Normal Subgroups of the
Symplectomorphism Group

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Abstract of the Dissertation

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This thesis considers the group of symplectomorphisms of $2n$-dimensional Euclidean space, a subgroup of the group of volume-preserving diffeomorphisms. Symplectomorphisms in this group can be generated by time-dependent Hamiltonian functions and, in the thesis, sub-collections of Hamiltonian functions are investigated as sources of normal subgroups of symplectomorphisms. An extension of the Hofer norm is introduced, and is used to show that some of these subgroups are proper subgroups. As the main result, this extended Hofer norm is used to show that another, rather unusual subgroup is also a proper subgroup. The result is interesting because a similarly-constructed subgroup of the group of volume-preserving diffeomorphisms has been shown to not be a proper subgroup in the cases where $n$ is greater than or equal to 2. So the result distinguishes the structure of the group of symplectomorphisms from that of the group of volume-preserving diffeomorphisms.
To my parents
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Chapter 1

Hamiltonian Symplectomorphisms
and subgroups of $\text{Symp}(\mathbb{R}^{2n}, \omega_0)$

1.1 Introduction and Notation

In this paper, we discuss the symplectomorphisms of $\mathbb{R}^{2n}$, denoted $\text{Symp}(\mathbb{R}^{2n})$. This set is the subset of the set $\text{Diff}(\mathbb{R}^{2n})$, consisting of those that preserve the standard symplectic form, $\omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i$. That is, $\phi \in \text{Diff}(\mathbb{R}^{2n})$ such that $\phi^*\omega_0 = \omega_0$. Since $\omega_0^n$ is a volume form, any such $\phi$ will be an element of the set of volume-preserving diffeomorphisms, $\text{Diff}_{vol}(\mathbb{R}^{2n})$, as well. With the operation of composition, each of the three sets we have mentioned is a group, and we have the sequence of subgroups, $\text{Symp}(\mathbb{R}^{2n}, \omega_0) < \text{Diff}_{vol}(\mathbb{R}^{2n}) < \text{Diff}(\mathbb{R}^{2n})$. These are topological groups, with the compact-open topology. We begin by considering symplectomorphisms generated by functions.
1.2 Hamiltonian Symplectomorphisms of Compact Support

Let $C$ be the vector space of all compactly supported smooth maps $H : \mathbb{R}^n \times [0, 1] \to \mathbb{R}$. $C$ is called the set of *compactly supported (time dependent) Hamiltonian functions*. Such a function *generates a Hamiltonian Symplectomorphism* via the sequence shown in Fig. 1.1 and whose maps and spaces are explained below.

**map a:** For a Hamiltonian function $H$, we define the *associated Hamiltonian vector field*, $X^H_t$, by the equation $\iota(X^H_t) \omega_0 = d(H_t)$. That $X^H_t$ is well-defined depends on the non-degeneracy of $\omega_0$. Note that if $H$ is time-dependent, then $X^H_t$ will be, as well. Such an $X^H_t$ will automatically be a *symplectic vector field* (meaning simply that $\iota(X^H_t) \omega_0$ is closed). $X^H_t$ is sometimes called the *symplectic gradient vector field* of $H_t$, denoted $X^H_t = sgrad(H_t)$.

**map b:** We will distinguish between the statement $H$ is *compactly-supported* and $H_t$ is *compactly supported at each* $t \in [0, 1]$, which is a weaker statement, and to which we will return later in this chapter. Since $H$ is compactly supported, $X^H_t$ will be, also. Therefore, there exists a unique family of diffeomorphisms, $h_t$ for $t \in [0, 1]$, with velocity vector field $\dot{h}_t$, that solves the first-order initial value problem

\[
\begin{align*}
  h_0 &= id \\
  \dot{h}_t &= X^H_t \circ h_t.
\end{align*}
\]
This family is a smooth path in the compact-open topology on $\text{Diff}(\mathbb{R}^{2n})$, and is called the *Hamiltonian isotopy associated to the function* $H$. We say that the function $H$ *generates* the isotopy $h_t$. For each $t \in [0,1]$, it turns out that $h_t \in \text{Symp}(\mathbb{R}^{2n})$ [2], so $h_t$ is also called a *Symplectic isotopy*. Yet another name for $h_t$ is the *symplectic gradient flow* of $H$.

-map c: When $h_t$ is evaluated at time $t = 1$, the result is a symplectomorphism, $h_1$, which is often denoted $\psi^H$ as a reminder that it was obtained from $H$ in the manner described above. We say that $\psi^H$ is the *Hamiltonian symplectomorphism generated by* $H$, or associated to $H$. We could write $\psi^C$ for the set of all symplectomorphisms generated by compactly supported Hamiltonian functions. A more common symbol for $\psi^C$ is $\text{Ham}^c(\mathbb{R}^{2n})$; it is a subset of $\text{Symp}^c(\mathbb{R}^{2n})$.

An element of $\text{Ham}^c(\mathbb{R}^{2n})$ is understood to be the endpoint, the time $t = 1$ value, of a Hamiltonian isotopy, such as $h_t$. As mentioned above, at the intermediate times, $0 < t < 1$, $h_t$ takes values in $\text{Symp}^c(\mathbb{R}^{2n})$. But closer inspection shows that simply by re-scaling the time variable and the height, any intermediate value, say $h_\lambda$, with $0 < \lambda < 1$, can be re-cast as the endpoint of an isotopy of its own. That is, if $H_t$ generates isotopy $h_t$ with endpoint $h_1$, then the function $t \mapsto \lambda H_t$ generates the isotopy $t \mapsto h_\lambda$, which has time $= 1$ value $h_\lambda$. So any compactly supported Hamiltonian isotopy is actually a path in $\text{Ham}^c(\mathbb{R}^{2n})$, starting at the identity. Hence, $\text{Ham}^c(\mathbb{R}^{2n})$ is path-connected. Note that although we have seen that any compactly supported Hamiltonian function generates a path in $\text{Ham}^c(\mathbb{R}^{2n})$, it is not immediately clear that the converse is true, that any path in $\text{Ham}^c(\mathbb{R}^{2n})$ can be realized as the isotopy of some compactly supported Hamiltonian. But this is in fact the case. [2]

That $\text{Ham}^c(\mathbb{R}^{2n})$ is a normal subgroup of $\text{Symp}^c(\mathbb{R}^{2n})$ follows from the following observations: Suppose that $f$ and $g$ are any two elements of $\text{Ham}^c(\mathbb{R}^{2n})$,
generated by Hamiltonian functions $F \in C$ and $G \in C$. Then the symplectomorphisms $f^{-1}$ and $g \circ f$ are generated by the functions $-F_t \circ f_t$ and $G_t + F_t \circ g_t^{-1}$, both of which also have compact support. This shows that $Ham^c(\mathbb{R}^{2n})$ is a subgroup. Next, note that for any $\phi \in Symp(\mathbb{R}^{2n})$, the symplectomorphism $\phi^{-1} \circ f \circ \phi$ is generated by the function $F_t \circ \phi$, whose support is compact. This shows that the subgroup is normal.

Summarizing this section, we have $Ham^c(\mathbb{R}^{2n}) \triangleleft Symp_0(\mathbb{R}^{2n}) \triangleleft Symp(\mathbb{R}^{2n})$.

### 1.3 Hamiltonian Symplectomorphisms of Arbitrary Support

Beginning as we did above, we could omit the restriction of compact support and consider the vector space of all smooth maps $F : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R}$. (Smooth in the compact-open $C^\infty$ topology.) But if a function $F : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R}$ has non-compact support, then a solution to the initial value problem

$$\begin{align*}
h_0 &= id \\
h_t &= X^F_t \circ h_t
\end{align*}$$

might not exist for the whole time interval $t \in [0, 1]$. Example 1 in section 1.4 gives a such a function. Only if a solution to the initial value problem does exist for the whole time interval do we call the function a Hamiltonian function. We denote by $\mathcal{H}$ the set of Hamiltonian functions of arbitrary support. It is worth noting that, while the set $C$ is a vector space, the set $\mathcal{H}$ is not. Example 2 in section 1.4 gives an example of two functions, $G, H \in \mathcal{H}$, such that $G + H \notin \mathcal{H}$.

For the Hamiltonian functions of arbitrary support, then, we have the same generation of vector fields, isotopies, and symplectomorphisms as in the compact case. We will have a diagram similar to the one shown in Fig. 1.1, depicting the
generation of Hamiltonian symplectomorphisms from functions, but with the qualification compact support omitted in each box.

We denote by $\psi^\#$ the set of associated Hamiltonian symplectomorphisms, but will also use the more common symbol, $Ham(\mathbb{R}^{2n})$. It shown to be a normal subgroup of $Symp(\mathbb{R}^{2n})$ in exactly the same way that $Ham^c(\mathbb{R}^{2n})$ was shown to be, above. But in fact it is straightforward to show that $Ham(\mathbb{R}^{2n}) = Symp(\mathbb{R}^{2n})$. So all symplectomorphisms of $\mathbb{R}^{2n}$ are Hamiltonian symplectomorphisms. The proof of this claim is postponed to section 1.4. It is a fairly simple, well-known proof that uses what is sometimes referred to as the Alexander trick. We include only to introduce notation and the idea of the trick, in preparation for more sophisticated uses of it in chapters 2 and 4.

1.4 Normal subgroups of $Symp(\mathbb{R}^{2n})$

Much about the structure of $Symp(\mathbb{R}^{2n})$ is unknown. Here we will explore one aspect: the existence of non-trivial normal subgroups. It might be expected, since $Symp(\mathbb{R}^{2n})$ is a topological group, that we would be most interested in its closed normal subgroups. But, because the compact-open topology gives no control over behavior at infinity, the closure of $Ham^c(\mathbb{R}^{2n})$ in $Symp(\mathbb{R}^{2n})$ is actually all of $Symp(\mathbb{R}^{2n})$. So, we will be interested in the algebraic structure of $Symp(\mathbb{R}^{2n})$ as a discrete group.

In the case of a compact symplectic manifold $(M, \omega)$ without boundary, we know from Banyaga [1] that $Ham(M, \omega)$ is simple, that is, has no non-trivial normal subgroups. Less is known for non-compact spaces such as $(\mathbb{R}^{2n}, \omega_0)$. We have already seen that the subgroup $Ham^c(\mathbb{R}^{2n}, \omega_0)$ is normal, and clearly non-trivial. Again from Banyaga, we know that a simple subgroup is found within $Ham^c(\mathbb{R}^{2n})$: the kernel of the Calabi homomorphism. Aside from $Ham^c(\mathbb{R}^{2n})$ and its simple subgroup, however, it is not clear where one might look for non-trivial normal subgroups of $Symp(\mathbb{R}^{2n})$. 

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One idea suggests itself when we look again at $Ham^c(\mathbb{R}^{2n})$. That subgroup could be thought of as being obtained from the larger group $Symp(\mathbb{R}^{2n})$ by imposing a restriction on the set of allowed Hamiltonian functions, in this case the restriction being that they be compactly supported. What other restrictions, milder than requiring compact support, might we place on the set of Hamiltonian functions and obtain, as a result, a non-trivial normal subgroup of $Symp(\mathbb{R}^{2n})$ that contains $Ham^c(\mathbb{R}^{2n})$?

One milder restriction immediately comes to mind: instead of requiring that the function be compactly supported, we could require only that it be compactly supported at each time $t \in [0, 1]$. We must be careful how we pose our restriction, though. The set $C$ was defined as the vector space of all compactly supported smooth maps $H : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R}$. Any such function generates a symplectomorphism.

If we merely defined a new set of functions as the set of smooth maps $F : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R}$ that are compactly supported at each time $t \in [0, 1]$, then our set would be too large: it would contain functions that don’t generate symplectomorphisms. These would not be called Hamiltonian functions, and are of no interest to us. In example 3 in section 1.4 of this chapter, a smooth function $F : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R}$ is presented that has compact support at each time $t \in [0, 1]$ and yet does not generate a symplectomorphism.

So, we want to only consider functions that are in $\mathcal{H}$ and that have compact support at each time. Let us denote by $\mathcal{H}_C$ the subset \[ \{ H \in \mathcal{H} : \text{support} (H) \text{ is compact at each } t \in [0, 1] \}, \] and denote by $\psi^{\mathcal{H}_C}$ the associated set of symplectomorphisms generated by these functions. One can quickly prove that $\psi^{\mathcal{H}_C}$ is a normal subgroup of $Symp(\mathbb{R}^{2n})$ by the same method that was used to show that $Ham^c(\mathbb{R}^{2n})$ is a normal subgroup. We consider two symplectomorphisms, $f$ and $g$, that are generated by Hamiltonian functions $F, G \in \mathcal{H}_C$, and any $\phi \in Symp(\mathbb{R}^{2n})$. Then the symplectomorphisms $f^{-1}$,
$g \circ f$, and $\phi^{-1} \circ f \circ \phi$ are generated by the functions $-F_t \circ f_t$, $G_t + F_t \circ g_t^{-1}$, and $F_t \circ \phi$ each of whose support is compact at each time. This shows that $\psi^{\mathcal{H}_C}$ is a normal subgroup.

Because of the inclusions $C \subset \mathcal{H}_C \subset \mathcal{H}$, we will have the following sequence of normal subgroups of symplectomorphisms: $\text{Ham}^c(\mathbb{R}^{2n}) = \psi^C \triangleleft \psi^\mathcal{H}_C \triangleleft \psi^\mathcal{H} = \text{Ham}(\mathbb{R}^{2n}) = \text{Symp}(\mathbb{R}^{2n})$

For now, we postpone the question of which of these subgroups is proper.

Other fairly obvious sub-classes of $\mathcal{H}$ that we should consider are those that are decaying at each time, bounded at each time, or uniformly bounded. These we will denote by $\mathcal{H}_D$, $\mathcal{H}_B$, and $\mathcal{H}_{UB}$:

$$\mathcal{H}_D = \left\{ H \in \mathcal{H} : \forall t \in [0,1], \lim_{R \to \infty} \sup \{|H_t(x)| : \|x\| > R\} = 0 \right\}$$

$$\mathcal{H}_B = \left\{ H \in \mathcal{H} : \forall t \in [0,1], \sup_{x \in \mathbb{R}^{2n}} |H_t(x)| < \infty \right\}$$

$$\mathcal{H}_{UB} = \left\{ H \in \mathcal{H} : \sup_{t \in [0,1]} \left( \sup_{x \in \mathbb{R}^{2n}} |H_t(x)| \right) < \infty \right\}$$

We denote by $\psi^{\mathcal{H}_D}$, $\psi^{\mathcal{H}_B}$, and $\psi^{\mathcal{H}_{UB}}$ the associated sets of symplectomorphisms generated by these functions. One shows that these sets are in fact normal subgroups of $\text{Symp}(\mathbb{R}^{2n})$ in the same way that we showed that $\text{Ham}^c(\mathbb{R}^{2n})$ and $\psi^\mathcal{H}_C$ are normal subgroups. Note that because we have the sequence of inclusions of sets of functions, $C \subset \mathcal{H}_C \subset \mathcal{H}_D \subset \mathcal{H}_B \subset \mathcal{H}$, we will have this sequence of corresponding normal subgroups:

$$\text{Ham}^c(\mathbb{R}^{2n}) = \psi^C \triangleleft \psi^\mathcal{H}_C \triangleleft \psi^{\mathcal{H}_D} \triangleleft \psi^{\mathcal{H}_B} \triangleleft \psi^\mathcal{H} = \text{Ham}(\mathbb{R}^{2n}) = \text{Symp}(\mathbb{R}^{2n})$$

We turn now to the question of which of these inclusions are proper. The requirement that the generating Hamiltonians have compact support at each time $t \in [0,1]$ seems a fairly strong restriction, and one might expect that the resulting normal subgroup, $\psi^{\mathcal{H}_C}$, would be a proper subgroup of $\text{Symp}(\mathbb{R}^{2n})$. It is rather surprising, then, that the subgroup is in fact the entire group $\text{Symp}(\mathbb{R}^{2n})$. In chapter 2,
we will prove the following theorem:

Theorem A: Each $f$ in $\text{Symp}(\mathbb{R}^{2n})$ can be generated by a Hamiltonian function $H_t$ with the property that $\text{support}(H_t)$ is compact for each $t \in [0,1)$ and $H_1$ is zero.

With that, our intriguing sequence of subgroups collapses to something far less interesting:

$$\text{Ham}^c(\mathbb{R}^{2n}) = \psi^C \triangleleft \psi^\mathcal{H}_C = \psi^\mathcal{H}_D = \psi^\mathcal{H}_B = \psi^\mathcal{H} = \text{Ham}(\mathbb{R}^{2n}) = \text{Symp}(\mathbb{R}^{2n})$$

The sequence contains no new non-trivial normal subgroups of $\text{Symp}(\mathbb{R}^{2n})$.

We have not considered the other sequence of inclusions of sets of functions, $C \subset \mathcal{H}_{UB} \subset \mathcal{H}$. As with the other sets of symplectomorphisms, we can show that $\psi^\mathcal{H}_{UB}$ is a normal subgroup of $\text{Symp}(\mathbb{R}^{2n})$, so that we have the sequence $\text{Ham}^c(\mathbb{R}^{2n}) \triangleleft \psi^\mathcal{H}_{UB} \triangleleft \text{Symp}(\mathbb{R}^{2n})$. Clearly $\text{Ham}^c(\mathbb{R}^{2n})$ is a proper subgroup of $\psi^\mathcal{H}_{UB}$, but is $\psi^\mathcal{H}_{UB}$ a proper subgroup of $\text{Symp}(\mathbb{R}^{2n})$? The answer to this question turns out to be yes, but its proof uses notation and mathematical tools developed in Chapter 3. In that chapter, we will introduce the extended Hofer infinity norm, $\widetilde{E}_\infty$. The chapter ends with the following corollary, which is what we need to prove that $\psi^\mathcal{H}_{UB}$ is a proper subgroup of $\text{Symp}(\mathbb{R}^{2n})$:

**Corollary to the energy-capacity inequality:** If a subset $\Omega \subset \text{Symp}(\mathbb{R}^{2n})$ has the property that $\widetilde{E}_\infty(\psi)$ is finite for all $\psi \in \Omega$, then $\Omega$ is a proper subset of $\text{Symp}(\mathbb{R}^{2n})$, for $\Omega$ will contain no rotations.

The set of symplectomorphisms for which $\widetilde{E}_\infty(\psi)$ is finite are precisely those that can
be generated by uniformly-bounded Hamiltonian functions. That is to say, elements of $\psi^{H_{UB}}$.

In Chapter 4, this corollary will be used in the proof that another normal subgroup of $\text{Symp}(\mathbb{R}^{2n})$ is in fact a proper normal subgroup. The description of that subgroup, which is quite a bit more complicated than those we have seen, is as follows.

Let $U = \bigcup_{i=1}^{\infty} f_i(B^{2n}(R)) \subset \mathbb{R}^{2n}$ be any disjoint union of symplectic balls of radius $R < \frac{1}{2}$. (By symplectic ball of radius $R$, we mean the image, $f(B^{2n}(R))$, of a symplectic embedding, $f: B^{2n}(R) \to \mathbb{R}^{2n}$, where $B^{2n}(R)$ is the closed ball of radius $R$.) Denote by $\text{Symp}_U(\mathbb{R}^{2n})$ the set of symplectomorphisms of $\mathbb{R}^{2n}$ that can be generated by Hamiltonian functions with support contained in $U$. Then $\text{Symp}_U(\mathbb{R}^{2n})$ is a subgroup of $\text{Symp}(\mathbb{R}^{2n})$, but it is not a normal subgroup: conjugation of an element of $\text{Symp}_U(\mathbb{R}^{2n})$ with a translation can produce an element of $\text{Symp}(\mathbb{R}^{2n})$ that is not supported in $U$. Define $G_U \triangleleft \text{Symp}(\mathbb{R}^{2n})$ to be the minimal normal subgroup of $\text{Symp}(\mathbb{R}^{2n})$ containing $\text{Symp}_U(\mathbb{R}^{2n})$. That is, $G_U$ contains $\text{Symp}_U(\mathbb{R}^{2n})$, is closed under conjugation by elements of $\text{Symp}(\mathbb{R}^{2n})$, and is closed under composition.

The theorem that we will prove is,

*Theorem B (Chapter 4):* $G_U$ is a proper subgroup of $\text{Symp}(\mathbb{R}^{2n})$.

In the proof of that theorem, we will show that if $\psi \in G_U$, then $E_{\infty}(\psi)$ is finite. Therefore, $G_U \triangleleft \psi^{H_{UB}}$, and hence, $G_U$ must be a proper subgroup of $\text{Symp}(\mathbb{R}^{2n})$. 

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Note that it is not clear whether $G_U$ is a proper subgroup of $\psi^U_B$ and, if it is, how it compares to $Ham^c(\mathbb{R}^{2n})$, which is also a proper normal subgroup of $\psi^U_B$. It is clear that $G_U$ and $Ham^c(\mathbb{R}^{2n})$ have non-trivial elements in common: symplectomorphisms supported in a single one of the symplectic balls of the union $U$ are compactly supported, so they are in both $G_U$ and $Ham^c(\mathbb{R}^{2n})$. And, $G_U$ contains elements of non-compact support that cannot be in $Ham^c(\mathbb{R}^{2n})$. But it is also possible that $Ham^c(\mathbb{R}^{2n})$ could contain elements that are not in $G_U$.

