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Abstract of the Dissertation
Analytic torsion and Faddeev-Popov ghosts

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The regularized determinant of the Laplacian on n-differentials on a hyperbolic Riemann surface is studied. The main result is an intrinsic characterization of the connection form for the determinant line bundle, endowed with the Quillen metric, over the Teichmüller space, in terms of the Green's function of the Cauchy-Riemann operator. Further, an explicit series representation of that Green's function, on a Schottky uniformization of the surface, is established. This is a rigorous version of physical heuristics due to Martinec and Verlinde & Verlinde, relating the determinant to the stress-energy tensor of Faddeev-Popov ghost fields on the Riemann

surface. One corollary is a simpler proof of the rigorous hyperbolic Belavin-Knizhnik formula, due to Zograf and Takhtajan, which is an intrinsic characterization of the curvature form of the determinant line bundle with Quillen metric. Another corollary is a proof of an explicit holomorphic factorization formula for $n = 1$ and genus greater than 1, due to Zograf, which generalizes the well-known formula for $n = 1$ and genus 1 relating the determinant of the Laplacian to the Dedekind eta function.

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But I will not try since to do so properly would double the length of this work.

Chapter 1

Introduction

1.1 History

Let Δ_n be the Laplacian acting on n-differentials $\phi(z)(\mathrm{d}z)^n$ on a compact Riemann surface X of genus $g \geq 1$ equipped with a Riemannian metric $\rho(z)|\mathrm{d}z|^2$, and let $\{\lambda_j\}_{j=1}^{\infty}$ be its positive eigenvalues. (See the next chapter for definitions.) Formally, if we form the function $\zeta(s) = \sum \lambda_j^{-s}$, then $-\zeta'(0) = \sum \log \lambda_j = \log \det \Delta_n$. Of course, the two equality signs are meaningless, but $-\zeta'(0)$ is not meaningless since $\zeta(s)$ has a meromorphic continuation to the whole s-plane, with a single simple pole at $s = 1$ [MP49]. Hence the regularized determinant of the Laplacian is defined to be $\exp(-\zeta'(0))$. (Ray and Singer [RS73] studied the regularized determinant of the Laplacian acting on differential forms with values in a unitary line bundle, on a Kähler manifold of any dimension; taking a graded sum gives the analytic torsion.)

These functions are of interest in string theory. In [Pol81], Polyakov showed how to perform a "sum over histories" for the bosonic free string (the simplest case). A string is produced from the vacuum, perhaps splits into two or more pieces, rejoins, and eventually pops out of existence. The graph of this process versus time is a metric surface. To calculate the "vacuum to vacuum amplitude", one must sum over all ways this can happen (over all metric surfaces), with an appropriate weighting factor. Polyakov showed that the weighting factor is of the form

$$W = \frac{\det \Delta_{-1}}{\det N_{-1} \det N_2} \left(\frac{\det \Delta_0}{\det N_0 \det N_1} \right)^{-D/2},$$

where D is the dimension of the spacetime and N_n is the Gram matrix (ϕ_j, ϕ_k) of inner products for some choice $\{\phi_j\}_{j=1}^d$ of basis for $\ker \Delta_n$. (By convention $\det N_n = 1$ when $\ker \Delta_n = \{0\}$.) This process is much more tractable if W can be made invariant under conformal changes of the metric, since then W descends to a function on the Teichmüller space T_g, so the "summation" may be performed over a finite-dimensional space. To this end, Polyakov calculates the "conformal anomaly":

$$\frac{\partial}{\partial \epsilon}\Big|_{\epsilon=0} \log \left(\frac{\det \Delta_n}{\det N_n \det N_{1-n}} \right) = -\frac{6n^2 - 6n + 1}{12\pi} \int_X \rho^{-1}(\delta\rho) R,$$

for a family of metrics $\big(\rho(z) + \epsilon(\delta\rho(z))\big)|\mathrm{d}z|^2$. R is the Gaussian curvature of the metric. From this it follows that the conformal dependence cancels in W exactly when $D = 26$.

It is then natural to ask if W has any structure with respect to the complex structure on T_g. Naively, one might hope that the property $\det AB = \det A \det B$ holds for regularized determinants, at least in this case; we could then use $\Delta_n = \overline{\partial}_n^* \overline{\partial}_n$ to factor the determinant as $\det \Delta_n = |\det \overline{\partial}_n|^2$, which would have the great advantage that, since $\overline{\partial}_n = \overline{\partial}$ does not depend on the metric, and since it depends holomorphically on the Teichmüller coordinates (see the next chapter for the precise meaning of that statement), $\det \overline{\partial}_n$ should presumably be a *holomorphic* function on T_g. The first problem is that we have thrown out zero eigenvalues in our regularized determinants, but this is easily remedied: the appropriate thing to do is divide by $\det N_n \det N_{1-n}$ as we have done above (see [Qui85] or [VV87] for the reasoning). Then the conjectured formula should be

$$\frac{\det \Delta_n}{\det N_n \det N_{1-n}} = |F_n|^2$$

for some choice of metric in each conformal class, and for some holomorphic function $F_n : T_g \to \mathbb{C}$. Belavin and Knizhnik showed that this is not quite true, by calculating the "holomorphic anomaly" [BK86]:

$$\frac{\partial}{\partial \overline{\epsilon}} \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \log \left(\frac{\det \Delta_n}{\det N_n \det N_{1-n}} \right) = \frac{6n^2 - 6n + 1}{12\pi} \int_X |\mu|^2 R\rho, \tag{1.1}$$

for a family of metrics $\rho(z)|\mathrm{d}z + \epsilon\mu \mathrm{d}\overline{z}|^2$, with μ a harmonic Beltrami differential on X. However, again when $D = 26$, this anomaly cancels in W so that

$$\frac{\partial}{\partial \overline{\epsilon}} \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \log W(\epsilon) = 0$$

and hence locally

$$W(\epsilon) = |G(\epsilon)|^2$$

where G is some holomorphic function of ϵ. This suggests that

$$W = |G|^2$$

for some holomorphic function $G : T_g \to \mathbb{C}$, which was later shown to be correct.

Since the dependence of $\dfrac{\det \Delta_n}{\det N_n \det N_{1-n}}$ on conformal changes of the metric is relatively well understood, it is reasonable to restrict to metrics of constant curvature -1 (or 0 when $g = 1$), and in this way consider $\det \Delta_n$ to be a function on T_g. We will do this for the remainder of the introduction (and throughout the rest of the thesis). The corresponding version of equation (1.1) should give some information about the geometry of T_g. The model for this is the paper [Qui85], in which Quillen studies the simpler case of the family of all Cauchy-Riemann operators D on a fixed vector bundle E over a fixed Riemann surface X. This is simpler because X is fixed, and because the parameter space $\mathcal{A}$ is linear (an affine space based on the End E-valued $(0, 1)$ forms on X). He shows that

$$\bar{\partial}\partial \log \frac{\det D^* D}{\det N \det N^*}$$

coincides with the natural Kähler form on $\mathcal{A}$ (see [Qui85] for a precise statement). Further, he shows that

$$\det D^* D = \exp(-\|D - D_0\|^2)|F(D, D_0)|^2$$

for a holomorphic function $F : \mathcal{A} \to \mathbb{C}$ and a choice of basepoint $D_0 \in \mathcal{A}$ (again, see [Qui85] for the exact theorem).

The case of $\det \Delta_n$ for constant curvature metrics is considerably more subtle, but Takhtajan and Zograf found that an analogous result holds [ZT87a]:

$$\partial\overline{\partial}\log\left(\frac{\det\Delta_n}{\det N_n \det N_{1-n}}\right)(\mu,\nu) = \frac{6n^2-6n+1}{12\pi}\int_X \mu\overline{\nu}\rho \tag{1.2}$$

for harmonic Beltrami differentials μ and ν on X; that is, the $(1,1)$ differential form $\dfrac{\det\Delta_n}{\det N_n \det N_{1-n}}$ on T_g is, up to a constant factor, the Kähler form of the Weil-Petersson metric on T_g. Further, guided by physical heuristics regarding this formula [Pol81], they were able to find a potential for the Weil-Petersson metric on T_g, that is, a function $S: T_g \to \mathbb{R}_{>0}$ (the "classical Liouville action") satisfying $\partial\overline{\partial}S = -2i\omega_{WP}$ [ZT87b]. S is given as an explicit integral expression over a Schottky uniformization of the surface. In fact, S is well-defined on the Schottky space S_g, which is a quotient of T_g.

The result (1.2) suggests that, at least locally,

$$\frac{\det\Delta_n}{\det N_n \det N_{1-n}} = \exp\left(-\frac{6n^2-6n+1}{12\pi}S\right)|F_n|^2$$

for some holomorphic function $F_n : U \subset T_g \to \mathbb{C}$ and some appropriate choice of basis for $\ker\Delta_n$ (or $\ker\Delta_{1-n}$). In the case $n=1$, Zograf was able to show that this is correct, and in fact holds true globally: [Zog89]:

$$\frac{\det\Delta_1}{\det N_1 \det N_0} = \exp\left(-\frac{1}{12\pi}S\right)|F_1|^2 \tag{1.3}$$

where the basis for Abelian differentials is chosen to be normalized in the usual way. The holomorphic function $F_1 : T_g \to \mathbb{C}$ is defined globally, and in fact it descends to a holomorphic function on S_g.

Zograf was able to go further, and find an explicit product formula for

F_1. There already existed an explicit product formula for $\det \Delta_n$, when $g > 1$ [DP86], [Vor87], [Sar87]:

$$\det \Delta_n = \begin{cases} c_{n,g} Z(n) & : n \geq 2 \\ c_{1,g} Z'(1) & : n = 1 \end{cases}$$

where

$$Z(n) = \prod_{[\gamma]} \prod_{m=0}^{\infty} (1 - \lambda_\gamma^{m+n})$$

is the Selberg zeta function. $[\gamma]$ runs over all primitive conjugacy classes (except the identity) in a Fuchsian group uniformizing X, and λ_γ is the multiplier of γ, that is, the unique complex number such that γ is conjugate to multiplication by λ_γ and $0 < |\lambda_\gamma| < 1$. This does not address the relation of $\det \Delta_n$ to the complex structure of T_g, though, since $Z(n)$ is only real-analytic and has no obvious relation to the complex structure. A better clue comes from the case $g = 1$, which (in different language) is due to Kronecker (see[Lan87]):

$$\det \Delta_n = (\mathrm{Im}\tau)^2 |\eta(\tau)|^4$$

where

$$\eta(\tau) = \lambda^{\frac{1}{24}} \prod_{m=1}^{\infty} (1 - \lambda^m)$$

is the classical Dedekind η-function, with $\tau \in \mathbb{H} \simeq T_1$ and $\lambda = e^{2\pi i \tau}$. Note that η is similar to the Selberg zeta function, except that λ is the multiplier for a *Schottky* group uniformizing the torus (it has one generator, and its fundamental domain is an annulus). Zograf found an analogue F_1 of the Dedekind zeta function for genus $g > 2$ that makes (1.3) true:

$$F_1 = \prod_{[\gamma]} \prod_{m=0}^{\infty} (1 - \lambda_\gamma^{m+1}), \tag{1.4}$$

where the product now goes over primitive conjugacy classes in a *Schottky* group uniformizing X! It is evident from this expression that F_1 is indeed holomorphic and well defined on S_g. The product only converges on a proper open subset of S_g (for those groups whose exponent of convergence is strictly less than 1), but (1.3) shows that there is an analytic continuation to all of S_g.

1.2 This thesis

The guiding problem for this work is to extend Zograf's results (1.3) and (1.4) to $n \geq 2$. This problem splits into three parts:

1. Find an intrinsic characterization of $\partial \log \dfrac{\det \Delta_n}{\det N_n \det N_{1-n}} \in T^*_{[X]} T_g$, in terms of the Riemann surface X associated to the point $[X] \in T_g$. Further, express it, up to certain known terms, in a Poincaré series with respect to a Schottky group uniformizing X.

2. Find an appropriate global normalization for a basis of holomorphic n-differentials.

3. Find a product formula for F_n.

To give a more specific idea of what is asked for in problem 1, note that the formula (1.2), together with the construction of the Weil-Petersson potential S, constitute a solution of this problem for $\partial\overline{\partial} \log \dfrac{\det \Delta_n}{\det N_n \det N_{1-n}} \in T^*_{[X]} T_g \otimes \overline{T^*_{[X]} T_g}$.

For $n = 1$, the solution to problem 1 was given in [ZT87a]:

$$\partial \log \frac{\det \Delta_n}{\det N_n \det N_{1-n}} = \frac{1}{6\pi} \left(R^F - R^B \right) \tag{1.1}$$

where R^F is the projective connection associated to the Fuchsian uniformization of X, and R^B is the projective connection associated to the classical Bergman kernel $B(w, w')$ on X,

$$R^B(w) = 6 \lim_{w' \to w} \left(B(w, w') - \frac{1}{\pi} \frac{1}{(w - w')^2} \right)$$

(see [ZT87a] for definitions). The equality in (1.1) means that the $(1, 0)$ form on T_g on the left is represented by the quadratic differential on X on the right. The kernel $B(w, w')$ has an explicit representation as a Poincaré series with respect to a Schottky group uniformizing X:

$$B(w, w') = \frac{1}{\pi} \sum_{\gamma} \frac{1}{(w - \gamma w')^2} \gamma'(w'). \tag{1.2}$$

The main results of this thesis are generalizations of (1.1) and (1.2) to $n \geq 2$, giving a complete solution to problem 1. The intrinsic characterization generalizing (1.1) is theorem 4.1, where the Bergman kernel is replaced by the Green's function of $\overline{\partial}_n$. The series representation of the Green's function, generalizing (1.2), is given in theorems 3.2 and 3.3. The appearance of the Weil-Petersson potential S is explained in theorem 5.1.

The idea for the statement of the main results comes from the physical heuristics developed by Verlinde & Verlinde, who used a functional integral to assert that $\partial \log \dfrac{\det \Delta_n}{\det N_n \det N_{1-n}}$ is given by a "b-c stress energy tensor" [VV87], and Martinec, who gave a computation of the b-c stress-energy tensor on a Schottky uniformization, up to certain undetermined "zero-mode" terms [Mar87]. We determine the missing zero-mode terms in [Mar87] and make rigorous the constructions given there, and show directly that the b-c stress energy tensor equals $\partial \log \dfrac{\det \Delta_n}{\det N_n \det N_{1-n}}$, bypassing the functional integral (and hence giving a rigorous justification for it).

Further, Martinec gives a rough calculation (again, not rigorously and up to some undetermined terms, in particular, ignoring problem 2) which indicates that the function F_n is what one would expect from the above discussion [Mar87]:

$$F_n = \prod_{[\gamma]} \prod_{m=0}^{\infty} (1 - K_\gamma^{m+n}).$$

The solutions to problems 2 and 3 do not appear in this work but will be addressed in a later publication.

Chapter 2

Background

In this chapter, we fix notations and definitions for use in the rest of the paper, and collect some results we will need. Everything in this chapter is known.

