Hyperbolic 3-manifolds and Geodesics in Teichmüller Space

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Abstract of the Dissertation

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We are interested in studying the geometry of hyperbolic 3-manifolds homotopy equivalent to a given compact 3-manifold M. When M is a surface bundle over a circle, or when M is an interval bundle over a surface, one can associate to such a hyperbolic 3-manifold a geodesic in $\mathcal{T}(S)$, the Teichmüller space of the surface S. The interplay between the geometry of hyperbolic 3-manifolds and geodesics in the Teichmüller space is the basic subject of our study. More specifically, our goal is to predict the behavior of geodesics in $\mathcal{T}(S)$ based on their end points, then apply the results in studying the geometry of hyperbolic 3-manifolds. Our results are, in summary:

1. A hyperbolic 3-manifold has bounded geometry if and only if the corresponding Teichmüller geodesic stays in the thick part of $\mathcal{T}(S)$.

2. In general, every curve that is short in a hyperbolic 3-manifold is also short for some metric in the corresponding Teichmüller geodesic.

3. The converse of the second statement above is not true.

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To Amy

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Chapter 1

Introduction

Let M be a hyperbolizable compact 3-manifold with incompressible boundary. Define $\mathbf{h}(M)$ to be the space of complete hyperbolic structures on the interior of M. Elements of $\mathbf{h}(M)$ are infinite volume hyperbolic 3-manifolds homeomorphic to the interior of M. We call the intersection of $N \in \mathbf{h}(M)$ with a sufficiently small neighborhood of a boundary component of M an *end* of N. The end of N corresponding to the boundary component S of M is homeomorphic to $S \times (0, \infty)$. To this end one can assign an invariant ν_S in an enlargement $\hat{\mathcal{T}}(S)$ of the Teichmüller space of S, related to Thurston's compactification of the Teichmüller space. If an end is *geometrically finite*, that is if the conformal structure on $S \times \{t\}$ stabilizes as t approaches infinity, ν_S is defined to be the limiting conformal structure on S, i.e., it is a point in $\mathcal{T}(S)$. The subset of $\mathbf{h}(M)$ containing all hyperbolic 3-manifolds whose ends are geometrically finite is well understood through the work of Ahlfors, Bers, Kra, Marden, Maskit and others (e.g., [1], [2], [12] and [14]). If an end is *simply degenerate*, ν_S is a geodesic lamination on S. The existence of end invariants for geometrically infinite ends is due to Bonahon and Thurston ([3],

[29]). Thurston's ending lamination conjecture states that these invariants are sufficient to determine the hyperbolic 3-manifold N.

Conjecture 1.1. *(Thurston) The topological type of* N *and its end invariants determine* N *up to isometry.*

The main theorem

For simplicity, we restrict our discussion to the case where $M = S \times I$ is a product of a surface and an interval and $N \in \mathbf{h}(M)$ has no geometrically finite end. With each such N we associate the following objects:

1. Its end invariants ν_+ and ν_-. Since M has two boundary components homeomorphic to S, both these invariants are laminations on S. We often write (N, ν_+, ν_-) to emphasize the end invariants.

2. The geodesic $g : \mathbb{R} \to \mathcal{T}(S)$ in the Teichmüller space of S connecting ν_+ and ν_-, where $\hat{\mathcal{T}}(S)$ is equipped with the Teichmüller metric. We represent this geodesic by (g, ν_+, ν_-).

3. The universal curve U_g over this geodesic. U_g is a 3-manifold homeomorphic to $S \times \mathbb{R}$ where for every $t \in \mathbb{R}$, $S \times \{t\}$ is isometric to $g(t)$.

4. A coefficient $d_Y(\nu_+, \nu_-)$ to every isotopy class of an essential subsurface Y of S. This coefficient measures the relative complexity of ν_+ and ν_- restricted to Y (see [16] for definition and discussion).

We say N has bounded geometry, if there exists a universal lower bound on the injectivity radius of N, and we say g has bounded geometry if there exists an ϵ such that g does not intersect the ϵ-thin part of the Teichmüller space (see chapter 5 for definition of ϵ-thin part of $\mathcal{T}(S)$).

In the case when ν_+ and ν_- are stable and unstable laminations of a pseudo-Anosov homeomorphism, both N and g have bounded geometry; in fact Cannon and Thurston ([7]) showed that U_g is quasi-isometric to N. We would like to compare the geometry of N and the universal curve U_g in the general case, to examine whether N and U_g have the same set of short curves. Our vehicle for making this connection is the combinatorial machinery introduced by Masur and Minsky ([16]), and the subsurface coefficients $d_Y(\nu_+, \nu_-)$ (see chapter 2).

Theorem 1.2. *The following are equivalent:*

1. *N has bounded geometry.*

2. *g has bounded geometry.*

3. *$\tilde{U}_g$, the universal cover of U_g, is bi-Lipschitz to $\mathbb{H}^3$.*

4. *There exists K such that for every essential subsurface Y of S we have*

$$d_Y(\nu_+, \nu_-) < K.$$

In [22] and [25] Minsky has shown that $1 \Rightarrow 2$, $1 \Rightarrow 3$ and $1 \Leftrightarrow 4$.

Theorem 1.3. *(Minsky [22], [25]) N has bounded geometry if and only if the collection of surface coefficients $\{d_Y(\nu_+, \nu_-)\}$ is bounded above. Moreover, in this case the universal curve U_g is quasi-isometric to N.*

Note that this theorem in particular proves the ending lamination conjecture in the case of bounded geometry, since if $N_1, N_2 \in \mathbf{h}(M)$ have the same end invariants ν_+ and ν_-, then N_1 and N_2 are both quasi-isometric to the same universal curve U_g, and therefore to each other. A theorem of Sullivan ([28]) implies that N_1 and N_2 are isometric.

To complete Theorem 1.2, we prove $2 \Leftrightarrow 4$ (Chapter 6) and $3 \Rightarrow 2$ (Chapter 8) (see also, [20]).

Short curves in 3-manifolds and Teichmüller geodesics

In the case that the geodesic g does not have bounded geometry, we would like to have a description of short curves in g. Such a description is available for hyperbolic 3-manifolds. The following theorem states that the short curves in (N, ν_+, ν_-) are exactly the boundary components of subsurfaces of S where $d_Y(\nu_+, \nu_-)$ is large.

Theorem 1.4. *(Minsky [26]) Let $(N, \nu_+, \nu_-) \in \mathbf{h}(M)$.*

1. *For every $\epsilon > 0$ there exists a $K > 0$ such that if Y is a subsurface of S and α is a component of the boundary of Y, then $d_Y(\nu_+, \nu_-) > K$ implies $l_N(\alpha) < \epsilon$.*

2. *Conversely, for every $K > 0$ there exists $\epsilon > 0$ such that if α is a curve on S and the length of α in N is less than ϵ, then there exists a subsurface Y of S with $\alpha \subset \partial Y$ such that $d_Y(\nu_+, \nu_-) > K$.*

In the case when S is the one-punctured torus, the "same" curves are short in both N and g. This allows for a canonical construction of a model manifold for N which implies the ending lamination conjecture in this case (see

Minsky [23], McMullen [17]). It is reasonable to try to prove an analogue of Theorem 1.4 for geodesics in $\mathcal{T}(S)$, i.e., to prove that the same combinatorial condition would predict which curves are short in the geodesic g.

We have proved the following theorem, which is an analogue of the first part of Theorem 1.4. Let (g, ν_+, ν_-) be a geodesic in $\mathcal{T}(S)$, Y be a subsurface of S, and α be a component of the boundary of Y. Define

$$l_g(\alpha) = \min_{\sigma \in g} l_\sigma(\alpha)$$

to be the shortest hyperbolic length of α over all hyperbolic metrics on g.

Theorem 1.5. *For every $\epsilon > 0$, there exists a $K > 0$ such that, if (g, ν_+, ν_-) is a geodesic in the Teichmüller space of S and if Y is a subsurface of X with $d_Y(\nu_+, \nu_-) > K$, then there exists $t \in \mathbb{R}$ such that for each boundary component α of Y, $l_{g(t)}(\alpha) < \epsilon$.*

The following is a corollary of the previous theorem and Theorem 1.4:

Corollary 1.6. *Curves that are "short" in N are also "short" in g.*

We construct a counterexample to show that the converse of the above corollary, an analogue of Theorem 1.4 Part 2, is not true:

Theorem 1.7. *There exist sequences of manifolds (N_n, ν_+^n, ν_-^n), corresponding Teichmüller geodesics g_i and a curve γ in S such that*

$$l_{N^n}(\gamma) \to c > 0 \qquad \text{as} \qquad n \to \infty, \tag{1.1}$$

but

$$l_{g_n}(\gamma) \to 0 \qquad as \qquad n \to \infty. \tag{1.2}$$

We are, however, able to prove the following weaker version of the converse:

Theorem 1.8. *For every K, there exists $\epsilon > 0$ such that, if (g, ν_+, ν_-) is a geodesic in the Teichmüller space of S, and if α is a curve in S with $l_{g(t)}(\alpha) < \epsilon$ for some $t \in \mathbb{R}$, then there exists a subsurface Z of S disjoint from α such that*

$$d_Z(\nu_+, \nu_-) > K.$$

Outline of the paper

We begin in chapter 2, by introducing the subsurface coefficients $d_Y(\nu_+, \nu_-)$ and recalling some of their properties. In chapter 3 we discuss the geometry of the quadratic differential metric on a Reimann surface. The describion of the ϵ-dicomposition of the quadratic differential metric is given in chapter 4. This is analugous to the thick-thin decomposition of the hyperbolic metric of a Reimann surface. As a corollary we drive an estimate for the quadratic differential length of intersecting curves. This estimate is used in an essential way in the proof of $2 \Leftrightarrow 4$ in the main theorem ,Theorem 1.2, which is given in chapter 6. The counter example, Theorem 1.7, is proven in chapter 7. In chapter 8 we finish the proof of Theorem 1.2 by provin $2 \Rightarrow 1$.

Notation

To simplify our presentation we use the notations O, $\succ$ and $\asymp$ (defined below) whenever the constants involved depend on the topology of S only. That is, for a function f, $O(f)$ represents a function that is bounded above by $C\,f$ for a constant C depending on the topology of S only. For two function f and g, $f \succ g$ means $f \geq c\,g$ and $f \asymp g$ means $c\,f \leq g \leq C\,f$, where C and c depend on the topology of S only.

Chapter 2

Simple Closed Curves

In this chapter we will introduce and study several structures involving simple closed curves on a surface S. Let S be an orientable surface of finite type excluding sphere an torus. We define the *complexity* of S as $\xi(S) = 3g + p$, where g is the genus of S and p is the number of boundary components of S.

By a *curve* we mean a non-trivial, non-peripheral, simple closed curve in S. The free homotopy class of a curve α is denoted by $[\alpha]$. By an *arc* ω we mean a non-trivial arc with endpoints on the boundary of S. Here non-trivial means that ω cannot be pushed to the boundary of S. In case S is not an annulus, $[\omega]$ represents the homotopy class of ω relative to the boundary of S. When S is an annulus, $[\omega]$ is defined to be the homotopy class of ω relative to the endpoints of ω.

Define $\mathcal{C}(S)$ to be the set of all homotopy classes of curves and arcs on the surface S. We equip $\mathcal{C}(S)$ with the following metric. For curves α and β on S, let $d_S(\alpha, \beta)$ be equal to one, if $[\alpha] \neq [\beta]$ and $[\alpha]$ and $[\beta]$ have representatives that are disjoint from each other (for two arcs, being disjoint means having disjoint interior). Let the metric in $\mathcal{C}(S)$ be the maximal metric having the above property. That is, $d_S(\alpha, \beta) = n$ if $\alpha = \alpha_0, \alpha_1, \dots, \alpha_n = \beta$ is the shortest sequence of curves on S such that α_{i-1} is distance one from α_i, $i = 1, \dots, n$.

