# Einstein 4-manifolds with circle actions

A Dissertation, Presented

by

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to

The Graduate School

in Partial Fulfillment of the

Requirements

for the Degree of

Doctor of Philosophy

in

Mathematics

State University of New York

at Stony Brook

August 2001

State University of New York

at Stony Brook

The Graduate School

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Graduate School

# Abstract of the Dissertation,
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Mathematics

State University of New York

at Stony Brook


2001


A classification of compact Einstein 4–manifolds arising as warped products is proved under some extra conditions.

# Contents

To my parents.

# Acknowledgements

I would like to thank my advisor, Michael T. Anderson, for his guidance and help during my graduate studies. His insightful remarks and encouragement have been of immense help to me. I would also like to thank Claude LeBrun and Yair Minsky from whom I learnt a lot of mathematics through their excellent courses and discussions. Special thanks go to Siddhartha Gadgil for several helpful conversations. I would like to express my deep gratitude to B.S.Pandu Ranga Rao whose infectious enthusiasm and deep understanding of mathematics gave me the initial impetus to study this subject.

I would like to give special thanks to Adelaide Pangemanan for her continuous encouragement, understanding and support. I would also like to thank my friends Ramón del Castillo, Kasra Rafi, Susana Alba Core Bianchi,Lee-Peng Teo, Jason Behrstock, Chanyoung Sung and Sergei Panafidin.

I would like to give special thanks to my parents for the freedom, moral support and encouragement I received from them all my life.

# Chapter 1

# Introduction

A Riemannian manifold $(N, h)$ is called Einstein if it has constant Ricci curvature i.e. if $Ricci(h) = \lambda h$ where $\lambda$ is a constant. In dimensions 2 and 3 this condition implies that $h$ has constant sectional curvature. In dimension 4 the Einstein condition is much weaker but still has topological implications. It is not known if any manifold of dimension at least 5 admits an Einstein metric. In this work we restrict ourselves to dimension 4.

For the metric study of these manifolds it appears hopeless to attempt a classification without further restrictions: note that in two dimensions this is the vast subject of Teichmüller theory. Hence it is natural to assume the presence of a sufficiently "non-trivial" isometry group $G$ to begin with. The case $\dim G \geq 3$ is well-understood [1]. If we just assume $\dim G \geq 1$, for compact $N$ it is known that either $N$ is a flat 4-manifold or $\lambda > 0$. After normalizing we can assume $\lambda = 3$ (this is the value of $\lambda$ for $S^4$ with the standard metric).

The known simply connected examples are $S^4$, $S^2 \times S^2$, $CP(2)\#k\overline{C}P(2)$ where $k = 0, 1, 3, 4$. Here the first two have the standard metrics and the last has the Fubini-Study metric for $k = 0$, the Page metric for $k = 1$ and the

Tian-Yau metrics for $k = 3, 4$.

We are interested in classifying, up to isometry, a sub-class of the compact Einstein 4-manifolds with isometric $S^1$ actions. A description of this sub-class is as follows: Let $M$ be the orbit space of the action. It can be shown that $M$ is actually a 3-manifold with boundary. We can put a Riemannian metric on $M$ so that the projection from $N$ to $M$ is a Riemannian submersion. We consider the case when the horizontal distribution of this submersion is integrable. $h$ is then called a *warped product* in Riemannian geometry or *static* in General Relativity. In this case one can prove that the fixed point set $F$ of the action can be identified, via the projection map, with the boundary of $M$.

There are only two known examples of simply connected static Einstein 4-manifolds: $(S^4, std)$ and $(S^2 \times S^2, std)$. We study the question of whether these are the only possibilities.
It turns out that the geometry of $F$ is important for determining the isometry type of $(N, h)$. Our main result is that if the Gaussian curvature of $F$ satisfies certain conditions then the answer to our question above is in the affirmative. The proof uses the Bochner-Weitzenböck formula and the theory of minimal surfaces in 3-manifolds.

In section 2 we deal with the case of connected $F$ which corresponds to $(S^4, std)$ and in section 3 the other case i..e $(S^2 \times S^2, std)$ is dealt with.

# Chapter 2

# Motivation

Let $(N, h)$ be a compact Einstein 4-manifold with $Ricci(h) = \lambda h$. As mentioned in Section 1 if the dimension of the isometry group $G$ is at least 3 the possibilities for $(N, h)$ are known except for one case. More precisely, we have the following theorem of Berard-Bergery and A. Derdzinski:

**Theorem 2.0.1** *(Berard-Bergery, A. Derdzinski [1]) Let (N,h) be a compact Einstein 4-manifold. Assume that either*

*a) the dimension of G is at least 4 or*

*b) G= $T^3$, the 3-torus or*

*c) G= SO(3) and the principal orbits are $S^2$ or $RP^2$*

*Then either (N,h) is locally symmetric or the Page metric or it's $Z_2$ quotient.*

The Page metric is an Einstein metric on the non-trivial $S^2$ bundle over $S^2$. The only case not covered by this theorem is when $\dim(G)= 3$ and $G = SU(2)$ or $SO(3)$ with a 3-dimensional principal orbit. This case currently remains open .

