Hyperbolic 3-manifolds and Geodesics in Teichmüller Space

A Dissertation, Presented
by
Kasra Rafi
to
The Graduate School
in Partial Fulfillment of the
Requirements
for the Degree of
Doctor of Philosophy
in
Mathematics
State University of New York
at Stony Brook
August 2001
State University of New York
at Stony Brook
The Graduate School

Kasra Rafi

We, the dissertation committee for the above candidate for the Doctor of Philosophy degree, hereby recommend acceptance of this dissertation.

Yair Minsky
Professor, Department of Mathematics
Dissertation Director

Detlef Gromoll
Leading Professor, Department of Mathematics
Chairman of Dissertation

Bernard Maskit
Leading Professor, Department of Mathematics

Folkert Tangerman
Professor, Department of Applied Mathematics and Statistics
Outside Member

This dissertation is accepted by the Graduate School.

Graduate School
Abstract of the Dissertation

Hyperbolic 3-manifolds and Geodesics in Teichmüller Space

by

Kasra Rafi

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

2001

We are interested in studying the geometry of hyperbolic 3-manifolds homotopy equivalent to a given compact 3-manifold $M$. When $M$ is a surface bundle over a circle, or when $M$ is an interval bundle over a surface, one can associate to such a hyperbolic 3-manifold a geodesic in $\mathcal{T}(S)$, the Teichmüller space of the surface $S$. The interplay between the geometry of hyperbolic 3-manifolds and geodesics in the Teichmüller space is the basic subject of our study. More specifically, our goal is to predict the behavior of geodesics in $\mathcal{T}(S)$ based on their end points, then apply the results in studying the geometry of hyperbolic 3-manifolds. Our results are, in summary:
1. A hyperbolic 3-manifold has bounded geometry if and only if the corresponding Teichmüller geodesic stays in the thick part of $T(S)$.

2. In general, every curve that is short in a hyperbolic 3-manifold is also short for some metric in the corresponding Teichmüller geodesic.

3. The converse of the second statement above is not true.
Contents

List of Figures vi
Acknowledgements viii

1 Introduction 1

2 Simple Closed Curves 8

3 Quadratic Differentials 12

4 Thick-thin Decomposition for a Quadratic Differential Metric on a Surface 22

5 The Teichmüller Space of $S$ 33

6 Subsurface Coefficients and Teichmüller Geodesics 35

7 A Counterexample 45

8 The Universal Curve and the Hyperbolic 3-space 57

Bibliography 64
List of Figures

3.1 Local pictures of a foliation by trajectories ............... 15

7.1 Quadratic differential $q_n$ ......................... 47
7.2 $t = n.$ ........................................... 49
7.3 $t = 2n.$ ........................................... 50
7.4 $X_\infty$ and $X'_\infty.$ ............................ 51
To Amy
Acknowledgements

I would like to express my enthusiastic gratitude to my advisor, Yair Minsky, for suggesting my thesis problem and for his guidance. He has been a wonderful mentor, devoting much time to working with me, and I have learned a great deal from him through our illuminating discussions. Bernie Maskit, Dennis Sullivan, and Misha Lyubich have also been instrumental to my mathematical growth. I am indebted to Slava Matveyev for many inspirational and enjoyable conversations. It has been a great pleasure working with my good friend Harish Seshadri, and I have benefitted from many fruitful discussions with Siddhartha Gadgil.

Ramón del Castillo has been a great friend and colleague throughout my years at Stony Brook. I would also like to express my appreciation and affection to Reza Chamanara, Amir Jafari, Jason Behrstock, Hossein Abbaspour, Mahmoud Zeinalian, Rûkmini Dey, Joe Coffey, Rodrigo Perez, Lee-Peng Teo and Sergei Panafidin for their friendship and mathematical insight.

Amy McKenzie has enriched my life by giving me love, companionship and encouragement. I would also like to acknowledge her assistance in editing my thesis. Finally, I thank my family for nurturing me and supporting my mathematical aspirations.
Chapter 1

Introduction

Let $M$ be a hyperbolizable compact 3-manifold with incompressible boundary. Define $h(M)$ to be the space of complete hyperbolic structures on the interior of $M$. Elements of $h(M)$ are infinite volume hyperbolic 3-manifolds homeomorphic to the interior of $M$. We call the intersection of $N \in h(M)$ with a sufficiently small neighborhood of a boundary component of $M$ an end of $N$. The end of $N$ corresponding to the boundary component $S$ of $M$ is homeomorphic to $S \times (0, \infty)$. To this end one can assign an invariant $\nu_S$ in an enlargement $\hat{T}(S)$ of the Teichmüller space of $S$, related to Thurston’s compactification of the Teichmüller space. If an end is geometrically finite, that is if the conformal structure on $S \times \{t\}$ stabilizes as $t$ approaches infinity, $\nu_S$ is defined to be the limiting conformal structure on $S$, i.e., it is a point in $T(S)$. The subset of $h(M)$ containing all hyperbolic 3-manifolds whose ends are geometrically finite is well understood through the work of Ahlfors, Bers, Kra, Marden, Maskit and others (e.g., [1], [2], [12] and [14]). If an end is simply degenerate, $\nu_S$ is a geodesic lamination on $S$. The existence of end invariants for geometrically infinite ends is due to Bonahon and Thurston ([3],
Thurston's ending lamination conjecture states that these invariants are sufficient to determine the hyperbolic 3-manifold $N$.

**Conjecture 1.1.** *(Thurston)* **The topological type of $N$ and its end invariants determine $N$ up to isometry.**

**The main theorem**

For simplicity, we restrict our discussion to the case where $M = S \times I$ is a product of a surface and an interval and $N \in \mathfrak{h}(M)$ has no geometrically finite end. With each such $N$ we associate the following objects:

1. Its end invariants $\nu_+$ and $\nu_-$. Since $M$ has two boundary components homeomorphic to $S$, both these invariants are laminations on $S$. We often write $(N, \nu_+, \nu_-)$ to emphasize the end invariants.

2. The geodesic $g : \mathbb{R} \to \mathcal{T}(S)$ in the Teichmüller space of $S$ connecting $\nu_+$ and $\nu_-$, where $\mathcal{T}(S)$ is equipped with the Teichmüller metric. We represent this geodesic by $(g, \nu_+, \nu_-)$.

3. The universal curve $U_g$ over this geodesic. $U_g$ is a 3-manifold homeomorphic to $S \times \mathbb{R}$ where for every $t \in \mathbb{R}$, $S \times \{t\}$ is isometric to $g(t)$.

4. A coefficient $d_Y(\nu_+, \nu_-)$ to every isotopy class of an essential subsurface $Y$ of $S$. This coefficient measures the relative complexity of $\nu_+$ and $\nu_-$ restricted to $Y$ (see [16] for definition and discussion).

We say $N$ has bounded geometry, if there exists a universal lower bound on the injectivity radius of $N$, and we say $g$ has bounded geometry if there
exists an $\epsilon$ such that $g$ does not intersect the $\epsilon$-thin part of the Teichmüller space (see chapter 5 for definition of $\epsilon$-thin part of $T(S)$).

In the case when $\nu_+$ and $\nu_-$ are stable and unstable laminations of a pseudo-Anosov homeomorphism, both $N$ and $g$ have bounded geometry; in fact Cannon and Thurston ([7]) showed that $U_g$ is quasi-isometric to $N$. We would like to compare the geometry of $N$ and the universal curve $U_g$ in the general case, to examine whether $N$ and $U_g$ have the same set of short curves. Our vehicle for making this connection is the combinatorial machinery introduced by Masur and Minsky ([16]), and the subsurface coefficients $d_Y(\nu_+, \nu_-)$ (see chapter 2).

**Theorem 1.2.** The following are equivalent:

1. $N$ has bounded geometry.

2. $g$ has bounded geometry.

3. $\tilde{U}_g$, the universal cover of $U_g$, is bi-Lipschitz to $\mathbb{H}^3$.

4. There exists $K$ such that for every essential subsurface $Y$ of $S$ we have 

   $$d_Y(\nu_+, \nu_-) < K.$$ 

In [22] and [25] Minsky has shown that $1 \Rightarrow 2$, $1 \Rightarrow 3$ and $1 \Leftrightarrow 4$.

**Theorem 1.3.** (Minsky [22], [25]) $N$ has bounded geometry if and only if the collection of surface coefficients $\{d_Y(\nu_+, \nu_-)\}$ is bounded above. Moreover, in this case the universal curve $U_g$ is quasi-isometric to $N$. 

3
Note that this theorem in particular proves the ending lamination conjecture in the case of bounded geometry, since if \( N_1, N_2 \in \mathcal{H}(M) \) have the same end invariants \( \nu_+ \) and \( \nu_- \), then \( N_1 \) and \( N_2 \) are both quasi-isometric to the same universal curve \( U_\rho \), and therefore to each other. A theorem of Sullivan ([28]) implies that \( N_1 \) and \( N_2 \) are isometric.

To complete Theorem 1.2, we prove \( 2 \Leftrightarrow 4 \) (Chapter 6) and \( 3 \Rightarrow 2 \) (Chapter 8) (see also, [20]).

**Short curves in 3-manifolds and Teichmüller geodesics**

In the case that the geodesic \( g \) does not have bounded geometry, we would like to have a description of short curves in \( g \). Such a description is available for hyperbolic 3-manifolds. The following theorem states that the short curves in \( (N, \nu_+, \nu_-) \) are exactly the boundary components of subsurfaces of \( S \) where \( d_Y(\nu_+, \nu_-) \) is large.

**Theorem 1.4. (Minsky [26])** Let \( (N, \nu_+, \nu_-) \in \mathcal{H}(M) \).

1. For every \( \epsilon > 0 \) there exists a \( K > 0 \) such that if \( Y \) is a subsurface of \( S \) and \( \alpha \) is a component of the boundary of \( Y \), then \( d_Y(\nu_+, \nu_-) > K \) implies \( l_N(\alpha) < \epsilon \).

2. Conversely, for every \( K > 0 \) there exists \( \epsilon > 0 \) such that if \( \alpha \) is a curve on \( S \) and the length of \( \alpha \) in \( N \) is less than \( \epsilon \), then there exists a subsurface \( Y \) of \( S \) with \( \alpha \subset \partial Y \) such that \( d_Y(\nu_+, \nu_-) > K \).

In the case when \( S \) is the one-punctured torus, the “same” curves are short in both \( N \) and \( g \). This allows for a canonical construction of a model manifold for \( N \) which implies the ending lamination conjecture in this case (see...
Minsky [23], McMullen [17]). It is reasonable to try to prove an analogue of Theorem 1.4 for geodesics in $T(S)$, i.e., to prove that the same combinatorial condition would predict which curves are short in the geodesic $g$.

We have proved the following theorem, which is an analogue of the first part of Theorem 1.4. Let $(g, \nu_+, \nu_-)$ be a geodesic in $T(S)$, $Y$ be a subsurface of $S$, and $\alpha$ be a component of the boundary of $Y$. Define

$$l_g(\alpha) = \min_{\sigma \in g} l_{\sigma}(\alpha)$$

to be the shortest hyperbolic length of $\alpha$ over all hyperbolic metrics on $g$.

**Theorem 1.5.** For every $\epsilon > 0$, there exists a $K > 0$ such that, if $(g, \nu_+, \nu_-)$ is a geodesic in the Teichmüller space of $S$ and if $Y$ is a subsurface of $X$ with $d_Y(\nu_+, \nu_-) > K$, then there exists $t \in \mathbb{R}$ such that for each boundary component $\alpha$ of $Y$, $l_{g(t)}(\alpha) < \epsilon$.

The following is a corollary of the previous theorem and Theorem 1.4:

**Corollary 1.6.** Curves that are "short" in $N$ are also "short" in $g$.

We construct a counterexample to show that the converse of the above corollary, an analogue of Theorem 1.4 Part 2, is not true:

**Theorem 1.7.** There exist sequences of manifolds $(N_n, \nu_+^n, \nu_-^n)$, corresponding Teichmüller geodesics $g$, and a curve $\gamma$ in $S$ such that

$$l_{N_n}(\gamma) \to c > 0 \quad \text{as} \quad n \to \infty,$$

(1.1)
but

\[ l_{g_n}(\gamma) \to 0 \quad \text{as} \quad n \to \infty. \quad (1.2) \]

We are, however, able to prove the following weaker version of the converse:

**Theorem 1.8.** For every \( K \), there exists \( \epsilon > 0 \) such that, if \((g, \nu_+, \nu_-)\) is a geodesic in the Teichmüller space of \( S \), and if \( \alpha \) is a curve in \( S \) with \( l_{g(t)}(\alpha) < \epsilon \) for some \( t \in \mathbb{R} \), then there exists a subsurface \( Z \) of \( S \) disjoint from \( \alpha \) such that

\[ d_Z(\nu_+, \nu_-) > K. \]

**Outline of the paper**

We begin in chapter 2, by introducing the subsurface coefficients \( d_Y(\nu_+, \nu_-) \) and recalling some of their properties. In chapter 3 we discuss the geometry of the quadratic differential metric on a Riemann surface. The description of the \( \epsilon \)-decomposition of the quadratic differential metric is given in chapter 4. This is analogous to the thick-thin decomposition of the hyperbolic metric of a Riemann surface. As a corollary we drive an estimate for the quadratic differential length of intersecting curves. This estimate is used in an essential way in the proof of \( 2 \Leftrightarrow 4 \) in the main theorem, Theorem 1.2, which is given in chapter 6. The counter example, Theorem 1.7, is proven in chapter 7. In chapter 8 we finish the proof of Theorem 1.2 by proving \( 2 \Rightarrow 1 \).