Theorem 2.1. *(Masur and Minsky, [15]) For $\xi(S) \neq 0, 1$ or 3, $\mathcal{C}(S)$ has the following properties:*

1. *The diameter of $\mathcal{C}(S)$ is infinite.*

2. *As a metric space, $\mathcal{C}(S)$ is δ-hyperbolic in the sense of Gromov.*

A curve system Γ is a non-empty set of elements of $\mathcal{C}(S)$ that are pairwise disjoint from each other. Let $\mathcal{CS}(S)$ be the space of all curve systems on S. For two curve systems Γ_1 and Γ_2 we define $d_S(\Gamma_1, \Gamma_2)$ to be the minimum distance in $\mathcal{C}(S)$ between members of Γ_1 and Γ_2. This defines a pseudo-metric on $\mathcal{CS}(S)$. For any isotopy class of a subsurface Y of S, we define a projection map

$$\pi_Y : \mathcal{CS}(S) \to \mathcal{CS}(Y) \cup \{\emptyset\}$$

as follows: First assume Y is not an annulus. Let $\Gamma = \{[\gamma_1], \dots, [\gamma_k]\}$ be a curve system on S. Fix a hyperbolic metric on S and consider the geodesic representatives of γ_i and boundary components of Y. We define $\pi(\Gamma)$ to be the union of homotopy classes of components of γ_i restricted to Y, $i = 1, \dots, k$. Since the γ_i are pairwise disjoint from each other, the resulting set is in fact a curve system in Y. If Γ does not intersect Y, π_Y is defined to be the empty set.

Since curves in an annulus are defined up to homotopy fixing the end points, we need a special definition for π_Y in the case where Y is an annulus. Fix a hyperbolic metric on S. Let $\hat{S}$ be the annular covering of S corresponding to

Y and let $\hat{\gamma}_i$ be lifts of γ_i intersecting the core of $\hat{S}$ non-trivially. Changing γ_i homotopically in S does not change the end points of $\hat{\gamma}_i$; therefore $\hat{\gamma}_i$ is a well defined element of $\mathcal{C}(Y)$. Now we define $\pi_Y(\Gamma)$ to be the union of $\hat{\gamma}_i$, $i = 0, \dots, n$.

One can also define the projection of a lamination λ onto a subsurface Y. Choose a hyperbolic metric σ on S and restrict the σ-representative of λ to the subsurface in the homotopy class of Y with σ-geodesic boundary. Then the projection of λ onto Y will be the set of homotopy classes of closed curves and compact arcs in the restriction of λ to Y (the projection cannot be defined for leaves that intersect the boundary of Y only once). To define π_Y for a measured foliation one can use the projection of the corresponding geodesic lamination instead.

Subsurface coefficients

For curves $\alpha, \beta \in \mathcal{C}(S)$ that intersect a subsurface Y non-trivially, we call the distance in $\mathcal{C}(Y)$ between the projections $\pi_Y(\alpha)$ and $\pi_Y(\beta)$ the Y*-coefficient* of α and β and denote it by $d_Y(\alpha, \beta)$, that is

$$d_Y(\alpha, \beta) = d_Y(\pi_Y(\alpha), \pi_Y(\beta)).$$

Also, $i(\alpha, \beta)$ denotes the geometric intersection number between α and β. The intersection number between two curves gives a bound on all their subset coefficients.

Theorem 2.2. *For $\alpha, \beta \in \mathcal{C}(S)$ we have*

$$d_Y(\alpha, \beta) \leq 2 \times i(\alpha, \beta) + 1.$$

Conversely, a bound on all subset coefficients gives a bound for the intersection number between two curves.

Theorem 2.3. *For every $K > 0$ there exists $D > 0$ such that for $\alpha, \beta \in \mathcal{C}(S)$, if $i(\alpha, \beta) > D$, then there exists a subsurface Y of S such that*

$$d_Y(\alpha, \beta) > K.$$

Chapter 3

Quadratic Differentials

In this chapter we review the geometry of quadratic differentials. For detailed discussion, see [21] and [27].

Let $\hat{X}$ be a Riemann surface with the conformal structure given by charts $\{(U_\alpha, h_\alpha)\}$. A *(meromorphic) quadratic differential* q on $\hat{X}$ is a set of meromorphic function elements q_α in the local parameter $z_\alpha = h_\alpha(P)$ for which the transformation law

$$q_\alpha(z_\alpha)dz_\alpha^2 = q_\beta(z_\beta)dz_\beta^2$$

holds whenever z_α and z_β are parameter values which correspond to the same point P of $\hat{X}$. The quadratic differential is called holomorphic if all the q_α are holomorphic.

While it clearly does not make sense to speak of the value of a quadratic differential q at a point $P \in \hat{X}$ (since it depends on the local parameter near P), it does make sense to speak of the zeroes and poles of q. In fact, the order of a zero or a pole is invariant under change of parameter.

Definition 3.1. The *critical points* of a quadratic differential q are its zeroes and poles. All other points of $\hat{X}$ are called *regular points* of q. Poles of the first order and zeroes of any order will be called *finite critical points*, and poles of order greater than or equal to two *infinite critical points.*

For the rest of this section let $\hat{X}$ be a compact Riemann surface and let $q \not\equiv 0$ be a meromorphic quadratic differential on $\hat{X}$ with finite critical points only. Also assume that the set of critical points of q is a finite discrete set. The surface $\hat{X}$ punctured at poles P_i of q is denoted by $X = \hat{X} \setminus \{P_i\}$.

Theorem 3.2. *In a neighborhood of every regular point P of q we can introduce a local parameter w, in terms of which the representation of q is identically equal to one. The parameter is given by the integral*

$$w = Q(z) = \int \sqrt{q(z)}\, dz.$$

It is uniquely determined up to a transformation $w \to \pm w + const$ and it will be called the natural parameter near P.

The length of a rectifiable curve can be computed by means of the differential $dw = \sqrt{q(z)}dz$ in terms of an arbitrary local parameter on X.

Definition 3.3. The differential $|dw| = |\sqrt{|q(z)|}|dz|$ is called the *length element* of the q-metric. The length of a curve γ in this metric is denoted by $l_q(\gamma)$ and is computed by

$$l_q(\gamma) = \int_{\gamma'} |dw| = \int_{\gamma} \sqrt{|q(z)|}\, |dz|.$$

The corresponding area element is $dA(q) = |q(z)|\, dx\, dy$; it is also invariant under a change of parameter. The total area of the Riemann surface X in this metric is the L^1-norm of q:

$$|X|_q = ||q|| = \iint\limits_X dA(q).$$

Horizontal and vertical foliations

A *straight arc* is a smooth curve γ whose image under the map Q is a straight line, that is, a curve along which

$$\arg dw^2 = \arg q(z)\, dz^2 = 2\theta = const, \quad 0 \leq \theta < \pi.$$

A maximal such straight arc is called a *θ-trajectory.* A straight arc connecting two critical points in (X, q) is called a *straight segment.*

Through every regular point of q there exists a uniquely determined θ-trajectory, $\pi > \theta \geq 0$. In particular, two θ-trajectories never have a common point, unless they coincide. A trajectory is called a critical trajectory if at least one of its ends tends to a critical point. The total q-area of critical trajectories is zero; in fact, there are only finitely many critical trajectories.

For each θ, the θ-trajectories foliate the set of regular points in X. In a neighborhood of a finite critical point of degree n, $n \geq -1$, this foliation has one of the forms shown in Fig. 3.1.

The foliation corresponding to $\theta = 0$ (respectively, $\theta = \pi/2$), is called the *horizontal foliation* (respectively, the *vertical foliation*) and is denoted by ν_- (respectively, ν_+).

For any closed curve or arc γ, the *horizontal length* of γ is defined to be

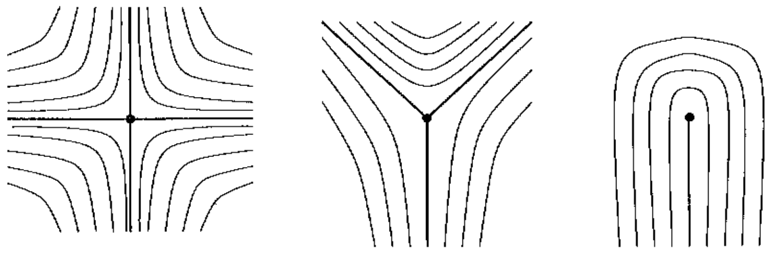

Figure 3.1: Local pictures of a foliation by trajectories

the infimum

$$h_q(\gamma) = \inf_{\hat{\gamma}\sim\gamma} \int_{\hat{\gamma}} |\mathrm{Re}\{q(z)^{\frac{1}{2}}\}|,$$

where $\hat{\gamma}$ ranges over all curves or arcs homotopic to γ in X.

The vertical length of γ is defined similarly and is denoted by $v_q(\gamma)$. Let γ be an arc that is transverse to the horizontal foliation. It is easy to see that the vertical length of γ does not change after a homotopic change through arcs that are transverse to the horizontal foliation and with end points on the same leaves as γ (this is because the vertical length of a horizontal arc is zero). Therefore the horizontal length defines a transverse measure on ν_+; similarly, the vertical length defines a transverse measure on ν_-, making ν_+ and ν_- into measured foliations.

Geodesics and almost geodesics in (X,q)

The *q-geodesic representative* of a curve γ is a curve in the free homotopy class of γ with the shortest q-length and is denoted by $[\gamma]_q$. If X has no punctures, the q-metric is complete and the geodesic representative always exists. In the case that there are poles, the q-metric on X is not complete, since the finite poles are finite distance from interior points of X; however, the metric on $\hat{X}$ is complete. Therefore in this case we allow $[\gamma]_q$ to pass through the poles of $\hat{X}$. That is, $[\gamma]_q$ is defined to be the shortest curve in $\hat{X}$ which is a limit of curves in X in the homotopy class of γ. The same definition works where γ is an arc; here by homotopy class we mean homotopy class relative to end points of γ. Note that if γ is simple, $[\gamma]_q$ is the limit of simple curves in X.

Theorem 3.4. *(see, [21], [27]) For (X, q) as above, we have*

1. *Let γ be an arc joining two given points x and y of X; then the q-geodesic representative of γ exists and is unique.*

2. *Let γ be a loop on X which is not contractible to a point or a punctured disk; then the q-geodesic representative of γ exists and is unique, except for the case where it is one of the continuous family of closed Euclidean geodesics in a flat annulus (see Page 19 for definition of a flat annulus).*

A geodesic is composed of straight segments which meet at critical points of X, making an angle of at least π on either side. When a geodesic passes through a finite pole, it makes at least a full turn around the pole.