Previous examples of normal subgroups of $Symp(\mathbb{R}^{2n})$ in this chapter arose through fairly natural restrictions on the set of generating Hamiltonian functions, whereas this last example, $G_U$, seems rather random. But remember that $Symp(\mathbb{R}^{2n})$ is a subgroup of the group $Diff_{vol}(\mathbb{R}^{2n})$ of volume-preserving diffeomorphisms of $\mathbb{R}^{2n}$. Our subgroup $G_U$ of $Symp(\mathbb{R}^{2n})$ becomes more interesting, when we compare it to a similarly-constructed subgroup of $Diff_{vol}(\mathbb{R}^{2n})$. That is, for the set $U$, described above, let $Diff_{vol}(U) \subset Diff_{vol}(\mathbb{R}^{2n})$ be the collection of volume-preserving diffeomorphisms of $\mathbb{R}^{2n}$ that are supported in $U$. Note that $Diff_{vol}(U)$ is a non-normal subgroup of $Diff_{vol}(\mathbb{R}^{2n})$. As above, define $G_{Diff_{vol}(U)} \triangleleft Diff_{vol}(\mathbb{R}^{2n})$ to be the minimal normal subgroup of $Diff_{vol}(\mathbb{R}^{2n})$ containing $Diff_{vol}(U)$. McDuff [5] showed that for $n \geq 2$, $G_{Diff_{vol}(U)} = Diff_{vol}(\mathbb{R}^{2n})$, so $G_{Diff_{vol}(U)}$ is not a proper subgroup. So Theorem B distinguishes the structure of $Symp(\mathbb{R}^{2n})$ from that of $Diff_{vol}(\mathbb{R}^{2n})$.

### 1.5 Examples and Proofs

#### 1.5.1 A non-Hamiltonian smooth function $F : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R}$

Our example is a time independent function, $F : \mathbb{R}^2 \to \mathbb{R}$ with these properties:

- $F$ is supported in the set $\mathbb{R} \times (0, 3)$
Figure 1.2: A non-Hamiltonian smooth function

- $F$ has height 1 on the set $\mathbb{R} \times \{1\}$

- $F$ has height 2 on the set $\mathbb{R} \times \{2\}$

- The height of $F$ decreases monotonically as $|y - 2|$ increases.

- The maximum gradient occurs at points of the form $(x, 1)$. At those points, the partial in the $y$ direction is $\left. \frac{\partial F(x, y)}{\partial y} \right|_{(x, 1)} = \frac{x}{2} (1 + x^2)$.

Figure 1.2 shows a cross section of the graph of $F$.

Associated to the function $F$ is the time-independent vector field $X^F = sgrad(F)$. Consider the solutions $f_t$ to the initial value problem

$$
\begin{cases}
  f_0 = id \\
  \dot{f}_t = X^F \circ f_t.
\end{cases}
$$

Let $p$ be a point of the form $p = (a, 1)$, and let $(x_p(t), y_p(t)) = f_t(p)$ be the coordinates of the point as it moves under the influence of $f_t$. Note that $p$ will move on the set $\mathbb{R} \times \{1\}$, because that is a level set of the function $F$, so $y_p(t)$ will be constant,
with value 1. The initial value problem simplifies to a the one-variable problem

\[
\begin{align*}
    x_p(0) &= a \\
    \frac{dx_p(t)}{dt} &= \frac{\partial F(x,y)}{\partial y} \bigg|_{(x_p(t),1)} = \frac{y}{2} \left( 1 + (x_p(t))^2 \right).
\end{align*}
\]

This has solution \( x_p(t) = \tan \left( \frac{t \pi}{2} + \arctan(a) \right) \). If \( p \) is the point \((0,1)\), then \( x_p(t) = \tan \left( \frac{t \pi}{2} \right) \), which is defined for \( t \in [0,1) \), but not for \( t = 1 \). So \( f_t \) is not defined on the interval \( t \in [0,1] \), and \( F \) does not generate a symplectomorphism; it is not Hamiltonian. Notice that in fact, \( f_t \) is not defined on any time interval \( t \in [0,b] \), where \( 0 < b < 1 \). To see this, let \( p \) be the point \((\tan \left( \frac{\pi}{2} (1 - b) \right), 1)\), and consider its evolution, of as it moves under the influence of \( f_t \). At time \( t \), its \( x \) coordinate will be \( x_p(t) = \tan \left( \frac{\pi}{2} (1 + t - b) \right) \), which is defined for \( t \in [0,b) \), but not for \( t = b \).

### 1.5.2 Two Hamiltonian functions whose sum is not Hamiltonian

We will construct two Hamiltonian functions, \( G \) and \( H \), whose sum is the function \( F \) from the previous example. Let \( \{ \chi_i \}_{i=1}^{\infty} \) be a partition of unity on \( \mathbb{R} \) subordinate to the cover \( \{ U_i \}_{i=1}^{\infty} \), where \( U_i = (i - 1, i + 1) \), and define \( \phi_i(x,y) = \chi_i(x) \cdot F(x,y) \). Because \( \phi_i \) is compactly supported, time-independent, and smooth, it is Hamiltonian. Notice that if \( i \neq j \), then the support of \( \phi_{2i} \) and \( \phi_{2j} \) are disjoint. So let \( G(x,y) = \sum_{i=-\infty}^{\infty} \phi_{2i} \). Then \( G \) is Hamiltonian. Similarly, let \( H(x,y) = \sum_{i=-\infty}^{\infty} \phi_{2i+1} \). Then \( H \) is Hamiltonian, as well, but the sum of \( G \) and \( H \) is \( F \), which is not Hamiltonian.

### 1.5.3 A smooth function \( G : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R} \), with compact support at each time \( t \in [0,1] \), that is not Hamiltonian.

In this example, let \( \chi_t(x) \) be a moving cutoff function whose height, at each time, is identically 1 on the interval \( x \in [\tan(t) - 1, \tan(t) + 1] \) and which is supported in
the interval \( x \in (\tan(t) - 2, \tan(t) + 2) \). Define \( G : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R} \) by

\[
G_t(x, y) = \begin{cases} 
\chi_t(x) \cdot F_t(x,y) & \text{for } t \in [0, 1) \\
0 & \text{for } t = 1,
\end{cases}
\]

where \( F \) is the function from example 1. Then \( G \) is smooth in the compact-open \( C^\infty \) topology. Associated to the function \( G \) is the time-dependent vector field \( X^G_t = sgrad(G_t) \).

For times \( t \in [0, 1) \), consider the solution \( g_t \) of the initial value problem

\[
\begin{aligned}
g_0 &= id \\
\dot{g}_t &= X^G_t \circ g_t
\end{aligned}
\]

Let \( p \) be the point \((0,1)\), and consider the evolution of the point \( p \) under the influence of \( g_t \). Notice that the moving cutoff function, \( \chi_t(x) \), defined above, is centered on the point \( x(t) = \tan \left( \frac{\pi t}{2} \right) \). But this is exactly the \( x \) coordinate with which the same point \( p = (0,1) \) moved in example 1, when under the influence of \( f_t \). Because of the way we have constructed the time-dependent function \( G \), at each time \( t \in [0, 1) \), in a neighborhood of the point \( (x,y) = (\tan \left( \frac{\pi t}{2} \right), 1) \), it will have exactly the same shape as the function \( F \). So, the point \( p \) will move in exactly the same way under the influence of \( g_t \) as it would under the influence of \( f_t \). That is to say, its \( x \) coordinate will go to infinity, as time \( t \) approaches 1. So, there is no way that \( g_t \) can extend to an isotopy defined on the entire time interval \([0, 1)\), and hence, \( G \) is not Hamiltonian.

It is worth articulating what's "wrong" with the function \( G \), because in Chapter 2, we will construct a function \( H \) that, like \( G \), has support that is compact at each time \( t \in [0, 1] \) and moves to infinity, in space, as time \( t \) approaches 1. Unlike \( G \), however, the function \( H \), will be Hamiltonian. What bad behavior of \( G \) will we avoid when constructing \( H \)?

Think again of the point \( p = (0,1) \), discussed above. It starts moving, at time \( t = 0 \), because it sits at a place where the gradient of \( G \) is non-zero. One could
think of the function $G$ as a wave; the point $p$, a surfer. Note, however, that $p$ moves not “down”, which would be in the direction of the negative gradient, but rather, along the level set - the direction of the symplectic gradient. Since such a “symplectic surfer” never moves down, off the wave: the surfer will keep moving as long as the wave stays up. Only if the wave falls (or levels off) does the surfer stop moving. Since the wave - our function $G$ -stays up and moves to infinity, in space, the surfer goes with it. We could call such a function gnarly. Fun as it may be for the surfer, it is of no interest to us, because it is not Hamiltonian: it does not give rise to a map that sends each point of $\mathbb{R}^2$ to some well-defined destination in $\mathbb{R}^2$.

In Chapter 2, we will be careful to build a function $H$ that could be thought of as a succession of waves, rising in rings from the level sea, radiating outward, then falling. Successive waves will rise from spots farther and farther out from a particular spot, and they will move to infinity. But, every point - every symplectic surfer - will at some time be picked up, moved around, and let down by some wave, and will not get picked up by any subsequent rising wave. Since every point gets moved from its initial location to a well-defined final destination sometime in the interval $t \in [0, 1)$, the result, at time $t = 1$, is a well-defined symplectomorphism.

1.5.4 Proof that $\text{Ham} \left( \mathbb{R}^{2n} \right) = \text{Symp} \left( \mathbb{R}^{2n} \right)$

Let $f$ be an arbitrary element of $\text{Symp} \left( \mathbb{R}^{2n} \right)$. With no loss of generality, we may assume that $f (0) = 0$.

We start by constructing a symplectic isotopy from a linear map to $f$, using the Alexander trick. Let $m_c : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n}$ be the multiplication by the scalar $c$. Then for each $t \in (0, 1)$, the map $m_1 \circ f \circ m_t$ is an element of $\text{Symp} \left( \mathbb{R}^{2n} \right)$. Note that when this symplectomorphism acts on an element $x \in \mathbb{R}^{2n}$, the result is just the
difference quotient
\[ m_t \circ f \circ m_t(x) = \frac{f(tx)}{t} = \frac{f(tx) - f(0)}{t}. \]

Because \( f \) is differentiable, the \( t \to 0 \) limit exists in \( \mathbb{R}^{2n} \):
\[
\lim_{t \to 0} m_t \circ \tau_{-f(0)} \circ f \circ m_t(x) = \lim_{t \to 0} \frac{f(tx) - f(0)}{t} = \mathcal{L}_0((df)\circ f)(x).
\]

Here, \( \mathcal{L}_0((df)\circ f) \) is the linear operator obtained by left multiplication by the matrix \( (df)_0 \in Sp(2n) \), and is called the linearization of \( f \) at zero.

If we define a path \( \beta_t \) in \( \text{Symp}(\mathbb{R}^{2n}) \) as
\[
\beta_t = \begin{cases} 
    m_t \circ \tau_{-f(0)} \circ f \circ m_t & \text{if } t \in (0, 1] \\
    \mathcal{L}_0((df)\circ f) & \text{if } t = 0
\end{cases}
\]
then \( \beta_t \) is a continuous path in the compact-open topology on \( \text{Symp}(\mathbb{R}^{2n}) \), with \( \beta_0 \) being the linear map \( \mathcal{L}_0((df)\circ f) \), and \( \beta_1 = f \). The map \( \beta_t \) is the isotopy we promised; it is the method of constructing \( \beta_t \) that is referred to as the Alexander trick.

Now we construct an isotopy from the identity to \( \mathcal{L}_0((df)\circ f) \). Because \( Sp(2n) \) is path connected, there is a path \( \sigma : [0, 1] \to Sp(2n) \) with \( \sigma_0 = I \) and \( \sigma_1 = (df)_0 \). Let \( \alpha : [0, 1] \to \text{Symp}(\mathbb{R}^{2n}) \) be the map defined by \( \alpha_t = L_{\sigma_t} \). Then \( \alpha \) is an isotopy with \( \alpha_0 = id \) and \( \alpha_1 = L_{(df)\circ f} \).

Concatenating these two paths in time and smoothing, we obtain a symplectic isotopy, \( \gamma = \alpha \circ \beta \), from the identity map to \( f \). From the symplectic isotopy \( \gamma_t \), we obtain the velocity vector field, \( \dot{\gamma}_t \), which will be a time-dependent symplectic vector field. This defines a family of closed 1-forms, \( \iota(\dot{\gamma}_t)\omega_0 \). Since \( H^1(\mathbb{R}^{2n}, \mathbb{R}) = 0 \), there is a smooth function, \( F : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R} \) such that at each \( t \in [0, 1] \), \( \iota(\dot{\gamma}_t)\omega_0 = d(F_t) \).

We see that \( F \) generates \( f \), so \( f \in \text{Ham}(\mathbb{R}^{2n}) \). Also note that since \( \text{Ham}(\mathbb{R}^{2n}) \) is contained in the path-component of the identity in \( \text{Symp}(\mathbb{R}^{2n}) \), which is in turn contained in the connected component of the identity, \( \text{Symp}_0(\mathbb{R}^{2n}) \), we conclude that
\[
\text{Ham}(\mathbb{R}^{2n}) = \text{Symp}_0(\mathbb{R}^{2n}) = \text{Symp}(\mathbb{R}^{2n}).
\]
Chapter 2

A surprising fact about $\text{Symp}(\mathbb{R}^{2n})$

2.1 Introduction and Theorem A

In Chapter 1, we denoted by $\mathcal{H}$ the collection of time-dependent Hamiltonian functions on $\mathbb{R}^{2n}$ with arbitrary support; we denoted by $\mathcal{H}_C$ the subset

$$\{H \in \mathcal{H} : \text{support}(H) \text{ is compact at each } t \in [0,1]\};$$

and we denoted by $\psi^{\mathcal{H}_C}$ the associated set of symplectomorphisms generated by these functions. We saw that the set was in fact a normal subgroup, $\psi^{\mathcal{H}_C} \subset \text{Symp}(\mathbb{R}^{2n})$, but the question of whether or not this subgroup was proper remained to be answered. In this chapter, we prove that the subgroup is not proper: it is the entire group. The proof exploits the sometimes-overlooked fact that the compact-open topologies used on $\text{Symp}(\mathbb{R}^{2n})$ and $\mathcal{H}$ allow some unusual paths to qualify as smooth.

More specifically, in $\text{Symp}(\mathbb{R}^{2n})$, it is relatively easy for a continuous map $g : [0,1) \to \text{Symp}(\mathbb{R}^{2n})$ to be extendable to the entire time interval $t \in [0,1]$. All that is necessary is that for each $x \in \mathbb{R}^{2n}$, there is some time $t_x < 1$ such that for all times $t > t_x$, $g_t(x) = g_{t_x}(x)$. That is, the point $x$ gets moved around during the time interval $0 \leq t \leq t_x$, and then stays put for the remainder of the time interval, $t_x < t < 1$. If that is the case, then $\lim_{t \to 1} g_t$ exists, and $g_1$ can be defined as this limit.
So, \( g_t \) may be defined on the entire time interval \([0, 1]\), even though it may get quite wild as \( t \to 1 \).

Analogously, in the compact-open \( C^\infty \) topology on the function space, for a continuous function \( H : \mathbb{R}^{2n} \times [0, 1] \to \mathbb{R} \) to be extendable to the entire time interval \( t \in [0, 1] \), one need only insure that the support of the function go to infinity (in space), as \( t \to 1 \). If that is the case, \( \lim_{t \to 1} H_t \) is the zero function, and \( H_1 \) can be defined to be this limit, thus extending \( H \) to the entire interval. So, as with \( g_t \), \( H_t \) may be defined on the entire time interval \([0, 1]\) even though it may get quite wild as \( t \to 1 \). When we exploit this fact in the proof, we will also be interested in insuring that the resulting function \( H \) is a Hamiltonian function, i.e. that it does generate a well-defined symplectomorphism. But this turns out to be possible in the cases of interest here. The key, as was discussed at the end of example 1.4.3, will be to make sure that the function \( H \) is constructed so that each point of \( \mathbb{R}^{2n} \), as it moves with the symplectic gradient flow of \( H \), eventually, at some time before \( t = 1 \), leaves the support set of \( H \) for good.

**Theorem A:** Each \( f \) in \( \text{Symp}(\mathbb{R}^{2n}) \) can be generated by a Hamiltonian function \( H \) with the property that \( \text{support}(H_t) \) is compact for each \( t \in [0, 1] \) and \( H_1 \) is zero.

**Proof of Theorem A:**

We will prove in **Lemma A1** that there is a sequence of Hamiltonian functions \( G_{k,t} \), \( k = 1, 2, \cdots \) with these properties:

1. \( \forall k, \text{support}(G_{k,t}) \) is compact for all \( t \in [0, 1] \).

2. \( \forall k, G_{k,0} \equiv G_{k,1} \equiv 0 \).

3. There is a sequence of radii, \( R_k \), \( k = 1, 2, \cdots \), with \( R_k \xrightarrow{k \to \infty} \infty \), such that for all \( t \in [0, 1] \), \( G_{k,t} | B^{2n}(R_k, f(0)) \equiv 0 \).
4. $G_{k,t}$ generates a Hamiltonian isotopy $g_{k,t}$ with these properties

(a) $g_{k,t} | B^{2n} (R_k, f(0)) = id$ for all $t \in [0, 1]$.

(b) Let $\pi_k = g_k \circ g_{k-1} \circ \cdots \circ g_2 \circ g_1$. Then $\pi_k | B^{2n} (k) = f | B^{2n} (k)$.

Notice that property (4a) follows immediately from property (3). Also, because of property (3), the sequence $G_{k,t}$ converges in the compact-open topology to the zero function, as $k$ approaches infinity. And, because of property (2), the functions can be concatenated in time to produce a smooth function. So we define a new function, $H_t$, for $t \in [0, 1]$ by concatenating and re-scaling height and time in the following way. We define the sequence of dyadic times, $t_0 = 0$, $t_1 = \frac{1}{2}$, $t_2 = \frac{3}{4}$, $\cdots$, $t_k = 1 - \left(\frac{1}{2}\right)^k$, $\cdots$. Then the $k^{th}$ interval, $t_{k-1} \leq t \leq t_k$, has length $\left(\frac{1}{2}\right)^k$. Define the function $H : \mathbb{R}^{2n} \times [0,1] \to \mathbb{R}$ piecewise in time by the formula

$$H_t = \begin{cases} 2^k G_{k,2^k(t-t_{k-1})} & \text{for } t_{k-1} \leq t \leq t_k \\ 0 & \text{for } t = 1. \end{cases}$$

Then $H$ is continuous in the compact-open topology, and for each $t \in [0,1]$, the support of $H_t$ is compact.

Let us now check that because of property (4), $H$ will generate a well-defined symplectic isotopy, which we will call $h_t$, defined on the time interval $t \in [0,1]$. That is, we check that $H$ is Hamiltonian. Certainly, for any number $b$, where $0 \leq b < 1$, $H$ generates an isotopy that is well-defined on the time interval $t \in [0, b]$. So, the isotopy is well-defined for $t \in [0, 1)$. Call this isotopy $h_t$. We show now that in fact $h_t$ extends to the entire time interval.

Examining the value of $h_t$ at one of the dyadic times, we find that because of property (4b) $h_{t_k} = \pi_k = g_k \circ g_{k-1} \circ \cdots \circ g_1$. That is, $h_{t_k}$ agrees with $f$ on the set $B^{2n} (k)$. Now consider the in-between times, $t_{k-1} < t < t_k$. Because of property (4a), progress of the isotopy $h_t$ during this time interval will not affect points in the ball $B^{2n} (R_k, f(0))$. So every point in $\mathbb{R}^{2n}$ gets moved around by $h_t$ during the interval
$t \in [0, 1)$, but each point eventually lands in one of the balls $B^{2n}(R_k, f(0))$. Where it remains for the rest of the time interval. For all time beyond $t_k$, all the activity caused by the isotopy $h_t$ occurs outside of the ball $B^{2n}(R_k, f(0))$. Inside that ball, all points are in the same configuration that they would be in if they had been moved by the symplectomorphism $f$. Since $R_k \to \infty$, we see that as time $t \to 1$, the isotopy $h_t$ converges in the compact-open topology to $f$. So in fact, $H_t$ generates an isotopy that is well defined for $t \in [0, 1]$, and the time $= 1$ value of that isotopy is the symplectomorphism $f$.

End of proof of Theorem A

2.2 Lemma A1

Lemma A1: For any $f \in \text{Symp}(\mathbb{R}^{2n})$, there is a sequence of Hamiltonian functions $G_{k,t}, k = 1, 2, \cdots$ with these properties:

1. $\forall k$, support $(G_{k,t})$ is compact for all $t \in [0, 1]$.

2. $\forall k, G_{k,0} \equiv G_{k,1} \equiv 0$.

3. There is a sequence of radii, $R_k, k = 1, 2, \cdots$, with $R_k \to \infty$, such that for all $t \in [0, 1], G_{k,t} | B^{2n}(R_k, f(0)) \equiv 0$.

4. $G_{k,t}$ generates a Hamiltonian isotopy $g_{k,t}$ with these properties

(a) $g_{k,t} | B^{2n}(R_k, f(0)) = \text{id}$ for all $t \in [0, 1]$.

(b) Let $\pi_k = g_k \circ g_{k-1} \circ \cdots \circ g_2 \circ g_1$. Then $\pi_k | B^{2n}(k) = f | B^{2n}(k)$.

Proof of Lemma A1

step i: Establish the sequence of radii, $R_k, k = 1, 2, \cdots$. 

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Let \( R_k = \sup \{ r : B^{2n}(r, f(0)) \subset f(B^{2n}(k)) \} \). Notice that \( R_k < R_{k+1} \) and that \( R_k \to \infty \) as \( k \to \infty \).