Now consider the case $\dim(G)= 1$ or 2. This class has not been classified yet. In our study of $S^1$ actions we make the following assumption of "staticity"

: Firstly assume that the action is semi-free i.e. the stabilizer of any point is either the whole group or just the identity element. Let $F$ be the fixed point set, $M$ the orbit space of the free action of $G$ on $N - F$ and $\pi : N - F \longrightarrow M$ the projection map. Give $M$ the metric which makes $\pi$ a Riemannian submersion. Assume that the horizontal distribution of $\pi$ is integrable. Then $h$ is a warped product over a 3 dimensional base with fiber either a circle.

**Remark:** Note that a $(N, h)$ can admit an $S^1$ action for which it is static and another $S^1$ action for which it is not. For example, $S^1$ acts on $(S^4, std)$ by $(\alpha, (z_1, z_2, t)) \mapsto (\alpha z_1, \alpha z_2, t)$ and also by $(\alpha, (z_1, z_2, t)) \mapsto (\alpha z_1, z_2, t)$ where $\alpha \in S^1 \subset C$ and $(z_1, z_2, t) \subset R^5 = C \times C \times R$ The former action is not static because it has isolated fixed points which cannot happen for static actions by Lemma 3.0.7 below while the latter action can be checked to be static: in fact, it is equivalent to example 1) in Section 3.1.

We have defined warped products with circle fibers above. Similarly one can study the case of warped prodcuts for $T^2$ (the 2-torus) isometric actions. In this case the fibers are 2-torii. This case can be easily analyzed as follows. We follow [1] for this section. Suppose

$$(N, h) = (B \times T, g + f^2 g_0)$$

with

$$Ricci(h) = \lambda h.$$

Then it follows from O'Neill's formulae for Riemannian submersions [1] that

$$D^2 f = -\frac{1}{2}(2\triangle f - \frac{\|df\|^2}{f} - \lambda f)g \tag{2.1}$$

The above type of equation admits non-constant solutions only if the metric $g$ admits a non-zero Killing field. More precisely

**Lemma 2.0.2** *On a 2-manifold (B,g) the equation $D^2 f = \phi g$ admits a non-constant solution $f$ if and only if, locally at points where $df \neq 0$ there exist coordinates $(t, \theta)$ such that $f$ is a function of $t$ alone, $g = dt^2 + f'(t)^2 d\theta^2$ and $\phi = f''$.*

**Proof:** Suppose $df \neq 0$ at $p$. Let $r = \|df\|$ and let $X$ be a tangent vector to $f^{-1}(p)$. Then

$$\begin{aligned} D^2 f(X, \nabla f) &= X\nabla f(f) - \nabla_X \nabla f(f) \\ &= Xr^2 - < \nabla_X \nabla f, \nabla f > \\ &= Xr^2 - \frac{1}{2}Xr^2 \\ &= \frac{1}{2}Xr^2. \end{aligned} \tag{2.2}$$

On the other hand,

$$D^2 f(X, \nabla f) = \phi < X, \nabla f >= X(f) = 0$$

for all $X, Y$ which implies that

$$\nabla_X \nabla f = \phi X. \tag{2.3}$$

Also note that this implies

$$\nabla_N N = 0. \tag{2.4}$$

This is because

$$\begin{aligned}
\nabla_N N &= \frac{1}{r}\nabla_{\nabla f}\frac{\nabla f}{r} \\
&= \frac{1}{r}\left(\nabla f(\frac{1}{r})\nabla f + \frac{1}{r}\nabla_{\nabla f}\nabla f\right) \\
&= \frac{1}{r}\left(\frac{-1}{r^2} < \nabla f, \nabla r > \nabla f + \frac{1}{r}\phi\nabla f\right) \\
&= \frac{1}{r}\left(\frac{-D^2(N,N)}{r}\nabla f + \frac{1}{r}\phi\nabla f\right) \\
&= 0.
\end{aligned} \tag{2.5}$$

Now

$$[X, N] = \frac{< [X, N], X >}{\|X\|^2}X + < [X, N], N > N.$$

But

$$\begin{aligned}
< [X, N], N > &= < \nabla_X N, N > - < \nabla_N X, N > \\
&= \frac{1}{2}X < N, N > -(N < N, X > - < \nabla_N N, X >) \quad = 0.
\end{aligned}$$

To get the last equation above we have used (2.5).

Now

$$\begin{aligned}
< [X, N], X > &= < \nabla_X N, X > - < \nabla_N X, X > \\
&= -\frac{X(u)}{r^2} < N, X > + \frac{1}{r} < \nabla_X \nabla f, X > - \frac{1}{2} N r^2 \\
&= \phi r - D^2(N, N) r \\
&= 0.
\end{aligned} \tag{2.6}$$

Hence

$$[X, N] = 0.$$

By Frobenius' Theorem we can find coordinates $t$ and $\theta$ such that $\frac{\partial}{\partial t} = N$ and $\frac{\partial}{\partial \theta} = X$.

It's clear that $g = dt^2 + f'(t)^2 d\theta^2$ and since

$$D^2 \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial t} \right) = f''(t)$$

we have

$$\phi = f''.$$

$\square$

The equation

$$Ricci\,(h) = \lambda h$$

now becomes

$$2ff'' + f'^2 + \lambda f^2 = 0$$

in a neighbourhood of any point where $df \neq 0$.

The above ODE can be solved to generate complete *noncompact* solutions. In fact all the complete Einstein metrics arising as warped products over 2-dimensional bases have been classified [1].

