and E is a line bundle over it. As we did before, we can define the singular sets Σ_k, associated maps f_k on $X \setminus \Sigma_k$ and the bundles V_k over $X \setminus \Sigma_k$. Unfortunately in general the bundles V_k can not be extended smoothly to all X.

Lemma 20 *The bundles V_k can be continued smoothly to $X \setminus S_k$, where $S_k \subset \Sigma_k$ is a finite point set.*

Proof. Because the bundle V_{k+1} is defined as the kernel of $e_k : V_0 \to E_k$, the problem of continuation of V_k is equivalent to the continuation of the image $e_k^*(E_k^*) \subset V_0^*$ across Σ_k. Let x be a nonisolated point in Σ_k. In some neighbourhood U of this point the bundle E_k can be trivialized, and the map $\wedge^{d_k} E_k \to \wedge^{d_k} V$ can be represented by a vector function g on U. On $X \setminus \Sigma_k$ it defines a map $U \to P(\wedge^{d_k} V)$. The intersection $U \cap \Sigma_k$ is an analytic subset of U defined by the vanishing of g. Outside of a finite set $sing(\Sigma_k)$, Σ_k is locally irreducible, defined by vanishing to the first order of a holomorphic function ψ. Then each component g_i of the vector function g can be represented as $g_i = \prod \psi^{n_i} \cdot h_i$ where h_i doesn't vanish identically on $U \cap \Sigma_k$. Therefore on U, $g = \phi \tilde{g}$ where $\tilde{g}$ does not vanish identically on $U \cap \Sigma_k$ and ϕ is a scalar holomorphic function. The function $\tilde{g}$ then vanishes on a discrete set and outside of it defines the map $X \to P(\wedge^{d_k} V)$. But on the open dense set $X \setminus S_k$ the image of this map belongs to $G_{d_k} \subset P(\wedge^{d_k} V)$. Therefore the extended map is also a map to the grassmannian. $\square$

Let m be the largest value of k for which e_k is generically surjective. Then all the bundles V_k, $k = 1, \cdots, m$ can be continued to $X \setminus S_m$.

Let $\pi : \tilde{X} \to X$ be the blowup of X at all points in S_m. Then the fact that the associated maps and bundles V_k, restricted to any curve C in X can be continued across $S_m \cap C$ implies that the pullback bundles $\tilde{V}_k = \pi^* V_k$ can be continued to all $\tilde{X}$.

If we define D_k as 1-degeneracy current of e_k and σ_k as 1-degeneracy current of $\tilde{e}_k$ then we have the following analog of Plücker identities for surfaces, which interestingly involves third differences.

Theorem 21 *On $\tilde{X}$ there is the relation between cohomology classes:*

$$c_1(V_{k+1}) - 3c_1(V_k) + 3c_1(V_{k-1}) - c_1(V_{k-2}) = D_{k+1} - 3D_k + 3D_{k-1} - D_{k-2} + c_1(\tilde{X}) \tag{3.54}$$

Proof. Because $\tilde{V}_{k+1} \subset ker(\tilde{e}_k)$ we can consider the equidimensional homomorphisms

$$\beta_k : \tilde{V}_k / \tilde{V}_{k+1} \to E \otimes S^k \Omega \tag{3.55}$$

After we blow up all isolated singular points, all the degeneracy sets in $\tilde{X}$ on which β_k drops rank by one have codimension one which is the "expected" dimension. Therefore we have the formula

$$c_1(V_k/V_{k+1}) - c_1(E \otimes S^k\Omega) = \sigma_k + dT_k \tag{3.56}$$

which together with the exact sequence

$$0 \to E \otimes S^{k+1}\Omega \to (E \otimes S^k\Omega) \otimes \Omega \to (E \otimes S^{k-1}\Omega) \otimes \wedge^2\Omega \to 0 \tag{3.57}$$

implies the equation (3.54). □

3.7 Equidimensional maps to projective space

In this section we consider equidimensional holomorphic maps $f : X \to P^*(V)$. This is a special case of maps $f : X \to M$ with $dim(X) \geqslant dim(M)$ considered in [6]. The general construction from sections 3 and 4 applies in this case. The diagram (3.19) terminates after V_1 and becomes

$$\begin{array}{ccccc} V_0 & \longleftarrow & V_1 & \longleftarrow & 0 \\ \downarrow{\scriptstyle \tilde{e}} & & \downarrow{\scriptstyle \tilde{e}_1} & & \\ E & & E \otimes \Omega & & \end{array} \tag{3.58}$$

The (singular) metric induced on $E \otimes \Omega$ from V_1 is

$$h(E \otimes \Omega) = h(E) \cdot \omega(E) \tag{3.59}$$

where $\omega(E)$ is understood as a hermitian form on TX. Outside of the singular subset where df fails to be injective, V_1 is isomorphic to $E \otimes \Omega$ and we have the exact sequence

$$0 \to E \otimes \Omega \to V_0 \to E \to 0 \tag{3.60}$$

Therefore outside of the singularities

$$\omega(E) = dd^c \log\,(\det h(E)) = -dd^c \log\,(\det(h(E) \cdot \omega(E))) = \tag{3.61}$$

$$-dd^c \log\; h(E)^n + dd^c \log\; \omega(E)^n$$

Taking into account singularities we obtain the current equation

$$dd^c \log\; \omega(E)^n = (n+1)\omega(E) - D_1 \tag{3.62}$$

the singular volume form $\omega(E)^n$ on X is just the pullback of invariant volume form on the projective space.

This formula is precisely the nonintegrated second main theorem for equidimensional maps into projective space. Notice that the same general construction produces a local second main theorem in both extreme cases of lowest (the case of holomorphic curves) and highest (considered in this section) dimension of X. These two cases are the simplest. First because for curves all bundles $E \otimes S^k\Omega$ remain one dimensional for all k and the second because the diagram (3.58) terminates after V_1 and therefore there are no higher order associated maps. This allows one to hope that other local formulas obtained in this way might also find some application.

Bibliography

[1] L.V. Ahlfors, The theory of meromorphic curves, Acta Soc. Sci. Fenn. Ser. A, vol. 3, No. 4.

[2] M.F. Atiyah, Complex analytic connections in fibre bundles, Trans. Amer. Math. Soc., 85 (1957), 181-207.

[3] Bott, R., and Chern, S.S., Hermitian vector bundles and the equidistribution of the zeroes of their holomorphic sections. Acta Math., Vol. 114 (1966), pp. 71-112.

[4] E. Calabi, Isometric imbeddngs of complex manifolds, Ann. of Math. 58 (1953), pp. 1-23.

[5] J. Carlson and P. Griffiths, A defect relation for equidimensional holomorphic mappings between algebraic varietes, Ann. Math. 95 (1972) 557-584.

[6] P. Griffiths and J. King, Nevanlinna theory and holomorphic mappings between algebraic varieties, Acta Mathematica 130 (1973), 145-220.

[7] F.R. Harvey and H. Blaine Lawson, A theory of characteristic currents associated with a singular connection, Asterisque 213 (1993), 1-268.

[8] F.R. Harvey and H. Blaine Lawson, Geometric residue theorems, American Journal of Math., 17 (1995) 829-873.

[9] W.F. Pohl, Differential geometry of higher order, Topology, 1 (1962), 169-211.

[10] Weyl, H., and Weyl, J., Meromorphic functions and analytic curves. Princeton University Press, Princeton (1943).

[11] Wu, H., The equidistribution theory of holomorphic curves. Princeton University Press, Princeton (1970).