2.1 Riemann surfaces, bundles, differential operators

Throughout, X will denote a compact Riemann surface of genus $g > 1$, endowed with its unique Poincaré metric of constant curvature -1, written locally as $\mathrm{d}s^2 = \rho(z)|\mathrm{d}z|^2$, $\rho > 0$. Let K be the holomorphic cotangent bundle of X, and let $\mathcal{D}^{p,q}(K^n \otimes \overline{K}^m)$ be the space of smooth differential forms of type (p, q) on X with values in $K^n \otimes \overline{K}^m$. An (n, m)-differential (abbreviated to n-differential when $m = 0$) is an element of $\mathcal{D}^{0,0}(K^n \otimes \overline{K}^m)$; symbolically, it is an expression of the form $\phi(z)(\mathrm{d}z)^n(\mathrm{d}\overline{z})^m$. Note that $\mathcal{D}^{p,q}(K^n \otimes \overline{K}^m) \simeq \mathcal{D}^{0,0}(K^{n+p} \otimes \overline{K}^{m+q})$.

The metric on X induces a pointwise Hermitian metric $\langle f, g\rangle = \rho^{-n-p-q} f\overline{g}$ and a global Hermitian metric $(f, g) = \int_X \rho^{1-n-p-q} f\overline{g}$ on $\mathcal{D}^{p,q}(K^n)$. (Here and in the sequel we abbreviate, for example,

$$\int_X \rho^{1-n-p-q} f\overline{g} = \int_X \rho(z)^{1-n-p-q} f(z)\overline{g(z)}\mathrm{d}^2 z,$$

where z is any local coordinate (summing over partitions of unity if necessary) and

$$\mathrm{d}^2 z = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\overline{z}$$

is the usual plane measure.)

There is a unique connection

$$D = \partial_n \oplus \overline{\partial}_n : \mathcal{D}^{0,0}(K^n) \to \mathcal{D}^{1,0}(K^n) \oplus \mathcal{D}^{0,1}(K^n)$$

compatible with the metric and complex structure; it is given by

$$\begin{aligned} \overline{\partial}_n = \overline{\partial} : \mathcal{D}^{0,0}(K^n) \to \mathcal{D}^{0,1}(K^n) \simeq \mathcal{D}^{0,0}(K^n \otimes \overline{K}) \\ \text{and} \quad \partial_n = \rho^n \partial \rho^{-n} : \mathcal{D}^{0,0}(K^n) \to \mathcal{D}^{1,0}(K^n) \simeq \mathcal{D}^{0,0}(K^{n+1}), \end{aligned}$$

where $\overline{\partial} = \partial/\partial\overline{z}$ and $\partial = \partial/\partial z$.

With respect to the Hermitian metric, the operator $\overline{\partial}_n : \mathcal{D}^{0,0}(K^n) \to \mathcal{D}^{0,1}(K^n)$ has adjoint $\overline{\partial}_n^* = -\rho^{-1}\partial_n : \mathcal{D}^{0,1}(K^n) \to \mathcal{D}^{0,0}(K^n)$, so the $\overline{\partial}$-Laplacian $\Delta = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$ is given by

$$\begin{aligned} \Delta_n^{0,0} = \overline{\partial}_n^*\overline{\partial}_n = -\rho^{-1}\partial_n\overline{\partial}_n : \mathcal{D}^{0,0}(K^n) \to \mathcal{D}^{0,0}(K^n) \\ \Delta_n^{0,1} = \overline{\partial}_n\overline{\partial}_n^* = -\overline{\partial}_n\rho^{-1}\partial_n : \mathcal{D}^{0,1}(K^n) \to \mathcal{D}^{0,1}(K^n). \end{aligned}$$

We abbreviate $\Delta_n = \Delta_n^{0,0}$.

Since Δ_n is self-adjoint, its eigenvalues are nonnegative real, and since X is compact, Hodge theory implies that they form a discrete set accumulating only at infinity. Further, each eigenspace is finite dimensional, and the L^2 closure of $\mathcal{D}^{0,0}(K^n)$ equals the direct sum of the eigenspaces.

The nonzero spectrum of Δ_{1-n} is identical to that of Δ_n, since $\Delta_n^{0,0} = \overline{\partial}_n^* \overline{\partial}_n$ and $\Delta_n^{0,1} = \overline{\partial}_n \overline{\partial}_n^*$ have the same nonzero spectrum, as does $\overline{\Delta_n^{0,1}}$, and

$$\Delta_{1-n}^{0,0} = \rho^{-n} \overline{\Delta_n^{0,1}} \rho^n.$$

Here $\overline{\Delta_n^{0,1}}$ means

$$\overline{\Delta_n^{0,1}} = -\partial \rho^{n-1} \overline{\partial} \rho^{-n} : \mathcal{D}^{1,0}(\overline{K}^n) \to \mathcal{D}^{1,0}(\overline{K}^n);$$

it has the same spectrum as $\Delta_n^{0,1}$ since the eigenvalues are real.

Write Δ_n^0 for the restriction of Δ_n to the orthogonal complement of $\ker \Delta_n$, and write $\overline{\partial}_n^0$ for the restriction of $\overline{\partial}_n$ to the same space. Then, for $n \geq 2$,

$$(\Delta_n^0)^{-1} \overline{\partial}_n^* = (\overline{\partial}_n^0)^{-1}.$$

2.2 Fuchsian and Kleinian groups

See [Ber72], [Ber81] for the material in this and the next section. The Riemann surface X may be realized as a quotient space of the upper half plane $\mathbb{H} = \{z = x + iy | y > 0\}$ by a Fuchsian group Γ, that is, a discrete group of hyperbolic linear fractional transformations with real coefficients. The hyperbolic metric $y^{-2}|\mathrm{d}z|^2$ then induces the Poincaré metric ρ on X. The space of (n,m)-automorphic forms is the space of smooth functions $\phi : \mathbb{H} \to \mathbb{C}$ satisfying

$$\phi(\gamma z)\gamma'(z)^n\overline{\gamma'(z)}^m = \phi(z)$$

for all $z \in \mathbb{H}$ and for all$\gamma \in \Gamma$; it is isomorphic to the space of (n,m)-differentials on X and the two spaces will be implicitly identified in the sequel.

More generally, X may be realized as the quotient of an open set $\Omega_0 \subset \mathbb{C}$ by a Kleinian group Σ, in the following sense. Suppose $\Sigma \subset \mathrm{PSL}(2,\mathbb{C})$ is a discrete group of linear fractional transformations. The limit set $\Lambda \subset \mathbb{C}$ of Σ is the set of accumulation points of orbits of Σ, and the ordinary set $\Omega \subset \mathbb{C}$ of Σ is the complement of the limit set. The group Σ is called a Kleinian group if Ω is nonempty, and it is called a function group if Ω has at least one nonempty connected component $\Omega_0 \subset \Omega$ invariant under the action of Σ. We say that X is *uniformized* by a function group Σ if $X \simeq \Omega_0/\Sigma$ for such an invariant component Ω_0. In this case the space of (n,m)-differentials on X is isomorphic to the space of smooth functions $\phi : \Omega_0 \to \mathbb{C}$ satisfying

$$\phi(\gamma w)\gamma'(w)^n\overline{\gamma'(w)}^m = \phi(w)$$

for all $w \in \Omega_0$ and for all $\gamma \in \Sigma$.

Of particular importance will be the case where Σ is a Schottky group, that is, a free Kleinian group, all of whose elements (except the identity element) are loxodromic (i.e. not elliptic or parabolic). In this case, $\Omega = \Omega_0$ has only one component. A fundamental domain may be found which is bounded by $2g$ Jordan curves $\{C_j, C_j'\}$, such that the g generators $\{\gamma_j\}$ of Σ satisfy

$\gamma_j(C_j) = -C'_j$. A classical theorem (the retrosection theorem) asserts that any Riemann surface X may be realized in this manner.

Suppose X is uniformized by both a Fuchsian group Γ and a Kleinian group Σ. By the uniformization theorem, there is a holomorphic, surjective, locally invertible map $J : \mathbb{H} \to \Omega_0$, such that $\gamma \circ J \in \Gamma$ if and only if $\gamma \in \Sigma$.

Naturally, (n, m)-differentials may be pulled back by J: given an (n, m)-automorphic form ϕ^Σ with respect to Σ on Ω_0, define an (n, m)-automorphic form ϕ^Γ with respect to Γ on $\mathbb{H}$ by

$$\phi^\Gamma(z) = \phi^\Sigma(J(z))J'(z)^n\overline{J'(z)}^m.$$

Less obviously, (n, m)-differentials may be pushed forward by J. Given ϕ^Γ an (n, m)-automorphic form with respect to Γ, define ϕ^Σ by

$$\phi^\Sigma(J(z)) = \frac{\phi^\Gamma(z)}{J'(z)^n\overline{J'(z)}^m}.$$

To show this is well-defined, let $J(\tilde{z}) = J(z)$ for some $\tilde{z}, z \in \mathbb{H}$. Then $\tilde{z} = \gamma z$ for some $\gamma \in \Gamma$ such that $J \circ \gamma = J$, and hence

$$\begin{aligned}\phi^\Sigma(J(\tilde{z})) &= \frac{\phi^\Gamma(z)}{\gamma'(z)^n\overline{\gamma'(z)}^m} \cdot \frac{1}{J'(\gamma z)^n\overline{J'(\gamma z)}^m} \\ &= \frac{\phi^\Gamma(z)}{(J\gamma)'(z)^n\overline{(J\gamma)'(z)}^m} \\ &= \frac{\phi^\Gamma(z)}{J'(z)^n\overline{J'(z)}^m} = \phi^\Sigma(J(z)).\end{aligned}$$

When there is no risk of confusion, we will omit the superscripts and let the pullback or pushforward be implied by the domain of the variable: for $z \in \mathbb{H}$

and $w \in \Omega_0$, $\phi(z) = \phi^\Gamma(z)$, while $\phi(w) = \phi^\Sigma(w)$.

2.3 Teichmüller and Schottky spaces

Let T_g be the Teichmüller space of compact, marked (i.e. with a distinguished standard basis $\{A_1, \ldots, A_g, B_1, \ldots, B_g\}$ of $\pi_1(X)$) Riemann surfaces of genus $g > 1$. This is realized as the Teichmüller space of a corresponding Fuchsian group with a distinguished set of standard generators. T_g has a natural structure of a complex manifold of dimension $3g-3$. Its holomorphic tangent space, at a point $[X]$ representing a surface X, is isomorphic to the space of harmonic Beltrami differentials on X, i.e. $(-1,1)$-differentials μ satisfying $\Delta^{0,1}_{-1}\mu = 0$. The dual holomorphic cotangent space of T_g consists of the holomorphic quadratic differentials on X, i.e. $(2,0)$-differentials ϕ satisfying $\overline{\partial}\phi = 0$.

Given a basepoint $[X] \in T_g$, we can give coordinates (Bers coordinates) for a neighbourhood of $[X]$, also parametrized by harmonic Beltrami differentials, as follows. Given a harmonic Beltrami differential μ on X satisfying $\|\mu\|_\infty = \sup_{z\in\mathbb{H}}|\mu(z)| < 1$, there exists a unique homeomorphism $f^\mu : \mathbb{H} \to \mathbb{H}$ fixing $0, 1, \infty$ and satisfying $\overline{\partial} f^\mu = \mu \partial f^\mu$. Set $\Gamma^\mu = f^\mu \Gamma (f^\mu)^{-1}$ and $X^\mu = \mathbb{H}/\Gamma^\mu$. Choosing a basis $\mu_1, \ldots, \mu_{3g-3}$, $\mu = \sum \epsilon^j \mu_j$, the $\epsilon^j \in \mathbb{C}$ form coordinates of T_g in a neighbourhood of $[X]$.

There is a canonical complex fibration $\pi : \mathcal{T}_g \to T_g$, (the Teichmüller universal curve), such that π is a holomorphic submersion and the fibre $\pi^{-1}([X])$ is isomorphic to the surface X. Over $\mathcal{T}_g$, there is a natural holomorphic line bundle, the vertical tangent bundle $T_V\mathcal{T}_g \to \mathcal{T}_g$, consisting of vectors in $T\mathcal{T}_g$

that are tangent to the fibres $\pi^{-1}([X])$. We can then define a family ϕ^ϵ of (n,m)-differentials on $X^{\epsilon\mu}$ to be a section of the bundle

$$((T_V \mathcal{T}_g)^*)^n \otimes (\overline{(T_V \mathcal{T}_g)^*})^m \to \mathcal{T}_g.$$

Instead of considering the deformation space of a marked Fuchsian group representing X, we can consider the deformation space of a marked Schottky group (with g free generators distinguished) representing X. This corresponds to fixing a normal subgroup $\mathcal{N} \subset \pi_1(X)$ generated by a set of "B-cycles", that is, the set $\{B_1, \ldots, B_g\}$ in a standard basis of $\pi_1(X)$. In this way we obtain the Schottky space S_g of Schottky-marked Riemann surfaces of genus g. S_g is a quotient space of T_g. All of the constructions described above work analogously for S_g; see [Ber75].

2.4 Zero modes

Suppose $n \geq 1$. The kernel of Δ_n consists of the holomorphic n-differentials on X, the space of which has finite dimension $d(n)$ ($d(n) = (2n-1)(g-1)$ for $n > 1$ and $d(1) = g$). A basis of the holomorphic n-differentials can actually be chosen, holomorphically and simultaneously, for all Riemann surfaces of a fixed genus; that is, there exist $d(n)$ global holomorphic sections $\phi_1^\epsilon, \ldots, \phi_{d(n)}^\epsilon$ of the bundle $((T_V \mathcal{T}_g)^*)^n \to \mathcal{T}_g$, which form a basis of $\ker \Delta_n$ on X when restricted to $\pi^{-1}([X])$ [Ber66]. When $n = 1$, the basis may be chosen so that it is normalized in the usual manner, with respect to the marking of X [Ber66].

Note that this shows that the vector bundle $\operatorname{ind} \overline{\partial}_n = \ker \overline{\partial}_n = \ker \Delta_n$ over

T_g is holomorphically trivial, as is the line bundle $\det \operatorname{ind} \overline{\partial}_n = \Lambda^{d(n)} \operatorname{ind} \overline{\partial}_n$. The Hermitian metric on n-differentials induces a metric on these line bundles. The metric on $\det \operatorname{ind} \overline{\partial}_n$ is $\det N_n$, where $[N_n]_{jk} = (\phi_j, \phi_k)$ is the Gram matrix of the chosen basis of zero modes.