Proposition 3.5. *(see, [27]) Let γ be a curve or an arc in X. Then the horizontal length of γ is realized by its q-geodesic representative, that is,*

$$h_q(\gamma) = \int_{[\gamma]_q} |Re\{q(z)^{\frac{1}{2}}\}|.$$

Note that part 1 of Theorem 3.4 implies in particular that the geodesic representatives of disjoint curves do not cross each other. However, they might be tangent to each other, that is, their geodesic representatives might have a geodesic arc in common (also, a geodesic representative of a simple curve might have self-tangencies). As a result of this, the homotopy type of γ might not be retrievable from the image of its geodesic representative. To avoid this problem we construct an embedded curve that is a union of straight arcs in X and approximates $[\gamma]_q$ as closely we like. In fact, we make a general construction for a curve system $\Gamma =' \{\gamma_1, \dots, \gamma_k\}$ on X (that is, a set of essential disjoint simple closed curves in X). Let $[\Gamma]_q$ be the union of $[\gamma_i]_q$. As a subset of $\hat{X}$, $[\Gamma]_q$ can be considered as a graph in X with vertices on the critical points of X and straight segments as its edges. The sequence of simple closed curves in X approximating γ_i gives rise to an immersion of a circle into $[\Gamma]_q$ for $i = 1, \dots k$. For a straight segment ω in $[\Gamma]_q$ connecting critical points P and Q that has been traversed by these circles n times, we replace ω with n disjoint segments parallel to ω in a small neighborhood of ω. Then we connect these arcs in a neighborhood of P and a neighborhood of Q such that the incoming and outgoing arcs match in the correct order. This results in an embedded piecewise smooth representation for Γ which we call the almost geodesic representative of Γ and denote by $\{\Gamma\}_q$. For every $\delta > 0$, $\{\Gamma\}_q$ can be chosen such that $\{\Gamma\}_q$ stays in a δ-neighborhood of $[\Gamma]_q$ and

$$l_q(\{\gamma\}_q) < (1 + \delta) l_q([\gamma]_q).$$

Subsurfaces of X

A subsurface Y of X is a connected open subset of X with piecewise-smooth boundary. Let γ be a boundary component of Y. The curvature of γ with respect to Y, $\kappa(\gamma)$, is well defined as a measure with atoms at the corners (that is, $(\pi - \theta)$, where θ is the interior angle at the corner). We choose the sign to be positive when the acceleration vector points into Y. If γ is curved positively (or negatively) with respect to Y at every point, we say it is *monotonically curved* with respect to Y.

Theorem 3.6. *(Gauss-Bonnet) Let Y be a subsurface of $\hat{X}$ with boundary components $\partial_1, \ldots, \partial_n$; also let $P_1, \ldots, P_k$ be the critical points in Y. Then we have*

$$\sum_{i=1}^{n} \kappa(\partial_i) = 2\pi\big(\chi(Y) + \sum_{i=1}^{k} deg(P_i)\big);$$

where $deg(P_i)$ is the order of the critical pint P_i. In particular, for every i, we have

$$\kappa(\partial_i) \leq 2\pi\big(1 - \chi(X)\big).$$

Given a subsurface Y, it is often desirable to define a representation for the isotopy class Y with q-geodesic boundary. This is not always possible since the geodesic representatives of boundary components of Y can be tangent to each other or themselves. Therefore we define Y_q to be a subsurface in the homotopy class of Y with almost geodesic boundaries. That is, for $\Gamma = \{\partial_1, \ldots, \partial_k\}$, let $\{\Gamma\}_q$ be an almost geodesic representative of Γ. We denote the open set in the complement of $\{\Gamma\}_q$ homotopically equivalent to Y by Y_q and the boundary components of Y_q by ∂_q^i.

Modulus of primitive annuli in X

Let A be a closed annulus in X with boundaries ∂_0 and ∂_1. Suppose both boundaries are monotonically curved with respect to A. Further, suppose that the boundaries are equidistant from each other, and that $\kappa(\partial_0) \leq 0$. We call A a *regular* annulus. If $\kappa(\partial_0) < 0$, we call A *expanding* and say that ∂_0 is the inner boundary and ∂_1 is the outer boundary. If the interior of A contains no zeroes, we say A is a primitive annulus, and we write $\kappa(A) = -\kappa(\partial_0)$. When $\kappa(A) = 0$, A is a *flat* annulus and is foliated by closed Euclidean geodesics homotopic to the boundaries.

Lemma 3.7. *For a regular annulus A we have the following inequality for the distance between the boundaries of A:*

$$d(\partial_0, \partial_1) \geq Mod(A) l_q(\partial_0).$$

Proof. Let $|q|$ be the Euclidean metric on A, $dA(q)$ be the corresponding area form, α be the free homotopy class of the core of A, and d be the distance between boundaries of A. For $0 \leq t \leq d$ define α_t to be the equidistant curve from ∂_0 with $d(\partial_0, \alpha_t) = t$. Define a new metric g_0 as follows: For a point P in α_t,

$$g_0 = \frac{|q|}{l_q^2(\alpha_t)}.$$

We have $l_{g_0}(\alpha_t) = 1$; therefore the area of A in the new metric is

$$Area_{g_0}(A) = \int_A \frac{1}{l_q^2(\alpha_t)} dA(q) = \int_0^d l_{g_0}(\alpha_t) \frac{dt}{l_q(\alpha_t)} = \int_0^d \frac{dt}{l_q(\alpha_t)}.$$

Let A_t be the annulus bounded by ∂_0 and α_t. The Gauss-Bonnet theorem implies that $\kappa(\alpha_t) + \kappa(\partial_0) \geq 0$. Since $\kappa(\partial_0) < 0$, $\kappa(\alpha_t)$, the curvature of α_t with respect to A_t, is positive. Therefore $\frac{d}{dt} l_q(\alpha_t) = \kappa(\alpha_t) > 0$ and $l_q(\alpha_t)$ is an increasing function of t. In particular, $l_q(\alpha_t) \geq l_q(\partial_0)$. Therefore

$$Area_{g_0}(A) \leq \int_0^d \frac{dt}{l_q(\alpha_0)} = \frac{d}{l_q(\partial_0)}.$$

On the other hand, by definition,

$$\frac{1}{Mod(A)} = \sup_g \frac{l_g^2(\alpha)}{Area_g(A)},$$

where the supremum is taken over all metrics g in the conformal class of $|q|$. Since $l_{g_0}(\alpha) = 1$, we have

$$\begin{aligned} \frac{1}{Mod(A)} &\geq \frac{l_{g_0}^2(\alpha)}{Area_{\rho_0}(A)} \\ &\geq \frac{l_q(\partial_0)}{d} \end{aligned}$$

That is,

$$d \geq Mod(A) l_q(\partial_0)$$

□

By a similar method one can also show the following (see [21]).

Lemma 3.8. *For a primitive annulus A, if $\kappa = \kappa(A) > 0$, then*

$$e^{\kappa Mod(A)} l_q(\partial_0) \geq d(\partial_0, \partial_1).$$

For $\kappa(A) = 0$ *we have*

$$Mod(A) l_q(\partial_0) = d(\partial_0, \partial_1).$$

Theorem 3.9. *(Minsky [21]) If* $A \subset X$ *is any homotopically nontrivial annulus with* $Mod(A) \geq m$*, then* A *contains a primitive annulus* B *such that*

$$Mod(B) \geq aMod(A) - b,$$

where m*,* a *and* b *depend only on the topology of* X*.*

We also recall the following theorem of Maskit.

Theorem 3.10. *(Maskit, [13]) Let* q *be a quadratic differential with area 1, and let* σ *be the hyperbolic metric in its conformal class. Then for every* $L > 0$ *there exist* $L' > 0$ *such that for every curve* α *on* X *we have:*

$$l_\sigma(\alpha) < L \Longrightarrow l_q(\alpha) < L'$$

Chapter 4

Thick-thin Decomposition for a Quadratic Differential Metric on a Surface

Here we describe a decomposition of (X, q) into pieces similar to the thick-thin decomposition of X with respect to its hyperbolic metric. Some components are flat, possibly degenerate, annuli. To a component Y that is not an annulus, we associate a number λ_{Y} called the size of Y. In Theorem 4.3 we give upper bounds for the diameter and the area of Y using its size, justifying the use of the term size for λ_{Y}. Then we show that two neighboring component have one of the following properties. Either they have very different sizes, or their common boundary is significantly shorter than the size of each of them or they are connected through a long flat annulus. Although the existence of this decomposition is interesting in itself, the only result that is going to be useful later is Corollary 4.7, which is an analogue of the collar lemma in hyperbolic geometry.

For a fixed ϵ, let $\{\alpha_1, \ldots, \alpha_n\}$ be the set of geodesics in the hyperbolic metric of X with hyperbolic length less than ϵ, and let $Y_1, \ldots, Y_m$ be components of the complement of α_i. For $i = 1, \ldots, n$, let A_i be the largest flat annulus (in the q-metric) whose core is homotopic to α_i and let $[\alpha_i]^+$ and $[\alpha_i]^-$ be the boundary components of A_i ($[\alpha_i]^+$ and $[\alpha_i]^-$ may be identical). For each Y_j, $j = 1, \ldots, m$, let γ_j be an essential curve in Y_j with the shortest q-length $\lambda_j = l_q([\gamma_j]_q)$ among all essential curves in Y_j. Since α_i and γ_j form a curve system, $[\alpha_i]^\pm$ and $[\gamma_j]_q$ have no essential crossings; therefore, one can represent them simultaneously with δ-almost geodesics. Let Y_j be the subsurface corresponding to Y_j with almost q-geodesic boundaries. Note that $\{\gamma_j\}_q$ is an embedded curve in Y_q^j. One should think of λ_j as an indication of how large Y_j is. In Theorem 4.3 we show that the area and the diameter of Y_j are bounded by constants depending on λ_j. We call λ_j the *size* of Y_j. We denote the almost geodesic representatives of $[\alpha_i]_q^\pm$ by $\partial_i^\pm$.

For the rest of this section, let Y be one of these Y_j, $\gamma = \gamma_j$ and $\lambda = \lambda_j$.

Lemma 4.1. *Let A be an annulus in Y with one of its boundary components in ∂Y. Then*

$$area_q(A) = O(l_q(\partial A)^2) \qquad \text{and} \qquad diam_q(A) = O(l_q(\partial A)).$$

Proof. Denote the boundary components of A by ∂_1 and ∂_2, where $\partial_1 \subset \partial Y$. Let A_r be the complement in A of a neighborhood of the boundary of A, $A_r = A - N_r(\partial A)$. For small r, A_r is still an annulus. For r in some interval $[0, r_0]$, A_r has one connected component that is an annulus whose boundary components we denote by ∂_1^r and ∂_2^r, and possibly some connected components that are topological disks. At $r = r_0$, the boundaries of the annulus component of A_r touch each other, and, for $r > r_0$, all components of A_r are disks. Finally, let $r = r_1$ be the infimum of the set of values of r for which A_r is empty. Since $\chi(A_r) \geq 0$, the Gauss-Bonnet theorem implies that the total curvature of the boundaries of A_r, $\kappa(\partial A_r)$, is non-negative. Also, $\frac{d}{dr} l_q(\partial A_r) = -\kappa(\partial A_r)$, and therefore $l_q(\partial A_r)$ is a decreasing function of r.

First we try to find an upper bound for r_0. For $r_0 > r > \delta$ (note that there might not be such r, but in that case δ is an upper bound for r_0), the curvature of ∂_1^r with respect to A_r is negative. This is because ∂_1 is δ-close to the boundary of A_i, and for $r > \delta$, ∂_1^r is disjoint from A_i. The Gauss-Bonnet theorem applied on the annulus component of A_r implies that the curvature of ∂_2^r should be positive, but the curvature of a curve in X is always a multiple of π; therefore,

$$\frac{d}{dr} l_q(\partial_2^r) = -\kappa(\partial_2^r) \leq -\pi.$$

Hence

$$r_0 - \delta \leq \frac{l_q(\partial_2)}{\pi} = O(l_q(\partial_2)).$$

For $r_1 > r > r_0$, A_r has only disk components, and it must have at least one since otherwise it would be empty. Using the Gauss-Bonnet theorem, we have $\frac{d}{dr} l_q(\partial A^r) = -\kappa(\partial A_r) \leq -2\pi$. But $l_q(\partial_2^r)$ is positive; hence

$$r_1 - r_0 \leq \frac{l_q(\partial A_{r_0})}{2\pi} \leq \frac{l_q(\partial_2)}{2\pi} = O(l_q(\partial_2)).$$

Therefore, $r_1 = O(l_q(\partial A))$, and

$$diam_q(A) \leq 2r_1 + \max(l_q(\partial_1), l_q(\partial_2)) = O(l_q(\partial A)).$$

Also, we have the following bound for the area of A:

$$area_q(A) = \int_0^{r_1} l_q(\partial A_r) \leq r_1 l_q(\partial_2 A) = O(l_q(\partial A)^2).$$

$\square$

Using a similar argument, we can show

Lemma 4.2. *Let D be a disk or a punctured disk in X with ∂D as its boundary; then*

$$area_q(D) = O(l_q(\partial D)^2) \qquad and \qquad diam_q(D) = O(l_q(\partial D)).$$

Theorem 4.3. *There exists $R > 0$, depending on ϵ and the topology of X, such that*

1. *If α_i is a boundary component of Y, then $l_q(\partial_i^+) = O(R\lambda)$.*
2. *$diam_q(\mathsf{Y}) = O(R\lambda)$.*
3. *$area_q(\mathsf{Y}) = O(R^2\lambda^2)$.*

Proof. Define N_r to be the open r-neighborhood of a point in $\{\gamma\}_q$ in Y, and let Z_r be the union of N_r with all the components of $Y - N_r$ that are disks, punctured disks or annuli parallel to $\partial\mathsf{Y}$. Let $\partial_1^r, \ldots, \partial_k^r$ be the boundary components of Z_r, and let ∂_r be the union of the ∂_r^i. Also, let $r = \hat{r}$ be the infimum of the set of values of r such that $Z_r = \mathsf{Y}$.