However there are no *compact* solutions because the same reasoning as in Lemmas 3.0.8 and 3.0.11 (note that a warped product with torus fiber over a two-dimensional basis can be regarded as a warped product with circle fiber over a three-dimensional basis) shows that $f = 0$ somewhere and $\|df\| = 1$ wherever $f = 0$. This clearly violates the above equation.

Hence, in the static situation, the only remaining case is that of one-dimensional fibers. In the context of static manifolds there is one class which has been studied extensively in General Relativity, namely complete non-compact static Ricci-flat Lorentzian 4-manifolds. The most significant example of such a metric is the *Schwarzschild metric* on $N = [2m, \infty) \times S^2 \times R$ (where $m$ is any positive number) given by

$$h = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 g_{S^2} - \left(1 - \frac{2m}{r}\right) dt^2$$

We get a Riemannian metric with the same properties by just changing the sign of the last term. The associated metric $g$ on $M$ is given by $g = (1 - \frac{2m}{r})^{-1} dr^2 + r^2 g_{S^2}$. This metric is asymptotically flat, with mass $m$ i.e. it approaches the Euclidean metric on $R^3$, in the following precise way. There exists a compact set $K$ such that $M - K$ is diffeomorphic to $R^3 - ball$ and in these coordinates

$$h \sim \left(1 + \frac{2m}{r}\right) g_e + O\left(r^{-2}\right)$$

$$f \sim 1 - \frac{m}{r} + O(r^{-2})$$

as $r \to \infty$. In the above $g_e$ is the flat metric on $R^3$.

It turns out that, under these conditions, the Schwarzschild metric is rigid. In fact we have the following theorem.

**Theorem 2.0.3** *([3] W. Israel, D. Robinson, G. Bunting and A. Massoud-ul-Alam) Let (N,h) be a complete static Ricci-flat Lorentzian or Riemannian 4-manifold such that the 3-manifold (M,g) is asymptotically flat with mass m. Then (N,h) is isometric to the Schwarzschild metric with mass m.*

This theorem is one of the important "Black Hole Uniqueness" theorems. These theorems essentially characterize certain static or stationary solutions of Einstein's field equations. All of them have dealt with the non-compact case, due to physical considerations.

Our main theorem can be regarded as an attempt at proving a black hole uniqueness theorem in the compact case. It relates the geometry of the fixed point set $F$ to that of $N$, and characterizes the known examples in these terms. Similar to one of the proofs of the Black Hole uniqueness theorem we use a divergence identity in the proof.

First we have the following simple topological characterization of closed static simply-connected manifolds.

**Theorem 2.0.4** *Let $(N, h)$ be a closed simply-connected static 4-manifold. Then*

1. *F is a non-empty disjoint union of 2-spheres and projective planes.*

2. *N is homeomorphic to* $\#n(S^2 \times S^2)$ *or* $S^4$.

Our main theorem is

**Theorem 2.0.5** *Let* $(N, h)$ *be a closed non-flat static Einstein 4-manifold. Then*

1. *F is totally geodesic and has Gaussian curvature* $K \geq 1$.
   *If* $K = 1$ *at every point of F then* $(N, h)$ *is isometric to* $(S^4, std)$.
   *If* $K \leq 3$ *at every point of F and F has more than one component, then* $(N, h)$ *is isometric to* $(S^2 \times S^2, std)$ *or* $(S^2 \times RP(2), std)$.

2. *The function* $f = u^2 + \|du\|^2$ *on M attains it's maximum value 1 on* $\partial M$. *If f attains an interior global maximum then* $(N, h)$ *is isometric to* $(S^4, std)$.

# Chapter 3

# Preliminary Observations

Throughout the paper $(N, h)$ will be a closed oriented non-flat Einstein 4-manifold with $Ricci\,(h) = \lambda h$ and an isometric semi-free static $S^1$ action.The first observation is that the scalar curvature $s = 4\lambda$ has to be positive. This is because of the following classical result of Bochner [1]:

**Theorem 3.0.6** *Let $(M, g)$ be a compact Riemannian n-manifold with $Ricci\,(g) \leq 0$. Then any Killing field on $M$ is parallel.*

In particular if a Killing field vanishes at a point, it has to be identically zero.As will be shown below any Killing field on $N$ has to vanish somewhere. Let $F = \{x \in M : gx = x \; \forall g \in S^1\}$ be the fixed point set.

**Lemma 3.0.7** *The action cannot have an isolated fixed point.*

**Proof:** Let $p$ in $N$ be an isolated fixed point and $B$ be an open geodesic ball with center $p$ which is homeomorphic to an Euclidean ball. $B$ is invariant under the isometric $S^1$ action. By staticity $B - \{p\}$ is homeomorphic to $S^1 \times U$ where $U$ is an open set in the interior of $M$. This is impossible since $B - \{p\}$ is simply connected. $\square$

**Lemma 3.0.8** *$F$ is a non-empty disjoint union of closed totally geodesic surfaces.*

**Proof:** If $F$ is empty then the Euler characteristic $\chi(M) = 0$. But the four-dimensional Gauss-Bonnet theorem for an Einstein manifold states that

$$\int_M \|R\|^2 = 8\pi^2\chi(M)$$

where $R$ is the Riemann curvature tensor. This implies that $R = 0$ i.e. $M$ is flat.