2.5 Variational formulas

We collect here some variational formulas which we will need later. Define the pullback of an (n, m)-differential ϕ^ϵ over $X^{\epsilon\mu}$ to an (n, m)-differential over X by

$$f_*^{\epsilon\mu}(\phi^\epsilon) = \phi^\epsilon \circ f^{\epsilon\mu} \cdot (\partial f^{\epsilon\mu})^n (\overline{\partial f^{\epsilon\mu}})^m,$$

where $f^{\epsilon\mu} : \mathbb{H} \to \mathbb{H}$ as defined above. Using the pullback, we can define Lie derivatives of ϕ^ϵ in the directions μ and $\overline{\mu}$:

$$\begin{aligned} \delta_\mu \phi &= \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} f_*^{\epsilon\mu}(\phi^\epsilon), \\ \overline{\delta_\mu} \phi &= \frac{\partial}{\partial \overline{\epsilon}}\bigg|_{\epsilon=0} f_*^{\epsilon\mu}(\phi^\epsilon). \end{aligned}$$

Similarly, for a family of operators A^ϵ taking (n, m)-differentials to (k, l)-differentials, we define the Lie derivatives by

$$\begin{aligned} \delta_\mu A &= \frac{\partial}{\partial \epsilon}\bigg|_{\epsilon=0} \left(f_*^{\epsilon\mu} A^\epsilon (f_*^{\epsilon\mu})^{-1} \right) \\ \overline{\delta_\mu} A &= \frac{\partial}{\partial \overline{\epsilon}}\bigg|_{\epsilon=0} \left(f_*^{\epsilon\mu} A^\epsilon (f_*^{\epsilon\mu})^{-1} \right). \end{aligned}$$

If $G : T_g \to \mathbb{C}$, then G may be naturally identified with a family of constant

0-differentials, and then the Lie derivative coincides with the usual derivative:

$$\delta_\mu G = \partial G(\mu)$$
$$\overline{\delta}_\mu G = \overline{\partial} G(\mu),$$

where ∂ and $\overline{\partial}$ are the $(1,0)$ and $(0,1)$ components, respectively, of the exterior differential d on T_g.

We collect here, without proof, some formulas we will need. Let μ be a harmonic Beltrami differential on X, representing an element of $T_{[X]}T_g$. If we let

$$F_\mu = \frac{\partial}{\partial \epsilon} f^{\epsilon\mu}|_{\epsilon=0}, \quad \Phi_\mu = \frac{\partial}{\partial \overline{\epsilon}} f^{\epsilon\mu}|_{\epsilon=0},$$

then we have [Ahl61]

$$\overline{\partial} F_\mu = \mu$$
$$\text{and} \quad \Phi'''_\mu(z) = -\frac{1}{2} y^{-2} \overline{\mu(z)}.$$

If ρ is the Poincaré metric, considered as a family of $(1,1)$-differentials, its first derivatives on T_g vanish [Ahl61]:

$$\delta_\mu \rho = \overline{\delta_\mu} \rho = 0.$$

Using this result and the chain rule, we can calculate

$$\delta_\mu \overline{\partial}_n = -\mu \partial_n \qquad \overline{\delta_\mu}\, \overline{\partial}_n = 0$$

$$\delta_\mu \partial_n = 0 \qquad \overline{\delta_\mu} \partial_n = -\overline{\mu} \overline{\partial}_n.$$

Using these formulas, we may then derive:

$$\delta_\mu \Delta_n = \rho^{-1} \mu \partial_{n+1} \partial_n$$

$$\overline{\delta_\mu} \Delta_n = \overline{\mu} \overline{\partial}_{n-1} \rho^{-1} \overline{\partial}_n.$$

We will also need a formula for the second variation of the metric [Wol86],

$$\overline{\delta}_\nu \delta_\mu \rho = \frac{1}{2} \rho \left(\Delta_0 + \frac{1}{2} \right)^{-1} (\mu \overline{\nu}),$$

and a formula for the Lie derivative of a vector field [Wol86],

$$\overline{\delta}_\nu \mu = -\overline{\partial} \rho^{-1} \overline{\partial} \left(\Delta_0 + \frac{1}{2} \right)^{-1} (\mu \overline{\nu});$$

consequently we have

$$\Delta_0(\rho^{-1} \overline{\delta}_\nu \delta_\mu \rho) = \frac{1}{2} (\mu \overline{\nu} - (\rho^{-1} \overline{\delta}_\nu \delta_\mu \rho))$$

and

$$\overline{\delta}_\nu \mu = -2 \overline{\partial} \rho^{-1} \overline{\partial} (\rho^{-1} \overline{\delta}_\nu \delta_\mu \rho)$$

Note that if ϕ is a holomorphically varying family of holomorphic n-differentials (for example, the elements of the basis chosen in the previous section), differentiating $\overline{\partial}\phi = 0$ yields

$$\overline{\partial}(\delta_\mu \phi) = \mu \partial_n \phi.$$

Finally, we will need a formula for the variation of the period matrix. Let $\{\phi_1^\epsilon, \ldots, \phi_g^\epsilon\}$ be a basis for the space of holomorphic 1-differentials on $X^{\epsilon\mu}$, normalized in the usual manner $\int_{A_j} \phi_k$ with respect to the marking of $X^{\epsilon\mu}$, and let $[N_1]_{jk} = (\phi_j, \phi_k)$. Then [Rau65]

$$\partial [N_1]_{jk} = -\phi_j \phi_k,$$

under the identification of $T_{[X]}T_g$ with the space of holomorphic $(2,0)$-differentials on X.

2.6 Green's function of Δ_n

The Green's operator for Δ_n is an operator $G_n : \mathcal{D}^{0,0}(K^n) \to \mathcal{D}^{0,0}(K^n)$ such that $\Delta_n G_n = G_n \Delta_n = I - P_n$, where P_n is the orthogonal projection onto holomorphic n-differentials, and such that $G_n P_n = 0$. (P_n is zero when $n < 0$.) For a realization of $X = \Omega_0/\Sigma$ by a Kleinian group Σ, the Green's function for Δ_n is the integral kernel for this operator, that is, a function $G_n(w, w')$, $w, w' \in \Omega_0$, smooth for $w \neq w'$, which is an $(n, 0)$-differential in w and a $(1-n, 1)$-differential in w', such that $G_n \phi(w) = \int_{\mathcal{D}} G_n(w, w') \phi(w') \mathrm{d}^2 w'$, where $\mathcal{D}$ is a fundamental domain for Σ, and $\mathrm{d}^2 w = \frac{i}{2} \mathrm{d}w \wedge \mathrm{d}\overline{w}$ is the usual volume form. The kernel $P_n(w, w')$ for the projection P_n is defined in the same way.

Fixing a basis $\{\phi_1, \ldots, \phi_d\}$ for the holomorphic n-differentials, we can write

$$P_n(w, w') = \sum_{j=1}^{d} \sum_{k=1}^{d} [N_n^{-1}]_{kj} \phi_j(w) \phi_k(w').$$

The Green's function $Q_n(z, z')$ for Δ_n on the upper half plane $\mathbb{H}$ is uniquely determined by the following properties:

1. $Q_n(z, z')$ is smooth for $z \neq z'$;
2. $Q_n(\gamma z, \gamma z')\gamma'(z)^n \gamma'(z')^{1-n}\overline{\gamma'(z')} = Q_n(z, z')$ for all $\gamma \in \mathrm{PSL}(2, \mathbb{R})$ and $z \neq z'$;
3. $Q_n(z, z') = -\frac{1}{\pi} y'^{-2} \log|z - z'|^2 + O(1)$ as $z \to z'$;
4. $\Delta_n Q_n(z, z') = 0$ for $z \neq z'$;

and an additional growth condition as $z \to \partial\mathbb{H}$.

The Green's function $G_n(z, z')$ for a Riemann surface X, expressed on the upper half plane, is then given by the Poincaré series

$$G_n(z, z') = \sum_{\gamma \in \Gamma} Q_n(z, \gamma z')\gamma'(z')^{1-n}\overline{\gamma'(z')},$$

where Γ is the Fuchsian group such that $X = \mathbb{H}/\Gamma$. $G_n(z, z')$ is uniquely determined by the following properties:

1. $G_n(z, z')$ is smooth for $z \neq \gamma z'$, $\gamma \in \Gamma$;
2. $G_n(\gamma_1 z, \gamma_2 z')\gamma_1'(z)^n \gamma_2'(z')^{1-n}\overline{\gamma_2'(z')} = G_n(z, z')$ for all $\gamma_1, \gamma_2 \in \Gamma$ and $z \neq \gamma z'$;

3. $G_n(z,z') = -\frac{1}{\pi} y'^{-2} \log|z-z'|^2 + O(1)$ as $z \to z'$;

4. $\Delta_n G_n(z,z') = -P_n(z,z')$ for $n \geq 0$, $\Delta_n G_n(z,z') = 0$ for $n < 0$, where $z \neq \gamma z'$;

5. $\int_{\mathcal{D}} G_n(z,z')\phi(z')\mathrm{d}^2 z' = 0$ for all holomorphic n-differentials ϕ and for all z.

$G_n(z,z')$ also satisfies the symmetry property

$$\rho'^{n-1} G_n(z,z') = \rho^{n-1} \overline{G_n(z',z)}.$$

The kernels $L_n = (y')^2 \partial'_{1-n} Q_n$ and $K_n = (y')^2 \partial'_{1-n} G_n$ will be needed later. For the former we can derive the formula

$$L_n(z,z') = \frac{1}{\pi} \cdot \frac{1}{z-z'} \left(\frac{\overline{z}-z'}{\overline{z}-z} \right)^{2n-1}$$

from the properties above, and it follows that

$$\lim_{z' \to z} \left(L_n(z,z') + L_{1-n}(z',z) \right) = 0.$$

For the kernel K_n, we have the corresponding formula for $n \neq 0, 1$:

$$K_n(z,z') + K_{1-n}(z',z) = 0,$$

which follows from the fact that the left hand side is regular on the diagonal (and hence everywhere), harmonic with respect to both variables (in the sense of Δ_n and Δ'_{1-n}), and automorphic with respect to both variables; since one of the spaces $\ker \Delta_n$, $\ker \Delta_{1-n}$ contains only zero, the left side must be identically zero.

Note that the definition given here of the Green's function disagrees with much of the literature, including [ZT87a]; there, the Green's function (call it $\widetilde{G_n}(w, w')$) on the upper half plane is defined so that

$$G_n \phi = \int_{\mathcal{D}} \widetilde{G_n}(z, z') \phi(z') (y')^{2n-2} \mathrm{d}^2 z',$$

so the two definitions are related by $G_n = (\rho')^{1-n} \widetilde{G_n}$. The notational change is convenient because we are considering Kleinian groups which are not necessarily Fuchsian.

2.7 Determinant of Δ_n

We review the zeta function regularization of the determinant of the laplacian operator, and the proper time regularization, and give the relation between the two approaches. We also show that $\det \Delta_n : T_g \to \mathbb{R}$ is a smooth function.

2.7.1 Zeta regularization

The Minakshisundaram-Pleijl zeta function of Δ_n is defined to be $\zeta(s) = \sum_{j=1}^{\infty} \lambda_j^{-s}$, where λ_j runs through the nonzero eigenvalues of Δ_n. It is initially defined for $\operatorname{Re} s > 1$, and has an analytic continuation to all $s \in \mathbb{C}$ except $s = 1$ where it has a simple pole [MP49].

Ignoring convergence, formal manipulation gives $-\zeta'(0) = \sum_j \log \lambda_j = \log \det \Delta_n$. This motivates the definition of the regularized determinant of Δ_n

as [RS73]

$$\det \Delta_n = e^{-\zeta'(0)}.$$

Corresponding to the zeta function, we may also define a theta function associated to Δ_n by $\theta(t) = \sum_{j=1}^{\infty} e^{-t\lambda_j}$ for $t > 0$, where again λ_j runs through nonzero eigenvalues. Note that

$$\mathrm{Tr}\left(e^{-t\Delta_n}\right) = \theta(t) + d(n),$$

where $d(n)$ is the dimension of the space of holomorphic n-differentials. We have the asymptotic expansion

$$\theta(t) \sim \frac{a_{-1}}{t} + a_0 + a_1 t + \cdots,$$

valid as $t \to 0$, with $a_{-1} = \frac{\text{area } X}{4\pi} = g - 1$ and $a_0 = \frac{1}{6}\int K = \frac{\pi}{3}(2-2g)$. The theta and zeta functions are related by

$$\Gamma(s)\zeta(s) = \int_0^\infty \theta(t) t^{s-1} \mathrm{d}t.$$

2.7.2 Proper time regularization

The material of this section is based on [Shv81]. Formally, we may write

$$-\zeta'(0) = -\int_0^\infty \theta(t) \left(\frac{t^s}{s\Gamma(s)}\right)\Bigg|_{s=0} t^{-1}\mathrm{d}t = -\int_0^\infty \theta(t) t^{-1} \mathrm{d}t,$$

but the integral does not converge. We could make an alternate definition of the regularized determinant by

$$\begin{aligned}\log \widetilde{\det}\Delta_n &= \text{“Finite part” of } -\int_0^\infty \theta(t)t^{-1}\mathrm{d}t \\ &= \lim_{\epsilon\to 0}\left(-\int_\epsilon^\infty \theta(t)t^{-1}\mathrm{d}t + \frac{a_{-1}}{\epsilon} - a_0\log\epsilon\right);\end{aligned}$$

this is the proper time regularized determinant. The two definitions vary by a constant:

$$\log\det\Delta_n = \log\widetilde{\det}\Delta_n + a_0\gamma,$$

where $\gamma = \Gamma'(1)$ is Euler's constant. The remainder of this section is devoted to a proof of this fact.