First observe that $l_q(\partial_r)$ and $area_q(Z_r)$ are differentiable functions of r almost everywhere and are continuous except at finitely many points where we add a disk, a punctured disk, or an annulus parallel to a boundary component of Y. For differentiable points we have $\frac{d}{dr}l_q(\partial_r) \leq \kappa(\partial_r)$ (the inequality is there because Z_r does not grow beyond Y). At any $r = r_u$ where we add a disk or a punctured disk D_u, the value of $l_q(\partial_r)$ decreases by the length of the boundary of D_u, c_u. Also, adding an annulus parallel to the boundary of Y replaces a boundary component ∂_r^i with the almost geodesic representative of ∂_r^i, therefore magnifying the value of $l_q(\partial_r^i)$ by at most a factor of $(1+\delta)$ (note that each ∂_r^i is magnified by $(1+\delta)$ at most once). We have the following inequality.

$$\frac{l_q(\partial_r)}{(1+\delta)} \leq \int_0^r \kappa(\partial_\rho)\, d\rho - \sum_u c_u.$$

Using the Gauss-Bonnet theorem, we have

$$\kappa(\partial_\rho) \leq 2\pi\big(1 - \chi(Z_r)\big) = O(1);$$

hence

$$l_q(\partial_r) = O(r) - \sum_u c_u. \tag{4.1}$$

To find an upper bound for $area_q(Z_r)$ we observe that

$$\frac{d}{dr} area_q(Z_r) \leq l_q(\partial_r).$$

Also, at any $r = r_u$ where we add a disk or a punctured disk, $area_q(Z_r)$ increases by $O(l_q^2(c_u))$ (Lemma 4.2), and replacing a boundary component of Z_r, ∂_r^i, by a boundary component of Y increases $area_q(Z_r)$ by $O(l_q^2(\partial_r^i))$ (Lemma 4.1). Therefore,

$$\begin{aligned}
area_q(Z_{\hat{r}}) &\leq \int_0^{\hat{r}} l_q(\partial_r)\,dr + \sum_u O(c_u^2) + \sum_i O(l_q(\partial_r^i)^2) \\
&\leq \int_0^{\hat{r}} \Big[O(r) - \sum_u c_u\Big]\,dr + \sum_u O(c_u^2) + \sum O(\hat{r}^2) \\
&\leq O(\hat{r}^2) + \sum_u O(c_u^2) \\
&\leq O(\hat{r}^2) + \Big[\sum_u O(c_u)\Big]^2
\end{aligned}$$

But Equation 4.1 implies

$$\sum_u c_u = O(\hat{r});$$

therefore,

$$area_q(Z_{\hat{r}}) = O(\hat{r}^2). \tag{4.2}$$

We claim that $\hat{r} \leq \lambda R$ for some R depending on ϵ and the topology of S only. Note that, assuming the claim, Equation (4.1) implies part 2 of our theorem and Equation (4.2) implies part 3. Also, part 1 follows from the fact that

$$diam_q(Y) \leq 2\hat{r}.$$

Let $r_1, \ldots, r_l$ be the points where the topology of Z_r changes or a critical point of X is added to Z_r. All boundary components of Z_r that are not boundary components of Y are essential in X. Therefore, for any such curve, $\partial_{r_j}^i$ and $\partial_{r_{j+1}}^i$ bound an essential primitive annulus A. In particular, the core of A is not homotopy equivalent to any of the α_i. But the α_i are all the curves with hyperbolic length less than ϵ; therefore, the modulus of A is bounded by some M depending on ϵ only. Lemma 3.8 implies

$$Mod(A) \geq \frac{1}{\kappa} \log(\frac{r_{j+1} - r_j}{l_q(\partial^i_{r_j})}).$$

But $l_q(\partial_r) = O(r)$; therefore,

$$r_{j+1} - r_j = e^{\kappa M} O(r_j).$$

This implies that the ratio between consecutive r_i is bounded. Also, l is bounded by the topology of X, and that gives a bound on the ratio $\frac{r_l}{r_1}$. But $r_1 \leq \lambda$ and $r_l = \hat{r}$; therefore there exists R depending on ϵ and the topology of X such that $\hat{r} \leq R\lambda$. □

Lemma 4.4. *Let A be a primitive annulus with ∂_1 and ∂_2 as its boundary components and $\kappa(A) > 0$. Then*

1. $dist_q(\partial_1, \partial_2)^2 \asymp area_q(A)$.
2. $l_q(\partial_1)^2 = O(\frac{area_q(A)}{Mod(A)^2})$.

Proof. Similar to that of Lemma 4.1. □

Lemma 4.5. *For every $C > 0$ there exists $\epsilon > 0$ such that, in the ϵ-decomposition of X, if A_i is the annulus connecting Y and Y' and if λ and λ' are the sizes of Y and Y', respectively, then one of the following holds.*

1. $\lambda \geq C\, l_q(\partial_i^+)$.

2. $\lambda' \geq C\, l_q(\partial_i^+)$.

3. $dist_q(\partial_i^+, \partial_i^-) > C\, l_q(\partial_i^+)$.

Proof. Let A be the primitive annulus corresponding to α_i given by Theorem 3.9, that is,

$$Mod(A) \geq a \log(\frac{1}{\epsilon}) - b.$$

We know that $[\alpha_i]_q^+$ and $[\alpha_i]_q^-$ are disjoint from the interior of A. Therefore, A is a subset of either A_i, Y or Y'. In the first case, Lemma 3.8 implies that

$$l_q(\partial_i^+) \leq Mod(A) dist_q(\partial_i^+, \partial_i^-).$$

If A is a subset of Y, we have that the length of any boundary component of A is at least $\frac{l_q(\partial_i^+)}{(1+\delta)}$. Part 2 of Lemma 4.4 implies

$$l_q(\partial_i^+)^2 \leq l_q(\partial_1 A)^2 = O(\frac{area_q(A)}{Mod(A)^2}) = O(\frac{R^2\lambda^2}{Mod(A)^2}).$$

A similar inequality holds if A is a subset of Y'. Since $Mod(A)$ approaches infinity as ϵ approaches zero, the above inequalities prove our lemma. □

Lemma 4.6. *For given C let ϵ be as in Lemma 4.5. Consider the ϵ-decomposition of X. Let Y be a component of this decomposition and let ∂_i^+ be one of its boundary components. Then there exists an annulus B_i with $\partial_i^+ \in \partial B_i$ such that.*

1. $Mod(B_i) \succ \log(C)$.

2. $Area_q(B_i) = O(\lambda^2)$.

3. $dist_q(\partial_1 B_i, \partial_2 B_i) \asymp \lambda$.

Proof. The previous lemma shows that a large neighborhood on at least one side of ∂_i^+ is still in the form of an annulus; therefore one can choose an annulus B_i which satisfies part 3 of our lemma. Now part 3 of Lemma 4.5 and Lemma 3.7 together imply that B_i satisfies part 1 of our lemma, and part 2 of Lemma 4.4 shows that B_i satisfies part 2 also. $\square$

Let σ be the hyperbolic metric on X.

Corollary 4.7. *For every $L > 0$, there exists D_L such that if α and β are two simple closed geodesics in the hyperbolic metric of X intersecting non-trivially with $l_\sigma(\alpha) < L$, then*

$$D_L l_q([\beta]_q) \geq l_q([\alpha]_q).$$

Proof. First assume α is different from α_i, $i = 1, \dots, n$. Let A_α be a tubular neighborhood of α. For some m and ϵ' depending on L only, one can choose this neighborhood such that A_α is an annulus of modulus larger than m and A_α is disjoint from the ϵ'-thin part of X. Let $\epsilon' > \epsilon > 0$. Consider the decomposition of (X, q) corresponding to ϵ. Since α is disjoint from all α_i, it is an essential curve in some Y_j.

We claim that, for an appropriate choice of ϵ and K depending on L only, $l_q(\alpha) < K\lambda_j$. For every boundary component ∂_i^+ of Y_j, let B_i be a regular annulus as in Lemma 4.6, and let σ_i be the complete hyperbolic metric on B_i. By choosing ϵ appropriately, we can make the modulus of B_i large enough so that the σ_i-length of $\hat{\alpha}_i$, the σ_i-geodesic core of B_i, is less than ϵ'. The Schwarz lemma implies that $l_\sigma(\hat{\alpha}_i)$ is also less than ϵ'. Therefore A_α is disjoint from $\hat{\alpha}_i$.

Let B'_i be the annulus connecting ∂_i^+ to $\hat{\alpha}_i$, and let $\hat{\mathsf{Y}}_j$ be the union of Y_j with all such B'_i. Using Theorem 4.3 and Lemma 4.6 we have

$$area_q(\hat{\mathsf{Y}}_j) \leq area_q(\mathsf{Y}_j) + \sum area_q(B_i) = O(R^2\lambda_j^2).$$

Since α is homotopic to a curve in $\hat{\mathsf{Y}}_j$, A_α should either be a subset of $\hat{\mathsf{Y}}_j$ or intersect one of its boundary components. But A_α is disjoint from $\hat{\alpha}_i$ and therefore stays in $\hat{Y}_j$. By the definition of modulus, we have

$$\frac{l_q^2(\alpha)}{area_q(\hat{Y}_j)} < \frac{1}{Mod(A_\alpha)} < \frac{1}{m};$$

hence

$$l_q(\alpha) = O(\frac{R\lambda}{\sqrt{m}}),$$

which proves our claim.

To finish the proof we have to show that $l_q([\beta]_q) \succ \lambda_j$. If β is an essential curve in Y, then $l_q([\beta]_q) \geq \lambda$. Otherwise since β intersects α non-trivially, β has to intersect some B_i essentially. Lemma 4.6 implies that

$$l_q([\beta]_q) \geq dist_q(\partial_1 B_i, \partial_2 B_i) \asymp \lambda_j.$$

In the case $\alpha = \alpha_i$ for some i, assume α_i is a boundary component of a subsurface Y and B_i is the annulus given by Lemma 4.6. Since in this case β has to intersect B_i, we have $l_q([\beta]_q) \asymp \lambda$. But Theorem 4.3 implies that

$$l_q([\alpha]_q) = l_q(\partial_i^+) = O(R\lambda).$$

Therefore $l_q([\alpha]_q) = O(R\, l_q([\beta]_q))$. $\square$

Chapter 5

The Teichmüller Space of S

The Teichmüller space of S is the space of conformal structures on S, where two structures are considered to be equivalent if there is a conformal map between them isotopic to the identity. There are several natural metrics defined on $\mathcal{T}(S)$, all inducing the same natural topology. We work with the Teichmüller metric, which assigns to $\Sigma_1, \Sigma_2 \in \mathcal{T}(S)$ the distance

$$d_{\mathcal{T}(S)}(\Sigma_1, \Sigma_2) = \frac{1}{2}\log(K),$$

where K is the smallest dilatation of a quasi-conformal homeomorphism from Σ_1 to Σ_2 that is isotopic to the identity.