Now we apply the following basic fact from Kobayashi [10].

*The connected components of the fixed point set of an isometric and semi-free $S^1$ action on a Riemannian n-manifold are totally geodesic submanifolds of even co-dimensions.*

In our case $F$ has to consist of points and surfaces. Combining this with the previous lemma we are done. □

**Remark:** Note that we have used the fact that $F$ is non-empty to conclude that *Ricci* $> 0$. If we know that *Ricci* $> 0$ we can prove that $F$ is non-empty as follows: If $F$ is empty $\chi(M) = 0$, as before. But $\chi(M) = 2 - \beta_1 + \beta_2$ where the $\beta_i$ are the Betti numbers. In particular $\beta_1 \neq 0$. This contradicts the fact that the fundamental group is finite.

**Lemma 3.0.9** *$F$ consists of 2-spheres and projective planes.*

**Proof:** Note that since $Ricci\,(h) \geq 3$, Myers's theorem implies that the universal cover $\tilde{N}$ is a finite cover of $N$. $\tilde{N}$ is again static with the pulled-back metric. If $\tilde{F}$ is the corresponding fixed point then $\tilde{F}$ is a finite cover of $F$ and it's enough to prove the result for $\tilde{F}$. Hence we can assume $N$ simply connected to begin with. First we prove that this implies that $M$ is also simply connected. This is clear since any curve in $M$ can be first homotoped to a curve in interior($M$) and then lifted to a curve in $N$ (since $N - F = M \times S^1$, $M$ can be regarded a submanifold of $N$) and finally the homotopy in $N$ contracting the lifted curve to a point can be pushed to a homotopy in $M$.

Now we prove that $H_1(M, Z/2) = 0 \Rightarrow H_1(\partial M, Z/2) = 0$. This follows from the long exact sequence for the pair $(M, \partial M)$:

$$\to H_2(M, \partial M) \to H_1(\partial M) \to H_1(M) \to$$

By Poincare Duality

$$H_2(M, \partial M) \cong H_1(M) = 0$$

and we are done. $\square$

**Remarks** :

1) Note that $N$ need not be simply connected - for example $S^2 \times RP(2)$ with the standard metric is Einstein with an isometric $S^1$ action given by rotation on the $S^2$ factor.

2) The above lemma fails completely, at least in the purely topological context, in case $N$ is not simply connected. Let $M$ be the 3-manifold with boundary

obtained by removing a ball from a closed 3-manifold and let $N$ be obtained by attaching $M \times S^1$ with $S^2 \times D^2$ along their boundaries. Note that $N$ has a $S^1$ action ( on the $S^2 \times D^2$ $S^1$ acts by rotating $D^2$ about it's center ) and $\pi_1(M) = \pi_1(N)$

**Corollary 3.0.10** *If $N$ is simply connected then it is homeomorphic to a connected sum of copies of $S^2 \times S^2$*

**Proof:** Since $N$ is simply connected so is $M$ by the proof of the previous lemma and hence $M$ has the homotopy type of $S^3$ with certain number of balls removed. $N$ is, up to homotopy type, $S^3 - balls \times S^1$ with a $S^2 \times D^2$ attached to every boundary component. But the latter is homotopy equivalent to a connected sum of copies of $S^2 \times S^2$. This can be seen by noting that this manifold is simply connected and has the intersection form on the second cohomology equivalent to that of $\#n(S^2 \times S^2)$ where $F$ has $n+1$ components and then using the following fundamental theorem of J.H.C.Whitehead [14]:

*If two closed simply-connected 4-manifolds have equivalent intersection forms, then they are homotopy equivalent.*

By the results of M.H.Freedman [6], homotopy type determines the homeomorphism type for this class of manifolds since both manifolds involved are spin. □

Let $M$ be the orbit space of the action i.e. $M = \frac{N}{S^1}$ Then by lemma 3.3 $M$ is a compact 3-manifold with boundary. The boundary of $M$ is exactly the fixed point set $F$ and staticity implies that $N - F = \text{interior}(M) \times S^1$. The metric on $N$ projects, under the projection map $N \longrightarrow M$, to a smooth metric

$g$ on $M$ Again by staticity

$$h = g + u^2 d\theta^2.$$

The equation $Ricci\,(h) = 3h$ is equivalent to the following two equations on $M$ [1]

$$Ricci\,(g) - \frac{D^2 u}{u} = 3g \tag{3.1}$$

$$\triangle u = -3u \tag{3.2}$$

where $D^2 u$ is the Hessian of $u : M \longrightarrow R$. Here $u$, as a function on $N$, can be interpreted as follows: $u(x)$ is the length of the orbit of $x$. In particular $u = 0$ on $F$. (3.1) then implies $D^2 u = 0$ on $F$. In particular $||\nabla u||$ is constant on each component of $F$.