**Notation**

To simplify our presentation we use the notations \( O, \succ \) and \( \asymp \) (defined
below) whenever the constants involved depend on the topology of $S$ only. That is, for a function $f$, $O(f)$ represents a function that is bounded above by $Cf$ for a constant $C$ depending on the topology of $S$ only. For two functions $f$ and $g$, $f \succ g$ means $f \geq cg$ and $f \asymp g$ means $cf \leq g \leq Cf$, where $C$ and $c$ depend on the topology of $S$ only.
Chapter 2

Simple Closed Curves

In this chapter we will introduce and study several structures involving simple closed curves on a surface $S$. Let $S$ be an orientable surface of finite type excluding sphere an torus. We define the complexity of $S$ as $\xi(S) = 3g + p$, where $g$ is the genus of $S$ and $p$ is the number of boundary components of $S$.

By a curve we mean a non-trivial, non-peripheral, simple closed curve in $S$. The free homotopy class of a curve $\alpha$ is denoted by $[\alpha]$. By an arc $\omega$ we mean a non-trivial arc with endpoints on the boundary of $S$. Here non-trivial means that $\omega$ cannot be pushed to the boundary of $S$. In case $S$ is not an annulus, $[\omega]$ represents the homotopy class of $\omega$ relative to the boundary of $S$. When $S$ is an annulus, $[\omega]$ is defined to be the homotopy class of $\omega$ relative to the endpoints of $\omega$.

Define $C(S)$ to be the set of all homotopy classes of curves and arcs on the surface $S$. We equip $C(S)$ with the following metric. For curves $\alpha$ and $\beta$ on $S$, let $d_S(\alpha, \beta)$ be equal to one, if $[\alpha] \neq [\beta]$ and $[\alpha]$ and $[\beta]$ have representatives that are disjoint from each other (for two arcs, being disjoint means having disjoint interior). Let the metric in $C(S)$ be the maximal metric having the
above property. That is, \( d_S(\alpha, \beta) = n \) if \( \alpha = \alpha_0, \alpha_1, \ldots, \alpha_n = \beta \) is the shortest sequence of curves on \( S \) such that \( \alpha_{i-1} \) is distance one from \( \alpha_i, i = 1, \ldots, n \).

**Theorem 2.1.** (Masur and Minsky, [15]) For \( \xi(S) \neq 0, 1 \) or 3, \( \mathcal{C}(S) \) has the following properties:

1. The diameter of \( \mathcal{C}(S) \) is infinite.

2. As a metric space, \( \mathcal{C}(S) \) is \( \delta \)-hyperbolic in the sense of Gromov.

A curve system \( \Gamma \) is a non-empty set of elements of \( \mathcal{C}(S) \) that are pairwise disjoint from each other. Let \( \mathcal{CS}(S) \) be the space of all curve systems on \( S \). For two curve systems \( \Gamma_1 \) and \( \Gamma_2 \) we define \( d_S(\Gamma_1, \Gamma_2) \) to be the minimum distance in \( \mathcal{C}(S) \) between members of \( \Gamma_1 \) and \( \Gamma_2 \). This defines a pseudo-metric on \( \mathcal{CS}(S) \). For any isotopy class of a subsurface \( Y \) of \( S \), we define a projection map

\[
\pi_Y : \mathcal{CS}(S) \to \mathcal{CS}(Y) \cup \{\emptyset\}
\]

as follows: First assume \( Y \) is not an annulus. Let \( \Gamma = \{[\gamma_1], \ldots, [\gamma_k]\} \) be a curve system on \( S \). Fix a hyperbolic metric on \( S \) and consider the geodesic representatives of \( \gamma_i \) and boundary components of \( Y \). We define \( \pi(\Gamma) \) to be the union of homotopy classes of components of \( \gamma_i \) restricted to \( Y, i = 1, \ldots, k \). Since the \( \gamma_i \) are pairwise disjoint from each other, the resulting set is in fact a curve system in \( Y \). If \( \Gamma \) does not intersect \( Y \), \( \pi_Y \) is defined to be the empty set.

Since curves in an annulus are defined up to homotopy fixing the end points, we need a special definition for \( \pi_Y \) in the case where \( Y \) is an annulus. Fix a hyperbolic metric on \( S \). Let \( \hat{S} \) be the annular covering of \( S \) corresponding to
Y and let \( \hat{\gamma}_i \) be lifts of \( \gamma_i \) intersecting the core of \( \hat{S} \) non-trivially. Changing \( \gamma_i \) homotopically in \( S \) does not change the end points of \( \hat{\gamma}_i \); therefore \( \hat{\gamma}_i \) is a well defined element of \( C(Y) \). Now we define \( \pi_Y(\Gamma) \) to be the union of \( \hat{\gamma}_i \), \( i = 0, \ldots, n \).

One can also define the projection of a lamination \( \lambda \) onto a subsurface \( Y \). Choose a hyperbolic metric \( \sigma \) on \( S \) and restrict the \( \sigma \)-representative of \( \lambda \) to the subsurface in the homotopy class of \( Y \) with \( \sigma \)-geodesic boundary. Then the projection of \( \lambda \) onto \( Y \) will be the set of homotopy classes of closed curves and compact arcs in the restriction of \( \lambda \) to \( Y \) (the projection cannot be defined for leaves that intersect the boundary of \( Y \) only once). To define \( \pi_Y \) for a measured foliation one can use the projection of the corresponding geodesic lamination instead.

**Subsurface coefficients**

For curves \( \alpha, \beta \in C(S) \) that intersect a subsurface \( Y \) non-trivially, we call the distance in \( C(Y) \) between the projections \( \pi_Y(\alpha) \) and \( \pi_Y(\beta) \) the \( Y \)-coefficient of \( \alpha \) and \( \beta \) and denote it by \( d_Y(\alpha, \beta) \), that is

\[
d_Y(\alpha, \beta) = d_Y(\pi_Y(\alpha), \pi_Y(\beta)).
\]

Also, \( i(\alpha, \beta) \) denotes the geometric intersection number between \( \alpha \) and \( \beta \). The intersection number between two curves gives a bound on all their subset coefficients.

**Theorem 2.2.** For \( \alpha, \beta \in C(S) \) we have

\[
d_Y(\alpha, \beta) \leq 2 \times i(\alpha, \beta) + 1.
\]
Conversely, a bound on all subset coefficients gives a bound for the intersection number between two curves.

**Theorem 2.3.** For every $K > 0$ there exists $D > 0$ such that for $\alpha, \beta \in C(S)$, if $i(\alpha, \beta) > D$, then there exists a subsurface $Y$ of $S$ such that $d_Y(\alpha, \beta) > K$. 


Chapter 3

Quadratic Differentials

In this chapter we review the geometry of quadratic differentials. For detailed discussion, see [21] and [27].

Let $\hat{X}$ be a Riemann surface with the conformal structure given by charts $\{(U_\alpha, h_\alpha)\}$. A (meromorphic) quadratic differential $q$ on $\hat{X}$ is a set of meromorphic function elements $q_\alpha$ in the local parameter $z_\alpha = h_\alpha(P)$ for which the transformation law

$$q_\alpha(z_\alpha) dz_\alpha^2 = q_\beta(z_\beta) dz_\beta^2$$

holds whenever $z_\alpha$ and $z_\beta$ are parameter values which correspond to the same point $P$ of $\hat{X}$. The quadratic differential is called holomorphic if all the $q_\alpha$ are holomorphic.

While it clearly does not make sense to speak of the value of a quadratic differential $q$ at a point $P \in \hat{X}$ (since it depends on the local parameter near $P$), it does make sense to speak of the zeroes and poles of $q$. In fact, the order of a zero or a pole is invariant under change of parameter.

Definition 3.1. The critical points of a quadratic differential $q$ are its zeroes
and poles. All other points of $\hat{X}$ are called regular points of $q$. Poles of the first order and zeroes of any order will be called finite critical points, and poles of order greater than or equal to two infinite critical points.

For the rest of this section let $\hat{X}$ be a compact Riemann surface and let $q \not= 0$ be a meromorphic quadratic differential on $\hat{X}$ with finite critical points only. Also assume that the set of critical points of $q$ is a finite discrete set. The surface $\hat{X}$ punctured at poles $P_i$ of $q$ is denoted by $X = \hat{X} \setminus \{P_i\}$.

**Theorem 3.2.** In a neighborhood of every regular point $P$ of $q$ we can introduce a local parameter $w$, in terms of which the representation of $q$ is identically equal to one. The parameter is given by the integral

$$w = Q(z) = \int \sqrt{|q(z)|} \, dz.$$

It is uniquely determined up to a transformation $w \to \pm w + \text{const}$ and it will be called the natural parameter near $P$.

The length of a rectifiable curve can be computed by means of the differential $dw = \sqrt{|q(z)|} \, dz$ in terms of an arbitrary local parameter on $X$.

**Definition 3.3.** The differential $|dw| = \sqrt{|q(z)|} \, |dz|$ is called the length element of the $q$-metric. The length of a curve $\gamma$ in this metric is denoted by $l_q(\gamma)$ and is computed by

$$l_q(\gamma) = \int_{\gamma} |dw| = \int_{\gamma} \sqrt{|q(z)|} \, |dz|.$$ 

The corresponding area element is $dA(q) = |q(z)| \, dx \, dy$; it is also invariant under a change of parameter. The total area of the Riemann surface $X$ in this
metric is the $L^1$-norm of $q$:

$$|X|_q = ||q|| = \iint_X d\Lambda(q).$$

**Horizontal and vertical foliations**

A *straight arc* is a smooth curve $\gamma$ whose image under the map $Q$ is a straight line, that is, a curve along which

$$\arg dw^2 = \arg q(z) dz^2 = 2\theta = \text{const., \hspace{1em} } 0 \leq \theta < \pi.$$ 

A maximal such straight arc is called a $\theta$-trajectory. A straight arc connecting two critical points in $(X, q)$ is called a *straight segment*.

Through every regular point of $q$ there exists a uniquely determined $\theta$-trajectory, $\pi > \theta \geq 0$. In particular, two $\theta$-trajectories never have a common point, unless they coincide. A trajectory is called a critical trajectory if at least one of its ends tends to a critical point. The total $q$-area of critical trajectories is zero; in fact, there are only finitely many critical trajectories.

For each $\theta$, the $\theta$-trajectories foliate the set of regular points in $X$. In a neighborhood of a finite critical point of degree $n, n \geq -1$, this foliation has one of the forms shown in Fig. 3.1.

The foliation corresponding to $\theta = 0$ (respectively, $\theta = \pi/2$), is called the *horizontal foliation* (respectively, the *vertical foliation*) and is denoted by $\nu_-$ (respectively, $\nu_+$).

For any closed curve or arc $\gamma$, the *horizontal length* of $\gamma$ is defined to be
the infimum

\[ h_q(\gamma) = \inf_{\hat{\gamma} \sim \gamma} \int_{\hat{\gamma}} |\text{Re}\{q(x)^{\frac{1}{2}}\}|, \]

where \( \hat{\gamma} \) ranges over all curves or arcs homotopic to \( \gamma \) in \( X \).

The vertical length of \( \gamma \) is defined similarly and is denoted by \( v_q(\gamma) \). Let \( \gamma \) be an arc that is transverse to the horizontal foliation. It is easy to see that the vertical length of \( \gamma \) does not change after a homotopic change through arcs that are transverse to the horizontal foliation and with end points on the same leaves as \( \gamma \) (this is because the vertical length of a horizontal arc is zero). Therefore the horizontal length defines a transverse measure on \( \nu_+ \); similarly, the vertical length defines a transverse measure on \( \nu_- \), making \( \nu_+ \) and \( \nu_- \) into measured foliations.