For $\operatorname{Re} s > 1$, write

$$\Gamma(s)\zeta(s) = \int_1^\infty \theta(t)t^{s-1}\mathrm{d}t + \int_0^1 \left(\theta(t) - \frac{a_{-1}}{t} - a_o\right)t^{s-1}\mathrm{d}t + \frac{a_{-1}}{s-1} + \frac{a_0}{s}.$$

The right side is defined for $\operatorname{Re} s > -1$, so this gives the analytic continuation to this line. Hence

$$\begin{aligned}\zeta'(s) = &-\frac{\Gamma'(s)}{\Gamma(s)^2}\left(\int_1^\infty \theta(t)t^{s-1}\mathrm{d}t + \int_0^1\left(\theta(t) - \frac{a_{-1}}{t} - a_0\right)t^{s-1}\mathrm{d}t\right) \\ &- \frac{\Gamma'(s)}{\Gamma(s)^2}\left(\frac{a_{-1}}{s-1} + \frac{a_0}{s}\right) \\ &+ \frac{1}{\Gamma(s)}\left(\int_1^\infty \theta(t)(\log t)t^{s-1}\mathrm{d}t + \int_0^1\left(\theta(t) - \frac{a_{-1}}{t} - a_0\right)(\log t)t^{s-1}\mathrm{d}t\right) \\ &+ \frac{1}{\Gamma(s)}\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{a_{-1}}{s-1} + \frac{a_0}{s}\right)\end{aligned}$$

From $\Gamma(s+1) = s\Gamma(s)$, we find

$$s\frac{\Gamma'(s+1)}{\Gamma(s+1)^2} = \frac{1}{s\Gamma(s)} + \frac{\Gamma'(s)}{\Gamma(s)^2},$$

and hence

$$\lim_{s\to 0} s\Gamma(s) = 1, \quad \lim_{s\to 0} \frac{\Gamma'(s)}{\Gamma(s)^2} = -1,$$

and

$$\gamma = \lim_{s\to 0} \frac{1}{s^2\Gamma(s)} + \frac{\Gamma'(s)}{s\Gamma(s)^2}.$$

It follows that, as $s \to 0$, the first line in the expression for $\zeta'(s)$ becomes

$$\int_1^\infty \theta(t)t^{-1}\mathrm{d}t + \int_0^1 \left(\theta(t) - \frac{a_{-1}}{t} - a_0\right) t^{-1}\mathrm{d}t.$$

The third line is $\frac{1}{\Gamma(s)}$ times a factor with a finite limit, so the third line goes to 0 as $s \to 0$. Combining the second and fourth line, as $s \to 0$ we obtain

$$\begin{aligned} & -a_{-1} - a_0 \lim_{s\to 0}\left(\frac{\Gamma'(s)}{s\Gamma(s)^2} + \frac{1}{s^2\Gamma(s)}\right) \\ & = -a_{-1} - a_0\gamma. \end{aligned}$$

So finally, we find that

$$-\zeta'(0) = -\int_1^\infty \theta(t)t^{-1}\mathrm{d}t - \int_0^1 \left(\theta(t) - \frac{a_{-1}}{t} - a_0\right) t^{-1}\mathrm{d}t + a_1 + a_0\gamma.$$

On the other hand,

$$
\begin{aligned}
&\lim_{\epsilon\to 0} -\int_\epsilon^\infty \theta(t)t^{-1}\mathrm{d}t - \frac{a_{-1}}{\epsilon} - a_0\log\epsilon \\
&\qquad = -\int_1^\infty \theta(t)t^{-1}\mathrm{d}t - \lim_{\epsilon\to 0}\int_\epsilon^1 \left(\theta(t) - \frac{a_{-1}}{t} - a_0\right)t^{-1}\mathrm{d}t \\
&\qquad\quad - \int_\epsilon^1 \left(\frac{a_{-1}}{t} + a_0\right)t^{-1}\mathrm{d}t + \frac{a_{-1}}{\epsilon} - a_0\log\epsilon \\
&\qquad = -\int_1^\infty \theta(t)t^{-1}\mathrm{d}t - \int_0^1 \left(\theta(t) - \frac{a_{-1}}{t} - a_0\right)t^{-1}\mathrm{d}t \\
&\qquad\quad + \lim_{\epsilon\to 0}\left(a_1 - \frac{a_{-1}}{\epsilon} + a_0\log\epsilon + \frac{a_{-1}}{\epsilon} - a_0\log\epsilon\right) \\
&\qquad = -\int_1^\infty \theta(t)t^{-1}\mathrm{d}t - \int_0^1 \left(\theta(t) - \frac{a_{-1}}{t} - a_0\right)t^{-1}\mathrm{d}t + a_1.
\end{aligned}
$$

So our final result is

$$
\log\det\Delta_n = \lim_{\epsilon\to 0}\left(-\int_\epsilon^\infty \theta(t)t^{-1}\mathrm{d}t + \frac{a_{-1}}{\epsilon} - a_0\log\epsilon + a_0\gamma\right),
$$

as claimed.

2.7.3 Smoothness of $\det\Delta_n$

Note that we can also continue $\zeta(s)$ by integrating by parts:

$$
\zeta(s) = \frac{1}{s\Gamma(s)}\frac{1}{s-1}\int_0^\infty \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t\theta(t)\right)t^s\mathrm{d}t
$$

for $\operatorname{Re} s > -1$. This gives

$$
\zeta'(0) = -\int_0^\infty \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(t\theta(t)\right)\log t\,\mathrm{d}t - (\gamma - 1)a_0.
$$

Now, the nonzero eigenvalues are real-analytic functions on T_g in any neighbourhood where none of them are repeated. If there are multiple eigenvalues, the eigenvalues are real-analytic along any real-analytic curve in T_g. (See [Bus92], theorem 14.9.1 and 14.9.3 for a careful statement and proof.) Consequently, $\theta(t)$ is a real-analytic function on T_g for $t > 0$, and hence $\det \Delta_n$ is real-analytic on T_g.

Chapter 3

Green's function of $\overline{\partial}_n$

The main result of this chapter is an explicit expression for the Green's function K_n of $\overline{\partial}_n$ ($n \geq 1$) as a Poincaré series with respect to a function group uniformizing X, plus a "zero mode correction" term constructed from a basis of holomorphic n-differentials. For $n \geq 2$ the function group is arbitrary; for $n = 1$ we prove the result only for a Schottky group for which the series converges. The definition of the Green's function of $\overline{\partial}_n$ is given in the first section, its relation to the Green's function of Δ_n is established in the second section, and the explicit expression is defined and proved in the third section. This completes and makes rigorous partial constructions of Martinec [Mar87].

3.1 Definition

Let n be any integer. The *Green's operator of* $\overline{\partial}_n$ is the unique operator

$$K_n : \mathcal{D}^{0,1}(K^n) \to \mathcal{D}^{0,0}(K^n)$$

satisfying

$$K_n\overline{\partial}_n = I_n - P_n$$
$$\text{and} \quad K_n\big|_{(\operatorname{Im}\overline{\partial}_n)^\perp} = 0,$$

where I_n is the identity and P_n is the orthogonal projection onto $\ker\overline{\partial}_n$.

Since our objective is to study $\det\Delta_n$, and $\det\Delta_{1-n} = \det\Delta_n$, we concentrate on the case $n \geq 1$. If $n \geq 2$, the second condition in the definition of K_n is vacuously satisfied, since

$$\begin{aligned}(\operatorname{Im}\overline{\partial}_n)^\perp &= \ker\overline{\partial}_n^* \\ &= \{\phi \in \mathcal{D}^{0,1}(K^n)\big|\rho^{-n}\overline{\phi} \text{ holomorphic}\},\end{aligned}$$

and there are no nonzero holomorphic $(1-n, 0)$-differentials. When $n = 1$,

$$(\operatorname{Im}\overline{\partial}_1)^\perp = \{k\rho\big|k \in \mathbb{C}\}.$$

Now, let Σ be a function group uniformizing the surface X, so $X \simeq \Omega_0/\Sigma$ for some invariant component Ω_0 of the ordinary set of Σ. Then the *Green's function for* $\overline{\partial}_n$ is the integral kernel $K_n(w, w')$ of the Green's operator for $\overline{\partial}_n$; that is, $K_n(w, w')$ is an automorphic form of type $(n, 0)$ in w and type $(1-n, 0)$ in w', with $w, w' \in \Omega_0$, smooth for $w \neq w'$, such that for all $\phi \in \mathcal{D}^{0,1}(K^n)$,

$$K_n\phi(w) = \int_D K_n(w, w')\phi(w')\mathrm{d}^2w',$$

where D is a fundamental domain for Ω_0 with respect to Σ, and $\mathrm{d}^2w = \frac{i}{2}\mathrm{d}w \wedge$

$d\overline{w}$ is the usual volume form.

3.2 Relation to Green's function of Δ_n

Proposition 3.1. *Let $n \geq 1$, and let $G_n(w, w')$ be the Green's function for Δ_n in some uniformization $X \simeq \Omega_0/\Sigma$ of X. The Green's function K_n for $\overline{\partial}_n$ on Ω_0 is given by*

$$K_n(w, w') = -(\overline{\partial}^*_{1-n})' G_n(w, w').$$

Proof. From

$$G_n \Delta_n = (G_n \overline{\partial}^*_n) \overline{\partial}_n = I_n - P_n$$
$$\text{and} \quad (G_n \overline{\partial}^*_n)\big|_{(\operatorname{Im} \overline{\partial}_n)^\perp} = (G_n \overline{\partial}^*_n)\big|_{\ker \overline{\partial}^*_n} = 0,$$

we see that $K_n = G_n \overline{\partial}^*_n$, so we need only show that the integral kernel of $G_n \overline{\partial}^*_n$ is $-(\overline{\partial}^*_{1-n})' G_n(w, w')$. This is essentially trivial integration by parts, except that we must check that the boundary contribution about the singularity goes to zero. Start by assuming that $\Sigma = \Gamma$ is a Fuchsian group, $\Omega_0 = \mathbb{H}$. Let

$\phi \in \mathcal{D}^{0,1}(K^n)$. Then, (abbreviating $G_n(z,z') = G_n$, $\phi(z') = \phi'$, etc.),

$$\begin{aligned}
\int_{\mathcal{D}} (-(\overline{\partial}_{1-n}^{*})' G_n)\phi' \mathrm{d}^2 z' &= \int_{\mathcal{D}} (\rho'^{-n} \partial' \rho'^{n-1} G_n)\phi' \mathrm{d}^2 z' \\
&= \int_{\mathcal{D}} G_n(-\rho'^{n-1}\partial'\rho'^{-n}\phi')\mathrm{d}^2 z' \\
&\quad + \frac{1}{2i}\lim_{\epsilon\to 0}\int_{|z'-z|=\epsilon} \rho'^{-1} G_n \phi' \mathrm{d}\overline{z'} \\
&= \int_{\mathcal{D}} G_n((\overline{\partial}_n^{*})'\phi')\mathrm{d}^2 z' \\
&\quad - \frac{1}{2\pi i}\lim_{\epsilon\to 0}\int_{|z'-z|=\epsilon} \log|z-z'|^2 \phi(z')\mathrm{d}\overline{z'} \\
&= G_n \overline{\partial}_n^{*}\phi,
\end{aligned}$$

since the integral under the limit is bounded by $C\epsilon\log\epsilon$.

The result now follows for any function group by simply changing coordinates. $\square$

3.3 Series expression

In this section, we define an explicit Poincaré series $\widehat{K}_n$ and a "zero mode" term Z_n such that the Green's function K_n of $\overline{\partial}_n$ is given by

$$K_n = \widehat{K}_n - Z_n,$$

with a slight modification when $n = 1$.

3.3.1 The series $\widehat{K}_n$, $n \geq 2$

Let the surface X be given a uniformization $X = \Omega_0/\Sigma$ by some function group Σ, and let $n \geq 2$. Choose $2n-1$ points $\{a_1, \ldots, a_{2n-1}\}$ in the limit set of Σ. Then, following Bers [Ber71], we define for $w, w' \in \Omega_0$, $w \neq \gamma w'$ for any $\gamma \in \Sigma$,

$$\widehat{K}_n(w, w') = \frac{1}{\pi} \sum_{\gamma \in \Sigma} \frac{1}{\gamma w - w'} \left(\prod_{j=1}^{2n-1} \frac{w' - a_j}{\gamma w - a_j} \right) \gamma'(w)^n.$$

$\widehat{K}_n$ converges absolutely provided $n \geq 2$; it converges for certain groups when $n = 1$ (for example, if Σ is a Schottky group, it converges when the Hausdorff dimension of the limit set is strictly less than 1). We will assume $n \geq 2$ in this section and return to the case $n = 1$ later. The series is a meromorphic n-differential in the first variable: for any $\alpha \in \Sigma$,

$$\widehat{K}_n(\alpha w, w')\alpha'(w)^n = \widehat{K}_n(w, w').$$

In the second variable it is not quite a $(1-n)$-differential, but rather a meromorphic Eichler integral of weight $1-n$; that is, for any $\alpha \in \Sigma$,

$$\widehat{K}_n(w, \alpha w')\alpha'(w')^{1-n} = \widehat{K}_n(w, w') + \Pi_\alpha(w, w'),$$

where $\Pi_\alpha(w, w')$ is a holomorphic n-differential in w, and a polynomial of degree at most $2n-2$ in w'. This follows from a straightforward calculation: first, using the formula

$$\frac{1}{(w-w')^2} = \frac{\gamma'(w)\gamma'(w')}{(\gamma w - \gamma w')^2},$$

valid for any linear fractional transformation γ, calculate

$$\widehat{K}_n(w, \alpha w')\alpha'(w')^{1-n} = \sum_{\gamma \in \Sigma} \frac{1}{\gamma w - w'} \left(\prod_{j=1}^{2n-1} \frac{w' - \alpha^{-1} a_j}{\gamma w - \alpha^{-1} a_j} \right) \gamma'(w)^n.$$

Then,

$$\begin{aligned} &\widehat{K}_n(w, \alpha w')\alpha'(w')^{1-n} - \widehat{K}_n(w, w') \\ &\qquad = \sum_{\gamma \in \Sigma} \frac{1}{\gamma w - w'} \left(\prod_{j=1}^{2n-1} \frac{w' - \alpha^{-1} a_j}{\gamma w - \alpha^{-1} a_j} - \prod_{j=1}^{2n-1} \frac{w' - a_j}{\gamma w - a_j} \right) \gamma'(w)^n, \end{aligned}$$

and the factor in parentheses vanishes when $\gamma w - w' = 0$, cancelling the first factor and leaving a polynomial of degree $2n - 2$ or less in w'.

3.3.2 The zero mode term Z_n, $n \geq 2$

We will define a term Z_n with the same transformation properties as $\widehat{K}_n$, so that the difference will be a differential of the required type.

Let $\{\phi_1, \ldots, \phi_{d(n)}\}$ be a basis for the holomorphic n-differentials on X, written in the local coordinate $w \in \Omega_0$. Let $\{a_1, \ldots, a_{2n-1}\}$ in the limit set of Σ be the *same* points as chosen in the definition of $\widehat{K}_n$. Again following Bers [Ber67], [Ber71], define potentials of the ϕ_k by

$$\begin{aligned} F_k(w) &= -\frac{1}{\pi} \int_{\Omega_0} \frac{\rho(\zeta)^{1-n} \overline{\phi_k}(\zeta)}{\zeta - w} \prod_{j=1}^{2n-1} \frac{w - a_j}{\zeta - a_j} \mathrm{d}^2\zeta \\ &= -\int_{\mathcal{D}} \rho(\zeta)^{1-n} \overline{\phi_k}(\zeta) \widehat{K}_n(\zeta, w) \mathrm{d}^2\zeta, \end{aligned}$$

where $\rho(\zeta)$ is the hyperbolic metric on Ω_0. The function F_k has the property

$$\overline{\partial} F_k = \rho^{1-n}\overline{\phi_k}$$

on Ω_0.

Recall that $[N_n]_{jk} = (\phi_j, \phi_k)$ denotes the Gram matrix of the chosen basis. Define

$$Z_n(w, w') = -\sum_{j=1}^{d}\sum_{k=1}^{d}\phi_j(w)[N_n^{-1}]_{kj}F_k(w');$$

from the property of F_k stated above, we see that

$$\overline{\partial}' Z_n(w, w') = -P_n(w, w').$$

Z_n, like $\widehat{K}_n$, is a holomorphic n-differential in the first variable, and an Eichler integral of weight $1-n$ (though not holomorphic) in the second variable. In fact, Z_n has the same Eichler periods as $\widehat{K}_n$:

$$\begin{aligned} Z_n(w, \alpha w')\alpha'(w')^{1-n} &- Z_n(w, w') \\ &= \sum_{j=1}^{d}\sum_{k=1}^{d}\phi_j(w)[N_n^{-1}]_{kj}\int_{\mathcal{D}}\rho(\zeta)^{1-n}\overline{\phi_k}(\zeta)\Pi_\alpha(\zeta, w')\mathrm{d}^2\zeta \\ &= \sum_{j=1}^{d}\sum_{k=1}^{d}\phi_j(w)[N_n^{-1}]_{kj}(\Pi_\alpha(\cdot, w'), \phi_k) \\ &= \Pi_\alpha(w, w'), \end{aligned}$$

the last line following because $\Pi_\alpha(w, w')$ is a holomorphic n-differential in w. Consequently, $\widehat{K}_n - Z_n$ is a $(1-n)$-differential in the second variable.