**Lemma 3.0.11** $||\nabla u|| = ||du|| = 1$ *at every point of* $F$

**Proof:** First note that $u$ is a real analytic function on $M$ since it satisfies an elliptic P.D.E. Now (3.1) implies that if $\nabla u(p) = 0$ then all the higher derivatives are zero at $p$ which is impossible by analyticity. Therefore $\nabla u(p) \neq 0$. Let $X = \frac{\nabla u}{||\nabla u||}$ and let $L$ be the integral curve of $X$ beginning at $p$. Since the induced metric (in the 4-manifold $N$) on the two plane $L \times S^1$ i.e. $d\sigma^2 + u(\sigma)^2 d\theta^2$ is smooth where $L = \sigma(t), 0 \leq t \leq \epsilon$ with $\sigma(0) = p$ and $\sigma'(t) = X(t)$ we have $X(u)(p) = 1$ and $X^{(even)}(u)(p) = 0$. Since $X(u) = ||\nabla u||$ we are done. $\square$

In the above result it is essential that $(M, g)$ along with equations (3.1) and (3.2) comes from a 4-manifold with a smooth metric. i.e. there are 3-manifolds

$M$ with metric $g$ satisfying equations (3.1) and (3.2) where the result above fails. The following example is from [11] where it was constructed as a counter-example to the Fischer-Marsden conjecture. This stated that the map from the space of metrics on a compact manifold $M$ to the space of functions given by taking the scalar curvature of a metric is a submersion at any metric $g$ if $(M, g)$ is not isometric to $(S^n, std)$ or $Ricci\,(g) = 0$. This example, in a slightly modified form, was also studied in Relativity as a static solution of the vacuum Einstein equations with negative cosmological constant [2].

Let $M = I \times S^2(a)$ and $g = dt^2 + h(t)^2 g_{S^2(a)}$ where $I = [0, \frac{T}{2}]$ (the value $T$ will be explained below), $S^2(a)$ is the 2-sphere of scalar curvature $a$ and $h$ satisfies the O.D.E which ensures that $g$ has constant scalar curvature equal to 6 i.e.

$$3h^2 + 2hh'' + (h')^2 = \tfrac{a}{2} \qquad (*)$$

It can be checked that if $u = h'$ then (3.1) and (3.2) are satisfied. It can also be proved that there is an interval $A$ such that if $a \in A$ then there is a periodic solution, with period $T$, with $h'(0) = 0, h''(0) = 1$. It follows that $h' > 0$ on $(0, \frac{T}{2})$, $h'(\frac{T}{2}) = 0, h > 0$ on $[0, \frac{T}{2}]$.

Note that $F = \partial M = 0 \times S^2(a) \cup \frac{T}{2} \times S^2(a)$ and $||du||^2 = h''^2$.

Now we claim that Lemma 3.0.12 fails for this example.

CLAIM: There is no solution $h$ such that $h' = 0$ and $(h'')^2 = 1$ on $F$

**Proof:** Suppose $h$ is such a solution. Consider the function $e = 3u^2 + ||du||^2 = 3(h')^2 + (h'')^2$. We have

$$e' = 2h''(3h' + h''').$$

Differentiating equation $(*)$ we get

$$3h' + h''' = \frac{-2}{h} h' h''$$

Therefore

$$e' = \frac{-2}{h} h' (h'')^2 \leq 0$$

Since $e = 1$ at $t = 0$ and $t = \frac{T}{2}$ we get $e' = 0$ everywhere. Hence $h' = 0$ or $h'' = 0$ everywhere which is impossible since $h$ is a non-constant periodic function.

We will refer to this example as the Lafontaine metric.

# Chapter 4

# Proof of the Main Theorem

## 4.1 The Bochner-Weitzenböck formula

We first look at the two known examples of static actions:

**1)** $(N,h) = (S^4, std)$. In this case $(M,g) = (I \times S^2, dt^2 + sin^2(t) g_{S^2})$ where $I$ is the closed interval $[0, \frac{\pi}{2}]$ and $g_{S^2}$ is the standard metric on $S^2$. Also $(N,h) = (M \times S^1, g + cos^2(t) d\theta^2)$. So $u = cos(t)$ and $du = sin(t)dt$. Note that $u^2 + \|du\|^2 = 1$ in this case.

**2)** $(N,h) = (S^2(a) \times S^2(a), g_{S^2(a)} + g_{S^2(a)})$ where $a = \frac{1}{\sqrt{3}}$. Now $(M,g) = (I \times S^2, dt^2 + g_{S^2})$ where $I$ is the closed interval $[0, a\pi]$. $(N,h) = (M \times S^1, g + asin(\frac{t}{a}) d\theta^2)$. So $u = asin(\frac{t}{a})$. In this case $3u^2 + \|du\|^2 = 1$.

Using the formula (all the quantities considered here are on the 3-manifold $M$ with metric $g$)

$$\frac{1}{2}\triangle \|du\|^2 = \|D^2 u\|^2 + < \nabla \triangle u, \nabla u > + Ricci\,(\nabla u, \nabla u) \qquad (4.1)$$

and in consideration of the examples above we will prove

**Lemma 4.1.1** *The function $f = u^2 + ||du||^2$ cannot attain it's global maximum in interior$(M)$ unless it is constant and $(M, g)$ is isometric to a hemisphere in $(S^3, std)$.*

Before proving the lemma we show how it implies the main theorem.

**Corollary 4.1.2** *If $K(p)$ is the Gaussian curvature of $F$ at $p$ then $K(p) \geq 1$ $\forall p \in F$.*

**Proof:** Since $f$ attains it's global maximum on $\partial M = F$ (note that lemma 3.7 implies that $f = 1$ on $F$) and since $u$ attains it's global minimum on $F$, it follows that $< \nabla f, \nabla u > (x) \leq 0$ for all $x \in M$ sufficiently close to $p$.