**Geodesics and almost geodesics in \((X, q)\)**

The \( q \)-geodesic representative of a curve \( \gamma \) is a curve in the free homotopy class of \( \gamma \) with the shortest \( q \)-length and is denoted by \([\gamma]_q\). If \( X \) has no punctures, the \( q \)-metric is complete and the geodesic representative always exists. In the case that there are poles, the \( q \)-metric on \( X \) is not complete, since the finite poles are finite distance from interior points of \( X \); however, the
metric on $\hat{X}$ is complete. Therefore in this case we allow $[\gamma]_q$ to pass through the poles of $\hat{X}$. That is, $[\gamma]_q$ is defined to be the shortest curve in $\hat{X}$ which is a limit of curves in $X$ in the homotopy class of $\gamma$. The same definition works where $\gamma$ is an arc; here by homotopy class we mean homotopy class relative to end points of $\gamma$. Note that if $\gamma$ is simple, $[\gamma]_q$ is the limit of simple curves in $X$.

**Theorem 3.4.** (see, [21], [27]) For $(X, q)$ as above, we have

1. Let $\gamma$ be an arc joining two given points $x$ and $y$ of $X$; then the $q$-geodesic representative of $\gamma$ exists and is unique.

2. Let $\gamma$ be a loop on $X$ which is not contractible to a point or a punctured disk; then the $q$-geodesic representative of $\gamma$ exists and is unique, except for the case where it is one of the continuous family of closed Euclidean geodesics in a flat annulus (see Page 19 for definition of a flat annulus).

A geodesic is composed of straight segments which meet at critical points of $X$, making an angle of at least $\pi$ on either side. When a geodesic passes through a finite pole, it makes at least a full turn around the pole.

**Proposition 3.5.** (see, [27]) Let $\gamma$ be a curve or an arc in $X$. Then the horizontal length of $\gamma$ is realized by its $q$-geodesic representative, that is,

$$h_q(\gamma) = \int_{[\gamma]_q} |Re\{q(z)^{\frac{1}{2}}\}|.$$

Note that part 1 of Theorem 3.4 implies in particular that the geodesic representatives of disjoint curves do not cross each other. However, they might
be tangent to each other, that is, their geodesic representatives might have a geodesic arc in common (also, a geodesic representative of a simple curve might have self-tangencies). As a result of this, the homotopy type of $\gamma$ might not be retrievable from the image of its geodesic representative. To avoid this problem we construct an embedded curve that is a union of straight arcs in $X$ and approximates $[\gamma]_q$ as closely we like. In fact, we make a general construction for a curve system $\Gamma = \{\gamma_1, \ldots, \gamma_k\}$ on $X$ (that is, a set of essential disjoint simple closed curves in $X$). Let $[\Gamma]_q$ be the union of $[\gamma_i]_q$. As a subset of $\hat{X}$, $[\Gamma]_q$ can be considered as a graph in $X$ with vertices on the critical points of $X$ and straight segments as its edges. The sequence of simple closed curves in $X$ approximating $\gamma_i$ gives rise to an immersion of a circle into $[\Gamma]_q$ for $i = 1, \ldots, k$. For a straight segment $\omega$ in $[\Gamma]_q$ connecting critical points $P$ and $Q$ that has been traversed by these circles $n$ times, we replace $\omega$ with $n$ disjoint segments parallel to $\omega$ in a small neighborhood of $\omega$. Then we connect these arcs in a neighborhood of $P$ and a neighborhood of $Q$ such that the incoming and outgoing arcs match in the correct order. This results in an embedded piecewise smooth representation for $\Gamma$ which we call the almost geodesic representative of $\Gamma$ and denote by $\{\Gamma\}_q$. For every $\delta > 0$, $\{\Gamma\}_q$ can be chosen such that $\{\Gamma\}_q$ stays in a $\delta$-neighborhood of $[\Gamma]_q$ and

$$l_q(\{\gamma\}_q) < (1 + \delta)l_q([\gamma]_q).$$

**Subsurfaces of $X$**

A subsurface $Y$ of $X$ is a connected open subset of $X$ with piecewise-smooth boundary. Let $\gamma$ be a boundary component of $Y$. The curvature of $\gamma$ with
respect to $Y$, $\kappa(\gamma)$, is well defined as a measure with atoms at the corners (that is, $(\pi - \theta)$, where $\theta$ is the interior angle at the corner). We choose the sign to be positive when the acceleration vector points into $Y$. If $\gamma$ is curved positively (or negatively) with respect to $Y$ at every point, we say it is monotonically curved with respect to $Y$.

**Theorem 3.6.** (Gauss-Bonnet) Let $Y$ be a subsurface of $\hat{X}$ with boundary components $\partial_1, \ldots, \partial_n$; also let $P_1, \ldots, P_k$ be the critical points in $Y$. Then we have

$$\sum_{i=1}^{n} \kappa(\partial_i) = 2\pi \left( \chi(Y) + \sum_{i=1}^{k} \text{deg}(P_i) \right);$$

where $\text{deg}(P_i)$ is the order of the critical point $P_i$. In particular, for every $i$, we have

$$\kappa(\partial_i) \leq 2\pi \left( 1 - \chi(X) \right).$$

Given a subsurface $Y$, it is often desirable to define a representation for the isotopy class $Y$ with $q$-geodesic boundary. This is not always possible since the geodesic representatives of boundary components of $Y$ can be tangent to each other or themselves. Therefore we define $Y_q$ to be a subsurface in the homotopy class of $Y$ with almost geodesic boundaries. That is, for $\Gamma = \{\partial_1, \ldots, \partial_k\}$, let $\{\Gamma\}_q$ be an almost geodesic representative of $\Gamma$. We denote the open set in the complement of $\{\Gamma\}_q$ homotopically equivalent to $Y$ by $Y_q$ and the boundary components of $Y_q$ by $\partial_q^i$.

**Modulus of primitive annuli in $X$**

Let $A$ be a closed annulus in $X$ with boundaries $\partial_0$ and $\partial_1$. Suppose both boundaries are monotonically curved with respect to $A$. Further, suppose that
the boundaries are equidistant from each other, and that \( \kappa(\partial_0) \leq 0 \). We call 
A a regular annulus. If \( \kappa(\partial_0) < 0 \), we call A expanding and say that \( \partial_0 \) is the inner boundary and \( \partial_1 \) is the outer boundary. If the interior of A contains no zeroes, we say A is a primitive annulus, and we write \( \kappa(A) = -\kappa(\partial_0) \). When \( \kappa(A) = 0 \), A is a flat annulus and is foliated by closed Euclidean geodesics homotopic to the boundaries.

**Lemma 3.7.** For a regular annulus \( A \) we have the following inequality for the distance between the boundaries of \( A \):

\[
d(\partial_0, \partial_1) \geq \text{Mod}(A) l_q(\partial_0).
\]

**Proof.** Let \( |q| \) be the Euclidean metric on \( A \), \( dA(q) \) be the corresponding area form, \( \alpha \) be the free homotopy class of the core of \( A \), and \( d \) be the distance between boundaries of \( A \). For \( 0 \leq t \leq d \) define \( \alpha_t \) to be the equidistant curve from \( \partial_0 \) with \( d(\partial_0, \alpha_t) = t \). Define a new metric \( g_0 \) as follows: For a point \( P \) in \( \alpha_t \),

\[
g_0 = \frac{|q|}{l_q^2(\alpha_t)}.
\]

We have \( l_{g_0}(\alpha_t) = 1 \); therefore the area of \( A \) in the new metric is

\[
\text{Area}_{g_0}(A) = \int_A \frac{1}{l_q^2(\alpha_t)} dA(q) = \int_0^d l_{g_0}(\alpha_t) \frac{dt}{l_q(\alpha_t)} = \int_0^d \frac{dt}{l_q(\alpha_t)}.
\]

Let \( A_t \) be the annulus bounded by \( \partial_0 \) and \( \alpha_t \). The Gauss-Bonnet theorem implies that \( \kappa(\alpha_t) + \kappa(\partial_0) \geq 0 \). Since \( \kappa(\partial_0) < 0 \), \( \kappa(\alpha_t) \), the curvature of \( \alpha_t \) with respect to \( A_t \), is positive. Therefore \( \frac{d}{dt} l_q(\alpha_t) = \kappa(\alpha_t) > 0 \) and \( l_q(\alpha_t) \) is an increasing function of \( t \). In particular, \( l_q(\alpha_t) \geq l_q(\partial_0) \). Therefore

19
\[ \text{Area}_{g_0}(A) \leq \int_0^d \frac{dt}{l_q(\alpha_t)} = \frac{d}{l_q(\partial_0)}. \]

On the other hand, by definition,

\[ \frac{1}{\text{Mod}(A)} = \sup_g \frac{l^2_g(\alpha)}{\text{Area}_g(A)}, \]

where the supremum is taken over all metrics \( g \) in the conformal class of \( |q| \).

Since \( l_{g_0}(\alpha) = 1 \), we have

\[ \frac{1}{\text{Mod}(A)} \geq \frac{l^2_{g_0}(\alpha)}{\text{Area}_{g_0}(A)} \geq \frac{l_q(\partial_0)}{d}. \]

That is,

\[ d \geq \text{Mod}(A)l_q(\partial_0) \]

By a similar method one can also show the following (see [21]).

**Lemma 3.8.** For a primitive annulus \( A \), if \( \kappa = \kappa(A) > 0 \), then

\[ e^{\kappa \text{Mod}(A)}l_q(\partial_0) \geq d(\partial_0, \partial_t). \]
For $\kappa(A) = 0$ we have

$$\text{Mod}(A)l_q(\partial_0) = d(\partial_0, \partial_1).$$

**Theorem 3.9.** (Minsky [21]) If $A \subset X$ is any homotopically nontrivial annulus with $\text{Mod}(A) \geq m$, then $A$ contains a primitive annulus $B$ such that

$$\text{Mod}(B) \geq a\text{Mod}(A) - b,$$

where $m$, $a$ and $b$ depend only on the topology of $X$.

We also recall the following theorem of Maskit.

**Theorem 3.10.** (Maskit, [13]) Let $q$ be a quadratic differential with area 1, and let $\sigma$ be the hyperbolic metric in its conformal class. Then for every $L > 0$ there exist $L' > 0$ such that for every curve $\alpha$ on $X$ we have:

$$l_\sigma(\alpha) < L \implies l_q(\alpha) < L'$$
Chapter 4

Thick-thin Decomposition for a Quadratic Differential Metric on a Surface

Here we describe a decomposition of \((X, g)\) into pieces similar to the thick-thin decomposition of \(X\) with respect to its hyperbolic metric. Some components are flat, possibly degenerate, annuli. To a component \(Y\) that is not an annulus, we associate a number \(\lambda_Y\) called the size of \(Y\). In Theorem 4.3 we give upper bounds for the diameter and the area of \(Y\) using its size, justifying the use of the term size for \(\lambda_Y\). Then we show that two neighboring component have one of the following properties. Either they have very different sizes, or their common boundary is significantly shorter than the size of each of them or they are connected through a long flat annulus. Although the existence of this decomposition is interesting in itself, the only result that is going to be useful later is Corollary 4.7, which is an analogue of the collar lemma in hyperbolic geometry.

For a fixed \(\epsilon\), let \(\{\alpha_1, \ldots, \alpha_n\}\) be the set of geodesics in the hyperbolic metric of \(X\) with hyperbolic length less than \(\epsilon\), and let \(Y_1, \ldots, Y_m\) be com-
ponents of the complement of $\alpha_i$. For $i = 1, \ldots, n$, let $A_i$ be the largest flat annulus (in the $q$-metric) whose core is homotopic to $\alpha_i$ and let $[\alpha_i^+]$ and $[\alpha_i^-]$ be the boundary components of $A_i$ ($[\alpha_i^+]$ and $[\alpha_i^-]$ may be identical). For each $Y_j$, $j = 1, \ldots, m$, let $\gamma_j$ be an essential curve in $Y_j$ with the shortest $q$-length $\lambda_j = l_q([\gamma_j])$ among all essential curves in $Y_j$. Since $\alpha_i$ and $\gamma_j$ form a curve system, $[\alpha_i]^\pm$ and $[\gamma_j]$ have no essential crossings; therefore, one can represent them simultaneously with $\delta$-almost geodesics. Let $Y_j$ be the subsurface corresponding to $Y_j$ with almost $q$-geodesic boundaries. Note that $\{\gamma_j\}_q$ is an embedded curve in $Y_q^2$. One should think of $\lambda_j$ as an indication of how large $Y_j$ is. In Theorem 4.3 we show that the area and the diameter of $Y_j$ are bounded by constants depending on $\lambda_j$. We call $\lambda_j$ the size of $Y_j$. We denote the almost geodesic representatives of $[\alpha_i]^\pm_q$ by $\partial_i^\pm$.