3.3.3 Equality of K_n and $\widehat{K}_n - Z_n$, $n \geq 2$

We come to the main theorem in this chapter, in the case $n \geq 2$. The case $n = 1$, which is slightly different, will be addressed in the next section.

Theorem 3.2. *Let $X \simeq \Omega_0/\Sigma$ be a uniformization of X by a function group Σ, let $w, w' \in \Omega_0$, $w \neq \gamma w'$ for any $\gamma \in \Sigma$, and let $n \geq 2$. Then, with notations as defined above, the Green's function $K_n(w, w')$ for $\overline{\partial}_n$ on X in Ω_0 is given by*

$$\begin{aligned} K_n(w,w') &= \widehat{K}_n(w,w') - Z_n(w,w') \\ &= \frac{1}{\pi} \sum_{\gamma \in \Sigma} \frac{1}{\gamma w - w'} \left(\prod_{j=1}^{2n-1} \frac{w' - a_j}{\gamma w - a_j} \right) \gamma'(w)^n \\ &\quad + \sum_{j=1}^{d} \sum_{k=1}^{d} \phi_j(w) [N_n^{-1}]_{kj} F_k(w'). \end{aligned}$$

Proof. It has been established above that K_n and $\widehat{K}_n - Z_n$ are bidifferentials of the same type. Since $n \geq 2$, it will be sufficient to show that, for any $\phi \in \mathcal{D}^{0,0}(K^n)$,

$$\int' (\widehat{K}_n - Z_n) \overline{\partial}' \phi' = (I_n - P_n) \phi$$

(with the abbreviations $\phi' = \phi(w')$, $\int' = \int_{\mathcal{D}} \mathrm{d}^2 w'$). Integrate by parts:

$$\int' (\widehat{K}_n - Z_n) \overline{\partial}' \phi' = \int' \overline{\partial}' ((\widehat{K}_n - Z_n) \phi') - \int' (\overline{\partial}' (\widehat{K}_n - Z_n)) \phi'.$$

Since $\widehat{K}_n(w, w')$ is holomorphic in w' for $w' \neq w$, the second integral becomes

$$- \int' P_n \phi' = -P_n \phi.$$

The first integral is a sum of an integral over $\partial\mathcal{D}$, which vanishes since $(\widehat{K}_n - Z_n)\phi'$ is a $(1,0)$-differential in w', and a boundary term around the singularity:

$$\begin{aligned}\int' \overline{\partial}'((\widehat{K}_n - Z_n)\phi') &= \lim_{\epsilon\to 0} \frac{i}{2}\int_{|w'-w|=\epsilon} (\widehat{K}_n - Z_n)(w,w')\phi(w')\mathrm{d}w' \\ &= \lim_{\epsilon\to 0}\frac{1}{2\pi i}\int_{|w'-w|=\epsilon}\frac{1}{w'-w}\phi(w')\mathrm{d}w' \\ &= \phi(w),\end{aligned}$$

where the circle $|w'-w| = \epsilon$ has the standard orientation. Here the singular part of $\widehat{K}_n - Z_n$ as $w' \to w$ comes from the term corresponding to $\gamma =$ identity in the series $\widehat{K}_n$. □

3.3.4 The case $n = 1$

When $n = 1$, we can formally write the same series $\widehat{K}_n$ as above:

$$\begin{aligned}\widehat{K}_1(w,w') &= \frac{1}{\pi}\sum_{\gamma\in\Sigma}\frac{1}{\gamma w - w'}\left(\frac{w'-a}{\gamma w - a}\right)\gamma'(w) \\ &= \frac{1}{\pi}\sum_{\gamma\in\Sigma}\left(\frac{1}{\gamma w - w'} - \frac{1}{\gamma w - a}\right)\gamma'(w).\end{aligned}$$

However, this series need not converge. We will restrict our attention to the case of Σ a Schottky group whose exponent of convergence (the supremum of δ such that $\sum|\gamma'(w)|^\delta$ converges) is strictly less than 1; δ can also be described as the Hausdorff dimension of the limit set in this case [Bow79]. These groups form an open set in the space of Schottky groups. However, even with these conditions, the series will converge only when a is not in the limit set, but in the ordinary set. (Note that if the series was to converge when a and infinity are in the limit set, it would represent a nonzero 1-differential with a single simple pole.) So we assume that Σ is a Schottky group of the type described, and that a is in the ordinary set of Σ.

Let $\{\phi_1, \ldots, \phi_g\}$ be a basis of the holomorphic 1-differentials, normalized so $\int_{A_k} \phi_j = \delta_{jk}$, where the A-cycles on X are taken to be represented by half the bounding curves C_k, $k = 1, \ldots, g$ of a fundamental domain $\mathcal{D}$ of Σ. (Recall that Σ has g free generators $\{\gamma_1, \ldots, \gamma_g\}$, and the fundamental domain is bounded by $2g$ Jordan curves C_k and $C_k' = -\gamma_k C_k$, $k = 1, \ldots, g$.) The construction of potentials for ϕ_k given above for $n \geq 2$ does not work in this case. It is natural to take an abelian integral $\overline{\int_a^w \phi_k}$ as the potential for ϕ_k, but this is not single-valued on the fundamental domain; hence we must take instead

$$F_k(w) = \overline{\int_a^w \phi_k(\zeta)\mathrm{d}\zeta} - \int_a^w \phi_k(\zeta)\mathrm{d}\zeta,$$

which also satisfies

$$\overline{\partial} F_k = \overline{\phi_k}$$

and is single-valued on Ω_0. Then we define

$$Z_1(w, w') = -\sum_{j=1}^{g} \sum_{k=1}^{g} \phi_j(w) [N_1^{-1}]_{kj} F_k(w')$$

as before, so

$$\overline{\partial}' Z_1(w, w') = -P_1(w, w').$$

Both $\widehat{K}_1(w, w')$ and $Z_1(w, w')$ are holomorphic 1-differentials in w and are abelian integrals in w'. We claim that in fact they have the same periods in w', so that the difference is a 0-differential (automorphic function) in w'.

First consider $\widehat{K}_1$. We will need the formula

$$\frac{\gamma'(w)}{\gamma w - w'} = \frac{1}{w - \gamma^{-1}w'} + \frac{1}{2}\frac{\gamma''(w)}{\gamma'(w)},$$

valid for any linear fractional transformation γ. Using this, the period $\Pi_\alpha(w) = \widehat{K}_n(w, \alpha w') - \widehat{K}_n(w, w')$ becomes a telescoping series:

$$\begin{aligned}
\Pi_\alpha(w) &= \frac{1}{\pi}\sum_{\gamma\in\Sigma}\left(\frac{1}{\gamma w - \alpha w'} - \frac{1}{\gamma w - w'}\right)\gamma'(w) \\
&= \frac{1}{\pi}\sum_{\gamma\in\Sigma}\frac{1}{w - \gamma^{-1}\alpha w'} - \frac{1}{w - \gamma^{-1}w'} \\
&= \frac{1}{\pi}\sum_{\gamma\in\langle\alpha\rangle\backslash\Sigma}\sum_{n=-\infty}^{\infty}\frac{1}{w - \gamma^{-1}\alpha^{n+1}w'} - \frac{1}{w - \gamma^{-1}\alpha^n w'} \\
&= \frac{1}{\pi}\sum_{\gamma\in\langle\alpha\rangle\backslash\Sigma}\frac{1}{w - \gamma^{-1}a_\alpha} - \frac{1}{w - \gamma^{-1}b_\alpha},
\end{aligned}$$

where a_α and b_α are the attracting and repelling fixed points of α respectively, and $\langle\alpha\rangle\backslash\Sigma$ denotes the quotient on the left of Σ by the cylic group generated by α, or equivalently, the set of all reduced words in Σ not beginning with a power of α. Consequently, for a group generator γ_r we have

$$\int_{C_j}\Pi_{\gamma_r}(w)\mathrm{d}w = 2i\delta_{jr},$$

which shows that

$$\Pi_{\gamma_r}(w) = 2i\phi_r.$$

Now consider Z_1. We have

$$\begin{aligned}
Z_1(w,\gamma_r w') - Z_1(w,w') &= \sum_{j=1}^{g}\sum_{k=1}^{g} \phi_j(w)[N_1^{-1}]_{kj}\left(\int_{B_r}\phi_k - \overline{\int_{B_r}\phi_k}\right) \\
&= 2i\sum_{j=1}^{g}\sum_{k=1}^{g} \phi_j(w)[N_1^{-1}]_{kj}[N_1]_{kr} \\
&= 2i\phi_r(w),
\end{aligned}$$

since $N_1 = N_1^T$.

Hence $\widehat{K}_1 - Z_1$ is a $(1,0)\times(0,0)$ bidifferential, and we may now state the main theorem of the chapter in the case $n = 1$:

Theorem 3.3. *Let X be uniformized by a Schottky group Σ with exponent of convergence strictly less than 1, $X \simeq \Omega_0/\Sigma$, and let $w, w', a \in \Omega_0$, with $w \neq \gamma w', \gamma a$ for any $\gamma \in \Sigma$. If $K_1(w,w')$ is the Green's function for $\overline{\partial}_1$ on X in Ω_0, then, with notations as defined above,*

$$\begin{aligned}
K_1(w,w') - K_1(w,a) &= \widehat{K}_1(w,w') - Z_1(w,w') \\
&= \frac{1}{\pi}\sum_{\gamma\in\Sigma}\frac{1}{\gamma w - w'}\left(\frac{w'-a}{\gamma w - a}\right)\gamma'(w) \\
&\quad + \sum_{j=1}^{g}\sum_{k=1}^{g}\phi_j(w)[N_1^{-1}]_{kj}\left(\overline{\int_a^{w'}\phi_k} - \int_a^{w'}\phi_k\right).
\end{aligned}$$

Proof. The proof that

$$\int'(\widehat{K}_1 - Z_1)\overline{\partial}'\phi' = (I_n - P_n)\phi$$

is identical to the proof given for $n \geq 2$ in the previous section. However, this only shows that $\widehat{K}_1 - Z_1$ and K_1 agree on $\operatorname{Im} \overline{\partial}_1$, or equivalently, that

$$K_1(w, w') = \widehat{K}_1(w, w') - Z_1(w, w') + \psi(w)$$

for some holomorphic 1-differential $\psi(w)$ not depending on w'. To evaluate $\psi(w)$, set $w' = a$, which shows $\psi(w) = K_1(w, a)$ and completes the proof of the theorem. $\square$

Remark 3.4. Taking ∂' of the equation in theorem 3.3 yields a formula of Fay relating the classical Schiffer and Bergman kernels; see [Fay77], pp. 160–161.

Chapter 4

First variation of $\log \det \Delta_n$

The purpose of this chapter is to prove the following expression for the derivative of the function $\log \det \Delta_n$ on the Teichmüller space T_g.

Theorem 4.1. *Let the surface X be uniformized by a Fuchsian group Γ, so $X \simeq \mathbb{H}/\Gamma$, and let $\mathcal{D} \in \mathbb{H}$ be a fundamental domain. Let μ be a harmonic Beltrami differential on X (representing an element of $T_{[X]}T_g$). Then for $n \geq 1$,*

$$\partial \log \det \Delta_n(\mu) = \int_{\mathcal{D}} \mu \left(\big(n\partial' - (1-n)\partial \big) \left(K_n - \frac{1}{\pi} \frac{1}{z - z'} \right) \right) \Big|_{\Delta},$$

where K_n is the Green's function of $\overline{\partial}_n$ defined in the previous chapter, and $|_\Delta$ means the limit as $z' \to z$.

The form of the differential operator appearing in the integral comes from considering b-c "ghosts": see [Mar87], [VV87].

Theorem 4.1 will follow from

Theorem 4.2. *With assumptions as above,*

$$\partial \log \det \Delta_n(\mu) = - \int_{\mathcal{D}} \mu \big(\partial_n (K_n - L_n) \big) \big|_{\Delta},$$

where $L_n(z,z') = -(\overline{\partial}^*_{1-n})' Q(z,z')$ *is the "Green's function of* $\overline{\partial}_n$ *on* $\mathbb{H}$*".*

Theorem 4.2 appears (in slightly different form) in [ZT87a].

It is not immediately obvious that the integrals appearing in theorems 4.1 and 4.2 are independent of the choice of fundamental domain $\mathcal{D}$; this will be proved below.

In the first section it will be shown that theorem 4.2 implies theorem 4.1, and that the integrals do not depend on the choice of $\mathcal{D}$. In the following two sections, we give two different proofs of theorem 4.2, both different than that given in [ZT87a]. The first uses the Green's function of Δ_n, while the second uses the Green's function of $\overline{\partial}_n$ ("bosonic" and "fermionic" proofs respectively).

4.1 Theorem 4.2 implies theorem 4.1

The fact that the integral in theorem 4.2 is independent of the choice of fundamental domain follows from

Lemma 4.3. $\big(\partial_n(K_n - L_n)\big)\big|_\Delta$ *is a* $(2,0)$*-automorphic form with respect to* Γ*.*

Proof. $\partial_n K_n$ is an $(n+1,0)\times(1-n,0)$-automorphic form. The same is not true of $\partial_n L_n$ in general, but it is true in the case of the diagonal action of Γ:

$$(\partial_n L_n)(\gamma z, \gamma z')\gamma'(z)^{n+1}\gamma'(z')^{1-n} = (\partial_n L_n)(z,z')$$

for all $\gamma \in \Gamma$. This is sufficient to prove the lemma. $\square$

Lemma 4.4. *For any smooth function $K(w, w')$ of two variables,*

$$\begin{aligned}((n\partial' - (1-n)\partial)K)|_\Delta &= n\partial_1(K|_\Delta) - (\partial_n K)|_\Delta \\ &= (n-1)\partial_1(K|_\Delta) + (\partial'_{1-n}K)|_\Delta\end{aligned}$$

Proof. Using $\partial_n = \partial + n(-\rho^{-1}(\partial\rho))$, we see that $\partial_1(K|_\Delta) = ((\partial'_{1-n} + \partial_n)K)|_\Delta$, and the result follows. □

The integral appearing in theorem 4.1 is also independent of the choice of $\mathcal{D}$:

Lemma 4.5. *Let T_n be defined by*

$$T_n(z) = \left(\left(n\partial' - (1-n)\partial\right)\left(K_n - \frac{1}{\pi}\frac{1}{z-z'}\right)\right)\Big|_\Delta.$$

Then T_n is a $(2,0)$-automorphic form with respect to Γ.

Proof. Follows from the previous two lemmas. □

Lemma 4.6. *With notations as above,*

$$\int_\mathcal{D} \mu\big(\partial_n(K_n - L_n)\big)\big|_\Delta = \int_\mathcal{D} \mu\Big(\big(n\partial' - (1-n)\partial\big)\big(K_n - L_n\big)\Big)\Big|_\Delta.$$

Proof. Using lemma 4.4, we find that the difference of the two sides is

$$n\int_\mathcal{D} \mu\partial_1\big((K_n - L_n)|_\Delta\big).$$

Reasoning as in lemma 4.3, we find that $(K_n - L_n)\big|_\Delta$ is a $(1,0)$-automorphic form with respect to Γ, and consequently we may integrate by parts. But

$\partial_{-1}\mu = 0$, so the integral vanishes. $\square$

We must now equate the regularization terms appearing in the two theorems.