Let $X = \frac{\nabla u}{||\nabla u||}$, as before.

We have

$$\begin{aligned} < \nabla f, \nabla u > &=< 2u\nabla u + 2||du||\nabla||du||, \nabla u > \\ &= 2u||du||^2(1 + \frac{< \nabla||du||, \nabla u >}{u||du||}) \\ &= 2u||du||^2(1 + \frac{< \nabla||du||, X >}{u}) \\ &= 2u||du||^2(1 + \frac{XX(u)}{u}) \end{aligned} \tag{4.2}$$

Therefore $\frac{XX(u)}{u} \leq -1$. Now $< X, X >= 1 \implies X < X, X >= 0 \implies < \nabla_X X, X >= 0$

If $X(x) \neq 0$ then $u^{-1}(u(x))$ is a smooth surface near $x$ and from the above equality we get: $\nabla_X X$ is tangent to $u^{-1}(u(x))$ and so $\nabla_X X(u) = 0$. Hence

$$\frac{XX(u)}{u} = \frac{D^2u(X,X)}{u}.$$

From (3.1) it follows that

$$\frac{-D^2u(X,X)}{u} = 3 - \mathit{Ricci}\,(X,X) = K(x)$$

where $K(x)$ is the (extrinsic) sectional curvature of the surface $u^{-1}(u(x))$ at $x$. Note that we have used the fact $g$ has scalar curvature 6 everywhere (which follows by taking trace on both sides of (3.1) and then using (3.2)) in the last equation.

Therefore $K(x) = \frac{-XX(u)}{u} \geq 1$. In the limit, as $x \to p$, we get $K(p) \geq 1$. Since $F = u^{-1}(0)$ is totally geodesic, $K(p)$ is also the intrinsic (Gaussian) curvature of $F$. $\square$

Now we prove Lemma 4.1.1.

**Proof:** Using (3.1) and (3.2) in (4.1),

$$\begin{aligned}
\frac{1}{2}\triangle\|du\|^2 &= \|D^2u\|^2 - 3\|\nabla u\|^2 + \mathit{Ricci}\,(\nabla u, \nabla u) \\
&= \|D^2u\|^2 + \frac{D^2u(\nabla u, \nabla u)}{u} \\
&= \|D^2u\|^2 + \frac{(\nabla u \nabla u(u) - \nabla_{\nabla u}\nabla u(u))}{u} \\
&= \|D^2u\|^2 + \frac{(< \nabla u, \nabla\|du\|^2 > -\frac{1}{2}\nabla u < \nabla u, \nabla u >)}{u} \\
&= \|D^2u\|^2 + \frac{1}{2u} < \nabla u, \nabla\|du\|^2 >
\end{aligned} \tag{4.3}$$

which implies that

$$\frac{1}{2}\, div\left(\frac{\nabla \|du\|^2}{u}\right) = \frac{\|D^2u\|^2}{u} \tag{4.4}$$

Now

$$\begin{aligned}\frac{1}{2}\Delta u^2 &= u\Delta u + \|du\|^2 \\ &= -3u^2 + \|du\|^2\end{aligned} \tag{4.5}$$

(using $\Delta u = -3u$)

This implies that

$$\frac{1}{2}\, div\left(\frac{\nabla u^2}{u}\right) = \frac{-3u^2}{u} \tag{4.6}$$

Adding equations (4.4) and (4.6) we get

$$\frac{1}{2}\, div\left(\frac{\nabla f}{u}\right) = \frac{1}{u}(\|D^2u\|^2 - 3u^2) \tag{4.7}$$

But

$$\|D^2u\|^2 \geq \frac{1}{3}(\Delta u)^2 = 3u^2$$

Hence $div\left(\frac{\nabla f}{u}\right) \geq 0$. Now, if $\phi : R \to R$ is any smooth, non-negative, increasing function, we have

$$div\left(\frac{\phi(f)}{u}\nabla f\right) = \phi(f)\ div\left(\frac{\nabla(f)}{u}\right) + \frac{\phi'(f)}{u}\|df\|^2 \geq 0$$

Suppose the maximum of $f$ is $m > 1$. Choose a smooth, non-negative, increasing $\phi$ so that $\phi(t) = 0$ if $t \leq 1$ and $\phi(t) = 1$ if $t \geq \frac{1+m}{2}$. Integrating the equation above, using Stokes' theorem and noting that the boundary integrals vanish because of the way we chose $\phi$, we see that $\nabla f = 0$ and $\|D^2(u)\|^2 = 3u^2$ on an open set. By analyticity $\nabla f = 0$ on $M$.

Hence $f = 1$ and $\|D^2(u)\|^2 = 3u^2$ on $M$.

To see the second part, note that $\|Ricci\,(g) - 2g\|^2 = 0$ from the above equation and (3.1). Next, we claim that $\partial M$ is connected. This follows from the following result of Frankel [5]:

*Let $(M, g)$ be a complete Riemannian manifold with positive Ricci curvature. Then any two compact minimal hypersurfaces in $M$ have to intersect.*

Hence $\partial M = S^2$ or $RP(2)$. Since the second Stiefel-Whitney class of $RP(2)$ is non-zero, it cannot bound a 3-manifold. So $\partial M = S^2$. Note that $M$ will have to simply connected. If not, consider the double of $M$. This inherits a $C^2$ metric, since $\partial M$ is totally geodesic, with $Ricci > 0$. The fundamental group of the double will be a free-product: in particular, it will be infinite which will contradict the fact that $Ricci > 0$, by Myer's theorem. Therefore $M$ is a hemisphere.