Geodesics in the Teichmüller space of S, $\mathcal{T}(S)$, are determined by the quadratic differentials. Given q holomorphic for some Riemann surface $\Sigma \in \mathcal{T}(S)$, for any $t \in \mathbb{R}$ we consider the conformal structure obtained by scaling the horizontal foliation of q by a factor of e^t, and the vertical by a factor of e^{-t}. The resulting family, which we denote by $g_q(t)$, is a geodesic in $\mathcal{T}(S)$ parametrized by arclength. The corresponding family of quadratic differentials is denoted by q_t.

Lemma 5.1. *Let α be a curve in S and g be a geodesic in $\mathcal{T}(S)$. The horizontal and vertical lengths of α vary with time as follows:*

$$h_{q_t}(\alpha) = h_q(\alpha)e^t \tag{5.1}$$

and

$$v_{q_t}(\alpha) = v_q(\alpha)e^{-t}. \tag{5.2}$$

For a fixed ϵ, there is a decomposition of $\mathcal{T}(S)$ into its thick part and its thin part. The thick part of $\mathcal{T}(S)$ is the set of hyperbolic metrics on S where the length of every closed geodesic is greater than ϵ, and the thin part of $\mathcal{T}(S)$ is the complement of this set. Given a Teichmüller geodesic g, we denote the length of the geodesic representative of a curve α in g(t) by $l_{g(t)}(\alpha)$. The infimum over all $t \in \mathbb{R}$ of $l_{g(t)}(\alpha)$ is denoted by $l_g(\alpha)$.

Chapter 6

Subsurface Coefficients and Teichmüller Geodesics

In this chapter we prove Theorem 1.5 and Theorem 1.8. Note that $2 \Leftrightarrow 4$ in Theorem 1.2 is a corollary of these two theorems.

Let X be a Riemann surface homeomorphic to S, and let q be a holomorphic quadratic differential on X with poles of order at most 1 on punctures of X and with ν_+ and ν_- as its horizontal and vertical foliations. Let Y be a subsurface of X and Y_q be the corresponding subsurface with δ-almost q-geodesic boundary. We denote the boundary components of Y_q by $\partial_1, \ldots, \partial_k$ and the union of the ∂_i by ∂Y_q. Define

$$l_q(\partial Y_q) = \sum_{i=1}^{k} l_q(\partial_i),$$
$$h_q(\partial Y_q) = \sum_{i=1}^{k} h_q(\partial_i)$$

and

$$v_q(\partial Y_q) = \sum_{i=1}^{k} v_q(\partial_i).$$

Definition 6.1. Let $f : [0,1]\times(0,1) \to Y_q$ be an embedding of the unit square in Y_q with the following properties:

1. For $s = 0, 1$, f maps $\{s\} \times (0,1)$ into ∂Y_q.

2. For $t \in (0,1)$, f maps $[0,1] \times \{t\}$ into a horizontal leaf of q.

Note that the image of f does not contain any critical point of q. We call $R = f([0,1] \times (0,1))$ a *horizontal strip.* We say a horizontal strip is *maximal* if it is not a proper subset of any other horizontal strip. The width $w(R)$ of a horizontal strip R is defined to be the vertical length of a transverse arc in the form $f(\{t\} \times I)$,

$$w(R) = v_q\big(f(\{t\} \times I)\big).$$

Vertical strips and *maximal vertical strips* are defined similarly. The width of a vertical strip R is the horizontal length of an arc transverse to R.

Lemma 6.2. *Any two maximal horizontal (respectively, vertical) strips in Y_q either have disjoint interiors or are identical. Furthermore, there are only finitely many distinct maximal strips, and the union of all maximal strips covers Y_q except for a measure zero set.*

Proof. If the interiors of two horizontal strips intersect non-trivially, their union is also a horizontal strip. But if they are both maximal, the union cannot be larger than either of them; therefore they are identical. To see the second part, note that for a generic point $P \in Y_q$ the horizontal leaf passing through P restricted to Y_q is compact and non-critical. Therefore a neighborhood of this leaf is also free of critical points, which implies that P is an interior point of some horizontal strip. Also, each maximal strip has a critical point on its boundary (otherwise one could extend it to a larger strip), and each critical point P of degree n can appear on the boundary of at most $n+2$ maximal strips. But the sum of the degrees of the critical points in X is finite, and so is the number of maximal strips. The proof for vertical strips is similar.

□

Corollary 6.3. *Let $R_1, \dots, R_{\hat{p}}$ be the decomposition of Y into maximal horizontal (respectively, vertical) strips. Then the sum of the widths of the R_i, $\sum_{i=1}^{\hat{p}} w(R_i)$, is equal to the total vertical (respectively, horizontal) length of the boundary of Y_q.*

If Y_q had geodesic boundary, then the restriction of a horizontal leaf to Y_q would either lie on the boundary of Y_q or be an essential arc in Y. This is because the q-geodesic representative of an arc connecting two points is unique. Since Y_q has almost geodesic boundary, some of our horizontal strips might be parallel to the boundary of Y_q. But since ∂Y_q is δ-close to its q-geodesic representative, the total vertical length of strips parallel to the boundary of Y_q is less than $\delta\, l_q(\partial Y_q)$. We call the strips that are not parallel to the boundary of Y_q non-trivial strips. Therefore, if $R_1, \dots, R_p$ are the non-trivial horizontal strips, then

$$\sum_{i=1}^{p} w(R_i) \geq (1-\delta) v_q(\partial Y_q). \tag{6.1}$$

A similar inequality holds for the vertical strips.

Let σ be the hyperbolic metric on X.

Theorem 6.4. *For every L, there exists C_L depending only on L and the topology of X, such that, if α is a curve in X intersecting a boundary component of Y non-trivially and $l_\sigma(\alpha) < L$, then*

$$\min\big(d_Y(\alpha,\nu_+), d_Y(\alpha,\nu_-)\big) < C_L.$$

Proof. First assume that Y is not an annulus. Since α intersects a boundary component of Y, Corollary 4.7 implies that

$$l_q([\alpha]_q) \le D_L l_q(\partial Y_q). \tag{6.2}$$

Also, we know that $l_q(\partial Y_q) \le v_q(\partial Y_q) + h_q(\partial Y_q)$. Without loss of generality we can assume that the boundary of Y_q is mostly vertical, that is,

$$l_q(\partial Y_q) \le 2v_q(\partial Y_q). \tag{6.3}$$

Let $\hat{\alpha}$ be a component of the restriction of $[\alpha]_q$ to Y_q, $\{R_1, \ldots R_p\}$ be the set of non-trivial maximal horizontal strips on Y_q, and b_i be the number of times $\hat{\alpha}$ crosses the strip R_i. Note that $\hat{\alpha}$ can have partial intersection with at most two strips, which we do not count as crossing. We have

$$v_q(\hat{\alpha}) \ge \sum_{i=1}^{p} b_i w(R_i). \tag{6.4}$$

Now let b be the minimum of the b_i, and let R be the corresponding strip. We have

$$\begin{aligned}
b\sum_{i=1}^{p} w(R_i) &< \sum_{i=1}^{p} b_i w(R_i) \\
&\leq v_q(\hat{\alpha}) && \text{Equation (6.4)} \\
&\leq l_q([\alpha]_q) \\
&\leq D_L l_q(\partial Y_q) && \text{Equation (6.2)} \\
&\leq 2D_L v_q(\partial Y_q) && \text{Equation (6.3)} \\
&= \frac{2D_L}{1-\delta}\sum_{i=1}^{p} w(R_i) && \text{Equation (6.1)}
\end{aligned}$$

Therefore, $\hat{\alpha}$ can cross R at most $\frac{2D_L}{1-\delta}$ times. This means that there exists a leaf of the horizontal foliation that intersects $\hat{\alpha}$ a bounded number of times, where the bound depends only on L and on the topology of S. A bound on this intersection number gives a bound on the distance in $\mathcal{C}(Y)$ between $\hat{\alpha}$ and the horizontal foliation (see, Lemma 2.2).

Now assume Y is an annulus and γ is its core. Let A be the annular covering of X with respect to γ; We represent the core curve of A by γ again. Let $\tilde{q}$ be the lift of quadratic differential q to A, $[\gamma]_q$ be the $\tilde{q}$-geodesic core of A, $[\tilde{\alpha}]_q$ be the geodesic representative of a lift of α to A intersecting $[\gamma]_q$ non-trivially. Also let h (respectively, v) be a leaf of the horizontal (respectively, vertical) foliation of $\tilde{q}$ intersecting $[\gamma]_q$ non-trivially. To prove the theorem in this case we have to show that $[\tilde{\alpha}]_q$ intersects one of h or v a bounded number of times, with bound depending on L only. Let B be the maximal flat annulus in A, and let B_1 and B_2 be the two annuli in the complement of B in A. Since B_1 and B_2 are expanding annuli, the Gauss-Bonnet theorem implies that h and v intersect $[\tilde{\alpha}]_q$ at most once in B_1 and B_2. Let s_h, s_v and s be the slopes of h, v and $[\tilde{\alpha}]_q$ in B, respectively (consider B as a vertical cylinder). Since h and v are perpendicular to each other, one of $|s_h|$ and $|s_v|$ (say $|s_h|$) is larger than or equal to 1. Let ω be the restriction of $[\tilde{\alpha}]_q$ to B. Since the projection map from A to X maps B into X injectively,

$$l_q([\alpha]_q) \geq l_{\tilde{q}}(\omega).$$

On the other hand, Theorem 4.7 implies

$$l_q([\gamma]_q) \leq \frac{1}{D_L} l_q([\alpha]_q).$$

Therefore $|s| \geq \frac{1}{D_L}$. The number of intersection points between h and ω is bounded by

$$\frac{dist(\partial_1 B, \partial_2 B)}{l_q(\tilde{\gamma})} \left|\frac{1}{s_h} - \frac{1}{s}\right| \leq Mod(B)(D_L + 1).$$

But γ intersects α, and $l_\sigma(\alpha) < L$; therefore, the hyperbolic length of γ is bounded below by a constant depending on L, and that gives an upper bound for the modulus of A. But, $Mod(B) \leq Mod(A)$; hence $Mod(B)$ is bounded by a constant depending only on L. Therefore we have a upper bound for the intersection number between h and ω, and hence, for the intersection number between h and $[\tilde{\alpha}]_q$. $\square$

Theorem 6.5. *Let (g, ν_+, ν_-) be a geodesic in the Teichmüller space of S. For every subsurface Y and $L > 0$, there exists $M > 0$, depending on Y, g, L and the topology of S, such that, if a curve β intersects Y non-trivially and the hyperbolic length of β is shorter than L at some time $t > M$ (respectively, $t < -M$), then*

$$d_Y(\beta, \nu_+) = 1$$
$$(\textit{respectively}, \quad d_Y(\beta, \nu_-) = 1).$$

Proof. Fix a time t_0, let $X_0 = g(t_0)$, and let q_0 be the corresponding quadratic differential on X_0. Let Y_0 be a subsurface of X in the isotopy class of Y whose boundaries are δ-almost geodesics in q_0, and let ∂_i be a boundary component of Y_0. Let R be a non-trivial horizontal strip in Y with a vertical width of d. For $t \in \mathbb{R}$, the q_t-area of X_t equals 1. Also, the hyperbolic length of β at time t is less than L. Theorem 3.10 implies that the q_t-length of β is less than L' for some L' depending on L. Therefore, the q_0-vertical length of β is less than $L'e^{t_0 - t}$. By choosing t large enough, we can make this quantity be less than d. Therefore, β does not cross R, and it is disjoint from some leaf of the horizontal foliation. But the horizontal leaves in R are essential arcs in Y; therefore, β is distance one from the horizontal foliation. □

Now we are ready to prove Theorem 1.5.