Lemma 4.7.

$$(n\partial' - (1-n)\partial)L_n = \frac{1}{\pi}\frac{1}{(z-z')^2} + O(z-z').$$

Proof. This is a direct computation, based on the formula

$$L_n(z,z') = \frac{1}{\pi}\frac{1}{z-z'}\left(\frac{\overline{z}-z'}{\overline{z}-z}\right)^{2n-1}.$$

Since

$$\begin{aligned}
\partial' L_n &= \frac{1}{\pi}\left(\frac{\overline{z}-z'}{\overline{z}-z}\right)^{2n-1}\left(\frac{1}{(z-z')^2} - (2n-1)\frac{1}{(z-z')(\overline{z}-z')}\right) \\
&= \frac{1}{\pi}\left(\frac{\overline{z}-z'}{\overline{z}-z}\right)^{2n-1}\Bigg(\frac{1}{(z-z')^2} - (2n-1)\frac{1}{(z-z')(\overline{z}-z)} \\
&\quad + (2n-1)\frac{1}{(\overline{z}-z)^2} + O(z-z')\Bigg)
\end{aligned}$$

and

$$\partial L_n = -\frac{1}{\pi}\left(\frac{\overline{z}-z'}{\overline{z}-z}\right)^{2n-1}\left(\frac{1}{(z-z')^2} - (2n-1)\frac{1}{(z-z')(\overline{z}-z)}\right),$$

we obtain

$$\begin{aligned}
(n\partial' - (1-n)\partial)L_n = \frac{1}{\pi}\left(\frac{\overline{z}-z'}{\overline{z}-z}\right)^{2n-1}\Bigg(&\frac{1}{(z-z')^2} - (2n-1)\frac{1}{(z-z')(\overline{z}-z)} \\
&+ n(2n-1)\frac{1}{(\overline{z}-z)^2} + O(z-z')\Bigg).
\end{aligned}$$

Expanding

$$\left(\frac{\overline{z}-z'}{\overline{z}-z}\right)^{2n-1} = 1+(2n-1)\frac{z-z'}{\overline{z}-z}+(2n-1)(n-1)\frac{(z-z')^2}{(\overline{z}-z)^2}+O((z-z')^3)$$

and multiplying out, we find that lower order terms cancel, yielding the required formula. $\square$

Now, collecting the results of the previous lemmas shows that theorem 4.2 implies theorem 4.1.

4.2 First proof of theorem 4.2 (using Green's function of Δ_n)

In this section we give a proof of theorem 4.2 using the Green's function G_n of Δ_n. Recall that $e^{-t\Delta_n}$ is the corresponding heat operator; write its integral kernel on $\mathbb{H}$ as $p_n(z,z',t)$. Start with the definition of $\det\Delta_n$ by proper time regularization:

$$\begin{aligned}
\delta_\mu \log\det\Delta_n &= \delta_\mu \lim_{\epsilon\to 0}\left(-\int_\epsilon^\infty \left(\mathrm{Tr}\left(e^{-t\Delta_n}\right)-d(n)\right)t^{-1}\mathrm{d}t+\frac{a_{-1}}{\epsilon}-a_0\log\epsilon+a_0\gamma\right)\\
&= -\lim_{\epsilon\to 0}\int_\epsilon^\infty \delta_\mu\left(\mathrm{Tr}\left(e^{-t\Delta_n}(I-P_n)\right)\right)\mathrm{d}t\\
&= \lim_{\epsilon\to 0}\int_\epsilon^\infty \mathrm{Tr}\left(\left(\delta_\mu\Delta_n\right)e^{-t\Delta_n}(I-P_n)\right)\mathrm{d}t\\
&= \lim_{\epsilon\to 0}\int_\epsilon^\infty \mathrm{Tr}\left(\rho^{-1}\mu\partial_{n+1}\partial_n e^{-t\Delta_n}(I-P_n)\right)\mathrm{d}t\\
&= \lim_{\epsilon\to 0}\int_\epsilon^\infty \mathrm{Tr}\left(\partial_n e^{-t\Delta_n}(I-P_n)\rho^{-1}\mu\partial_{n+1}\right)\mathrm{d}t\\
&= \lim_{\epsilon\to 0}\mathrm{Tr}\left(\partial_n e^{-\epsilon\Delta_n}G_n\rho^{-1}\mu\partial_{n+1}\right).
\end{aligned}$$

Now, write $k_n(z,z',\epsilon)$ for the integral kernel of $e^{-\epsilon\Delta_n}G_n$. For $\epsilon > 0$, $k_n(z,z',\epsilon)$ is regular as $z' \to z$ because $p_n(z,z',\epsilon)$ is. We can therefore use integration by parts and the fact that $\partial_{-1}\mu = 0$ to show that the integral kernel of

$$\partial_n e^{-\epsilon\Delta_n} G_n \rho^{-1} \mu \partial_{n+1}$$

is

$$\mu(z')\partial_n(\overline{\partial}^*_{1-n})' k_n(z,z',\epsilon).$$

Hence

$$\delta_\mu \log\det\Delta_n = \lim_{\epsilon\to 0}\int_{\mathcal{D}} \mu(z)\left(\partial_n(\overline{\partial}^*_{1-n})' k_n(z,z',\epsilon)\right)\big|_{z'=z}.$$

Now, let $p_n^0(z,z',t)$ be the heat kernel for the upper half plane (see, for example, [DP86]), and introduce

$$k_n^0(z,z',\epsilon) = \int_{\mathcal{D}} p_n^0(z,z'',\epsilon) Q_n(z'',z')\mathrm{d}^2 z''.$$

Note that $k_n - k_n^0$ is regular as $z' \to z$ even for $\epsilon = 0$, and that $k_n(z,z',0) = G_n(z,z')$, $k_n^0(z,z',0) = Q_n(z,z')$. Then

$$\begin{aligned}
\delta_\mu \log\det\Delta_n &= \lim_{\epsilon\to 0}\int_{\mathcal{D}} \mu(z)\left(\partial_n(\overline{\partial}^*_{1-n})'(k_n(z,z',\epsilon) - k_n^0(z,z',\epsilon))\right)\big|_{z'=z} \\
&\quad + \lim_{\epsilon\to 0}\int_{\mathcal{D}} \mu(z)\left(\partial_n(\overline{\partial}^*_{1-n})' k_n^0(z,z',\epsilon)\right)\big|_{z'=z} \\
&= -\int_{\mathcal{D}} \mu(z)\big(\partial_n(K_n - L_n)\big)\big|_{\Delta} \\
&\quad + \lim_{\epsilon\to 0}\int_{\mathcal{D}} \mu(z)\left(\partial_n(\overline{\partial}^*_{1-n})' k_n^0(z,z',\epsilon)\right)\big|_{z'=z},
\end{aligned}$$

so we need to show that the second integral goes to 0 as $\epsilon \to 0$. Note that

$$\left(\partial_n(\overline{\partial}^*_{1-n})' k_n^0(z, z', \epsilon)\right)\big|_{z'=z} = -\int_{\mathcal{D}} \partial_n p_n^0(z, z'', \epsilon) L_n(z'', z) \mathrm{d}^2 z''.$$

We will show that this approaches 0 as $\epsilon \to 0$.

Write the integral on D_1, the unit disc model of the hyperbolic plane, so for the rest of the paragraph p_n, L_n and $\mathcal{D}$ are the appropriate pullbacks onto D_1. Let γ_0 be a linear fractional transformation preserving D_1 and moving z to 0. Since

$$\begin{aligned} \partial_n p_n^0(\gamma z, \gamma z', \epsilon)\gamma'(z)^{n+1}\gamma'(z')^{1-n}\overline{\gamma'(z')} &= \partial_n p_n^0(z, z', \epsilon) \\ \text{and} \qquad L_n(\gamma z, \gamma z')\gamma'(z)^n \gamma'(z')^{1-n} &= L_n(z, z') \end{aligned}$$

for all linear fractional γ preserving D_1, the integral becomes

$$-\gamma_0'(z)^2 \int_{\gamma_0 \mathcal{D}} \partial_n p_n^0(0, z'', \epsilon) L_n(z'', 0) \mathrm{d}^2 z''.$$

Now if we rotate D_1 about 0, the integrand transforms as

$$\partial_n p_n^0(0, e^{i\theta} z'', \epsilon) L_n(e^{i\theta} z'', 0) i e^{2i\theta} = \partial_n p_n^0(0, z'', \epsilon) L_n(z'', 0)$$

(by applying the rotation to both variables). Hence in polar coordinates the integral takes the form

$$-\gamma_0'(z)^2 \int_{\gamma_0 \mathcal{D}} f(r) e^{2i\theta} \mathrm{d}r \mathrm{d}\theta$$

where $f(r)$ is a function only of the distance from the origin. Taken over a disc $D_a \subset \mathcal{D}$, the integral is exactly zero, and the integral over the rest of $\mathcal{D}$ goes to 0 as $\epsilon \to 0$, since p_n approaches 0 uniformly in that region. This completes the proof of theorem 4.2.

4.3 Second proof of theorem 4.2 (using Green's function of $\overline{\partial}_n$)

We now give another proof of theorem 4.2, obtaining the Green's function K_n of $\overline{\partial}_n$ directly rather than indirectly through G_n. This proof follows closely the proof (in the simpler situation with a linear parameter space and no zero eigenvalues) in [Qui85].

Let D be the restriction of $\overline{\partial}_n$ to $(\ker \overline{\partial}_n)^\perp$. Then D is invertible, with $D^{-1} = K_n$. The operator D^*D is the restriction of Δ_n to $(\ker \overline{\partial}_n)^\perp$ in both domain and range. Temporarily abbreviate $P_n = P$. Then

$$\overline{\partial}_n = D(I - P),$$
$$\overline{\partial}_n^* = (I - P)D^*.$$

Varying these gives

$$\delta_\mu \overline{\partial}_n + D(\delta_\mu P) = (\delta_\mu D)(I - P),$$
$$(\delta_\mu P)D^* = (I - P)(\delta_\mu D^*).$$

Varying the equation $P^2 = P$ shows that

$$P(\delta_\mu P) = (\delta_\mu P)(I - P),$$
$$(\delta_\mu P)P = (I - P)(\delta_\mu P).$$

Note further that

$$\Delta_n = (I - P)D^*D(I - P),$$
$$e^{-t\Delta_n} - P = (I - P)e^{-tD^*D}(I - P),$$

and, if we denote by Tr^0 the trace on $(\ker\overline{\partial}_n)^\perp$,

$$\mathrm{Tr}^0 A = \mathrm{Tr}\big((I - P)A(I - P)\big).$$

Finally also note that $PK_n = 0$.

Now,

$$\zeta_n(s) = \mathrm{Tr}^0\big((D^*D)^{-s}\big),$$

which implies that

$$\begin{aligned}
-\delta_\mu\zeta_n(s) &= s\,\mathrm{Tr}^0\big((D^*D)^{-s-1}\delta_\mu(D^*D)\big)\\
&= s\,\mathrm{Tr}^0\Big((D^*D)^{-s-1}\big((\delta_\mu D^*)D + D^*(\delta_\mu D)\big)\Big)\\
&= s\,\mathrm{Tr}^0\big((\delta_\mu D^*)(D^*)^{-1}(D^*D)^{-s} + (D^*D)^{-s}D^{-1}(\delta_\mu D)\big)\\
&= \frac{s}{\Gamma(s)}\int_0^\infty \mathrm{Tr}^0\big((\delta_\mu D^*)(D^*)^{-1}e^{-tD^*D} + e^{-tD^*D}D^{-1}(\delta_\mu D)\big)t^{s-1}\mathrm{d}t\\
&= s\lim_{t\to 0}\mathrm{Tr}^0\big((\delta_\mu D^*)(D^*)^{-1}e^{-tD^*D} + e^{-tD^*D}D^{-1}(\delta_\mu D)\big) + O(s),
\end{aligned}$$

and hence

$$\delta_\mu \log\det\Delta_n = \lim_{t\to 0} \mathrm{Tr}^0\big((\delta_\mu D^*)(D^*)^{-1}e^{-tD^*D}\big) + \lim_{t\to 0} \mathrm{Tr}^0\big(e^{-tD^*D}D^{-1}(\delta_\mu D)\big).$$

The first term on the right is

$$\begin{aligned} &\lim_{t\to 0} \mathrm{Tr}\big((I-P)(\delta_\mu D^*)(D^*)^{-1}e^{-tD^*D}(I-P)\big) \\ &= \lim_{t\to 0} \mathrm{Tr}\big((I-P)(\delta_\mu P)e^{-tD^*D}(I-P)\big) \\ &= \lim_{t\to 0} \mathrm{Tr}\big((\delta_\mu P)Pe^{-tD^*D}(I-P)\big) = 0. \end{aligned}$$

The second term is

$$\begin{aligned} &\lim_{t\to 0} \mathrm{Tr}\big((I-P)e^{-tD^*D}D^{-1}(\delta_\mu D)(I-P)\big) \\ &= \lim_{t\to 0} \mathrm{Tr}\big((I-P)e^{-tD^*D}D^{-1}(\delta_\mu\overline{\partial}_n + D(\delta_\mu P))(I-P)\big) \\ &= \lim_{t\to 0} \mathrm{Tr}\big((e^{-t\Delta_n}-P)D^{-1}\delta_\mu\overline{\partial}_n\big) + \lim_{t\to 0} \mathrm{Tr}\big((I-P)e^{-tD^*D}P(\delta_\mu P)\big) \\ &= -\lim_{t\to 0} \mathrm{Tr}\big(\mu\partial_n e^{-t\Delta_n}K_n\big), \end{aligned}$$

so in summary we have

$$\delta_\mu \log\det\Delta_n = -\lim_{t\to 0} \mathrm{Tr}\big(\mu\partial_n e^{-t\Delta_n}K_n\big).$$

Applying the same regularization subtraction as before,

$$\begin{aligned} \delta_\mu \log\det\Delta_n = &-\lim_{t\to 0}\int_{\mathcal{D}}\int_{\mathcal{D}} \mu(z)\big(\partial_n p_n(z,z',t)\big)\big(K_n(z',z)-L_n(z',z)\big)\mathrm{d}^2z\,\mathrm{d}^2z' \\ &-\lim_{t\to 0}\int_{\mathcal{D}}\int_{\mathcal{D}} \mu(z)\big(\partial_n p_n(z,z',t)\big)L_n(z',z)\mathrm{d}^2z\,\mathrm{d}^2z', \end{aligned}$$

and the second integral, exactly the one appearing in the first proof, goes to zero as before. Integrate by parts, use $\partial_{-1}\mu = 0$, and let $t \to 0$ to find

$$\delta_\mu \log\det\Delta_n = \int_{\mathcal{D}} \mu(z)\Big(\partial'_{1-n}\big(K_n(z,z') - L_n(z,z')\big)\Big)\big|_{z'=z} \mathrm{d}^2 z.$$

Now applying lemma 4.4 yields theorem 4.2.