For the rest of this section, let $Y$ be one of these $Y_j$, $\gamma = \gamma_j$ and $\lambda = \lambda_j$.

**Lemma 4.1.** Let $A$ be an annulus in $Y$ with one of its boundary components in $\partial Y$. Then

$$\text{area}_q(A) = O(l_q(\partial A)^2) \quad \text{and} \quad \text{diam}_q(A) = O(l_q(\partial A)).$$

**Proof.** Denote the boundary components of $A$ by $\partial_1$ and $\partial_2$, where $\partial_1 \subset \partial Y$. Let $A_r$ be the complement in $A$ of a neighborhood of the boundary of $A$, $A_r = A - N_r(\partial A)$. For small $r$, $A_r$ is still an annulus. For $r$ in some interval $[0, r_0)$, $A_r$ has one connected component that is an annulus whose boundary components we denote by $\partial^+_1$ and $\partial^-_2$, and possibly some connected components that are topological disks. At $r = r_0$, the boundaries of the annulus component of $A_r$ touch each other, and, for $r > r_0$, all components of $A_r$ are disks. Finally,
let $r = r_1$ be the infimum of the set of values of $r$ for which $A_r$ is empty. Since $\chi(A_r) \geq 0$, the Gauss-Bonnet theorem implies that the total curvature of the boundaries of $A_r$, $\kappa(\partial A_r)$, is non-negative. Also, $\frac{d}{dr} l_q(\partial A_r) = -\kappa(\partial A_r)$, and therefore $l_q(\partial A_r)$ is a decreasing function of $r$.

First we try to find an upper bound for $r_0$. For $r_0 > r > \delta$ (note that there might not be such $r$, but in that case $\delta$ is an upper bound for $r_0$), the curvature of $\partial_t^r$ with respect to $A_r$ is negative. This is because $\partial_t$ is $\delta$-close to the boundary of $A_i$, and for $r > \delta$, $\partial_t^r$ is disjoint from $A_i$. The Gauss-Bonnet theorem applied on the annulus component of $A_r$ implies that the curvature of $\partial_2^r$ should be positive, but the curvature of a curve in $X$ is always a multiple of $\pi$; therefore,

$$\frac{d}{dr} l_q(\partial_2^r) = -\kappa(\partial_2^r) \leq -\pi.$$

Hence

$$r_0 - \delta \leq \frac{l_q(\partial_2)}{\pi} = O(l_q(\partial_2)).$$

For $r_1 > r > r_0$, $A_r$ has only disk components, and it must have at least one since otherwise it would be empty. Using the Gauss-Bonnet theorem, we have $\frac{d}{dr} l_q(\partial A^r) = -\kappa(\partial A_r) \leq -2\pi$. But $l_q(\partial_2^r)$ is positive; hence

$$r_1 - r_0 \leq \frac{l_q(\partial A_{r_0})}{2\pi} \leq \frac{l_q(\partial_2)}{2\pi} = O(l_q(\partial_2)).$$

Therefore, $r_1 = O(l_q(\partial A))$, and

$$\text{diam}_q(A) \leq 2r_1 + \max(l_q(\partial_1), l_q(\partial_2)) = O(l_q(\partial A)).$$
Also, we have the following bound for the area of $A$:

$$\text{area}_q(A) = \int_0^{r_1} l_q(\partial A_r) \leq r_1 l_q(\partial_2 A) = O(l_q(\partial A)^2).$$

Using a similar argument, we can show

**Lemma 4.2.** Let $D$ be a disk or a punctured disk in $X$ with $\partial D$ as its boundary; then

$$\text{area}_q(D) = O(l_q(\partial D)^2) \quad \text{and} \quad \text{diam}_q(D) = O(l_q(\partial D)).$$

**Theorem 4.3.** There exists $R > 0$, depending on $\epsilon$ and the topology of $X$, such that

1. If $\alpha_i$ is a boundary component of $Y$, then $l_q(\partial_1^+) = O(R\lambda)$.

2. $\text{diam}_q(Y) = O(R\lambda)$.

3. $\text{area}_q(Y) = O(R^2\lambda^2)$.

**Proof.** Define $N_r$ to be the open $r$-neighborhood of a point in $\{\gamma\}_q$ in $Y$, and let $Z_r$ be the union of $N_r$ with all the components of $Y - N_r$ that are disks, punctured disks or annuli parallel to $\partial Y$. Let $\partial_1, \ldots, \partial_k$ be the boundary components of $Z_r$, and let $\partial_r$ be the union of the $\partial_i$. Also, let $r = \hat{r}$ be the infimum of the set of values of $r$ such that $Z_r = Y$.

First observe that $l_q(\partial_r)$ and $\text{area}_q(Z_r)$ are differentiable functions of $r$ almost everywhere and are continuous except at finitely many points where we
add a disk, a punctured disk, or an annulus parallel to a boundary component of $Y$. For differentiable points we have $\frac{d}{dr} l_q(\partial_r) \leq \kappa(\partial_r)$ (the inequality is there because $Z_r$ does not grow beyond $Y$). At any $r = r_u$ where we add a disk or a punctured disk $D_u$, the value of $l_q(\partial_r)$ decreases by the length of the boundary of $D_u$, $c_u$. Also, adding an annulus parallel to the boundary of $Y$ replaces a boundary component $\partial_r^k$ with the almost geodesic representative of $\partial_r^k$, therefore magnifying the value of $l_q(\partial_r^k)$ by at most a factor of $(1 + \delta)$ (note that each $\partial_r^k$ is magnified by $(1 + \delta)$ at most once). We have the following inequality.

$$\frac{l_q(\partial_r)}{1 + \delta} \leq \int_0^r \kappa(\partial_\rho) \, d\rho - \sum_u c_u.$$ 

Using the Gauss-Bonnet theorem, we have

$$\kappa(\partial_\rho) \leq 2\pi(1 - \chi(Z_r)) = O(1);$$

hence

$$l_q(\partial_r) = O(r) - \sum_u c_u. \quad (4.1)$$

To find an upper bound for $area_q(Z_r)$ we observe that

$$\frac{d}{dr} area_q(Z_r) \leq l_q(\partial_r).$$

Also, at any $r = r_u$ where we add a disk or a punctured disk, $area_q(Z_r)$ increases by $O(l_q^2(c_u))$ (Lemma 4.2), and replacing a boundary component
of $Z_r$, $\partial_t^i$, by a boundary component of $Y$ increases $\text{area}_q(Z_r)$ by $O(l_q^2(\partial_r^i))$ (Lemma 4.1). Therefore,

$$\text{area}_q(Z_r) \leq \int_0^r l_q(\partial_r) \, dr + \sum_u O(c_u^2) + \sum_t O(l_q(\partial_t^i)^2)$$

$$\leq \int_0^r [O(r) - \sum c_u] \, dr + \sum_u O(c_u^2) + \sum_t O(r^2)$$

$$\leq O(r^2) + \sum_u O(c_u^2)$$

$$\leq O(r^2) + [\sum_u O(c_u)]^2$$

But Equation 4.1 implies

$$\sum_u c_u = O(\hat{r})$$

therefore,

$$\text{area}_q(Z_r) = O(\hat{r}^2). \quad (4.2)$$

We claim that $\hat{r} \leq \lambda R$ for some $R$ depending on $\epsilon$ and the topology of $S$ only. Note that, assuming the claim, Equation (4.1) implies part 2 of our theorem and Equation (4.2) implies part 3. Also, part 1 follows from the fact that

$$\text{diam}_q(Y) \leq 2\hat{r}.$$

Let $r_1, \ldots, r_t$ be the points where the topology of $Z_r$ changes or a critical point of $X$ is added to $Z_r$. All boundary components of $Z_r$ that are not boundary components of $Y$ are essential in $X$. Therefore, for any such curve, $\partial_{t_j}^i$ and $\partial_{t_{j+1}}^i$ bound an essential primitive annulus $A$. In particular, the core
of $A$ is not homotopy equivalent to any of the $\alpha_i$. But the $\alpha_i$ are all the curves with hyperbolic length less than $\epsilon$; therefore, the modulus of $A$ is bounded by some $M$ depending on $\epsilon$ only. Lemma 3.8 implies

$$\text{Mod}(A) \geq \frac{1}{\kappa} \log\left(\frac{r_{j+1}}{l_q(\partial_j^i)}\right).$$

But $l_q(\partial_r) = O(r)$; therefore,

$$r_{j+1} - r_j = e^{\epsilon M} O(r_j).$$

This implies that the ratio between consecutive $r_i$ is bounded. Also, $l$ is bounded by the topology of $X$, and that gives a bound on the ratio $\frac{r_i}{r_1}$. But $r_1 \leq \lambda$ and $r_1 = \hat{r}$; therefore there exists $R$ depending on $\epsilon$ and the topology of $X$ such that $\hat{r} \leq R\lambda$.

\[\Box\]

**Lemma 4.4.** Let $A$ be a primitive annulus with $\partial_1$ and $\partial_2$ as its boundary components and $\kappa(A) > 0$. Then

1. $\text{dist}_q(\partial_1, \partial_2)^2 \asymp \text{area}_q(A)$.
2. $l_q(\partial_1)^2 = O\left(\frac{\text{area}_q(A)}{\text{Mod}(A)}\right)$.

**Proof.** Similar to that of Lemma 4.1. \[\Box\]

**Lemma 4.5.** For every $C > 0$ there exists $\epsilon > 0$ such that, in the $\epsilon$-decomposition of $X$, if $A_i$ is the annulus connecting $\gamma$ and $\gamma'$ and if $\lambda$ and $\lambda'$ are the sizes of $\gamma$ and $\gamma'$, respectively, then one of the following holds.

1. $\lambda \geq C l_q(\partial_i^+)$.
2. \( \lambda \geq C l_q(\partial_i^+) \).

3. \( \text{dist}_q(\partial_i^+, \partial_i^-) > C l_q(\partial_i^+) \).

**Proof.** Let \( A \) be the primitive annulus corresponding to \( \alpha_i \) given by Theorem 3.9, that is,

\[
\text{Mod}(A) \geq a \log\left(\frac{1}{\epsilon}\right) - b.
\]

We know that \([\alpha_i]_q^+\) and \([\alpha_i]_q^-\) are disjoint from the interior of \( A \). Therefore, \( A \) is a subset of either \( A_i \), \( Y \) or \( Y' \). In the first case, Lemma 3.8 implies that

\[
l_q(\partial_i^+) \leq \text{Mod}(A)\text{dist}_q(\partial_i^+, \partial_i^-) .
\]

If \( A \) is a subset of \( Y \), we have that the length of any boundary component of \( A \) is at least \( \frac{l_q(\partial_i^+)}{l_q(\partial_1)} \). Part 2 of Lemma 4.4 implies

\[
l_q(\partial_i^+)^2 \leq l_q(\partial_1 A)^2 = O\left(\frac{\text{area}_q(A)}{\text{Mod}(A)^2}\right) = O\left(\frac{R^2\lambda^2}{\text{Mod}(A)^2}\right) .
\]

A similar inequality holds if \( A \) is a subset of \( Y' \). Since \( \text{Mod}(A) \) approaches infinity as \( \epsilon \) approaches zero, the above inequalities prove our lemma. \( \square \)

**Lemma 4.6.** For given \( C \) let \( \epsilon \) be as in Lemma 4.5. Consider the \( \epsilon \)-decomposition of \( X \). Let \( Y \) be a component of this decomposition and let \( \partial_i^+ \) be one of its boundary components. Then there exists an annulus \( B_i \) with \( \partial_i^+ \in \partial B_i \) such that

1. \( \text{Mod}(B_i) \geq \log(C) \).

2. \( \text{Area}_q(B_i) = O(\lambda^2) \).
3. $\text{dist}_q(\partial_1 B_i, \partial_2 B_i) \asymp \lambda$.

Proof. The previous lemma shows that a large neighborhood on at least one side of $\partial_i^+$ is still in the form of an annulus; therefore one can choose an annulus $B_i$ which satisfies part 3 of our lemma. Now part 3 of Lemma 4.5 and Lemma 3.7 together imply that $B_i$ satisfies part 1 of our lemma, and part 2 of Lemma 4.4 shows that $B_i$ satisfies part 2 also.

Let $\sigma$ be the hyperbolic metric on $X$.

**Corollary 4.7.** For every $L > 0$, there exists $D_L$ such that if $\alpha$ and $\beta$ are two simple closed geodesics in the hyperbolic metric of $X$ intersecting non-trivially with $l_\sigma(\alpha) < L$, then

$$D_L l_q([\beta]_q) \geq l_q([\alpha]_q).$$

**Proof.** First assume $\alpha$ is different from $\alpha_i$, $i = 1, \ldots, n$. Let $A_\alpha$ be a tubular neighborhood of $\alpha$. For some $m$ and $\epsilon'$ depending on $L$ only, one can choose this neighborhood such that $A_\alpha$ is an annulus of modulus larger than $m$ and $A_\alpha$ is disjoint from the $\epsilon'$-thin part of $X$. Let $\epsilon' > \epsilon > 0$. Consider the decomposition of $(X, q)$ corresponding to $\epsilon$. Since $\alpha$ is disjoint from all $\alpha_i$, it is an essential curve in some $Y_j$.