**step ii:** Introduce \( F_t, f_t, S_1, N_1, \chi_1, G_{1,t}, g_{1,t} \), and \( g_1 \).

Since \( f \in Symp(\mathbb{R}^{2n}) \), there is a Hamiltonian function \( F_t \) that generates \( f \). Without loss of generality, we may assume that \( F_0 \equiv 0 \equiv F_1 \). Let \( f_t \) be the corresponding isotopy, so that \( f_t = f_1 \). We consider the evolution of the balls \( B^{2n}(1) \) and \( B^{2n}(2) \), as they move under the influence of isotopy \( f_t \). In particular, we define the sets \( S_1 = \bigcup_{t=0}^{1} f_t(B^{2n}(1)) \) and \( N_1 = \bigcup_{t=0}^{1} f_t(B^{2n}(2)) \). We will think of \( S_1 \) as the swath of \( \mathbb{R}^{2n} \) that the ball \( B^{2n}(1) \) moves through, under the influence of the isotopy \( f_t \), and we will think of \( N_1 \) as a larger neighborhood. Using these, we define a smooth cutoff function, \( \chi_1 \), to be identically one on \( S_1 \) and to be zero outside \( N_1 \). We then produce a new Hamiltonian function, \( G_{1,t} = F_t \cdot \chi_1 \), call the corresponding isotopy \( g_{1,t} \), and call its time = 1 value \( g_1 \).

Notice that since \( G_t \) agrees with \( F_t \) everywhere in the set \( S_1 \), the evolution of the ball \( B^{2n}(1) \), under the influence of \( g_{1,t} \), will be the same as its evolution would have been under the influence of \( f_t \). So \( g_1 | B^{2n}(1) = f | B^{2n}(1) \). Also, since \( G_{1,t} \) is supported in \( N_1 \) at all times, we conclude that

\[
g_1 = \begin{cases} 
  f & \text{on } B^{2n}(1) \\
  \text{non-trivial} & \text{on } N_1 \\
  \text{id} & \text{outside } N_1 
\end{cases}
\]

**step iii:** Introduce \( S_2, N_2, \chi_2, G_{2,t}, g_{2,t} \), and \( g_2 \).

We have produced a Hamiltonian symplectomorphism, \( g_1 \), that agrees with \( f \) on a ball of radius 1. Now, we will produce an isotopy that will enlarge the region of agreement with \( f \). It will be important that this new isotopy have no effect in the region where \( g_1 \) agrees with \( f \), but that the new one corrects some of the disagreement between the two in outer regions. To see that disagreement more clearly, we observe
the following behavior of the product $f \circ g_1^{-1}$:

$$f \circ g_1^{-1} = \begin{cases} id & \text{on } f (B^{2n} (1)) \\ non - trivial & \text{on } N_1 \\ f & \text{outside } N_1. \end{cases}$$

We can say a priori that there exists a Hamiltonian function, which we will call $F_{2,t}$, that generates $f \circ g_1^{-1}$. But in fact, we want to require more of $F_{2,t}$ than that it simply generate the desired symplectomorphism. We would like in addition to be able to say that for all $t \in [0, 1]$, $F_{2,t}$ is identically zero on some ball of known radius.

In Lemma A2, we will show that there is in fact a Hamiltonian function $F_{2,t}$ that generates $f \circ g_1^{-1}$ and such that for all $t \in [0, 1]$, $F_{2,t}$ is identically zero on the ball $B^{2n} (R_1, f (0))$. We call the corresponding Hamiltonian isotopy $f_{2,t}$, and call its corresponding time $= 1$ value $f_2$.

There is a problem with $F_{2,t}$, however, in that its support is not compact. So our next step is to produce a cutoff version of $F_{2,t}$, which we will call $G_{2,t}$. We will do this in a manner completely analogous to the way that we produced the Hamiltonian $G_{1,t}$ by multiplying $F_t$ by a cutoff function. More specifically, we define the sets $S_2$ and $N_2$ as

$$S_2 = \bigcup_{t=0}^{1} f_{2,t} (N_1) = \bigcup_{t=0}^{1} f_{2,t} \left( \bigcup_{t=0}^{1} f_t (B^{2n} (2)) \right)$$

$$N_2 = \bigcup_{t=0}^{1} f_{2,t} \left( \bigcup_{t=0}^{1} f_t (B^{2n} (3)) \right)$$

So $S_2$ is the swath of of $\mathbb{R}^{2n}$ that the set $N_1$ moves through, under the influence of the isotopy $f_{2,t}$, and $N_2$ is a larger neighborhood. Using these, we define a smooth cutoff function, $\chi_2$, to be identically one on $S_2$ and to be zero outside $N_2$. We then produce a new Hamiltonian function, $G_{2,t} = F_{2,t} \cdot \chi_2$, call the corresponding isotopy $g_{2,t}$, and call its time $= 1$ value, $g_2$.

Notice that since $G_{2,t}$ agrees with $F_{2,t}$ everywhere in the set $S_2$, the evolution of the set $N_1$, under the influence of $g_{2,t}$, will be the same as its evolution would
have been under the influence of $f_{2,t}$. So $g_2 | N_1 = f_2 | N_1 = f \circ g_1^{-1} | N_1$. Considering the behavior of the composition, $g_2 \circ g_1$, we find

$$g_2 \circ g_1 | B^{2n}(2) = g_2 | \underbrace{g_1 \left( B^{2n}(2) \right)}_{\text{contained in } N_1} \circ g_1 | B^{2n}(2)$$

(note that $g_2 | N_1 = (f \circ g_1^{-1}) | N_1$)

$$= (f \circ g_1^{-1}) | g_1 \left( B^{2n}(2) \right) \circ g_1 | B^{2n}(2)$$

$$= f \circ g_1^{-1} \circ g_1 | B^{2n}(2)$$

$$= f | B^{2n}(2).$$

As for the discrepancy between $g_2 \circ g_1$ and $f$, we observe that

$$f \circ (g_2 \circ g_1)^{-1} = \begin{cases} 
  \text{id} & \text{on } f \left( B^{2n}(2) \right) \\
  \text{non-trivial} & \text{on } N_2 \\
  f & \text{outside } N_2.
\end{cases}$$

*step iv: (inductive step)*

Let symplectomorphisms, $g_1, g_2, \cdots, g_k$ be given, and let $\pi_k = g_k \circ g_{k-1} \circ \cdots \circ g_2 \circ g_1$. Further suppose that a set of Hamiltonian functions $F_{1,t}, F_{2,t}, \cdots, F_{k,t}$ has been defined, with corresponding isotopies $f_{1,t}, f_{2,t}, \cdots, f_{k,t}$, and time $= 1$ values $f_1, f_2, \cdots, f_k$. Suppose that the collection of sets $S_1, S_2, \cdots, S_k$ and $N_1, N_2, \cdots, N_k$, called *swaths* and *neighborhoods*, has been defined in the following way. First, for $j = 1, \cdots, k$, recursively define the operation $\text{sweep}_j : \text{subsets of } \mathbb{R}^{2n} \rightarrow \text{subsets of } \mathbb{R}^{2n}$ as

$$\text{sweep}_j (A) = \begin{cases} 
  \bigcup_{t=0}^1 f_t (A) & \text{if } j = 1 \\
  \bigcup_{t=0}^1 f_j (\text{sweep}_{j-1} (A)) & \text{if } 2 \leq j \leq k.
\end{cases}$$

Then the $j^{th}$ swath and $j^{th}$ neighborhood, $S_j$ and $N_j$, are defined by the formulas

$$S_j = \text{sweep}_j \left( B^{2n} (j) \right)$$

$$N_j = \text{sweep}_j \left( B^{2n} (j+1) \right).$$
Further, suppose that the product $\pi_k$ has the property

$$
\pi_k = \begin{cases} 
  f & \text{on } B^{2n} (k) \\
  \text{non-trivial} & \text{on } N_k \\
  \text{id} & \text{outside } N_k.
\end{cases}
$$

We will define $F_{k+1}, f_{k+1,t}, f_{k+1}, \text{swhath}_{k+1}, S_{k+1}, N_{k+1}, \chi_{k+1}, G_{k+1,t}, g_{k+1,t},$ and $g_{k+1}$. Note that because of the behavior observed above for $\pi_k$, we know that

$$
f \circ \pi_k^{-1} = \begin{cases} 
  \text{id} & \text{on } f (B^{2n} (k)) \\
  \text{non-trivial} & \text{on } N_k \\
  f & \text{outside } N_k.
\end{cases}
$$

Using Lemma A2, we produce a Hamiltonian function, $F_{k+1,t}$ that generates $f \circ \pi_k^{-1}$ and that has the additional important property that for all $t \in [0,1]$, $F_{k+1,t}$ is identically zero on the ball $B^{2n} (R_k, f(0))$. The function $F_{k+1,t}$ will not have compact support, so we will produce a cutoff function that will allow us to retain the important properties of $F_{k+1,t}$ in a Hamiltonian that does have compact support.

To do that, we denote by $f_{k+1,t}$ the corresponding Hamiltonian isotopy, and denote by $f_{k+1}$, the time $=1$ value of that isotopy. That is, $f_{k+1} = f_{k+1,1} = f \circ \pi_k^{-1}$. Now that we have $f_{k+1}$, we can define the map $\text{sweep}_{k+1} : \text{subsets of } \mathbb{R}^{2n} \rightarrow \text{subsets of } \mathbb{R}^{2n}$, as well as the $(k+1)^{\text{st}}$ swath and $(k+1)^{\text{st}}$ neighborhood $S_{k+1}$ and $N_{k+1}$, as

$$
\text{sweep}_{k+1} = \bigcup_{t=0}^{1} f_{k+1} (\text{sweep}_t (A))
$$

$$
S_{k+1} = \text{sweep}_{k+1} (B^{2n} (k + 1))
$$

$$
N_{k+1} = \text{sweep}_{k+1} (B^{2n} (k + 2)).
$$

We define the cutoff function, $\chi_{k+1}$, to be identically one on $S_{k+1}$ and to be zero outside $N_{k+1}$. Finally, we define the Hamiltonian function $G_{k+1,t} = F_{k+1,t} \cdot \chi_{k+1}$. The corresponding isotopy will be called $g_{k+1,t}$, and its time $=1$ value will be called $g_{k+1}$.
Notice that since $G_{k+1,t}$ agrees with $F_{k+1,t}$ everywhere in the set $S_{k+1}$, the evolution of the set $N_k$, under the influence of $g_{k+1,t}$, will be the same as its evolution would have been under the influence of $f_{k+1,t}$. So $g_{k+1} | N_k = f_{k+1} | N_k = f \circ \pi_k^{-1} | N_k$. Considering the behavior of the composition, $\pi_{k+1} \circ g_{k+1} \circ g_k \circ \cdots \circ g_2 \circ g_1$, we find

$$
\pi_{k+1} | B^{2n} (k + 1) = g_{k+1} \circ g_k \circ \cdots \circ g_2 \circ g_1
$$

$$
= g_{k+1} \circ \pi_{k-1} | B^{2n} (k + 1)
$$

$$
= g_{k+1} | \underbrace{\pi_{k-1} \circ B^{2n} (k + 1)}_{\text{contained in } N_k} \circ \pi_{k-1} | B^{2n} (k + 1)
$$

(note that $g_{k+1} | N_k = (f \circ \pi_k^{-1}) | N_k$)

$$
= (f \circ \pi_k^{-1}) | \pi_{k-1} \circ B^{2n} (k + 1) \circ \pi_{k-1} | B^{2n} (k + 1)
$$

$$
= (f \circ \pi_k^{-1} \circ \pi_{k-1}) | B^{2n} (k + 1)
$$

$$
= f | B^{2n} (k + 1).
$$

By induction, we can produce the promised sequence of Hamiltonian functions and isotopies.


\textit{End of proof of Lemma A1.}

\section*{2.3 \textbf{Lemma A2}}

\textbf{Lemma A2:} If $\psi \in \text{Symp} (\mathbb{R}^{2n})$ and there is some point $p$ and radius $R > 0$ such that $\psi | B^{2n} (R, p) \equiv id$, then there is a Hamiltonian function $F_t$ that generates $\psi$ and which has the additional properties that $F_0 \equiv 0 \equiv F_1$ and for all $t \in [0, 1]$, $F_1 | B^{2n} (R, p) \equiv 0$.

\textit{Proof of Lemma A2:}

We will use the Alexander trick at the point $p$. For $t \in (0, 1]$, consider the symplectomorphism $\gamma_t$ defined by $\gamma_t = \tau_p \circ m_t \circ \tau_{-p} \circ \psi \circ \tau_p \circ m_t \circ \tau_{-p}$, where $\tau_p$ is the translation that sends the origin to point $p$, and $m_t$ is multiplication by $t$. When we apply $\gamma_t$ to a point $x$, the result is a difference quotient plus a translation: $\gamma_t (x) = \frac{\psi (x + t)}{t} - \psi (x) - \tau_p (t)$.
\[ p + \frac{\psi(p + t(x - p)) - p}{t} \] (This expression has been simplified by the fact that \( \psi(p) = p \).) At time \( t = 1 \), the expression simplifies to \( \psi(x) \). In the limit, as \( t \) approaches zero, the expression converges to:

\[
\lim_{t \to 0} \gamma_t(x) = p + L_{(\partial \psi)}_p(x - p) = p + (x - p) = x.
\]

(Here, we have used the fact that because \( \psi | B^{2n}(R, P) \equiv id \), the differential of \( \psi \) at \( p \) will be the identity matrix. So, \( L_{(\partial \psi)}_p = id \.) Because of this convergence, we know that as \( t \to 0 \), \( \gamma_t \) converges in the compact open topology to the identity map. So we can extend the time domain of \( \gamma_t \):

\[
\gamma_t = \begin{cases} 
id & \text{for } t = 0 \\
\tau_p \circ m_{\frac{1}{t}} \circ \tau_{-p} \circ \psi \circ \tau_p \circ m_t \circ \tau_{-p} & \text{for } 0 < t \leq 1.
\end{cases}
\]

Then \( \gamma_t \) is a symplectic isotopy, starting at the identity, and ending at \( \psi \).

Moreover, we can see that for all \( t \in [0, 1] \), \( \psi \circ B^{2n}(R, p) \equiv id \). To see this, let \( x \) be any point in \( B^{2n}(R, p) \) and let \( t \in (0, 1] \). Then \( \tau_p \circ m_t \circ \tau_{-p}(x) \) will also be in \( B^{2n}(R, p) \). (This sentence is the reason that we needed to introduce \( B^{2n}(R, p) \). We needed a star-shaped neighborhood of the point \( p \).) Since \( \psi \circ B^{2n}(R, p) \equiv id \), we see that the expression for \( \psi(x) \) simplifies to

\[
\gamma_t(x) = \tau_p \circ m_{\frac{1}{t}} \circ \tau_{-p} \circ \psi \circ \tau_p \circ m_t \circ \tau_{-p}(x) = \tau_p \circ m_{\frac{1}{t}} \circ \tau_{-p} \circ \tau_p \circ m_t \circ \tau_{-p}(x) = x.
\]

There exists a Hamiltonian function \( F \) that generates the symplectic isotopy \( \gamma_t \). Since, for each time \( t \), \( \gamma_t \) restricted to \( B^{2n}(R, p) \) is the identity map, at each time \( t \), we know that \( F_t \) will be constant on that region. We may impose the normalization requirement that at each time, \( F_t \) be in fact identically zero on that region. Furthermore, we can
re-scale the speed of the isotopy $\gamma_t$ so that at times $t = 0$ and $t = 1$, it has zero speed.

This will cause $F_t$ to be constant (in space) at those two times. This, with the fact that at each time, $F_t$ must be identically zero on $B^{2n}(R,p)$, tells us that $F_0 \equiv 0 \equiv F_1$.

*End of proof of Lemma A2*
Chapter 3

Extending Hofer's Infinity Norm

3.1 Introduction and Definitions

The group $Ham^c(\mathbb{R}^{2n})$ of Hamiltonian symplectomorphisms of compact support was introduced in Chapter 1. There, it was shown how elements of this group are generated by the family $C$, of compactly supported Hamiltonian functions.

Hofer [4, 3] has introduced two norms for $C$ that can be used to give a norm for $Ham^c(\mathbb{R}^{2n})$ $Ham^c(\mathbb{R}^{2n})$. One of these, which we will call the infinity norm, $E_\infty$, will prove particularly useful in this paper, and we introduce it here.

Define the map $|| \cdot ||_\infty : C \to [0, \infty)$ by

$$||H||_\infty = \max_{t \in [0,1]} \left\{ \max_{x \in \mathbb{R}^{2n}} \{ H(x,t) \} - \min_{x \in \mathbb{R}^{2n}} \{ H(x,t) \} \right\}$$

Then $|| \cdot ||_\infty$ is a norm on $C$.

Define a map $E_\infty : Ham^c(\mathbb{R}^{2n}) \to [0, \infty)$ by

$$E_\infty(\psi) = \inf \{ ||H||_\infty : H \in C \text{ and } H \text{ generates } \psi \}$$
Then it is easy to verify for that for any \( \theta, \psi \in Ham^c(\mathbb{R}^{2n}) \) and any \( \phi \in Symp(\mathbb{R}^{2n}) \),

\[
E_\infty(\psi) = 0 \iff \psi = id \quad \text{ (non-degeneracy)}
\]

\[
E_\infty(\theta \psi) \leq E_\infty(\theta) + E_\infty(\psi) \quad \text{ (triangle inequality)}
\]

\[
E_\infty(\psi) = E_\infty(\psi^{-1}) = E_\infty(\phi \psi \phi^{-1}) \quad \text{ (inversion and conjugation invariance)}.
\]

So \( E_\infty \) is a norm on \( Ham^c(\mathbb{R}^{2n}) \), invariant under conjugation by elements of \( Symp(\mathbb{R}^{2n}) \).

It is useful to extend this norm to symplectomorphisms of arbitrary support. In Chapter 1, we denoted by \( \mathcal{H} \) the set of Hamiltonian functions. These are smooth functions \( H: \mathbb{R}^{2n} \times [0,1] \to \mathbb{R} \) (smooth in the compact-open \( C^\infty \) topology) that generate symplectomorphisms by the maps described in chapter 1.

We define the map \( \|\|_\infty: \mathcal{H} \to [0, \infty] \) by

\[
\|H\|_\infty = \sup_{t \in [0,1]} \left\{ \sup_{x \in \mathbb{R}^{2n}} \{ H(x,t) \} - \inf_{x \in \mathbb{R}^{2n}} \{ H(x,t) \} \right\}
\]

Then \( \|\|_\infty \) is what we might call an \textit{extended norm} on \( \mathcal{H} \). By this we mean that it retains the properties of a norm, except that it can be infinite-valued.

As discussed in Chapter 1, the set of symplectomorphisms generated by \( \mathcal{H} \), which we denoted \( \psi^H \) or \( Ham(\mathbb{R}^{2n}) \), was actually the entire group \( Symp(\mathbb{R}^{2n}) \).

So we can define a map \( \overline{E}_\infty: Symp(\mathbb{R}^{2n}) \to [0, \infty] \) by

\[
\overline{E}_\infty(\psi) = \inf \left\{ \|H\|_\infty: H \in \mathcal{H} \text{ and } H \text{ generates } \psi \right\}
\]

Then it is easy to verify that \( \overline{E}_\infty \) is an extended norm on \( Symp(\mathbb{R}^{2n}) \), invariant under inversion and conjugation.

Notice that \( \overline{E}_\infty \) extends \( E_\infty \) in two ways. First, \( \overline{E}_\infty \) allows functions of arbitrary support in the infimum. Secondly, the domain of \( \overline{E}_\infty \) includes symplectomorphisms of arbitrary support. When restricted to symplectomorphisms \( \phi \) of compact support, we will have the inequality \( \overline{E}_\infty(\phi) \leq E_\infty(\phi) = \text{finite} \), because every function allowed in the infimum on the right side is also allowed in the infimum on the left side. (One immediately wonders whether in fact \( \overline{E}_\infty(\phi) = E_\infty(\phi) \), but
we will not address this question.) When applied to symplectomorphisms of non-compact support, however, $\overline{E}_\infty$ can give either a real number or infinity as a result. Of course, $E_\infty$ is not defined on these symplectomorphisms.

### 3.2 An Energy-Capacity Inequality

The question of whether or not $\overline{E}_\infty$ is finite will be important in this paper. We will show that certain subgroups of $\text{Symp}(\mathbb{R}^{2n})$ contain only symplectomorphisms for which $\overline{E}_\infty$ is finite, and this fact will distinguish those subgroups from the larger group. That the finiteness of $\overline{E}_\infty$ does distinguish a subgroup from the larger group will follow from an inequality that we present in this section.

Define the Gromov width, $w_G$, for subsets of $\mathbb{R}^{2n}$ by

$$ w_G(A) = \sup \{ r^2 | B^{2n}(r) \text{ embeds symplectically in } A \} $$

Observe that $w_G(A) \in [0, \infty]$. For symplectomorphisms $\psi \in \text{Symp}(\mathbb{R}^{2n})$, we will consider the Gromov width of compact sets $A$ that can be moved off themselves, or displaced by $\psi$. That is, $\psi(A) \cap A = \emptyset$. We will prove the following energy-capacity inequality, so called because the norm of a symplectomorphism is sometimes referred to as its energy.