Combining the facts above we see that up to isometry $(M, g) = ([0, \frac{\pi}{2}] \times S^2, dt^2 + sin^2(t)g_{S^2})$.

From (3.1) we get $D^2(u) = -ug$. Restricting this any geodesic of the form

$\sigma(t) = (t, x)$ where $x \in S^2$ we get, after noting that

$$u\left(\sigma\left(\frac{\pi}{2}\right)\right) = 0, \qquad u(t,x) = cos(t)$$

and hence

$$(N,h) = \left(\left[0, \frac{\pi}{2}\right] \times S^2 \times S^1, dt^2 + sin^2(t)g_{S^2} + cos^2(t)g_{S^1}\right)$$

which is isometric to $(S^4, std)$ $\square$

**Remarks :**

**1)** An alternative proof of Lemma 4.1.1 is via the following "Robinson" type identity (these identities essentially state that a divergence quantity equal to a non-negative quantity, the latter of which is zero iff the metric is conformally flat) in [12]. :

$$div\left(\frac{\nabla f}{u}\right) = \frac{u^3}{4\|du\|^2}\|S\|^2 + \frac{3}{4u\|du\|^2}\|\nabla f\|^2$$

where $S$ is the "Weyl-Schouten" tensor of $g$ i.e. $S = dRic$. It is the analogue of the Weyl tensor in dimensions 4 and above and is zero if and only if $g$ is conformally flat.

In the case of interior maximum, note that the above proof used only the fact that $div(\frac{\nabla f}{u}) \geq 0$. Hence using the same argument will show that $f$ is constant and $S = 0$ on $M$. i.e. $(M, g)$ is conformally flat.

**2)**We get the following bound : $\|du\|^2 \leq f \leq 1$.

**Corollary 4.1.3** *If $K = 1$ at every point of $F$ then $(N, h)$ is isometric to $(S^4, std)$*

**Proof:** Using (3.1) and (3.2) in (4.7) gives

$$u\|Ric - 2g\|^2 = \frac{1}{2}\, div\left(\frac{\nabla f}{u}\right)$$

Integrating the above equation on $M$ and using the divergence theorem we get

$$\begin{aligned} \int_M u\|Ricci\,(g) - 2g\|^2 dV &= \frac{1}{2}\int_F \frac{<\nabla f, -\nabla u>}{u} dA \\ &= \int_F (K-1)dA \end{aligned} \tag{4.8}$$

Now if $K = 1$ at every point of $F$ then the right integral is zero and hence $Ricci\,(g) = 2g$ on $M$. So, again $g$ is locally isometric to $(S^3, std)$.
The rest of the argument is as above. □

## 4.2 The Case of Disconnected $F$ and Minimal Surfaces

The next theorem, which covers the remaining part of Theorem 2.4 is proved using techniques different from the above, namely the theory of minimal surfaces in 3-manifolds. This section relies heavily on the work of Galloway [7] and Cai and Galloway [4]. The arguments presented below are very slight variants of those in [7], [4].

**Theorem 4.2.1** *If $K \leq 3$ and $F = \partial M$ has more than one component then $(N, h)$ is isometric to $(S^2 \times S^2, std)$ or $(S^2 \times RP(2), std)$.*

**Proof:** The proof is based on the following two results:

A) Let $M$ be a Riemannian 3-manifold-with-boundary. If $M$ is not a handlebody and if no component of the boundary is stable minimal then $M$ contains a stable minimal surface in it's interior. In our case we can take this surface to be a 2-sphere or projective 2-plane.

B) Let $(M, g)$ satisfy (3.1) and (3.2). Then it cannot contain a stable minimal 2-sphere in it's interior unless it's a Riemannian product $(S^2, std) \times [a, b]$.

We sketch the proof (for details refer to [7] Lemma 3) of B) first.

Let $N$ be an unit normal vector field to $\Sigma$. Define the following variation of $\Sigma :$ for each $t \in (-\epsilon, \epsilon), \Sigma_t$ is the surface obtained by travelling in the direction of $N$ along the geodesics in the metric $u^2 g$ for time $t$. Let

$H = H_t =$ mean curvature of $\Sigma_t$,

$B = B_t =$ second fundamental form of $\Sigma_t$ and

$N = N_t =$ normal to $\Sigma_t$.

We have the following evolution equation for $B$, in Gaussian normal coordinates (in $u^2 g$) $t, x^1, x^2$. All the quantities are in metric $g$ :

$$\frac{\partial b_{ij}}{\partial t} = u^{-1} R_{i3j3} + u b_i{}^m b_{mj} + u_{;ij}, \quad 1 \leq i, j \leq 2 \tag{4.9}$$

where $R$ is the Riemann curvature tensor of $(M, g)$, $b_{ij}$ and $u_{;ij}$ are the components of $B$ and $D^2 u$ respectively.

Taking the trace of equation (3.2) we get

$$\frac{\partial H}{\partial t} = uRic_M(N,N) + u\|B\|^2 + \triangle_{\Sigma_t} u \tag{4.10}$$

where $\triangle_{\Sigma_t}$ is the Laplacian on $\Sigma_t$ with the metric induced from $M$.