We claim that, for an appropriate choice of $\epsilon$ and $K$ depending on $L$ only, $l_q(\alpha) < K\lambda_j$. For every boundary component $\partial_i^+$ of $Y_j$, let $B_i$ be a regular annulus as in Lemma 4.6, and let $\sigma_i$ be the complete hyperbolic metric on $B_i$. By choosing $\epsilon$ appropriately, we can make the modulus of $B_i$ large enough so that the $\sigma_i$-length of $\hat{\alpha}_i$, the $\sigma_i$-geodesic core of $B_i$, is less than $\epsilon'$. The Schwarz lemma implies that $l_{\sigma_i}(\hat{\alpha}_i)$ is also less than $\epsilon'$. Therefore $A_\alpha$ is disjoint from $\hat{\alpha}_i$. 

30
Let $B'_i$ be the annulus connecting $\partial_i^+ \to \hat{a}_i$, and let $\hat{Y}_j$ be the union of $\gamma_j$ with all such $B'_i$. Using Theorem 4.3 and Lemma 4.6 we have

$$\text{area}_{q}(\hat{Y}_j) \leq \text{area}_{q}(Y_j) + \sum \text{area}_{q}(B_i) = O(R^2\lambda_j^2).$$

Since $\alpha$ is homotopic to a curve in $\hat{Y}_j$, $A_\alpha$ should either be a subset of $\hat{Y}_j$ or intersect one of its boundary components. But $A_\alpha$ is disjoint from $\hat{a}_i$ and therefore stays in $\hat{Y}_j$. By the definition of modulus, we have

$$\frac{l_q^2(\alpha)}{\text{area}_{q}(\hat{Y}_j)} < \frac{1}{\text{Mod}(A_\alpha)} < \frac{1}{m},$$

hence

$$l_q(\alpha) = O\left(\frac{R\lambda}{\sqrt{m}}\right),$$

which proves our claim.

To finish the proof we have to show that $l_q([\beta]_q) \geq \lambda_j$. If $\beta$ is an essential curve in $Y$, then $l_q([\beta]_q) \geq \lambda$. Otherwise since $\beta$ intersects $\alpha$ non-trivially, $\beta$ has to intersect some $B_i$ essentially. Lemma 4.6 implies that

$$l_q([\beta]_q) \geq \text{dist}_q(\partial_1 B_i, \partial_2 B_i) \times \lambda_j.$$

In the case $\alpha = \alpha_i$ for some $i$, assume $\alpha_i$ is a boundary component of a subsurface $Y$ and $B_i$ is the annulus given by Lemma 4.6. Since in this case $\beta$ has to intersect $B_i$, we have $l_q([\beta]_q) \simeq \lambda$. But Theorem 4.3 implies that

$$l_q([\alpha_i]_q) = l_q(\partial_i^+) = O(R\lambda).$$
Therefore $l_q(\lceil \alpha \rceil_q) = O(Rl_q(\lceil \beta \rceil_q))$. 

\qed
Chapter 5

The Teichmüller Space of $S$

The Teichmüller space of $S$ is the space of conformal structures on $S$, where two structures are considered to be equivalent if there is a conformal map between them isotopic to the identity. There are several natural metrics defined on $\mathcal{T}(S)$, all inducing the same natural topology. We work with the Teichmüller metric, which assigns to $\Sigma_1, \Sigma_2 \in \mathcal{T}(S)$ the distance

$$d_{\mathcal{T}(S)}(\Sigma_1, \Sigma_2) = \frac{1}{2} \log(K),$$

where $K$ is the smallest dilatation of a quasi-conformal homeomorphism from $\Sigma_1$ to $\Sigma_2$ that is isotopic to the identity.

Geodesics in the Teichmüller space of $S$, $\mathcal{T}(S)$, are determined by the quadratic differentials. Given $q$ holomorphic for some Riemann surface $\Sigma \in \mathcal{T}(S)$, for any $t \in \mathbb{R}$ we consider the conformal structure obtained by scaling the horizontal foliation of $q$ by a factor of $e^t$, and the vertical by a factor of $e^{-t}$. The resulting family, which we denote by $g_q(t)$, is a geodesic in $\mathcal{T}(S)$ parametrized by arclength. The corresponding family of quadratic differentials
is denoted by \( q_t \).

**Lemma 5.1.** Let \( \alpha \) be a curve in \( S \) and \( g \) be a geodesic in \( \mathcal{T}(S) \). The horizontal and vertical lengths of \( \alpha \) vary with time as follows:

\[
h_{q_t}(\alpha) = h_q(\alpha)e^t
\]

and

\[
v_{q_t}(\alpha) = v_q(\alpha)e^{-t}.
\]

For a fixed \( \epsilon \), there is a decomposition of \( \mathcal{T}(S) \) into its thick part and its thin part. The thick part of \( \mathcal{T}(S) \) is the set of hyperbolic metrics on \( S \) where the length of every closed geodesic is greater than \( \epsilon \), and the thin part of \( \mathcal{T}(S) \) is the complement of this set. Given a Teichmüller geodesic \( g \), we denote the length of the geodesic representative of a curve \( \alpha \) in \( g(t) \) by \( l_{g(t)}(\alpha) \). The infimum over all \( t \in \mathbb{R} \) of \( l_{g(t)}(\alpha) \) is denoted by \( l_g(\alpha) \).
Chapter 6

Subsurface Coefficients and Teichmüller Geodesics

In this chapter we prove Theorem 1.5 and Theorem 1.8. Note that 2 $\Leftrightarrow$ 4 in Theorem 1.2 is a corollary of these two theorems.

Let $X$ be a Riemann surface homeomorphic to $S$, and let $q$ be a holomorphic quadratic differential on $X$ with poles of order at most 1 on punctures of $X$ and with $\nu_+$ and $\nu_-$ as its horizontal and vertical foliations. Let $Y$ be a subsurface of $X$ and $Y_q$ be the corresponding subsurface with $\delta$-almost $q$-geodesic boundary. We denote the boundary components of $Y_q$ by $\partial_1, \ldots, \partial_k$ and the union of the $\partial_i$ by $\partial Y_q$. Define

$$l_q(\partial Y_q) = \sum_{i=1}^{k} l_q(\partial_i),$$

$$h_q(\partial Y_q) = \sum_{i=1}^{k} h_q(\partial_i)$$
and

\[ v_q(\partial Y_q) = \sum_{i=1}^{k} v_q(\partial_i). \]

**Definition 6.1.** Let \( f : [0, 1] \times (0, 1) \to Y_q \) be an embedding of the unit square in \( Y_q \) with the following properties:

1. For \( s = 0, 1 \), \( f \) maps \( \{s\} \times (0, 1) \) into \( \partial Y_q \).
2. For \( t \in (0, 1) \), \( f \) maps \([0, 1] \times \{t\}\) into a horizontal leaf of \( q \).

Note that the image of \( f \) does not contain any critical point of \( q \). We call \( R = f([0, 1] \times (0, 1)) \) a **horizontal strip**. We say a horizontal strip is **maximal** if it is not a proper subset of any other horizontal strip. The width \( w(R) \) of a horizontal strip \( R \) is defined to be the vertical length of a transverse arc in the form \( f(\{t\} \times I) \),

\[ w(R) = v_q(f(\{t\} \times I)). \]

**Vertical strips** and **maximal vertical strips** are defined similarly. The width of a vertical strip \( R \) is the horizontal length of an arc transverse to \( R \).

**Lemma 6.2.** Any two maximal horizontal (respectively, vertical) strips in \( Y_q \) either have disjoint interiors or are identical. Furthermore, there are only finitely many distinct maximal strips, and the union of all maximal strips covers \( Y_q \) except for a measure zero set.

**Proof.** If the interiors of two horizontal strips intersect non-trivially, their union is also a horizontal strip. But if they are both maximal, the union cannot be larger than either of them; therefore they are identical. To see the
second part, note that for a generic point $P \in Y_q$ the horizontal leaf passing through $P$ restricted to $Y_q$ is compact and non-critical. Therefore a neighborhood of this leaf is also free of critical points, which implies that $P$ is an interior point of some horizontal strip. Also, each maximal strip has a critical point on its boundary (otherwise one could extend it to a larger strip), and each critical point $P$ of degree $n$ can appear on the boundary of at most $n + 2$ maximal strips. But the sum of the degrees of the critical points in $X$ is finite, and so is the number of maximal strips. The proof for vertical strips is similar.

\[ \square \]

**Corollary 6.3.** Let $R_1, \ldots, R_p$ be the decomposition of $Y$ into maximal horizontal (respectively, vertical) strips. Then the sum of the widths of the $R_i$, $\sum_{i=1}^{p} w(R_i)$, is equal to the total vertical (respectively, horizontal) length of the boundary of $Y_q$.

If $Y_q$ had geodesic boundary, then the restriction of a horizontal leaf to $Y_q$ would either lie on the boundary of $Y_q$ or be an essential arc in $Y$. This is because the $q$-geodesic representative of an arc connecting two points is unique. Since $Y_q$ has almost geodesic boundary, some of our horizontal strips might be parallel to the boundary of $Y_q$. But since $\partial Y_q$ is $\delta$-close to its $q$-geodesic representative, the total vertical length of strips parallel to the boundary of $Y_q$ is less than $\delta l_q(\partial Y_q)$. We call the strips that are not parallel to the boundary of $Y_q$ non-trivial strips. Therefore, if $R_1, \ldots, R_p$ are the non-trivial horizontal strips, then

\[
\sum_{i=1}^{p} w(R_i) \geq (1 - \delta)v_q(\partial Y_q).
\]  
(6.1)
A similar inequality holds for the vertical strips.

Let $\sigma$ be the hyperbolic metric on $X$.

**Theorem 6.4.** For every $L$, there exists $C_L$ depending only on $L$ and the topology of $X$, such that, if $\alpha$ is a curve in $X$ intersecting a boundary component of $Y$ non-trivially and $l_\sigma(\alpha) < L$, then

$$\min\left(d_Y(\alpha, \nu_+), d_Y(\alpha, \nu_-)\right) < C_L.$$ 

**Proof.** First assume that $Y$ is not an annulus. Since $\alpha$ intersects a boundary component of $Y$, Corollary 4.7 implies that

$$l_q([\alpha]_q) \leq D_L l_q(\partial Y_q). \quad (6.2)$$

Also, we know that $l_q(\partial Y_q) \leq v_q(\partial Y_q) + h_q(\partial Y_q)$. Without loss of generality we can assume that the boundary of $Y_q$ is mostly vertical, that is,

$$l_q(\partial Y_q) \leq 2v_q(\partial Y_q). \quad (6.3)$$

Let $\hat{\alpha}$ be a component of the restriction of $[\alpha]_q$ to $Y_q$, $\{R_1, \ldots, R_p\}$ be the set of non-trivial maximal horizontal strips on $Y_q$, and $b_i$ be the number of times $\hat{\alpha}$ crosses the strip $R_i$. Note that $\hat{\alpha}$ can have partial intersection with at most two strips, which we do not count as crossing. We have

$$v_q(\hat{\alpha}) \geq \sum_{i=1}^{p} b_i \omega(R_i). \quad (6.4)$$
Now let $b$ be the minimum of the $b_i$, and let $R$ be the corresponding strip. We have

$$b \sum_{i=1}^{p} w(R_i) < \sum_{i=1}^{p} b_i w(R_i)$$

$$\leq v_q(\hat{\alpha}) \quad \text{Equation (6.4)}$$

$$\leq l_q([\alpha]_q)$$

$$\leq D_L l_q(\partial Y_q) \quad \text{Equation (6.2)}$$

$$\leq 2 D_L v_q(\partial Y_q) \quad \text{Equation (6.3)}$$

$$= \frac{2 D_L}{1 - \delta} \sum_{i=1}^{p} w(R_i) \quad \text{Equation (6.1)}$$

Therefore, $\hat{\alpha}$ can cross $R$ at most $\frac{2 D_L}{1 - \delta}$ times. This means that there exists a leaf of the horizontal foliation that intersects $\hat{\alpha}$ a bounded number of times, where the bound depends only on $L$ and on the topology of $S$. A bound on this intersection number gives a bound on the distance in $C(Y)$ between $\hat{\alpha}$ and the horizontal foliation (see, Lemma 2.2).