Claim (energy-capacity inequality): For any $\psi \in \text{Symp}(\mathbb{R}^{2n})$, the following holds.

$$ \sup \{ w_G(A) \text{ such that } A \subset \mathbb{R}^{2n}, \text{ compact and } \psi(A) \cap A = \emptyset \} \leq \overline{E}_\infty(\psi) $$

**Proof of the energy-capacity inequality:**

The claim is automatically true if $\overline{E}_\infty(\psi) = \infty$, so assume that $\overline{E}_\infty(\psi)$ is finite, and let $A$ be any compact set displaced by $\psi$. Let $\varepsilon > 0$ be any positive number. We will show that $w_G(A) \leq \overline{E}_\infty(\psi) + \varepsilon$. Since $\varepsilon$ was arbitrary, this will prove the
claim. Our method will be to produce a compactly supported symplectomorphism \( \theta \) such that \( w_G (A) \leq E_\infty (\theta) \leq \widehat{E}_\infty (\psi) + \epsilon \). The inequality will be obtained by the following string of inequalities, whose terminology and justifications will be given in the steps that follow:

\[
\begin{align*}
    w_G (A) & \leq c_{HZ} (A) && (Hofer - Zehnder capacity, step i) \\
    & \leq e (A) && (displacement energy, step ii) \\
    & \leq E_H (\theta) && (standard Hofer norm, step iii) \\
    & = E_\infty (\theta) && (step iv) \\
    & \leq \widehat{E}_\infty (\psi) + \epsilon && (step v).
\end{align*}
\]

**step i:** A Hamiltonian function \( H \) is called admissible if its associated Hamiltonian isotopy has no non-constant periodic orbits. That is, for all \( x \in \mathbb{R}^{2n} \), if \( h_{t_a} (x) = h_{t_b} (x) \) for some \( 0 \leq t_a < t_b \leq 1 \), then in fact \( h_t (x) = x \) for all \( t \in [0, 1] \). For a set \( A \subset \mathbb{R}^{2n} \), define \( H_{ad} (A, \omega_0) \) to be the set of admissible Hamiltonian functions on \( \mathbb{R}^{2n} \) whose support is contained in \( A \). The Hofer-Zehnder capacity of \( A \) is defined as \( c_{HZ} (A) = \sup \{ \| H \|_\infty : H \in H_{ad} (A, \omega_0) \} \). From [3], we have the inequality \( w_G (A) \leq c_{HZ} (A) \).

**step ii:** We have seen the Hofer infinity norm, \( E_\infty \), on \( \text{Ham}^c (\mathbb{R}^{2n}) \). We will denote by \( E_H \) a more common norm, also introduced by Hofer [3]:

\[
E_H (\phi) = \inf \left\{ \int_{t=0}^{1} \left( \max_{x \in \mathbb{R}^{2n}} \{ H (x, t) \} - \min_{x \in \mathbb{R}^{2n}} \{ H (x, t) \} \right) \, dt : H \in C \text{ and } \psi^H = \phi \right\}
\]

We define the displacement energy, \( e (A) \), of a compact set \( A \subset \mathbb{R}^{2n} \) by

\[
e (A) = \inf \{ E_H (\xi) : \xi \in \text{Ham}^c (\mathbb{R}^{2n}, \omega_0) \text{ and } \xi \text{ displaces } A \}.
\]

From [3], we have the inequality \( c_{HZ} (A) \leq e (A) \).

**step iii:** In this step, we will introduce the symplectomorphism \( \theta \in \text{Ham}^c (\mathbb{R}^{2n}) \) and prove the third inequality.
The Hamiltonian symplectomorphism $\psi$ might not be compactly supported, but we are assuming that its infinity norm, $E_\infty(\psi)$, is finite. So, there is some Hamiltonian function $H$ that generates $\psi$ such that

$$\|H\|_\infty = \sup_{t \in [0,1]} \left\{ \sup_{x \in \mathbb{R}^n} H(t, x) - \inf_{x \in \mathbb{R}^n} H(t, x) \right\} \leq E_\infty(\psi) + \varepsilon,$$

where $\varepsilon > 0$ was chosen at the beginning of the proof. Denote by $h_t$ the Hamiltonian isotopy generated by $H$. We can multiply $H$ by a moving cutoff function, where the cutoff function is identically 1 on the moving image of the set $A$, as the set $A$ evolves under the influence of the isotopy $h_t$, and is identically zero outside a compact set. The resulting function we call $F$; the compactly supported Hamiltonian symplectomorphism that it generates, $\theta$. Notice that because $\psi$ displaces $A$, $\theta$ also displaces $A$. This means that

$$e(A) = \inf \{ E_H(\xi) : \xi \in \text{Ham}^c(\mathbb{R}^n, \omega_0) \text{ and } \xi \text{ displaces } A \} \leq E_H(\theta),$$

because $\theta$ is a particular symplectomorphism that displaces $A$. This proves the third inequality.

**step iv:** The following inequality follows immediately from the definitions of the two norms:

$$E_H(\theta) = \inf \left\{ \int_{t=0}^1 \left( \max_{x \in \mathbb{R}^n} \{H(x,t)\} - \min_{x \in \mathbb{R}^n} \{H(x,t)\} \right) dt : H \in C \text{ and } \psi^H = \theta \right\}$$

$$\leq \inf \left\{ \max_{t \in [0,1]} \{H(x,t)\} - \min_{x \in \mathbb{R}^n} \{H(x,t)\} : H \in C \text{ and } \psi^H = \theta \right\}$$

$$= E_\infty(\theta).$$

But Polterovich showed in [6] that the two norms are in fact equal.

**step v:** Because $\theta$ is generated by a Hamiltonian function, $F$, that is a cut-off version
of the function that generates $\psi$, we have the fifth inequality:

\[
E_\infty (\theta) = \inf \{ \| G \|_\infty : G \in C \text{ and } \psi^G = \theta \}
\leq \| F \|_\infty \text{ because } F \text{ is a particular function that generates } \theta
\leq \max_{t \in [0,1]} \left\{ \max_{x \in \mathbb{R}^{2n}} F(t, x) - \min_{x \in \mathbb{R}^{2n}} F(t, x) \right\}
\leq \sup_{t \in [0,1]} \left\{ \sup_{x \in \mathbb{R}^{2n}} H(t, x) - \inf_{x \in \mathbb{R}^{2n}} H(t, x) \right\}
F \text{ was obtained by cutting off } H
= \| H \|_\infty
\leq \overline{E}_\infty (\psi) + \varepsilon.
\]

\textit{End of proof of the energy-capacity inequality}

### 3.3 Using the $e$-$c$ inequality to detect proper subsets of $\text{Symp} (\mathbb{R}^{2n})$

The energy-capacity inequality tells us that there must be elements of $\text{Symp} (\mathbb{R}^{2n})$ whose infinity norms are not finite. For example, let $\psi \in \text{Symp} (\mathbb{R}^{2n})$ be the counterclockwise rotation in the $x_1 \times x_2$ plane, about the origin, through an angle of $\frac{\pi}{2}$. Then for every $M > 0$, the ball $B^{2n}(M, (2M, 0, \cdots, 0))$ is displaced by $\psi$. Since $w_G(B^{2n}(M, (2M, 0, \cdots, 0))) = \pi M^2$, we see that

\[
\sup \{ w_G (A) \text{ such that } A \subset \mathbb{R}^{2n}, \text{ compact and } \psi (A) \cap A = \phi \} = \infty
\]

By the energy-capacity inequality, $\overline{E}_\infty (\psi)$ must be infinite as well. Thus, we can state the following

\textit{Corollary of the energy-capacity inequality}: If some subset $\Omega \subset \text{Symp} (\mathbb{R}^{2n})$ has the property that $\overline{E}_\infty (\psi)$ is finite for all $\psi \in \Omega$, then $\Omega$ is a proper subset of $\text{Symp} (\mathbb{R}^{2n})$, for $\Omega$ will contain no rotations.
For example, in chapter 1, we discussed the normal subgroup $\psi^{HU_B}$ in $Symp(\mathbb{R}^{2n})$ consisting of symplectomorphisms that can be generated by Hamiltonian functions that are uniformly-bounded. We see that for any such $\psi$, $\overline{E_\infty}(\psi)$ will be finite. So $\psi^{HU_B}$ is a proper subgroup of $Symp(\mathbb{R}^{2n})$. 
Chapter 4

A new non-trivial normal subgroup of $\text{Symp}(\mathbb{R}^{2n})$

4.1 Introduction, Theorem B, and Proposition B

In Section 1.4 of Chapter 1, the groups $\text{Ham}^c(\mathbb{R}^{2n})$ and $\psi^H \circ B$ were found to be proper normal subgroups of $\text{Symp}(\mathbb{R}^{2n})$. Each was obtained from the larger group by imposing restrictions on the set of generating Hamiltonian functions. In this chapter, we will describe a rather more complicated normal subgroup of $\text{Symp}(\mathbb{R}^{2n})$ and prove that it is also a proper subgroup.

The description of the group and statement of the theorem follow; the motivation for our considering the group in the first place will be in a remark following the statement of the theorem.

Theorem B:

Let $U = \bigcup_{i=1}^{\infty} f_i(B^{2n}(R)) \subset \mathbb{R}^{2n}$ be any disjoint union of symplectic balls of radius $R < \frac{1}{2}$. (By symplectic ball of radius $R$, we mean the image, $f(B^{2n}(R))$, of a symplectic embedding, $f : B^{2n}(R) \to \mathbb{R}^{2n}$, where $B^{2n}(R)$ is the closed ball of radius
Denote by $\text{Symp}_U(\mathbb{R}^{2n})$ the set of symplectomorphisms of $\mathbb{R}^{2n}$ that can be generated by Hamiltonian functions with support contained in $U$. Then $\text{Symp}_U(\mathbb{R}^{2n})$ is a subgroup of $\text{Symp}(\mathbb{R}^{2n})$, but it is not a normal subgroup: conjugation of an element of $\text{Symp}_U(\mathbb{R}^{2n})$ with a translation can produce an element of $\text{Symp}(\mathbb{R}^{2n})$ that is not supported in $U$. Define $G_U \triangleleft \text{Symp}(\mathbb{R}^{2n})$ to be the minimal normal subgroup of $\text{Symp}(\mathbb{R}^{2n})$ containing $\text{Symp}_U(\mathbb{R}^{2n})$. That is, $G_U$ contains $\text{Symp}_U(\mathbb{R}^{2n})$, is closed under conjugation by elements of $\text{Symp}(\mathbb{R}^{2n})$, and is closed under composition.

**Claim (Theorem B):** $G_U$ is a proper subgroup of $\text{Symp}(\mathbb{R}^{2n})$

**Comment:** Why is this surprising? Let $\text{Diff}_{vol}(U) \subset \text{Diff}_{vol}(\mathbb{R}^n)$ be the collection of volume-preserving diffeomorphisms of $\mathbb{R}^n$ that are supported in $U$. Note that $\text{Diff}_{vol}(U)$ is a non-normal subgroup of $\text{Diff}_{vol}(\mathbb{R}^n)$. As above, define $G_{\text{Diff}_{vol}(U)} \triangleleft \text{Diff}_{vol}(\mathbb{R}^n)$ to be the minimal normal subgroup of $\text{Diff}_{vol}(\mathbb{R}^n)$ containing $\text{Diff}_{vol}(U)$. McDuff [5] showed that for $n \geq 3$, $G_{\text{Diff}_{vol}(U)} = \text{Diff}_{vol}(\mathbb{R}^n)$. So the present claim distinguishes the structure of $\text{Symp}(\mathbb{R}^{2n})$ from that of $\text{Diff}_{vol}(\mathbb{R}^{2n})$

A flow chart illustrating the structure of the proof of Theorem B is shown in Figure 4.1.

**Proof of Theorem B:**

If $g \in G_U$, then $g = g_1 g_2 \cdots g_k$, where $g_j = h_j \circ \psi_j \circ h_j^{-1}$, with $h_j \in \text{Symp}(\mathbb{R}^{2n})$ and $\psi_j \in \text{Symp}_U(\mathbb{R}^{2n})$. Note that $\psi_j$ is supported in $U$, which is a disjoint union of symplectic balls. In Section 3.1, we introduced the conjugation-invariant extended Hofer infinity norm, $\overline{E}_\infty$ for elements of $\text{Symp}(\mathbb{R}^{2n})$. In Proposition B, below, we will prove that for any element $\psi_j \in \text{Symp}_U(\mathbb{R}^{2n})$, this norm is bounded: $\overline{E}_\infty(\psi_j) \leq 32$. By conjugation invariance, $\overline{E}_\infty(g_j) \leq 32$, and by the triangle in-
**Lemma B2:** Given a radius $r_n \geq 0$ and $k$ symplectic balls of radius $R < \frac{1}{2}$ that do not intersect $B^{2n}(r_n)$, there is a larger radius $r_b > r_n$ and a Hamiltonian symplectomorphism supported in the annulus $r_n < \| \cdot \| < r_b$ that will turn the $k$ symplectic balls into round balls and move them to integer lattice positions.

**Lemma B1:** Any disjoint union of symplectic balls of radius $R < \frac{1}{2}$ can be parsed into two sets, red and green, such that there is a symplectomorphism, $move_{red}$, that will turn the red balls into round balls and move them to integer lattice positions, and a symplectomorphism, $move_{green}$, that will do likewise to the green balls.

**Lemma B3:** A Hamiltonian symplectomorphism that shuffles a collection of round balls of radius $R < \frac{1}{2}$ located at integer lattice positions in a way that they all end up at lattice positions with $x_{2n} = 0$.

**Lemma B4:** If $\psi \in \text{Symp}_0(\mathbb{R}^{2n})$, where $\nu$ is a disjoint union of round balls of radius $R < \frac{1}{2}$, centered at integer lattice positions $x_i$, such that $x_{i,2n} = 0$, then $\tilde{E}_\infty(\nu) \leq 16$.

**Lemma B5:** If $\psi \in \text{Symp}_{\beta_{n;\alpha}}(\mathbb{R}^{2n})$, where $0 < R < \frac{1}{2}$, and if tube is the set $\{ y : \sum_{i=1}^{2n-1} (y_i - x_i)^2 < (\frac{3}{4} + \frac{1}{4})^2 \}$, then $\tilde{E}_\infty(\psi) \leq 8$.

**Proposition B:** If $\psi \in \text{Symp}_0(\mathbb{R}^{2n})$, where $U$ is a disjoint union of symplectic balls of radius $R < \frac{1}{2}$, then $\tilde{E}_\infty(\psi) \leq 32$.

**Chapter 3:** Definitions, an energy-capacity inequality, and a corollary

**Proof of Theorem B:** If $U$ is a disjoint union of symplectic balls of radius $R < \frac{1}{2}$, $\text{Symp}_0(\mathbb{R}^{2n})$ is the set of symplectomorphisms of $\mathbb{R}^{2n}$ generated by Hamiltonians with support contained in $U$, and $G_U$ is the minimal normal subgroup of $\text{Symp}(\mathbb{R}^{2n})$ containing $\text{Symp}_0(\mathbb{R}^{2n})$, then $G_U \neq \text{Symp}(\mathbb{R}^{2n})$.

**Figure 4.1:** Structure of the Proof of Theorem B
equality, $\overline{E}_\infty (g) \leq 32k$. So we see that for any $g \in G_U$, the norm $\overline{E}_\infty (g)$ will be finite.

But in Section 3.3, we proved the following corollary.

Corollary of the energy-capacity inequality: If a subset $\Omega \subset \text{Symp}(\mathbb{R}^{2n})$ has the property that $\overline{E}_\infty (\psi)$ is finite for all $\psi \in \Omega$, then $\Omega$ is a proper subset of $\text{Symp}(\mathbb{R}^{2n})$, for $\Omega$ will contain no rotations.

With that corollary, we see that $G_U$ is a proper subgroup of $\text{Symp}(\mathbb{R}^{2n})$.

End of proof of Theorem B

Proposition B: If $\psi \in \text{Symp}(U)$, where $U = \bigcup_{i=1}^\infty f_i (B^{2n}(R))$ is a disjoint union of symplectic balls, then $\overline{E}_\infty (\psi) \leq 32$

Proof of Proposition B:

First, recall that a symplectic ball is the image, $f (B^{2n}(R))$, of a symplectic embedding. We write round ball to denote $B^{2n}(R, x)$ or $B^{2n}(R)$.

If we were to parse the disjoint union into two sets of symplectic balls, say red ones and green ones, then we could write $\psi$ as a composition of symplectomorphisms supported in these two sets, $\psi = \psi_r \circ \psi_g$. Then by the triangle inequality, $\overline{E}_\infty (\psi) \leq \overline{E}_\infty (\psi_r) + \overline{E}_\infty (\psi_g)$. We will parse the union, and we will do it in a particular way, by using the result of Lemma B1, which we state here and prove in Section 4.2.1.

Lemma B1: Any disjoint union of symplectic balls $\bigcup_{i=1}^\infty f_i (B^{2n}(R))$ can be parsed into two sets, red and green, such that there are Hamiltonian
symplectomorphisms of $\mathbb{R}^{2n}$, $\text{move}_{\text{red}}$ and $\text{move}_{\text{green}}$, with the following properties: The symplectomorphism $\text{move}_{\text{red}}$ will turn the red symplectic balls into round balls and move them to integer lattice positions; $\text{move}_{\text{green}}$ will turn the green symplectic balls into round balls and move them to integer lattice positions.

The significance of a lattice of round balls is that they can be shuffled into a new arrangement where they are each located at lattice positions whose $x_{2n}$ coordinate is zero. In fact, the shuffling can be accomplished with a symplectomorphism that we will construct in Lemma B3. We will state that lemma here and prove it in Section 4.2.3.

**Lemma B3:** Given a lattice of round balls, meaning a union $\bigcup_{i=1}^{\infty} B^{2n}(R, x_i)$, where $0 < R < \frac{1}{2}$ and each $x_i \in \mathbb{R}^{2n}$ has integer coordinates, we claim that there is a Hamiltonian symplectomorphism, which we will call shuffle, that rearranges the balls so that they are again centered at integer lattice points, but now only at points such that $x_{i,2n} = 0$.

With the symplectomorphisms $\psi_r$, $\psi_g$, $\text{move}_r$, $\text{move}_g$, and shuffle, we can define two new symplectomorphisms, $\phi_r$ and $\phi_g$ in the following way:

$$\phi_r = \text{shuffle} \circ \text{move}_r \circ \psi_r \circ \text{move}_r^{-1} \circ \text{shuffle}^{-1}$$

$$\phi_g = \text{shuffle} \circ \text{move}_g \circ \psi_g \circ \text{move}_g^{-1} \circ \text{shuffle}^{-1}$$

Notice that both $\phi_r$ and $\phi_g$ are supported in disjoint unions of round balls centered at integer lattice positions such that $x_{2n} = 0$. So, we may apply the following Lemma, proven in Section 4.2.4 to give an estimate on the extended norm of each of them.
Lemma B4: If \( \psi \in \text{Symp}_V (\mathbb{R}^{2n}) \), where \( V = \bigcup_{i=1}^{\infty} B^{2n} (R, x_i) \) is a disjoint union of round balls with \( 0 < R < \frac{1}{2} \), centered at integer lattice positions \( x_i \) such that \( x_{i,2n} = 0 \), then its extended Hofer infinity norm is bounded: 
\[
\overline{E}_\infty (\psi) \leq 16
\]

With this result, we can make the estimates \( E_\infty (\phi_r) \leq 16 \) and \( E_\infty (\phi_g) \leq 16 \). By conjugation invariance of the extended norm \( \overline{E}_\infty \), we know that \( \overline{E}_\infty (\psi_r) = \overline{E}_\infty (\phi_r) \) and \( \overline{E}_\infty (\psi_g) = \overline{E}_\infty (\phi_g) \). Therefore, \( \overline{E}_\infty (\psi) \leq \overline{E}_\infty (\psi_r) + \overline{E}_\infty (\psi_g) \leq 32 \).

*End of Proof of Proposition B*

4.2 Lemmas B1 through B5

4.2.1 Lemma B1

*Lemma B1:* Any disjoint union of symplectic balls, \( \bigcup_{i=1}^{\infty} f_i (B^{2n} (R)) \), can be parsed into two sets, *red* and *green*, such that there are Hamiltonian symplectomorphisms of \( \mathbb{R}^{2n} \), \( \text{move}_{red} \) and \( \text{move}_{green} \), with the following properties. The symplectomorphism \( \text{move}_{red} \) will turn the *red* symplectic balls into round balls and move them to integer lattice positions; \( \text{move}_{green} \) will turn the *green* symplectic balls into round balls and move them to integer lattice positions.

*Proof of Lemma B1:*

The lemma will follow immediately from the following *claim*.

*claim:* There exists a sequence of radii, \( 0 = r_0 < r_1 < r_2 < \cdots \), a sequence of Hamiltonian symplectomorphisms \( h_1, h_2, h_3, \cdots \), a re-numbering of the disjoint union of symplectic balls, and a sequence of important indices in that numbering \( k_1 < k_2 < k_3 < \cdots \), with the following properties:
1. The open ball $||x|| < r_1$ contains the symplectic balls numbered $1, \cdots, k_1$, which we will call the 1st batch of balls. For $i \geq 2$, the $i^{th}$ annulus, $r_{i-1} < ||x|| < r_{i+1}$, contains the symplectic balls numbered $k_{i-1} + 1, \cdots, k_i$, which we will refer to as the $i^{th}$ batch. None of the annuli are empty: For each $i \geq 2$, the ball $B^{2n}(r_i)$ intersects at least one symplectic ball that does not intersect $B^{2n}(r_{i-1})$, and all of those symplectic balls are completely contained in the annulus $r_{i-1} < ||x|| < r_{i+1}$.

2. The $i^{th}$ batch of symplectic balls is red if $i$ is odd; green, if $i$ is even.

3. The symplectomorphism $h_i$ is supported in the open ball $||x|| < r_2$. For $i \geq 2$, the symplectomorphism $h_i$ is supported in the $i^{th}$ annulus, $r_{i-1} < ||x|| < r_{i+1}$.