Chapter 5

From Fuchsian to function groups (holomorphic anomaly)

The purpose of this chapter is to extend theorem 4.1, which is valid for the Fuchsian uniformization of the surface X, to a statement which is valid for any uniformization $X \simeq \Omega_0/\Sigma$ by a function group Σ.

Since K_n is automorphic, the only discrepancy when changing coordinates ("holomorphic anomaly") comes from the non-automorphic regularization term $\frac{1}{\pi}\frac{1}{z-z'}$.

Theorem 5.1. *Let $X \simeq \mathbb{H}/\Gamma$ be a Fuchsian uniformization of X with fundamental domain $\mathcal{D}$, and let $X \simeq \Omega_0/\Sigma$ be another uniformization by some function group Σ. Let $J : \mathbb{H} \to \Omega_0$ be the conformal, surjective, locally invertible covering map which respects the group actions. Let μ be a harmonic Beltrami differential on X (representing an element of $T_{[X]}T_g$). Then for $n \geq 1$,*

$$\partial \log\det\Delta_n(\mu) = \int_{J(\mathcal{D})} \mu^{\Sigma}(w)\Big(n\partial_{w'} - (1-n)\partial_w\Big)\left(K_n^{\Sigma}(w,w') - \frac{1}{\pi}\frac{1}{w-w'}\right)\Bigg|_{\Delta} \mathrm{d}^2 w$$
$$+ \frac{6n^2-6n+1}{6\pi}\int_{\mathcal{D}} \mu^{\Gamma}(z)\mathcal{S}(J)(z)\mathrm{d}^2 z,$$

where $\mathcal{S}$ represents the Schwarzian derivative,

$$\mathcal{S}(J) = \left(\frac{J''}{J'}\right)' - \frac{1}{2}\left(\frac{J''}{J'}\right)^2.$$

Proof. Begin by splitting the integral:

$$\begin{aligned}
&\int_{\mathcal{D}} \mu^{\Gamma}(z)(n\partial_{z'} - (1-n)\partial_z)(K_n^{\Gamma}(z,z') - \frac{1}{\pi}\frac{1}{z-z'})|_{\Delta}\mathrm{d}^2 z \\
&= \int_{\mathcal{D}} \mu^{\Gamma}(z)(n\partial_{z'} - (1-n)\partial_z)((K_n^{\Sigma}(J(z),J(z')) - \frac{1}{\pi}\frac{1}{J(z)-J(z')})J'(z)^n J'(z')^{1-n})|_{\Delta}\mathrm{d}^2 z \\
&\qquad + \frac{1}{\pi}\int_{\mathcal{D}} \mu^{\Gamma}(z)(n\partial_{z'} - (1-n)\partial_z)(\frac{J'(z)^n J'(z')^{1-n}}{J(z)-J(z')} - \frac{1}{z-z'})|_{\Delta}\mathrm{d}^2 z
\end{aligned}$$

The second integral will give the term involving the Schwarzian, and will be discussed below. The first integral is

$$\begin{aligned}
&\int_{\mathcal{D}} \mu^{\Sigma}(J(z))\frac{\overline{J'(z)}}{J'(z)}(nJ'(z')\partial_{w'} - (1-n)J'(z)\partial_w)(K_n^{\Sigma}(w,w') - \frac{1}{\pi}\frac{1}{w-w'})|_{\Delta}J'(z)\mathrm{d}^2 z \\
&+ \int_{\mathcal{D}} \mu^{\Gamma}(z)(K_n^{\Sigma}(w,w') - \frac{1}{\pi}\frac{1}{w-w'})(n\partial_{z'} - (1-n)\partial_z)(J'(z)^n J'(z')^{1-n})|_{\Delta}\mathrm{d}^2 z,
\end{aligned}$$

where $w = J(z)$, $w' = J(z')$. The second term vanishes, since

$$(n\partial_{z'} - (1-n)\partial_z)(J'(z)^n J'(z')^{1-n}) = n(1-n)J'(z)^n J'(z')^{1-n}(\frac{J''}{J'}(z') - \frac{J''}{J'}(z)),$$

which vanishes on the diagonal, is multiplied by a factor which is nonsingular as z approaches z'. For the same reason, the first term becomes

$$\int_{J(\mathcal{D})} \mu^{\Sigma}(w)(n\partial_{w'} - (1-n)\partial_w)(K_n^{\Sigma}(w,w') - \frac{1}{\pi}\frac{1}{w-w'})|_{\Delta}\mathrm{d}^2 w$$

as required.

Turning to the second integral, we must show that

$$\lim_{z\to z'}(n\partial_{z'} - (1-n)\partial_z)(\frac{J'(z)^n J'(z')^{1-n}}{J(z)-J(z')} - \frac{1}{z-z'}) = \frac{6n^2-6n+1}{6}\mathcal{S}(J)(z).$$

For $n = 1$, this assertion becomes

$$\frac{J'(z)J'(z')}{(J(z)-J(z'))^2} = \frac{1}{(z-z')^2} + \frac{1}{6}\mathcal{S}(J) + O(z-z'),$$

a classical result. We will use the $n = 1$ case to prove the formula for $n > 1$. For $n = 1$, simply expand:

$$\begin{aligned}
\frac{J'(z)J'(z')}{(J(z)-J(z'))^2} &= \frac{1}{(z-z')^2}\frac{1+\frac{J''}{J'}\cdot(z-z')+\frac{1}{2}\frac{J'''}{J'}\cdot(z-z')^2+\ldots}{(1+\frac{1}{2}\frac{J''}{J'}\cdot(z-z')+\frac{1}{6}\frac{J'''}{J'}\cdot(z-z')^2+\ldots)^2} \\
&= \frac{1}{(z-z')^2}(1+\frac{J''}{J'}\cdot(z-z')+\frac{1}{2}\frac{J'''}{J'}\cdot(z-z')^2+\ldots) \\
&\quad \cdot(1-\frac{J''}{J'}\cdot(z-z')+(\frac{3}{4}\left(\frac{J''}{J'}\right)^2-\frac{1}{3}\frac{J'''}{J'})(z-z')^2+\ldots) \\
&= \frac{1}{(z-z')^2}(1+(\frac{1}{6}\frac{J'''}{J'}-\frac{1}{4}\left(\frac{J''}{J'}\right)^2)(z-z')^2+\ldots) \\
&= \frac{1}{(z-z')^2}+\frac{1}{6}\mathcal{S}(J)+O(z-z').
\end{aligned}$$

(The dependence of J''/J', always on z', has been suppressed.) The formula for $n > 1$ may be proved similarly by brute force, but it saves some work to add and subtract a term so that the $n = 1$ formula may be used:

$$\begin{aligned}
&\lim_{z\to z'}(n\partial_{z'} - (1-n)\partial_z)(\frac{J'(z)^n J'(z')^{1-n}}{J(z)-J(z')} - \frac{1}{z-z'}) \\
&\quad = \lim_{z\to z'}\left(n(n-1)J'(z)^n J'(z')^{1-n}\frac{\frac{J''}{J'}(z') - \frac{J''}{J'}(z)}{J(z)-J(z')} \right. \\
&\qquad \left. + \frac{nJ'(z') + (1-n)J'(z)}{(J(z)-J(z'))^2}J'(z)^n J'(z')^{1-n} - \frac{1}{(z-z')^2} \right) \\
&\quad = n(n-1)\left(\frac{J''}{J'}\right)' + \frac{1}{6}\mathcal{S}(J) \\
&\qquad + \lim_{z\to z'}\left(\frac{nJ'(z') + (1-n)J'(z)}{(J(z)-J(z'))^2}J'(z)^n J'(z')^{1-n} - \frac{J'(z)J'(z')}{(J(z)-J(z'))^2} \right).
\end{aligned}$$

Now all that remains is to show that the term under the limit goes to

$$-\frac{n(n-1)}{2}\left(\frac{J''}{J'}\right)^2.$$

Use the $n = 1$ case again, and expand:

$$\begin{aligned}
&\lim_{z\to z'}\left(\frac{nJ'(z') + (1-n)J'(z)}{(J(z)-J(z'))^2}J'(z)^n J'(z')^{1-n} - \frac{J'(z)J'(z')}{(J(z)-J(z'))^2}\right)\\
&\quad= \lim_{z\to z'}\frac{J'(z)J'(z')}{(J(z)-J(z'))^2}(nJ'(z)^{n-1}J'(z')^{1-n} + (1-n)J'(z)^n J'(z')^{-n} - 1)\\
&\quad= \lim_{z\to z'}\left(\frac{1}{(z-z')^2} + O(1)\right)\\
&\qquad\cdot\left(n\left(1 + \frac{J''}{J'}\cdot(z-z') + \frac{1}{2}\frac{J'''}{J'}\cdot(z-z')^2 + \dots\right)^{n-1}\right.\\
&\qquad\left.+(1-n)\left(1 + \frac{J''}{J'}\cdot(z-z') + \frac{1}{2}\frac{J'''}{J'}\cdot(z-z')^2 + \dots\right)^{n} - 1\right)\\
&\quad= \lim_{z\to z'}\left(\frac{1}{(z-z')^2} + O(1)\right)\\
&\qquad\cdot\left(n\left(1 + (n-1)\frac{J''}{J'}\cdot(z-z')\right.\right.\\
&\qquad\quad\left.+\left(\frac{n-1}{2}\frac{J'''}{J'} + \frac{(n-1)(n-2)}{2}\left(\frac{J''}{J'}\right)^2\right)(z-z')^2 + \dots\right)\\
&\qquad+(1-n)\left(1 + n\frac{J''}{J'}\cdot(z-z')\right.\\
&\qquad\quad\left.\left.+\left(\frac{n}{2}\frac{J'''}{J'} + \frac{n(n-1)}{2}\left(\frac{J''}{J'}\right)^2\right)(z-z')^2 + \dots\right) - 1\right)\\
&\quad= -\frac{n(n-1)}{2}\left(\frac{J''}{J'}\right)^2.
\end{aligned}$$

This completes the proof of the theorem. $\square$

Chapter 6

Second variation of $\log \det \Delta_n$

This chapter is devoted to an alternate proof of the rigorous Belavin-Knizhnik formula due to Zograf and Takhtajan:

Theorem 6.1 ([ZT87a]). *Let μ and ν be harmonic Beltrami differentials on the surface X (representing elements of $T_{[X]}T_g$). Recall that $[N_n]_{jk} = (\phi_j, \phi_k)$ for a choice $\{\phi_1, \ldots, \phi_d\}$ of a basis for holomorphic n-differentials, varying holomorphically on T_g in a neighbourhood of $[X] \in T_g$. Recall also that $(\mu, \nu) = \int_X \rho\mu\overline{\nu}$ is the Weil-Petersson inner product. Then*

$$\overline{\partial}\partial \log \left(\frac{\det \Delta_n}{\det N_n} \right) (\mu, \nu) = \frac{6n^2 - 6n + 1}{12\pi} (\mu, \nu).$$

The proof in [ZT87a] is essentially different in the cases $n = 1$ and $n \geq 2$; we give here a simpler proof of the $n \geq 2$ case along the lines of the $n = 1$ proof in [ZT87a]. (This is possible since we have established a "good" description of the field of quadratic differentials corresponding to the $(1, 0)$ form $\delta \log \det \Delta_n$ on T_g; see [ZT87a], p.184.)

It will be convenient to define

$$\omega_j = \sum_{k=1}^{d} [N_n^{-1}]_{kj} \rho^{1-n} \overline{\phi}_k,$$

where $\{\phi_1, \ldots, \phi_d\}$ is our choice of basis of the space of holomorphic n-differentials varying holomorphically on T_g, so we may write the kernel of the projector on holomorphic n-differentials as

$$P_n(w, w') = \sum_{j=1}^{d} \phi_j(w) \omega_j(w').$$

Define also

$$\psi_j = \sum_{k=1}^{d} [N_n^{-1}]_{kj} F_k,$$

where F_j are Bers potentials for the ϕ_j as defined earlier, so we have $\overline{\partial}\psi_j = \omega_j$, and $K_n(w, w') = \widehat{K}_n(w, w') + \sum_{j=1}^{d} \phi_j(w)\psi_j(w')$. Recall the definition

$$T_n(w) = \Big(n\partial' - (1-n)\partial\Big) \left(K_n(w, w') - \frac{1}{\pi}\frac{1}{w - w'}\right)\Big|_{w'=w},$$

and make the abbreviation $|_\Delta = |_{w'=w}$.

Now, recall that we have

$$\delta_\mu \log \det \Delta_n = \int_{J(\mathcal{D})} \mu^\Sigma T_n^\Sigma + \frac{6n^2 - 6n + 1}{6\pi} \int_{\mathcal{D}} \mu^\Gamma \mathcal{S}(J),$$

where Σ is some function group uniformizing X. We will take Σ to be a Schottky group uniformizing X, since in that case all of the group parameters are holomorphic functions on T_g.

$$\overline{\delta}_\nu \delta_\mu \log\det\Delta_n = \int (\overline{\delta}_\nu \mu) T_n + \int \mu(\overline{\delta}_\nu T_n) + \frac{6n^2 - 6n + 1}{6\pi} \overline{\delta}_\nu \int \mu \mathcal{S}(J).$$

We compute each term in this expression in turn.