Now assume $Y$ is an annulus and $\gamma$ is its core. Let $A$ be the annular covering of $X$ with respect to $\gamma$. We represent the core curve of $A$ by $\gamma$ again. Let $\bar{q}$ be the lift of quadratic differential $q$ to $A$, $[\gamma]_q$ be the $\bar{q}$-geodesic core of $A$, $[\tilde{\alpha}]_q$ be the geodesic representative of a lift of $\alpha$ to $A$ intersecting $[\gamma]_q$ non-trivially. Also let $h$ (respectively, $v$) be a leaf of the horizontal (respectively, vertical) foliation of $\bar{q}$ intersecting $[\gamma]_q$ non-trivially. To prove the theorem in this case we have to show that $[\tilde{\alpha}]_q$ intersects one of $h$ or $v$ a bounded number of times, with bound depending on $L$ only. Let $B$ be the maximal flat annulus.

39
in \( A \), and let \( B_1 \) and \( B_2 \) be the two annuli in the complement of \( B \) in \( A \). Since \( B_1 \) and \( B_2 \) are expanding annuli, the Gauss-Bonnet theorem implies that \( h \)
and \( v \) intersect \( [\tilde{\alpha}]_q \) at most once in \( B_1 \) and \( B_2 \). Let \( s_h \), \( s_v \) and \( s \) be the slopes of \( h \), \( v \) and \([\tilde{\alpha}]_q \) in \( B \), respectively (consider \( B \) as a vertical cylinder). Since \( h \)
and \( v \) are perpendicular to each other, one of \( |s_h| \) and \( |s_v| \) (say \( |s_h| \) is larger
than or equal to 1. Let \( \omega \) be the restriction of \([\tilde{\alpha}]_q \) to \( B \). Since the projection
map from \( A \) to \( X \) maps \( B \) into \( X \) injectively,

\[
l_q([\alpha]_q) \geq l_q(\omega).
\]

On the other hand, Theorem 4.7 implies

\[
l_q([\gamma]_q) \leq \frac{1}{D_L} l_q([\alpha]_q).
\]

Therefore \( |s| \geq \frac{1}{D_L} \). The number of intersection points between \( h \) and \( \omega \) is
bounded by

\[
\frac{\text{dist}(\partial_1 B, \partial_2 B)}{l_q(\gamma)} \left| \frac{1}{s_h} - \frac{1}{s} \right| \leq \text{Mod}(B)(D_L + 1).
\]

But \( \gamma \) intersects \( \alpha \), and \( l_\omega(\alpha) < L \); therefore, the hyperbolic length of \( \gamma \) is
bounded below by a constant depending on \( L \), and that gives an upper bound
for the modulus of \( A \). But, \( \text{Mod}(B) \leq \text{Mod}(A) \); hence \( \text{Mod}(B) \) is bounded
by a constant depending only on \( L \). Therefore we have a upper bound for the
intersection number between \( h \) and \( \omega \), and hence, for the intersection number
between \( h \) and \([\tilde{\alpha}]_q \). 

\[\square\]
Theorem 6.5. Let \((g, \nu_+, \nu_-)\) be a geodesic in the Teichmüller space of \(S\). For every subsurface \(Y\) and \(L > 0\), there exists \(M > 0\), depending on \(Y, g, L\) and the topology of \(S\), such that, if a curve \(\beta\) intersects \(Y\) non-trivially and the hyperbolic length of \(\beta\) is shorter than \(L\) at some time \(t > M\) (respectively, \(t < -M\)), then

\[d_Y(\beta, \nu_+) = 1\]
\(\text{(respectively, } d_Y(\beta, \nu_-) = 1)\).

Proof. Fix a time \(t_0\), let \(X_0 = g(t_0)\), and let \(q_0\) be the corresponding quadratic differential on \(X_0\). Let \(Y_0\) be a subsurface of \(X\) in the isotopy class of \(Y\) whose boundaries are \(\delta\)-almost geodesics in \(q_0\), and let \(\partial_i\) be a boundary component of \(Y_0\). Let \(R\) be a non-trivial horizontal strip in \(Y\) with a vertical width of \(d\). For \(t \in \mathbb{R}\), the \(q_0\)-area of \(X_t\) equals 1. Also, the hyperbolic length of \(\beta\) at time \(t\) is less than \(L\). Theorem 3.10 implies that the \(q_0\)-length of \(\beta\) is less than \(L'\) for some \(L'\) depending on \(L\). Therefore, the \(q_0\)-vertical length of \(\beta\) is less than \(L'e^{t_0-t}\). By choosing \(t\) large enough, we can make this quantity be less than \(d\). Therefore, \(\beta\) does not cross \(R\), and it is disjoint from some leaf of the horizontal foliation. But the horizontal leaves in \(R\) are essential arcs in \(Y\); therefore, \(\beta\) is distance one from the horizontal foliation. \(\square\)

Now we are ready to prove Theorem 1.5.

Theorem 1.5. For every \(\epsilon > 0\), there exists a \(K > 0\) such that, if \((g, \nu_+, \nu_-)\) is a geodesic in the Teichmüller space of \(S\) and if \(Y\) is a subsurface of \(X\) with \(d_Y(\nu_+, \nu_-) > K\), then there exists \(t \in \mathbb{R}\) such that for each boundary
component $\alpha$ of $Y$, $l_{g(t)}(\alpha) < \epsilon$.

Proof. We prove the contrapositive of the above statement. Let $\epsilon$ be such that for every $t \in \mathbb{R}$ there exists a boundary component $\alpha_t$ of $Y$ with $l_{g(t)}(\alpha) > \epsilon$.

We have to find an upper bound for $d_Y(\nu_+, \nu_-)$. For every $t$, since $l_{g(t)}(\alpha_t) > \epsilon$, there exists a curve $\beta_t$, intersecting $\alpha_t$ non-trivially, whose hyperbolic length in $g(t)$ is less than $L$, for some $L > 0$ depending on $\epsilon$ only. By increasing the value of $L$ (say doubling the value of $L$) we can assume that the hyperbolic length of $\beta_t$ is less than $L$ in a neighborhood of $t$. Therefore, we can find a covering of $\mathbb{R}$ with intervals $[t_i, t_{i+1}]$ and a sequence of curves $\beta_i$ such that the $\beta_i$ intersect $\alpha$ non-trivially and, for $t \in [t_i, t_{i+1}]$, $l_{g(t)}(\beta_i) < L$.

For $C_L$ as in Theorem 6.4, we have that each $\beta_i$ is $C_L$ close to either $\nu_+$ or $\nu_-$. Assume $d_Y(\nu_+, \nu_-) > 2C_L$ (otherwise we are done). Then $\beta_i$ cannot be $C_L$ close to $\nu_+$ and $\nu_-$ at the same time. Define $I_+$ (respectively, $I_-$) to be the set of all integers $i$ such that $d_Y(\beta_i, \nu_+) \leq C_L$ (respectively, $d_Y(\beta_i, \nu_-) \leq C_L$). $I_+$ and $I_-$ are disjoint and non-empty (Theorem 6.5), and $\mathbb{Z} = I_+ \cup I_-$. Therefore there exists $j \in I_-$ such that $j + 1 \in I_+$ (or we could have $j - 1 \in I_+$. We have

$$d_Y(\beta_j, \nu_-) \leq C_L, \quad \text{and} \quad d_Y(\beta_{j+1}, \nu_+) \leq C_L.$$ 

Also, at $t_{j+1}$, both $\beta_j$ and $\beta_{j+1}$ have hyperbolic length less than $L$. Hence

$$d_Y(\beta_j, \beta_{j+1}) = O(1).$$

The combination of the above inequalities gives an upper bound on $d_Y(\nu_+, \nu_-)$ and therefore proves the theorem. \qed
Theorem 1.8. For every $K$, there exists $\epsilon > 0$ such that, if $(g, \nu_+, \nu_-)$ is a geodesic in the Teichmüller space of $S$, and if $\alpha$ is a curve in $S$ with $l_{g(t)}(\alpha) < \epsilon$ for some $t \in \mathbb{R}$, then there exists a subsurface $Z$ of $S$ disjoint from $\alpha$ such that

$$d_Z(\nu_+, \nu_-) > K.$$ 

Proof. Let $t$ be as above, $X_t = g(t)$, and $q_t$ be the corresponding quadratic differential on $X_t$. Also, let $Y = S \setminus \alpha$, and let $Y_q$ be the corresponding subsurface with almost geodesic boundaries. We claim that there exists a horizontal arc $h$ and a vertical arc $v$ with endpoints on the boundary of $Y_q$ such that $v$ and $h$ intersect each other $D$ times, where $D$ is a constant depending on $\epsilon$ and $D \to \infty$ as $\epsilon \to 0$. Since $h$ and $v$ are elements of $C(Y)$, if we choose $D$ large enough, then Theorem 2.3 implies that $Y$ has a subsurface $Z$ such that the distance between the projections of $h$ and $v$ to $C(Z)$ is larger than $K + 2$. But $Y = S \setminus \alpha$; therefore, $Z$ is disjoint from $\alpha$, and we have

$$d_Z(\nu_+, \nu_-) \geq d_Z(h, v) - 2 \geq K.$$ 

Now we prove the claim. Since the hyperbolic length of $\alpha$ at $g(t)$ is less than $\epsilon$, $\alpha$ has an annular collar $A$ with $\text{Mod}(A) \geq \log(\frac{1}{\epsilon})$. Theorem 3.9 implies that there exists a primitive annulus $B$ in $X$ having boundary components $\partial_0$ and $\partial_1$, with $\kappa(\partial_0) \leq 0$, such that the core of $B$ is homotopy equivalent to $\alpha$ and

$$\text{Mod}(B) \geq a \text{Mod}(A) - b.$$
For any $D'$ we can pick $\epsilon$ small enough such that

$$\text{Mod}(B) \geq a \log \left( \frac{1}{\epsilon} \right) - b > D'.$$

Theorem 3.7 implies that

$$d(\partial_0, \partial_1) \geq \text{Mod}(B) \times l_q(\partial_0) \geq D' l_q([\alpha]_q).$$

Since $\kappa(\partial_0) \leq 0$, we can assume that the interior of $B$ is disjoint from $[\alpha]_q$ (otherwise we would have a disk with boundary curvature less than $2\pi$ or an annulus with negative total boundary curvature, which contradicts the Gauss-Bonnet theorem). Let $\omega$ be a vertical arc connecting the boundaries of $B$ ($\omega$ might not exist if $B$ is a flat annulus foliated by closed vertical loops. In this case, let $\omega$ be a horizontal arc; then a similar proof still works). The length of $\omega$ is larger than $D' l_q([\alpha]_q)$. Let $\{R_1, \ldots, R_p\}$ be the set of maximal horizontal strips in $Y_q$. We have (see, Equation 6.1)

$$\sum_p w(R_i) \geq (1 - \delta) 2v_q([\alpha]_q),$$

By the pigeonhole principle, $\omega$ has to cross some $R_i$ at least $D = 2(1 - \delta) D'$ times (note that since trivial strips are parallel to the boundary, $\omega$ can intersect a trivial strip at most once; therefore the strip with large intersection is non-trivial). Let $h$ be a horizontal arc in $R_i$ and $v$ be the vertical arc in $Y_q$ that includes $\omega$. We have $h$ and $v$ intersecting each other at least $D$ times, which was our claim. \qed
Chapter 7

A Counterexample

In this chapter we will describe a sequence of geodesics \((g_n, \nu^n_1, \nu^n_2)\) in the Teichmüller space of \(S\). They all start in a neighborhood of a given point \(X_\infty \in \mathcal{T}(S)\), but each \(g_n\) penetrates deeper and deeper into the thin part of a fixed curve \(\gamma\). On the other hand we show that the subsurface coefficients \(d_Y(\nu^n_1, \nu^n_2)\), where \(Y\) is any subsurface with \(\gamma\) as a boundary component, are bounded above; in fact the bound is 3. This proves that the converse of Theorem 1.5 is not true. Then we study the sequence of hyperbolic 3-manifolds \(N_n\) with end invariants \(g_n(0)\) and \(g_n(2n)\), and we will show that the hyperbolic length of \(\gamma\) in \(N_n\) is bounded below, which proves the following theorem.

**Theorem 7.1.** There exist sequences of 3-manifolds \(N_n\), corresponding Teichmüller geodesics \(g_n\) and a curve \(\gamma\) in \(S\) such that

\[
  l_{N_n}(\gamma) > c > 0 \quad \text{for all} \quad n, \tag{7.1}
\]
but

\[ l_{n}(\gamma) \to 0 \quad \text{as} \quad n \to \infty. \]  \hspace{1cm} (7.2)

Our example is constructed on a compact surface of genus 2. At the end of this chapter we will explain how similar examples exist for every surface of finite type.