4. The symplectomorphism $h_i$ turns the $i^{th}$ batch of symplectic balls into round balls and moves them to integer lattice positions.

This claim will be proven below. First, however, we show how the lemma follows immediately from the claim.

Since the odd-numbered $h_i$ have disjoint support, their product, $\prod_{i=1}^{\infty} h_{2i-1}$, is a well-defined Hamiltonian symplectomorphism, which we call $move_r$. This symplectomorphism moves all the red symplectic balls to integer lattice positions, while turning them into round balls. Similarly, the even-numbered $h_i$ have disjoint support, so their product $\prod_{i=1}^{\infty} h_{2i}$ is a well-defined Hamiltonian symplectomorphism, which we call $move_g$ because it moves all the green symplectic balls to integer lattice positions, while turning them into round balls. This proves the lemma.

Proof of claim:

Denote by $sball_i$ the $i^{th}$ symplectic ball, $f_i(B^{2n}(R))$. 

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Basis step, part a: Choose \( r_0, r_1, r_2, h_1, k_1 \) and the first batch of red balls.

Let \( r_0 = 0 \). Choose the smallest \( r_1 \) such that \( B^{2n}(r_1) \) intersects at least one of the symplectic balls from the union. This ball \( B^{2n}(r_1) \) may intersect more than one symplectic ball. Renumber the union and choose an integer \( k_1 \) so that it is the first balls in the numbering of the union, \( \text{ball}_{1}, \ldots, \text{ball}_{k_1} \), that intersect \( B^{2n}(r_1) \). Color these first \( k_1 \) balls red. In section 4.2.2, we will prove the following lemma:

**Lemma B2:** If \( \bigcup_{i=1}^{k} f_i(B^{2n}(R)) \), with \( 0 < R < \frac{1}{2} \), is a disjoint union of symplectic balls, then there exists \( r_b > 0 \) and a Hamiltonian symplectomorphism \( h \), supported in the open ball \( \|x\| < r_b \) such that

\[
h \left( \bigcup_{i=1}^{k} f_i(B^{2n}(R)) \right) = \bigcup_{i=1}^{k} B^{2n}(R, x_i), \text{ a disjoint union of round balls centered on integer lattice points } x_i.
\]

(Actually, the statement of Lemma B2 is more general; we have extracted from the general statement a simpler one that suffices for our present need.) Applying this result to our present case, we will use our \( k_1 \) for the \( k \) in the lemma, and our union, \( \bigcup_{i=1}^{k_1} f_i(B^{2n}(R)) \), for the union in the lemma. We will define the radius \( r_2 \) to be the radius \( r_b \) produced by the lemma, and define the symplectomorphism \( h \) to be the Hamiltonian symplectomorphism \( h \) produced by the lemma. With no loss of generality, we can choose a larger \( r_2 \), if necessary, in order to insure that the ball \( B^{2n}(r_2) \) intersects at least one symplectic ball that is not among the first \( k_1 \) red balls that we picked above.

Note that the re-ordering of the union of symplectic balls, along with the \( r_0, r_1, r_2, h_1, k_1 \) that we have chosen, have properties (1) - (4) listed above.

Basis step, part b: Choose \( r_3, h_2, k_2 \), and the first batch of green balls.
In the previous step, we chose \( r_2 \) large enough that the ball \( B^{2n}(r_2) \) intersects at least one symplectic ball that is not red. (Keep in mind that we are still regarding the union of symplectic balls in their original state, although some have been colored red. We have proven the existence of a Hamiltonian symplectomorphism, \( h_1 \), that could manipulate these red balls - and possibly alter some other, uncolored, balls in the process - but we have not used it. So far, we have only considered the operation of \( h_1 \) in order to choose an appropriate radius \( r_2 \).) This ball \( B^{2n}(r_2) \) may intersect more than one symplectic ball that is not red, not among the \( k_1 \) balls that we have picked so far and colored red. Renumber the union and choose an integer \( k_2 > k_1 \) so that it is the next balls in the numbering of the union, \( sball_{k_1+1}, \ldots, sball_{k_2} \), that intersect \( B^{2n}(r_2) \) and are not red. (In the renumbering, do not alter the numbering of the first \( k_1 \) balls.)

Note that there are \( k_2 - k_1 \) symplectic balls in this new set, none of which intersect \( B^{2n}(r_1) \), and none of which have yet been colored, because in the previous step, it was those symplectic balls that did intersect \( B^{2n}(r_1) \) that were numbered as the first \( k_1 \) balls of the union and colored red. Color green these \( k_2 - k_1 \) new symplectic balls that we have just chosen.

In the above section, Basis step, part a, we stated a simple version of the claim of Lemma B2. The full-strength version of the claim is applicable to our current situation, and we state it here.

**Lemma B2:** If \( r_a > 0 \) and \( \bigcup_{i=1}^{k} f_i(B^{2n}(R)) \), with \( 0 < R < \frac{1}{2} \), is a disjoint union of symplectic balls that do not intersect the closed ball \( B^{2n}(r_a) \), then there exists \( r_b > r_a \) and a Hamiltonian symplectomorphism \( h \), supported in the open annulus \( r_a < ||x|| < r_b \) such that \( h \left( \bigcup_{i=1}^{k} f_i(B^{2n}(R)) \right) = \bigcup_{i=1}^{k} B^{2n}(R, x_i) \), a disjoint union of round balls centered on integer lattice points \( x_i \).
We will apply this result, using our \( r_1 \) for the \( r_a \) of the lemma, our integer \( k_2 - k_1 \) for the integer \( k \) in the lemma, and our union, \( \bigcup_{i=k_1+1}^{k_2} f_i(B^{2n}(R)) \), for the union in the lemma. The symplectomorphism \( h \) and the outer radius \( r_0 \) produced by the lemma we will call \( h_2 \) and \( r_3 \). If needed, we can choose a larger \( r_3 \), large enough that the ball \( B^{2n}(r_3) \) intersects at least one symplectic ball that is not among the first \( k_2 \) balls that we have picked so far.

Note that the new ordering of the union of symplectic balls, along with the \( r_0, r_1, r_2, r_3, h_1, h_2, k_1, \) and \( k_2 \) that we have chosen so far, have properties (1) - (4) listed above.

\textit{Inductive step:}

Suppose one is given a list of radii, \( 0 = r_0 < r_1 < \cdots < r_j \); a list of Hamiltonian symplectomorphisms, \( h_1, \cdots, h_{j-1} \); a numbering of the disjoint union of symplectic balls; and a list of important indices in that numbering, \( k_1, \cdots, k_{j-1} \); which have properties (1) - (4) listed above.

\textit{Inductive claim:} One can choose a new numbering of the union of symplectic balls, and choose \( r_{j+1}, h_j, k_j \), and the next batch of colored balls in a way that the new numbering; the new list of radii, \( 0 = r_0 < r_1 < \cdots < r_{j+1} \); the new list of Hamiltonian symplectomorphisms, \( h_1, \cdots, h_j \); and the new list of important indices, \( k_1, \cdots, k_j \); also have properties (1) - (4). (If \( j \) is an odd number, the batch of balls that we choose in this step will be colored red. If \( j \) is an even number, we will be coloring balls green.)

\textit{Proof of inductive claim:}

The radius \( r_j \) was chosen large enough that the ball \( B^{2n}(r_j) \) intersects at
least one symplectic ball that is not among the first $k_{j-1}$ balls in the ordering of the union. (As before, keep in mind that we are regarding the union of symplectic balls in their original state, although some have been colored. We have proven the existence of Hamiltonian symplectomorphisms, $h_1, \cdots, h_{j-1}$, that can manipulate these balls - and possibly alter some other, uncolored, balls in the process - but we have not used them. So far, we have only considered the operation of the $h_j$ in order to choose appropriate radii $r_j$.) This ball $B^{2n}(r_j)$ may intersect more than one symplectic ball that is not among the first $k_{j-1}$ balls. Renumber the union and choose an integer $k_j > k_{j-1}$ so that it is the next balls in the numbering of the union, $sball_{k_{j-1}+1}, \cdots, sball_{k_j}$, that intersect $B^{2n}(r_j)$ and are not already colored red or green.

Note that there are $k_j - k_{j-1}$ symplectic balls in this new set, none of which intersect $B^{2n}(r_{j-1})$, and none of which are colored, because in the previous steps, it was those symplectic balls that did intersect $B^{2n}(r_{j-1})$ that were numbered as the first $k_{j-1}$ balls of the union and colored either red or green. Color these new $k_j - k_{j-1}$ symplectic balls that we have just chosen: color them red if $j$ is odd; green, if $j$ is even.

We will apply Lemma B2, using $r_j$ for the $r_a$ of the lemma, our integer $k_j - k_{j-1}$ for the integer $k$ in the lemma, and our union, $\bigcup_{i=k_{j-1}+1}^{k_j} f_i(B^{2n}(R))$, for the union in the lemma. The symplectomorphism $h$ and the outer radius $r_0$, produced by the lemma we will call $h_j$ and $r_{j+1}$. If needed, we can choose a larger $r_{j+1}$, large enough that the ball $B^{2n}(r_{j+1})$ intersects at least one symplectic ball that is not among the first $k_j$ balls that we have picked so far. Note that the new numbering; the new list of radii, $0 = r_0 < r_1 < \cdots < r_{j+1}$; the new list of Hamiltonian symplectomorphisms, $h_1, \cdots, h_{j}$; and the new list of important indices, $k_1, \cdots, k_j$; have properties (1) - (4) listed above.

End of proof of the inductive claim.
Conclusion: By induction, we have proved the claim.

End of Proof of Lemma B1

4.2.2 Lemma B2

Lemma B2: If $r_a > 0$ and $\bigcup_{i=1}^{k} f_i(B^{2n}(R))$, with $0 < R < \frac{1}{2}$, is a disjoint union of symplectic balls that do not intersect the closed ball $B^{2n}(r_a)$, then there exists $r_b > r_a$ and a Hamiltonian symplectomorphism $h$, supported in the open annulus $r_a < ||x|| < r_b$, such that $h\left(\bigcup_{i=1}^{k} f_i(B^{2n}(R))\right) = \bigcup_{i=1}^{k} B^{2n}(R, x_i)$, a disjoint union of round balls centered on integer lattice points $x_i$. (In the case that $r_a = 0$, we mean simply that the $k$ symplectic balls are disjoint, and claim that there exists some $r_b > 0$, with $h$ supported in the open ball $||x|| < r_b$.)

Proof of Lemma B2:

step i: Establish neighborhoods.

A symplectic ball is the image, $f_i(B^{2n}(R))$, of a symplectic embedding, $f_i : B^{2n}(R) \to \mathbb{R}^{2n}$ of the closed round ball, $B^{2n}(R)$. By the usual definition of embedding, we know that there is a smooth map $\widehat{f}_i : U_i \to \mathbb{R}^{2n}$, where $U_i$ is some open set, $B^{2n}(R) \subset U_i \subset \mathbb{R}^{2n}$, and $\widehat{f}_i|U_i = f_i$. From now on, we will use the symbol $f_i$ for both $f_i$ and $\widehat{f}_i$. We have $k$ of these symplectic balls, which are closed, disjoint, and which do not intersect $B^{2n}(r_a)$. (Again, this last condition is omitted if $r_a = 0$.) Therefore, there is some $\varepsilon > 0$ such that $B^{2n}(R + \varepsilon) \subset U_i$ for each $i = 1, \ldots, k$, and such that the $k$ sets, $f_i(B^{2n}(R + \varepsilon))$, do not intersect each other or $B^{2n}(r_a)$. We will refer to the set $f_i(B^{2n}(R + \varepsilon))$ as the $i$th neighborhood, $N_i$. Note that the composition $\widehat{f}_i^{-1} \circ f_i$ is just the inclusion map, $i : B^{2n}(R) \to \mathbb{R}^{2n}$, which we will suppress.
step ii: Introduce $g_{i,t}$.

In *Detail 1*, found in section 4.3.1, a symplectic isotopy $g_{i,t} : [0,1] \to \text{Symp}(\mathbb{R}^{2n})$ is constructed. It has two important properties: $g_{i,0} = \text{id}$, and $g_{i,1}(f_i(B^{2n}(R))) = B^{2n}(R, x_i)$, where $x_i$ is an integer lattice point. So the isotopy $g_{i,t}$ transforms the $i^{th}$ symplectic ball into a round ball, centered on an integer lattice point. Also important for us will be the fact that in the process, the $i^{th}$ ball - in fact the entire $i^{th}$ neighborhood - stays outside the ball $B^{2n}(r_0)$. (If $r_0 = 0$, this sentence is omitted.)

It would be convenient if we could transform the entire union,

$$\bigcup_{i=1}^{k} f_i(B^{2n}(R)),$$

of symplectic balls at once by simply acting on the union with the product of the corresponding $k$ isotopies, $g_{k,t} \circ \cdots \circ g_{2,t} \circ g_{1,t}$, but this will not work. The supports of the various $g_{i,t}$ are not disjoint, indeed the supports are not even compact. As a result, their operations would interfere with each other (The operation of $g_{1,t}$, designed to transform the $1^{st}$ symplectic ball, would affect the $2^{nd}$ one as well, so $g_{2,t}$ would not have its intended affect on the $2^{nd}$ ball, etc.), and their supports would not be confined to the desired annular region. One way to get the various $g_{i,t}$ to cooperate with one another is to apply moving cutoff functions to the corresponding Hamiltonian functions, with the cutoff functions identically 1 on the images of the evolving symplectic balls, and supported inside the (slightly larger) evolving neighborhoods. This, we will do in the next step.

step iii: Introduce $G_{i,t}$, $\chi_{i,t}$.

We let $G_{i,t}$ be a Hamiltonian function that generates $g_{i,t}$. For each $i = 1, \ldots, k$, we define a cutoff function $\chi_{i,t}$ as follows. $\chi_{i,t}$ is a non-negative smooth function on $\mathbb{R}^{2n}$ that is identically 1 on the moving image $g_{i,t}(f_i(B^{2n}(R)))$ of the $i^{th}$ symplectic ball, as it is transformed by the isotopy $g_{i,t}$, and is identically zero outside
the moving image $g_{i,t}(f_1(B^{2n}(R + \varepsilon)))$ of the slightly larger $i^{th}$ neighborhood. This insures that the support of $\chi_{i,t}$ is compact. (Not merely compact at each $t \in [0,1]$.) Moreover, if $i \neq j$, the supports of $\chi_{i,t}$ and $\chi_{j,t}$ will be disjoint at each time $t$, because of the way that $g_{i,t}$ is constructed. That is, in Detail 1, the isotopy $g_{i,t}$ is constructed in a way that insures that the evolving image of the $i^{th}$ neighborhood remains within the confines of a linear magnification of the original $i^{th}$ neighborhood. But the original neighborhoods are disjoint, so their linear magnifications will also be disjoint and, therefore, the evolving images of the various neighborhoods will remain disjoint. Thus, the supports of the various cutoff functions, contained in those evolving images, will remain disjoint.

Multiplying the $i^{th}$ Hamiltonian function by this $i^{th}$ cutoff function, we obtain a new Hamiltonian function, $\chi_{i,t}G_{i,t}$. This function agrees with the function $G_{i,t}$ on the moving image $g_{i,t}(f_1(B^{2n}(R)))$ of the $i^{th}$ symplectic ball, as it is transformed by the isotopy $g_{i,t}$, but for each $t \in [0,1]$, the support of $\chi_{i,t}G_{i,t}$ is compact and if $i \neq j$, the supports of $\chi_{i,t}G_{i,t}$ and $\chi_{j,t}G_{j,t}$ are disjoint. Moreover, because the support of each $\chi_i$ is not merely compact at each time $t \in [0,1]$, but actually compact, the same can be said of the functions $\chi_{i,t}G_{i,t}$.

*step iv*: Introduce $H_t$, $h_t$, and $h$.

Define the Hamiltonian function $H$ by $H_t = \sum_{i=1}^{k} \chi_{i,t}G_{i,t}$, let $h_t$ be the symplectic isotopy it generates, and let $h = h_1$ be the time $= 1$ value of that isotopy. Notice that for each $i = 1, \ldots, k$, and for each $t \in [0,1]$, this function agrees with the function $G_{i,t}$ on the moving image $g_{i,t}(f_1(B^{2n}(R)))$ of the $i^{th}$ symplectic ball, as it is transformed by the isotopy $g_{i,t}$. Therefore, the isotopy $h_t$ will transform the $i^{th}$ symplectic ball in precisely the same way that $g_{i,t}$ would. So, the isotopy $h_t$ turns the $i^{th}$ symplectic ball into a round ball and moves it to an integer lattice position, and it does this to the whole batch of balls, $i = 1, \ldots, k$, simultaneously.
step v: Describe $r_b$, the outer radius.

We defined the function $H$ as the sum $H_t = \sum_{i=1}^{k} \chi_{i,t} G_{i,t}$, where the support of each $\chi_{i,t} G_{i,t}$ is compact. Therefore, the support of $H$ is also compact. So we can choose some number $r_b > r_a$ such that the support of $H$ is contained in the open ball $\|x\| < r_b$.

End of proof of Lemma B2

4.2.3 Lemma B3

Lemma B3: Given a lattice of round balls, meaning a union $\bigcup_{i=1}^{\infty} B^{2n}(R, x_i)$, where $0 < R < \frac{1}{2}$ and each $x_i \in \mathbb{R}^{2n}$ has integer coordinates, we claim that there is a Hamiltonian symplectomorphism, which we will call shuffle, that rearranges the balls so that they are again centered at integer lattice points, but now only at points such that $x_{i,2n} = 0$.

Proof of Lemma B3:

The symplectomorphism shuffle will be achieved by the composition, $\text{shuffle} = h \circ g \circ f$, of three Hamiltonian symplectomorphisms, $f$, $g$, and $h$, whose incremental effects on the lattice of balls are shown in Figures 4.2, 4.3, and 4.4. These symplectomorphisms effect movement only in the $x_{2n-1} \times x_{2n}$ plane, which is the plane shown. The three symplectomorphisms shown can be generated by three time-independent Hamiltonian functions, $F$, $G$, and $H$. We construct these Hamiltonian functions using the wedge and step functions described in Detail 6, found in section 4.3.6.
Figure 4.2: Effect of $f$

Figure 4.3: Effect of $g$

Figure 4.4: Effect of $h$
step i: Construction of the function $F$

Define the $j^{th}$ column of balls to be the set of balls whose centers have $x_{2n-1} = k$. Define the top half of the $j^{th}$ column to be the subset whose centers have $x_{2n} > 0$; the bottom half, the subset whose centers have $x_{2n} \leq -1$. From Fig. 4.2, showing the effect of symplectomorphism $f$, we see that for $j \geq 1$, the top half of the $j^{th}$ and $-j^{th}$ columns must move up $j$ units, while the bottom halves must move down $j - 1$ units. (The $0^{th}$ column does not move.) Observe that the Hamiltonian function $\text{wedge} (x_{2n-1} - j)$ generates a symplectomorphism that moves the $j^{th}$ column one unit in the $x_{2n}$ direction. We construct the Hamiltonian function $F$ by applying cutoff functions to this function, in order to restrict action to the top half or bottom half of the column, and multiplying it by scalars appropriate to achieve the desired displacements. (For typesetting reasons, we will write $w$ and $s$ for the functions $\text{wedge}$ and step in the expression for the function $F$)

$$F(x) = \sum_{j=1}^{\infty} \left( \frac{w(x_{2n-1} - j) + w(x_{2n-1} + j)}{\text{moves } j^{th} \text{ and } -j^{th} \text{ columns}} \right) \left( \frac{j \cdot s(x_{2n}) - (j - 1) \cdot s(1 - x_{2n})}{\text{top half up} \quad \text{bottom half down}} \right)$$

step ii: Construct the function $G$

Define the $k^{th}$ row of balls to be the set of balls whose centers have $x_{2n} = k$. From Fig. 4.3, showing the effect of symplectomorphism $g$, we see that for rows $k \geq 1$, the initial position of the left-most ball in the $k^{th}$ row has $x_{2n-1} = -k$. The symplectomorphism $g$ must move that left-most ball to the right, to a final position with $x_{2n-1} = 1 + 3 + \cdots + (2k - 1) = k^2$. This means that for $k \geq 1$, the $k^{th}$ row must move to the right an amount $k^2 - k = k(k + 1)$ units. Similarly, the $-k^{th}$ row must move to the left by the same amount. Accordingly, we construct the Hamiltonian function $G$:

$$G(x) = \sum_{k=1}^{\infty} \left( \frac{\text{wedge}(x_{2n} + k) - \text{wedge}(x_{2n} - k)}{\text{sends } -k^{th} \text{ row left} \quad \text{sends } k^{th} \text{ row right}} \right) k(k + 1)$$
step iii: Construct the function $H$:

From Fig. 4.4, showing the effect of symplectomorphism $h$, we see that it causes columns 1 through 3 move down 1 unit, columns 5 through 9 move down 2 units, columns 10 through 16 move down 3 units, etc. (At the same time, the corresponding negatively numbered columns move up the same amounts.) Let $left_k$ denote the number of the left-most column in the collection of columns that move down $k$ units, and let $right_k$ denote the right-most column in that collection. We will obtain explicit formulas for $left_k$ and $right_k$ by first describing them recursively.

$$
left_k = \begin{cases} 
1 & \text{for } k = 1 \\
left_{k-1} + (2k - 1) & \text{for } k \geq 2
\end{cases}
$$

$$
right_k = \begin{cases} 
3 & \text{for } k = 1 \\
right_{k-1} + (2k + 1) & \text{for } k \geq 2
\end{cases}
$$

The solutions to these recursive formulas are $left_k = k^2 - 2k + 2$ and $right_k = k^2$, for all $k \geq 1$. We will use these values as the lower and upper limits of a sum that defines the function $H$:

$$
H(x) = \sum_{k=1}^{\infty} k \left( \sum_{j=k^2-2k+2}^{k^2} \left( \underbrace{\text{wedge } (x_{2n-1} - j)}_{\text{moves } j^{th} \text{ column down}} - \underbrace{\text{wedge } (x_{2n-1} + j)}_{\text{moves } -j^{th} \text{ column up}} \right) \right)_{\text{the collection of columns that must be moved up or down by the amount } k}
$$

End of proof of Lemma B3

4.2.4 Lemma B4

Lemma B4: If $\psi \in \text{Symp}_V(\mathbb{R}^{2n})$, where $V = \bigcup_{i=1}^{\infty} B^{2n}(R, x_i)$ is a disjoint union of round balls with $0 < R < \frac{1}{2}$, centered at integer lattice positions $x_i$ such that
$x_{i:2n} = 0$, then its extended Hofer infinity norm is bounded: $\overline{E}_\infty(\psi) \leq 16$

**Proof of Lemma B4:**

Recall that the symbol $\psi \in \text{Sym}_{V}^{\infty}(\mathbb{R}^{2n})$ means that $\psi$ can be generated by a Hamiltonian function supported in $V$. Therefore, we can write $\psi$ as a product $\psi = \prod_{i=1}^{\infty} \psi_i$, where $\text{support}(\psi_i) \subset B^{2n}(R,x_i)$ and $\psi_i$ can be generated by a Hamiltonian function supported in $B^{2n}(R,x_i)$. Because both the ball $B^{2n}(R,x_i)$ and the time interval $[0,1]$ are compact, such a function will be uniformly-bounded, so the norms $E_\infty(\psi_i)$ and $\overline{E}_\infty(\psi_i)$ will be finite. But we can say more than this. We know that the norms will be finite even if we take the infimum over only those Hamiltonian functions supported in some particular set $U$, so long as that set $U$ contains the ball $B^{2n}(R,x_i)$.