Lemma 6.2.

$$\int (\overline{\delta}_\nu \mu) T_n = (n-1)\,\mathrm{Tr}((\mu\overline{\nu} - (\rho^{-1}\overline{\delta}_\nu \delta_\mu \rho))P_n).$$

Proof.

$$\begin{aligned}
\int (\overline{\delta}_\nu \mu) T_n &= -2 \int T_n \overline{\partial} \rho^{-1} \overline{\partial}(\rho^{-1}\overline{\delta}_\nu \delta_\mu \rho) \\
&= 2 \int (\overline{\partial} T_n) \rho^{-1} \overline{\partial}(\rho^{-1}\overline{\delta}_\nu \delta_\mu \rho) \\
&= 2(n-1) \int \partial_1 (P_n|_\Delta) \rho^{-1} \overline{\partial}(\rho^{-1}\overline{\delta}_\nu \delta_\mu \rho) \\
&= 2(n-1) \int (P_n|_\Delta) \Delta_0 (\rho^{-1}\overline{\delta}_\nu \delta_\mu \rho) \\
&= (n-1) \int (P_n|_\Delta)(\mu\overline{\nu} - \rho^{-1}\overline{\delta}_\nu \delta_\mu \rho).
\end{aligned}$$

In the third line we have used

$$\overline{\partial} T_n = (n-1)\partial_1 (P_n|_\Delta),$$

which follows from

$$\begin{aligned}
\overline{\partial} T_n(w) &= \left(\left(\overline{\partial} + \overline{\partial}'\right)\left(n\partial' - (1-n)\partial\right) \left(K_n(w,w') - \frac{1}{\pi}\frac{1}{w-w'} \right) \right) \Big|_\Delta \\
&= \left(n\partial' - (1-n)\partial\right)\overline{\partial}' K_n \big|_\Delta \\
&= (n-1)\partial_1(P_n\big|_\Delta) + (\partial'_{1-n}P_n)\big|_\Delta
\end{aligned}$$

since the second term is zero. □

Lemma 6.3.

$$\int \mu(\overline{\delta}_\nu T_n) = -n\,\mathrm{Tr}((\mu\overline{\nu}P_n)) - \int \mu(\partial_n\overline{\delta}_\nu K_n^\Sigma)|_\Delta.$$

Proof.

$$\begin{aligned}
\int \mu(\overline{\delta}_\nu T_n) &= \int \mu\overline{\delta}_\nu(((n\partial' - (1-n)\partial)(K_n - \frac{1}{\pi}\frac{1}{w-w'}))|_\Delta) \\
&= \int \mu((-n\overline{\nu}\overline{\partial}' + (1-n)\overline{\nu}\overline{\partial})(K_n - \frac{1}{\pi}\frac{1}{w-w'}))|_\Delta \\
&\quad + \int \mu((n\partial' - (1-n)\partial)(\overline{\delta}_\nu K_n))|_\Delta \\
&= -n\int \mu\overline{\nu}(\overline{\partial}' K_n)|_\Delta + n\int \mu\partial_1((\overline{\delta}_\nu K_n)|_\Delta) - \int \mu(\partial_n\overline{\delta}_\nu K_n)|_\Delta \\
&= -n\,\mathrm{Tr}((\mu\overline{\nu})P_n) - n\int (\partial_{-1}\mu)(\overline{\delta}_\nu K_n)|_\Delta - \int \mu(\partial_n\overline{\delta}_\nu K_n)|_\Delta,
\end{aligned}$$

and $\partial_{-1}\mu = 0$. □

Lemma 6.4.

$$\overline{\delta}_\nu \int \mu\mathcal{S}(J) = \frac{1}{2}(\mu,\nu).$$

Proof. This is proved in [ZT87b]; for the convenience of the reader, we reproduce the proof here. Remembering that $\mathcal{S}(J)$ is a quadratic differential with respect to Γ,

$$\overline{\delta}_\nu \int \mu \mathcal{S}(J) = \int (\overline{\delta}_\nu \mu) \mathcal{S}(J) + \int \mu (\overline{\delta}_\nu \mathcal{S}(J)).$$

Since $\mathcal{S}(J)$ is holomorphic, it represents a differential form of type $(1,0)$ on T_g. On the other hand, $\overline{\delta}_\nu \mu$ is a vector field of type $(0,-1)$ on T_g, so the two pair to give zero, showing that the first integral vanishes. To calculate the second integral, let $\tilde{f}^{\epsilon\nu}$ be the quasiconformal deformation of the ordinary set Ω of Σ satisfying

$$\tilde{f}^{\epsilon\nu} \circ J = J^{\epsilon\nu} \circ f^{\epsilon\nu}.$$

Since Σ is a Schottky group, $\tilde{f}^{\epsilon\nu}$ depends holomorphically on ϵ. Taking the Schwarzian derivative of this equation gives

$$\mathcal{S}(\tilde{f}^{\epsilon\nu}) \circ J \cdot (J')^2 + \mathcal{S}(J) = \mathcal{S}(J^{\epsilon\nu}) \circ f^{\epsilon\nu} (\partial f^{\epsilon\nu})^2 + \mathcal{S}(f^{\epsilon\nu}),$$

and taking $\frac{\partial}{\partial \overline{\epsilon}}\Big|_{\epsilon=0}$ yields

$$0 = \overline{\delta}_\nu(\mathcal{S}(J)) + \Phi_\nu''' = \overline{\delta}_\nu(\mathcal{S}(J)) - \frac{1}{2}\rho\overline{\nu},$$

and the result follows. $\square$

Now for the other side of the equation.

Lemma 6.5.

$$\overline{\delta}_\nu \delta_\mu \log \det N_n = -\int \mu(\partial_n \overline{\delta}_\nu K_n)|_\Delta + \mathrm{Tr}((-\mu\overline{\nu} + (1-n)(\rho^{-1}\overline{\delta}_\nu \delta_\mu \rho))P_n).$$

Proof. First note that

$$\begin{aligned}
\int \sum_j (\delta_\mu \phi_j)(\overline{\delta}_\nu \omega_j) &= \int \sum_j (\delta_\mu \phi_j)\overline{\partial}(\overline{\delta}_\nu \psi_j) \\
&= -\int \sum_j (\overline{\partial}\delta_\mu \phi_j)(\overline{\delta}_\nu \psi_j) + \int \overline{\partial}(\sum_j (\delta_\mu \phi_j)(\overline{\delta}_\nu \psi_j)) \\
&= -\int \mu \sum_j (\partial_n \phi_j)(\overline{\delta}_\nu \psi_j) + 0 \\
&= -\int \mu (\partial_n \overline{\delta}_\nu K_n)|_\Delta.
\end{aligned}$$

The integration by parts is justified because, although ψ_j is an Eichler integral, $\overline{\delta}_\nu \psi_j$ is a genuine $(1-n,0)$-differential: since all quantities in $\widehat{K}_n$ depend on Schottky parameters, $\overline{\delta}\widehat{K}_n = 0$, and consequently the Eichler periods of Z_n vary holomorphically as well:

$$\sum_j \phi_j(w)(\overline{\delta}_\nu \psi_j)(\gamma w')\gamma'(w')^{1-n} = \sum_j \phi_j(w)(\overline{\delta}_\nu \psi_j)(w').$$

and integrating this equation against $\psi_k(w)$ gives

$$(\overline{\delta}_\nu \psi_k)(\gamma w')\gamma'(w')^{1-n} = (\overline{\delta}_\nu \psi_k)(w')$$

Now calculate:

$$
\begin{aligned}
\overline{\delta}_\nu \delta_\mu \log \det N_n &= \overline{\delta}_\nu \mathrm{tr}(N_n^{-1}(\delta_\mu N_n)) \\
&= \overline{\delta}_\nu \sum_j \sum_k [N_n^{-1}]_{kj} [\delta_\mu N_n]_{jk} \\
&= \int \sum_j (\delta_\mu \phi_j)(\overline{\delta}_\nu \omega_j) + (1-n) \int \phi_j \omega_j (\rho^{-1} \overline{\delta}_\nu \delta_\mu \rho) - \int \phi_j \omega_j (\mu \overline{\nu}) \\
&= -\int \mu (\partial_n \overline{\delta}_\nu K_n)|_\Delta + \mathrm{Tr}((-\mu\overline{\nu} + (1-n)(\rho^{-1}\overline{\delta}_\nu \delta_\mu \rho))P_n).
\end{aligned}
$$

$\square$

Collecting the results of the previous four lemmas establishes the theorem.

Chapter 7

Zograf's product formula for $\det \Delta_1$

In this chapter we prove the following result, due to Zograf [Zog89], [Zog97]:

Theorem 7.1. *Let the surface X be uniformized by a Fuchsian group, $X \simeq \mathbb{H}/\Gamma$, and by a Schottky group, $X \simeq \Omega_0/\Sigma$, and let $J : \mathbb{H} \to \Omega_0$ be the locally invertible holomorphic surjection respecting the group actions. Let $S : S_g \to \mathbb{R}_{>0}$ be the function on Schottky space defined in [ZT87b], (the "classical Liouville action", a Kähler potential for the Weil-Petersson metric) satisfying $\delta_\mu S = 2\int_X \mu \mathcal{S}(J)$, where $\mathcal{S}$ is the schwarzian derivative. Let $\{\phi_1, \dots, \phi_g\}$ be a basis of holomorphic 1-differentials on X, normalized with respect to the Schottky marking (taking the bounding circles of a fundamental domain as A-cycles), and let $[N_1]_{jk} = (\phi_j, \phi_k)$. Denote the multiplier of an element $\gamma \in \Sigma$ by λ_γ, $0 < |\lambda_\gamma| < 1$.*

Now, suppose Σ is such that $\sum_{\gamma\in\Sigma} |\gamma'(w)|$ converges. Then

$$\det \Delta_1 = c_g |F|^2 \det N_1 \exp(\frac{1}{12\pi} S),$$
$$\textit{where} \quad F = \prod_{[\gamma]} \prod_{m=1}^{\infty} (1 - \lambda_\gamma^m),$$

the product being taken over primitive conjugacy classes in Σ. c_g is a constant on T_g, depending only on g, which we do not determine.

Remark 7.2.

1. The set of groups satisfying the convergence condition is a proper open subset of the Schottky space S_g, except for $g = 1$, when it is the entire space.

2. F is a holomorphic function on a subset of S_g, because the multipliers λ_g are holomorphic on T_g and (obviously) well-defined on S_g.

3. The result is actually true not only on an open set of T_g but descends to S_g, because $\det N_1$ turns out to be single valued on S_g (see [Zog89]).

4. The function F actually admits analytic continuation to the whole of S_g, by [Zog89], and the formula for $\det \Delta_1$ remains valid, though the product formula for F is only valid under the convergence hypothesis in the theorem.

Proof. It has been established so far that

$$\delta_\mu \log \det \Delta_1 = \int \mu \partial' \left(K_1(w, w') - \frac{1}{\pi} \frac{1}{w - w'} \right) \Big|_{w'=w} + \frac{1}{6\pi} \int \mu \mathcal{S}(J)$$

where

$$K_1(w, w') = \sum_{\gamma \in \Sigma} \frac{1}{\gamma w - w'} \gamma'(w) + \sum_{j=1}^{g} \sum_{k=1}^{g} \phi_j(w) [N_1^{-1}]_{kj} F_k(w').$$

We would therefore like to express each term on the right as the variation of some function on the Teichmüller space T_g.

Zograf and Takhtajan have found a positive real-valued function S on T_g (the "classical Liouville action") such that

$$\delta_\mu \frac{1}{2} S = \int \mu \mathcal{S}(J),$$

and moreover they give a formula for S as an explicit integral defined on the Schottky uniformization of X. We do not reproduce their arguments here, but refer the reader to [ZT87b].

Next we show that

$$\partial F = \partial' \left(K_1(w, w') - \frac{1}{\pi} \frac{1}{w - w'} \right) \Big|_{w'=w}.$$

The equation should be understood to mean that the holomorphic derivative on T_g on the left is represented by the holomorphic quadratic differential on X given on the right, so that for any harmonic Beltrami differential μ on X,

$$\partial F(\mu) = \int \mu \partial' \left(K_1(w, w') - \frac{1}{\pi} \frac{1}{w - w'} \right) \Big|_{w'=w}.$$

We will need the standard formula ([Zog89])

$$\frac{\partial \lambda_\gamma}{\lambda_\gamma} = -\frac{1}{\pi} \sum_{\alpha \in \langle \gamma \rangle \backslash \Sigma} \frac{(a_\gamma - b_\gamma)^2}{(\alpha w - a_\gamma)^2 (\alpha w - b_\gamma)^2} \alpha'(w)^2,$$

where a_γ, b_γ are the attracting and repelling fixed points respectively of γ, and the sum is extended over the quotient, on the left, of Σ by the cyclic group generated by γ.

First we do the usual manipulation

$$\begin{aligned}
\log F &= \sum_{[\gamma]} \sum_{m=1}^{\infty} \log(1-\lambda_\gamma^m) \\
&= -\sum_{[\gamma]} \sum_{m=1}^{\infty} \sum_{r=1}^{\infty} \frac{\lambda_\gamma^{mr}}{m} \\
&= -\sum_{[\gamma]} \sum_{m=1}^{\infty} \frac{1}{m} \frac{\lambda_\gamma^m}{1-\lambda_\gamma^m},
\end{aligned}$$

then take the derivative on T_g (again identified with a quadratic differential on X),

$$\begin{aligned}
\partial \log F &= -\sum_{[\gamma]} \sum_{m=1}^{\infty} \frac{\lambda_\gamma^m}{(1-\lambda_\gamma^m)^2} \frac{\partial \lambda_\gamma}{\lambda_\gamma} \\
&= \frac{1}{\pi} \sum_{[\gamma]} \sum_{m=1}^{\infty} \frac{\lambda_\gamma^m}{(1-\lambda_\gamma^m)^2} \sum_{\alpha \in \langle\gamma\rangle \backslash \Sigma} \frac{(a_\gamma - b_\gamma)^2}{(\gamma w - a_\gamma)^2 (\gamma w - b_\gamma)^2} \alpha'(w)^2 \\
&= \frac{1}{\pi} \sum_{[\gamma]} \sum_{m=1}^{\infty} \sum_{\alpha \in \langle\gamma\rangle \backslash \Sigma} \frac{(\gamma^n)'(\alpha w)}{(\gamma^n \alpha w - \alpha w)^2} \alpha'(w)^2 \\
&= \frac{1}{\pi} \sum_{[\gamma]} \sum_{m=1}^{\infty} \sum_{\alpha \in \langle\gamma\rangle \backslash \Sigma} \frac{1}{(\alpha^{-1}\gamma^n \alpha w - w)^2} (\alpha^{-1}\gamma^n \alpha w)'(w)^2 \\
&= \frac{1}{\pi} \sum_{\gamma} \frac{1}{(\gamma w - w)^2} \gamma'(w)^2 \\
&= \partial' \widehat{K}_1(w, w')|_{w'=w}
\end{aligned}$$

where in the third line we have combined the definition

$$\lambda_\gamma = \frac{w - b_\gamma}{w - a_\gamma} \cdot \frac{\gamma w - a_\gamma}{\gamma w - b_\gamma}$$

and its derivative

$$\lambda_\gamma = \left(\frac{w - b_\gamma}{\gamma w - b_\gamma}\right)^2 \gamma'(w)$$

to obtain

$$\frac{\lambda_\gamma^m}{(1-\lambda_\gamma^m)^2}\frac{(a_\gamma - b_\gamma)^2}{(w-a_\gamma)^2(w-b_\gamma)^2} = \frac{(\gamma^n)'(w)}{(\gamma^n w - w)^2}.$$

As for the zero mode term,

$$\begin{aligned}
\partial \log \det N_1 &= \operatorname{tr}(\partial \log N_1) \\
&= \operatorname{tr}(N_1^{-1}\partial N_1) \\
&= \sum_{j=1}^{g}\sum_{k=1}^{g}[N_1^{-1}]_{kj}\partial[N_1]_{jk} \\
&= -\sum_{j=1}^{g}\sum_{k=1}^{g}[N_1^{-1}]_{kj}\phi_j(w)\phi_k(w) \\
&= \sum_{j=1}^{g}\sum_{k=1}^{g}[N_1^{-1}]_{kj}\phi_j(w)\partial F_k(w),
\end{aligned}$$

where we have used Rauch's formula, and have used that

$$F_k(w) = \overline{\int_a^w \phi_k(\zeta)\mathrm{d}\zeta} - \int_a^w \phi_k(\zeta)\mathrm{d}\zeta$$

when $n = 1$.

Summarizing, we have

$$\delta_\mu \log \det \Delta_1 = \delta_\mu \log\left(F \det N_1 \exp(\frac{1}{12\pi}S)\right).$$

Further, noting that all quantities are real except F,

$$\overline{\delta_\mu} \log \det \Delta_1 = \overline{\delta_\mu} \log \left(\overline{F} \det N_1 \exp(\frac{1}{12\pi} S) \right).$$

Now since F is holomorphic, we can integrate up to an overall constant:

$$\det \Delta_1 = c_g |F|^2 \det N_1 \exp(\frac{1}{12\pi} S).$$

This establishes Zograf's formula. $\square$

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