**Construction of the example**

Let \( n \) be a positive integer, \( \epsilon = \epsilon_{n} = \epsilon^{-n} \) and \( Q \) be the unit square equipped with the usual Euclidean metric (see Fig 7.1). Define the horizontal foliation \( \nu_{+}^{n} \) to be the foliation of \( Q \) with straight lines of slope \( \epsilon \) and the vertical foliation \( \nu_{-}^{n} \) to be the perpendicular foliation with straight lines of slope \( -\frac{1}{\epsilon} \). Note that the Euclidean metric on \( Q \) defines a transverse measure on \( \nu_{+}^{n} \) and \( \nu_{-}^{n} \), turning them into measured foliations; therefore we can talk about the vertical and horizontal length of a segment transverse to these foliations. Now, let \( \omega_{n} \) be a segment passing through the center of \( Q \), with horizontal length \( \frac{1}{2} \) and vertical length \( \frac{\epsilon^{2}}{2} \) (note again that horizontal and vertical length are measured with respect to \( \nu_{+} \) and \( \nu_{-} \)). Also let \( Q' \) be a second copy of \( Q \) with a segment \( \omega_{n}' \) and horizontal and vertical foliations. We construct a singular Euclidean metric on a surface of genus 2 as follows. Identify the parallel edges of \( Q \) and \( Q' \) to make them into two Euclidean tori. Then cut open arcs \( \omega_{n} \) and \( \omega_{n}' \), making a slit at the middle of each of \( Q \) and \( Q' \), and glue \( Q \) to \( Q' \) through these slits, gluing the opposite sides of slits to each other. Denote the resulting Riemann surface by \( X_{n} \), also let \( X_{\infty} \) be the Riemann surface constructed as above with \( \epsilon = 0 \). Since the slits have the same angle in \( Q \) and in \( Q' \), the resulting foliations
match to give two perpendicular measured foliations on $X_n$, which we denote again by $\nu_+^n$ and $\nu_-^n$. These foliations define a quadratic differential $q_n$ on $X_n$.

Note that $\nu_+^n$ and $\nu_-^n$ have two critical points at the ends of the gluing slit with the total angle around either critical point equal to $4\pi$.

Let $\alpha$ be the curve connecting the top edge to the bottom edge of $Q$, $\beta$ be the curve connecting two sides of $Q$ and $\gamma$ be curve going once around $\omega_n$ in $Q$. (see Fig. 7.1). Let $\alpha'$ and $\beta'$ be similar curves in $Q'$, note that the curve going around $\omega'_n$ is homotopy equivalent to $\gamma$ in $X_n$.

These curve defines a marking of $X_n$. Therefore, we can consider $X_n$ as a point in the Teichmüller space of $S$. We have $X_n \to X_{\infty}$ in $\mathcal{T}(S)$; in particular, we have for some $B > 0$

$$d_{\mathcal{T}(S)}(X_{\infty}, X_n) < B.$$  \hfill (7.3)

![Figure 7.1: Quadratic differential $q_n$.](image)

Let $g_n$ be the geodesic in the Teichmüller space of $S$ corresponding to this quadratic differential, parameterized such that $g_n(0) = q_n$. Let $u_t(\cdot)$ and $h_t(\cdot)$ represent the vertical length and the horizontal length of a curve at $g_n(t)$. At
At $t = 0$ we have

$$v_0(\alpha) = 1 + O(\epsilon), \quad v_0(\beta) = \epsilon, \quad v_0(\gamma) = \epsilon^2,$$

$$h_0(\alpha) = \epsilon, \quad h_0(\beta) = 1 + O(\epsilon), \quad h_0(\gamma) = 1.$$  

Note that since $\gamma$ goes around the slit $\omega_n$ the horizontal length of $\gamma$ is twice the horizontal length of $\omega_n$, therefore $h_0(\gamma) = 1$ not $\frac{1}{2}$. The same is true also for the vertical length of $\gamma$. Recall that

$$v_t(\cdot) = e^t v_0(\cdot) \quad \text{and} \quad h_t(\cdot) = e^{-t} h_0(\cdot).$$

Therefore, at $t = n$, we have

$$v_n(\alpha) = \frac{1}{e} + O(1), \quad v_n(\beta) = 1, \quad v_n(\gamma) = \epsilon,$$

$$h_n(\alpha) = \epsilon^2, \quad h_n(\beta) = \epsilon + O(\epsilon^2), \quad h_n(\gamma) = \epsilon.$$

The Euclidean length of $\beta$ at $g_n(n)$ is almost equal to 1. Let $\tilde{\alpha}$ be a perpendicular curve to $\beta$. Since the area of $Q$ is still 1, the length of $\tilde{\alpha}$ is also almost equal to 1. Let $\tilde{\alpha}'$ be the similar curve in $Q'$ (see Fig. 7.2). We have

$$v_n(\tilde{\alpha}) = \epsilon \quad \text{and} \quad h_n(\tilde{\alpha}) = 1 + O(\epsilon).$$

Since there is an annulus of large modulus around $\gamma$, the hyperbolic length of $\gamma$ at $g(n)$ approaches zero as $n \to \infty$. We claim that for every subsurface
Figure 7.2: $t = n$.

$Y$ with $\gamma$ as its boundary component and every $n$,

$$d_Y(\nu^n_+, \nu^n_-) < 3.$$ 

The subsurface $Y$ is not allowed to be a thrice-punctured sphere because in that case the complex of curves is trivial. There are only two other possibilities. If $Y$ is the annulus with core homotopy equivalent to $\gamma$, then $d_Y(\nu^n_+, \nu^n_-) = 2$ (this is because there is no flat annulus whose core is $\gamma$; see arguments in the proof of Theorem 6.4). Finally, suppose $Y$ is one of the punctured tori bounded by $\gamma$. Consider the marking on $Y$ given by $\beta$ and $\bar{\alpha}$. We can parameterize simple arcs with end points on $\partial Y$ with pairs of relatively prime integers. A leaf of the horizontal foliation is represented by $(1, e^n)$, and a leaf of the vertical foliation is represented by $(e^n, 1)$. We can connect them in $C(Y)$ with a path of length 3 as follows:

$$(e^n, 1), \ (1, 0), \ (0, 1), \ (1, e^n).$$
That is, $d_Y(\nu_+^n, \nu_-^n) = 3$. This proves our claim.

Now let $t = 2n$; we have

$$v_{2n}(\bar{\alpha}) = 1, \quad v_{2n}(\beta) = \frac{1}{\epsilon}, \quad v_{2n}(\gamma) = \frac{1}{2},$$
$$h_{2n}(\bar{\alpha}) = \epsilon + O(\epsilon^2), \quad h_{2n}(\beta) = \epsilon^2 + O(\epsilon^3), \quad h_{2n}(\gamma) = \frac{\epsilon^2}{2}.$$

The Euclidean length of $\bar{\alpha}$ at $g_n(2n)$ is almost equal to 1. Hence, as before, the perpendicular curve to $\bar{\alpha}$, which we denote by $\bar{\beta}$, has length almost equal to 1. Let $\bar{\beta}'$ be the similar curve in $Q'$. We have the following picture (see Fig. 7.3).

![Figure 7.3: $t = 2n$.](image)

We also have

$$\bar{\alpha} = D^\alpha_{\bar{\beta}} \bar{\alpha}, \quad \bar{\alpha}' = D^\alpha_{\bar{\beta}'} \bar{\alpha'},$$

and

$$\bar{\beta} = D^\alpha_{\bar{\beta}} \beta, \quad \bar{\beta}' = D^\alpha_{\bar{\beta}'} \beta',$$

where $D_\alpha$ denotes the Dehn twist around curve $\alpha$. Let $\phi_n$ be the following
element of the mapping class of $S$:

$$\phi_n = D^n_\alpha D^n_{\alpha'} D^n_{\beta} D^n_{\beta'},$$

The sequence $\phi_n^{-1}(g_n(2n))$ converges to the Riemann surface $X'_n$ shown in Fig. 7.4. Therefore we can choose $B$ such that

$$d_{T(S)}(\phi_n^{-1}(g_n(2n)), X'_n) \leq \frac{B}{2}.$$

By choosing $B$ large enough, we can also assume

$$d_{T(S)}(X'_n, X'_{n'}) \leq \frac{B}{2}.$$

Therefore we have

$$d_{T(S)}(\phi_n(X'_n), g_n(2n)) < B. \quad (7.4)$$

![Diagram](image)

Figure 7.4: $X'_n$ and $X'_{n'}$.

We have to show that the length of $\gamma$ in the hyperbolic 3-manifold $N_n$ with $g_n(0)$ and $g_n(2n)$ as its end invariants is bounded below. To do this we compare
that to the length of $\gamma$ in the hyperbolic $3$-manifold with end invariants $X_\infty$ and $\phi_n(X_\infty)$. First we recall a few results in hyperbolic geometry. The following theorem is due to Bers and Thurston.

**Theorem 7.2.** Let $M$ be a compact manifold with boundary. If $M$ is aspherical, atoroidal, Haken and with incompressible boundary, then the space of all geometrically finite complete hyperbolic structures on the interior of $M$ is parameterized by

$$T(\partial_1 M) \times \ldots \times T(\partial_k M),$$

where $\partial_1 M, \ldots, \partial_k M$ are boundary components of $M$.

For $\sigma_1 \in T(\partial_1 M), \ldots, \sigma_k \in T(\partial_k M)$, let $M(\sigma_1, \ldots, \sigma_k)$ denote the unique hyperbolic $3$-manifold homeomorphic to the interior of $M$ having $\sigma_i$ as the end invariant corresponding to the boundary component $\partial_i M$.

The second result is a generalization of Thurston’s Dehn-filling theorem due to T.D. Comar and K. Bromberg (see [4], [6], [9]). We adopt the notation of [6].

**Theorem 7.3.** Let $M$ be a compact $3$-manifold with $k$ torus boundary components and assume $M$ has a geometrically finite hyperbolic structure without rank one cusps. We have the following:

1. For each collection of relatively prime pairs

$$(p, q) = (p_1, q_1; \ldots; p_k, q_k)$$

(except perhaps finitely many), there exists a geometrically finite hyperbolic metric on the $(p, q)$-Dehn filling $M(p, q)$ of $M$ with $M(p, q)$ having
the same end invariants as $M$.

2. If $X$ is a submanifold of $M$ such that the complement of $X$ is a neighborhood of cusps, then $f_{p,q}|x$ is $K_{p,q}$-bi-Lipschitz. Also, $K_{p,q}$ converges to 1 if $\min_i(p_i^2 + q_i^2)$ goes to infinity.

We will also need the following theorem of McMullen (see [19]). Note that here a map is a $K$-quasi-isometry if the norm of its derivative is bounded from above by $K$ and from below by $\frac{1}{K}$.

**Theorem 7.4.** Let $N_i = \mathbb{H}^3/\Gamma_i$, $i = 1, 2$, be a pair of hyperbolic 3-manifolds, and let $\psi$ be a $K$-quasi-conformal conjugacy between $\Gamma_0$ and $\Gamma_1$. Then $\psi$ extends to an equivariant $L$-quasi-isometry $\psi : \mathbb{H}^3 \to \mathbb{H}^3$, where $L = K^{\frac{3}{2}}$. In particular, $N_1$ and $N_2$ are quasi-isometric manifolds.

**Corollary 7.5.** Let $N_1 = M(\sigma_1, \ldots, \sigma_k)$ and $N_1 = M(\tau_1, \ldots, \tau_k)$ be two hyperbolic manifolds. For every $B > 0$ there exists a constant $C > 0$ such that, if

$$d_{\mathcal{T}(M)}(\sigma_i, \tau_i) < B \quad i = 1, \ldots, k,$$

then, for every closed curve $\gamma$ in $M$,

$$\frac{1}{C}l_{N_1}(\gamma) < l_{N_2}(\gamma) < Cl_{N_1}(\gamma).$$

For the rest of this section, let $M = S \times I$ where $S$ is a hyperbolic surface of finite type. We denote $S \times \{0\}$ as $\partial_0 M$ and $S \times \{1\}$ as $\partial_1 M$. Also let $t_1, \ldots, t_n \in [0, 1]$ be an increasing sequence of distinct numbers in the unit interval, and for $i = 1, \ldots, n$, let $\alpha_i$ be a simple closed curve in $S$ with the
property that if $i$ is different from $j$, $\alpha_i$ and $\alpha_j$ are not freely homotopic to each other. Define

$$\hat{M} = M \cup \alpha_i \times \{t_i\}.$$ 

**Theorem 7.6.** $\hat{M}$ is hyperbolizable.

**Proof.** Using Thurston's geometrization theorem, it is sufficient to show that $\hat{M}$ is aspherical, atoroidal and Haken. $\hat{M}$ is clearly Haken since it has a non-spherical boundary. We denote $\alpha_i \times \{t_i\}$ by $\bar{\alpha_i}$.