This notion of only taking the infimum over functions supported in a particular set will be used again, so we introduce notation for it in the case of both of the infinity norms that we have defined.

$$
E^U_\infty(\psi) = \inf \{ ||H||_\infty : H \in C, \text{support}(H) \subset U, \text{and } H \text{ generates } \psi \}
$$

$$
\overline{E}^U_\infty(\psi) = \inf \{ ||H||_\infty : H \in \mathcal{H}, \text{support}(H) \subset U, \text{and } H \text{ generates } \psi \}
$$

With this notation, we will have $\overline{E}_\infty^U(\psi_i) \leq E^U_\infty(\psi_i)$, with both being finite so long as $U$ contains $B^{2n}(R,x_i)$.

In this first use of the notation, our choice for the set $U$ will be a *tube* containing the $i^{\text{th}}$ ball, pointing in the $x_{2n}$ direction, defined as follows. First, we define $R^+ = \frac{R + \frac{1}{2}}{2} = \frac{R}{2} + \frac{1}{4}$. (We just need $R^+$ to be a number between $R$ and $\frac{1}{2}$.)

Then we define the $i^{\text{th}}$ tube as the set

$$
tube_i = \{ B^{2n-1}(R^+,x_i) \times \mathbb{R}^1 \} = \left\{ y \in \mathbb{R}^{2n} : \sum_{k=1}^{2n-1} (y_k - x_{i,k})^2 < (R^+)^2 \right\}.
$$

Since the $i^{\text{th}}$ tube will contain the $i^{\text{th}}$ ball, in our new notation we can say that
$E_{\infty}^{\text{tube}_i}(\psi_i)$ is finite. But in fact, in Section 4.2.5, we will prove the following lemma:

Lemma B5: If $\psi$ is a symplectomorphism generated by a Hamiltonian function supported in the ball $B^{2n}(R, x)$, $0 < R < \frac{1}{2}$, and if $\text{tube}$ is the set $\left\{ y : \sum_{i=1}^{2n-1} (y_i - x_i)^2 < \left( \frac{R}{2} + \frac{1}{4} \right)^2 \right\}$, then $E_{\infty}^{\text{tube}}(\psi) \leq 8$.

We apply this lemma to our present situation, using our $\psi_i$ for the symplectomorphism $\psi$ in the lemma, and our set $\text{tube}_i$ for the set $\text{tube}$ in the lemma. As a result, we can say that for each $i = 1 \cdots \infty$, $E_{\infty}^{\text{tube}_i}(\psi_i) \leq 8$. Put another way, $\max_{i=1, \cdots, \infty} E_{\infty}^{\text{tube}_i}(\psi_i) \leq 8$.

Recall that in Lemma B3, we shuffled the balls into an arrangement where the $2n^{th}$ coordinate of each ball was zero. Because of this shuffling, we know that the $i^{th}$ tube will contain the $i^{th}$ ball and no others. Moreover, if $i \neq j$ then $\text{tube}_i$ and $\text{tube}_j$ are disjoint. (These last two statements are the motivation for the shuffling that we did in Lemma B.) In Detail 5, found in Section 4.3.5, we prove the following fact about the extended Hofer infinity norm:

Detail 5: Let $\psi_i, i \in I$ be a finite or countable collection of Hamiltonian symplectomorphisms with support $(\psi_i) \subset U_i \subset \mathbb{R}^{2n}$, where $U_i \cap U_j = \phi$ if $i \neq j$. Further, assume that for each $i \in I$, $E_{\infty}^{U_i}(\psi_i)$ is finite and that $\max_{i \in I} E_{\infty}^{U_i}(\psi_i)$ exists. Then $E_{\infty}(\psi) \leq 2 \max_{i \in I} E_{\infty}^{U_i}(\psi_i)$.

Applying this fact to our current situation, we will use the sets $\text{tube}_i$, for $i = 1 \cdots \infty$,
as the sets $U_i$. The result is
\[
E_{\infty} \left( \prod_{i=1}^{\infty} \psi_i \right) \leq 2 \max_{i=1}^{\infty} E_{\infty}^{\text{tube}_i} (\psi_i) \quad \text{(by Detail 5)}
\]
\[
\leq 2 \max_{i=1}^{\infty} E_{\infty}^{\text{tube}_i} (\psi_i) \quad \text{(by definition)}
\]
\[
\leq 2 \cdot (8) \quad \text{(by Lemma B5)}
\]
\[
= 16.
\]
We have shown that $E_{\infty} (\psi) \leq 16$.

*End of Proof of Lemma B4*

4.2.5 Lemma B5

*Lemma B5*: If $\psi$ is a symplectomorphism generated by a Hamiltonian function supported in the ball $B^{2n} (R, x)$, $0 < R < \frac{1}{2}$, and if tube is the set
\[
\{ y : \sum_{i=1}^{2n-1} (y_i - x_i)^2 < \left( \frac{R}{2} + \frac{1}{4} \right)^2 \},
\]
then $E_{\infty}^{\text{tube}} (\psi) \leq 8$.

*Proof of Lemma B5:*

Let $\epsilon > 0$. We will show that $E_{\infty}^{\text{tube}} (\psi) \leq 8 + 2\epsilon$. Since $\epsilon$ is arbitrary, this will prove the claim.

Without loss of generality, we may assume that the point $x$ is the origin. To see why, suppose $x$ is not the origin. Then the symplectomorphism $\phi = \tau_x^{-1} \circ \psi \circ \tau_x$, where $\tau_x$ is the translation that sends the origin to the point $x$, will be supported in the ball $\tau_{-x} (B^{2n} (R, x)) = B^{2n} (R)$, which is contained in the set $\tau_{-x} (\text{tube})$, another tube. We could use the present claim to estimate $E_{\infty}^{\tau_{-x}(\text{tube})} (\phi)$. Then, by conjugation invariance, $E_{\infty}^{\text{tube}} (\psi) = E_{\infty}^{\tau_{-x}(\text{tube})} (\phi)$.

With that assumption, the description of our tube becomes simpler:
\[
tube = \left\{ y : \sum_{i=1}^{2n-1} y_i^2 < \left( \frac{R}{2} + \frac{1}{4} \right)^2 \right\}
\]

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We will consider translates of the ball $B^{2n}(R)$ along the length of the tube. (That is, in the $x_{2n}$ direction.) With that in mind, we introduce the following notation:

\[ \text{ball}_0 = \text{the original ball} = B^{2n}(R, 0) \]
\[ \text{ball}_k = \text{ball translated } k \text{ units in the } x_{2n} \text{ direction} = B^{2n}(R, (0, \cdots, 0, k)) \]
\[ \text{cell}_k = \left\{ y : \sum_{i=1}^{2n-1} y_i^2 < \left( \frac{R}{2} + \frac{1}{4} \right)^2 \text{ and } k - 1 - \left( \frac{R}{2} + \frac{1}{4} \right) < y_{2n} < k + \left( \frac{R}{2} + \frac{1}{4} \right) \right\} . \]

The set $\text{cell}_k$ is designed to be large enough to support a Hamiltonian function that will generate a symplectomorphism that will translate a ball of radius $R$ centered at $(0, \cdots, 0, k - 1)$ to the position $(0, \cdots, 0, k)$. That symplectomorphism, called $\sigma_k$, is described in Detail 4, below. By the construction shown there, $\sigma_k$ will have three important properties:

\[ \sigma_k(\text{ball}_{k-1}) = \text{ball}_k \]
\[ \text{support}(\sigma_k) \subset \text{cell}_k \]
\[ E^{\text{cell}_k}_\infty(\sigma_k) \leq 1. \]

Also note that if $|j - k| \geq 2$ then $\text{cell}_j$ and $\text{cell}_k$ do not intersect.

By hypothesis, $\psi$ can be generated by a Hamiltonian function supported in $\text{ball}_0$. By compactness, such a function will be uniformly-bounded. So, $\psi$ has finite extended Hofer infinity norm, even when we restrict the support of the functions used in the infimum. That is, $E^{\text{tube}}_\infty(\psi) \leq E^{\text{cell}_0}_\infty(\psi) \leq E^{\text{ball}_0}_\infty(\psi) < \infty$. Therefore, there is an integer $N \geq 1$ and a finite sequence $\psi = \psi_0, \psi_1, \ldots, \psi_N = \text{id}$, with each $\psi_i \in \text{Ham}(B^{2n}(\mathbb{R}^{2n}))$, such that $d_\infty(\psi_i, \psi_{i+1}) < \varepsilon$ for $i = 0, \ldots, N - 1$, even if we take the infimum over only those Hamiltonian functions supported in $\text{ball}_0$. That is,

\[ E^{\text{tube}}_\infty(\psi_i^{-1}\psi_{i+1}) \leq E^{\text{cell}_0}_\infty(\psi_i^{-1}\psi_{i+1}) \leq E^{\text{ball}_0}_\infty(\psi_i^{-1}\psi_{i+1}) < \varepsilon, \text{ for } i = 0, \ldots, N - 1. \]

For $k = 0, \ldots, N$ we define the symplectomorphism $\text{even}_k$ by

\[ \text{even}_k = \begin{cases} 
\psi_0 & \text{if } k = 0 \\
\sigma_{2k} \sigma_{2k-1} \cdots \sigma_2 \sigma_1 \psi_k \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{2k-1}^{-1} \sigma_{2k}^{-1} & \text{if } 1 \leq k \leq N.
\end{cases} \]
Note that the symplectomorphism \( even_k \) is supported in \( ball_{2k} \), which is centered at the point \((0, \cdots, 2k)\). Also note that \( even_N = id \) because \( \psi_N = id \).

For \( k = 0, \cdots, N - 1 \) define the symplectomorphism \( odd_k \) by

\[
odd_k = \sigma_{2k+1} \sigma_{2k} \cdots \sigma_2 \sigma_1 \psi_k \sigma_1^{-1} \sigma_2^{-1} \cdots \sigma_{2k}^{-1} \sigma_{2k+1}^{-1}.
\]

Note that \( odd_k \) is supported in \( ball_{2k+1} \), which is centered at the point \((0, \cdots, 0, 2k + 1)\).

Using these terms, we rewrite \( \psi \).

\[
\psi = \psi_0 = even_0 = even_0 \left( \prod_{i=0}^{N-1} odd_i odd_i^{-1} \right) \left( \prod_{j=1}^{N} even_j even_j^{-1} \right) even_N^{-1}
\]

In this last step, we have simply inserted a bunch of terms, along with their inverses, and one final term that equals the identity map. Notice that for any two factors in the above expression, either the supports of the two factors are disjoint, or the supports are the same and the two factors are inverses of one another. Either way, we see that the entire collection of factors commutes. This allows us to rearrange them:

\[
\psi = \left( \prod_{i=0}^{N-1} even_i odd_i^{-1} \right) \left( \prod_{j=1}^{N} even_j^{-1} odd_j^{-1} \right).
\]

Estimating the extended infinity norm, we find

\[
E_{\infty}^{tube} (\psi) = E_{\infty}^{tube} \left( \left( \prod_{i=0}^{N-1} even_i odd_i^{-1} \right) \left( \prod_{j=1}^{N} even_j^{-1} odd_j^{-1} \right) \right) \\ \leq E_{\infty}^{tube} \left( \prod_{i=0}^{N-1} even_i odd_i^{-1} \right) + E_{\infty}^{tube} \left( \prod_{j=1}^{N} even_j^{-1} odd_j^{-1} \right). \tag{4.1}
\]

Consider the expression on the left in the right side of Equation 4.1. Since \( even_i \) is supported in \( ball_{2i} \) and \( odd_i \) is supported in \( ball_{2i+1} \), we see that \( even_i odd_i \) is supported in \( cell_{2i+1} \). Recall that if \( |j - k| \geq 2 \) then \( cell_j \) and \( cell_k \) do not intersect. Therefore, if \( i \neq j \), then \( cell_{2i+1} \) and \( cell_{2j+1} \) do not intersect. This allows us to use the trick described in Detail 5, below to say that

\[
E_{\infty}^{tube} \left( \prod_{i=0}^{N-1} even_i odd_i^{-1} \right) \leq 2 \max_{i=0, \cdots, N-1} E_{\infty}^{cell_{2i+1}} (even_i odd_i^{-1}). \tag{4.2}
\]
For $i = 0$, we have
\[
E_{\infty}^{\text{cell}_i} \left( \text{even}_0 \text{odd}^{-1}_0 \right) = E_{\infty}^{\text{cell}_i} \left( \psi_0 \sigma_1 \psi_0^{-1} \sigma_1^{-1} \right) \\
\leq E_{\infty}^{\text{cell}_i} \left( \psi_0 \sigma_1 \psi_0^{-1} \right) + E_{\infty}^{\text{cell}_i} \left( \sigma_1^{-1} \right) \\
= E_{\infty}^{\text{cell}_i} \left( \sigma_1 \right) + E_{\infty}^{\text{cell}_i} \left( \sigma_1 \right) \text{ (conjugation invariance)} \\
\leq 1 + 1 = 2. \tag{4.3}
\]

For $i = 1 \cdots N - 1$, we have $E_{\infty}^{\text{cell}_{2i+1}} \left( \text{even}_i \text{odd}^{-1}_i \right) =$
\[
= E_{\infty}^{\text{cell}_{2i+1}} \left( \sigma_{2i} \sigma_{2i-1} \cdots \sigma_2 \sigma_1 \psi_i \psi_i^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1} \cdots \sigma_{2i-1}^{-1} \sigma_{2i}^{-1} \sigma_{2i+1}^{-1} \right) \\
\leq E_{\infty}^{\text{cell}_{2i+1}} \left( \sigma_{2i} \sigma_{2i-1} \cdots \sigma_2 \sigma_1 \psi_i \psi_i^{-1} \sigma_1^{-1} \sigma_2^{-1} \sigma_3^{-1} \cdots \sigma_{2i-1}^{-1} \sigma_{2i}^{-1} \sigma_{2i+1}^{-1} \right) + \tag{4.4}
E_{\infty}^{\text{cell}_{2i+1}} \left( \sigma_{2i+1}^{-1} \right) \text{ (triangle inequality)} \\
= E_{\infty}^{\text{cell}_{2i+1}} \left( \sigma_{2i+1} \right) + E_{\infty}^{\text{cell}_{2i+1}} \left( \sigma_{2i+1} \right) \text{ (conjugation invariance)} \\
\leq 1 + 1 = 2
\]

Summarizing, for the expression on the left in the right side of Equation 4.1, we have the estimate
\[
E_{\infty}^{\text{tube}} \left( \prod_{i=0}^{N-1} \text{even}_{i} \text{odd}^{-1}_{i} \right) \leq 2 \max_{i=0 \cdots N-1} E_{\infty}^{\text{cell}_{2i+1}} \left( \text{even}_{i} \text{odd}^{-1}_{i} \right) \leq 2 \, 2 = 4. \tag{4.5}
\]

Next, we estimate the expression on the right in the right side of Equation 4.1. As we did above, we will consider the supports of the factors in the product in order to be able to exploit the trick of Detail 5. Since $\text{even}_j$ is supported in $\text{ball}_{2j}$ and $\text{odd}_{j-1}$ is supported in $\text{ball}_{2j-1}$, we see that $\text{even}_j \text{odd}_{j-1}$ is supported in $\text{cell}_{2j}$. As above, we know that if $i \neq j$, then $\text{cell}_{2i}$ and $\text{cell}_{2j}$ do not intersect, and that will allow us to use the trick from Detail 5.

\[
E_{\infty}^{\text{tube}} \left( \prod_{j=1}^{N} \text{even}^{-1}_{j} \text{odd}_{j-1} \right) \leq 2 \max_{j=1 \cdots N} E_{\infty}^{\text{cell}_{2j}} \left( \text{even}^{-1}_{j} \text{odd}_{j-1} \right) 
\]

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For $j = 1 \cdots N$, we make the estimate $E_{\infty}^{cell_{2j}} (even_{j}^{-1}odd_{j-1}) =
\begin{align*}
&= E_{\infty}^{cell_{2j}} (\sigma_{2j}^{-1} \cdots \sigma_{1, \psi_{j}^{-1}}^{-1} \sigma_{1}^{-1} \sigma_{2}^{-1} \cdots \sigma_{2j}^{-1} \sigma_{2j-1} \cdots \sigma_{1, \psi_{j}^{-1}}^{-1} \sigma_{1}^{-1} \cdots \sigma_{2j-1}^{-1}) \\
&= E_{\infty}^{cell_{2j}} (\sigma_{2j}^{-1} \cdots \sigma_{1, \psi_{j}^{-1}}^{-1} \sigma_{1}^{-1} \sigma_{2}^{-1} \cdots \sigma_{2j-1}^{-1} \sigma_{2j} \cdots \sigma_{1, \psi_{j}^{-1}}^{-1} \sigma_{1}^{-1} \cdots \sigma_{2j-1}^{-1}) \\
&\leq E_{\infty}^{cell_{2j}} (\sigma_{2j}) + \\
&\quad + E_{\infty}^{cell_{2j}} (\sigma_{2j-1}^{-1} \cdots \sigma_{1, \psi_{j}^{-1}}^{-1} \sigma_{1}^{-1} \sigma_{2}^{-1} \cdots \sigma_{2j-1}^{-1} \sigma_{2j} \cdots \sigma_{1, \psi_{j}^{-1}}^{-1} \sigma_{1}^{-1} \cdots \sigma_{2j-1}^{-1}) \\
&\quad + E_{\infty}^{cell_{2j}} (\sigma_{2j-1}^{-1} \cdots \sigma_{1, \psi_{j}^{-1}}^{-1} \sigma_{1}^{-1} \sigma_{2}^{-1} \cdots \sigma_{2j-1}^{-1}) \\
&= E_{\infty}^{cell_{2j}} (\sigma_{2j}) + E_{\infty}^{cell_{2j}} (\sigma_{2j}) + E_{\infty}^{ball_{j}} (\psi_{j}^{-1} \psi_{j-1}) \text{ (conjugation inv.)} \\
&\leq 1 + 1 + \varepsilon = 2 + \varepsilon.
\end{align*}

Summarizing, for the expression on the right in the right side of Equation 4.1, we have the estimate

$$E_{\infty}^{tube} \left( \prod_{j=1}^{N} even_{j}^{-1}odd_{j-1} \right) \leq 2 \max_{j=1 \cdots N} E_{\infty}^{cell_{2j}} (even_{j}^{-1}odd_{j-1}) \leq 2 (2 + \varepsilon) = 4 + 2\varepsilon.$$  

(4.6)

Plugging Equations 4.5 and 4.6 back into Equation 4.1, we have

$$E_{\infty}^{tube} (\psi) \leq E_{\infty}^{tube} \left( \prod_{i=0}^{N-1} even_{i}odd_{i}^{-1} \right) + E_{\infty}^{tube} \left( \prod_{j=1}^{N} even_{j}^{-1}odd_{j-1} \right) \leq 4 + (4 + 2\varepsilon) = 8 + 2\varepsilon.$$

*End of Proof of Lemma B5*

### 4.3 Details

#### 4.3.1 Detail 1: An isotopy that transforms a symplectic ball

*step i: Introduce $g_{i,t}$, $g_{i,a}$, $g_{i,b}$, $g_{i,c}$, and $g_{i,d}$.*
In this section, we describe an isotopy $g_{i,t}$ that turns the $i^{th}$ symplectic ball into a round ball centered at an integer lattice point. Each $g_{i,t}$ is obtained by a time-concatenation of four isotopies, $g_i = g_{ia} * g_{ib} * g_{ic} * g_{id}$. The operation of these isotopies is as follows: $g_{ia}$ moves the $i^{th}$ symplectic ball radially outward from the origin, while making it more elliptical (in the sense that will be explained below). Then, $g_{ib}$ turns the symplectic ball into a symplectic ellipse, while holding it in place. Next, $g_{ic}$ turns the symplectic ellipse into a round ball, while holding it in place. Finally, $g_{id}$ translates the round ball to the nearest integer lattice position.

step ii: Describe $g_{ia}$.