Theorem 1.5. *For every $\epsilon > 0$, there exists a $K > 0$ such that, if (g, ν_+, ν_-) is a geodesic in the Teichmüller space of S and if Y is a subsurface of X with $d_Y(\nu_+, \nu_-) > K$, then there exists $t \in \mathbb{R}$ such that for each boundary* *component α of Y, $l_{g(t)}(\alpha) < \epsilon$.*

Proof. We prove the contrapositive of the above statement. Let ϵ be such that for every $t \in \mathbb{R}$ there exists a boundary component α_t of Y with $l_{g(t)}(\alpha) > \epsilon$. We have to find an upper bound for $d_Y(\nu_+, \nu_-)$. For every t, since $l_{g(t)}(\alpha_t) > \epsilon$, there exists a curve β_t, intersecting α_t non-trivially, whose hyperbolic length in $g(t)$ is less than L, for some $L > 0$ depending on ϵ only. By increasing the value of L (say doubling the value of L) we can assume that the hyperbolic length of β_t is less than L in a neighborhood of t. Therefore, we can find a covering of $\mathbb{R}$ with intervals $[t_i, t_{i+1}]$ and a sequence of curves β_i such that the β_i intersect α non-trivially and, for $t \in [t_i, t_{i+1}]$, $l_{g(t)}(\beta_i) < L$.

For C_L as in Theorem 6.4, we have that each β_i is C_L close to either ν_+ or ν_-. Assume $d_Y(\nu_+, \nu_-) > 2C_L$ (otherwise we are done). Then β_i cannot be C_L close to ν_+ and ν_- at the same time. Define I_+ (respectively, I_-) to be the set of all integers i such that $d_Y(\beta_i, \nu_+) \leq C_L$ (respectively, $d_Y(\beta_i, \nu_-) \leq C_L$). I_+ and I_- are disjoint and non-empty (Theorem 6.5), and $\mathbb{Z} = I_+ \cup I_-$. Therefore there exists $j \in I_-$ such that $j + 1 \in I_+$ (or we could have $j - 1 \in I_+$). We have

$$d_Y(\beta_j, \nu_-) \leq C_L, \qquad \text{and} \qquad d_Y(\beta_{j+1}, \nu_+) \leq C_L.$$

Also, at t_{j+1}, both β_j and β_{j+1} have hyperbolic length less than L. Hence

$$d_Y(\beta_j, \beta_{j+1}) = O(1).$$

The combination of the above inequalities gives an upper bound on $d_Y(\nu_+, \nu_-)$ and therefore proves the theorem. $\square$

Theorem 1.8. *For every K, there exists $\epsilon > 0$ such that, if (g, ν_+, ν_-) is a geodesic in the Teichmüller space of S, and if α is a curve in S with $l_{g(t)}(\alpha) < \epsilon$ for some $t \in \mathbb{R}$, then there exists a subsurface Z of S disjoint from α such that*

$$d_Z(\nu_+, \nu_-) > K.$$

Proof. Let t be as above, $X_t = g(t)$, and q_t be the corresponding quadratic differential on X_t. Also, let $Y = S \setminus \alpha$, and let Y_q be the corresponding subsurface with almost geodesic boundaries. We claim that there exists a horizontal arc h and a vertical arc v with endpoints on the boundary of Y_q such that v and h intersect each other D times, where D is a constant depending on ϵ and $D \to \infty$ as $\epsilon \to 0$. Since h and v are elements of $\mathcal{C}(Y)$, if we choose D large enough, then Theorem 2.3 implies that Y has a subsurface Z such that the distance between the projections of h and v to $\mathcal{C}(Z)$ is larger than $K + 2$. But $Y = S \setminus \alpha$; therefore, Z is disjoint from α, and we have

$$d_Z(\nu_+, \nu_-) \geq d_Z(h, v) - 2 \geq K.$$

Now we prove the claim. Since the hyperbolic length of α at $g(t)$ is less than ϵ, α has an annular collar A with $Mod(A) \geq \log(\frac{1}{\epsilon})$. Theorem 3.9 implies that there exists a primitive annulus B in X having boundary components ∂_0 and ∂_1, with $\kappa(\partial_0) \leq 0$, such that the core of B is homotopy equivalent to α and

$$Mod(B) \geq aMod(A) - b.$$

For any D' we can pick ϵ small enough such that

$$Mod(B) \geq a\log(\frac{1}{\epsilon}) - b > D'.$$

Theorem 3.7 implies that

$$d(\partial_0, \partial_1) \geq Mod(B) \times l_q(\partial_0) \geq D' \, l_q([\alpha]_q).$$

Since $\kappa(\partial_0) \leq 0$, we can assume that the interior of B is disjoint from $[\alpha]_q$ (otherwise we would have a disk with boundary curvature less than 2π or an annulus with negative total boundary curvature, which contradicts the Gauss-Bonnet theorem). Let ω be a vertical arc connecting the boundaries of B (ω might not exist if B is a flat annulus foliated by closed vertical loops. In this case, let ω be a horizontal arc; then a similar proof still works). The length of ω is larger than $D' \, l_q([\alpha]_q)$. Let $\{R_1, \ldots, R_p\}$ be the set of maximal horizontal strips in Y_q. We have (see, Equation 6.1)

$$\sum_p w(R_i) \geq (1-\delta) 2 v_q([\alpha]_q),$$

By the pigeonhole principle, ω has to cross some R_i at least $D = 2(1-\delta)D'$ times (note that since trivial strips are parallel to the boundary, ω can intersect a trivial strip at most once; therefore the strip with large intersection is non-trivial). Let h be a horizontal arc in R_i and v be the vertical arc in Y_q that includes ω. We have h and v intersecting each other at least D times, which was our claim.

□

Chapter 7

A Counterexample

In this chapter we will describe a sequence of geodesics (g_n, ν_+^n, ν_-^n) in the Teichmüller space of S. They all start in a neighborhood of a given point $X_\infty \in \mathcal{T}(S)$, but each g_n penetrates deeper and deeper into the thin part of a fixed curve γ. On the other hand we show that the subsurface coefficients $d_Y(\nu_+^n, \nu_-^n)$, where Y is any subsurface with γ as a boundary component, are bounded above; in fact the bound is 3. This proves that the converse of Theorem 1.5 is not true. Then we study the sequence of hyperbolic 3-manifolds N_n with end invariants $g_n(0)$ and $g_n(2n)$, and we will show that the hyperbolic length of γ in N_n is bounded below, which proves the following theorem.

Theorem 7.1. *There exist sequences of 3-manifolds N_n, corresponding Teichmüller geodesics g_n and a curve γ in S such that*

$$l_{N_n}(\gamma) > c > 0 \qquad \textit{for all} \quad n, \tag{7.1}$$

but

$$l_{g_n}(\gamma) \to 0 \qquad as \qquad n \to \infty. \tag{7.2}$$

Our example is constructed on a compact surface of genus 2. At the end of this chapter we will explain how similar examples exist for every surface of finite type.

Construction of the example

Let n be a positive integer, $\epsilon = \epsilon_n = e^{-n}$ and Q be the unit square equipped with the usual Euclidean metric (see Fig 7.1). Define the horizontal foliation ν_-^n to be the foliation of Q with straight lines of slope ϵ and the vertical foliation ν_+^n to be the perpendicular foliation with straight lines of slope $-\frac{1}{\epsilon}$. Note that the Euclidean metric on Q defines a transverse measure on ν_+^n and ν_-^n, turning them into measured foliations; therefore we can talk about the vertical and horizontal length of a segment transverse to these foliations. Now, let ω_n be a segment passing through the center of Q, with horizontal length $\frac{1}{2}$ and vertical length $\frac{\epsilon^2}{2}$ (note again that horizontal and vertical length are measured with respect to ν_+ and ν_-). Also let Q' be a second copy of Q with a segment ω_n' and horizontal and vertical foliations. We construct a singular Euclidean metric on a surface of genus 2 as follows. Identify the parallel edges of Q and Q' to make them into two Euclidean tori. Then cut open arcs ω_n and ω_n', making a slit at the middle of each of Q and Q', and glue Q to Q' through these slits, gluing the opposite sides of slits to each other. Denote the resulting Riemann surface by X_n, also let X_∞ be the Riemann surface constructed as above with $\epsilon = 0$. Since the slits have the same angle in Q and in Q', the resulting foliations match to give two perpendicular measured foliations on X_n, which we denote again by ν_+^n and ν_-^n. These foliations define a quadratic differential q_n on X_n. Note that ν_+^n and ν_-^n have two critical points at the ends of the gluing slit with the total angle around either critical point equal to 4π.

Let α be the curve connecting the top edge to the bottom edge of Q, β be the curve connecting two sides of Q and γ be curve going once around ω_n in Q. (see Fig. 7.1). Let α' and β' be similar curves in Q', note that the curve going around ω_n' is homotopy equivalent to γ in X_n.

These curve defines a marking of X_n. Therefore, we can consider X_n as a point in the Teichmüller space of S. We have $X_n \to X_\infty$ in $\mathcal{T}(S)$; in particular, we have for some $B > 0$

$$d_{\mathcal{T}(S)}(X_\infty, X_n) < B. \tag{7.3}$$

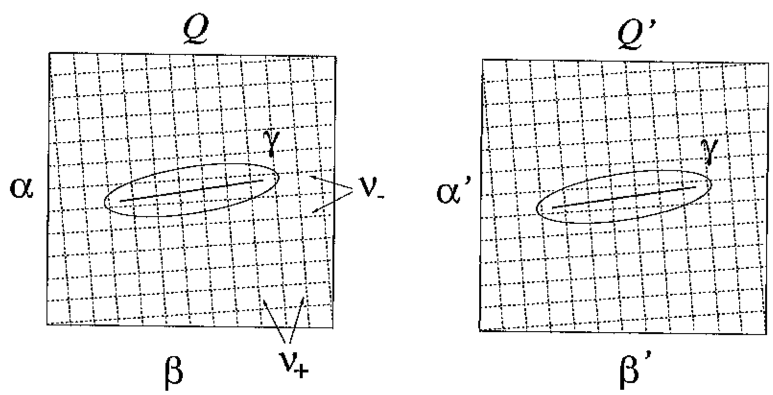

Figure 7.1: Quadratic differential q_n.