$\hat{M}$ is aspherical: Assume $\hat{M}$ contains an essential sphere; then, by the sphere theorem, $M$ contains an embedded essential sphere, $\Sigma$. Since $M$ is homotopy equivalent to $S$, $\pi_2(M)$ is trivial. Therefore $\Sigma$ bounds a 3-ball, $B$, in $M$. We know that $\Sigma$ does not intersect any of $\bar{\alpha_i}$, hence either the intersection of $B$ with $\bar{\alpha_i}$ is empty or $B$ contains $\bar{\alpha_i}$. But $B$ is contractible; therefore it can not contain any of $\bar{\alpha_i}$ which are non-trivial in $M$. Hence $B$ misses all $\alpha_i$, and therefore $\Sigma$ is not essential.

$\hat{M}$ is atoroidal: We have to prove that any essential torus in $\hat{M}$ can be pushed off to the boundary. Let $f : T^2 \rightarrow M$ be essential. Using the torus theorem we can assume $f$ is an embedding. Since $M$ does not have any essential tori, $f(T^2)$ bounds a solid torus, $K$, in $M$. Since $f(T^2)$ is disjoint from $\bar{\alpha_i}$, either $K \cap \bar{\alpha_i}$ is empty or $K$ contains $\bar{\alpha_i}$. The annulus connecting $\bar{\alpha_i}$ to $\alpha_i \times \{0\}$ intersects $f(T^2)$ (after pushing off the trivial intersections) in a simple closed curve. Therefore $\alpha_i$ can be pushed to a curve in $f(T^2)$, that is, $\alpha_i$ is not knotted in $K$. Let $\gamma$ be a curve in $S$ homotopy equivalent to the core of $K$. Since $\bar{\alpha_i}$ is non-trivial in $M$, it is non-trivial in $K$ also; therefore $\bar{\alpha_i}$ is homotopy equivalent to a power of $\gamma$. But $\alpha_i$ is a simple curve in $S$; hence it
is in fact homotopy equivalent to $\gamma$ itself. This implies that $\tilde{\alpha}_i$ is homotopy equivalent in $K$ to the core of $K$, and therefore the $f(T^2)$ can be pushed off to $\tilde{\alpha}_i$. 

Embedding $\hat{M} \subset M$ gives a homotopy equivalence between the two boundaries of $\hat{M}$, $\partial_{b}\hat{M}$ and $\partial_{t}\hat{M}$. We identify the Teichmüller spaces of the boundary components of $\hat{M}$ using this homotopy equivalence and denote it by $T(S)$. For Dehn-filling coefficients

$$(p_n, q_n) = (1, n; 1, n; \ldots; 1, n),$$

$M_n = \hat{M}(p_n, q_n)$ is homeomorphic to $M$. Therefore there is a natural homeomorphism $\phi_n$ between boundary components of $M_n$. Since $\partial_b M$ and $\partial_t M$ are identified, we can think of $\phi_n : \partial_b M \to \partial_t M$ as an element of the mapping class group of $S$. It is easy to see that

$$\phi_n = D_{\tilde{\alpha}_n}^n \circ \ldots \circ D_{\tilde{\alpha}_2}^n \circ D_{\tilde{\alpha}_1}^n,$$

where $D_\alpha$ represents the Dehn twist around the curve $\alpha$ and

$$\tilde{\alpha}_i = D_{\tilde{\alpha}_{i-1}}^n \circ \ldots \circ D_{\tilde{\alpha}_2}^n \circ D_{\tilde{\alpha}_1}^n \alpha_i.$$

Therefore, for $\sigma \in T(S)$, $M_n(\sigma, \sigma) = M(\sigma, \phi_n(\sigma))$.

Going back to our example, let $S$ be a surface of genus two, and $\alpha_1, \ldots, \alpha_4$ be respectfully $\alpha, \alpha', \beta$ and $\beta'$. Define $M$, $\hat{M}$, $M_n$ and $\phi_n$ as above and let $N_n = M(g_n(0), g_n(2n))$. Note that the above definition for $\phi_n$ matches our
previous definition. Also recall (Equations (7.3) and (7.4)) that

\[ d_{T(S)}(X_\infty, g_n(0)) < B \quad \text{and} \quad d_{T(S)}(\phi_n(X_\infty), g_n(2n)) < B. \]  

(7.5)

**Proposition 7.7.** The length of \( \gamma \) in \( N_n \) is bounded below by a constant independent of \( n \).

**Proof.** Theorem 7.3 shows that \( \tilde{M}(X_\infty, \phi_n(X_\infty)) = M_n(\sigma, \sigma) \) converges to the cusp manifold \( M(X_\infty, X_\infty) \) as \( n \to \infty \) (note that \( \gamma \) is not one of the cusp curves); therefore there is a lower bound for the length of \( \gamma \) in \( M_n(X_\infty, X_\infty) \). On the other hand, Equation (7.5) and Corollary 7.5 imply that the ratio of the length of \( \gamma \) in \( N_n \) and in \( M_n(\sigma, \sigma) \) is bounded by a constant independent of \( n \). This proves that there is a lower bound, independent of \( n \), on the length of \( \gamma \) in \( N_n \). \( \square \)
Chapter 8

The Universal Curve and the Hyperbolic 3-space

In this chapter we prove $2 \Rightarrow 1$ in Theorem 1.2. Let $(g, \nu_+, \nu_-)$ be a geodesic in the Teichmüller space of $S$ and $q_t$ be the corresponding quadratic differential on $g(t)$. Define $U_g$, the universal curve over $g$, to be the 3-manifold $S \times \mathbb{R}$ equipped with metric $d\rho^2$ defined by the equation

$$d\rho^2 = |q_t| + dt^2.$$

For each $t \in R$ we denote the surface $S \times \{t\}$ equipped with the metric $|q_t|$ by $X_t$. Let $\tilde{U}_g$ be the universal cover of $U_g$.

**Lemma 8.1.** If $\tilde{U}_g$ is bi-Lipschitz to $\mathbb{H}^3$, then $U_g$ is bi-Lipschitz to a hyperbolic 3-manifold $\hat{N}$ with $\nu_+$ and $\nu_-$ as its end invariants.

**Proof.** Consider the action of $G = \pi_1(S)$ on $\tilde{U}_g$. Conjugating this action by the bi-Lipschitz map between $\tilde{U}_g$ and $\mathbb{H}^3$ generates an action of $G$ on $\mathbb{H}^3$ by bi-Lipschitz maps, and therefore a quasi-conformal action on $\hat{C}$. By a theorem of
Sullivan [28], one can conjugate this action by a quasi-conformal map of \( \hat{C} \) to an action of \( G \) by a Möbius transformation on \( \hat{C} \) and therefore by isometries on \( \mathbb{H}^3 \). Let \( \Gamma \) denote this group of isometries of \( \mathbb{H}^3 \), and let \( \hat{N} = \mathbb{H}^3 / \Gamma \). Note that \( \hat{N} \) is bi-Lipschitz to \( U_g \). Therefore short curves in \( \hat{N} \) are also short in \( U_g \) (Corollary 1.6). Any end invariant of \( \hat{N} \) is a limit of a sequence of short geodesics in \( \hat{N} \) exiting on that end of \( \hat{N} \); therefore \( \hat{N} \) has the same end invariants as \( g \) does. \( \square \)

**Lemma 8.2.** Let \( (g, \nu_+, \nu_-) \) be a geodesic in the Teichmüller space of \( S \) and \( (N, \nu_+, \nu_-) \) be a hyperbolic 3-manifold with the same end invariants as \( g \) such that \( U_g \) and \( N \) are bi-Lipschitz. Then \( N \) has bounded geometry.

**Proof.** Let \( f: U_g \to N \) be the \( K \)-bi-Lipschitz map between \( U_g \) and \( N \). Assume \( N \) does not have bounded geometry, therefore, for every \( \epsilon > 0 \) there exists a curve \( \alpha \) in \( S \) such that the hyperbolic length of \( \alpha \) in \( N \) is less than \( \epsilon \), in fact, by making \( \epsilon \) smaller we can assume \( l_N(\alpha) = \epsilon \). Denote the geodesic representative of \( \alpha \) in \( N \) by \( \bar{\alpha} \). Let \( M \) be the Margulis tube corresponding to \( \bar{\alpha} \). The distance between \( \bar{\alpha} \) to the boundary of \( M \) is \( r_0 = \log(\epsilon \mu_M) \), where \( \epsilon \mu_M \) is the Margulis constant for dimension 3. For \( r \leq r_0 \), the \( r \) neighborhood of \( \bar{\alpha} \) in \( N \) \( V(r) \) is inside \( M \) and

\[
\text{vol}_N(V(r)) \asymp r^2(e^\epsilon - 1).
\]

(8.1)

Let \( \bar{t} \in \mathbb{R} \) be the time at which the \( q_t \)-length of \( \alpha \) is the shortest; Let \( \alpha_{\bar{t}} \) be the geodesic representative of \( \alpha \) in \( X_{\bar{t}} \). If \( \alpha \) has more than one geodesic representative, choose \( \alpha_{\bar{t}} \) to be one of the boundary components of the flat annulus in \( X_{\bar{t}} \) corresponding to \( \alpha \). Since \( U_g \) and \( N \) are \( K \)-bi-Lipschitz, we
have \( \frac{c}{K} \leq l_{q_t}(\alpha_t) \leq K \epsilon \), therefore, the length of \( f(\alpha_t) \) is less than or equal to \( K^2 \epsilon \). Hence

\[
f(\alpha_t) \subset V(K^2 \epsilon).
\] (8.2)

Let \( W(R) \) be the \( R \)-neighborhood of \( \alpha_t \) in \( U_g \). Using (8.2) we have

\[
f(W(R)) \subset V(KR + K^2 \epsilon).
\] (8.3)

Let \( A_t = A \times \{ \bar{t} \} \) be the largest regular annulus in \( X_t \) with \( \partial_1 A_t = \alpha_t \) and \( \kappa(\partial_1 A_t) > 0 \). Let \( R_0 \) be the \( q_t \)-distance between the boundary components of \( A_t \). The closure of \( A_t \) in \( X_t \) contains a non-trivial curve that is different from \( \alpha \) otherwise \( A_t \) would not be maximal. Therefore \( f(A_t) \) has to exit the Margulis tube corresponding to \( \bar{\alpha} \) because a non-trivial curve in \( M \) is homotopy equivalent to \( \bar{\alpha} \). Now (8.3) implies

\[
KR_0 + K^2 \epsilon \geq r_0.
\] (8.4)

Therefore for \( R \leq R_1 = \frac{r_0}{K} - K \epsilon \) we have

\[
f(W(R)) \subset M.
\]

Let \( B_t = B \times \{ \bar{t} \} \) be the intersection of the \( \frac{R_1}{3} \)-neighborhood of \( \alpha_t \) in \( X_t \) with \( A_t \). Since \( B_t \) is expanding \( area_{q_t}(B_t) \propto R_1^2 \). For \( \epsilon \) small enough and \( t \in [\bar{t} - 1, \bar{t} + 1] \), \( B_t = B \times \{ t \} \) is in \( R_1 \) neighborhood of \( \alpha_t \) (the distance between a point in \( B_t \) and \( \alpha_t \) is at most \( e^1 \frac{B_0}{3} + 1 \)). Also we have \( area_{q_t}(B_t) \propto R_1^2 \) therefore
\[ \text{vol}_{U_0}(W(R_1)) \asymp R_1^2. \text{ But } f(W(R_1)) \subset M \text{ therefore} \]
\[ \text{vol}_n(M) \geq \frac{\text{vol}_{U_0}(W(R_1))}{K^3}. \]

Also \( R_1 \asymp r_0 \). Using Equation (8.1) we have
\[ r_0^2(e^\epsilon - 1) \asymp \frac{r_0^2}{K^3}. \]

But as \( \epsilon \) approaches zero, the above inequality cannot remain true. The contradiction proves our lemma. \( \Box \)

**Theorem 8.3.** If \( \bar{U}_g \) is bi-Lipschitz to \( \mathbb{H}^3 \), then any hyperbolic 3-manifold \( N \) with end invariants \( \nu_+ \) and \( \nu_- \) has bounded geometry.

**Proof.** Lemma 8.1 states that \( U_g \) is bi-Lipschitz to some \( (\hat{N}, \nu_+, \nu_-) \). Lemma 8.2 implies that \( \hat{N} \) has bounded geometry. Now, since \( \hat{N} \) and \( N \) have the same end invariants and \( \hat{N} \) has bounded geometry, Theorem 1.3 (Minsky) proves that \( N \) also has bounded geometry. \( \Box \)
Bibliography