Recall notation from step i of the proof of Lemma B2. The $i^{th}$ symplectic ball is the image, $f_i(B^{2n}(R))$, of the symplectic embedding, $f_i : B^{2n}(R) \to \mathbb{R}^{2n}$. The set $f_i(B^{2n}(R+\varepsilon))$ is called the $i^{th}$ neighborhood, with $\varepsilon$ chosen for the whole set of $k$ neighborhoods in order that they be disjoint and do not intersect $B^{2n}(r_a)$.

In Detail 2, below, the workings of a smooth family of symplectomorphisms $\gamma_{f,t} \in \text{Symp}(\mathbb{R}^{2n})$, for $t \in [0,1]$, referred to as a partial linearization with translation applied to a symplectic ball, are explained. Here, we will apply such a symplectomorphism to the $i^{th}$ ball and describe its effect at some $t \in [0,1]$.

$$\gamma_{f,t} \left( f_i \left( \text{Ball}^{2n}(R) \right) \right) = \underbrace{\tau_{f_i(0)}}_{\text{translation}} \circ \underbrace{m_{1-t}}_{\text{symplectic ball becoming more elliptical; center fixed at origin}} \circ \underbrace{f_i }_{\text{symplectic ellipse}} \circ \underbrace{f_i^{-1} \circ \left( f_i \left( \text{Ball}^{2n}(R) \right) \right) }_{\text{round ball}}$$

In this expression, $\tau_x$ is the translation that sends the origin to the point $x \in \mathbb{R}^{2n}$, and $m_c$ is multiplication by the non-zero real number $c$. At time $t = 0$, this expression simplifies to $\gamma_{f_i,0} \left( \text{Ball}^{2n}(R) \right) = f_i \left( \text{Ball}^{2n}(R) \right)$, the original symplectic ball. As discussed in Detail 1, for times $0 < t < 1$, the above expression can be thought of as a difference quotient and, in the $t \to 1$ limit, it approaches the composition of a linear map with a translation.

A linear image of a ball is an ellipse; so as $t$ approaches 1, one could say
that the ball is becoming more elliptical. Furthermore, if \( f_i (0) \neq 0 \), the image gets translated farther and farther from the origin in the \( f_i (0) \) direction. In Detail 3 we will choose an ending time, \( 0 < \lambda < 1 \), for \( \gamma_{f_i,t} \). (The trick there will be to find an ending time that will work for the whole set, \( i = 1, \ldots, k \), of symplectic balls.) We use this \( \lambda \) to re-scale the time in \( \gamma_{f_i,t} \), and call the resulting map, \( g_{ia} \):

\[
g_{ia,t} = \tau_{\frac{f_i(0)}{1-\lambda t}} \circ \tau_{-\frac{f_i(0)}{1-\lambda t}} \circ f_i \circ m_{1-\lambda t} \circ f_i^{-1} \quad \text{for } t \in [0, 1].
\]

Figures 4.5 and 4.6 show the \( i^{th} \) symplectic ball evolving under the influence of \( g_{ia} \). Also shown is the \( j^{th} \) ball evolving under the influence of \( g_{ja} \). The symplectic balls are the darker shapes, and are described by the formulas

\[
g_{ia,t} (f_i (B^{2n}(R))) = \tau_{\frac{f_i(0)}{1-\lambda t}} \circ m_{\frac{1}{1-\lambda t}} \circ \tau_{-\frac{f_i(0)}{1-\lambda t}} \circ f_i \circ m_{1-\lambda t} \circ f_i^{-1} \circ f_i (B^{2n}(R))
\]

\[
= \tau_{\frac{f_i(0)}{1-\lambda t}} \circ m_{\frac{1}{1-\lambda t}} \circ \tau_{-\frac{f_i(0)}{1-\lambda t}} \circ f_i \circ m_{1-\lambda t} (B^{2n}(R)).
\]

Surrounding the original symplectic balls are dotted regions, the \( i^{th} \) and \( j^{th} \) neighborhoods, \( f_i (B^{2n}(R+\varepsilon)) \) and \( f_j (B^{2n}(R+\varepsilon)) \), that were described in Lemma B2. Note that as the symplectic balls evolve, they will always be confined to regions (the larger dotted shapes in the figures below) that are simply a linear magnification of these neighborhoods. That is,

\[
g_{ia,t} (f_i (B^{2n}(R))) = \tau_{\frac{f_i(0)}{1-\lambda t}} \circ m_{\frac{1}{1-\lambda t}} \circ \tau_{-\frac{f_i(0)}{1-\lambda t}} \circ f_i \circ m_{1-\lambda t} (B^{2n}(R))
\]

\[
= m_{\frac{1}{1-\lambda t}} \circ f_i \circ m_{1-\lambda t} (B^{2n}(R))
\]

\[
\subseteq m_{\frac{1}{1-\lambda t}} \circ f_i (B^{2n}(R))
\]

\[
\subseteq \frac{m \left( B^{2n}(R+\varepsilon) \right)}{\bar{f}_i}.
\]

What is slightly misleading about these figures is that the two isotopies do not really work simultaneously in this way. That is because the isotopy \( g_{ia} \) would also
Figure 4.5: Zoomed-in view of the start of isotopies $g_{ia}$ and $g_{ja}$
As the isotopy continues, the symplectic balls continue moving radially outward, becoming "more elliptical" while staying within the confines of the (growing) magnified neighborhoods. By the end of the isotopy, the neighborhoods have grown large enough to accommodate future transformations that the balls will undergo.

Figure 4.6: Zoomed-out view of the end of isotopies $g_{ia}$ and $g_{ja}$.
affect the $j^{th}$ symplectic ball, while the isotopy $g_{j;i}$ would also affect the $i^{th}$ symplectic ball, etc. But in Lemma B2, where the results of our present work are actually used, the individual isotopies $g_{i}$ with $i = 1 \cdots k$, will be incorporated into a more refined isotopy, $h$, that will affect the $i^{th}$ symplectic ball in precisely the same way that $g_{i;i}$ does, while simultaneously affecting the $j^{th}$ symplectic ball in precisely the same way that $g_{j;i}$ does, etc.

*step iii: Describe $g_{ib,t}$.\*

At the end of the operation of the isotopy $g_{i;i}$, the $i^{th}$ symplectic ball has been transformed into another symplectic ball,

$$g_{i;i} \left( f_{i} \left( B^{2n} (R) \right) \right) = \frac{\tau_{f_{i}(0)} \circ m_{1 - \lambda}}{1 - \lambda} \circ \tau_{-f_{i}(0)} \circ f_{i} \circ m_{1 - \lambda} \circ \beta_{i}^{-1} \left( f_{i} \left( B^{2n} (R) \right) \right),$$

that is more elliptical than the original and is centered at the point

$$g_{i;i} \left( f_{i} (0) \right) = \frac{\tau_{f_{i}(0)} \circ m_{1 - \lambda}}{1 - \lambda} \circ \tau_{-f_{i}(0)} \circ f_{i} \circ m_{1 - \lambda} (0) \circ \beta_{i} (0) = \frac{1}{1 - \lambda} \cdot \beta_{i} (0),$$

where $\lambda$, $0 \leq \lambda < 1$, is the *ending time* described in Detail 3. For simplicity of notation, we introduce the abbreviation $\theta_{i}$ for the composition $g_{i;i} \circ f_{i}$. With this notation, at the end of the operation of the isotopy $g_{i;i}$, the $i^{th}$ symplectic ball has been transformed to the symplectic ball $\theta_{i} \left( B^{2n} (R) \right)$, centered at the point $\theta_{i} (0) = \frac{1}{1 - \lambda} \cdot \beta_{i} (0)$.

In Detail 2, the workings of another family of symplectomorphisms, $\beta_{f,t} \in \operatorname{Symp} (\mathbb{R}^{2n})$, for $t \in [0, 1]$, referred to as a *linearization applied to a symplectic ball*, are explained. Here, we wish apply such a symplectomorphism to the transformed $i^{th}$ symplectic ball, $\theta_{i} \left( B^{2n} (R) \right)$. To accomplish that, we will substitute the new symbol $\theta_{i}$ for the symbol $f$ in the expression for $\beta_{f,t}$. The result will be the isotopy $g_{i;i}$:

$$g_{i;i} = \beta_{i;i} = \begin{cases} \tau_{\theta_{i}(0)} \circ m_{1 - \lambda} \circ \tau_{-\theta_{i}(0)} \circ \theta_{i} \circ m_{1 - \lambda} \circ \theta_{i}^{-1} & \text{if } t \in [0, 1) \\ \tau_{\theta_{i}(0)} \circ L_{(d\theta_{i})_{0}} \circ \theta_{i}^{-1} & \text{if } t = 1. \end{cases}$$
In this expression, \( L_{(d\theta_i)} \) is the linear operator obtained by left multiplication by the matrix \((d\theta_i)\). Because \( \theta \) is a symplectomorphism, \((d\theta_i)\) is a symplectic matrix, and we could describe \( L_{(d\theta_i)} \) \((\text{Ball}^{2n}(R))\) as a symplectic ellipse. The results of applying the isotopy \( g_{ib,t} \) to the transformed \( i^{th} \) symplectic ball \( \theta_i \) \((B^{2n}(R))\) are as follows. Because \( g_{ib,0} = \beta_{\theta_i,0} = id \), at time \( t = 0 \) the ball is, of course, unchanged. By time \( t = 1 \), the ball has been transformed to

\[
g_{ib,1}(\theta_i(\text{Ball}^{2n}(R))) = \beta_{\theta_i,1}(\theta_i(\text{Ball}^{2n}(R))) \\
= \tau_{\theta_i(0)} \circ L_{(d\theta_i)} \circ \theta_i^{-1}(\theta_i(\text{Ball}^{2n}(R))) \\
= \tau_{\theta_i(0)} \circ L_{(d\theta_i)} \left( \text{Ball}^{2n}(R) \right).
\]

So the isotopy \( g_{ib,t} \), applied to the \( i^{th} \) symplectic ball (after that ball has already been translated and deformed by isotopy \( g_{ia,t} \)) has the effect of turning the symplectic ball into a symplectic ellipse, while holding it in place. Figure 4.7 shows the effect of the isotopies \( g_{ib} \) and \( g_{jb} \) on the \( i^{th} \) and \( j^{th} \) symplectic balls: They have become symplectic ellipses, centered at the spots where the symplectic balls were sitting at the end of isotopies \( g_{ia} \) and \( g_{ja} \). Also shown on the figures are the dotted regions that are the magnification of the original \( i^{th} \) and \( j^{th} \) neighborhoods. As shown in the figures, the images of the \( i^{th} \) and \( j^{th} \) symplectic balls, as they evolve under the influence of isotopies \( g_{ib} \) and \( g_{jb} \), remain within these regions. This is not automatic. Rather, it is because, in Detail 3, we will be careful to choose an ending time \( \lambda \) sufficient to make it happen.

**step iv:** Describe \( g_{ic,t} \).

By the end of the operation of \( g_{ia} \ast g_{ib} \), the \( i^{th} \) symplectic ball has been
The symplectic balls, which had become "more elliptical" by the end of the previous isotopy, have now become actual symplectic ellipses. During this isotopy, the symplectic balls have remained within the confines of the magnified neighborhoods. The balls' centers, and the magnified neighborhoods, have not changed.

*Figure 4.7: Effect of the isotopies $g_{ib}$ and $g_{jb}$*
turned into a symplectic ellipse centered at $\frac{1}{1-\lambda} f_i (0)$:

$$(g_{ia} * g_{ib})_1 (i^{th}\text{symplectic ball}) = g_{ib,1} \circ g_{ia,1} (f_i (B^{2n} (R)))$$

$$= g_{ib,1} (B^{2n} (R))$$

$$= \tau_{\theta_i (0)} \circ L_{(d\theta_i)^0} (B^{2n} (R)),$$

where $\theta_i (0) = \frac{1}{1-\lambda} f_i (0)$.

Since $(d\theta_i)^0 \in Sp (\mathbb{R}^{2n})$, and $Sp (\mathbb{R}^{2n})$ is path connected, we know that there is a path $\sigma_{i,t} \in Sp (2n)$, $t \in [0, 1]$ connecting the identity map to $(d\theta_i)^0$. That is, $\sigma_{i,0} = id$ and $\sigma_{i,1} = (d\theta_i)^0$. So we can define $g_{ic,t} = \tau_{\theta_i (0)} \circ L_{\sigma_{i,t}} \circ \tau_{-\theta_i (0)}$. At time $t = 0$, this expression reduces to $g_{ic,0} = \tau_{\theta_i (0)} \circ L_{\sigma_{i,0}} \circ \tau_{-\theta_i (0)} = id$. Considering the time $t = 1$ expression, $g_{ic,1}$, applied to the $i^{th}$ symplectic ellipse, we find

$$g_{ic,1} (i^{th}\text{ symplectic ellipse}) = g_{ic,1} ((g_{ia} * g_{ib})_1 (i^{th}\text{ symplectic ball}))$$

$$= g_{ic,1} (\tau_{\theta_i (0)} \circ L_{(d\theta_i)^0} (B^{2n} (R)))$$

$$= \tau_{\theta_i (0)} \circ L_{\sigma_{i,1}} \circ \tau_{-\theta_i (0)} (\tau_{\theta_i (0)} \circ L_{(d\theta_i)^0} (B^{2n} (R)))$$

$$= \tau_{\theta_i (0)} \circ L_{(d\theta_i)^0} \circ \tau_{-\theta_i (0)} \circ L_{(d\theta_i)^0} (B^{2n} (R))$$

$$= \tau_{\theta_i (0)} (B^{2n} (R)).$$

So we see that $g_{ic,t}$ turns the $i^{th}$ symplectic ellipse into a round ball while holding it in place, with its center located at $\theta_i (0) = \frac{1}{1-\lambda} f_i (0)$. Figure 4.8 shows the effect of the isotopies $g_{ia}$ and $g_{ja}$ on the $i^{th}$ and $j^{th}$ symplectic balls. They have become round balls, centered at the spots where the symplectic ellipses were sitting at the end of isotopies $g_{ia}$ and $g_{ja}$. Also shown on the figures are the dotted regions that are the magnification of the original $i^{th}$ and $j^{th}$ neighborhoods. As shown in the figures, the images of the $i^{th}$ and $j^{th}$ symplectic balls, as they evolve under the influence of isotopies $g_{ic}$ and $g_{jc}$, remain within these regions. Again, this is not automatic, but rather because, in Détail 8, we will be careful to choose an ending time $\lambda$ sufficient to make it happen.

**step v:** Describe $g_{id}$. 

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During this isotopy, the symplectic ellipses have been turned into round balls. Their centers have not moved, and the transformation has taken place within the confines of the magnified neighborhoods.

Figure 4.8: Effect of the isotopies $g_{1c}$ and $g_{2c}$. 

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At the end of the operation of \( g_{i_0} \ast g_{i_0} \ast g_{i_0} \), the \( i^{th} \) symplectic ball has been turned into a round ball, centered at \( \theta_i(0) = \frac{1}{1-\lambda} f_i(0) \). Define \( g_{i_0,t} \) to be the translation \( g_{i_0,t} = \tau_{(x_i - \frac{1}{1-\lambda} f_i(0))} \), where \( x_i \) is the integer lattice point nearest \( \frac{1}{1-\lambda} f_i(0) \). So \( g_{i_0,0} = \tau_0 = id \), while \( g_{i_0,1} \) applied to the \( i^{th} \) ball is
\[
\begin{align*}
g_{i_0,1} \left( i^{th} \text{ball} \right) &= g_{i_0,1} \left( \tau_{\frac{1}{1-\lambda} f_i(0)} \left( B^{2n}(R) \right) \right) \\
&= \tau_{(x_i - \frac{1}{1-\lambda} f_i(0))} \left( \tau_{\frac{1}{1-\lambda} f_i(0)} \left( B^{2n}(R) \right) \right) \\
&= \tau_{x_i} \left( B^{2n}(R) \right).
\end{align*}
\]

We see that the isotopy \( g_{i_0,t} \) moves the \( i^{th} \) ball to the nearest integer lattice point. Figure 4.9 shows the effect of the isotopies \( g_{i_0} \) and \( g_{j_0} \) on the \( i^{th} \) and \( j^{th} \) balls. As above, during this isotopy, the balls remains confined to the dotted regions shown, because of our choice of the ending time, \( \lambda \).

**End of Detail 1**

### 4.3.2 Detail 2: Linearizations, moving and fixed

**Part i: The linearization at zero (The Alexander trick)**

Let \( f \in Diff(\mathbb{R}^{2n}) \) be any diffeomorphism of \( \mathbb{R}^{2n} \). Let \( m_c : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \) be the multiplication by the scalar \( c \) and let \( \tau_b : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n} \) be the translation by \( b \). Then for each \( t \in (0,1] \), the map \( m_1 \circ \tau_{-f(0)} \circ f \circ m_t \) is an element of \( Diff(\mathbb{R}^{2n}) \). Note that when this diffeomorphism acts on an element \( x \in \mathbb{R}^{2n} \), the result is just the difference quotient,
\[
m_1 \circ \tau_{-f(0)} \circ f \circ m_t(x) = \frac{f(tx) - f(0)}{t}.
\]

Because \( f \) is differentiable, the \( t \rightarrow 0 \) limit exists in \( \mathbb{R}^{2n} \):
\[
\lim_{t \rightarrow 0} m_1 \circ \tau_{-f(0)} \circ f \circ m_t(x) = \lim_{t \rightarrow 0} \frac{f(tx) - f(0)}{t}
= (df)_0 \cdot x
= L_{(df)_0}(x)
\]

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Figure 4.9: Effect of the isotopies \( g_{id} \) and \( g_{jd} \)

During this isotopy, the round balls get translated to the nearest integer lattice point, staying within their neighborhoods during the process.
Here, $L_{(df)}$ is the linear operator obtained by left multiplication by the matrix $(df)_0$, and is called the linearization of $f$ at zero. If we define a path $\alpha_{f,t}$ in $\text{Diff}(\mathbb{R}^{2n})$ as

$$\alpha_{f,t} = \begin{cases} m_{-1} \circ \tau_{f(0)} \circ f \circ m_{1-t} & \text{if } t \in [0,1) \\ L_{(df)} & \text{if } t = 1, \end{cases}$$

then $\alpha_{f,t}$ is smooth in the compact-open $C^\infty$ topology on $\text{Diff}(\mathbb{R}^{2n})$, $\alpha_{f,0} = \tau_{f(0)} \circ f$, and $\alpha_{f,1}$ is a linear map. We could say that $\alpha$ is a path from a translation of $f$ to a linearization of $f$. Acting on a round ball with these maps, we have $\alpha_{f,0}(\text{Ball}^{2n}(R)) = f(\text{Ball}^{2n}(R)) - f(0)$, a diffeomorphic image of the ball, which we could call a diffeomorphic ball, and $\alpha_{f,1}(\text{Ball}^{2n}(R)) = L_{(df)}(\text{Ball}^{2n}(R))$, a linear image of the ball, i.e. an ellipse. Note that the center of the image, the image of $x = 0$, remains fixed at zero:

$$\alpha_{f,t}(0) = \begin{cases} f(0) - f(0) = 0 & \text{if } t \in [0,1) \\ L_{(df)}(0) = 0 & \text{if } t = 1 \end{cases}$$

When we actually use paths like the one above in this paper, the map $f$ will be a symplectomorphism. In that case, we will have $\alpha_{f,t} \in \text{Symp}(\mathbb{R}^{2n})$ for all $t \in [0,1]$, and we will say symplectic ball, and symplectic ellipse, the latter because $(df)_0 \in Sp(2n)$. For the remainder of this section, we will assume that $f$ is a symplectic map, though the techniques work for any diffeomorphism.

Part ii: The linearization applied to a symplectic ball

The path that we constructed above was used to transform a round ball. At time $t = 0$, the result was a symplectic ball, and at time $t = 1$, the image was a symplectic ellipse, both centered at the origin. However, our need will be to transform not a round ball, but rather an existing symplectic ball, $f(\text{Ball}^{2n}(R))$, and to do it not at the origin, but rather at the spot where the symplectic ball sits. For that reason, we compose $\alpha_{f,t}$ with a translation and with $f^{-1}$ to achieve a new path, a
path which we will call $\beta_{f,t}$:

$$
\beta_{f,t} = \begin{cases} 
\tau_{f(0)} \circ m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t} \circ f^{-1} & \text{if } t \in [0,1) \\
\tau_{f(0)} \circ L_{(d')_0} \circ f^{-1} & \text{if } t = 1
\end{cases}
$$

Then $\beta_{f,t}$ is a path in $\text{Symp}(\mathbb{R}^{2n})$, with $\beta_{f,0} = \text{id}$.

When $\beta_{f,t}$ acts on the symplectic ball $f(\text{Ball}^{2n}(R))$, the result at time $t = 0$ is

$$
\beta_{f,0}(f(\text{Ball}^{2n}(R))) = \text{id}(f(\text{Ball}^{2n}(R))) = f(\text{Ball}^{2n}(R)),
$$

the original symplectic ball. At time $t = 1$, the result is

$$
\beta_{f,1}(f(\text{Ball}^{2n}(R))) = \tau_{f(0)} \circ L_{(d')_0} \circ f^{-1}(f(\text{Ball}^{2n}(R))) = \tau_{f(0)} \circ L_{(d')_0}(\text{Ball}^{2n}(R)),
$$

a symplectic ellipse, with center located at the same spot where the center of the original symplectic ball was located.