Let g_n be the geodesic in the Teichmüller space of S corresponding to this quadratic differential, parameterized such that $g_n(0) = q_n$. Let $v_t(\cdot)$ and $h_t(\cdot)$ represent the vertical length and the horizontal length of a curve at $g_n(t)$. At

$t = 0$ we have

$$v_0(\alpha) = 1 + O(\epsilon), \qquad v_0(\beta) = \epsilon, \qquad v_0(\gamma) = \epsilon^2,$$
$$h_0(\alpha) = \epsilon, \qquad h_0(\beta) = 1 + O(\epsilon), \qquad h_0(\gamma) = 1.$$

Note that since γ goes around the slit ω_n the horizontal length of γ is twice the horizontal length of ω_n, therefore $\mathbf{h}_0(\gamma) = 1$ not $\frac{1}{2}$. The same is true also for the vertical length of γ. Recall that

$$v_t(\cdot) = e^t v_0(\cdot) \qquad \text{and} \qquad h_t(\cdot) = e^{-t} h_0(\cdot).$$

Therefore, at $t = n$, we have

$$v_n(\alpha) = \frac{1}{\epsilon} + O(1), \qquad v_n(\beta) = 1, \qquad v_n(\gamma) = \epsilon,$$
$$h_n(\alpha) = \epsilon^2, \qquad h_n(\beta) = \epsilon + O(\epsilon^2), \qquad h_n(\gamma) = \epsilon.$$

The Euclidean length of β at $g_n(n)$ is almost equal to 1. Let $\bar{\alpha}$ be a perpendicular curve to β. Since the area of Q is still 1, the length of $\bar{\alpha}$ is also almost equal to 1. Let $\bar{\alpha}'$ be the similar curve in Q' (see Fig. 7.2). We have

$$v_n(\bar{\alpha}) = \epsilon \qquad \text{and} \qquad h_n(\bar{\alpha}) = 1 + O(\epsilon).$$

Since there is an annulus of large modulus around γ, the hyperbolic length of γ at $g(n)$ approaches zero as $n \to \infty$. We claim that for every subsurface

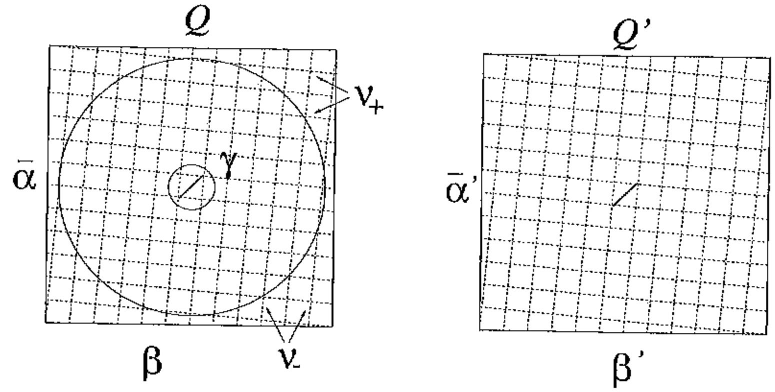

Figure 7.2: $t = n$.

Y with γ as its boundary component and every n,

$$d_Y(\nu_+^n, \nu_-^n) < 3.$$

The subsurface Y is not allowed to be a thrice-punctured sphere because in that case the complex of curves is trivial. There are only two other possibilities. If Y is the annulus with core homotopy equivalent to γ, then $d_Y(\nu_+^n, \nu_-^n) = 2$ (this is because there is no flat annulus whose core is γ; see arguments in the proof of Theorem 6.4). Finally, suppose Y is one of the punctured tori bounded by γ. Consider the marking on Y given by β and $\bar{\alpha}$. We can parameterize simple arcs with end points on ∂Y with pairs of relatively prime integers. A leaf of the horizontal foliation is represented by $(1, e^n)$, and a leaf of the vertical foliation is represented by $(e^n, 1)$. We can connect them in $\mathcal{C}(Y)$ with a path of length 3 as follows:

$$(e^n, 1), \quad (1, 0), \quad (0, 1), \quad (1, e^n).$$

That is, $d_Y(\nu_+^n, \nu_-^n) = 3$. This proves our claim.

Now let $t = 2n$; we have

$$v_{2n}(\bar{\alpha}) = 1, \qquad v_{2n}(\beta) = \frac{1}{\epsilon}, \qquad v_{2n}(\gamma) = \frac{1}{2},$$
$$h_{2n}(\bar{\alpha}) = \epsilon + O(\epsilon^2), \qquad h_{2n}(\beta) = \epsilon^2 + O(\epsilon^3), \qquad h_{2n}(\gamma) = \frac{\epsilon^2}{2}.$$

The Euclidean length of $\bar{\alpha}$ at $g_n(2n)$ is almost equal to 1. Hence, as before, the perpendicular curve to $\bar{\alpha}$, which we denote by $\bar{\beta}$, has length almost equal to 1. Let $\bar{\beta}'$ be the similar curve in Q'. We have the following picture (see Fig. 7.3).

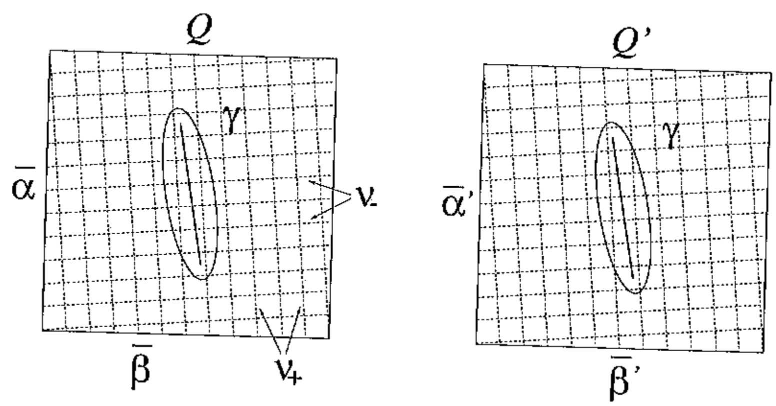

Figure 7.3: $t = 2n$.

We also have

$$\bar{\alpha} = D_\beta^n \alpha, \qquad \bar{\alpha}' = D_{\beta'}^n \alpha',$$

and

$$\bar{\beta} = D_{\bar{\alpha}}^n \beta, \qquad \bar{\beta}' = D_{\bar{\alpha}'}^n \beta',$$

where D_α denotes the Dehn twist around curve α. Let ϕ_n be the following element of the mapping class of S:

$$\phi_n = D^n_{\bar{\alpha}'} D^n_{\bar{\alpha}} D^n_{\beta'} D^n_{\beta},$$

The sequence $\phi_n^{-1}(g_n(2n))$ converges to the Riemann surface X'_∞ shown in Fig. 7.4. Therefore we can choose B such that

$$d_{\mathcal{T}(S)}(\phi_n^{-1}(g_n(2n)), X'_\infty) \leq \frac{B}{2}.$$

By choosing B large enough, we can also assume

$$d_{\mathcal{T}(S)}(X_\infty, X'_\infty) \leq \frac{B}{2}.$$

Therefore we have

$$d_{\mathcal{T}(S)}(\phi_n(X_\infty), g_n(2n)) < B. \tag{7.4}$$

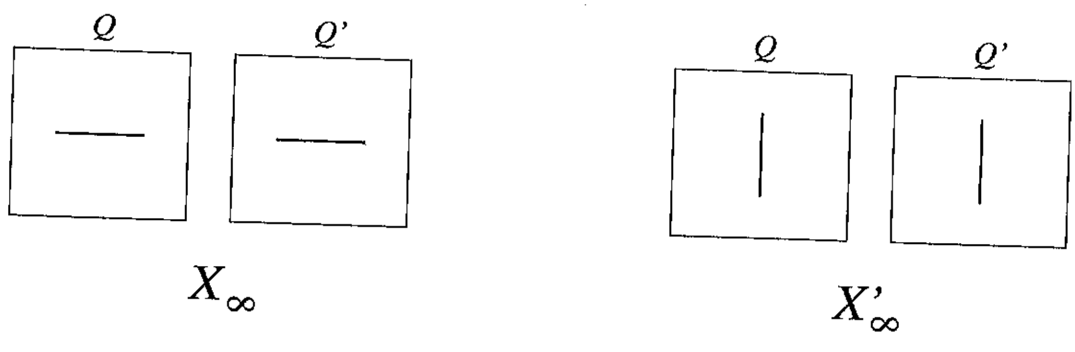

Figure 7.4: X_∞ and X'_∞.

We have to show that the length of γ in the hyperbolic 3-manifold N_n with $g_n(0)$ and $g_n(2n)$ as its end invariants is bounded below. To do this we compare that to the length of γ in the hyperbolic 3-manifold with end invariants X_∞ and $\phi_n(X_\infty)$. First we recall a few results in hyperbolic geometry. The following theorem is due to Bers and Thurston.

Theorem 7.2. *Let M be a compact manifold with boundary. If M is aspherical, atoroidal, Häken and with incompressible boundary, then the space of all geometrically finite complete hyperbolic structures on the interior of M is parameterized by*

$$\mathcal{T}(\partial_1 M) \times \ldots \times \mathcal{T}(\partial_k M),$$

where $\partial_1 M, \ldots, \partial_k M$ are boundary components of M.

For $\sigma_1 \in \mathcal{T}(\partial_1 M), \ldots, \sigma_k \in \mathcal{T}(\partial_k M)$, let $M(\sigma_1, \ldots, \sigma_k)$ denote the unique hyperbolic 3-manifold homeomorphic to the interior of M having σ_i as the end invariant corresponding to the boundary component $\partial_i M$.

The second result is a generalization of Thurston's Dehn-filling theorem due to T.D. Comar and K. Bromberg (see [4], [6], [9]). We adopt the notation of [6].

Theorem 7.3. *Let M be a compact 3-manifold with k torus boundary components and assume M has a geometrically finite hyperbolic structure without rank one cusps. We have the following:*

1. *For each collection of relatively prime pairs*

$$(\mathbf{p}, \mathbf{q}) = (p_1, q_1; \ldots; p_k, q_k)$$

(except perhaps finitely many), there exists a geometrically finite hyperbolic metric on the $(\mathbf{p}, \mathbf{q})$-Dehn filling $M(\mathbf{p}, \mathbf{q})$ of M with $M(\mathbf{p}, \mathbf{q})$ having

the same end invariants as M.

2. *If X is a submanifold of M such that the complement of X is a neighborhood of cusps, then $f_{\mathbf{p},\mathbf{q}}|_X$ is $K_{\mathbf{p},\mathbf{q}}$-bi-Lipschitz. Also, $K_{\mathbf{p},\mathbf{q}}$ converges to 1 if $\min_i(p_i^2 + q_i^2)$ goes to infinity.*

We will also need the following theorem of McMullen (see [19]). Note that here a map is a K-quasi-isometry if the norm of its derivative is bounded from above by K and from below by $\frac{1}{K}$.

Theorem 7.4. *Let $N_i = \mathbb{H}^3/\Gamma_i, i = 1, 2$, be a pair of hyperbolic 3-manifolds, and let ψ be a K-quasi-conformal conjugacy between Γ_0 and Γ_1. Then ψ extends to an equivariant L-quasi-isometry $\psi : \mathbb{H}^3 \to \mathbb{H}^3$, where $L = K^{\frac{3}{2}}$. In particular, N_1 and N_2 are quasi-isometric manifolds.*

Corollary 7.5. *Let $N_1 = M(\sigma_1, \dots, \sigma_k)$ and $N_1 = M(\tau_1, \dots, \tau_k)$ be two hyperbolic manifolds. For every $B > 0$ there exists a constant $C > 0$ such that, if*

$$d_{\mathcal{T}(S)}(\sigma_i, \tau_i) < B \qquad i = 1, \dots, k,$$

then, for every closed curve γ in M,

$$\frac{1}{C} l_{N_1}(\gamma) < l_{N_2}(\gamma) < C l_{N_1}(\gamma).$$

For the rest of this section, let $M = S \times I$ where S is a hyperbolic surface of finite type. We denote $S \times \{0\}$ as $\partial_0 M$ and $S \times \{1\}$ as $\partial_1 M$. Also let $t_1, \dots, t_n \in [0, 1]$ be an increasing sequence of distinct numbers in the unit interval, and for $i = 1, \dots, n$, let α_i be a simple closed curve in S with the property that if i is different from j, α_i and α_j are not freely homotopic to each other. Define

$$\hat{M} = M \setminus \bigcup \alpha_i \times \{t_i\}.$$

Theorem 7.6. *$\hat{M}$ is hyperbolizable.*

Proof. Using Thurston's geometrization theorem, it is sufficient to show that $\hat{M}$ is aspherical, atoroidal and Häken. $\hat{M}$ is clearly Häken since it has a non-spherical boundary. We denote $\alpha_i \times \{t_i\}$ by $\bar{\alpha}_i$.