If $A(t)$ =the area of $\Sigma_t$, then the first variation formula gives

$$A'(t) = -\int_{\Sigma_t} uH dA \tag{4.11}$$

We now obtain an evolution equation for $\frac{H}{u}$. Firstly

$$\triangle u = \triangle_{\Sigma_t} u + D^2u(N,N) - \frac{H}{u}\frac{\partial u}{\partial t} \tag{4.12}$$

Using (3.2)

$$\triangle_{\Sigma_t} u = -3u - D^2u(N,N) + \frac{H}{u}\frac{\partial u}{\partial t} \tag{4.13}$$

(3.1), (4.10) and (4.13) imply

$$\frac{\partial H}{\partial t} = u\|B\|^2 + \frac{H}{u}\frac{\partial u}{\partial t}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\frac{H}{u}\right) = \|B\|^2$$

$\Rightarrow \frac{H}{u}$ is increasing. Since $\Sigma$ is minimal $H_0 = 0$. Therefore $H_t \geq 0$ for $t > 0$. If $H > 0$ for some $t$ then $A'(t) < 0$ which contradicts the fact that $\Sigma$ is stable.

Hence

$$\frac{\partial H}{\partial t} = ||B||^2 = 0 \tag{4.14}$$

for all $t$ sufficiently close to 0.

Equations (3.1) and (4.14) imply that for all vectors $X, Y \in T_p\Sigma$

$$u^{-1}D^2_\Sigma u(X,Y) = Ricci_M(X,Y) - 3 < X, Y >= - < R(X,N)N, Y > \tag{4.15}$$

The last equation above and the definition of Ricci curvature imply that $Ricci(e,e) = \frac{1}{2}(K+3)$ where $e$ is any unit vector tangent to $\Sigma$ and $K$ is the Gaussian curvature of $\Sigma$.

By polarization we get

$$Ricci(X,Y) = \frac{1}{2}(K+3)g_\Sigma(X,Y) \tag{4.16}$$

Combining this with (4.15) gives

$$D^2_\Sigma u = \frac{1}{2}(K-3)ug_\Sigma \tag{4.17}$$

Galloway shows ( see [7] page 61 ) that the only solutions to the above equation are $u =$ constant (on $\Sigma$) which occur when $K = 3$.

Hence it follows, for example by the De Rham decomposition theorem, that a neighbourhood of $\Sigma$ is isometric to $S^2(\frac{1}{\sqrt{3}}) \times (-\delta, \delta)$ where $S^2(\frac{1}{\sqrt{3}})$ is the 2-sphere with the standard metric of radius $\frac{1}{\sqrt{3}}$. By analyticity of the metric this product structure extends globally and we are done.

In case one of the boundary components $\Sigma$ of $M$ is a stable minimal surface, we proceed as follows: First we note that $K \leq 3$ is equivalent to $Ricci\,(N, N) \geq 0$ on $\Sigma$ where $N$ is the inward pointing unit normal vector field on $\Sigma$. This follows from the fact the scalar curvature of $g$ is 6. We claim that in fact $Ricci\,(N, N) = 0$ on $\Sigma$. If not consider the variation of $\Sigma$ obtained by pushing it along the geodesics along the vector field $e^f N$ where $f$ is a smooth function on $\Sigma$ to be determined below. Using the same notation as above the derivative of the mean curvature at $t = 0$ is given by

$$\frac{\partial H}{\partial t} = Ricci\,(N, N) + \Delta f + ||df||^2$$

Now if $w = Ricci\,(N, N)$ then by Hodge theory we can solve the equation

$$\Delta f = c - w$$

where $c = \int_\Sigma w dA$. Note that $c > 0$ by our assumption. Hence

$$\frac{\partial H}{\partial t} = c + ||df||^2$$

In particular $\dfrac{\partial H}{\partial t} \geq c > 0$ at every point of $\Sigma$. So we derive the same contradiction as above.

Hence we have $Ricci\,(N, N) = 0$ on $F$.

Now by slightly modifying the proof of Cai and Galloway ( [4] Theorem B and the Claim on page 391) we can prove

**Theorem 4.2.2** *Let $(M, g)$ be a smooth Riemannian 3-manifold with constant scalar curvature 6. Suppose $\Sigma$ is an embedded stable minimal surface with area at least $\frac{2\pi}{3}$. If $N$ is a normal vector field on $\Sigma$, then, for the variation defined by $N$,*

$$(\|B\|^2)^{(n)}(0, x) = 0 \quad \forall x \in \Sigma$$

*where $f^{(n)}$ is the $n$-th derivative along the geodesics along $N$.*

The Claim of Cai and Galloway referred to above is :

*Let $(M, g)$ be a smooth Riemannian 3-manifold with nonnegative scalar curvature. Suppose $\Sigma$ is an embedded stable minimal torus. If $N$ is a normal vector field on $\Sigma$, then, for the variation defined by $N$,*

$$(\|B\|^2)^{(n)}(0, x) = 0 \quad \forall x \in \Sigma$$

*where $f^{(n)}$ is the $n$-th derivative along the geodesics along $N$.*

In our case, $g$ is analytic and it will follow that $B = 0$ for all the surfaces in the variation and we can proceed as earlier. $\square$

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