---

**Part iii: Partial linearization with translation applied to a symplectic ball**

In the construction of $\beta_{f,t}$, above, we inserted an additional translation into the expression for $\alpha_{f,t}$. As a result, the images of the symplectic ball, $\beta_{f,t}(f(\text{Ball}^{2n}(R)))$, remained fixed at $f(0)$. If, instead, we remove the translation from the expression that describes $\alpha_{f,t}$ for $t \in [0,1)$, we create a new path in $\text{Symp}(\mathbb{R}^{2n})$ that has the effect of moving the center of the image radially outward, in the $f(0)$ direction. We call this path $\gamma_{f,t}$:

$$
\gamma_{f,t} = m_{\frac{1}{1-t}} \circ f \circ m_{1-t} \circ f^{-1}
$$

$$
= m_{\frac{1}{1-t}} \circ \tau_{f(0)} \circ \tau_{-f(0)} \circ f \circ m_{1-t} \circ f^{-1}
$$

$$
= \tau_{f(0)} \circ m_{\frac{1}{1-t}} \circ \tau_{-f(0)} \circ f \circ m_{1-t} \circ f^{-1}.
$$

This expression looks just like the one above for $\beta_{f,t}$ in the time interval $t \in [0,1)$, except that the final translation is not fixed. Rather, the amount of translation increases as time $t$ approaches 1.
Consider the effect of the symplectomorphism $\gamma_{f,t}$ on the symplectic ball $f(Ball^{2n}(R))$:

$$\gamma_{f,t}(f(Ball^{2n}(R))) = \frac{m_1}{1-t} \circ \tau_{-f(0)} \circ f \circ m_{\tau - t} \circ f^{-1}(f(Ball^{2n}(R))) .$$

At time $t = 0$, this simplifies to $\gamma_{f,0}(Ball^{2n}(R)) = f(Ball^{2n}(R))$, the original symplectic ball. As $t$ approaches 1, the image of the ball becomes more elliptical and, if $f(0) \neq 0$, the image gets translated farther and farther from the origin in the $f(0)$ direction, with distance from the origin going to infinity. For that reason, we cannot extend the definition of $\gamma_{f,t}$ from the time interval $t \in [0,1)$ to the entire interval $[0,1]$. But in our use of this symplectomorphism, we will choose an ending time, $\tau$ that is less than 1. The result, then, will be that as $t$ goes from 0 to $\tau$, the image of the ball will evolve from the original symplectic ball, $f(Ball^{2n}(R))$, to another symplectic ball, $\gamma_{f,\tau}(Ball^{2n}(R))$, that is more elliptical than the original and is translated radially outward from the origin in the $f(0)$ direction. (Of course, if $f(0)$ does equal zero, then we could extend the definition of $\gamma_{f,t}$ to the time interval $t \in [0,1]$, but we don't need to. For our purposes, $\gamma_{f,t}$ will be stopped at the ending time, $\tau < 1$.)

End of Detail 2

4.3.3 Detail 3: The ending time

Given a disjoint union of symplectic balls, $\bigcup_{i=1}^{k} f_i(B^{2n}(R))$, we choose an ending time, $\lambda \geq 0$.

step i: Choose $R_1$ for one symplectic ball.

A symplectic ball is the image $f(B^{2n}(R))$, of a symplectic embedding, $f : B^{2n}(R) \rightarrow \mathbb{R}^{2n}$, where $B^{2n}(R)$ is the open ball of radius $R$. The center of this symplectic ball is the image of the origin, $f(0)$. Choose an $R_1$ such that
\( f(B^{2n}(R)) \subset B^{2n}(R_1, f(0)) \). Note that of course \( R \leq R_1 \), because the map \( f \) is volume-preserving.

**step ii:** Choose \( R_2 \) for one symplectic ball.

In **detail 2**, we described the linearization applied to a symplectic ball, a technique based on the Alexander trick. Using that technique, we can construct a path in \( \text{Symp}(\mathbb{R}^{2n}) \) which we will call \( \beta_{f,t} \), where \( t \in [0, 1] \). When we apply \( \beta_{f,t} \) to our symplectic ball, the important results are:

\[
\beta_{f,t} (f(B^{2n}(R))) = \begin{cases} 
\tau_{f(0)} \circ L_{\text{id}}(B^{2n}(R)) , & \text{the original symplectic ball,} \\
\tau_{f(0)} \circ \tau_{-f(0)} \circ f \circ m_{1-t} (B^{2n}(R)) & \text{if } 0 < t < 1 \\
\tau_{f(0)} \circ L_{(df)^*_0} (B^{2n}(R)) , & \text{a symplectic ellipse,} \\
\end{cases}
\]

Note that in each case, the result is a symplectic image of \( B^{2n}(R) \), with center located at \( f_1(0) \). Choose radius \( R_2 \) such that for all \( t \in [0, 1] \), \( \beta_{f,t} (f(B^{2n}(R))) \subset B^{2n}(R_2, f(0)) \). Also note that \( R_1 \leq R_2 \).

**step iii:** Choose \( R_3 \) for one symplectic ball.

Because \( (df)^* \in S p(\mathbb{R}^{2n}) \), and \( S p(\mathbb{R}^{2n}) \) is path connected, we know that there is a path \( \sigma_t \in S p(2n) \), \( t \in [0, 1] \) connecting the identity map to \( (df)^{-1}_0 \). That is, \( \sigma_0 = \text{id} \) and \( \sigma_1 = (df)^{-1}_0 \). Define a map \( g_t = \tau_{f(0)} \circ L_{\sigma_t} \circ \tau_{-f(0)} \). Then when \( g_t \) applied to the symplectic ellipse that we obtained at the end of **step ii**, above, the result will be:

\[
g_t (\tau_{f(0)} \circ L_{(df)^*_0} (B^{2n}(R))) = \begin{cases} 
\tau_{f(0)} \circ L_{(df)^*_0} (B^{2n}(R)) , & \text{if } t = 0 \\
\tau_{f(0)} \circ L_{\sigma_t} (B^{2n}(R)) & \text{if } 0 < t < 1 \\
\tau_{f(0)} (B^{2n}(R)) & \text{if } t = 1.
\end{cases}
\]

The \( t = 0 \) result is simply the original symplectic ellipse. The \( 0 < t < 1 \) result is another symplectic ellipse, centered at the same location. We could think of these
ellipses as becoming more spherical as time approaches 1. Finally, the $t = 1$ result is a round ball, centered at the same location. Choose radius $R_3$ such that for all $t \in [0, 1]$, $g_t \left( \tau_{f(0)} \circ L_{df_0} (B^{2n} (R)) \right) \subset B^{2n} (R_3, f(0))$. Note that $R_2 \leq R_3$.

*step iv:* Choose $R_4$ for one symplectic ball.

We will be interested in determining the radius $R_4$ necessary to ensure that a ball centered at $f(0)$ has room in its interior for all the activities of the previous three steps, and is also large enough to contain a copy of $B^{2n} (R)$ that is centered at an integer lattice point. But since $0 < R < \frac{1}{2}$, we know that a ball of radius 2 centered at any point will contain some $B^{2n} (R, x_k)$, where $x_k$ denotes an integer lattice point. Therefore, let $R_4 = \max \{2, R_3\}$.

*step v:* Choose $\lambda$ for one symplectic ball.

In *Detail 2*, we described the *partial linearization with translation*. It was a path $\gamma_{f,t}$ in $\text{Symp} (\mathbb{R}^{2n})$, defined by $\gamma_{f,t} = m_{\frac{1}{1-t}} \circ f \circ m_{1-t} \circ f^{-1}$ When we applied the symplectomorphism $\gamma_{f,t}$ to the symplectic ball $f(Ball^{2n} (R))$, the result was

$$
\gamma_{f,t} \left( f \left( Ball^{2n} (R) \right) \right) = \tau_{f(0)} \circ \frac{1}{1-t} \circ f \circ \tau_{-f(0)} \circ f \circ m_{1-t} \circ f^{-1} \left( f \left( Ball^{2n} (R) \right) \right).
$$

As time $t$ approaches 1, the image becomes more elliptical, with center fixed at origin outward from $f(0)$.

If we examine again the defining expression for $\gamma_{f,t}$, we will notice that the evolving image of the symplectic ball remains within the confines of a simple
magnification of the original neighborhood that was described in Lemma B2:

$$\gamma_{f,t}(B^{2n}(R)) = m_{\frac{1}{1-t}} \circ f \circ m_{1-t} \circ f^{-1}(f(B^{2n}(R)))$$

$$\subseteq m_{\frac{1}{1-t}} \circ f \circ m_{1-t}(B^{2n}(R))$$

$$\subseteq m_{\frac{1}{1-t}} \circ f \circ m_{\left(B^{2n}(R) + \varepsilon\right)}$$

Also note that as it evolves under the influence of the partial linearization with translation, the image of the symplectic ball has precisely the same shape as it would if it were evolving under the influence of the linearization with fixed center, the only difference being the translation. This is easy to see if we examine the expressions for the two linearizations during the time interval $0 \leq t < 1$.

$$\beta_{f,t}(f(B^{2n}(R))) = \tau_{\frac{t}{1-t}}(0) \circ m_{\frac{1}{1-t}} \circ \tau_{f(0)} \circ f \circ m_{1-t}(B^{2n}(R))$$

$$\gamma_{f,t}(f(B^{2n}(R))) = \tau_{\frac{t}{1-t}}(0) \circ m_{\frac{1}{1-t}} \circ \tau_{f(0)} \circ f \circ m_{1-t}(B^{2n}(R))$$

In step ii, above, we found a radius $R_2$ such that during the entire evolution of the linearization with fixed center, the evolving image remained within the confines of a ball of radius $R_2$ with fixed center. That is, for all $t \in [0, 1]$, $\beta_{f,t}(f(B^{2n}(0))) \subset B^{2n}(R_2, f(0))$. Now we see that during the evolution of the partial linearization with moving center, the evolving image will remain within the confines of a moving ball of radius $R_2$, centered at $\frac{f(0)}{1-t}$. That is, for all $t \in [0, 1]$, $\gamma_{f,t}(f(B^{2n}(R_0))) \subset B^{2n}\left(R_2, \frac{f(0)}{1-t}\right)$. Since, as time $t$ grows from 0 towards 1, the evolving symplectic ball is remaining within a moving ball of fixed radius, while the evolving neighborhood it lies in is growing without bound, we know that there is some $\lambda > 0$ such that at time $t = \lambda$, the neighborhood will be large enough so that the following will be true.

$$\gamma_{f,\lambda}(f(B^{2n}(R))) \subset B^{2n}\left(R_2, \frac{f(0)}{1-\lambda}\right) \subset B^{2n}\left(R_4, \frac{f(0)}{1-\lambda}\right) \subset m_{\frac{1}{1-\lambda}} \circ f \circ m_{\left(B^{2n}(R + \varepsilon)\right)}$$

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This is illustrated in Fig. 4.10 and Fig. 4.11.

**step vi:** Choose $\lambda$ for the entire collection of balls, $\bigcup_{i=1}^{k} f_{i}(B^{2n}(R))$.

For $i = 1 \cdots k$, let $\lambda_{i}$ be the number chosen by steps (i) through (v) above.

Then let $\lambda = \max \{\lambda_{1}, \lambda_{2}, \cdots, \lambda_{k}\}$.

**Summarize our construction of $\lambda$.**

Given a collection of symplectic balls, $\bigcup_{i=1}^{k} f_{i}(B^{2n}(R))$, we have chosen $\lambda$ in a way that, if each symplectic ball is subjected to a partial linearization with translation for time $\lambda$, then by the end of that time, each will have evolved to a state similar to that shown in Fig. 4.10 and Fig. 4.11.

**End of Detail 3**

### 4.3.4 Detail 4: A Symplectomorphism that moves a round ball

Recall the sets described in Lemma B5.

$ball_{0} =$ the original ball $= B^{2n}(R, 0)$

$ball_{k} =$ ball translated $k$ units $= B^{2n}(R, (0, \cdots, 0, k))$, for $k \in \mathbb{Z}$

$cell_{k} = \left\{y : \sum_{i=1}^{2n-1} y_{i}^{2} < \left(\frac{R}{2} + \frac{1}{4}\right)^{2} \text{ and } k - 1 - \left(\frac{R}{2} + \frac{1}{4}\right) < y_{2n} < k + \left(\frac{R}{2} + \frac{1}{4}\right)\right\}$

$tube = \left\{y : \sum_{i=1}^{2n-1} y_{i}^{2} < \left(\frac{R}{2} + \frac{1}{4}\right)^{2}\right\}$

Note that for each $k \in \mathbb{Z}$, we have $cell_{k} \subset tube$, and that if $|j - k| \geq 2$, then $cell_{j}$ and $cell_{k}$ do not intersect. The set $cell_{k}$ is designed to be large enough to support a Hamiltonian function $F_{k}$ that will generate a symplectomorphism that will translate a ball centered at $(0, \cdots, 0, k - 1)$ to the position $(0, \cdots, 0, k)$. We will describe that function now.
simple magnification of original neighborhood

moving ball of radius R4, still unable to fit inside the growing neighborhood

"center" of the original symplectic ball

the original symplectic ball

original neighborhood

Ball of radius R4 centered at center of original ball

Figure 4.10: Zoomed-in view of a growing neighborhood not yet large enough
Figure 4.11: Zoomed-out view of a growing neighborhood sufficiently large
For $k \in \mathbb{Z}$, define the Hamiltonian function $F_k$ by

$$F_k(\xi) = \text{bump}(\xi_1) \cdots \text{bump}(\xi_{2n-2}) \text{wedge}(\xi_{2n-1}) \text{widebump}(\xi_{2n} - k),$$

where the functions bump, wedge, and widebump are described in Detail 6. Notice that $\text{support}(F_k) \subset \text{cell}_k$ and $\|F_k\|_\infty \leq 1$. Let $\sigma_k$ be the symplectomorphism generated by $F_k$. By the construction of $F_k$, we see that $\sigma_k$ will have these three important properties:

$$\sigma_k(\text{ball}_{k-1}) = \text{ball}_k$$

$$\text{support}(\sigma_k) \subset \text{cell}_k \subset \text{tube}$$

$$E_\infty(\sigma_k) \leq E_\infty^{\text{tube}}(\sigma_k) \leq E_\infty^{\text{cell}_k}(\sigma_k) \leq \sup_{\xi \in \mathbb{R}^{2n}} |F_k(\xi)| \leq 1.$$

\textit{End of Detail 4}

\textbf{4.3.5 Detail 5: A fact about the Hofer infinity norms}

Let $\psi_i$, $i \in I$ be a finite or countable collection of Hamiltonian symplectomorphisms with $\text{support}(\psi_i) \subset U_i \subset \mathbb{R}^{2n}$, where $U_i \cap U_j = \phi$ if $i \neq j$. Further, assume that the Hofer norm of each $\psi_i$ is finite when the infimum in the Hofer norm is taken over only those Hamiltonian functions supported in $U_i$. (Hofer and Zehnder [3] have proven the same claim that we will make below, but without this additional condition. Their proof is more difficult, however. Since the weaker claim - with the additional assumption - is sufficient for us, and is easy to prove, we will state and prove it here.) Recall that this norm was introduced at the start of Lemma B4, where it was denoted by $E_\infty^U$ and $\overline{E}_\infty^U$:

$$E_\infty^U(\psi) = \inf \{ \|H\|_\infty : H \in \mathcal{C}, \text{support}(H) \subset U, \text{and } H \text{ generates } \psi \}$$

$$\overline{E}_\infty^U(\psi) = \inf \left\{ \|H\|_\infty : H \in \mathcal{H}, H \text{ generates } \psi, \text{and support}(H) \subset U \right\}$$

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Let $\psi = \prod_{i \in I} \psi_i$.

Claim: If $\max_{i \in I} \widetilde{E}^{-U_i}_\infty (\psi_i)$ exists, then $\widetilde{E}_\infty (\psi) \leq 2 \max_{i \in I} \widetilde{E}^{U_i}_\infty (\psi_i)$.

Remark: An analogous claim can be made for the norm $E_\infty$: That is, if $\max_{i \in I} E^{U_i}_\infty (\psi_i)$ exists, then $E_\infty (\psi) \leq 2 \max_{i \in I} E^{U_i}_\infty (\psi_i)$. Note, however, that this norm is defined only for symplectomorphisms of compact support. Therefore, the product $\psi = \prod_{i \in I} \psi_i$ will have to be a finite product. We will use both inequalities in this section, but will prove the result only for the extended norm, $\widetilde{E}_\infty$; The proof for the norm $E_\infty$ is identical.

Proof of claim:

Let $\varepsilon > 0$. We will show that $\widetilde{E}_\infty (\psi) \leq 2 \max_{i \in I} \widetilde{E}^{-U_i}_\infty (\psi_i) + 2 \varepsilon$. Since $\varepsilon$ is arbitrary, this will prove the claim.

By our assumption, for each $i \in I$, there is a Hamiltonian function $H_{i,t}$, supported in $U_i$, such that

$$\sup_{t \in [0,1]} \left\{ \sup_{x \in U_i} (H_{i,t}(x)) - \inf_{x \in U_i} (H_{i,t}(x)) \right\} \leq \widetilde{E}^{-U_i}_\infty (\psi_i) + \varepsilon.$$

Note that for such a function, we will have the equality

$$\| H_i \|_\infty = \sup_{t \in [0,1]} \left\{ \sup_{x \in \mathbb{R}^{2n}} (H_{i,t}(x)) - \inf_{x \in \mathbb{R}^{2n}} (H_{i,t}(x)) \right\}$$

$$= \sup_{t \in [0,1]} \left\{ \sup_{x \in U_i} (H_{i,t}(x)) - \inf_{x \in U_i} (H_{i,t}(x)) \right\} \text{ because } H_i \text{ is supported in } U_i.$$

Define the Hamiltonian function $H_t = \sum_{i \in I} H_{i,t}$. Note that the sum exists, because the supports of the various $H_{i,t}$ are disjoint, and that that $H_t$ generates $\psi$. That is, $\psi = \psi^H$. 

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Computing the extended Hofer infinity norm of $\psi$, we find

\[
\widetilde{E}_\infty(\psi) = E_\infty(\prod_{i \in I} \psi_i)
\]

\[
= \inf_{K \in H} \left\{ \|K\|_\infty : K \text{ generates } \prod_{i \in I} \psi_i \right\}
\]

\[
\leq \|H\|_\infty \text{ because } H \text{ is a particular such function}
\]

\[
= \sup_{t \in [0,1]} \left\{ \sup_{x \in \mathbb{R}^{2n}} \left( \sum_{i \in I} H_{i,t} (x) \right) - \inf_{x \in \mathbb{R}^{2n}} \left( \sum_{i \in I} H_{i,t} (x) \right) \right\} \text{ (definition of } H_t) \]

\[
= \sup_{t \in [0,1]} \left\{ \sup_{i \in I} \left\{ \sup_{x \in U_i} (H_{i,t} (x)) \right\} - \inf_{i \in I} \left\{ \inf_{x \in U_i} (H_{i,t} (x)) \right\} \right\}
\]

(because the supports of the Hamiltonian functions are disjoint.)

\[
\leq \sup_{t \in [0,1]} \left\{ \sup_{i \in I} \left\{ \sup_{x \in U_i} (H_{i,t} (x)) - \inf_{x \in U_i} (H_{i,t} (x)) \right\}
\right.

\[
- \inf_{i \in I} \left\{ \inf_{x \in U_i} (H_{i,t} (x)) - \sup_{x \in U_i} (H_{i,t} (x)) \right\}
\}
\]

(subtracted something non-positive, added something non-negative.)

\[
= \sup_{t \in [0,1]} \left\{ \sup_{i \in I} \left\{ \sup_{x \in U_i} (H_{i,t} (x)) - \inf_{x \in U_i} (H_{i,t} (x)) \right\}
\right.

\[
+ \inf_{i \in I} \left\{ \sup_{x \in U_i} (H_{i,t} (x)) - \inf_{x \in U_i} (H_{i,t} (x)) \right\}
\}
\]

(just reversed the subtraction.)

\[
\leq \sup_{t \in [0,1]} \left\{ 2 \sup_{i \in I} \left\{ \sup_{x \in U_i} (H_{i,t} (x)) - \inf_{x \in U_i} (H_{i,t} (x)) \right\} \right\}
\]

\[
= 2 \sup_{i \in I} \left\{ \sup_{t \in [0,1]} \left\{ \sup_{x \in U_i} (H_{i,t} (x)) - \inf_{x \in U_i} (H_{i,t} (x)) \right\} \right\}
\]

\[
= 2 \sup_{i \in I} \left\{ \|H_i\|_\infty \right\}
\]

\[
\leq 2 \sup_{i \in I} \left\{ \widetilde{E}_\infty^{U_i} (\psi_i) + \varepsilon \right\}
\]

\[
= 2 \sup_{i \in I} \left\{ \widetilde{E}_\infty^{U_i} (\psi_i) \right\} + 2\varepsilon.
\]

End of proof of claim

End of Detail 5
4.3.6 **Detail 6: Four useful functions of one variable.**

For $0 < R < \frac{1}{2}$, we define $R^+ = \frac{R + \frac{1}{2}}{2} = \frac{R}{2} + \frac{1}{4}$. (We just need $R^+$ to be a number between $R$ and $\frac{1}{2}$.) Figure 4.12 shows four functions of one variable that are used throughout this paper.
Bibliography