$\hat{M}$ is aspherical: Assume $\hat{M}$ contains an essential sphere; then, by the sphere theorem, M contains an embedded essential sphere, Σ. Since M is homotopy equivalent to S, $\pi_2(M)$ is trivial. Therefore Σ bounds a 3-ball, B, in M. We know that Σ does not intersect any of $\bar{\alpha}_i$, hence either the intersection of B with $\bar{\alpha}_i$ is empty or B contains $\bar{\alpha}_i$. But B is contractible; therefore it can not contain any of $\bar{\alpha}_i$ which are non-trivial in M. Hence B misses all α_i, and therefore Σ is not essential.

$\hat{M}$ is atoroidal: We have to prove that any essential torus in $\hat{M}$ can be pushed off to the boundary. Let $f : T^2 \to M$ be essential. Using the torus theorem we can assume f is an embedding. Since M does not have any essential tori, $f(T^2)$ bounds a solid torus, K, in M. Since $f(T^2)$ is disjoint from $\bar{\alpha}_i$, either $K \cap \bar{\alpha}_i$ is empty or K contains $\bar{\alpha}_i$. The annulus connecting $\bar{\alpha}_i$ to $\alpha_i \times \{0\}$ intersects $f(T^2)$ (after pushing off the trivial intersections) in a simple closed curve. Therefore α_i can be pushed to a curve in $f(T^2)$, that is, α_i is not knotted in K. Let γ be a curve in S homotopy equivalent to the core of K. Since $\bar{\alpha}_i$ is non-trivial in M, it is non-trivial in K also; therefore $\bar{\alpha}_i$ is homotopy equivalent to a power of γ. But α_i is a simple curve in S; hence it is in fact homotopy equivalent to γ itself. This implies that $\bar{\alpha}_i$ is homotopy equivalent in K to the core of K, and therefore the $f(T^2)$ can be pushed off to $\bar{\alpha}_i$.

□

Embedding $\hat{M} \subset M$ gives a homotopy equivalence between the two boundaries of $\hat{M}$, $\partial_0 \hat{M}$ and $\partial_1 \hat{M}$. We identify the Teichmüller spaces of the boundary components of $\hat{M}$ using this homotopy equivalence and denote it by $\mathcal{T}(S)$. For Dehn-filling coefficients

$$(\mathbf{p_n}, \mathbf{q_n}) = (1, n; 1, n; \ldots ; 1, n),$$

$M_n = \hat{M}(\mathbf{p_n}, \mathbf{q_n})$ is homeomorphic to M. Therefore there is a natural homeomorphism ϕ_n between boundary components of M_n. Since $\partial_0 M$ and $\partial_1 M$ are identified, we can think of $\phi_n : \partial_0 M \to \partial_1 M$ as an element of the mapping class group of S. It is easy to see that

$$\phi_n = D^n_{\bar{\alpha}_k} \circ \ldots \circ D^n_{\bar{\alpha}_2} \circ D^n_{\alpha_1},$$

where D_α represents the Dehn twist around the curve α and

$$\bar{\alpha}_i = D^n_{\bar{\alpha}_{i-1}} \circ \ldots \circ D^n_{\bar{\alpha}_2} \circ D^n_{\alpha_1} \alpha_i.$$

Therefore, for $\sigma \in \mathcal{T}(S)$, $M_n(\sigma, \sigma) = M(\sigma, \phi_n(\sigma))$.

Going back to our example, let S be a surface of genus two, and $\alpha_1, \ldots, \alpha_4$ be respectfully α, α', β and β'. Define M, $\hat{M}$, M_n and ϕ_n as above and let $N_n = M(g_n(0), g_n(2n))$. Note that the above definition for ϕ_n matches our previous definition. Also recall (Equations (7.3) and (7.4)) that

$$d_{\mathcal{T}(S)}(X_\infty, g_n(0)) < B \qquad \text{and} \qquad d_{\mathcal{T}(S)}(\phi_n(X_\infty), g_n(2n)) < B. \tag{7.5}$$

Proposition 7.7. *The length of γ in N_n is bounded below by a constant independent of n.*

Proof. Theorem 7.3 shows that $\hat{M}(X_\infty, \phi_n(X_\infty)) = M_n(\sigma, \sigma)$ converges to the cusp manifold $M(X_\infty, X_\infty)$ as $n \to \infty$ (note that γ is not one of the cusp curves); therefore there is a lower bound for the length of γ in $M_n(X_\infty, X_\infty)$. On the other hand, Equation (7.5) and Corollary 7.5 imply that the ratio of the length of γ in N_n and in $M_n(\sigma, \sigma)$ is bounded by a constant independent of n. This proves that there is a lower bound, independent of n, on the length of γ in N_n.

$\square$

Chapter 8

The Universal Curve and the Hyperbolic 3-space

In this chapter we prove $2 \Rightarrow 1$ in Theorem 1.2. Let (g, ν_+, ν_-) be a geodesic in the Teichmüller space of S and q_t be the corresponding quadratic differential on $g(t)$. Define U_g, the universal curve over g, to be the 3-manifold $S \times \mathbb{R}$ equipped with metric $d\rho^2$ defined by the equation

$$d\rho^2 = |q_t| + dt^2.$$

For each $t \in R$ we denote the surface $S \times \{t\}$ equipped with the metric $|q_t|$ by X_t. Let $\tilde{U}_g$ be the universal cover of U_g.

Lemma 8.1. *If $\tilde{U}_g$ is bi-Lipschitz to $\mathbb{H}^3$, then U_g is bi-Lipschitz to a hyperbolic 3-manifold $\hat{N}$ with ν_+ and ν_- as its end invariants.*

Proof. Consider the action of $G = \pi_1(S)$ on $\tilde{U}_g$. Conjugating this action by the bi-Lipschitz map between $\tilde{U}_g$ and $\mathbb{H}^3$ generates an action of G on $\mathbb{H}^3$ by bi-Lipschitz maps, and therefore a quasi-conformal action on $\hat{\mathbb{C}}$. By a theorem of

Sullivan [28], one can conjugate this action by a quasi-conformal map of $\hat{C}$ to an action of G by a Möbius transformation on $\hat{\mathbb{C}}$ and therefore by isometries on $\mathbb{H}^3$. Let Γ denote this group of isometries of $\mathbb{H}^3$, and let $\hat{N} = \mathbb{H}^3/\Gamma$. Note that $\hat{N}$ is bi-Lipschitz to U_g. Therefore short curves in $\hat{N}$ are also short in U_g (Corollary 1.6). Any end invariant of $\hat{N}$ is a limit of a sequence of short geodesics in $\hat{N}$ exiting on that end of $\hat{N}$; therefore $\hat{N}$ has the same end invariants as g does.

□

Lemma 8.2. *Let (g, ν_+, ν_-) be a geodesic in the Teichmüller space of S and (N, ν_+, ν_-) be a hyperbolic 3-manifold with the same end invariants as g such that U_g and N are bi-Lipschitz. Then N has bounded geometry.*

Proof. Let $f : U_g \to N$ be the K-bi-Lipschitz map between U_g and N. Assume N does not have bounded geometry, therefore, for every $\epsilon > 0$ there exists a curve α in S such that the hyperbolic length of α in N is less than ϵ, in fact, by making ϵ smaller we can assume $l_N(\alpha) = \epsilon$. Denote the geodesic representative of α in N by $\bar{\alpha}$. Let M be the Margülis tube corresponding to $\bar{\alpha}$. The distance between $\bar{\alpha}$ to the boundary of M is $r_0 = \log(\frac{\epsilon_M}{\epsilon})$, where ϵ_M is the Margülis constant for dimension 3. For $r \leq r_0$, the r neighborhood of $\bar{\alpha}$ in N $V(r)$ is inside M and

$$vol_N(V(r)) \asymp r^2(e^{\epsilon} - 1). \tag{8.1}$$

Let $\bar{t} \in \mathbb{R}$ be the time at which the $q_{\bar{t}}$-length of α is the shortest; Let $\alpha_{\bar{t}}$ be the geodesic representative of α in $X_{\bar{t}}$. If α has more than one geodesic representative, choose $\alpha_{\bar{t}}$ to be one of the boundary components of the flat annulus in $X_{\bar{t}}$ corresponding to α. Since U_g and N are K-bi-Lipschitz, we have $\frac{\epsilon}{K} \leq l_{q_{\bar{t}}}(\alpha_{\bar{t}} \leq K\,\epsilon$, therefore, the length of $f(\alpha_{\bar{t}})$ is less than or equal to $K^2\epsilon$. Hence

$$f(\alpha_{\bar{t}}) \subset V(K^2\epsilon). \tag{8.2}$$

Let $W(R)$ be the R-neighborhood of $\alpha_{\bar{t}}$ in U_g. Using (8.2) we have

$$f(W(R)) \subset V(K\,R + K^2\,\epsilon). \tag{8.3}$$

Let $A_{\bar{t}} = A \times \{\bar{t}\}$ be the largest regular annulus in $X_{\bar{t}}$ with $\partial_1 A_{\bar{t}} = \alpha_{\bar{t}}$ and $\kappa(\partial_1 A_{\bar{t}}) > 0$. Let R_0 be the $q_{\bar{t}}$-distance between the boundary components of $A_{\bar{t}}$. The closure of $A_{\bar{t}}$ in $X_{\bar{t}}$ contains a non-trivial curve that is different form α otherwise $A_{\bar{t}}$ would not be maximal. Therefore $f(A_{\bar{t}})$ has to exit the Margülis tube corresponding to $\bar{\alpha}$ because a non-trivial curve in M is homotopy equivalent to $\bar{\alpha}$. Now (8.3) implies

$$K\,R_0 + K^2\,\epsilon \geq r_0. \tag{8.4}$$

Therefore for $R \leq R_1 = \frac{r_0}{K} - K\,\epsilon$ we have

$$f(W(R)) \subset M.$$

Let $B_{\bar{t}} = B \times \{\bar{t}\}$ be the intersection of the $\frac{R_1}{3}$-neighborhood of $\alpha_{\bar{t}}$ in $X_{\bar{t}}$ with $A_{\bar{t}}$. Since $B_{\bar{t}}$ is expanding $area_{q_{\bar{t}}}(B_{\bar{t}}) \asymp R_1^2$. For ϵ small enough and $t \in [\bar{t}-1, \bar{t}+1]$, $B_t = B \times \{t\}$ is in R_1 neighborhood of $\alpha_{\bar{t}}$ (the distance between a point in B_t and $\alpha_{\bar{t}}$ is at most $e^1\,\frac{R_1}{3} + 1$). Also we have $area_{q_t}(B_t) \asymp R_1^2$ therefore

$vol_{U_g}(W(R_1)) \succ R_1^2$. But $f(W(R_1)) \subset M$ therefore

$$vol_n(M) \geq \frac{vol_{U_g}(W(R_1))}{K^3}.$$

Also $R_1 \asymp r_0$. Using Equation (8.1) we have

$$r_0^2(e^\epsilon - 1) \succ \frac{r_0^2}{K^3}.$$

But as ϵ approaches zero, the above inequality can not remain true. The contradiction proves our lemma. $\square$

Theorem 8.3. *If $\tilde{U}_g$ is bi-Lipschitz to $\mathbb{H}^3$, then any hyperbolic 3-manifold N with end invariants ν_+ and ν_- has bounded geometry.*

Proof. Lemma 8.1 states that U_g is bi-Lipschitz to some $(\hat{N}, \nu_+, \nu_-)$. Lemma 8.2 implies that $\hat{N}$ has bounded geometry. Now, since $\hat{N}$ and N have the same end invariants and $\hat{N}$ has bounded geometry, Theorem 1.3 (Minsky) proves that N also has bounded geometry. $\square$

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